The geometry of conformally invariant random objects

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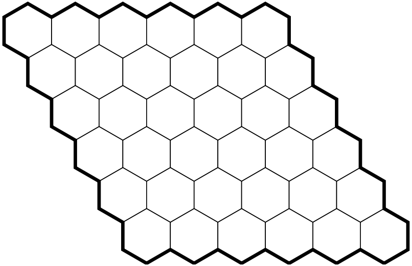
- The development of *conformal field theory* in the 1980s (Belavin, Polyakov, Zamolodchikov, Cardy, . . .) leads to a detailed non-rigorous understanding of these models.
- Oded Schramm's discovery of Schramm–Loewner evolutions in 2000 initiates a rapid expansion in the rigorous understanding of these models (Schramm, Werner, Lawler, Sheffield, Smirnov, . . .).

Outline

- What are examples of the models we wish to understand? What is meant precisely by conformal invariance?
- What are Schramm–Loewner Evolutions, and what are their properties?
- What are *conformal loop ensembles*, and what can be said about their geometry?
- How can one understand the geometry of random surfaces? What about random processes on those surfaces?

Criticality phenomena: percolation



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- Fix half the boundary white, and half black
- Flip a coin independently for each hexagon coloring it white with probability p, and black with probability $1 - p$.

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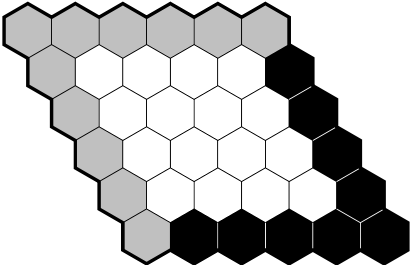

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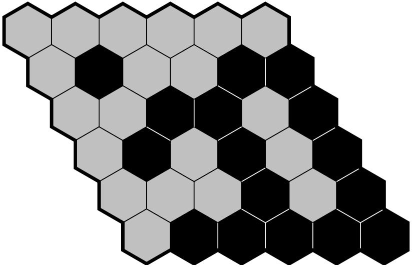

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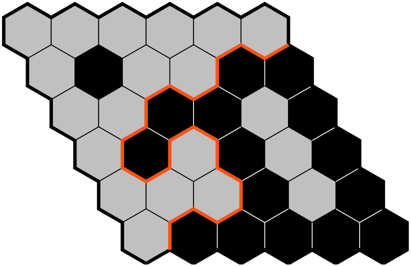
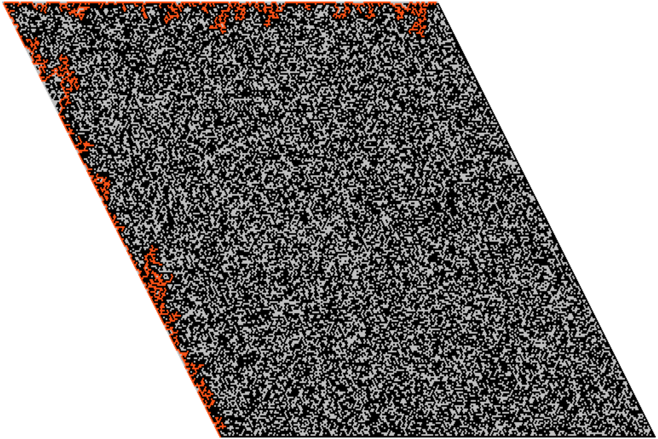
Macroscopic structure: the interface

Nothing interesting occurs with individual hexagons, however macroscopic structures can appear when you consider them jointly.

- We consider the *interface* which keeps white hexagons on the left and black hexagons on the right.



Percolation $p = 0.4$



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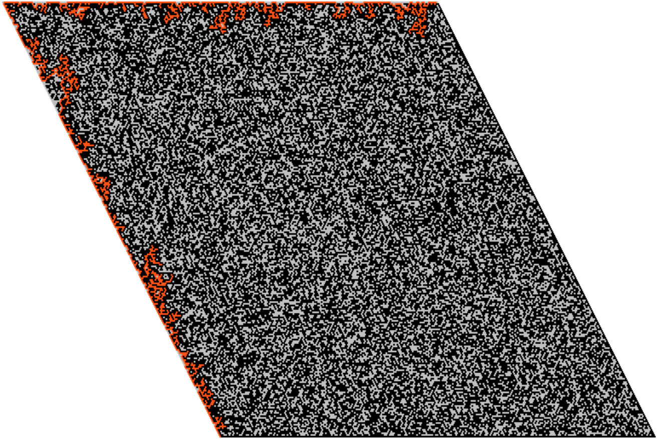
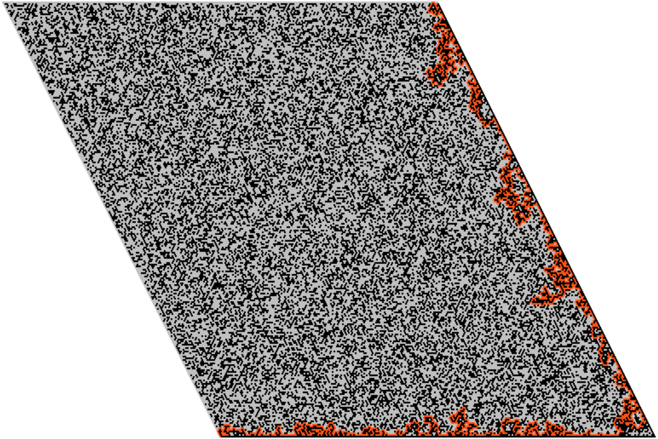


- p small forces the interface along the white boarder.

Percolation $p = 0.6$



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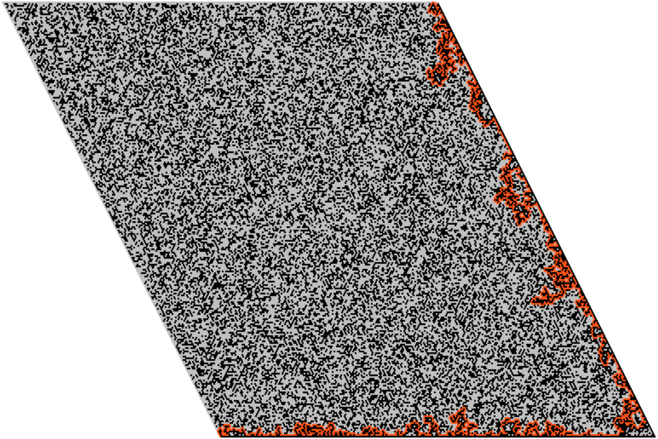
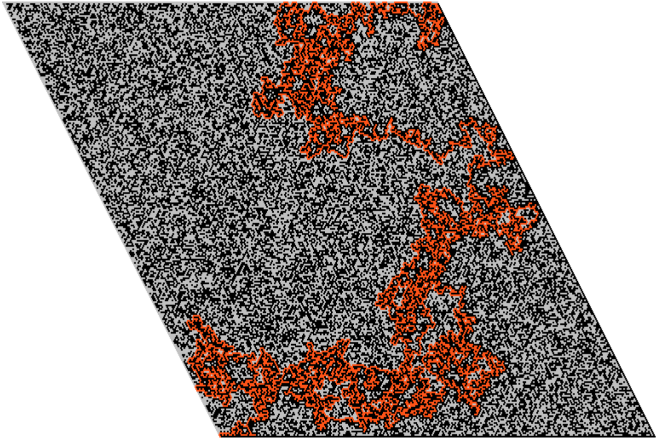


- p large forces the interface along the black boarder.

Percolation: $p = \frac{1}{2}$



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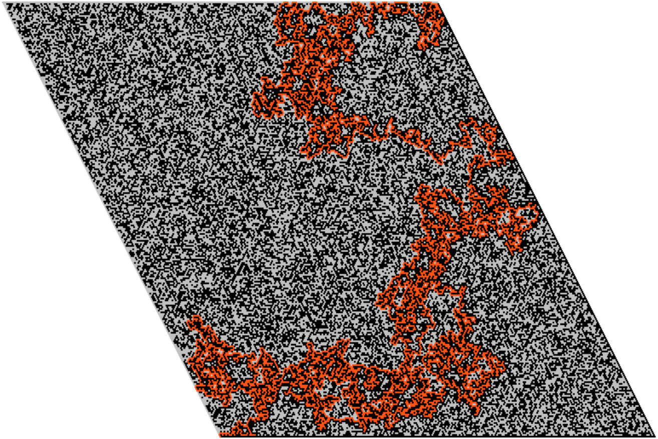


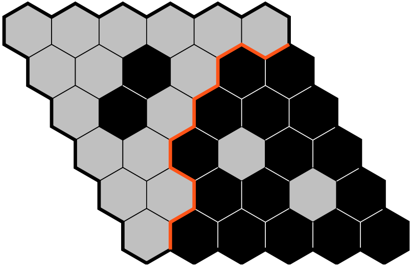
- p critical creates a non-trivial interface.

PERCOLATION: A PHASE TRANSITION

The picture changes dramatically as p changes:

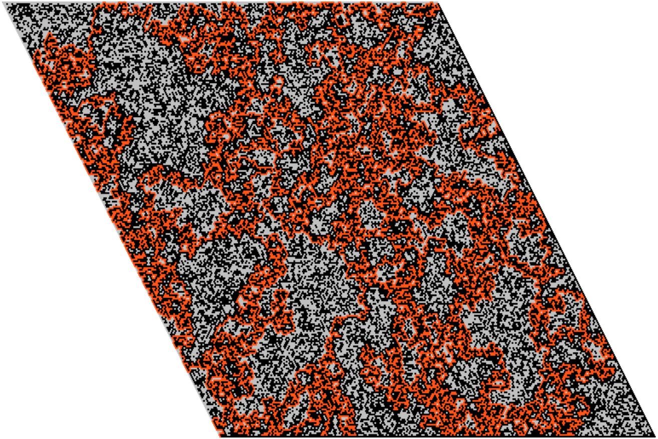
- When $p \neq \frac{1}{2}$, and the grid dimensions are sent to infinity, the interface converges to one of the two boundaries.
- When $p = \frac{1}{2}$, and the grid dimensions are sent to infinity, the interface converges (as probability measures on the space of curves) to a non-trivial limiting measure.
- This is an example of a *phase transition* where the observed quantity changes sharply as a function of p. The value of p where the change occurs is a *critical point*, and this value is often the most interesting.

Critical phenomena: Ising model

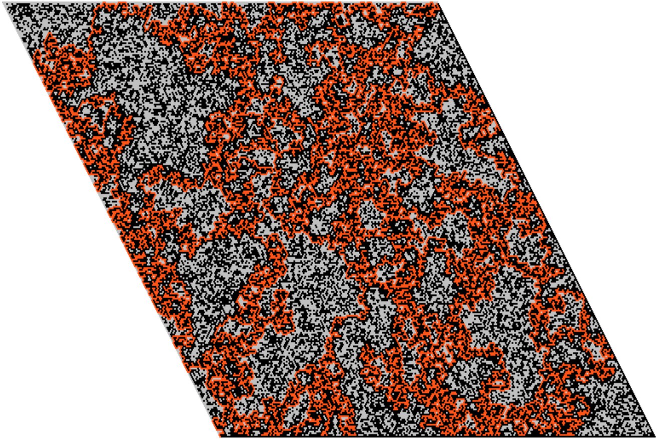


- Work on the hexagonal lattice with boundary
- Let N denote the number of edges where the two incident hexagons have different colors
- Now take the probability of a configuration to be proportional to $q^{N/2}$ for $q \in (0,1)$
- Again consider the interface

Ising model: $q = 0.8$

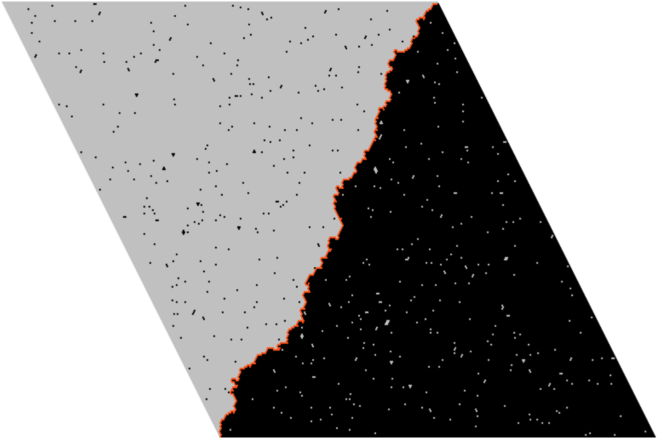


ISING MODEL: $q = 0.8$



- $q \approx 1$ couples weakly and behaves like percolation.

Ising model: $q = 0.2$



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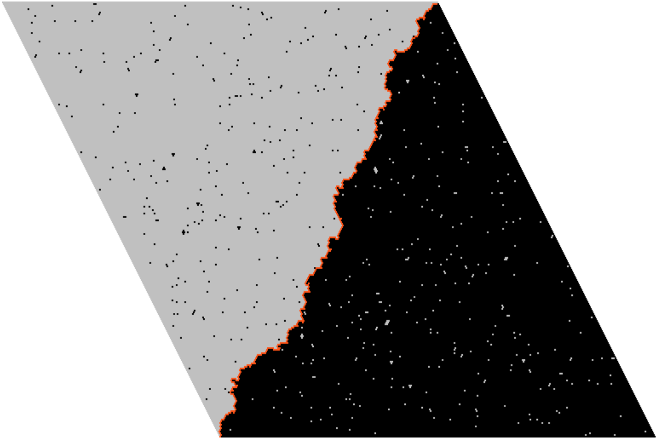
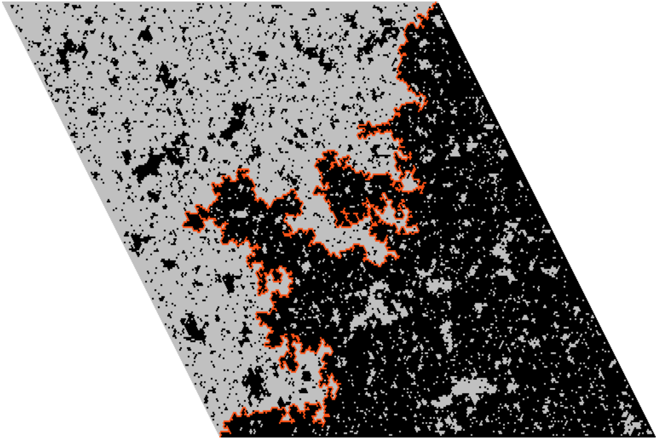


- $q \approx 0$ couples strongly and creates a trivial interface.

Ising model: $q = \frac{1}{3}$



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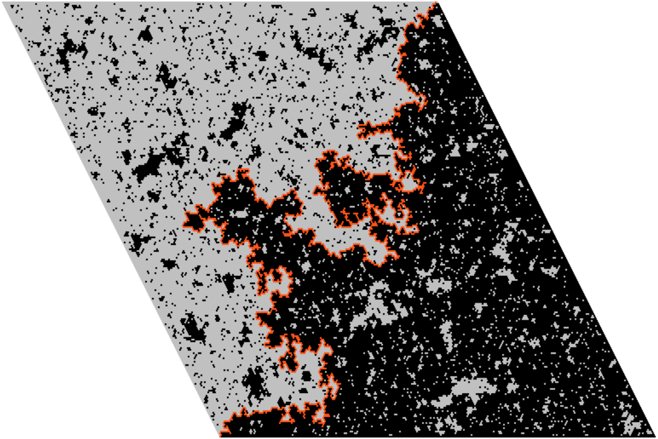


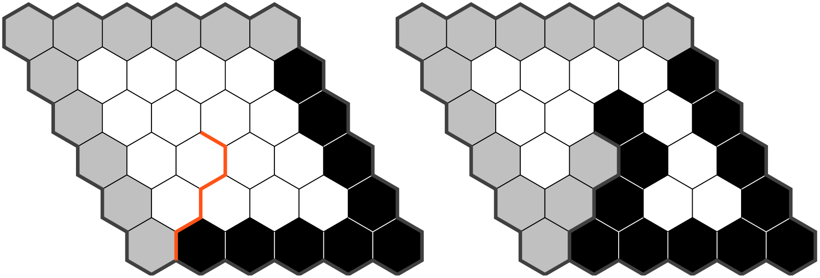
- q critical creates a new non-trivial measures on curves.

Scaling Limits: the setup

We will be concerned in this talk with *scaling limits*.

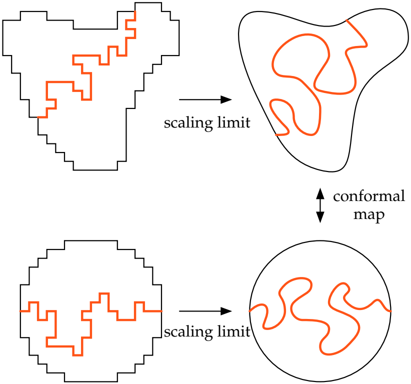
- We work (first) with probability measures $\mu_\Omega(z, w)$ on non-crossing curves within simply connected domains Ω connecting pairs of boundary points $z, w \in \partial\Omega$.
- These measures are designed to describe the limit of the measures discussed above as the lattice spacing is sent to zero.

Scaling Limits: domain Markov property



- Given a curve γ, the measure $\mu_\Omega(z, w)$ conditioned to begin with the segment $\gamma[0, t]$ is the same as $\mu_{\Omega \setminus \gamma[0,t]}(\gamma(t), w)$.

Scaling Limits: conformal invariance



- Taking a scaling limit followed by a conformal map is the same as taking the scaling limit in the mapped domain.
- For $f : \Omega \to f(\Omega)$ a conformal map:

$$\mu_{f(\Omega)}(f(z), f(w)) = f \circ \mu_\Omega(z, w)$$

Schramm's Breakthrough: Schramm–Loewner Evolutions (SLE)

Assuming these two axioms, these measures on curves may be completely characterized.

Theorem (Schramm 2000)

Suppose $\mu_\Omega(z, w)$ is a measure of non-crossing curves in a simply connected $\Omega \subset \mathbb{C}$ connecting $z, w \in \partial\Omega$ which satisfy the domain Markov property and conformal invariance. Then μ is one of a one parameter family of probability measures SLE_κ for $\kappa \geq 0$.

Menagerie of Examples

The collection of models that converge to SLE is quite diverse:

- Percolation exploration process ($\kappa = 6$) (Smirnov 2001)
- Loop-erased random walk ($\kappa = 2$) (Lawler, Schramm, Werner 2004)
- Uniform spanning tree ($\kappa = 8$) (Lawler, Schramm, Werner 2004)
- Level line of the Harmonic Explorer ($\kappa = 4$) (Schramm, Sheffield 2005)
- Level line of the Gaussian free field ($\kappa = 4$) (Schramm, Sheffield 2006)
- Ising model interface ($\kappa = 3$) (Smirnov, Chelkak 2011)
- FK-cluster boundaries ($\kappa = 16/3$) (Smirnov, Chelkak 2011)
- Self-avoiding walk ($\kappa = 8/3$) (conjectural)
- Double dimer model ($\kappa = 4$) (conjectural)
- Q-state Potts' model (conjectural)

The definition

- Given a curve γ from 0 to ∞ in $\mathbb{H}$, define $H_t = \mathbb{H} \setminus \gamma[0,t]$.
- Let $g_t : H_t \to \mathbb{H}$ be the conformal uniformizing map normalized so $g_t(z) = z + tz^{-1} + O(z^{-2})$ as $z \to \infty$.
- Then g_t satisfies the Loewner differential equation

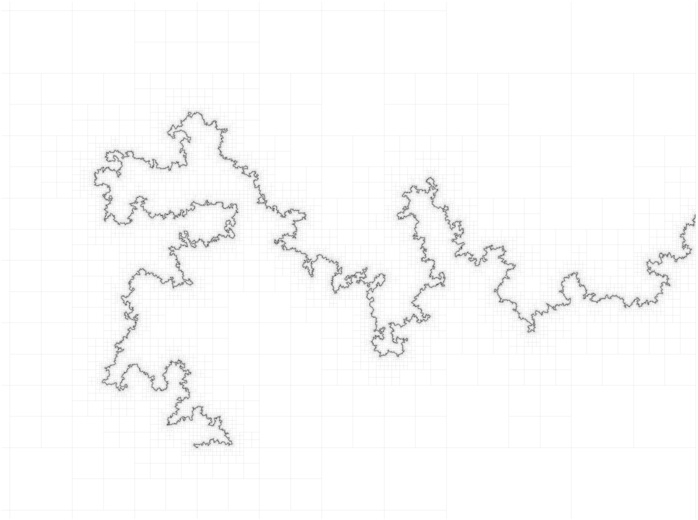
$$\dot{g}_t(z) = \frac{2}{g_t - W_t}, \qquad g_0(z) = z,$$

for some driving function $W_t : [0,\infty) \to \mathbb{R}$.

Definition

The *(chordal)* SLE_κ is the random curve produced when W_t is taken as a $\sqrt{\kappa}$ times a standard Brownian motion.

Simulation of an SLE_3



- A (two-sided whole-plane) SLE_3

Some Properties

Much is known about the geometry of these curves:

- SLE_κ curves exist and are simple for $\kappa \leq 4$, self-intersecting but non-crossing for $4 < \kappa < 8$, and space-filling for $\kappa \geq 8$ (Rohde, Schramm 2001).
- SLE_κ has almost sure Hausdorff dimension $d = 1 + \kappa/8$ (Beffara 2008).
- SLE_κ curves are reversible for $\kappa < 8$ (Zhan 2008; Miller, Sheffield 2012).

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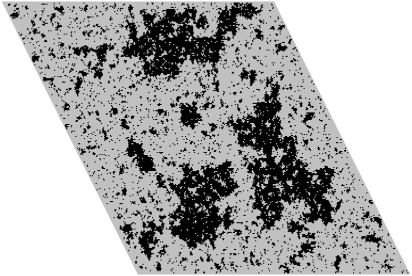
- SLE_κ has almost sure Hausdorff dimension $d = 1 + \kappa/8$ (Beffara 2008).
- SLE_κ curves are reversible for $\kappa < 8$ (Zhan 2008; Miller, Sheffield 2012).
- If γ is an SLE_κ curve, then the following limit exists (Lawler, W. 2013):

$$G(z, w) := \lim_{\varepsilon,\delta 0} \varepsilon^{d-2}\delta^{d-2}\mathbb{P}\{r_\gamma(z) < \varepsilon,\ r_\gamma(w) < \delta\}.$$

- SLE_κ curves may be reparametrized to be Hölder of any order $\alpha < 1/d$ for $\kappa \leq 4$. (W. 2012)

A related object: Conformal loop ensembles



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- Consider outer connected component of white hexagons
- This provides a random fractal carpet

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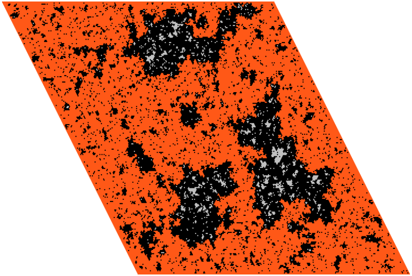

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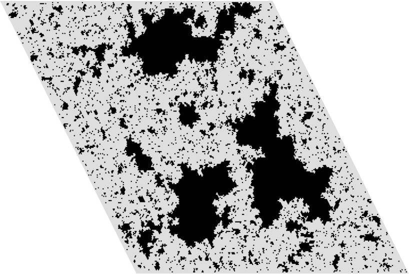

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- This provides a random fractal carpet

Conformal loop ensembles (CLE)

We will again have a pair of axioms on a probability measure μ on disjoint, non-nested, Jordan loops.

- *Conformal invariance:* The measure on the collection of loops satisfies $(\mu(f(\Omega)) = f \circ \mu(\Omega))$ for conformal f.
- *Restriction:* Given a simply connected subdomain $\Omega' \subseteq \Omega$, the collection of loops inside each component, Ω_i, of $\Omega' \setminus \{\overline{\text{int}(\gamma)} \mid \gamma \cap (\Omega \setminus \Omega') \neq \emptyset\}$ is $\mu(\Omega_i)$.

Theorem (Sheffield, Werner 2011)

There is precisely a one parameter family of such measures on loops parametrized by $\kappa \in (8/3, 4]$.

Carpets

To study the geometry of these random carpets, we look to the study of the quasi-conformal geometry of deterministic carpets for motivation.

Definition

A set T is a *carpet* if

- $\text{int}(T) = \emptyset$
- $T = \hat{\mathbb{C}} \setminus \bigcup_i int(D_i)$ where $\{D_i\}$ is a countable collection of pairwise disjoint closed Jordan regions with $\text{diam}(D_i) \to 0$.

Quasiconformal Maps

Given a function f, define

$$L_f(r,x) = \sup\{d(f(y),f(x)) \ : \ d(y,x) = r\}$$
$$l_f(r,x) = \inf\{d(f(y),f(x)) \ : \ d(y,x) = r\}.$$

Definition

A function f is *K-quasiconformal* if

$$\frac{L_f(r,x)}{l_f(r,x)} \leq K.$$

These can be thought of as weakened versions of conformal maps.

The uniformization of (nice) carpets

Theorem (Bonk 2011)

Suppose that $T = \hat{\mathbb{C}} \setminus \bigcup_i \operatorname{int}(D_i)$ *is a carpet such that:*

- *The* ∂D_i *are uniform quasicircles: the image of a circle under a K-quasiconformal map for a uniform choice of K,*
- *The* ∂D_i *are s-separated:*

$$\frac{\operatorname{dist}(\partial D_i, \partial D_j)}{\min\{\operatorname{diam}(\partial D_i), \operatorname{diam}(\partial D_j)\}} \geq s, \quad i \neq j.$$

Then there exists a quasiconformal map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ *with* $f(\partial D_i)$ *a circle for all i. If T moreover has measure zero, then such an f is unique up to Möbius transform.*

The rigidity of carpets from dynamics

Theorem (Bonk, Lyubich, Merenkov 2014)

Suppose $\mathcal{J}$ is a carpet obtained as the Julia set of a post-critically finite rational map. Then any quasisymmetry of $\mathcal{J}$ is the restriction of a Möbius transformation on $\hat{\mathbb{C}}$.

Deterministic uniformization of CLE?

How close are CLE_κ to satisfying Bonk's theorem?

- The boundary components of loops in CLE_κ are not quasi-circles.
- The boundary components of loops in CLE_κ are not s-separated.
- There cannot be a quasiconformal map uniformizing the CLE_κ carpet.

DETERMINISTIC UNIFORMIZATION OF CLE?

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- There cannot be a quasiconformal map uniformizing the CLE_κ carpet.

Intuitively, they should still somehow "hold on average."
Can a form of weakened a uniformization theorem be proven for CLE_κ carpets?

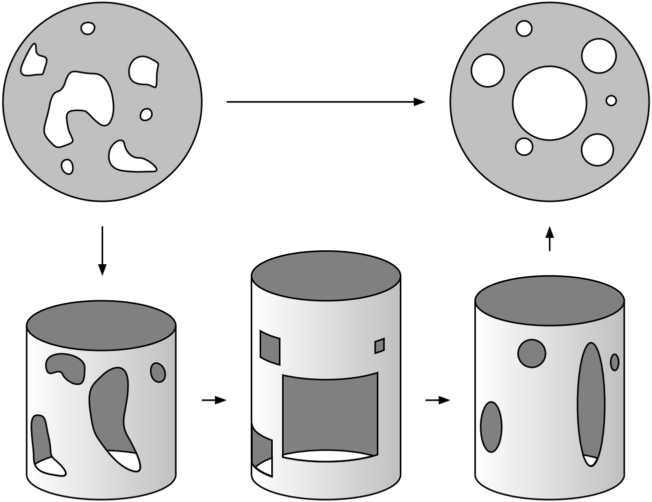
Partial uniformization of CLE

Given a carpet T with an enumeration of the complementary components $\{D_j\}_{j=1}^{\infty}$, let f_k be the conformal map uniformizing $\mathbb{D} \setminus \bigcup_{i=1}^{k} D_i$ to a circle domain in $\mathbb{D}$ sending $f(\partial D_1)$ to a circle centered at the origin and $f(\partial D_2)$ to a circle centered on the positive real axis (which exists and is unique by the Koebe Theorem).

Theorem (Rohde, W.)

Let T be a CLE_κ carpet in the unit disk for $\kappa \leq 4$. Then, with probability one, the following holds. Let $\{D_j\}_{j=1}^{\infty}$ be an enumeration of the complementary components of the CLE inside the disk. Then, for any n, $\lim_{k\to\infty}(f_k(\partial D_1), \ldots, f_k(\partial D_n))$ exists subsequentially (as centers and radii) and is non-degenerate (closures of the circles are disjoint from each other and the boundary of the unit disk).

THE STRUCTURE OF THE PROOF



The primary estimate

To obtain the control over the above forms of modulus, we must provide some form of average roundness of CLE loops.

Lemma (Rohde, W.)

Let γ_z *denote the* CLE_κ *loop surrounding the point* $z \in \mathbb{D}$*. Then we have that*

$$\mathbb{E}\left[\frac{\mathrm{diam}(\gamma_z)^2}{\mathrm{area}(\gamma_z)}\right]$$

is finite and integrable in z *over* $\mathbb{D}$*.*

The future: random surfaces

Given n unit squares, glue these squares together on the edges to produce a surface homeomorphic to the $\mathbb{S}^2$. There are finitely many ways to do this, so we may pick one such surface at random. What can be said about:

- The metric space structure? (Le Gall, Miermont 2010-)
- The conformal structure? (Polyakov 1981; Sheffield, Miller 2010-)
- Random models on the surfaces? (KPZ* 1988)

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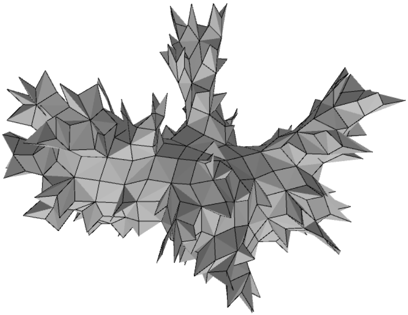
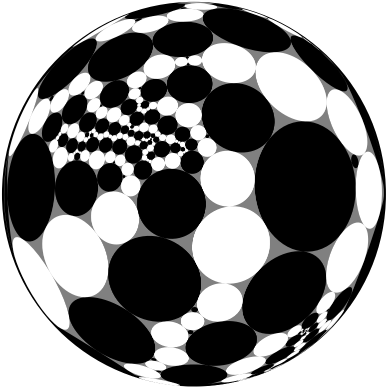
*Knizhnik, Polyakov, and Zamolodchikov

What do random surfaces look like?





Left image by personal code, right produced using CirclePack (Ken Stephenson).

Conformal structure: processes on surfaces

One of the primary philosophies in this area can be summarized with the following conjecture:

Conjecture (informal)

Any reasonable random surface model when properly coupled with a critical statistical physics model, and then conformally uniformized should yield the same random object (SLE *or* CLE) *as it would if it were on a deterministic geometry.*

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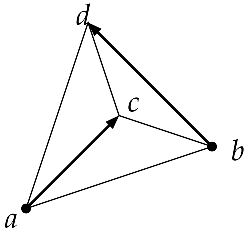
Many of the proofs of convergence in deterministic geometry need to prove that some function of the random model is *discrete holomorphic* and then show that these discrete holomorphic functions converge to true holomorphic functions.

Discrete holomorphicity

The following is dates to Isaacs in the 1940s.



Definition

A function f from a quadrilateral lattice Q to $\mathbb{C}$ is discrete holomorphic if for each face

$$\frac{f(c) - f(a)}{c - a} = \frac{f(d) - f(b)}{d - b}.$$

Convergence without global control

Existing results (Chelkak, Smirnov 2011; Skopenkov 2013) require global control on the size of faces. The following theorem requires only local control.

Definition

We will say that a quadrilateral is K-round if the ratio of all pairs of lengths is bounded above by K, and all angles are bounded below by $1/K$.

Theorem (W. 2014)

Fix K. Let $\{Q_n\}$ be a sequence of K-round orthogonal quadrilateral lattices approximating a domain Ω. Then, for any $C^1(\mathbb{C})$ boundary values g, the sequence of solutions to the discrete Dirichlet problem with boundary values g on Q_n converge uniformly to to solution to the Dirichlet problem in Ω with boundary values g.

Open Questions

- What more can be said about the conformal geometry of CLE gaskets? Is there a natural class of maps under which we have convergence of the Koebe maps?
- To what degree can local control be weakened in the discrete holomorphic convergence results? Is there even a sequence of lattices for which convergence fails?
- Many models of random surfaces are obtained by welding pairs of random trees. Can techniques from the conformal mating of dendrites or conformally embedded trees be applied to help understand random surfaces?

Thank you for your attention!