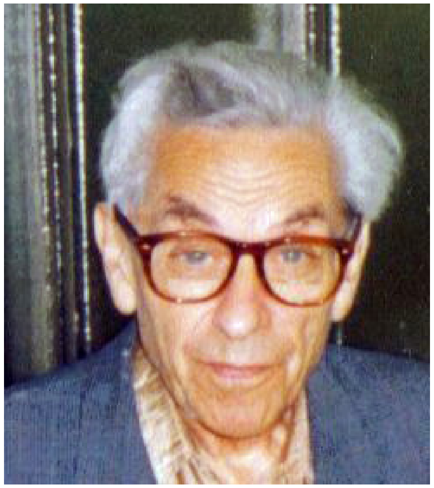
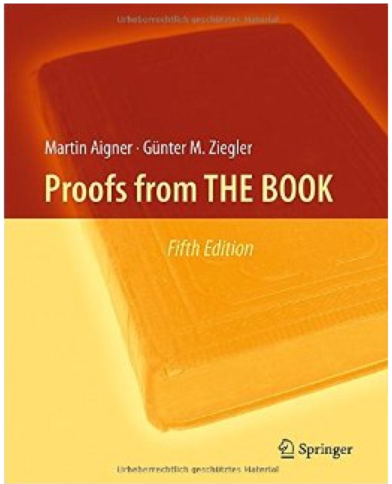
Proofs (not) from The Book

Stony Brook University, February 2015



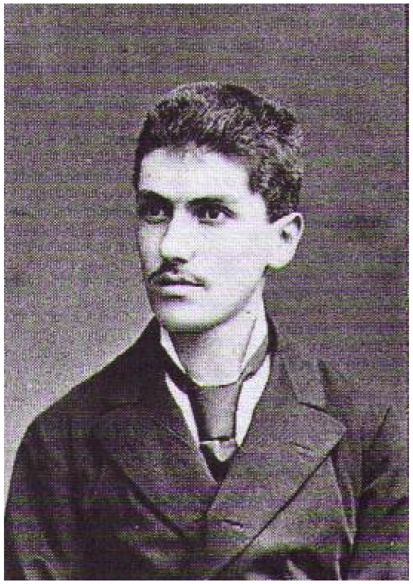


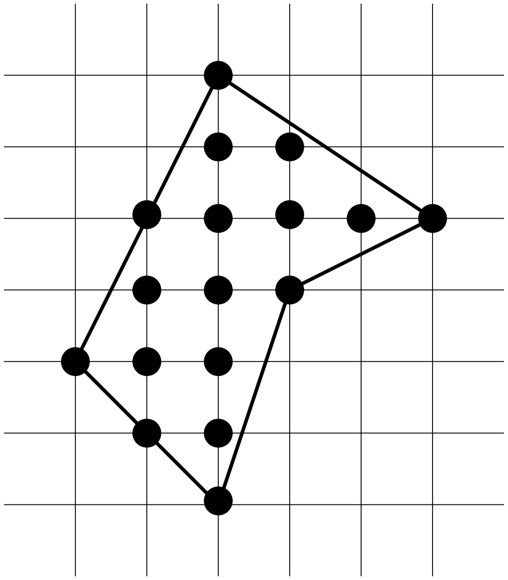
Paul Erdős and M. Aigner & G. Ziegler's book.

(My *Erdős Number* is 3).

Pick's Formula

Georg Alexander Pick (1859 – 1942).

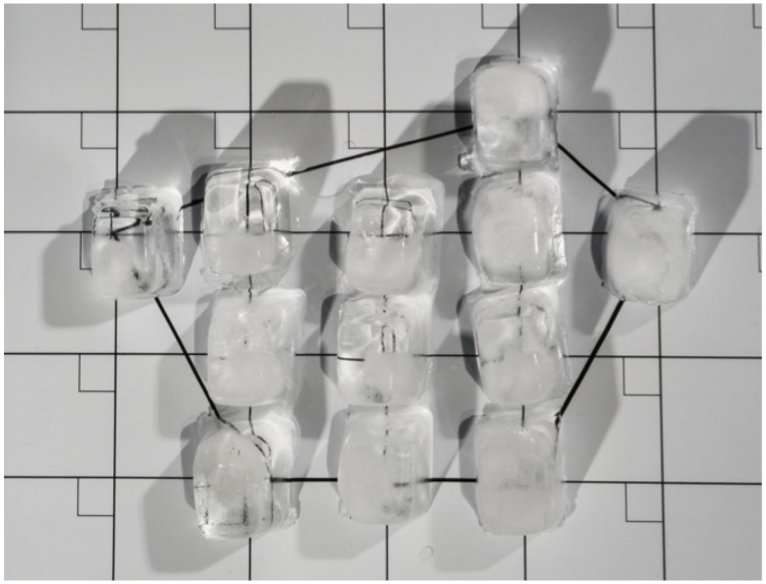




Theorem (1899): Area $= I + \frac{B}{2} - 1$.

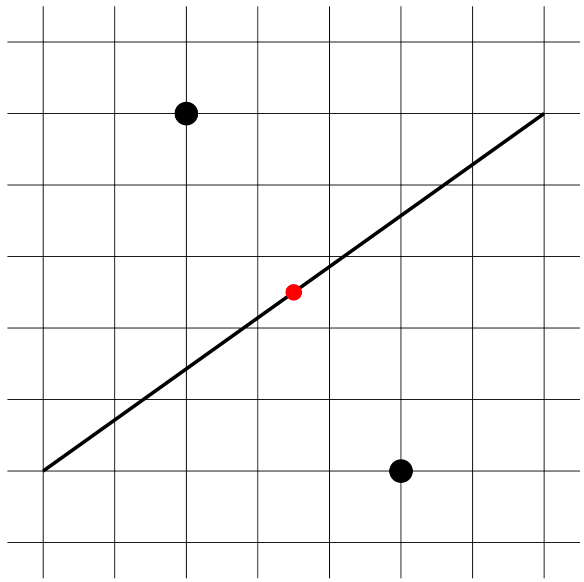
In the example, $I = 10, B = 7$, and Area $= 12.5$.

Proof by ice melting (C. Blatter, 1997).



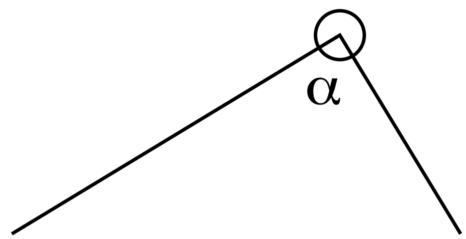
Where does this water come from?

Claim: *the mid-point of a side is a center of symmetry of the lattice*.



Hence the sides might have been water-proof as well!

Let's count: points inside the polygon contribute everything, points inside the sides contribute a half, and the vertices contribute $\alpha/2\pi$:



Now,

$$\alpha_1 + \alpha_2 + \ldots + \alpha_n = (n-2)\pi,$$

hence

$$\frac{\alpha_1 + \ldots + \alpha_n}{2\pi} = \frac{n}{2} - 1,$$

as needed.

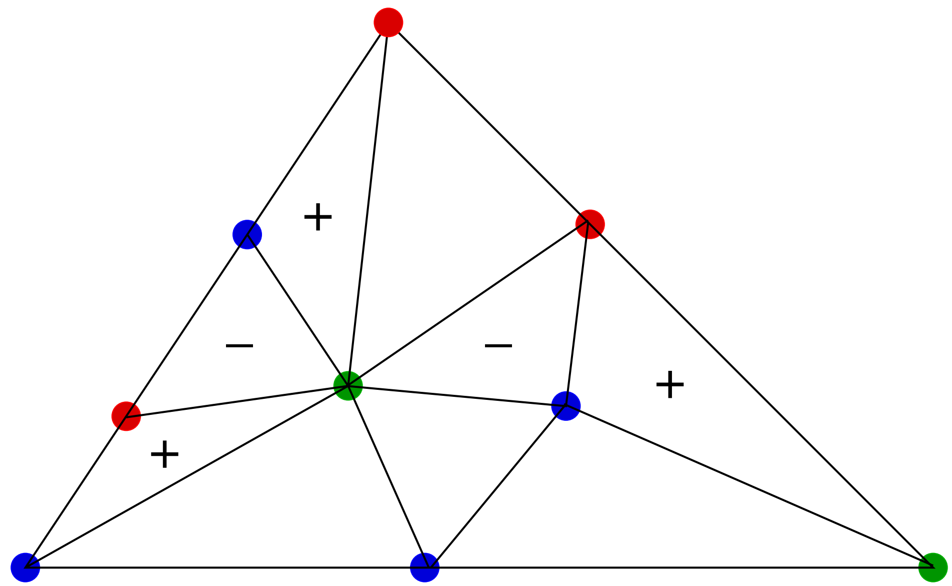
Multi-dimensional bonus

Pick's formula does not hold in higher dimensions, starting with 3-space (can you come up with an example?)

However, the ice melting argument works for polytopes whose facets are centrally symmetric (A. Barvinok and J. Pommersheim, 1999).

Sperner's Lemma

Emanuel Sperner (1905 – 1980).



Theorem (1928): *the algebraic number of triangles is 1*.

Proof by areas (A. McLennan and R. Tourky, 2008).

The signed area of triangle ABC is

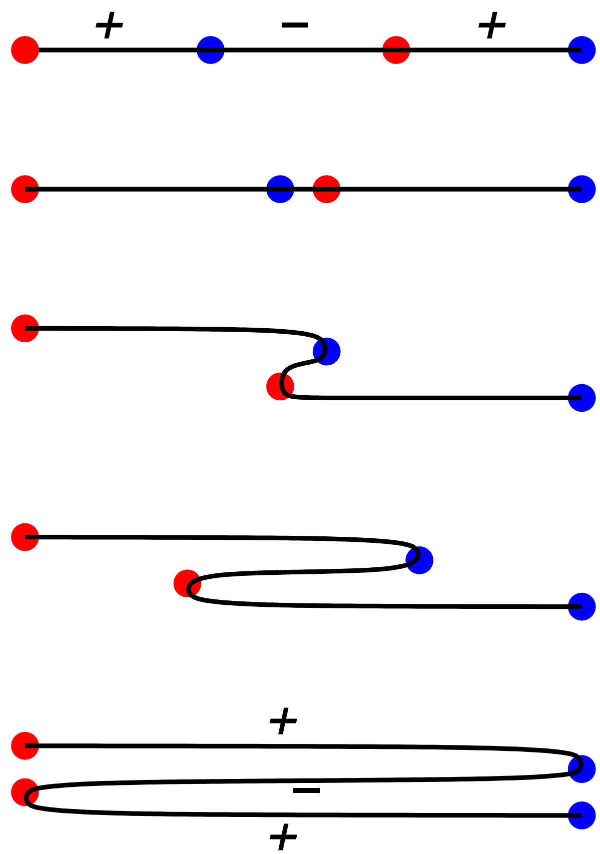
$$\frac{1}{2}\det|B-A,C-A|.$$

If A, B and C depend linearly on time t, then Area $(\triangle ABC)$ is a quadratic polynomial in t. And so is the sum of areas of all triangles involved, say, $P(t)$.

What is the value of $P(t)$ for small t? It's 1. Hence $P(t) \equiv 1$.

And what is $P(1)$? It's the number of positive triangles minus the number of negative ones.

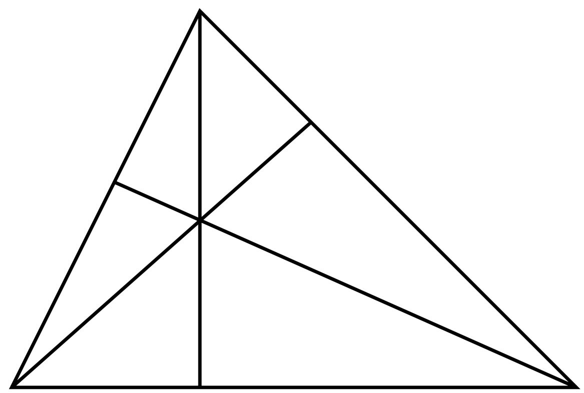
For example, in dimension 1:



And likewise in any dimension.

Altitudes of a spherical triangle

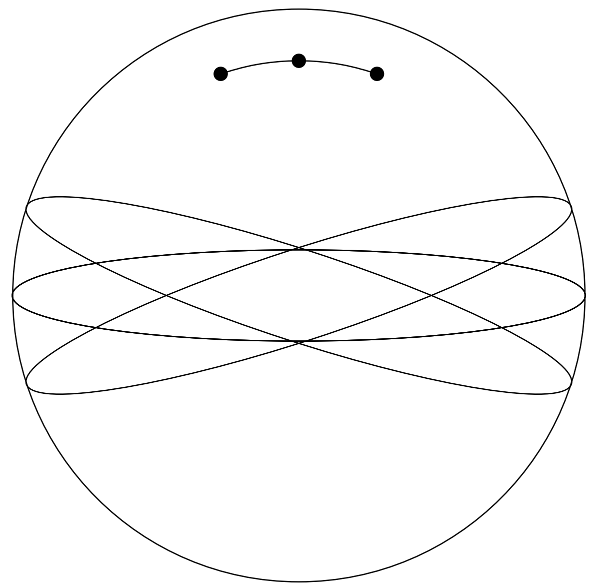
In the Euclidean plane:



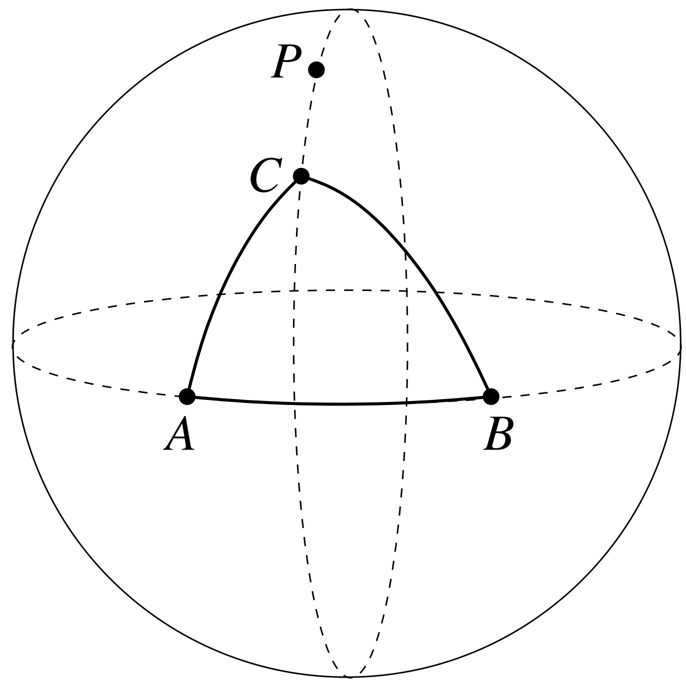
(Do you remember how to prove it, in Euclidean geometry?)

Proof by the Jacobi identity by Vladimir Arnold (1937 – 2010).

Spherical duality: points $\longleftrightarrow$ great circles.



The points are collinear if and only if the lines are concurrent.



Thus $P \sim A \times B$, and the altitude $PC \sim (A \times B) \times C$.
The three altitudes are concurrent iff the (normalized) points

$$(A \times B) \times C,\ (B \times C) \times A,\ \text{and } (C \times A) \times B$$

are collinear, or these three vectors are linearly dependent.

And indeed,

$$(A \times B) \times C + (B \times C) \times A + (C \times A) \times B = 0,$$

the Jacobi identity for the cross-product (the Lie algebra of motions of the sphere)!

In the hyperbolic plane, a similar argument works, with the Lie algebra $so(3)$ replaced by the Lie algebra of motions of the hyperbolic plane, $sl(2, \mathbf{R})$. Surprisingly, this *does not* work in the Euclidean plane.

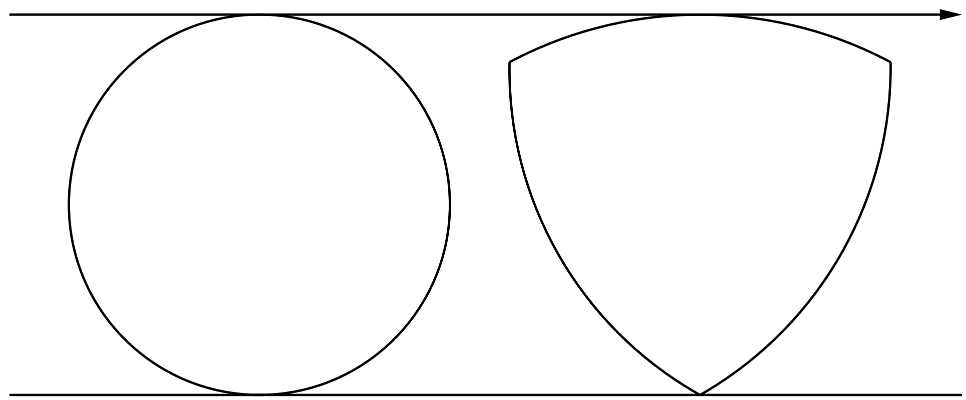
Barbier's Theorem

Joseph-Émile Barbier (1839 –1889).

Theorem (1860): *The perimeter length of a curve of constant width* w *equals* πw.

Proof by rolling.

Move the upper line with speed v.



The angular velocities of the "wheels" are equal to v/w.
The displacement of the line equals the perimeter length of the wheel times the number of turns. The latter is the same, since their angular speeds are equal, hence the perimeter lengths are the same.

Two mechanical realizations:

and

(how does it work?!)

Fáry's Inequality

István Fáry (1922 – 1984).

The *average absolute curvature* of a plane curve γ is

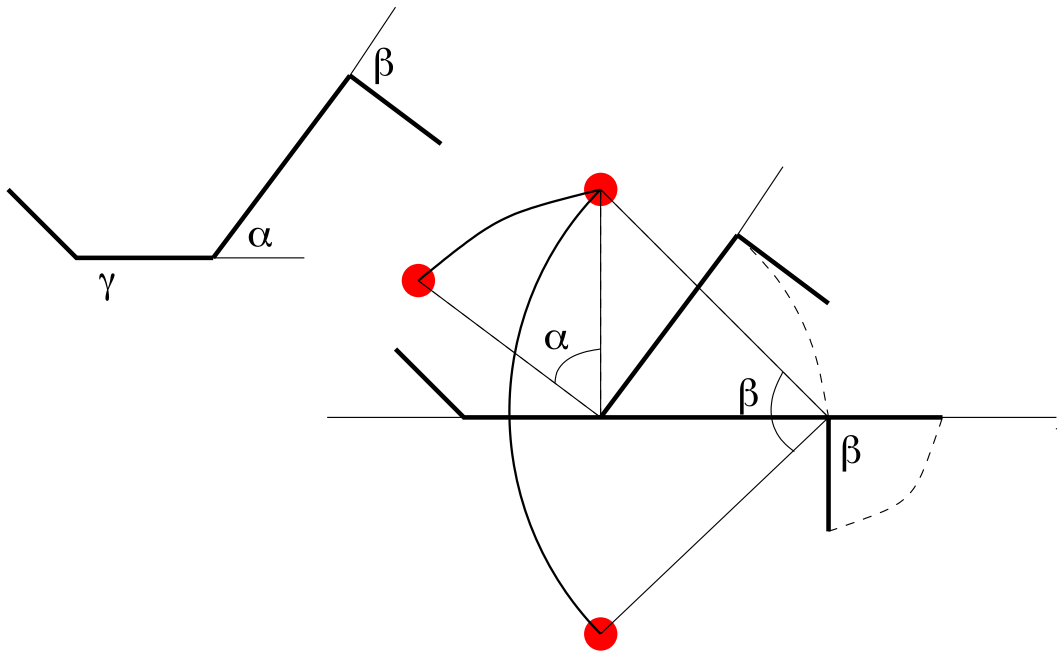
$$\frac{\int_\gamma |\kappa(s)|\ ds}{L(\gamma)}$$

(the amount of turning per unit of length).

NB: not the same as $\int_\gamma \kappa(s)\ ds$, which is a multiple of 2π!

Theorem (1950): *if a curve lies inside a unit disk then the average absolute curvature is not less than 1.*

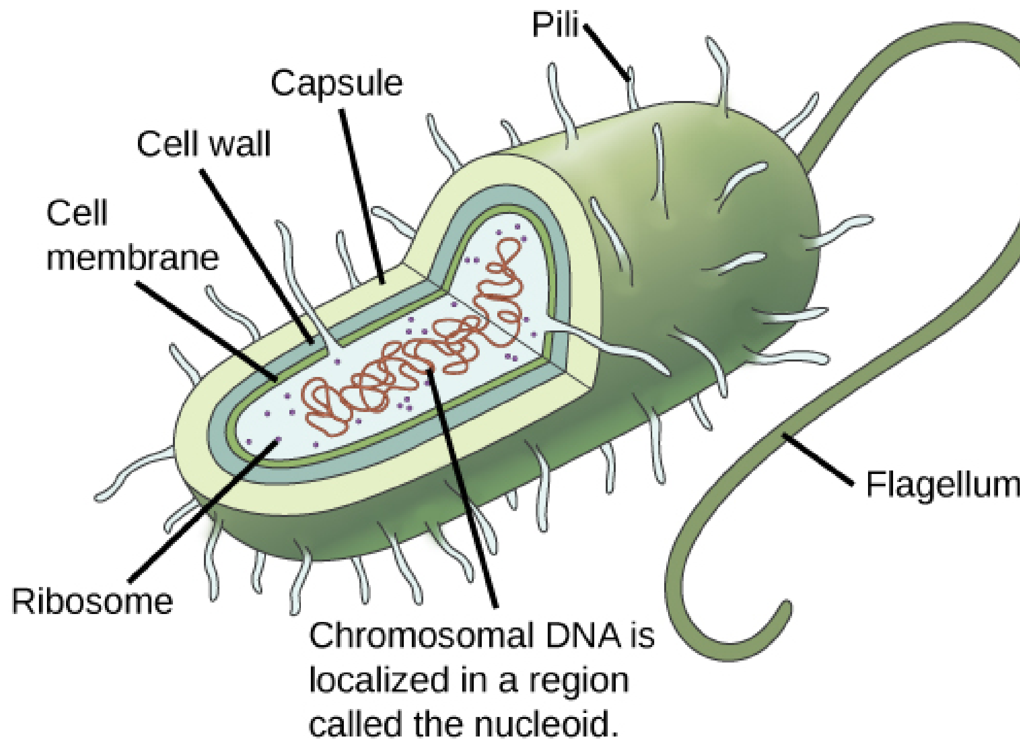
Proof by unfolding (for polygonal curves) (MMO 1973).



The total displacement of the center is $L(\gamma)$. The trajectory of the center consists of arcs, each not greater than α_i. Hence

$$\int_\gamma |\kappa(s)| = \sum \alpha_i \geq L(\gamma).$$

Generalization (DNA Theorem): no 'Book proof' available yet...



If γ lies inside a convex Γ, then the average absolute curvature of γ is not less than that of Γ (J. Lagarias, T. Richardson, 1997).

Sturm-Hurwitz Theorem

Jacques Charles François Sturm (1803 – 1855) and Adolf Hurwitz (1859 – 1919).

A trigonometric polynomial of degree N is a 2π-periodic function

$$f(x) = c + \sum_{k=1}^{N} a_k \cos kx + b_k \sin kx.$$

Easy fact: a trigonometric polynomial of degree N has at most $2N$ roots.

Theorem (Sturm 1836, Hurwitz 1903): *If*

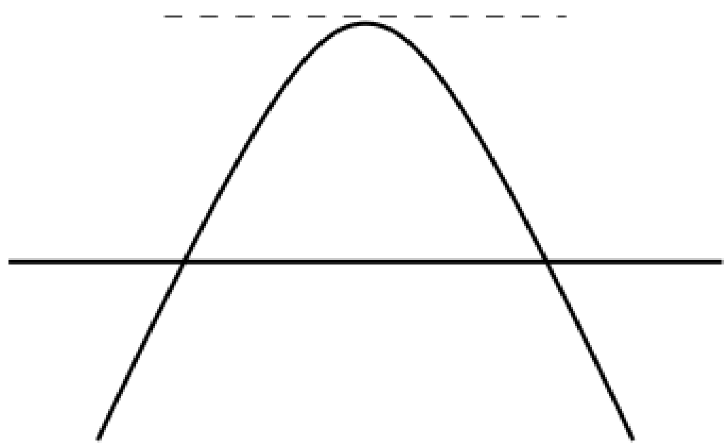
$$f(x) = \sum_{k=n}^{N} a_k \cos kx + b_k \sin kx,$$

then the number of sign changes of f is at least $2n$.

More generally, and in words: *the number of roots of a periodic function is not less than the number of roots of its first harmonic*.

Proof by Rolle's theorem (G. Katriel, Y. Martinez-Maure 2003).

Rolle's theorem for periodic functions: $Z(f') \geq Z(f)$



(the derivative changes sign between consecutive sign changes of a function).

Equivalently,

$$Z(f) \geq Z(D^{-1}f),$$

where D^{-1} is the inverse derivative, defined on the space of functions with zero average (only such functions are the derivative of *periodic* functions).

What does D^{-2} do to kth harmonic? Divides by $-k^2$.

After rescaling (keeping nth harmonic fixed), we get a sequence

$$f_m(x) = (a_n \cos nx + b_n \sin nx) + \sum_{k=n+1}^{N} \left(\frac{n}{k}\right)^{2m} (a_k \cos kx + b_k \sin kx)\,.$$

By Rolle's theorem, $Z(f) \geq Z(f_m)$.

As $m \to \infty$, the function $f_m(x)$ tends to

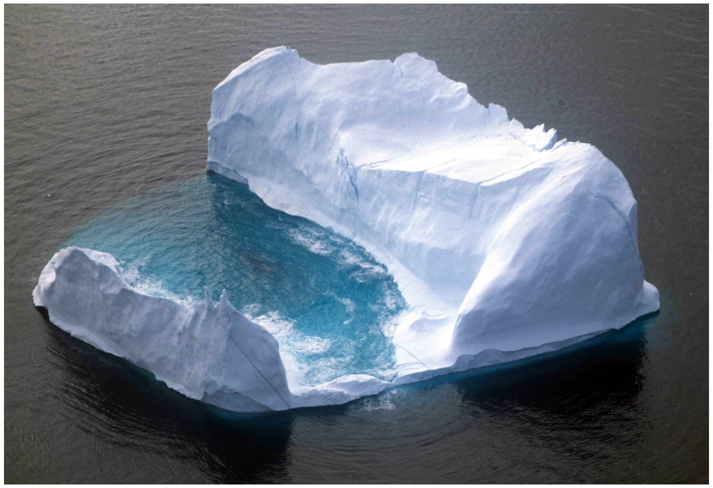
$$a_n \cos nx + b_n \sin nx.$$

This pure harmonic changes sign exactly $2n$ times, and so does f_m, for m large enough. Therefore $Z(f) \geq 2n$.

Heat equation version: Let $f(x)$ be the initial distribution of heat on the circle. Consider the propagation of heat:

$$\frac{\partial F(x,t)}{\partial t} = \frac{\partial^2 F(x,t)}{\partial x^2}, \qquad F(x,0) = f(x).$$

The number of sign changes of $F(x,t)$ (as a function of x) does not increase with t: an iceberg can melt down in a warm sea but cannot appear out of nowhere (the *maximum principle* in PDE).



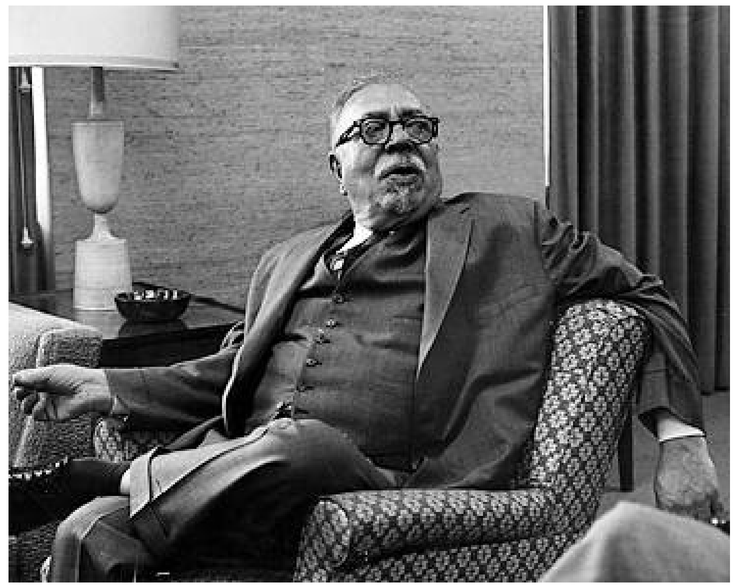
But one can solve the heat equation explicitly:

$$F(x,t) = \sum_{k \geq n} e^{-k^2 t} \left(a_k \cos kx + b_k \sin kx \right).$$

The higher harmonics tend to zero faster than the first non-trivial one. As $t \to \infty$, the (renormalized) function tends to its first non-trivial harmonic, that has exactly $2n$ zeroes. Hence $f(x) = F(x,0)$ has at least $2n$ zeroes.

Thus D^{-1} is a discrete analog of the heat operator, and Rolle's theorem plays the role of the maximum principle!

A similar argument yields a Polya-Wiener theorem (1941): If the number of zeros of the derivatives of a periodic function is uniformly bounded then the function is a trigonometric polynomial.



George Pólya (1887 – 1985) and Norbert Wiener (1894 – 1964).

Some questions to mull over:

- Does every theorem have a ‘Book proof’? For example, what about the Four Color Theorem or the Collatz $3n+1$ Conjecture?

- Is a ‘Book proof’ of a theorem unique? For example, I’d be hard pressed to choose one for the four vertex theorem.

- The original proof of a theorem is not necessarily a ‘Book proof’. At average, how long does it take for a ‘Book proof’ to emerge?

Thank you!