Chinese dragons and mating trees

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Overview

Part I: Cast of Characters

1. **Fractals from complex dynamics:** background, motivation, Julia sets, matings
2. **Canonical random trees:** Brownian motion, continuum random tree
3. **Canonical random surfaces:** quantum gravity, planar maps, string theory
4. **Canonical random paths:** walks, interfaces, Schramm-Loewner evolution
5. **Canonical random growth:** Eden model, DLA, DBM

Part II: Drama

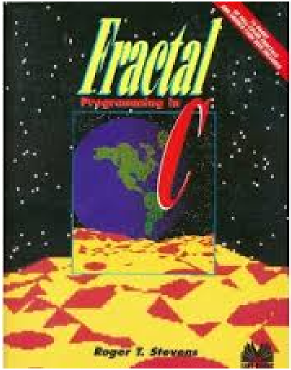
1. **Welding random surfaces:** a calculus of random surfaces and SLE seams
2. **Mating random trees:** tree plus tree (conformally mated) equals surface plus path
3. **Random growth on random surfaces:** dendrites, dragons, surprising tractability

References:

1. *Conformal weldings of random surfaces: SLE and the quantum gravity zipper* (2010)
2. *Imaginary Geometry I-IV* (Miller, S., 2012-2013)
3. *Quantum Loewner Evolution* (Miller, S. 2013)
4. *Liouville quantum gravity as a mating of trees* (Duplantier, Miller, S. 2014)

FRACTALS FROM COMPLEX DYNAMICS

- Julia sets (Julia, 1918), popularized in 1980's



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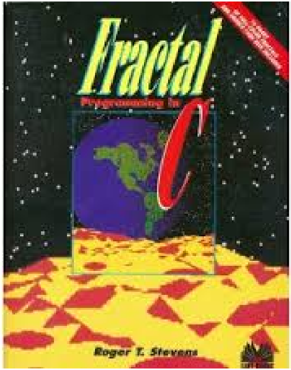

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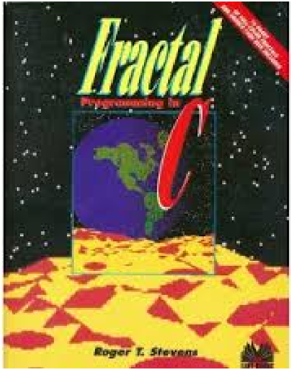

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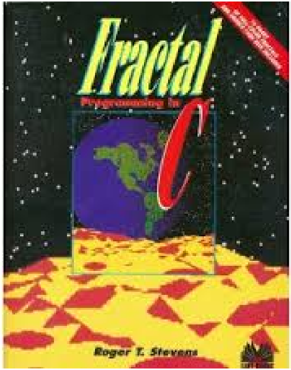

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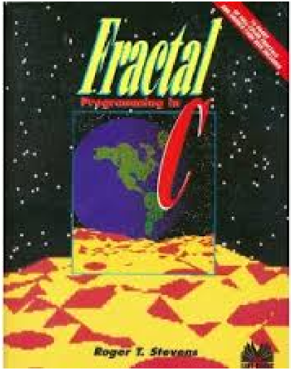

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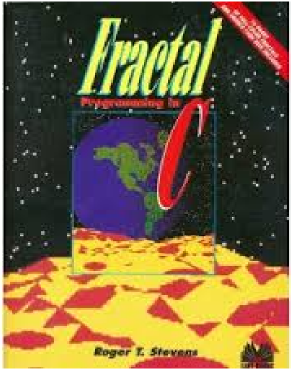

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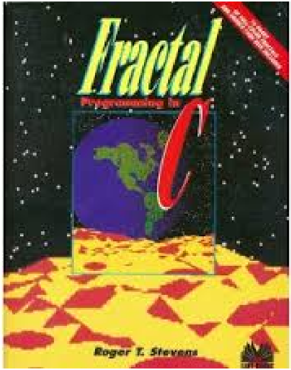

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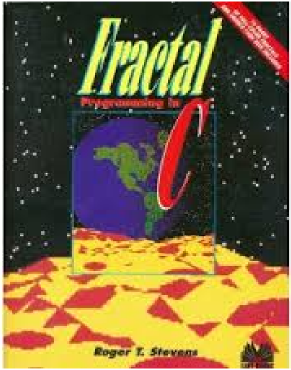

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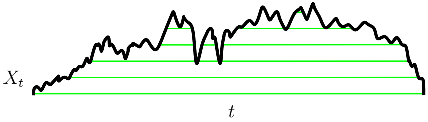
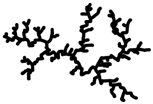
- Popular lexicon: chaos, butterly effect, fractal, self-similar.
- What about *random* fractals, only self similar in law?

RANDOM TREES





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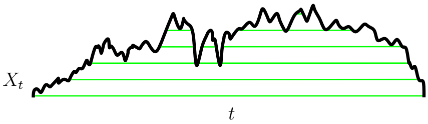

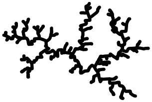

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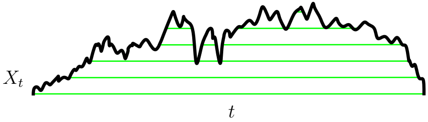

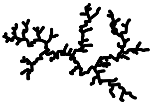

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- Discrete analog: Consider a tree embedded in the plane with n edges and a distinguished root. As one traces the outer boundary of the tree clockwise, distance from root performs a simple walk on $\mathbf{Z}_+$ with $2n$ steps, starting and ending at 0.

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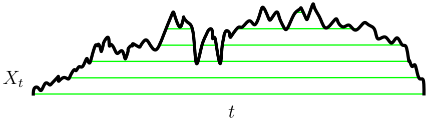

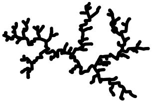

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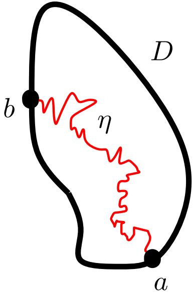
- Simple bijection rooted planar trees and walks of this type.

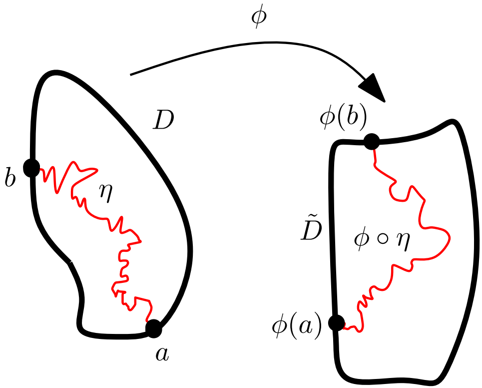
RANDOM PATHS

Given a simply connected planar domain D with boundary points a and b and a parameter $\kappa \in [0, \infty)$, the **Schramm-Loewner evolution** SLE_κ is a random non-self-crossing path in $\overline{D}$ from a to b.



The parameter κ roughly indicates how "windy" the path is. Would like to argue that SLE is in some sense the "canonical" random non-self-crossing path. What symmetries characterize SLE?

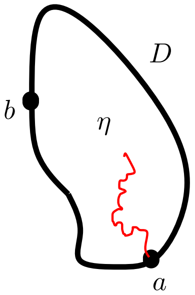
Conformal Markov property of SLE



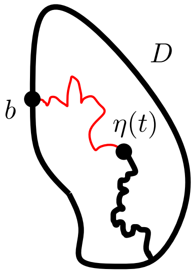
If ϕ conformally maps D to $\tilde{D}$ and η is an SLE_κ from a to b in D, then $\phi \circ \eta$ is an SLE_κ from $\phi(a)$ to $\phi(b)$ in $\tilde{D}$.

Markov Property

Given η up to a stopping time t...



law of remainder is SLE in $D \setminus \eta[0,t]$ from $\eta(t)$ to b.



Chordal Schramm-Loewner evolution (SLE)

- **THEOREM [Oded Schramm]:** Conformal invariance and the Markov property completely determine the law of SLE, up to a single parameter which we denote by $\kappa \geq 0$.

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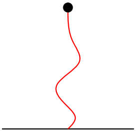
- **THEOREM [Oded Schramm]:** Conformal invariance and the Markov property completely determine the law of SLE, up to a single parameter which we denote by $\kappa \geq 0$.
- **Explicit construction:** An SLE path γ from 0 to ∞ in the complex upper half plane $\mathbf{H}$ can be defined in an interesting way: given path γ one can construct conformal maps $g_t : \mathbf{H} \setminus \gamma([0,t]) \to \mathbf{H}$ (normalized to look like identity near infinity, i.e., $\lim_{z\to\infty} g_t(z) - z = 0$). In SLE$_\kappa$, one defines g_t via an ODE (which makes sense for each fixed z):

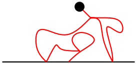
$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z,$$

where $W_t = \sqrt{\kappa} B_t =_{LAW} B_{\kappa t}$ and B_t is ordinary Brownian motion.

SLE phases [Rohde, Schramm]



$\kappa \leq 4$



$\kappa \in (4, 8)$

$\kappa \geq 8$

Radial Schramm-Loewner evolution (SLE)

- **Radial SLE:** $\partial_t g_t(z) = g_t(z)\frac{\xi_t + g_t(z)}{\xi_t - g_t(z)}$ where $\xi_t = e^{i\sqrt{\kappa}B_t}$.

Radial Schramm-Loewner evolution (SLE)

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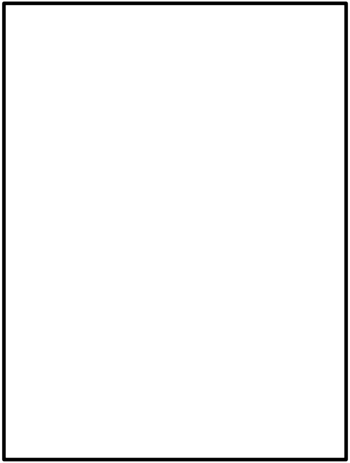
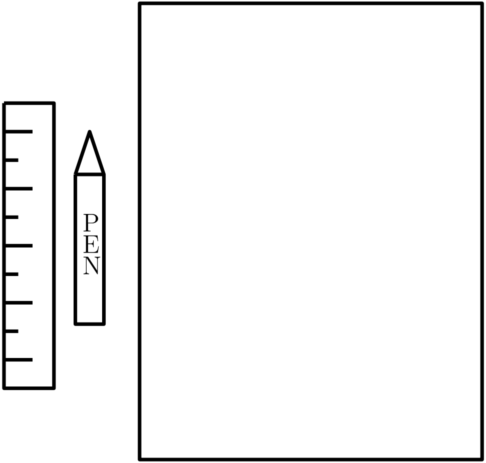
- **Radial measure-driven Loewner evolution:** $\partial_t g_t(z) = \int g_t(z)\frac{x + g_t(z)}{x - g_t(z)} dm_t(x)$ where, for each g, m_t is a measure on the complex unit circle.

RANDOM SURFACES



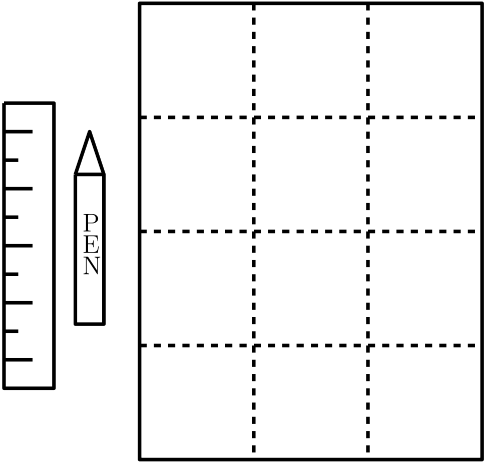
Start out with a sheet of paper

RANDOM SURFACES



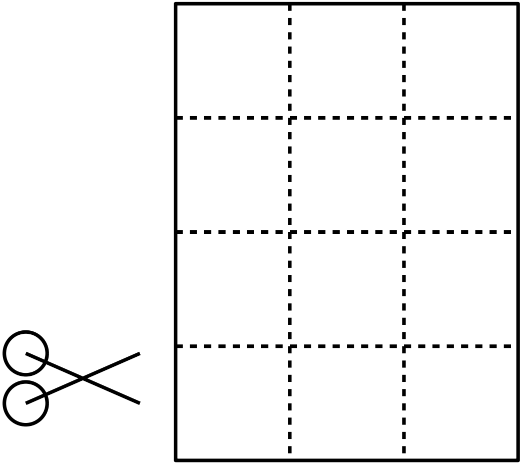
Get out pen and ruler

RANDOM SURFACES



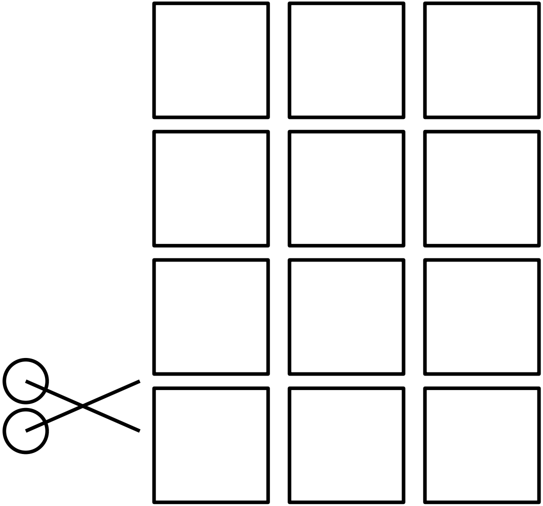
Measure and mark squares squares of equal size

RANDOM SURFACES



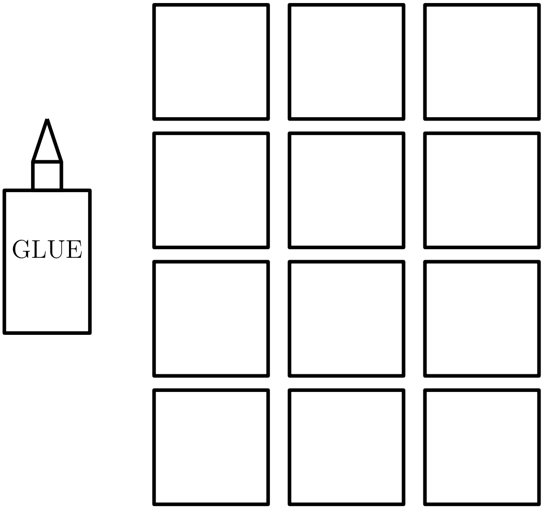
Get out scissors

RANDOM SURFACES



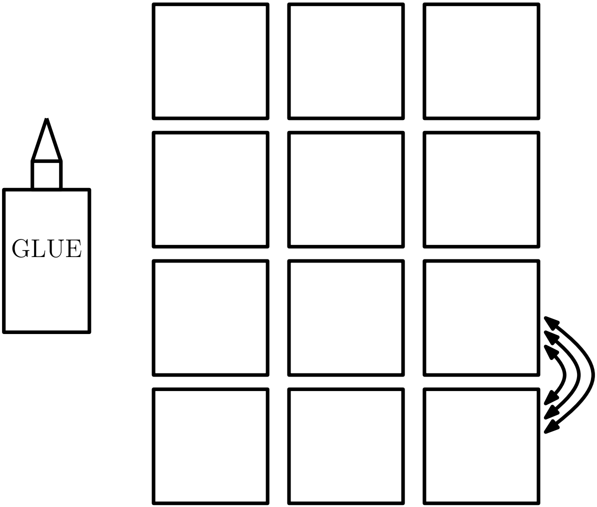
Cut into squares

RANDOM SURFACES

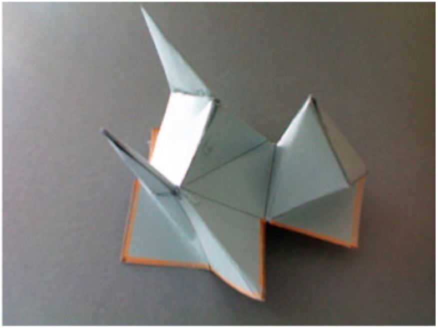


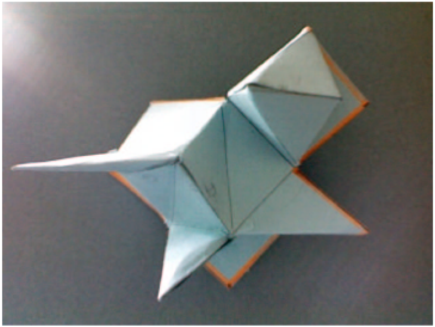
Get out bottle of glue

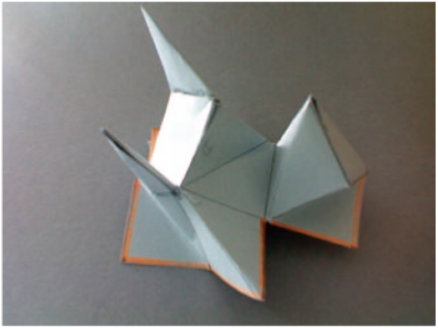
RANDOM SURFACES

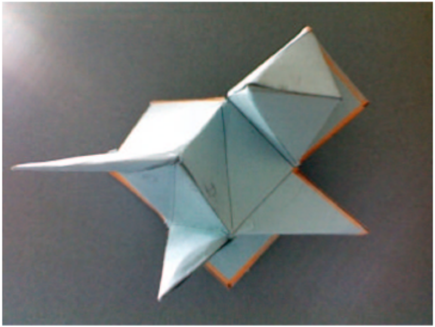


Attach squares along boundaries with glue to form a surface "without holes."



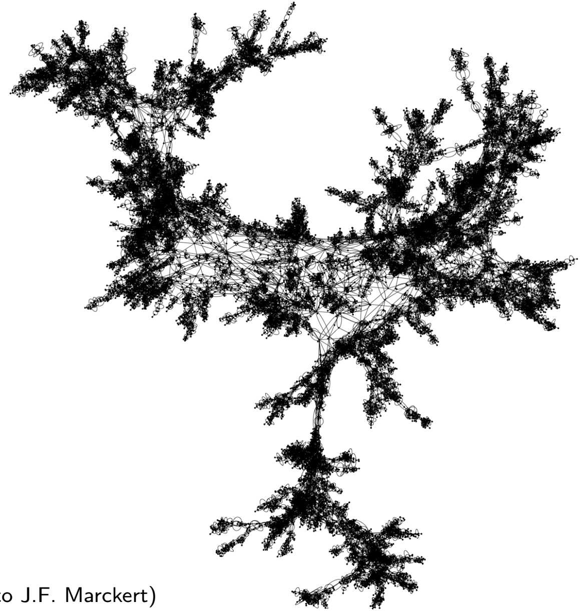






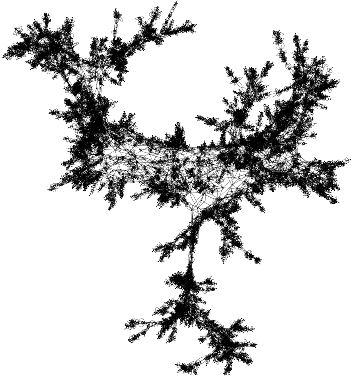
What is the structure of a typical quadrangulation when the number of faces is large?

Random quadrangulation with 25,000 faces



(Simulation due to J.F. Marckert)

Background



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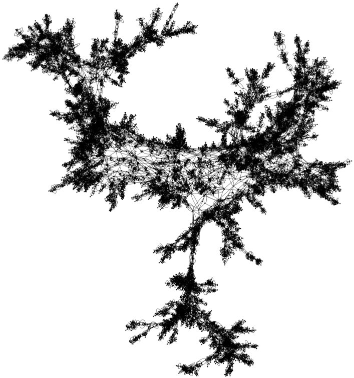

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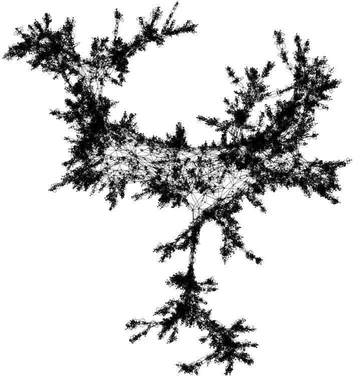

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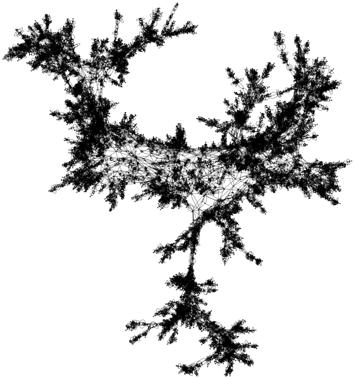

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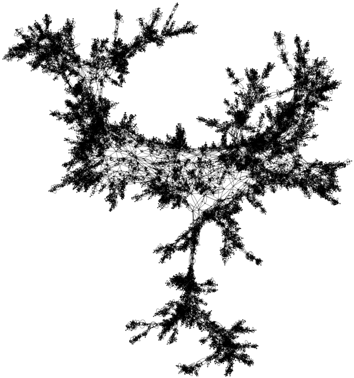

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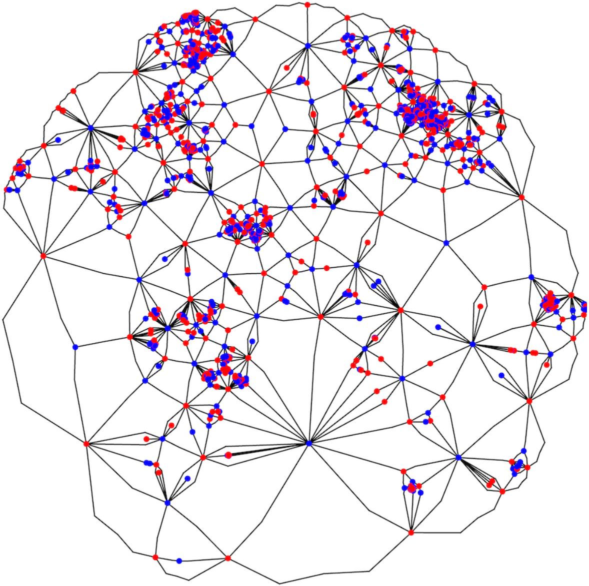
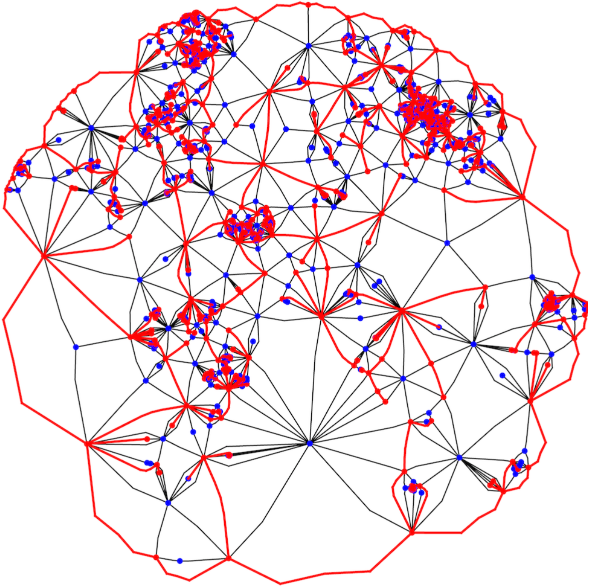
5. Important tool: Bijections encoding surface via pair of trees.

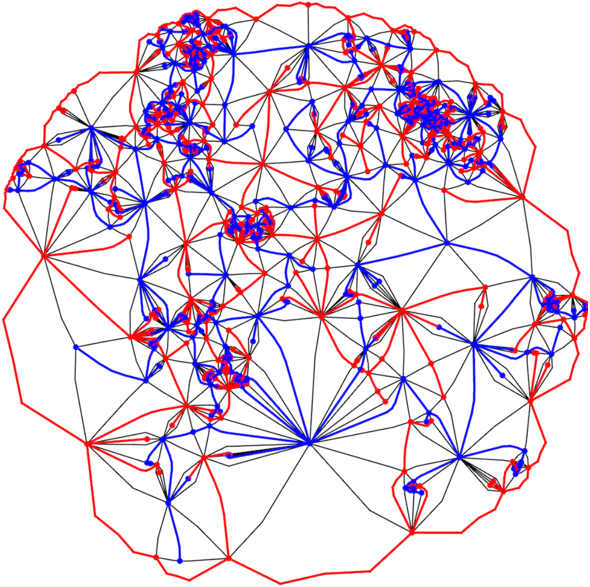
Random quadrangulation



Red tree

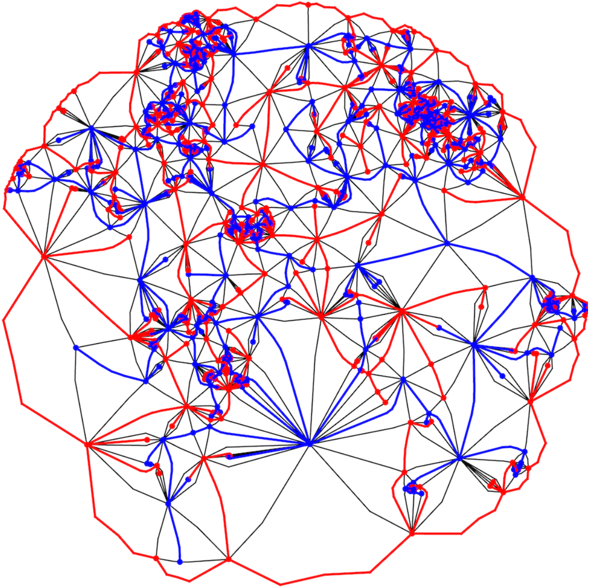


Red and blue trees

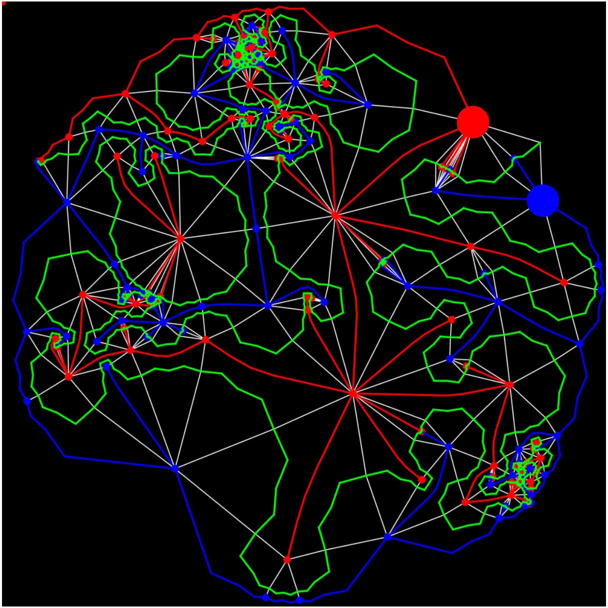


Red and blue trees alone do not determine the map structure

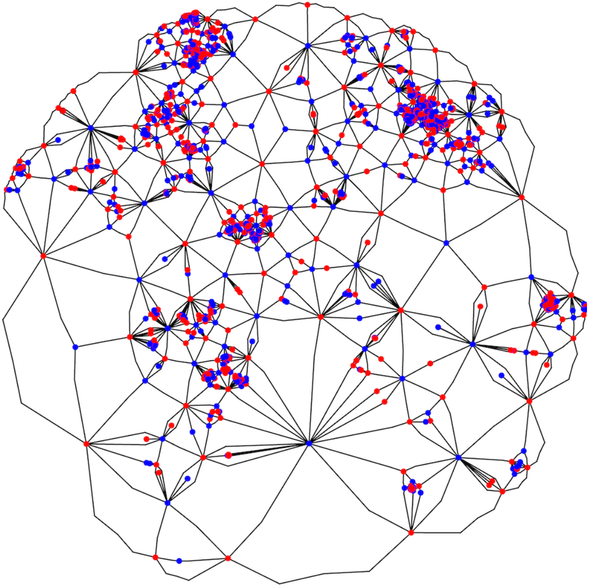
Random quadrangulation with red and blue trees



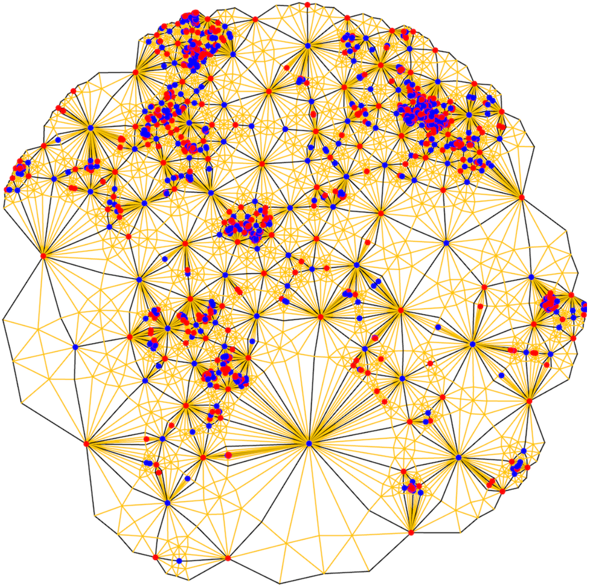
Path snaking between the trees. Encodes the trees and how they are glued together.



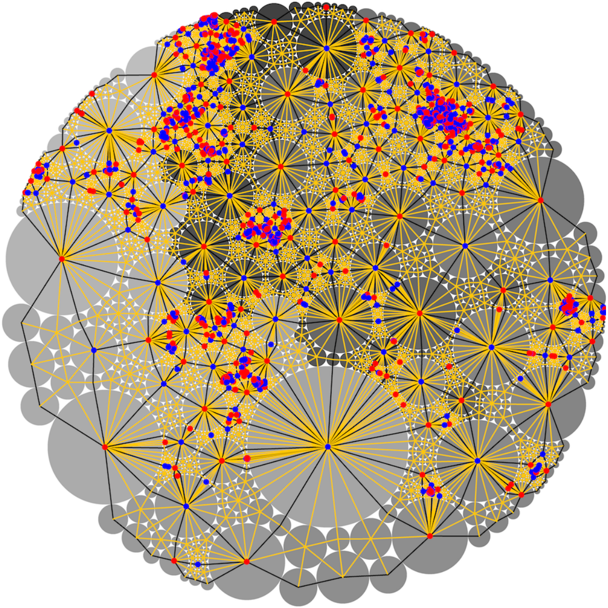
How was the graph embedded into $\mathbf{R}^2$?



Can subivide each quadrilateral to obtain a triangulation without multiple edges.



Circle pack the resulting triangulation.



Packed with Stephenson's CirclePack.

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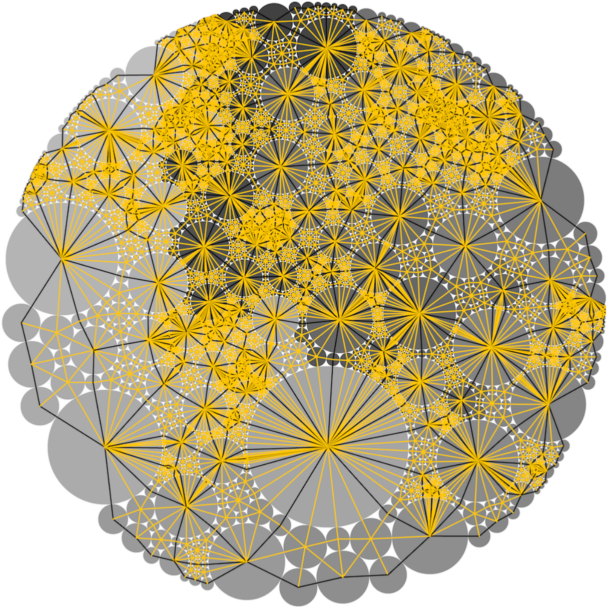

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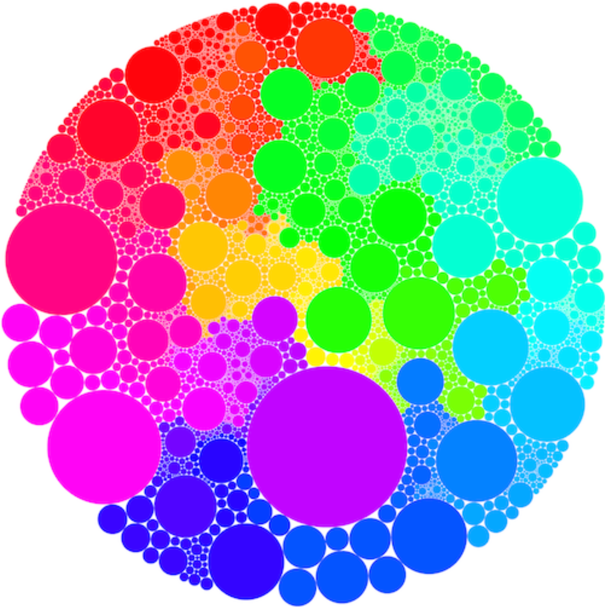

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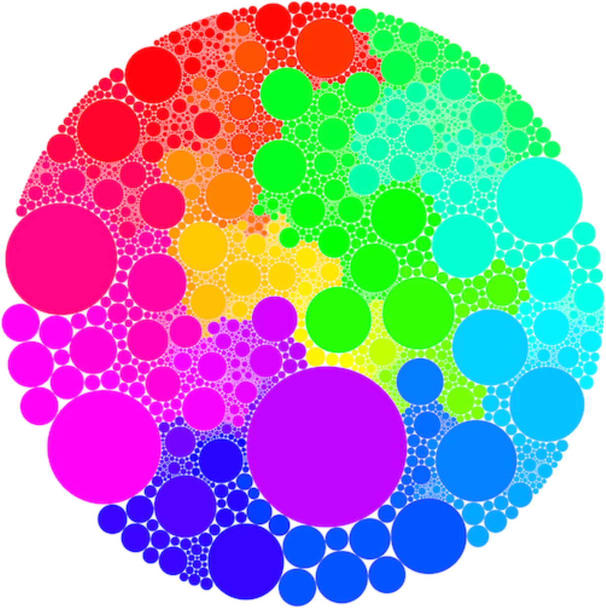
What is the "limit" of this embedding? Circle packings are related to conformal maps.



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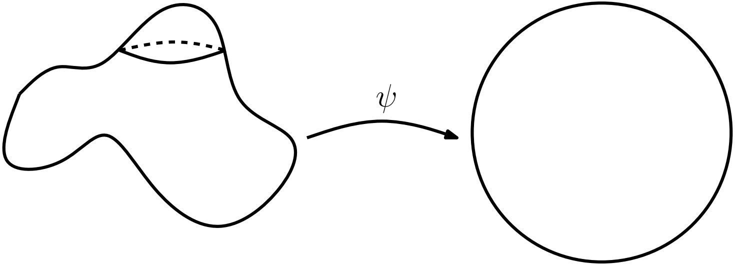
Picking a surface at random in the continuum

Uniformization theorem: every simply connected Riemannian surface can be conformally mapped to either the unit disk, the plane, or the sphere $\mathbf{S}^2$ in $\mathbf{R}^3$



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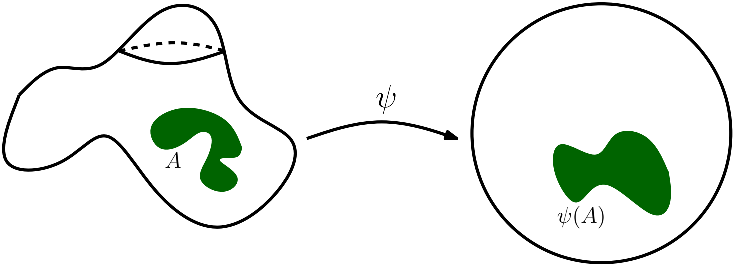

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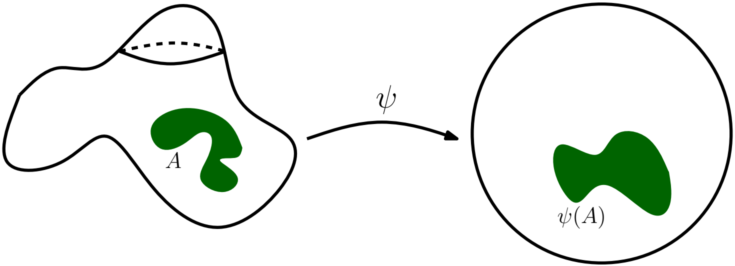

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$\Rightarrow$ Can parameterize the space of surfaces with smooth functions.

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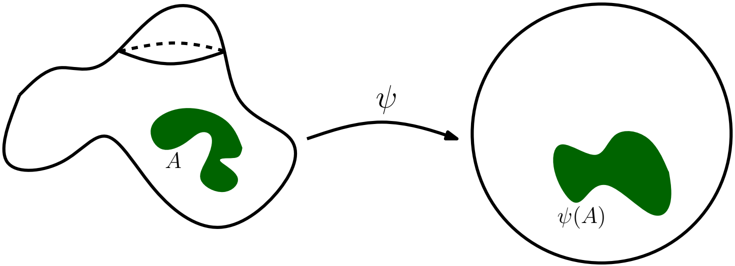

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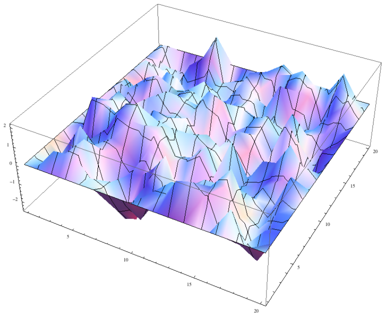
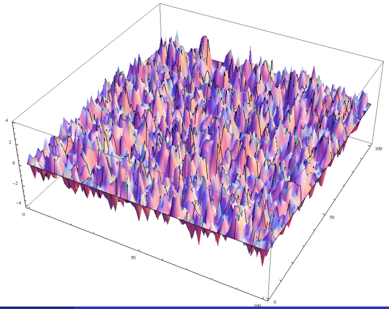
Question: Which measure on ρ? If we want our surface to be a perturbation of a flat metric, natural to choose ρ as the canonical perturbation of a harmonic function.

The Gaussian free field

- The **discrete Gaussian free field** (DGFF) is a **Gaussian random surface** model.





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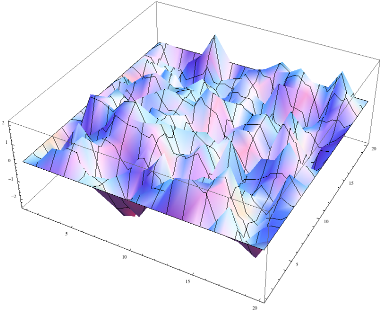
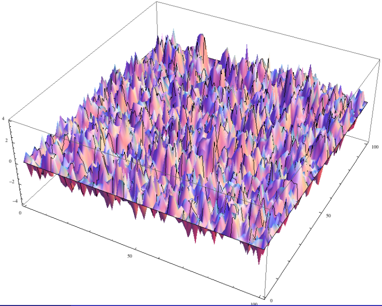
- Measure on functions $h \colon D \to \mathbf{R}$ for $D \subseteq \mathbf{Z}^2$ and $h|_{\partial D} = \psi$ with density respect to Lebesgue measure on $\mathbf{R}^{|D|}$:

$$\frac{1}{\mathcal{Z}} \exp\left(-\frac{1}{2} \sum_{x \sim y} (h(x) - h(y))^2 \right)$$





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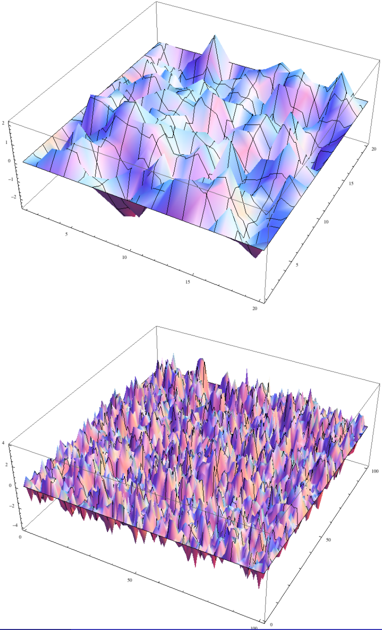

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- Fine mesh limit: converges to the continuum GFF, i.e. the standard Gaussian wrt the **Dirichlet inner product**

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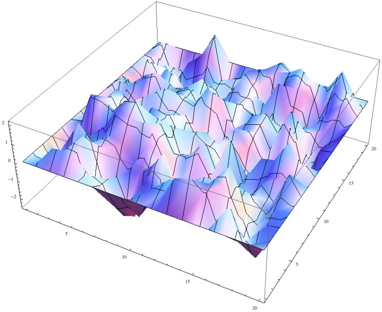

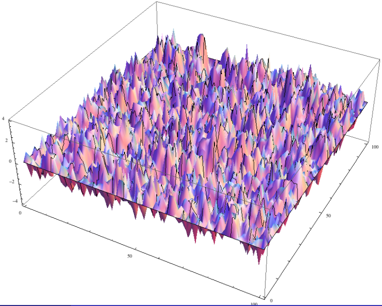

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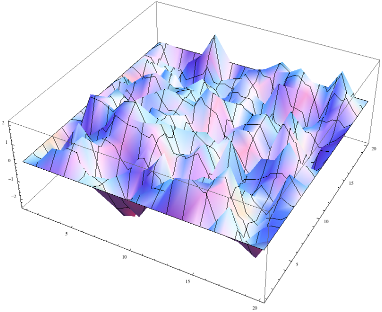

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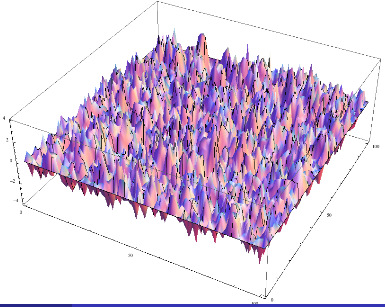

- Continuum GFF not a function — only a generalized function

Liouville quantum gravity

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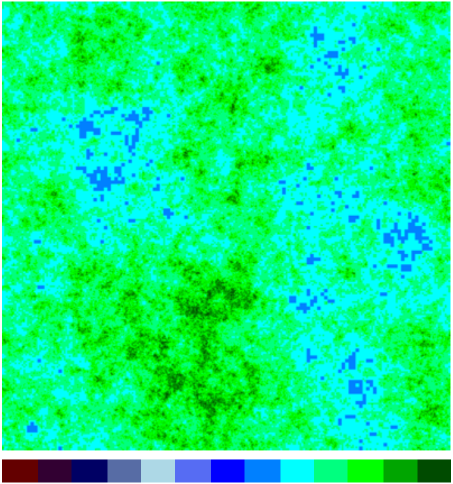
$\gamma = 0.5$



(Number of subdivisions)

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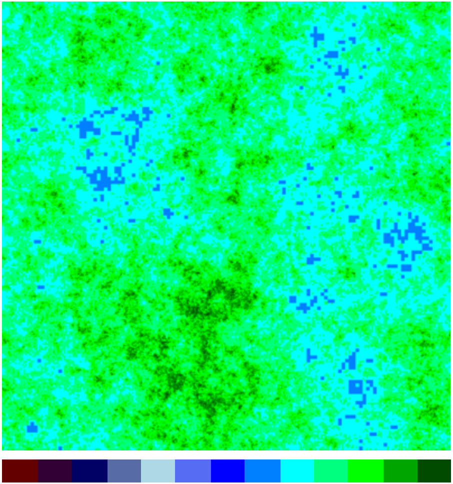

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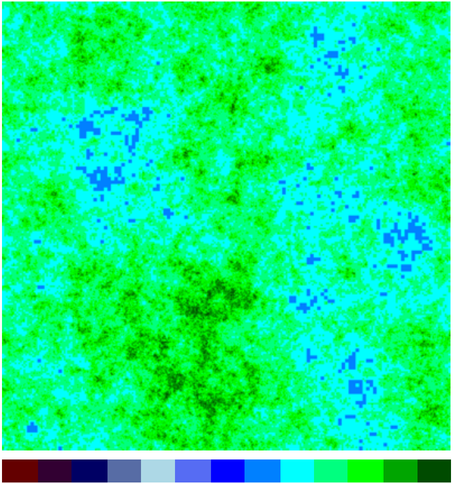

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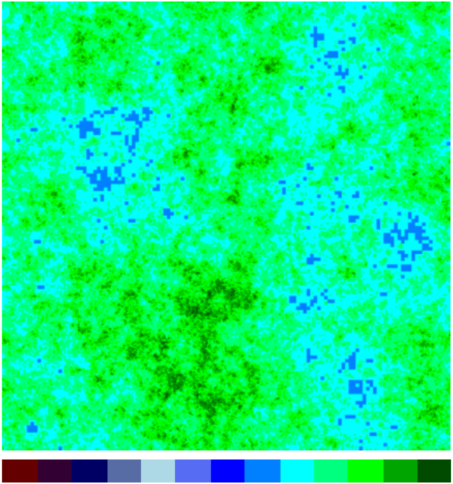

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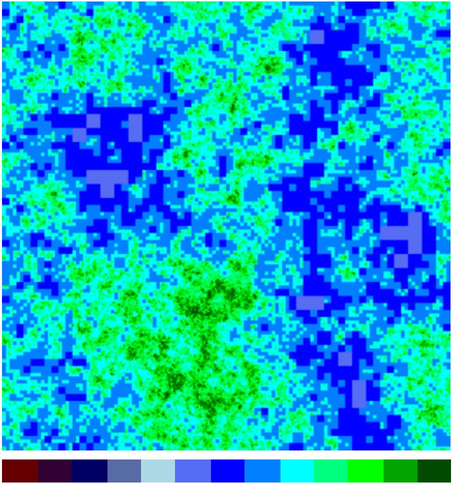

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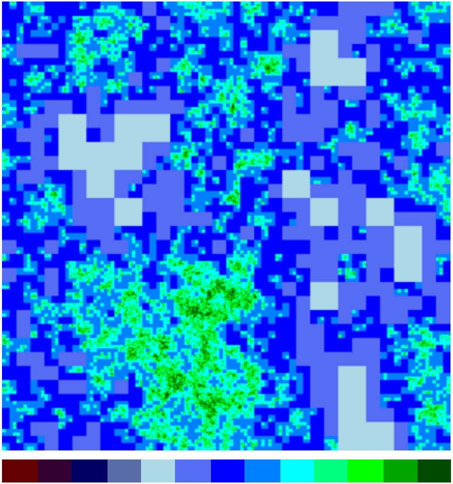

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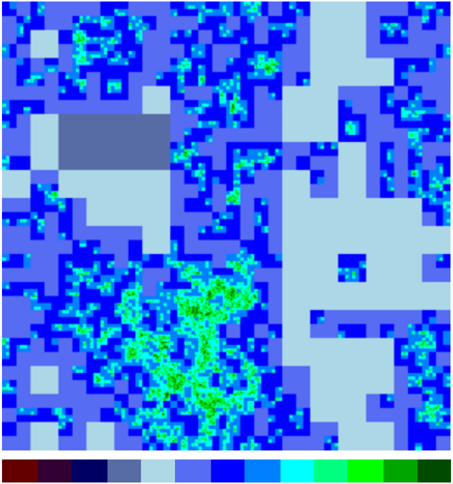
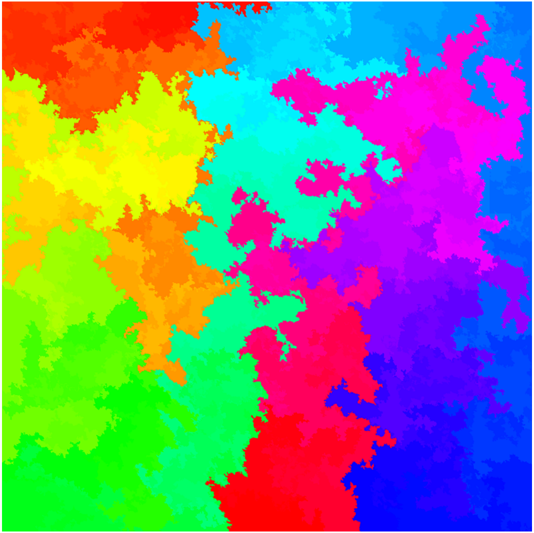


(Number of subdivisions)

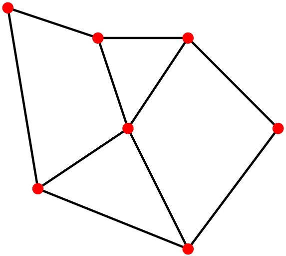
Continuum space-filling path



Space-filling SLE_6 on a LQG surface. Random path which encodes the limit of a RPM.

RANDOM GROWTH

- Associate with a graph (V, E) i.i.d. $\exp(1)$ edge weights



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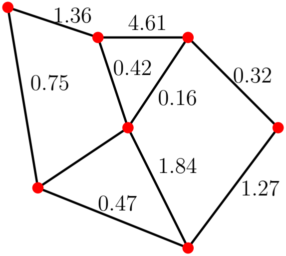

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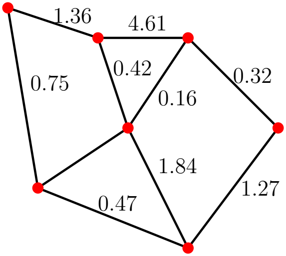

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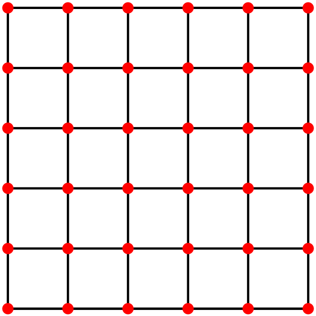

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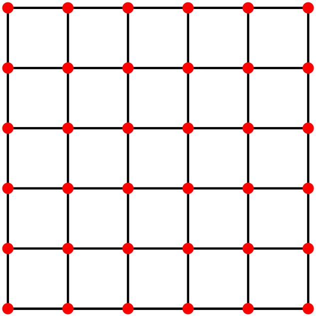

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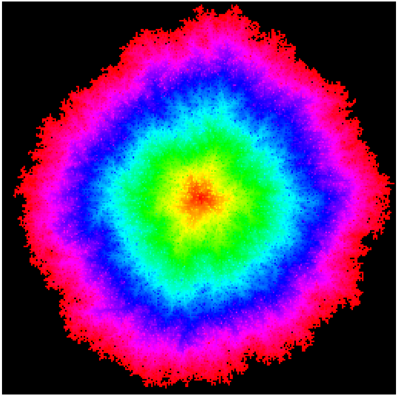

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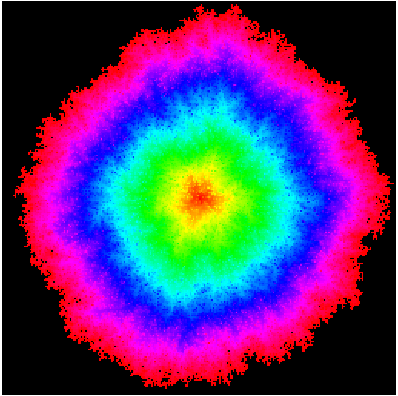

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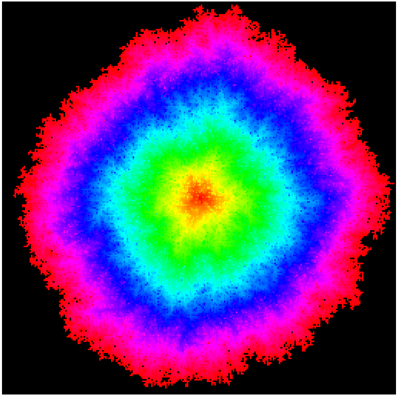

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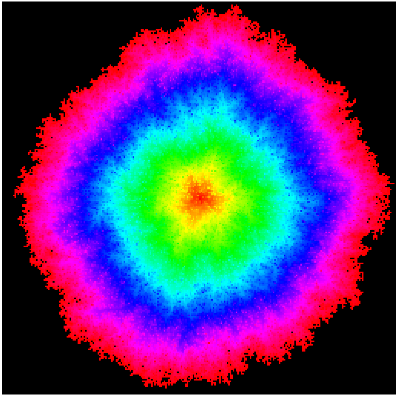

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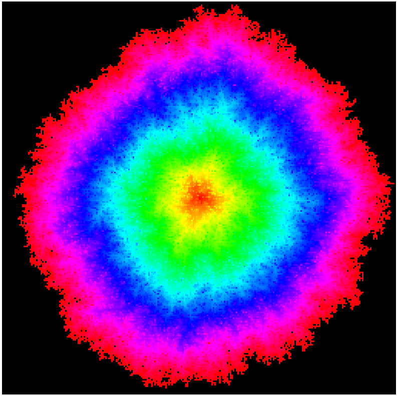
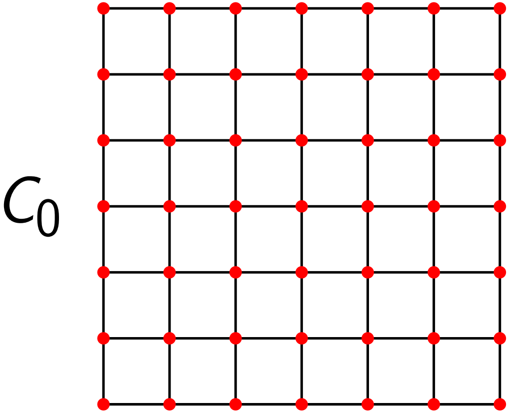
- Vahidi-Asl and Weirmann (1990) showed that the rescaled ball converges to a disk if $\mathbf{Z}^2$ is replaced by the Voronoi tesselation associated with a Poisson process



Markovian formulation

Eden exploration



Sample the cluster C_{n+1} from C_n by selecting an edge uniformly at random on ∂C_n, and then adding the vertex which is attached to it. **VARIANT:** Choose locations from harmonic measure (DLA) or harmonic measure to η power (η-DBM).

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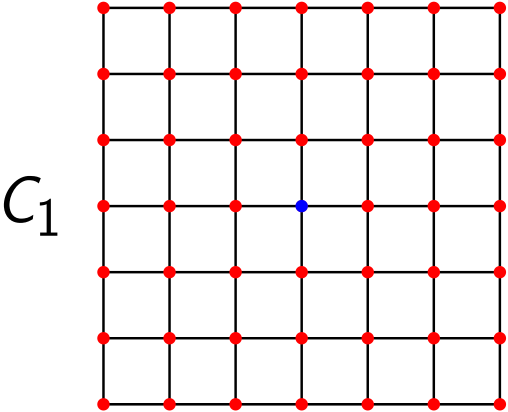

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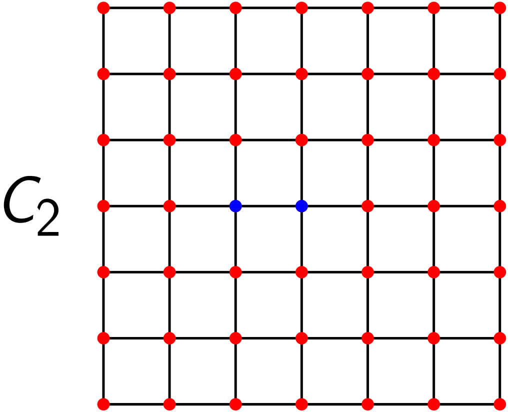

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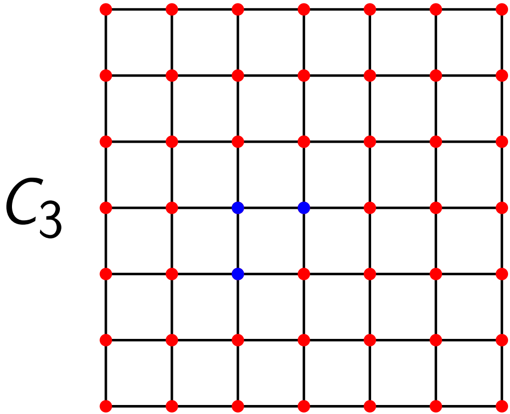

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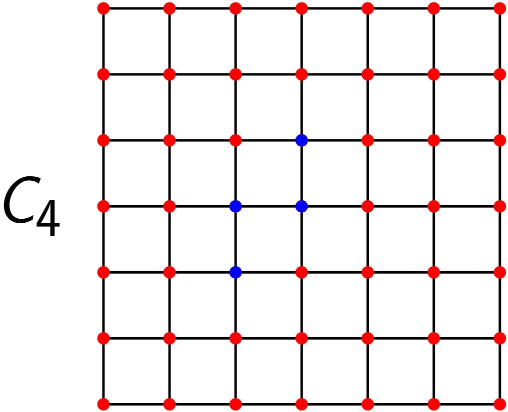

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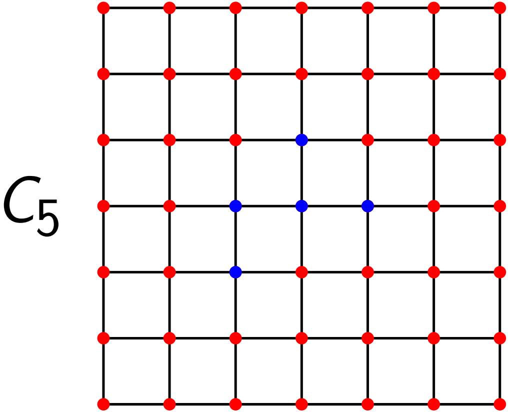

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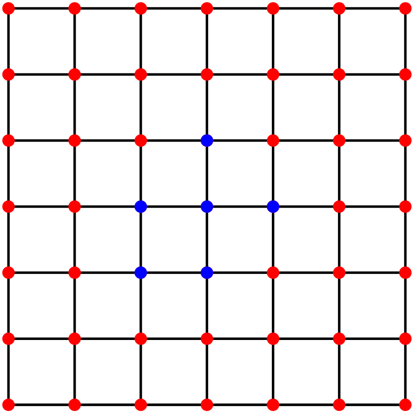
C_6



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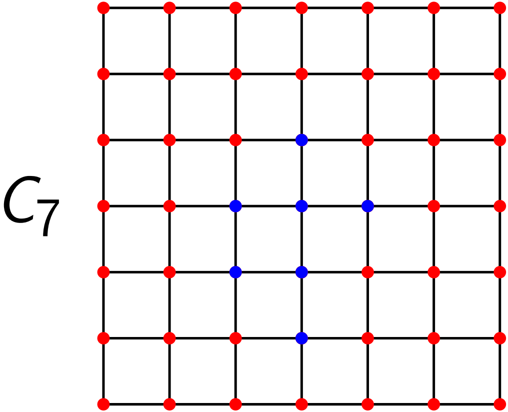

Sample the cluster C_{n+1} from C_n by selecting an edge uniformly at random on ∂C_n, and then adding the vertex which is attached to it. **VARIANT:** Choose locations from harmonic measure (DLA) or harmonic measure to η power (η-DBM).

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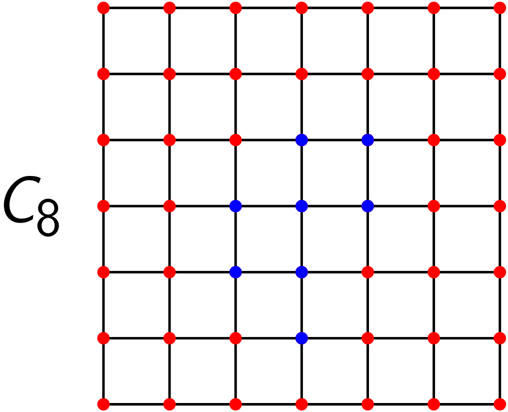

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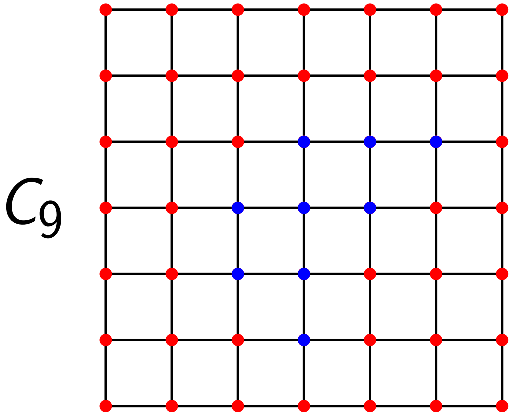

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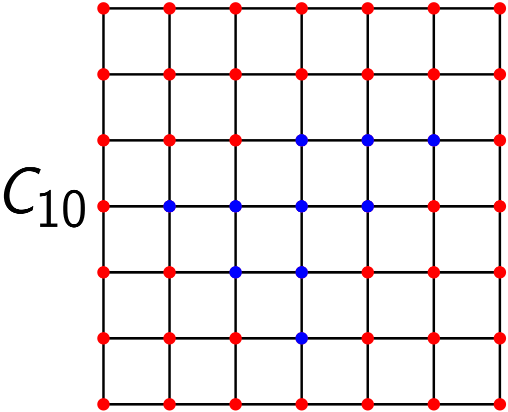

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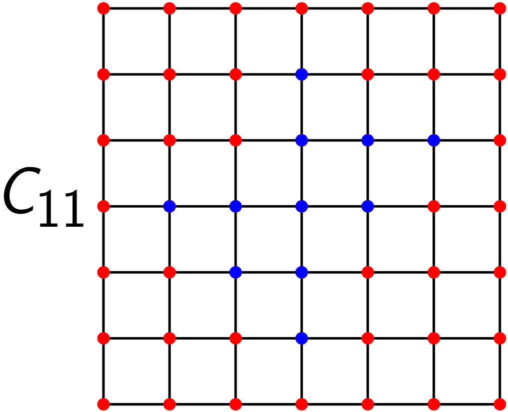

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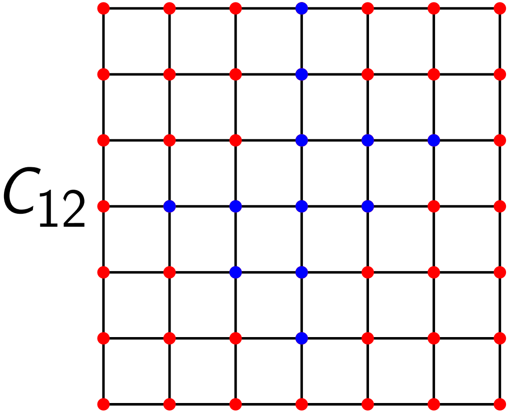

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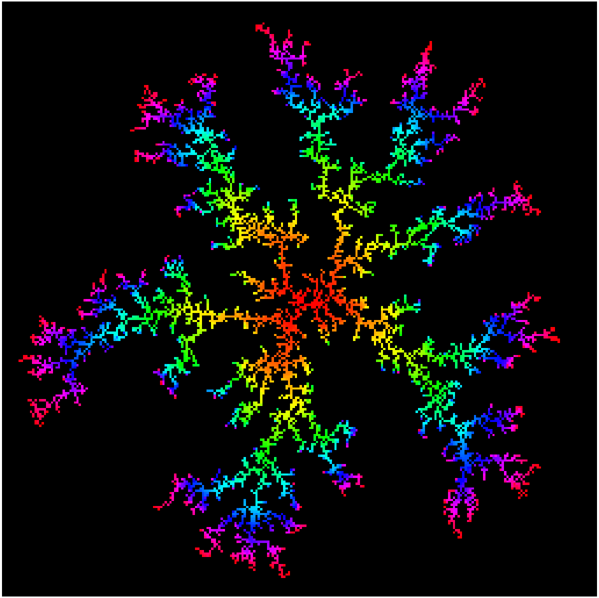
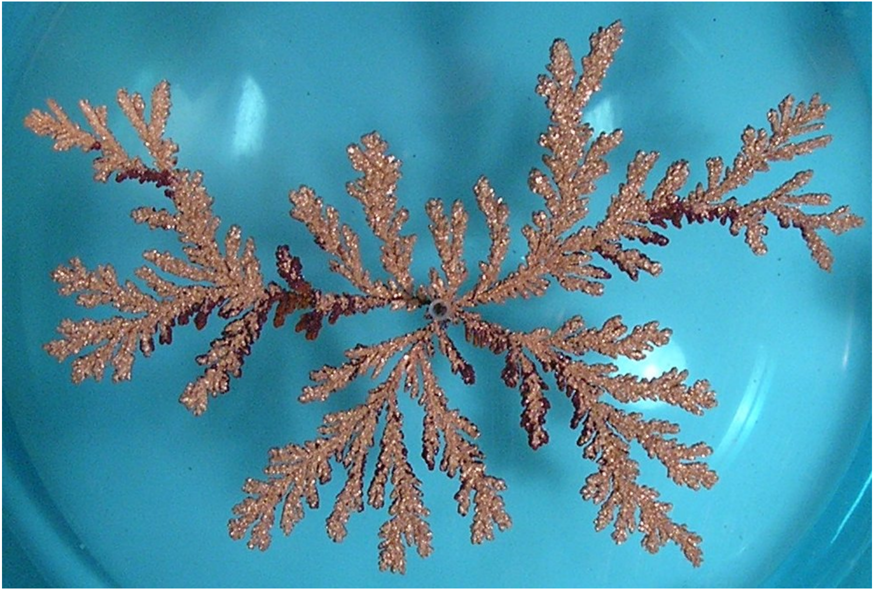


Euclidean Diffusion Limited Aggregation (DLA) introduced by Witten-Sander 1981.



DLA in nature: "A DLA cluster grown from a copper sulfate solution in an electrodeposition cell" (from Wikipedia)

DLA in nature: Magnese oxide patterns on the surface of a rock. (Halsey, Physics Today 2000)

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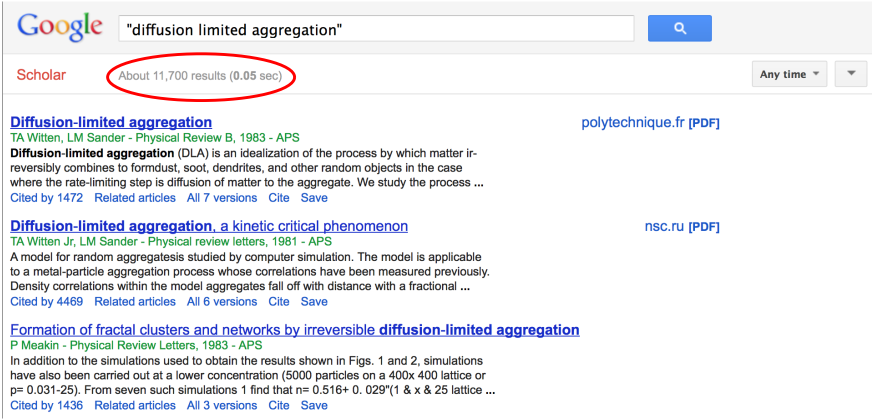
DLA in art: “High-voltage dielectric breakdown within a block of plexiglas” (from Wikipedia)

DLA in physics

Introduced by Witten and Sander in 1981 as a model for crystal growth. (Mineral deposits, Hele-Shaw flow, electrodeposition, lichen growth, lightning paths, coral, etc.)

An active area of research in physics for the last 33 years:



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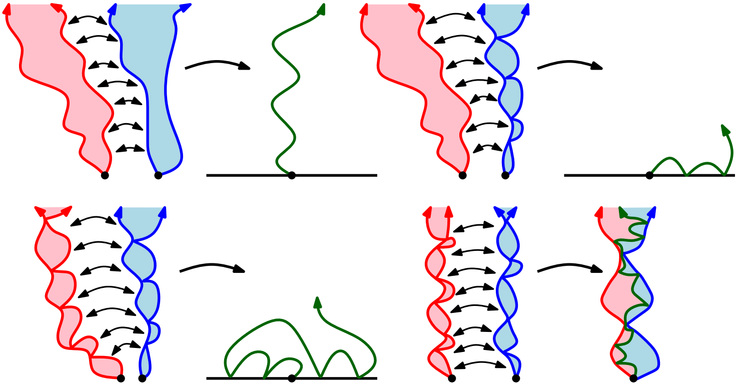
What about DLA on random planar maps and Liouville quantum gravity surfaces?

Part II:
DRAMA

WELDING RANDOM SURFACES

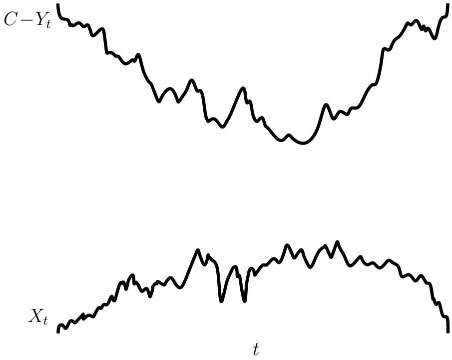
Can "weld" and "slice" special quantum surfaces called quantum wedges (with "weight" parameters indicating thickness) to obtain wedges (with other weights).



- Weight parameter $W = \gamma(\gamma + \frac{2}{\gamma} - \alpha)$ is additive under the welding operation.
- Interface between welding of independent wedges $\mathcal{W}_1, \mathcal{W}_2$ of weight W_1 and W_2 is an $\mathrm{SLE}_\kappa(W_1 - 2; W_2 - 2)$ on combined surface.
- Glue **canonical random surfaces**, seam becomes **canonical random path**.

MATING RANDOM TREES

X, Y independent Brownian excursions on $[0,1]$. Pick $C > 0$ large so that the graphs of X and $C - Y$ are disjoint.



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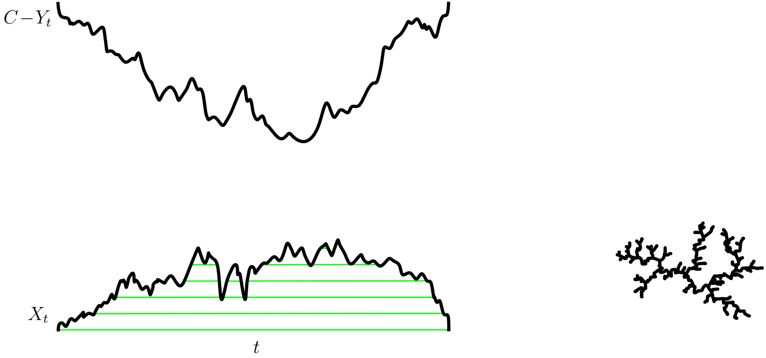

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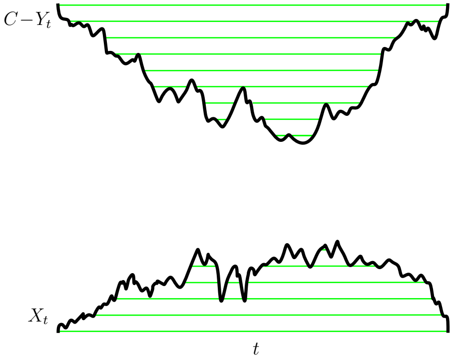

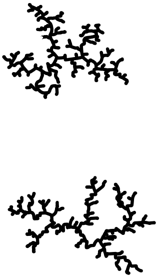

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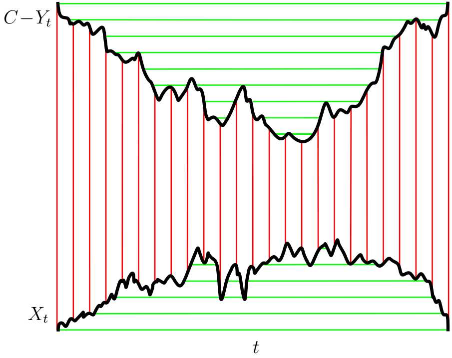

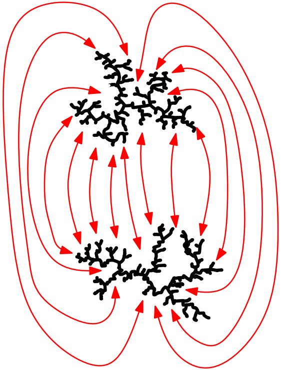

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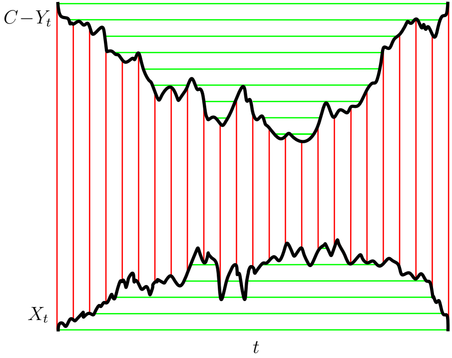

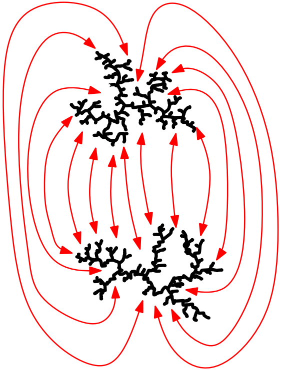

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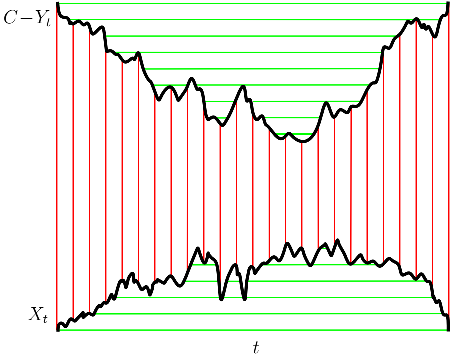

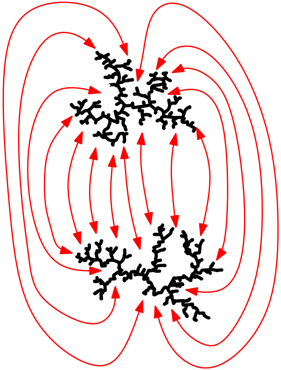

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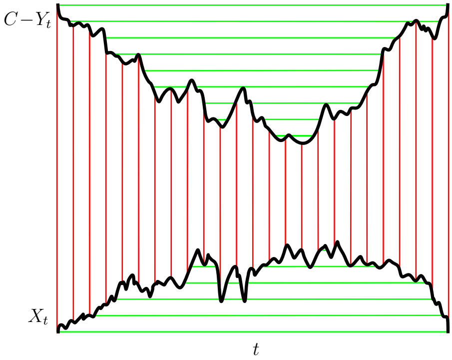

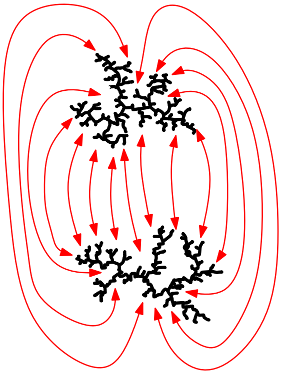

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Q: What is the resulting structure? **A:** Sphere with a space-filling path. A **peanosphere**.

How to check this?

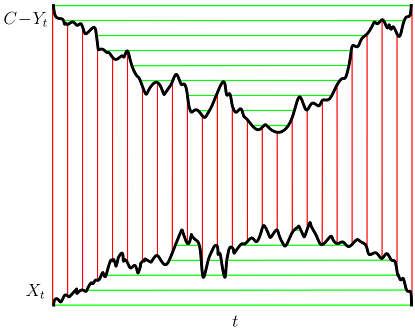
Theorem (Moore 1925)

Let $\cong$ be any topologically closed equivalence relation on the sphere $\mathbf{S}^2$. Assume that each equivalence class is connected and not equal to all of $\mathbf{S}^2$. Then the quotient space $\mathbf{S}^2/\cong$ is homeomorphic to $\mathbf{S}^2$ if and only if no equivalence class separates the sphere into two or more connected components.

- An equivalence relation is topologically closed iff for any two sequences (x_n) and (y_n) with
  - $x_n \cong y_n$ for all n
  - $x_n \to x$ and $y_n \to y$
- we have that $x \cong y$.

Constructing a sphere from a pair of trees

- X, Y ind. Brownian excursions on $[0, 1]$
- **Red**/**green** lines give an $\cong$-relation on $\mathbf{S}^2$



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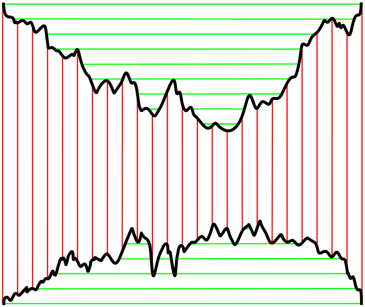

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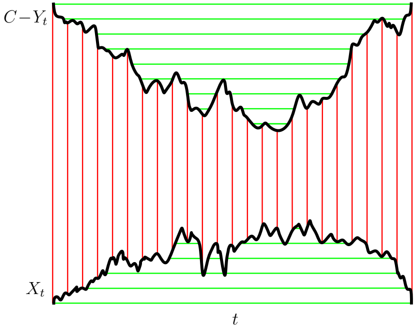

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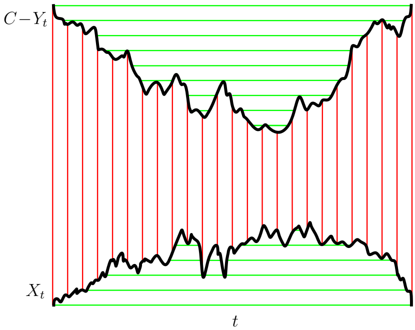

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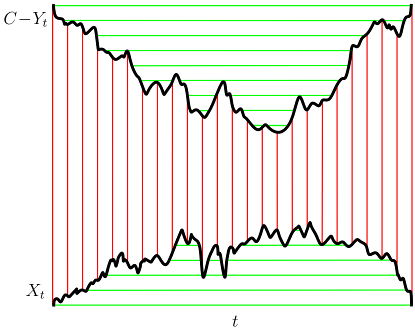

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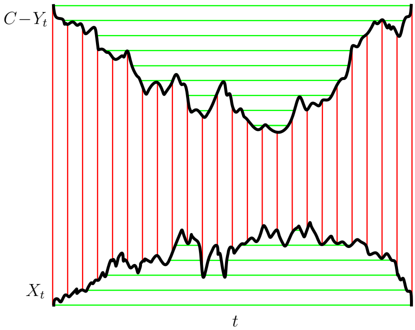

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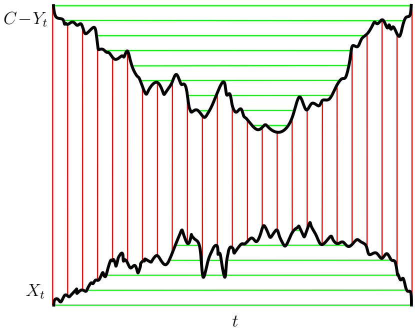

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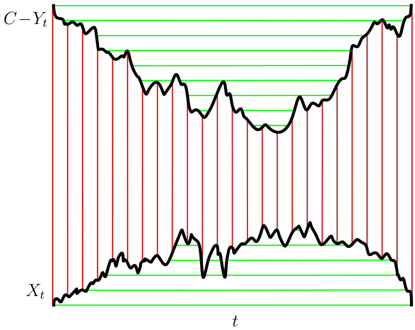

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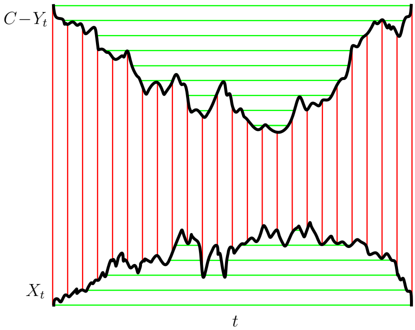

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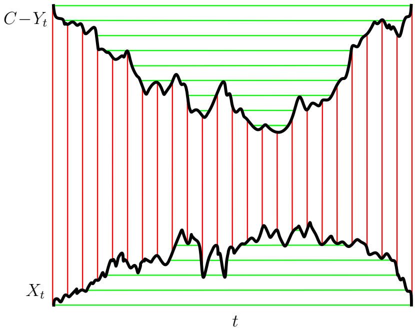
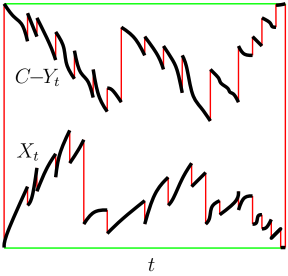


The sphere/space-filling path pair is a **peanoshere**

Peanophere has **canonical embedding** in Euclidean sphere as **LQG, space-filling SLE**.

Gluing independent Lévy trees

Can view $\mathrm{SLE}_{\kappa'}$ process, $\kappa' \in (4,8)$ as a gluing of two $\frac{\kappa'}{4}$-stable Lévy trees.



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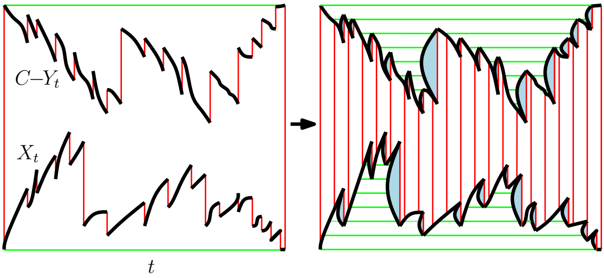

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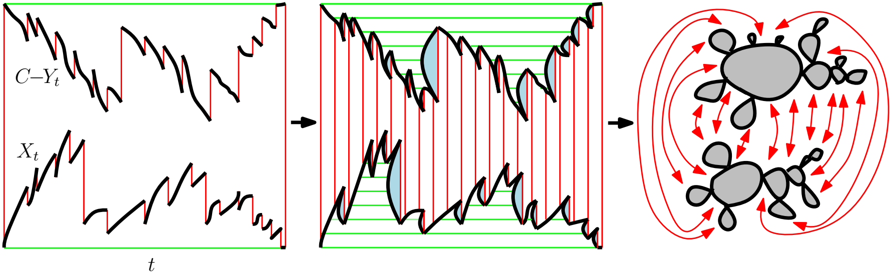

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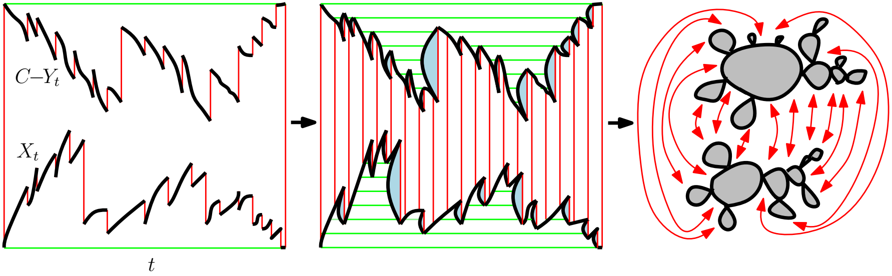

- The two trees of quantum disks almost surely determine both the $\mathrm{SLE}_{\kappa'}$ and the LQG surface on which it is drawn

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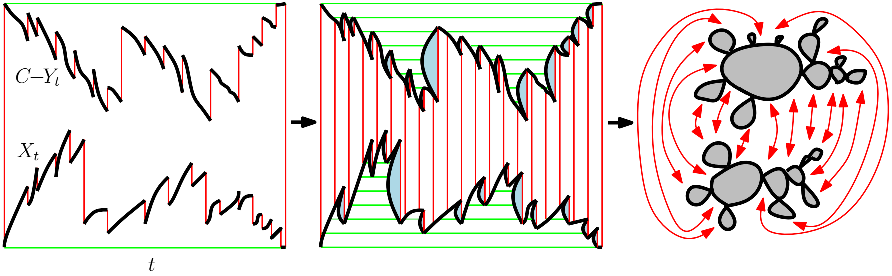

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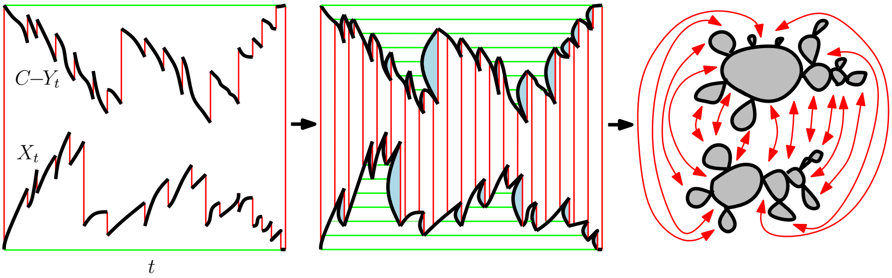

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- Can convert questions about $\mathrm{SLE}_{\kappa'}$ into questions about $\frac{\kappa'}{4}$-stable processes.
- Scaling limit of "exploration path" on random planar map should be SLE_6 on a $\sqrt{8/3}$-LQG. Using welding machinery, we can understand well the "bubbles" cut out by such an exploration process. We can understand conditional law of unexplored region given what we have seen.

RANDOM GROWTH ON RANDOM SURFACES

- Can we make sense of η-DBM on a γ-LQG? We have shown how to tile an LQG surface with diadic squares of "about the same size" so we could run a DLA on this set of squares and try to take a fine mesh limit.

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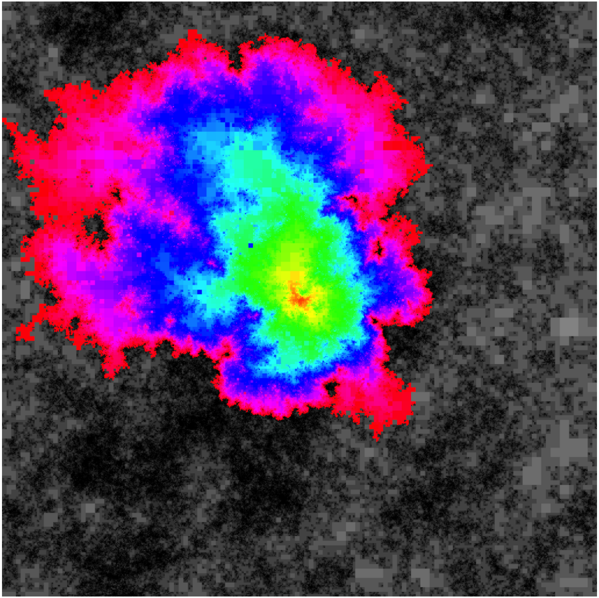
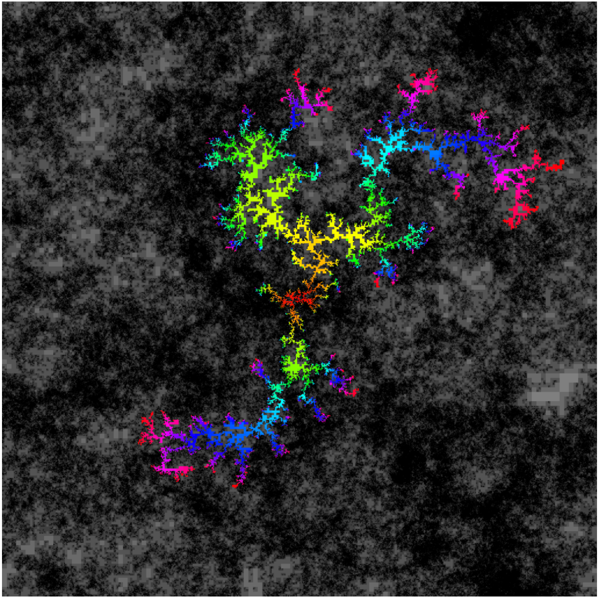
- We will ultimately want to construct a candidate for the scaling limit, which we will call (for reasons explained later) **quantum Loewner evolution:** QLE(γ^2, η).
- But first let's look at some computer generated images (and some animations), starting with an Eden exploration.



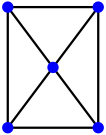
Eden model on $\sqrt{8/3}$-LQG



DLA on a $\sqrt{2}$-LQG

Eden model on planar map

- Random planar map, random vertex x. Perform FPP from x.



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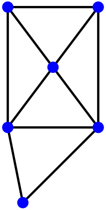

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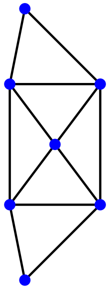

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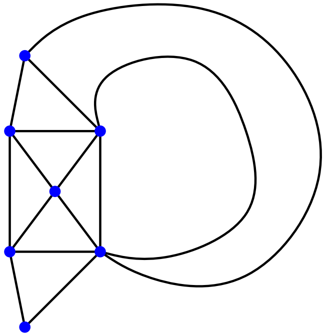

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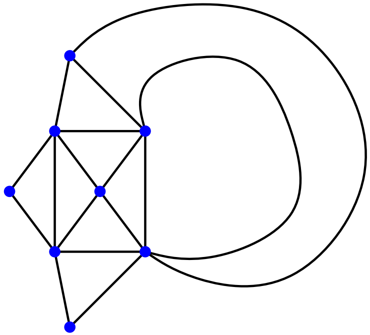

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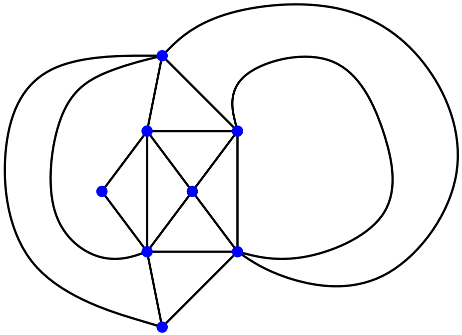

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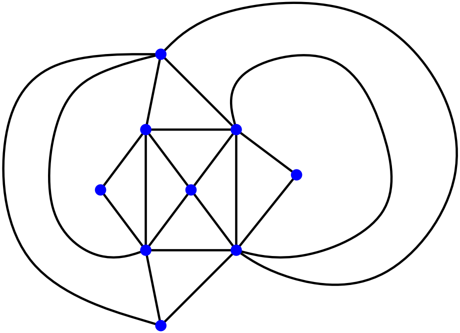

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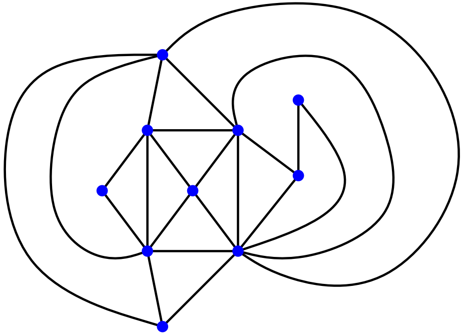

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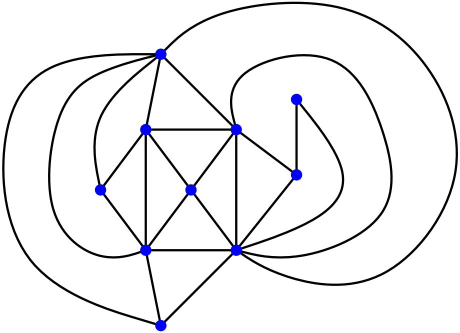

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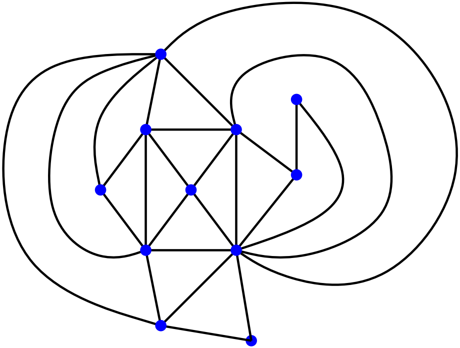

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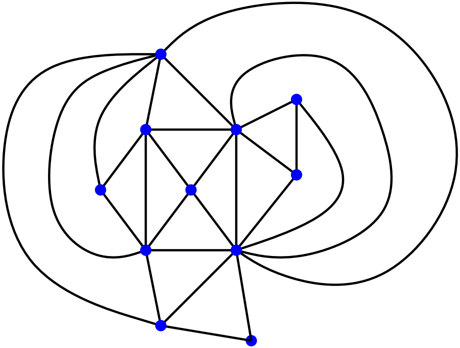

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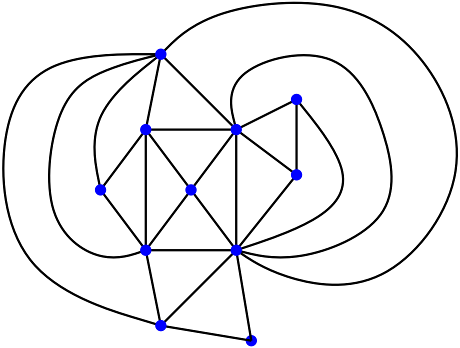

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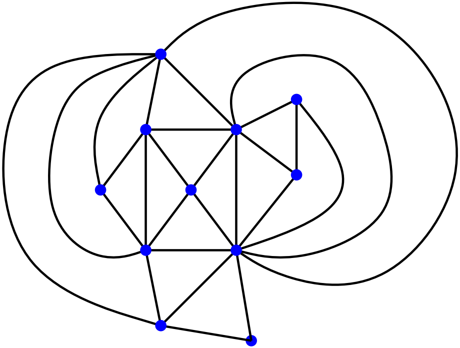

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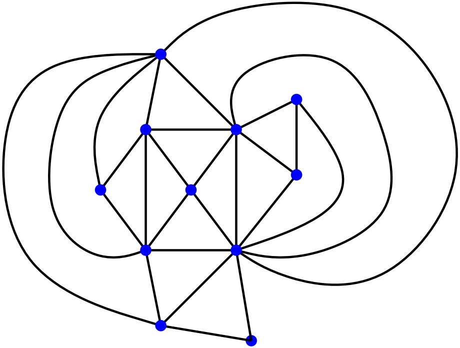

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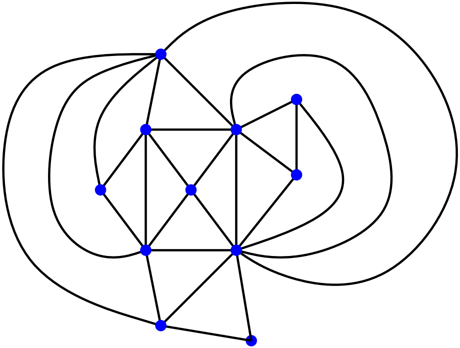

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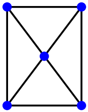
Belief: Isotropic enough so that at large scales this is close to a ball in the graph metric

First passage percolation on random planar maps III

Variant:

- Pick two edges on outer boundary of cluster



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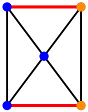

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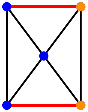

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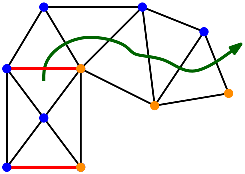

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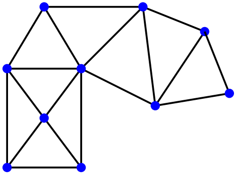

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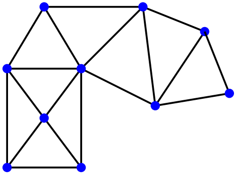

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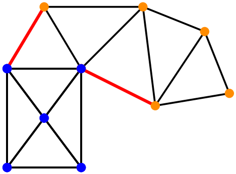

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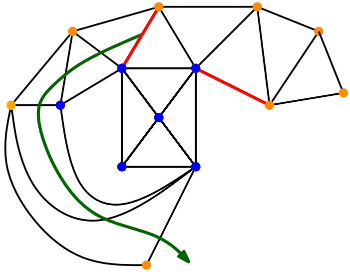

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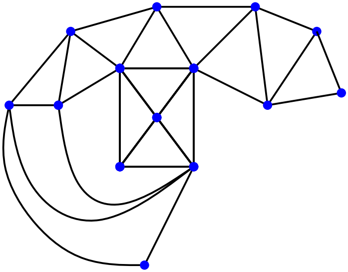

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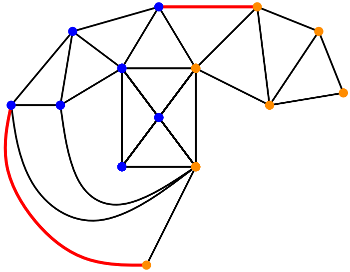

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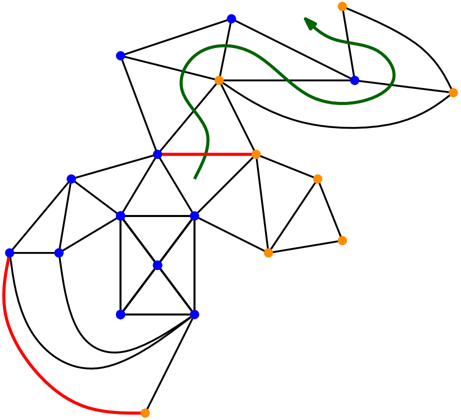

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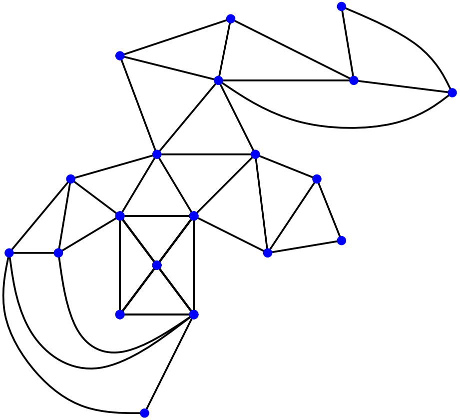

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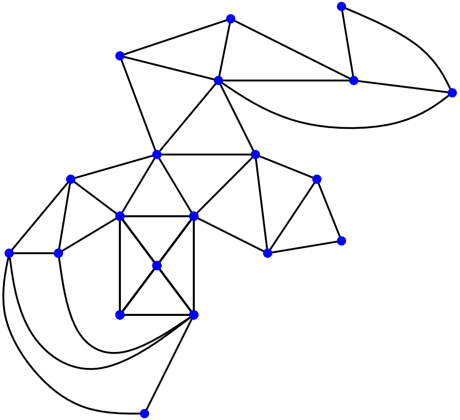

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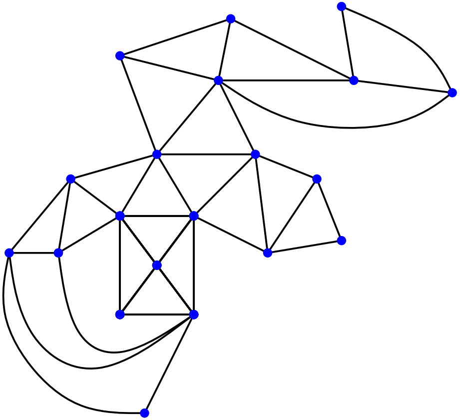

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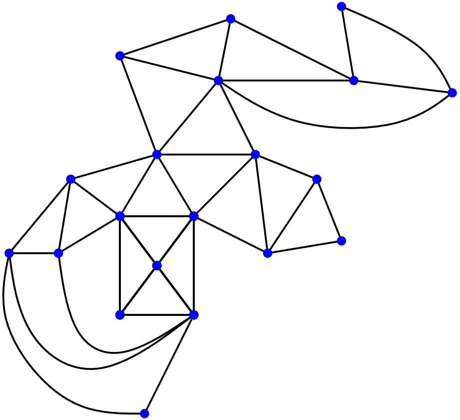
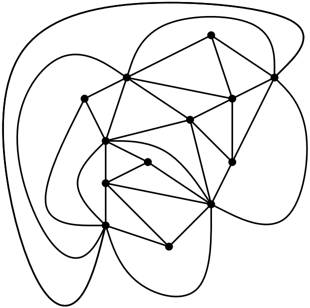


- *This exploration also respects the Markovian structure of the map.*
- If we work on an "infinite" planar map, the conditional law of the map in the unbounded component only depends on the boundary length.
- Expect that at large scales this growth process looks the same as FPP, hence the same as the graph metric ball

Continuum limit ansatz



- Sample a random planar map

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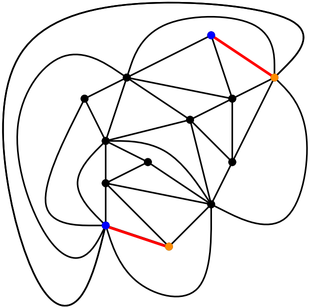

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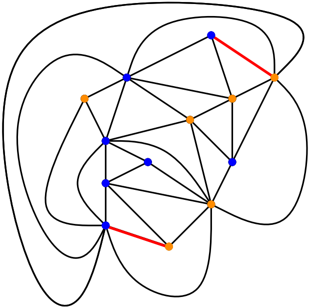

- Sample a random planar map and two edges uniformly at random
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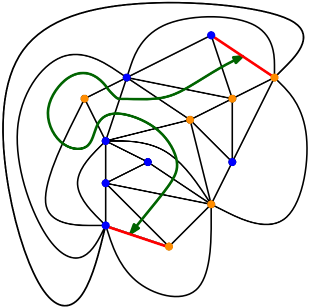

- Sample a random planar map and two edges uniformly at random
- Color vertices blue/yellow with probability $1/2$ and draw percolation interface

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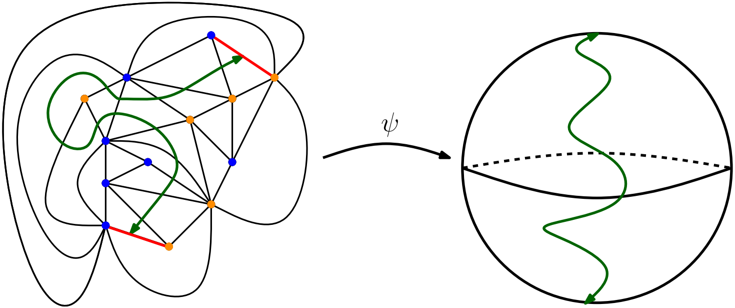

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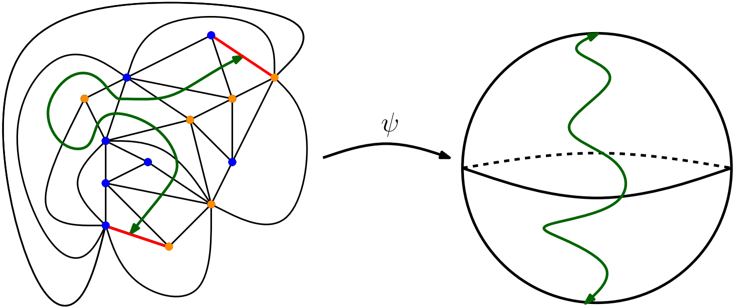

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Ansatz Image of random map converges to a $\sqrt{8/3}$-LQG surface and the image of the interface converges to an independent SLE_6.

Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point x

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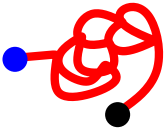

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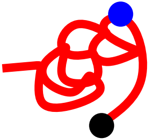

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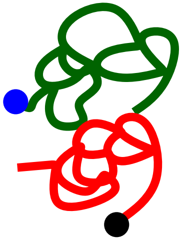

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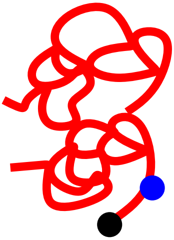

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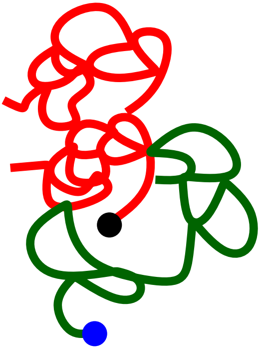

Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
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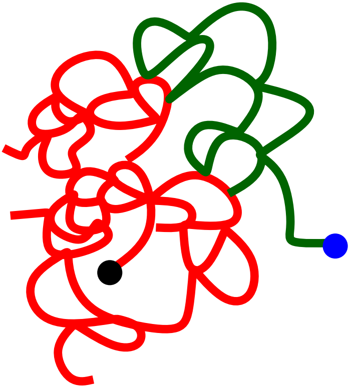

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$\mathrm{QLE}(8/3, 0)$ is SLE_6 with **tip re-randomization**. It can be understood as a "reshuffling" of the exploration procedure associated to the peanosphere.

What is $\mathrm{QLE}(\gamma^2, \eta)$?

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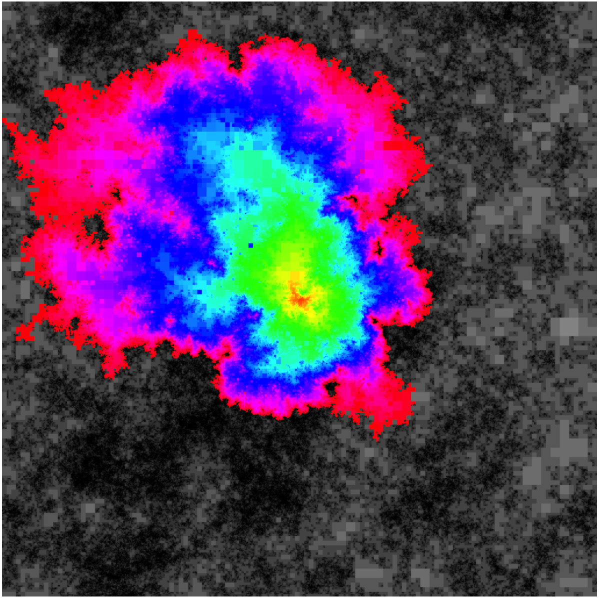
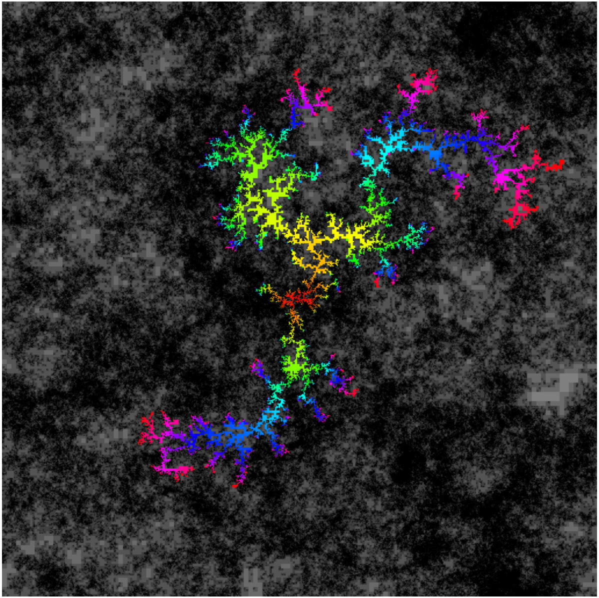
- **η-dieletric breakdown model:** general values of η

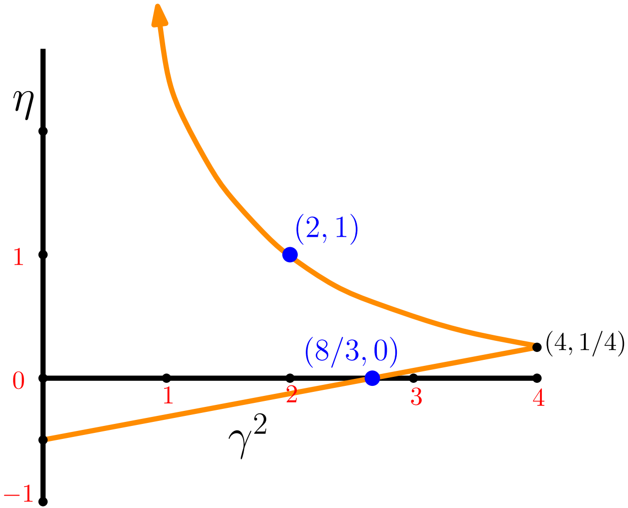


Discrete approximation of $\mathrm{QLE}(8/3, 0)$. Metric ball on a $\sqrt{8/3}$-LQG



Discrete approximation of $\mathrm{QLE}(2,1)$. DLA on a $\sqrt{2}$-LQG

$\mathrm{QLE}(\gamma^2, \eta)$ processes we can construct



Each of the $\mathrm{QLE}(\gamma^2, \eta)$ processes with (γ^2, η) on the orange curves is built from an SLE_κ process using tip re-randomization.

Results

What we can do:

- Existence of $\mathrm{QLE}(\gamma^2, \eta)$ on the orange curves as a Markovian exploration of a γ-LQG surface.
- Derive an SPDE which the measure valued diffusion satisfies
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Work in progress:

- Results on phases for sample path behavior: which QLEs are trees, have holes, and fill space (joint also with Ewain Gwynne and Xin Sun)
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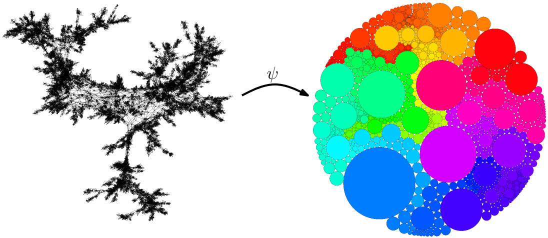
What we would like to do: construct and study $\mathrm{QLE}(\gamma^2, \eta)$ for (γ^2, η) pairs off the orange curves



Thanks!