Decay of scalar and electromagnetic waves on black hole space-times

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Stony Brook, Oct. 2014

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Outline.

1. Asymptotically flat space-times
2. Decay estimates for scalar waves
3. Local energy decay
4. Electromagnetic waves

Asymptotically flat nontrapping space-times

Domain: $\mathbb{R}^{3+1}$.

Lorenzian metric (signature $(3,1)$):

$$g = g_{\alpha\beta} dx^\alpha dx^\beta.$$

Space-like foliation: $t = const$, time-like normal $N = \nabla t$.

Asymptotically flat:

$$g = m + O(r^{-\epsilon}) \qquad \nabla g = O(r^{-\epsilon-1}).$$

$$m = -dt^2 + dx^2, \qquad \text{Minkowski}$$

Stationary: Killing field $X = \partial_t$, time-like.

Slowly varying: $\nabla_{\partial_t} g = O(\epsilon)$.

Nontrapping: all null geodesics escape to infinity.

Asymptotically flat black hole space-times (e.g. Schwarzschild, Kerr)

Domain: $\mathbb{R}^{3+1} \supset \mathcal{M} = \{r > r_0\}$,

Lorenzian metric: $g = g_{\alpha\beta} dx^\alpha dx^\beta$.

Space-like foliation: $t = const$, normal $N = \nabla t$, time-like.

Outgoing time-like inner boundary $r = r_0$.

Event horizon: $\mathcal{H} = \{r = r_{\mathcal{H}}\}$, $r_{\mathcal{H}} > r_0$.

Asymptotically flat: $g = m + O_{rad}(1/r) + O(1/r^2)$.

Null generator: $L = \nabla r$, tangent to $\mathcal{H}$, $\nabla_L L = \sigma L$.

Trapped set: $\mathcal{T} \subset \{r > r_{\mathcal{H}}\}$, compact.

Stationary: Killing field $X = \partial_t$, time-like outside a compact set.

Slowly varying: $\nabla_{\partial_t} g = O(\epsilon)$.

Scalar waves

Inhomogeneous wave equation:

$$\Box_g u = f, \qquad u[0] := (u(0), Nu(0)) = (u_0, u_1).$$

Also with magnetic field and/or potential.

Energy momentum tensor:

$$T_{\alpha\beta} = \partial_\alpha u \partial_\beta u - \frac{1}{2} g_{\alpha\beta} \partial^\nu u \partial_\nu u,$$

$$\nabla^\alpha T_{\alpha\beta} = 0, \qquad \nabla^\alpha (T_{\alpha\beta} X^\beta) = 0.$$

Conserved energy:

$$E = \int T(X, N) dV.$$

Positive definite in nontrapping case, positive definite outside a compact set in black hole case.

Decay estimates for wave equations

Equation:

$$(\Box_g + V)u = f, \qquad u[0] = (u_0, u_1).$$

Decay estimates for linear waves:

- Uniform energy bounds
- Local energy decay
- Strichartz estimates
- Pointwise decay

Goals:

- Do such properties hold for physically relevant space-times ?
- Characterization in terms of spectral properties
- Stability with respect to (time dependent) perturbations

Possible obstructions:

- Low frequency: eigenvalues, resonances
- High frequency: trapping

Uniform energy bounds and the resolvent

$$(E) \qquad \|u[t]\|_{\dot H^1\times L^2} \lesssim \|u[0]\|_{\dot H^1\times L^2}.$$

To define the resolvent take a time Fourier transform

$$\Box_g u = f \longrightarrow P_\tau \hat u(\tau) = \hat f(\tau) \longleftrightarrow \hat u(\tau) = R_\tau f(\tau)$$

In product case, $g = -dt^2 + g_0, \quad R_\tau = (\Delta_{g_0} + \tau^2)^{-1}$.
A-priori we have exponential bounds

$$\|u[t]\|_{\dot H^1\times L^2} \lesssim e^{Mt}\|u[0]\|_{\dot H^1\times L^2}.$$

so resolvent is well defined and holomorphic for $\Im\tau < -M$.

Proposition

Uniform energy bounds are equivalent to the resolvent bound

$$\|R_\tau\|_{L^2\to\dot H^1} \lesssim |\Im\tau|^{-1}, \qquad \Im\tau < 0$$

Eigenvalues (Must be on imaginary axis in product case.):

$$P_\tau u = 0, \qquad \Im\tau < 0$$

Local energy decay in Minkowski space-time

$$\Box\phi = 0 \qquad \text{in } \mathbb{R}^{n+1}, \qquad \phi[0] = (\phi_0, \phi_1).$$

Local energy decay (also known as *Morawetz estimates*):

$$\|\nabla_{x,t}\phi(x,t)\|_{L^2(\mathbb{R}\times B_R)} \lesssim R^{\frac{1}{2}}\|\nabla_{x,t}\phi(x,0)\|_{L^2}.$$

Heuristics: A speed 1 wave spends at most $O(R)$ time inside B_R.
Morawetz's proof uses the positive commutator method. If P and Q are selfadjoint, respectively skewadjoint operators then

$$2\Re\langle P\phi, Q\phi\rangle = \langle [Q,P]\phi, \phi\rangle$$

Apply this with

$$P = \Box, \qquad Q = \partial_r + \frac{n-1}{2r},$$

to obtain

$$\|r^{-\frac{1}{2}}\not\nabla\phi(x,t)\|_{L^2} + \|\phi(0,t)\|_{L^2} \lesssim \|\nabla_{x,t}\phi(x,0)\|_{L^2}, \qquad n = 3$$

The local energy norms

At the L^2 level we set

$$\|u\|_{LE} = \sup_k \|\langle r\rangle^{-\frac{1}{2}} u\|_{L^2(\mathbb{R}\times A_k)}, \qquad A_k = \{|x| \approx 2^k\} \times \mathbb{R}$$

We also define its H^1 counterpart, as well as the dual norm

$$\|u\|_{LE^1} = \|\nabla u\|_{LE} + \|\langle r\rangle^{-1} u\|_{LE} \quad \|f\|_{LE^*} = \sum_k \|\langle r\rangle^{\frac{1}{2}} f\|_{L^2(\mathbb{R}\times A_k)}$$

Sharp formulation of local energy decay:

$$(LE) \qquad \|u\|_{LE^1} + \|\nabla u\|_{L^\infty L^2} \lesssim \|\Box u\|_{LE^* + L^1 L^2} + \|\nabla u(0)\|_{L^2}$$

Proposition

Assume uniform energy bounds. Then local energy decay is equivalent to the uniform resolvent bound

$$\|R_\tau f\|_{LE^1_0} \lesssim \|f\|_{LE^*_0} \ , \qquad \Im \tau \le 0$$

Embedded resonances

These are obstructions to the resolvent local energy decay estimate,

$$\|R_\tau f\|_{LE_0^1} \lesssim \|f\|_{LE_0^*} \quad , \qquad \Im\tau \leq 0$$

On real axis R_τ is defined as the limit as $\Im\tau \to 0$. This implies the outgoing radiation condition

$$r^{-\frac{1}{2}}(\partial_r - i\tau)u \in L^2, \qquad u = R_\tau f.$$

Definition

$u \in LE_0^1$ is an embedded resonance associated to the real time frequency τ if it satisfies the outgoing radiation condition and $P_\tau u = 0$.

Local energy decay in geometries with trapping

Example: Schwarzschild space-time, with trapped set = all null geodesics tangent to the photon sphere $r = 3M$.

Redeeming feature: hyperbolic flow around trapped null geodesics.

Heuristics: frequency λ waves will stay localized up to time $\log \lambda$ (Ehrenfest time) near the trapped set, then disperse.

Consequence: $|\log \lambda|^{\frac{1}{2}}$ loss in (LE) at frequency λ on trapped set.

Modified local energy norm has log losses on the trapped set,

$$LE^1 \subset LE^1_{\mathcal{T}}, \qquad LE^*_{\mathcal{T}} \subset LE^*$$

with equality away from $\mathcal{T}$. Local energy decay:

$$(LE) \qquad \|u\|_{LE^1_{\mathcal{T}}} + \|\nabla u\|_{L^\infty L^2} \lesssim \|\Box u\|_{LE^*_{\mathcal{T}}+L^1L^2} + \|\nabla u(0)\|_{L^2}$$

Similar modification in resolvent bounds.

Strichartz estimates (averaged decay)

Range of indices in $3+1$ dimensions:

$$2 < p \leq \infty, \qquad \frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$$

Direct estimate for $\Box_g u = 0$:

$$\| |D_x|^{-\rho} \nabla u \|_{L^p L^q} \lesssim \|\nabla u_0\|_{L^2} + \|u_1\|_{L^2}, \quad \rho = \frac{3}{2} - \frac{1}{p} - \frac{3}{q}$$

Inhomogeneous estimate for $\Box_g u = f$, $u[0] = 0$

$$\|\nabla u\|_{L^\infty L^2} \lesssim \| |D_x|^{\rho} f \|_{L^{p'} L^{q'}}$$

Retarded estimate for $\Box_g u = f$, $u[0] = (u_0, u_1)$:

$$\| |D_x|^{-\rho} \nabla u \|_{L^p L^q} + \|\nabla u\|_{L^\infty L^2} \lesssim \|f\|_{|D_x|^{-\rho} L^{p'} L^{q'} + L^1 L^2} + \|u[0]\|_{\dot H^1 \times L^2}$$

Pointwise decay estimates (Price Law)

Set-up at infinity:

$$g = m + O_{rad}(r^{-1}) + O(r^{-2}), \qquad V = O_{rad}(r^{-3}) + O(r^{-4}).$$

(Improved) Price Law:

$$|u(t,x)| \lesssim \frac{1}{\langle t\rangle\langle t-|x|\rangle^2}\|\nabla u(0)\|_{H^{m,k}},$$

$$|\partial_t u(t,x)| \lesssim \frac{1}{\langle t\rangle\langle t-|x|\rangle^3}\|\nabla u(0)\|_{H^{m,k}}.$$

$$|\partial_x u(t,x)| \lesssim \frac{1}{\langle r\rangle\langle t-|x|\rangle^3}\|\nabla u(0)\|_{H^{m,k}}.$$

Remark

When true, the above decay rates are sharp due to the contribution of the leading order radial terms in the metric or potential.

Local Energy Decay as a central concept

Connection to Strichartz estimates:

Theorem (Metcalfe-T. '07 (nontrapping, nonstationary))

Assume that uniform energy bounds and local energy decay hold. Then the Strichartz estimates hold.

Idea: Outgoing parametrix with good pointwise decay estimates. The same method applies in the black hole setting, provided one has only hyperbolic trapping, and a good result near the trapped set $\mathcal{T}$ (e.g. Burq - Guillarmou-Hassell).

Connection to pointwise decay estimates:

Theorem (T. '09 (stationary), Metcalfe-T.-Tohaneanu '11 (non-stat.))

Assume that uniform energy bounds and local energy decay hold. Then the pointwise decay bounds hold (Price's Law).

Idea: Combine Klainerman's vector field method near the light cone with local energy decay inside the cone, reiterate.

Local energy decay in the nontrapping case

Theorem (Metcalfe-T.'08 (nonstationary))

Local energy decay holds if g is a small perturbation of Minkowski.

Theorem (Sterbenz-T. '14, also Marzuola-Metcalfe-T.'07 for Schrödinger)

(stationary)] Assume that no negative eigenfunctions and zero resonances exist for $\Box_g$. Then local energy decay holds.

A key element here is

Theorem (Kato '59, Agmon '69, ..., Koch-T. '05)

There are no (nonzero) resonances embedded in the continuous spectrum.

Theorem (Sterbenz-T. '14)

a) (stationary) Bifurcations to negative eigenfunctions for $\Box_g$ can occur only via zero resonances.
b) The stationary LE result above extends to slowly varying metrics.

The geometry of black hole space-times

Three distinct regions:

(i) Exterior region $r \gg 1$.
Assumption: asymptotically flat, $g = m + O(r^{-1})$.

(ii) Trapped set $\mathcal{T}$.
Assumptions: (a) hyperbolic trapping (e.g. Zworski-Wunsch),
(b) separate from horizon, and
(c) $\tau \neq 0$ on the trapped set (i.e. ∂_t energy positive there).

(iii) The event horizon $\mathcal{H}$.
Assumption: smooth, nondegenerate red shift and convexity.

Challenges:

- understand the coupling of three regions at high frequency
- the separation between the three regions is blurred at medium and low frequency.

A conditional local energy decay result

Theorem (Sterbenz-T., in progress)

For black hole space-times as above, assume that there are no eigenvalues in $\Im\tau < 0$, and no resonances on $\Im\tau = 0$. Then local energy decay holds. The converse is also true.

A key intermediate step in the above proof is to establish a high frequency local energy decay estimate,

$$(LE) \qquad \|u\|_{LE^1_{\mathcal{T}}} + \|\nabla u\|_{L^\infty L^2} \lesssim \|\Box u\|_{LE^*_{\mathcal{T}} + L^1L^2} + \|\nabla u(0)\|_{L^2} + \|u\|_{L^2_{loc}}.$$

We can also characterize eigenvalues and resonances:

Proposition

a) Eigenvalues and resonances can only occur in a compact subset of $\{\Im\tau \leq 0\}$.
b) Eigenvalues in $\Im\tau < 0$ are smooth and decay exponentially at infinity.
c) Resonances in $\Im\tau = 0$ are smooth and decay like r^{-1} at infinity.

A less conditional local energy decay

Here we make an additional assumption***, that the null generator L extends to a Killing vector field which is time-like near the horizon.

Theorem (Sterbenz-T., in progress)

For black hole space-times as above, assume that there are no eigenvalues in $\Im\tau < 0$, and no zero resonances. Then local energy decay holds.

Ideas:

- Absence of eigenvalues in $\Im\tau < 0 \implies$ subexponential decay.
- The extra assumption above guarantees via Carleman estimates from both infinity and from the horizon, that we have a weaker form of local energy decay for solutions in $[0, T]$, namely

$$(LE) \qquad \|u\|_{LE^1_{\mathcal{T}}} + \|\nabla u\|_{L^\infty L^2} \lesssim \|\Box u\|_{LE^*_{\mathcal{T}} + L^1 L^2} + \|\nabla u(0)\|_{L^2} + \|\nabla u(T)\|_{L^2}.$$

- Coupling the two pieces of information above leads to uniform energy bounds, and thus to local energy decay.

Continuity and stability of local energy decay

Theorem (Sterbenz-T., work in progress)

a) For continuous families of black hole space-times as above, eigenvalues can only bifurcate via a zero resonance.
b) The local energy decay result above is stable with respect to small stationary perturbations.
c) The local energy decay result above extends to slowly varying metrics.

*** Some extra condition is needed here near the trapped set.

- One can get local energy decay for Kerr with large a by continuity only by knowing that no zero resonances exist in Kerr.
- The trapped set dynamics are a-priori unstable with respect to small nonstationary nondecaying perturbations.

The Maxwell system

Electromagnetic field F = two form on (M, g).
1. Via differential forms:

$$dF = 0, d * F = 0$$

2. Using covariant differentiation:

$$\nabla^\alpha F_{\alpha\beta} = 0, \qquad \nabla_{[\gamma} F_{\alpha\beta]} = 0$$

3. Using electromagnetic potential A, $F = dA$:

$$\nabla^\alpha \nabla_\alpha A_\beta = 0, \qquad \nabla^\alpha A_\alpha = 0 \quad \text{(gauge condition)}$$

4. Expressed in a reference frame (Neumann-Penrose formalism)

The Maxwell energy

Energy-momentum tensor

$$T_{ij} = g^{kl} F_{ik} F_{lj} + \frac{1}{4} g_{ij} F_{kl} F^{kl}$$

$$\nabla^i T_{ij} = 0$$

If X is Killing then

$$\nabla^i (T_{ij} X^j) = 0$$

and one obtains a conserved energy,

$$E_X(F) = \int_{\Sigma_t} * i_X T = \int_{\Sigma_t} \nu^i T_{ij} X^j dV_\Sigma$$

Positive definite if X is timelike and Σ is space-like. Then

$$E_X(F) \approx \|F\|^2_{L^2(\Sigma_t)}$$

General considerations

- the same three high frequency regions: (i) the exterior region, (ii) the trapped region and (iii) the event horizon, with the same high frequency energy dynamics
- the red shift effect is effective at the level of L^2 solutions for familiar space-times (e.g. Schwarzschild/Kerr)
- additional difficulty at zero frequency arising from charges.
- Modified form of local energy decay, to account for charges.

The low frequencies and charges

For a closed two dimensional surface S define the electric charge inside S by

$$Q = \int_S F$$

Magnetic charge inside S:

$$Q^* = \int_S F^*$$

It is natural to take S which includes the black hole inside. Then these are conserved quantities for the homogeneous problem.
Hodge dual stationary solutions in Schwarzschild:

$$F_0 = \frac{Q}{4\pi} d\omega_{\mathbb{S}^2}, \qquad F_0^* = \frac{Q^*}{4\pi} r^{-2} dr \wedge dt$$

There is a straightforward modification for Kerr.

Local energy decay

Bound for the homogeneous equation:

$$\|F\|_{LE_{\mathcal{T}} \cap L^\infty L^2} \lesssim \|F(0)\|_{L^2}$$

for charge free solutions.

Inhomogeneous equation:

$$dF = G, \qquad dF^* = G^*$$

Modified local energy decay:

$$\|F\|_{LE_{\mathcal{T}}} + \|rF_{rad}\|_{LE} \lesssim \|F(0)\|_{L^2} + \|(G, G^*)\|_{LE^*_{\mathcal{T}}} + \|r(G, G^*)_{rad}\|_{LE^*}$$

Poinwise decay

Price law:

$$|F| \lesssim \frac{1}{\langle r\rangle\langle t-r\rangle^3}$$

Peeling estimates (Penrose, Klainerman)

$$|F(\bar{L},e)| \lesssim \frac{1}{\langle r\rangle\langle t-r\rangle^3}$$

$$|F(\bar{L},L)| + |F(e,e)| \lesssim \frac{1}{\langle r\rangle\langle t\rangle\langle t-r\rangle^2}$$

$$|F(L,e)| \lesssim \frac{1}{\langle r\rangle\langle t\rangle^2\langle t-r\rangle}$$

Here $L = \partial_t + \partial_r$, $L^* = \partial_t - \partial_r$.
Null frame $(L, \bar{L}, e_A, e_B)$.

The results so far

Theorem (Sterbenz-T ’13)

Consider a spherically symmetric black hole space-time as above. Then:
a) Uniform energy estimates hold for Maxwell.
b) Local energy decay holds for Maxwell.

Ongoing work: Spectral characterization of local energy decay for nonradial metrics, similar to the scalar case

Theorem ((Price Law) Metcalfe-Tohaneanu-T. ’14)

Assume that uniform energy estimates and local energy decay hold for Maxwell. Then pointwise decay estimates hold.

This last result does not require the metric to be radial or stationary.