Energy cascades and weak turbulence for nonlinear dispersive equations

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Outline

1. Introduction: The general problematic and our model equation
2. First approach: Energy cascades and growth of Sobolev norms
   - Recent progress on Bourgain's infinite Sobolev norm growth conjecture
3. Second Approach: Deriving effective equations (Weak turbulence theory)
   - The weakly nonlinear large-box limit of NLS (Faou-Germain-H.)
   - Properties of the limiting equation
   - Rigorous approximation results
4. Further Directions

Outline

Nonlinear dispersive equations

- What are dispersive equations? (e.g. Nonlinear Schrödinger (NLS), nonlinear wave (NLW), water waves, Einstein's equations of GR, etc.).
- Dispersion= Solutions (or waves packets) with different frequencies travel with different velocities.
- On unbounded domains like $\mathbb{R}^d$, dispersion is a mechanism of *non-dissipative* decay:

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \lesssim t^{-\alpha} \qquad \text{for some} \quad \alpha > 0.$$

- Decay $\Rightarrow$ Nonlinear Asymptotic stability of equilibrium sol'ns on $\mathbb{R}^d$ (e.g. small-data scattering, stability of Minkowski (or black hole?) spaces in general relativity, theory of elasticity, etc.)

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Out-of-equilibrium dynamics on bounded domains

- On compact domains, wave packets interact indefinitely.
- Dispersion $\not\Rightarrow$ Decay neither at the linear nor the nonlinear level.
- Consequence: Loss of all asymptotic stability results of equilibrium solutions for nonlinear dispersive equations on compact domains.
- Out-of-equilibrium dynamics is anticipated.

Problematic: How to understand, explain, and capture this out-of-equilibrium dynamics?

Energy cascades

- One main aspect of this out-of-equilibrium dynamics is Energy-cascade: Energies of the system (while remaining conserved) move their concentration zones between characteristically different length-scales.
- Direct Cascade of energy: Migration of energy from low to arbitrarily high frequency concentration zones (small scales).
- **Question:** How to capture this cascade? More generally, how to understand the out-of-equilibrium dynamics?

Two approaches

1. Growth of Sobolev Norm Approach: Search for solutions whose high Sobolev norms grow in time.

$$\|u(t)\|_{H^s(\mathbb{T}^d)} = \sum_{|\alpha|\leqslant s} \|\nabla^\alpha u\|_{L^2(\mathbb{T}^d)} \sim \left(\sum_{n\in\mathbb{Z}^d} (1+|n|^2)^s |\widehat{u}(n)|^2\right)^{1/2}.$$

   Bourgain, Staffilani, Kuksin, Tao, Colliander, Keel, Takaoka, H., Kaloshin, Guardia, etc.

2. Effective dynamics approach: Derive effective equations for the dynamics by taking various limits of the original system.
   Weak (wave) turbulence theory. Peierls (1929), Hasselman (1962), Zakharov et al., Majda-Mclaughlin-Tabak (1997–), etc.

Our model: Cubic NLS on a periodic box

We consider the 2D cubic nonlinear Schrödinger equation (NLS) on $\mathbb{T}_L^2 := [0,L] \times [0,L]$ with periodic boundary conditions (dimension could be higher):

$$\begin{cases} -i\partial_t v(t,x) + \Delta v(t,x) & = \lambda |v(t,x)|^2 v(t,x), \quad \lambda \in \{+1,-1\}, \\ v(0) & = v_0, \end{cases}$$

- Solutions exist globally at least for $\|v_0\|_{L^2} \leqslant \epsilon$ (Bourgain '93).
- Aim: Understand out-of-equilibrium dynamics of small initial data.
- Ansatz $v(t,x) = \epsilon u(t,x)$ with $\|u_0\|_{L^2(\mathbb{T}_L^2)} \sim 1 \rightsquigarrow$ Weak nonlinearity.

$$\begin{cases} -i\partial_t u(t,x) + \Delta u(t,x) & = \epsilon^2 \lambda |u(t,x)|^2 u(t,x) \\ u(0) & = u_0, \end{cases} \qquad (\text{NLS}_\epsilon)$$

Fourier Picture

- Functions on $\mathbb{T}_L^2$ can be expanded in Fourier series ($K \in \mathbb{Z}_L^2 := \mathbb{Z}^2/L$)

$$f(x) = \frac{1}{L^2} \sum_{K \in \mathbb{Z}^2/L} a_K \, e^{2\pi i K.x}, \qquad a_K := \int_{\mathbb{T}_L^2} f(x) e^{-2\pi i K.x} \, dx.$$

- Expanding the solution $u(t) = \frac{1}{L^2} \sum_{K \in \mathbb{Z}_L^2} a_K(t) \, e^{2\pi i K.x}$. We get that (up to a phase factor):

$$-i\partial_t a_K(t) = \lambda \frac{\epsilon^2}{L^4} \sum_{(K_1,K_2,K_3) \in \mathcal{S}_K} a_{K_1}(t) \overline{a_{K_2}(t)} a_{K_3}(t) e^{4\pi^2 i \Omega t} \qquad \text{(NLS)}$$

$$\mathcal{S}_K = \{(K_1, K_2, K_3) \in (\mathbb{Z}_L^2)^3 \; : \; K_1 - K_2 + K_3 = K\}$$

$$\Omega = |K_1|^2 - |K_2|^2 + |K_3|^2 - |K|^2.$$

- Resonant interactions: $\mathcal{R}(K) = \mathcal{S}(K) \cap \{\Omega = 0\}$ are most important.

$$-i\partial_t r_K = \lambda \frac{\epsilon^2}{L^4} \sum_{(K_1,K_2,K_3) \in \mathcal{R}_K} r_{K_1}(t) \overline{r_{K_2}(t)} r_{K_3}(t) \qquad \text{(RNLS)}$$

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Strichartz Estimates

- Strichartz Estimates: $L^p_{t,x}$-estimates on linear solutions.

$$(-i\partial_t + \Delta)u = 0, \quad u(0) = \varphi; \qquad \boxed{u_{lin}(t) := e^{-it\Delta}\varphi}.$$

Crucial for low-regularity existence questions (Bourgain, K-P-V, Tao, etc.).

- The relevant Strichartz estimate in 2D is:

$$\|e^{-it\Delta}P_N\varphi\|_{L^4_{t,x}([-1,1]\times\mathbb{T}^2)} \leqslant C(N)\|\varphi\|_{L^2}.$$

  - $C(N) \leqslant C_\epsilon \exp(\frac{c\log N}{\log\log N}) \ll N^\epsilon$ for all $\epsilon > 0$ (Bourgain '93).
  - $C(N) \geqslant C(\log N)^{1/4}$. Counterexample: $\widehat{\phi}(k) = \mathbf{1}_{B(0,10N)}$, $k \in \mathbb{Z}^2$.
  - **Question:** What is the sharp dependence of $C(N)$ on N?

$$\boxed{\|e^{it\Delta}P_N\phi\|_{L^4_{t,x}(\mathbb{T}^2\times[0,1])} \lesssim (\log N)^{1/4}\|\phi\|_{L^2(\mathbb{T}^2)}??}$$ Bourgain '93, '96.

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Quest for unbounded Sobolev orbits

- Movement of energy to high-frequency regions leads to the increase in the H^s Sobolev norms for $s > 1$

$$\|u(t)\|_{H^s(\mathbb{T}^d)} = \left(\sum_{n\in\mathbb{Z}^d} (1+|n|^2)^s |\widehat{u}(n)|^2\right)^{1/2}$$

Conjecture (Bourgain GAFA 2000)

There exists (many) global solutions to cubic NLS whose H^s norm ($s > 1$) exhibits infinite growth in time, i.e.

$$\limsup_{t\to+\infty} \|u(t)\|_{H^s} = +\infty$$

- This is sometime called the "unbounded orbits conjecture".

First progress

- Colliander, Keel, Staffilani, Takaoka, and Tao constructed solutions with arbitrary large but finite growth:

Theorem (CKSTT; Inventiones 2008)

Let $s > 1$ and $d \geqslant 2$. For any $\delta \ll 1$ and $K \gg 1$, there exists a solutions $u(t)$ of cubic NLS on $\mathbb{T}^d$ and a time T such that

$$\|u(0)\|_{H^s} \leqslant \delta \quad \text{but} \quad \|u(T)\|_{H^s} \geqslant K.$$

- Regard as long-time strong instability of the zero solution.

Theorem (H.; ARMA 2012)

There exists solutions to the resonant cubic NLS (RNLS) on $\mathbb{T}^d$ $(d \geqslant 2)$ that exhibit infinite growth of high Sobolev norms. The same is true for a family of systems approximating (NLS) arbitrarily closely.

Outline

Unbounded orbits for cubic NLS on $\mathbb{R} \times \mathbb{T}^d$

- Consider the cubic NLS equation posed on $\mathbb{R} \times \mathbb{T}^d$ ($d \geqslant 2$).
- Modified scattering to the resonant dynamics: Sol'ns to NLS converge to solutions of its resonant system [H.-Pausader-Tzvetkov-Visciglia].
- Combining this to [H. 2012] gives

Theorem (H.-Pausader-Tzvetkov-Visciglia 2013)

For any $d \geqslant 2$ and $\varepsilon > 0$, there exists global H^s ($s > 1$) solutions to the cubic NLS equation on $\mathbb{R} \times \mathbb{T}^d$ satisfying

$$\|u(0)\|_{H^s(\mathbb{R}\times\mathbb{T}^d)} \leqslant \varepsilon \qquad \text{and} \qquad \limsup_{t\to\infty} \|u(t)\|_{H^s(\mathbb{R}\times\mathbb{T}^d)} = +\infty.$$

- This gives the first rigorous results on infinite energy cascade for *any* natural nonlinear dispersive equation.

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Outline

Weak (or wave) turbulence theory

- Aim: Statistical description of out-of-equilibrium dynamics of solutions [Zakharov 60's after Kolmogorov 41].
- Setup: Random initial data (random phase and amplitude RPA): $a_K(0)$ are independent random variable ($K \in \mathbb{Z}_L^2$).
- Key quantity: $n(K,t) := \mathbb{E}|a_K(t)|^2$. wave spectrum/ mass density.
- Limits taken in the formal derivation of the effective eq'n for $n(K)$:
  1. Statistical and time averaging (particularly non-rigorous).
  2. Large-box limit ($L \to \infty$).
  3. weak-nonlinearity limit ($\epsilon \to 0$).

This gives an effective equation for $n(K,t)$ ($K \in \mathbb{R}^2$): The Kolmogorov-Zakharov equation.

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The weakly nonlinear large-box limit (jointly with E. Faou and P. Germain)

- Infinite-volume approximation A physical system might be more effectively modeled $\mathbb{R}^d$ rather than the box of size L.
- For well-localized data (frequency ~ 1) of size ϵ, there are two relevant time scales
  1. Time to reach the boundary is $T_1 \sim L$ (wave moves with speed ~ 1).
  2. Time for the nonlinearity to take effect $T_{nl} \sim \epsilon^{-2}$.
- Compare!
  - If $T_1 \gg T_{nl} \Leftrightarrow L \gg \epsilon^{-2} \rightarrow$ Use NLS on $\mathbb{R}^2$.
  - If $T_1 \ll T_{nl} \Leftrightarrow L \ll \epsilon^{-2}$, the wave feels the "boundary" before the nonlinearity kicks in. We are interested in this weakly nonlinear regime.
- In this regime, we will see that a new equation dictates the nonlinear dynamics for (NLS).

Deriving the weakly nonlinear large-box limit

- Argue formally. Consider the (NLS) for $a_K(t)$ (Now $K \in \mathbb{Z}_L^2$!).
- Due to the weak nonlinearity regime we are in, one can approximate the NLS flow with the resonant flow.

$$\boxed{-i\partial_t a_K(t) = \frac{\epsilon^2}{L^4} \sum_{(K_1,K_2,K_3)\in\mathcal{R}(K)} a_{K_1}(t)\overline{a_{K_2}(t)}a_{K_3}(t)} \qquad \text{(RNLS)}$$

where $\mathcal{R}(K) = \{(K_1, K_2, K_3) \in \mathbb{Z}_L^2 : K_1 - K_2 + K_3 = K,\ \Omega := |K_1|^2 - |K_2|^2 + |K_3|^2 - |K|^2 = 0\}$.

- Now we want to take the large box limit $\rightsquigarrow$ Let $L \to \infty$.
- Let $(K_1, K_2, K_3) \in \mathcal{R}(K)$. Set $N_i = K_i - K$ $(i = 1, 2, 3)$ $\rightsquigarrow N_2 = N_1 + N_3$ & $|N_2|^2 = |N_1|^2 + |N_3|^2 \Rightarrow N_1 \perp N_3$.

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Parametrization of rectangles in $\mathbb{Z}^2/L$

$$-i\partial_t a_K(t) = \frac{\epsilon^2}{L^4} \sum_{\substack{N_1, N_3 \in \mathbb{Z}_L^2 \\ N_1 \perp N_3}} a_{K+N_1}(t)\overline{a_{K+N_1+N_3}(t)}a_{K+N_3}(t)$$

- $N_1 = \alpha(p,q)/L$ with $\alpha \in \mathbb{N}$ and $(p,q) \in \mathbb{Z}^2$ satisfying $\mathrm{g.c.d}(|p|,|q|) = 1$. Then $N_3 = \beta(-q,p)/L$ for some $\beta \in \mathbb{Z}$.
- A lattice point $J \in \mathbb{Z}_L^2$ is called visible if $J = (p,q)/L$ with $\mathrm{g.c.d}(|p|,|q|) = 1$.
- Writing $N_1 = \alpha J$ and $N_3 = \beta J^\perp$, with J visible, one obtains

Resonant NLS in new coordinates

$$-i\partial_t a(K) = \frac{\epsilon^2}{L^4} \sum_{\alpha\in\mathbb{N},\beta\in\mathbb{Z}} \sum_{\substack{J\in\mathbb{Z}_L^2 \\ \textit{visible}}} a(K + \overbrace{\alpha J}^{N_1}) a(K + \overbrace{\beta J^{\perp}}^{N_3}) \bar{a}(K + \overbrace{\alpha J + \beta J^{\perp}}^{N_2})$$

- Passing to the large box limit ($L \to \infty$) amounts to replacing the above sums by integrals.
- To do this we need information about the equidistribution of visible lattice points+quantitative error estimates.

Co-prime equidistribution

- Equidistribution: $L^{-2}\sum_{K\in\mathbb{Z}^2_L} u(K) \to \int_{\mathbb{R}^2} u(z)dz$ as $L\to\infty$ provided say that u is sufficiently well-behaved (like $u\in L^1$, $\nabla u \in L^1$).
- Key point: Density of visible lattice points in $\mathbb{Z}^2_L$ is $\frac{1}{\zeta(2)}=\frac{6}{\pi^2}$. I.e. $L^{-2}\#\{J\in\mathbb{Z}^2_L\cap\Omega : J \text{ visible}\}\to\frac{\mathrm{Vol}\,\Omega}{\zeta(2)}$ as $L\to\infty$ (classical).

Proposition (Co-prime equidistribution)

Suppose that u is sufficiently nice (say $|u|+|\nabla u|\in\langle K\rangle^{-2-\delta}L^\infty(\mathbb{R}^2)$), then for $L\gg 1$

$$\left|L^{-2}\sum_{\substack{J\in\mathbb{Z}^2/L\\ J\ \text{visible}}} u(J)-\frac{1}{\zeta(2)}\int_{\mathbb{R}^2}u(z)dz\right| = O\left(\frac{\log L}{L}\right),\quad \zeta(2)=\frac{\pi^2}{6}.$$

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Continuum limit

- Using this info., we get (formally) that a_K satisfies:

$$-i\partial_t a(K,t) = \frac{1}{T^*}\int_{-1}^{1}\int_{\mathbb{R}^2} a(K+\lambda z)\bar{a}(K+\lambda z+z^\perp)a(K+z^\perp)dz\,d\lambda$$

where $T^* \stackrel{def}{=} \frac{\zeta(2)L^2}{2\epsilon^2\log L} \sim \frac{L^2}{\epsilon^2\log L}$ $(\gg \epsilon^{-2}!)$.

- Reparametrizing time $t = T^*\tau$, we get formally that

$$-i\partial_\tau a(K,\tau) = \int_{-1}^{1}\int_{\mathbb{R}^2} a(K+\lambda z)\bar{a}(K+\lambda z+z^\perp)a(K+z^\perp)\,dz\,d\lambda.$$

The Continuous Resonant equation (CR)

$$\boxed{\begin{aligned} -i\partial_t g(\xi,t) &= \mathcal{T}(g,g,g)(\xi,t); \qquad \xi \in \mathbb{R}^2 \\ \mathcal{T}(g,g,g)(\xi,t) &= \int_{-1}^{1}\int_{\mathbb{R}^2} g(\xi+\lambda z,t)\overline{g}(\xi+\lambda z+z^\perp)g(\xi+z^\perp)\,dz\,d\lambda. \end{aligned}} \tag{CR}$$

- $g : \mathbb{R}_t \times \mathbb{R}^2_\xi \to \mathbb{C}$.
- Analogue of K-Z equation.
- It is Hamiltonian (like NLS):

$$\mathcal{H}(g) = \frac{1}{2}\int_{-1}^{1}\int_{\mathbb{R}^2_\xi\times\mathbb{R}^2_z} \overline{g}(\xi)g(\xi+\lambda z)\overline{g}(\xi+\lambda z+z^\perp)g(\xi+z^\perp)\,dz\,d\lambda$$

$$= \frac{1}{2}\int_{\mathbb{R}_s}\int_{\mathbb{R}^2_x} |e^{is\Delta_{\mathbb{R}^2}} g(x)|^4 dsdx \quad \to L^4_{t,x} \text{ Strichartz norm!}$$

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Conserved quantities

Proposition

The following quantities are conserved by the flow of (CR)*:*

- *The Hamiltonian* $\mathcal{H}$,
- *Mass:* $\int |g(x)|^2 \, dx$.
- *Momentum:* $\int \xi \, |\widehat{g}(\xi)|^2 \, d\xi$.
- *Position:* $\int x|g(x)|^2 \, dx$.
- *Second moment* $\int |x|^2|g(x)|^2 \, dx$.
- *Kinetic energy:* $\int |\nabla g(x)|^2 \, dx$.
- *Angular momentum* $\int (x \times \nabla) g(x)\overline{g(x)} \, dx$.

Fourier transform

Theorem (Invariance under Fourier transform)

If $g(t)$ is a solution of (CR), *the so is $\widehat{g}(t) := \mathcal{F}g(t)$. Moreover,*

$$\mathcal{H}(f) = \mathcal{H}(\widehat{f}) \qquad \textit{for any function } f \in L^2.$$

Invariance of Harmonic oscillator eigenspaces

- The quantum harmonic oscillator $H = -\Delta + |x|^2$ admits an orthonormal basis of eigenvectors for $L^2(\mathbb{R}^2)$.
- The eigenspaces E_k correspond to the eigenvalue $2k$ $(k = 1, 2, ...)$. They are k-dimensional and are spanned by $k - th$ order Hermite functions (e.g. $E_0 = \text{Span}\{e^{-\frac{|x|^2}{2}}\}$).

Theorem

The spaces E_k are invariant by the nonlinear flow of (CR), i.e. if $g_0 \in E_k$, then $g(t) \in E_k$ for all $t \in \mathbb{R}$.

Global well-posedness

Global well-posedness= global existence+uniqueness+continuous dependence on initial data.

Theorem (Global well-posedness)

- *Equation* (CR) *is globally well-posed in* $L^2(\mathbb{R}^2)$, *i.e. for any* $g_0 \in L^2(\mathbb{R}^2)$, *there exists a unique global solution* $g(t) \in C_t L^2(\mathbb{R}_t \times \mathbb{R}^2)$.
- *Equation* (CR) *is globally well-posed in* $H^s(\mathbb{R}^2)$ *for any* $s \geqslant 0$.
- *Equation* (CR) *is globally well-posed in* $H^{0,s}(\mathbb{R}^2) := \langle x \rangle^{-s} L^2$ *for any* $s \geqslant 0$.
- *Equation* (CR) *is globally well-posed in* $X^\sigma(\mathbb{R}^2) = \langle x \rangle^{-\sigma} L^\infty$ *for any* $\sigma > 2$.

Explicit Stationary Solutions

- Gaussian Family: For any $\alpha \in \mathbb{C}$ satisfying $\mathrm{Re}\,\alpha > 0$, there exists a constant $\omega = \omega(\alpha)$ such that

$$g(t,\xi) = e^{i\omega t} e^{-\alpha|\xi|^2} \quad \text{solves} \quad (\mathrm{CR}).$$

  Applying the symmetry group of the equation we obtain a 7-dim. manifold of stationary solutions +Orbital Stability.

- "Raleigh-Jeans" solution

$$g(t,\xi) = \frac{e^{i\omega' t}}{|\xi|} \quad \text{solves} \quad (\mathrm{CR}) \qquad \text{corresponds to } n(\xi) = |\xi|^{-2} \text{of (KZ).}$$

- Many other explicit stationary solutions at higher energy levels of the harmonic oscillator.

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Outline

From Equation (CR) to NLS

- Ultimately, we would like to project the dynamics of (CR) onto that of (NLS) on the box $\mathbb{T}^2_L$ of finite size. Suppose we have a solution $g(t,\xi)$ of (CR) on an interval $[0, M]$ (with M arbitrarily large). We would like to construct a solution of (NLS) that carries the dynamics of $g(t)$.
- For this, we start with a solution of NLS with initial data $a_K(0) = g(0, K)$.
- Recall that formal derivation of (CR) tells us that

$$-i\partial_t a_K \overset{formally}{``="} \frac{1}{T^*}\mathcal{T}(a,a,a) \qquad T^* = \frac{\zeta(2)L^2}{2\epsilon^2 \log L}.$$

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Difficulties

1. Passing to the resonant system: Tools from dynamical systems (Normal forms transformations). $\rightsquigarrow$

$$-i\partial_t a_K = \underbrace{\epsilon^2 L^{-4} \sum_{\mathcal{R}(K)} a_{K_1}\overline{a_{K_2}}a_{K_3}}_{resonant\ interactions} + \underbrace{O(\epsilon^4 L^{0+})}_{\substack{contribution\ of \\ non-resonant\ interactions}}$$

2. Obtaining good disc. to cont. error estimates: Tools from analytic number theory (Möbius inversion formula).
3. Sharp estimates on resonant sums: Tools from Harmonic analysis and analytic number theory (Periodic Strichartz estimates).

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2. Obtaining good disc. to cont. error estimates: Tools from analytic number theory (Möbius inversion formula).
3. Sharp estimates on resonant sums: Tools from harmonic analysis and analytic number theory (Periodic Strichartz estimates).

Estimates on resonant sums

$$\Big\| \sum_{\mathcal{R}(K)} a_{K_1} \overline{b_{K_2}} c_{K_3} \Big\|_{X(\mathbb{Z}^2_L)} \leqslant C(L) \|a_K\|_{X(\mathbb{Z}^2_L)} \|b_K\|_{X(\mathbb{Z}^2_L)} \|c_K\|_{X(\mathbb{Z}^2_L)} \qquad (*)$$

- Formal argument gives that $C(L) \sim L^2 \log L$ if $\{a_K\}, \{b_K\}, \{c_K\}$ are "smooth".
- If $X = \langle K \rangle^{-\sigma} \ell^2_L$ (Sobolev space), (*) is equivalent to the (still open!)

$$\Big\| e^{it\Delta_{\mathbb{T}^2}} P_N \phi \Big\|_{L^4_{t,x}([0,1]\times\mathbb{T}^2)} \overbrace{\leqslant}^{???} C(\log N)^{1/4} \|\phi\|_{L^2(\mathbb{T}^2)} \quad \text{[Bourgain 93, 96].}$$

- We prove (*) in the space $X^\sigma = \langle K \rangle^{-\sigma} \ell^\infty$ for $\sigma > 2$ with the sharp constant $L^2 \log L$. Corollary: Periodic Strichartz estimates at critical scaling.

Discrete weak turbulence regime

$$-i\partial_t a_K = \underbrace{\epsilon^2 L^{-4} \sum_{\mathcal{R}(K)} a_{K_1}\overline{a_{K_2}}a_{K_3}}_{O(\frac{\epsilon^2 \log L}{L^2}) \leftarrow in\ X^\sigma\ by\ (*)} + O(\epsilon^4 L^{0+})$$

- For the resonant sum to drive the dynamics, we need

$$\text{Resonant Inter.} \sim \frac{\epsilon^2 \log L}{L^2} \gg \epsilon^4 L^{0+} \quad \text{i.e.} \quad \boxed{\epsilon \ll \frac{1}{L^{1+}}}.$$

Discrete wave turbulence regime.

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Discrete wave turbulence regime.

Discrete to continuous error estimates

Proposition

Assuming that $g : \mathbb{R}^2 \to \mathbb{C}$ *is "reasonably nice"* $(g, \nabla g \in \langle \xi \rangle^{-3-\delta} L^\infty$ *is enough), then*

$$\left\| \frac{1}{L^2 \log L} \sum_{\mathcal{R}(K)} g(K_1)\overline{g}(K_2)g(K_3) - \frac{1}{\zeta(2)} \mathcal{T}(g,g,g)(K) \right\|_{X^\sigma} \leqslant \frac{C}{\log L}$$

- Proof relies on some analytic number theory (Möbuis inversion formula).

Approximation theorem on $\mathbb{T}_L^2$

Theorem (Faou-Germain-H. 2013)

- *Fix $\sigma > 2$. Suppose that $g(t,\xi)$ is a solution of the* (CR) *on an interval* $[0, M]$ *such that $g_0, \nabla g_0 \in X^{\sigma+1}$. Recall that* (CR) *is globally well-posed for such initial data.*
- *For any $L \geqslant L_0(M)$ and any $\epsilon \ll L^{-1}$, let $a_K(t)$ be the solution of NLS with initial data $\underbrace{a_K(0)}_{=\widehat{u}_0(K)} = g_0(K)$ (so $\|u_0\|_{L^2} \sim 1$).*
- *THEN*

$$\left\| a_K(t) - g\left(\frac{t}{T^*}, K\right) \right\|_{X^\sigma(\mathbb{Z}_L^2)} \leqslant \frac{C}{\log L}.$$

for all $0 \leqslant t \leqslant MT^*$, where $T^* = \frac{\zeta(2)L^2}{2\epsilon^2 \log L}$.

Approximate NLS solutions on $\mathbb{T}^2$

Corollary (Faou-Germain-H. 2013)

- *Fix $s > 1$. Suppose that $g(t,\xi)$ is a solution of* (CR) *over an interval* $[0, M]$ *with initial data $g_0 = g(0,\xi)$ such that $g_0, \nabla g_0 \in X^{s+3}(\mathbb{R}^2)$. Recall that* (CR) *is globally well-posed for such initial data.*
- *Let $N \geqslant N_0(M)$. Define $v(t)$ to be the solution to (NLS) with initial data $\widehat{v}(t=0,k) := N^{-1-s} g_0(\frac{k}{N})$ for all $k \in \mathbb{Z}^2$ (so that $\|v(0)\|_{H^s(\mathbb{T}^2)} \sim \|g_0\|_{H^{0,s}(\mathbb{R}^2)} \sim 1$ uniformly in N).*
  THEN

$$\left\| v(t) - \mathcal{F}^{-1}\left\{ e^{4\pi^2 i|k|^2 t} N^{-1-s} g\left(\frac{t}{T_0}, \frac{k}{N}\right)\right\}\right\|_{H^s(\mathbb{T}^2)} \leqslant \frac{C}{\log N}$$

  for all $0 \leqslant t \leqslant T_0 M$ where $T_0 = \frac{\zeta(2) N^{2s}}{2\log N}$.

Remarks

- The time interval of approximation allows to transfer all information from $g(t, \xi)$ over the interval $[0, M]$ and M can arbitrarily large.
- The last theorem gives explicit examples of coherent out-of-equilibrium dynamics for NLS on $\mathbb{T}^2$.
- This answers what happens in the regime $\epsilon \ll L^{-1}$. What happens in the rest of the weakly nonlinear regime $L^{-1} \lesssim \epsilon \lesssim L^{-1/2}$???.

Outline

1. Introduction: The general problematic and our model equation

2. First approach: Energy cascades and growth of Sobolev norms
   - Recent progress on Bourgain's infinite Sobolev norm growth conjecture

3. Second Approach: Deriving effective equations (Weak turbulence theory)
   - The weakly nonlinear large-box limit of NLS (Faou-Germain-H.)
   - Properties of the limiting equation
   - Rigorous approximation results

4. Further Directions

Further Directions

- Analysis of (CR)
  1. Numerical study of (CR)$\rightsquigarrow$ long-time dynamics of NLS.
  2. Analytical study of (CR): Properties and dynamics of its solutions (in progress with P. Germain and L. Thomann).
  3. Relation of (CR) to NLS with harmonic potential $\rightsquigarrow$ another justification (with L. Thomann).
  4. Is (CR) completely integrable?
- Deriving the weakly nonlinear large-box limit for other equations:
  1. 1D cubic and quintic NLS (with J. Shatah) leading to water wave equations.
  2. Klein-Gordon equations on spheres (with P. Germain and B. Pausader).
  3. Geophysical flows.
- Can one pass from weakly nonlinear large-box limit equations like (CR) to KZ equations of weak turbulence by an appropriate randomization?

Thanks!

Thank you for your attention!