The functional equation $f(P) = g(Q)$ in dynamics, number theory, analysis and algebraic geometry

Michael Zieve

University of Michigan

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Joint work with Alex Carney, Thao Do, Jared Hallett, Ruthi Hortsch, Xiangyi Huang, Yuwei Jiang, Qingyun Sun, Ben Weiss, Elliot Wells, Susan Xia

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- Abel, 1826: $(X+1) \circ P = P \circ Q$
- Schröder 1871, ..., Yoccoz 1995: $\lambda X \circ P = P \circ Q$
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- and many more.

We know all *polynomials* f, P, g, Q such that $f \circ P = g \circ Q$ (Ritt, 1922).

But we aren't close to knowing all solutions in rational functions: the most general published result is if f, g are polynomials and $P, Q \in \mathbb{C}[X, 1/X]$ are Laurent polynomials (Pakovich, Z 2007).

Today I'll present all solutions when f, g are polynomials and P, Q are rational functions (or more generally, meromorphic functions on $\mathbb{C}$), and give several consequences.

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A dynamics result

Theorem (Ghioca–Tucker–Z, 2008 & 2012): *For $\alpha, \beta \in \mathbb{C}$ and nonlinear $f, g \in \mathbb{C}[X]$, if the orbits $\{\alpha, f(\alpha), f(f(\alpha)) \dots\}$ and $\{\beta, g(\beta), g(g(\beta)), \dots\}$ have infinite intersection, then f and g have a common iterate.*

Proof sketch:

- Writing $f^k(X)$ for the k-th iterate of f, we have $f^k(\alpha) = g^\ell(\beta)$ for infinitely many pairs (k, ℓ).
- For any n, m, the equation $f^m(X) = g^n(Y)$ has infinitely many solutions $X = f^{k-m}(\alpha)$, $Y = g^{\ell-n}(\beta)$.
- Every $f^i(\alpha)$ and $g^j(\beta)$ lies in the ring R generated by α, β and the coefficients of f and g.
- Hence (Siegel, 1929; Lang, 1960) there are nonconstant Laurent polynomials $P, Q \in \mathbb{C}[X, 1/X]$ such that $f^m \circ P = g^n \circ Q$.
- Solve this for each m, n, then piece together the solutions.

Summary: From dynamics to number theory to $F(P) = G(Q)$ to QED.

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Reformulate: the set of pairs (m, n) such that $(f^m(\alpha), g^n(\beta))$ lies on the diagonal $X = Y$ consists of finitely many "arithmetic progressions" (cosets of cyclic subsemigroups of $\mathbb{N}^2$).

This resembles the Mordell–Lang conjecture (proved by Faltings and Vojta): the intersection of a subvariety V of a (semi-)abelian variety J and a finitely-generated subgroup G of $J(\mathbb{C})$ consists of finitely many cosets of subgroups of G.

It also resembles the Skolem–Mahler–Lech theorem: if $a_1, a_2, \dots$ is a sequence of complex numbers satisfying a linear recurrence relation, then the n's for which $a_n = 0$ comprise finitely many arithmetic progressions.

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A common framework

Question: if J is a variety with a subvariety V and a point $\alpha \in J(\mathbb{C})$, and S is a finitely-generated commutative semigroup of endomorphisms of J, then does the set of $s \in S$ for which $s(\alpha) \in V$ consist of finitely many cosets of subsemigroups of S?

- Yes if $J = \mathbb{A}^2$ and V is a line and S is generated by the maps $(u, v) \mapsto (f(u), v)$ and $(u, v) \mapsto (u, g(v))$ for some nonlinear $f, g \in \mathbb{C}[X]$ (Ghioca, Tucker, Z)
- Yes if J is a (semi-)abelian variety and S consists of translations (Faltings, Vojta)
- Yes if $J = \mathbb{C}^* \times \mathbb{C}$ (Skolem–Mahler–Lech)
- Yes in several other situations (Benedetto, Ghioca, Kurlberg, Scanlon, Tucker, Vojta, Zannier, Z)
- No sometimes.

Note that the proofs in the various cases seem completely unrelated, so a common proof would shed much light.

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This result is best possible:

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- The "6" cannot be improved: for $f(X) := (X^3 - X)^2$,

$$f\left(\pm\frac{2t-1}{t^2-t+1}\right) = f\left(\pm\frac{t^2-1}{t^2-t+1}\right) = f\left(\pm\frac{t^2-2t}{t^2-t+1}\right)$$

for each $t \in \mathbb{Q}$.

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$$f\left(\pm\frac{2t-1}{t^2-t+1}\right) = f\left(\pm\frac{t^2-1}{t^2-t+1}\right) = f\left(\pm\frac{t^2-2t}{t^2-t+1}\right)$$

  for each $t \in \mathbb{Q}$.

$f(P) = g(Q)$ and polynomials over the rational numbers

Theorem (Carney–Hortsch–Z) *For any $f \in \mathbb{Q}[X]$, the function $\mathbb{Q} \to \mathbb{Q}$ defined by $c \mapsto f(c)$ is at most 6-to-1 outside a finite set.*

Proof sketch:

- If f is (≥ 7)-to-1 infinitely often, then there are infinitely many rational points on some subvariety of $f(X_1) = f(X_2) = \cdots = f(X_7)$ which is not contained in any diagonal $X_i = X_j$ (with $i \neq j$).
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A plausible generalization

Theorem (Mazur, 1977): *Any elliptic curve* $Y^2 = X^3 + aX + b$ *over* $\mathbb{Q}$ *has at most* 16 *rational torsion points.*

Reformulation: *For any nonconstant morphism* $f : E_1 \to E_2$ *between genus-*1 *curves over* $\mathbb{Q}$*, the induced map* $f : E_1(\mathbb{Q}) \to E_2(\mathbb{Q})$ *is at most* 16*-to-*1*.*

Our result: *For any morphism* $f : \mathbb{A}^1 \to \mathbb{A}^1$ *over* $\mathbb{Q}$*, the induced map* $\mathbb{A}^1(\mathbb{Q}) \to \mathbb{A}^1(\mathbb{Q})$ *is at most* 6*-to-*1 *outside a finite set.*

Speculation: *Perhaps, for any morphism* $f : V_1 \to V_2$ *between* d*-dimensional varieties over* $\mathbb{Q}$*, the map* $f : V_1(\mathbb{Q}) \to V_2(\mathbb{Q})$ *is at most* $c(d)$*-to-*1 *outside a lower-dimensional locus ("proper Zariski-closed subset of* V_2*").*

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Theorem (CDHHJSWWXZ): *For $f(X), g(X) \in \overline{\mathbb{Q}}[X] \setminus \mathbb{Q}$, the equation $f(X) = g(Y)$ has infinitely many solutions in a number field K is and only if...*

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- $F = X^n$ *and* G *is either* $X^i H(X)^n$ *or* $X^i (X+1)^{n-i} H(X)^n$ *or ...*
- $F = T_n(X)$ *and* $G(X)^2 - 4 = D(X)H(X)^2$ *with* D *squarefree of degree* ≤ 6
- $F = X^i (X+1)^j$ *and* $G = cX^i (X+1)^j$ *for some* $c \in \overline{\mathbb{Q}} \setminus \{0, 1\}$
- $\max(\deg(F), \deg(G)) \leq 16$ *and* F, G *are on an explicit list.*

When $F = T_n$:

- We can count the number of corresponding $G \in \overline{\mathbb{Q}}[X]$ with fixed degree and fixed critical values.
- Solutions $G \in K[X]$ of degree N are in bijection with triples (C, σ, P) where C is a curve$/K$ of genus ≤ 2, σ is a "hyperelliptic involution" on C, and $P \in C(K)$ satisfies $N([P] - [\sigma(P)]) = 0$ in $\mathrm{Jac}(C)$.

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Proof: by Faltings' theorem and Picard's theorem (see the next slide), the hypotheses are equivalent to asserting that $f \circ P = g \circ Q$ has a solution with P, Q being nonconstant meromorphic functions on $\mathbb{C}$. So "just" find all such solutions (which is very difficult).

Meromorphic functions

An *entire function* is a function on $\mathbb{C}$ given by a power series which converges everywhere.

A *meromorphic function* is the ratio of two entire functions.

Theorem (Picard, 1887) *For any nonconstant $F(X, Y) \in \mathbb{C}[X, Y]$, there exist nonconstant meromorphic $p(t)$ and $q(t)$ with $F(p(t), q(t)) = 0$ if and only if some irreducible factor of $F(X, Y)$ defines a curve of genus 0 or 1.*

Recall Faltings' theorem: *For any nonconstant $F(X, Y) \in \mathbb{Q}[X, Y]$, the equation $F(X, Y) = 0$ has infinitely many solutions in some number field if and only if some irreducible factor of $F(X, Y)$ defines a curve of genus 0 or 1.*

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Theorem (Nevanlinna, 1926): *If nonconstant meromorphic functions $P(t)$ and $Q(t)$ satisfy $P^{-1}(\alpha_i) = Q^{-1}(\alpha_i)$ for five distinct values $\alpha_i \in \mathbb{C}$, then $P = Q$.*

A much-studied question: What if $P^{-1}(S_i) = Q^{-1}(T_i)$ for several pairs (S_i, T_i) of finite subsets of $\mathbb{C}$?

Remark: If $f \circ P = g \circ Q$ with $f, g \in \mathbb{C}(X) \setminus \mathbb{C}$, then $P^{-1}(f^{-1}(\gamma)) = Q^{-1}(g^{-1}(\gamma))$ for every $\gamma \in \mathbb{C}$.

Question: Does this account for all pairs (P, Q) such that $P^{-1}(S_i) = Q^{-1}(T_i)$ for infinitely many pairs (S_i, T_i) of finite subsets of $\mathbb{C}$?

- Yes if $P, Q \in \mathbb{C}(X)$ (Beals–Wetherell–Z, 2009 +...)
- Yes if the polynomials $\prod_{s \in S_i}(X - s)$ and $\prod_{s \in T_i}(X - s)$ have "few" critical points (Weiss–Z)

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A much-studied question: What if $P^{-1}(S_i) = Q^{-1}(T_i)$ for several pairs (S_i, T_i) of finite subsets of $\mathbb{C}$?

Remark: If $f \circ P = g \circ Q$ with $f, g \in \mathbb{C}(X) \setminus \mathbb{C}$, then $P^{-1}(f^{-1}(\gamma)) = Q^{-1}(g^{-1}(\gamma))$ for every $\gamma \in \mathbb{C}$.

Question: Does this account for all pairs (P, Q) such that $P^{-1}(S_i) = Q^{-1}(T_i)$ for infinitely many pairs (S_i, T_i) of finite subsets of $\mathbb{C}$?

- Yes if $P, Q \in \mathbb{C}(X)$ (Beals–Wetherell–Z, 2009 +...)
- Yes if the polynomials $\prod_{s \in S_i}(X - s)$ and $\prod_{s \in T_i}(X - s)$ have "few" critical points (Weiss–Z)

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Value sharing and functional equations

Sample theorem (Weiss–Z): *If nonconstant meromorphic functions $P(t)$ and $Q(t)$ and nonempty finite $S, T \subset \mathbb{C}$ satisfy $P^{-1}(S) = Q^{-1}(T)$, and at most $\min(\#S, \#T) - 13$ complex numbers are critical points of $f(X) := \prod_{s\in S}(X - s)$ and/or $g(X) := \prod_{s\in T}(X - s)$, then*

$$f(P(t)) = \frac{g(Q(t))}{c \cdot g(Q(t)) + d} \qquad (*)$$

for some $c, d \in \mathbb{C}$.

Theorem (CDHHHJSWWXZ): *We know all $f, g \in \mathbb{C}[X]$ and meromorphic P, Q satisfying $(*)$.*

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How we solved $f(P) = g(Q)$ in polynomials f, g and meromorphic P, Q

By Picard's theorem and uniqueness of meromorphic parametrizations, the problem amounts to determining when $f(X) = g(Y)$ has a component of genus 0 or 1.

First classify $f, g \in \mathbb{C}[X]$ for which $f(X) - g(Y)$ is *irreducible* and defines a curve of genus ≤ 1.

The genus $\mathfrak{g}$ of $f(X) = g(Y)$ can be expressed in terms of the factorization types of all $f(X) - \lambda$ and $g(X) - \lambda$ in $\mathbb{C}[X]$ (with $\lambda \in \mathbb{C}$).

Use this to determine all numerical plausibilities for the factorization types of all $f(X) - \lambda$, assuming $f(X) - g(Y)$ irreducible and $\mathfrak{g} \in \{0, 1\}$.

Then determine all corresponding polynomials via computations in fundamental groups, Riemann's existence theorem, and solutions of differential equations.

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The reducible case

We cannot immediately resolve the reducible case after solving the irreducible case, since factors of $f(X) - g(Y)$ generally cannot be written in this form. Instead we pass from the decomposable case to the indecomposable case, using several ingredients including:

Theorem (Hallett–Wells–Xia–Z, building on Fried, 1973; Feit, 1973; Feit, 1980; Müller, 1993; Cassou-Noguès–Couveignes, 1999; Elkies, 2012) *We explicitly know all indecomposable $f(X) \in \mathbb{C}[X]$ for which the Galois group of $f(X) - t$ over $\mathbb{C}(t)$ is neither S_n nor A_n (where $n := \deg(f)$).*

Note: the proof of this Theorem crucially uses consequences of the classification of finite simple groups.

Corollary: *If $f, g \in \mathbb{C}[X]$ are indecomposable and $f(X) - g(Y)$ is reducible then either $g = f \circ h$ (with h linear) or $\deg(f) = \deg(g) \leq 31$ and f, g are explicitly known.*

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Summary

Solutions of instances of $f \circ P = g \circ Q$ in polynomials f, g and meromorphic P, Q have been applied to:

- Describing intersections of orbits of complex polynomials
- Showing that for $f \in \mathbb{Q}[X]$ the function $f : \mathbb{Q} \to \mathbb{Q}$ is at most 6-to-1 outside a finite set
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