Non-Archimedean Methods in Complex Dynamics

Araceli Bonifant

University of Rhode Island

Stony Brook University

March 7, 2013

The setting

Any cubic polynomial map with marked critical point is affinely conjugate to one of the form

$$F(z) = F_{a,v}(z) = z^3 - 3a^2 z + (2a^3 + v)$$

The setting

Any cubic polynomial map with marked critical point is affinely conjugate to one of the form

$$F(z) \;=\; F_{a,v}(z) \;=\; z^3 - 3a^2 z + (2a^3 + v)$$

where

- a is the marked critical point

The setting

Any cubic polynomial map with marked critical point is affinely conjugate to one of the form

$$F(z) \;=\; F_{a,v}(z) \;=\; z^3 - 3a^2 z + (2a^3 + v)$$

where

- a is the marked critical point
- $v = F(a)$ is the marked critical value

The setting

Any cubic polynomial map with marked critical point is affinely conjugate to one of the form

$$F(z) \;=\; F_{a,v}(z) \;=\; z^3 - 3a^2 z + (2a^3 + v)$$

where

- a is the marked critical point
- $v = F(a)$ is the marked critical value
- $-a$ is the free critical point

The setting

Any cubic polynomial map with marked critical point is affinely conjugate to one of the form

$$F(z) \;=\; F_{a,v}(z) \;=\; z^3 - 3a^2 z + (2a^3 + v)$$

where

- a is the marked critical point
- $v = F(a)$ is the marked critical value
- $-a$ is the free critical point

The set of all such maps $F = F_{a,v}$ will be identified with the parameter space, consisting of all pairs $(a, v) \in \mathbb{C}^2$.

The Period p Curve, $\mathcal{S}_p$

Definition: The period p curve

$$\mathcal{S}_p = \Big\{(a,\ v) \in \mathbb{C}^2\ ;\ F^{\circ p}(a) = a\Big\} \qquad \text{with} \quad p \quad \text{minimal}$$

The Period p Curve, $\mathcal{S}_p$

Definition: The period p curve

$$\mathcal{S}_p = \Big\{(a,\ v) \in \mathbb{C}^2\ ;\ F^{\circ p}(a) = a\Big\} \qquad \text{with} \quad p \quad \text{minimal}$$

Theorem (Milnor 1991). $\mathcal{S}_p$ is a smooth affine algebraic curve in $\mathbb{C}^2$ (conjecturally always connected).

The Period p Curve, $\mathcal{S}_p$

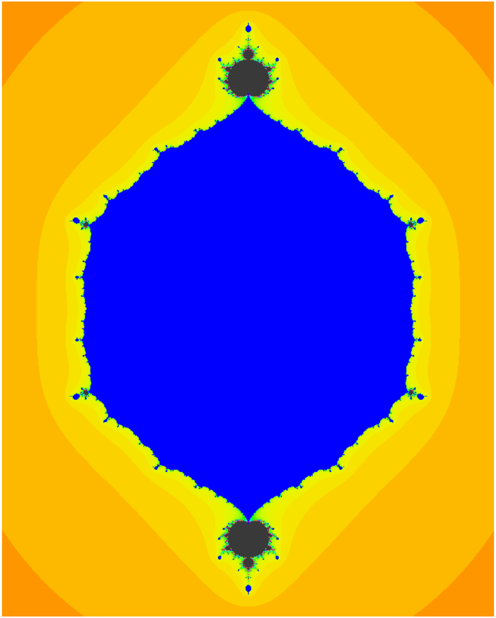
Definition: The period p curve

$$\mathcal{S}_p = \Big\{(a,\ v) \in \mathbb{C}^2\ ;\ F^{\circ p}(a) = a\Big\} \qquad \text{with} \quad p \quad \text{minimal}$$

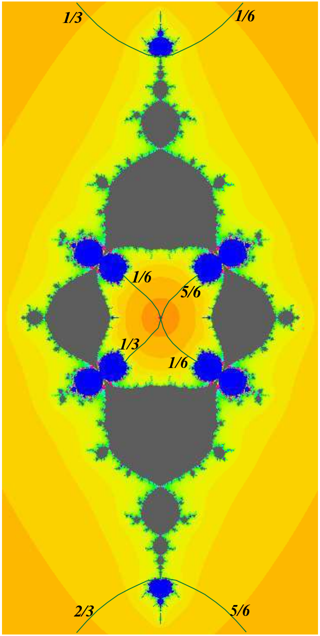
Theorem (Milnor 1991). $\mathcal{S}_p$ is a smooth affine algebraic curve in $\mathbb{C}^2$ (conjecturally always connected).

For most periods p, $\mathcal{S}_p$ is a many times punctured surface of high genus.

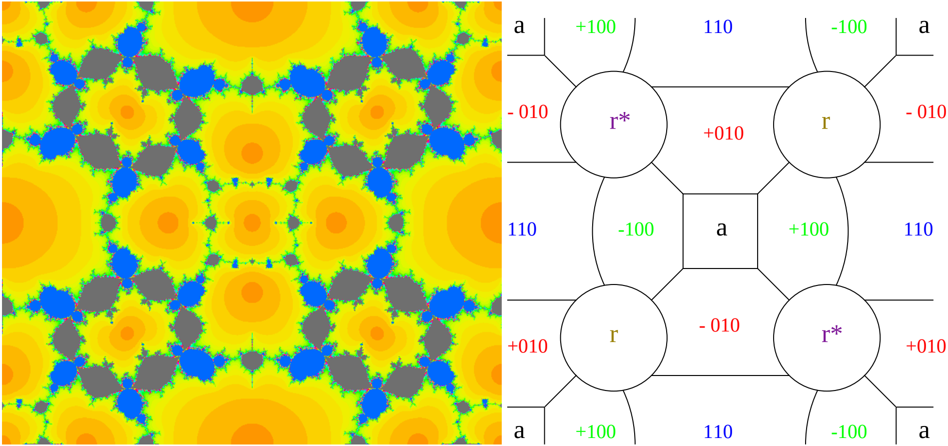
$\mathcal{S}_1$ has genus zero with one puncture ($\cong \mathbb{C}$)



$\mathcal{S}_2$ has genus zero with two punctures



$\mathcal{S}_3$ has genus one with eight punctures



Universal covering of $\overline{S}_3$.

Escape Regions $\mathcal{E}_h \subset \mathcal{S}_p$

Definition. An **escape region** $\mathcal{E}_h \subset \mathcal{S}_p$ is a connected component of the open set consisting of maps $F \in \mathcal{S}_p$

Escape Regions $\mathcal{E}_h \subset \mathcal{S}_p$

Definition. An **escape region** $\mathcal{E}_h \subset \mathcal{S}_p$ is a connected component of the open set consisting of maps $F \in \mathcal{S}_p$ for which the orbit of the critical point $-a$ escapes to ∞.

Escape Regions $\mathcal{E}_h \subset \mathcal{S}_p$

Definition. An **escape region** $\mathcal{E}_h \subset \mathcal{S}_p$ is a connected component of the open set consisting of maps $F \in \mathcal{S}_p$ for which the orbit of the critical point $-a$ escapes to ∞.

Lemma (Milnor 1991). Each escape region $\mathcal{E}_h$ is canonically diffeomorphic to the μ-fold covering of the complement $\mathbb{C} \smallsetminus \overline{\mathbb{D}}$,

Escape Regions $\mathcal{E}_h \subset \mathcal{S}_p$

Definition. An **escape region** $\mathcal{E}_h \subset \mathcal{S}_p$ is a connected component of the open set consisting of maps $F \in \mathcal{S}_p$ for which the orbit of the critical point $-a$ escapes to ∞.

Lemma (Milnor 1991). Each escape region $\mathcal{E}_h$ is canonically diffeomorphic to the μ-fold covering of the complement $\mathbb{C} \smallsetminus \overline{\mathbb{D}}$, where $\mu = \mu_h \geq 1$ is an integer called the **multiplicity** of $\mathcal{E}_h$.

Escape Regions $\mathcal{E}_h \subset \mathcal{S}_p$

Definition. An **escape region** $\mathcal{E}_h \subset \mathcal{S}_p$ is a connected component of the open set consisting of maps $F \in \mathcal{S}_p$ for which the orbit of the critical point $-a$ escapes to ∞.

Lemma (Milnor 1991). Each escape region $\mathcal{E}_h$ is canonically diffeomorphic to the μ-fold covering of the complement $\mathbb{C} \smallsetminus \overline{\mathbb{D}}$, where $\mu = \mu_h \geq 1$ is an integer called the **multiplicity** of $\mathcal{E}_h$.

Compactification of $\mathcal{S}_p$

The curve $\mathcal{S}_p$ can be compactified by adding finitely many ideal points ∞_h,

Compactification of $\mathcal{S}_p$

The curve $\mathcal{S}_p$ can be compactified by adding finitely many ideal points ∞_h, one for each escape region $\mathcal{E}_h$,

Compactification of $\mathcal{S}_p$

The curve $\mathcal{S}_p$ can be compactified by adding finitely many ideal points ∞_h, one for each escape region $\mathcal{E}_h$, thus yielding a smooth complex 1-manifold $\overline{\mathcal{S}}_p$.

Compactification of $\mathcal{S}_p$

The curve $\mathcal{S}_p$ can be compactified by adding finitely many ideal points ∞_h, one for each escape region $\mathcal{E}_h$, thus yielding a smooth complex 1-manifold $\overline{\mathcal{S}}_p$.

Caution. $\overline{\mathcal{S}}_p$ is NOT the closure of $\mathcal{S}_p$ in the projective space $\mathbb{CP}^2$. (The closure has singular points at infinity.)

Compactification of $\mathcal{S}_p$

The curve $\mathcal{S}_p$ can be compactified by adding finitely many ideal points ∞_h, one for each escape region $\mathcal{E}_h$, thus yielding a smooth complex 1-manifold $\overline{\mathcal{S}}_p$.

Caution. $\overline{\mathcal{S}}_p$ is NOT the closure of $\mathcal{S}_p$ in the projective space $\mathbb{CP}^2$. (The closure has singular points at infinity.)

With this compactification, each escape region, together with its ideal point, is conformally isomorphic to the open unit disk.

Degree and the number N_p of escape regions.

Degree and the number N_p of escape regions.

The degree of S_p is equal to the sum

$$\deg(\mathcal{S}_p) \;=\; \mu_1 + \cdots + \mu_{N_p} \;,$$

where μ_h is the multiplicity of $\mathcal{E}_h$.

Degree and the number N_p of escape regions.

The degree of S_p is equal to the sum

$$\deg(\mathcal{S}_p) \;=\; \mu_1 + \cdots + \mu_{N_p} \;,$$

where μ_h is the multiplicity of $\mathcal{E}_h$. Thus

$$1 \le N_p \le \deg(S_p) \;.$$

Degree and the number N_p of escape regions.

The degree of S_p is equal to the sum

$$\deg(\mathcal{S}_p) \;=\; \mu_1 + \cdots + \mu_{N_p} \;,$$

where μ_h is the multiplicity of $\mathcal{E}_h$. Thus

$$1 \le N_p \le \deg(S_p)\;.$$

Here the degree grows exponentially fast with p,

$$\deg(\mathcal{S}_p) \;\sim\; 3^{p-1} \qquad \text{as} \qquad p \to \infty\;.$$

Degree and the number N_p of escape regions.

The degree of S_p is equal to the sum

$$\deg(\mathcal{S}_p) \ = \ \mu_1 + \cdots + \mu_{N_p} \ ,$$

where μ_h is the multiplicity of $\mathcal{E}_h$. Thus

$$1 \le N_p \le \deg(S_p) \ .$$

Here the degree grows exponentially fast with p,

$$\deg(\mathcal{S}_p) \ \sim \ 3^{p-1} \qquad \text{as} \qquad p \to \infty \ .$$

More precisely, $\deg(S_p)$ can be computed from the equation

$$\deg(\mathcal{S}_p) \ = \ 3^{p-1} - \sum_{n|p,\ n<p} \deg(\mathcal{S}_n).$$

Degree and the number N_p of escape regions.

The degree of S_p is equal to the sum

$$\deg(\mathcal{S}_p) \;=\; \mu_1 + \cdots + \mu_{N_p} \;,$$

where μ_h is the multiplicity of $\mathcal{E}_h$. Thus

$$1 \le N_p \le \deg(S_p)\;.$$

Here the degree grows exponentially fast with p,

$$\deg(\mathcal{S}_p) \;\sim\; 3^{p-1} \qquad \text{as} \qquad p \to \infty\;.$$

More precisely, $\deg(S_p)$ can be computed from the equation

$$\deg(\mathcal{S}_p) \;=\; 3^{p-1} - \sum_{n|p,\; n<p} \deg(\mathcal{S}_n).$$

Example. $\deg(\mathcal{S}_1) \;=\; 3^0 - 0 \;=\; 1$

Degree and the number N_p of escape regions.

The degree of S_p is equal to the sum

$$\deg(\mathcal{S}_p) \;=\; \mu_1 + \cdots + \mu_{N_p} \;,$$

where μ_h is the multiplicity of $\mathcal{E}_h$. Thus

$$1 \le N_p \le \deg(S_p) \;.$$

Here the degree grows exponentially fast with p,

$$\deg(\mathcal{S}_p) \;\sim\; 3^{p-1} \qquad \text{as} \qquad p \to \infty \;.$$

More precisely, $\deg(S_p)$ can be computed from the equation

$$\deg(\mathcal{S}_p) \;=\; 3^{p-1} - \sum_{n|p,\; n<p} \deg(\mathcal{S}_n).$$

Example.

$$\deg(\mathcal{S}_1) \;=\; 3^0 - 0 \;=\; 1$$
$$\deg(\mathcal{S}_2) \;=\; 3^1 - 1 \;=\; 2$$

Degree and the number N_p of escape regions.

The degree of S_p is equal to the sum

$$\deg(\mathcal{S}_p) \;=\; \mu_1 + \cdots + \mu_{N_p} \;,$$

where μ_h is the multiplicity of $\mathcal{E}_h$. Thus

$$1 \le N_p \le \deg(S_p)\;.$$

Here the degree grows exponentially fast with p,

$$\deg(\mathcal{S}_p) \;\sim\; 3^{p-1} \qquad \text{as} \qquad p \to \infty\;.$$

More precisely, $\deg(S_p)$ can be computed from the equation

$$\deg(\mathcal{S}_p) \;=\; 3^{p-1} - \sum_{n|p,\; n<p} \deg(\mathcal{S}_n).$$

Example.

$$\begin{aligned}
\deg(\mathcal{S}_1) &= 3^0 - 0 = 1\\
\deg(\mathcal{S}_2) &= 3^1 - 1 = 2\\
\deg(\mathcal{S}_3) &= 3^2 - 1 = 8
\end{aligned}$$

Degree and the number N_p of escape regions.

The degree of S_p is equal to the sum

$$\deg(\mathcal{S}_p) \;=\; \mu_1 + \cdots + \mu_{N_p} \;,$$

where μ_h is the multiplicity of $\mathcal{E}_h$. Thus

$$1 \le N_p \le \deg(S_p) \;.$$

Here the degree grows exponentially fast with p,

$$\deg(\mathcal{S}_p) \;\sim\; 3^{p-1} \qquad \text{as} \qquad p \to \infty \;.$$

More precisely, $\deg(S_p)$ can be computed from the equation

$$\deg(\mathcal{S}_p) \;=\; 3^{p-1} - \sum_{n|p,\; n<p} \deg(\mathcal{S}_n).$$

Example.

$$\begin{aligned}
\deg(\mathcal{S}_1) &= 3^0 - 0 = 1\\
\deg(\mathcal{S}_2) &= 3^1 - 1 = 2\\
\deg(\mathcal{S}_3) &= 3^2 - 1 = 8\\
\deg(\mathcal{S}_4) &= 3^3 - 2 - 1 = 24.
\end{aligned}$$

Local Parametrization of $\mathcal{S}_p \subset \mathbb{C}^2$.

Local Parametrization of $\mathcal{S}_p \subset \mathbb{C}^2$.

Define $\phi_p : \mathbb{C}^2 \to \mathbb{C}$ by

$$\phi_p(a, v) \;=\; F^{\circ p}(a) - a\,, \qquad \text{where} \quad F \;=\; F_{a,v}\,.$$

Local Parametrization of $\mathcal{S}_p \subset \mathbb{C}^2$.

Define $\phi_p : \mathbb{C}^2 \to \mathbb{C}$ by

$$\phi_p(a, v) \;=\; F^{\circ p}(a) - a\,, \qquad \text{where} \quad F \;=\; F_{a,v}\,.$$

This vanishes everywhere on $\mathcal{S}_p$, with $d\phi_p \neq 0$ on $\mathcal{S}_p$.

Local Parametrization of $\mathcal{S}_p \subset \mathbb{C}^2$.

Define $\phi_p : \mathbb{C}^2 \to \mathbb{C}$ by

$$\phi_p(a, v) \;=\; F^{\circ p}(a) - a\,, \qquad \text{where} \quad F \;=\; F_{a,v}\,.$$

This vanishes everywhere on $\mathcal{S}_p$, with $d\phi_p \neq 0$ on $\mathcal{S}_p$.

Let $t \mapsto (a,\ v)$ be any solution to the Hamiltonian differential equation

$$\frac{da}{dt} \;=\; \frac{\partial \phi_p}{\partial v}\,, \qquad \frac{dv}{dt} \;=\; -\frac{\partial \phi_p}{\partial a}\,.$$

Local Parametrization of $\mathcal{S}_p \subset \mathbb{C}^2$.

Define $\phi_p : \mathbb{C}^2 \to \mathbb{C}$ by

$$\phi_p(a, v) \;=\; F^{\circ p}(a) - a\,, \qquad \text{where} \quad F \;=\; F_{a,v}\,.$$

This vanishes everywhere on $\mathcal{S}_p$, with $d\phi_p \neq 0$ on $\mathcal{S}_p$.

Let $t \mapsto \big(a,\; v\big)$ be any solution to the Hamiltonian differential equation

$$\frac{da}{dt} \;=\; \frac{\partial \phi_p}{\partial v}\,, \qquad \frac{dv}{dt} \;=\; -\,\frac{\partial \phi_p}{\partial a}\,.$$

The local solutions $t \;\mapsto\; (a, v) \;=\; \big(a(t),\; v(t)\big)$ are holomorphic, with $\frac{d\phi_p}{dt} \;=\; \frac{\partial \phi_p}{\partial a}\frac{da}{dt} \;+\; \frac{\partial \phi_p}{\partial v}\frac{dv}{dt} \;\equiv\; 0.$

Local Parametrization of $\mathcal{S}_p \subset \mathbb{C}^2$.

Define $\phi_p : \mathbb{C}^2 \to \mathbb{C}$ by

$$\phi_p(a, v) \;=\; F^{\circ p}(a) - a\,, \qquad \text{where} \quad F \;=\; F_{a,v}\,.$$

This vanishes everywhere on $\mathcal{S}_p$, with $d\phi_p \neq 0$ on $\mathcal{S}_p$.

Let $t \mapsto \big(a,\ v\big)$ be any solution to the Hamiltonian differential equation

$$\frac{da}{dt} \;=\; \frac{\partial \phi_p}{\partial v}\,, \qquad \frac{dv}{dt} \;=\; -\,\frac{\partial \phi_p}{\partial a}\,.$$

The local solutions $t \;\mapsto\; (a, v) \;=\; \big(a(t),\ v(t)\big)$ are holomorphic, with $\frac{d\phi_p}{dt} \;=\; \frac{\partial \phi_p}{\partial a}\frac{da}{dt} \;+\; \frac{\partial \phi_p}{\partial v}\frac{dv}{dt} \;\equiv\; 0.$

Hence they lie in curves $\phi_p \;=\; \text{constant}\,.$

Local Parametrization of $\mathcal{S}_p \subset \mathbb{C}^2$.

Define $\phi_p : \mathbb{C}^2 \to \mathbb{C}$ by

$$\phi_p(a, v) \;=\; F^{\circ p}(a) - a\,, \qquad \text{where} \quad F \;=\; F_{a,v}\,.$$

This vanishes everywhere on $\mathcal{S}_p$, with $d\phi_p \neq 0$ on $\mathcal{S}_p$.

Let $t \mapsto (a,\ v)$ be any solution to the Hamiltonian differential equation

$$\frac{da}{dt} \;=\; \frac{\partial \phi_p}{\partial v}\,, \qquad \frac{dv}{dt} \;=\; -\frac{\partial \phi_p}{\partial a}\,.$$

The local solutions $t \;\mapsto\; (a, v) \;=\; \big(a(t),\ v(t)\big)$ are holomorphic, with $\frac{d\phi_p}{dt} \;=\; \frac{\partial \phi_p}{\partial a}\frac{da}{dt} \;+\; \frac{\partial \phi_p}{\partial v}\frac{dv}{dt} \;\equiv\; 0.$

Hence they lie in curves $\phi_p \;=\; \text{constant}\,.$

Those solutions which lie in $\mathcal{S}_p$ provide a local holomorphic parametrization, unique up to a translation, $t \mapsto t + \text{constant}$.

Local Parametrization of $\mathcal{S}_p \subset \mathbb{C}^2$.

Define $\phi_p : \mathbb{C}^2 \to \mathbb{C}$ by

$$\phi_p(a, v) = F^{\circ p}(a) - a\,, \qquad \text{where} \quad F = F_{a,v}\,.$$

This vanishes everywhere on $\mathcal{S}_p$, with $d\phi_p \neq 0$ on $\mathcal{S}_p$.

Let $t \mapsto (a,\ v)$ be any solution to the Hamiltonian differential equation

$$\frac{da}{dt} = \frac{\partial \phi_p}{\partial v}\,, \qquad \frac{dv}{dt} = -\frac{\partial \phi_p}{\partial a}\,.$$

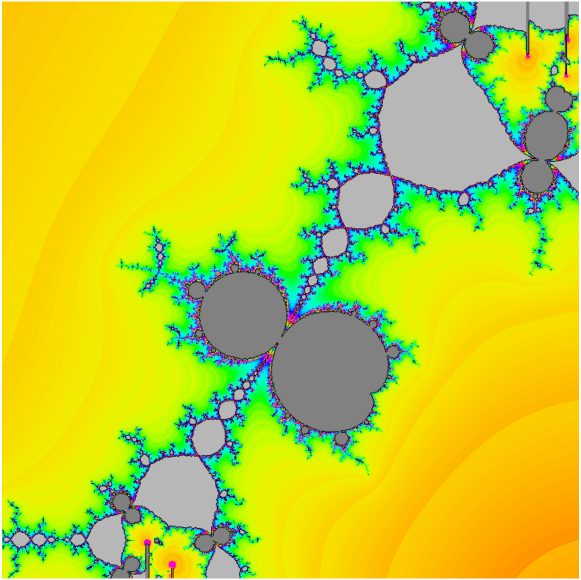
The local solutions $t \mapsto (a, v) = \big(a(t),\ v(t)\big)$ are holomorphic, with $\frac{d\phi_p}{dt} = \frac{\partial \phi_p}{\partial a}\frac{da}{dt} + \frac{\partial \phi_p}{\partial v}\frac{dv}{dt} \equiv 0.$

Hence they lie in curves $\phi_p = \text{constant}\,.$

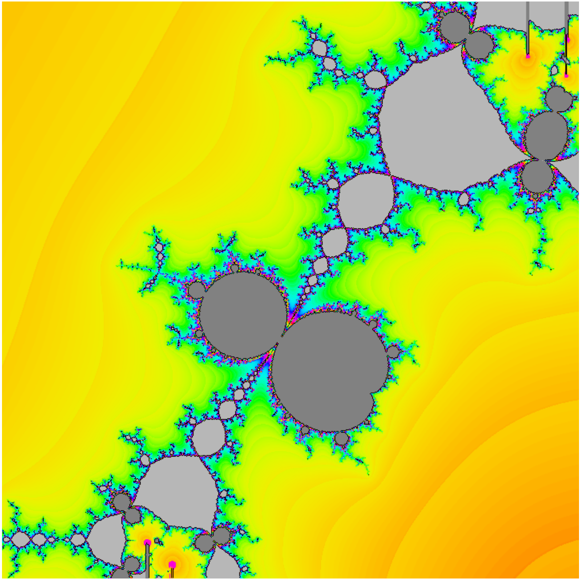
Those solutions which lie in $\mathcal{S}_p$ provide a local holomorphic parametrization, unique up to a translation, $t \mapsto t + \text{constant}$.

Equivalently, the holomorphic 1-form dt is well defined and non-zero everywhere on $\mathcal{S}_p$.

Sample parametrization: A small part of S_4 .

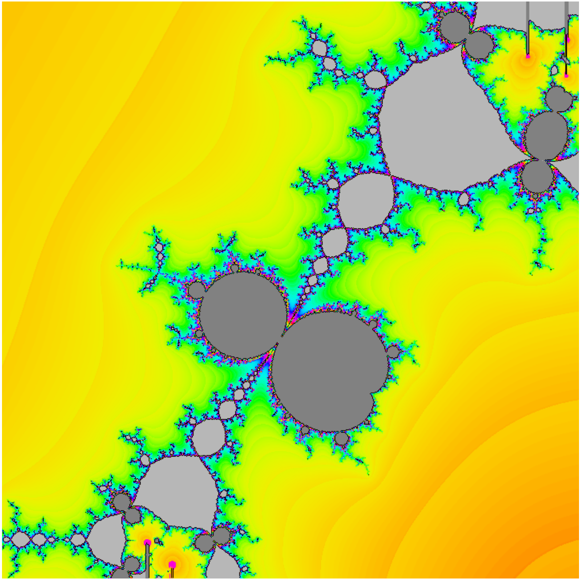


Sample parametrization: A small part of S_4 .



$\mathcal{S}_4$ has genus fifteen with twenty punctures

Sample parametrization: A small part of S_4 .



$\mathcal{S}_4$ has genus fifteen with twenty punctures (5 visible here).

Local Uniformizing Parameter near ∞_h

For each escape region $\mathcal{E}_h \subset \mathcal{S}_p$ the projection map

$$(a, v) \mapsto a$$

has a pole of order $\mu \geq 1$ at the ideal point ∞_h.

Local Uniformizing Parameter near ∞_h

For each escape region $\mathcal{E}_h \subset \mathcal{S}_p$ the projection map

$$(a, v) \mapsto a$$

has a pole of order $\mu \geq 1$ at the ideal point ∞_h.

We will work with the variable

$$\xi = \frac{1}{a}$$

Local Uniformizing Parameter near ∞_h

For each escape region $\mathcal{E}_h \subset \mathcal{S}_p$ the projection map

$$(a, v) \mapsto a$$

has a pole of order $\mu \geq 1$ at the ideal point ∞_h.

We will work with the variable

$$\xi = \frac{1}{a}$$

which is a bounded holomorphic function throughout a neighborhood of ∞_h in $\mathcal{S}_p$.

Local Uniformizing Parameter near ∞_h

For each escape region $\mathcal{E}_h \subset \mathcal{S}_p$ the projection map

$$(a, v) \mapsto a$$

has a pole of order $\mu \geq 1$ at the ideal point ∞_h.

We will work with the variable

$$\xi = \frac{1}{a}$$

which is a bounded holomorphic function throughout a neighborhood of ∞_h in $\mathcal{S}_p$.

Since ξ has a zero of order μ at ∞_h, we can choose some μ-th root $\eta = \xi^{1/\mu}$ as a local uniformizing parameter near the ideal point.

The Winding Number w

In any escape region, the residue $\oint dt$ is zero, so that t is locally a meromorphic function.

The Winding Number w

In any escape region, the residue $\oint dt$ is zero, so that t is locally a meromorphic function. After adjusting by an additive constant we can write

$$\begin{aligned} t &= c\,\eta^w + (\text{higher} \quad \text{order} \quad \text{terms}) \\ &\sim c\,\eta^w \qquad \text{with} \qquad c \neq 0 \end{aligned}$$

where $w = w_h \in \mathbb{Z} \setminus \{0\}$ is a new invariant called the winding number.

The Winding Number w

In any escape region, the residue $\oint dt$ is zero, so that t is locally a meromorphic function. After adjusting by an additive constant we can write

$$\begin{aligned} t &= c\,\eta^{w} + (\text{higher} \quad \text{order} \quad \text{terms}) \\ &\sim c\,\eta^{w} \qquad \text{with} \qquad c \neq 0 \end{aligned}$$

where $w = w_h \in \mathbb{Z} \setminus \{0\}$ is a new invariant called the winding number. As we wind once around the ideal point in S_p, we wind w times around zero in the t plane.

The Winding Number w

In any escape region, the residue $\oint dt$ is zero, so that t is locally a meromorphic function. After adjusting by an additive constant we can write

$$\begin{aligned} t &= c\,\eta^w + (\text{higher order terms}) \\ &\sim c\,\eta^w \qquad \text{with} \qquad c \neq 0 \end{aligned}$$

where $w = w_h \in \mathbb{Z} \setminus \{0\}$ is a new invariant called the winding number. As we wind once around the ideal point in S_p, we wind w times around zero in the t plane. Hence,

$$\frac{dt}{d\eta} \sim c'\eta^{w-1} + (\text{higher order terms}).$$

The Winding Number w

In any escape region, the residue $\oint dt$ is zero, so that t is locally a meromorphic function. After adjusting by an additive constant we can write

$$\begin{aligned} t &= c\,\eta^w + (\text{higher order terms}) \\ &\sim c\,\eta^w \qquad \text{with} \qquad c \neq 0 \end{aligned}$$

where $w = w_h \in \mathbb{Z} \smallsetminus \{0\}$ is a new invariant called the winding number. As we wind once around the ideal point in S_p, we wind w times around zero in the t plane. Hence,

$$\frac{dt}{d\eta} \sim c'\eta^{w-1} + (\text{higher order terms}).$$

Thus, dt has a zero of order $w-1$ or a pole of order $1-w$ at the ideal point ∞_h.

Euler Characteristic of $\mathcal{S}_p$

Given any meromorphic 1-form on an smooth compact curve $\mathcal{S}$,

$$\chi(\mathcal{S}) \;=\; \#(\textit{poles}) - \#(\textit{zeros})$$

Euler Characteristic of $\mathcal{S}_p$

Given any meromorphic 1-form on an smooth compact curve $\mathcal{S}$,

$$\chi(\mathcal{S}) \;=\; \#(poles) \;-\; \#(zeros)$$

Proposition. The Euler characteristic of the compact curve $\overline{\mathcal{S}}_p$ can be expressed as

$$\chi(\overline{\mathcal{S}}_p) \;=\; \sum_h (1 \;-\; w_h)\,,$$

Here w_h is the winding number of $\mathcal{E}_h$.

Euler Characteristic of $\mathcal{S}_p$

Given any meromorphic 1-form on an smooth compact curve $\mathcal{S}$,

$$\chi(\mathcal{S}) \;=\; \#(\textit{poles}) - \#(\textit{zeros})$$

Proposition. The Euler characteristic of the compact curve $\overline{\mathcal{S}}_p$ can be expressed as

$$\chi(\overline{\mathcal{S}}_p) \;=\; \sum_h (1 - w_h)\,,$$

Here w_h is the winding number of $\mathcal{E}_h$.

It follows that the Euler characteristic of the affine curve $\mathcal{S}_p$ is given by

$$\chi(\mathcal{S}_p) \;=\; -\sum_h w_h\,.$$

Euler Characteristic of $\mathcal{S}_p$

Given any meromorphic 1-form on an smooth compact curve $\mathcal{S}$,

$$\chi(\mathcal{S}) \;=\; \#(poles) - \#(zeros)$$

Proposition. The Euler characteristic of the compact curve $\overline{\mathcal{S}}_p$ can be expressed as

$$\chi(\overline{\mathcal{S}}_p) \;=\; \sum_h (1 - w_h)\,,$$

Here w_h is the winding number of $\mathcal{E}_h$.

It follows that the Euler characteristic of the affine curve $\mathcal{S}_p$ is given by

$$\chi(\mathcal{S}_p) \;=\; -\sum_h w_h\,.$$

If $\mathcal{S}_p$ is connected, then

$$\mathrm{genus}(\mathcal{S}_p) \;=\; \mathrm{genus}(\overline{\mathcal{S}}_p) \;=\; 1 - \chi(\overline{\mathcal{S}}_p)/2\,.$$

The Euler characteristic and the degree d_p of $\mathcal{S}_p$

Main Theorem. The Euler characteristic of the affine curve $\mathcal{S}_p$ is given by

$$\chi(\mathcal{S}_p) = (2 - p)\, d_p\,.$$

The Euler characteristic and the degree d_p of $\mathcal{S}_p$

Main Theorem. The Euler characteristic of the affine curve $\mathcal{S}_p$ is given by

$$\chi(\mathcal{S}_p) \;=\; (2 - p)\, d_p\,.$$

Hence the Euler characteristic of $\overline{\mathcal{S}_p}$ is

$$\chi(\overline{\mathcal{S}_p}) \;=\; N_p + (2 - p)\, d_p\,,$$

where N_p is the number of escape regions (= number of puncture points) and $1 \le N_p \le d_p$.

Examples $p \leq 4$

p	d_p	$\chi(S_p)$	N_p	$\chi(\overline{\mathcal{S}}_p)$
1	1	1×1	1	2
2	2	0×2	2	2
3	8	-1×8	8	0
4	24	-2×24	20	-28

Examples $p \leq 4$

p	d_p	$\chi(S_p)$	N_p	$\chi(\overline{\mathcal{S}}_p)$
1	1	1×1	1	2
2	2	0×2	2	2
3	8	-1×8	8	0
4	24	-2×24	20	-28

(Using the equation $\chi(S_p) \;=\; (2-p)\, d_p\ .$)

Some Euler Characteristics of $\overline{\mathcal{S}}_p$ (DeMarco)

- Period 5: -184
- Period 6: -784
- Period 7: -3236
- Period 8: -11848
- Period 9: -42744
- Period 10: -147948
- Period 11: -505876
- Period 12: -1694848
- Period 13: -5630092
- Period 14: -18491088
- Period 15: -60318292
- Period 16: -195372312
- Period 17: -629500300
- Period 18: -2018178780
- Period 19: -6443997852
- Period 20: -20498523320

Computation of the Euler characteristic

Step 1. The Branner-Hubbard Puzzle for maps in $\mathcal{E}_h$.

Computation of the Euler characteristic

Step 1. The Branner-Hubbard Puzzle for maps in $\mathcal{E}_h$.
In order to understand asymptotic behavior near ∞_h, we must first study the dynamics of maps $F \in \mathcal{E}_h$.

Computation of the Euler characteristic

Step 1. The Branner-Hubbard Puzzle for maps in $\mathcal{E}_h$.
In order to understand asymptotic behavior near ∞_h, we must first study the dynamics of maps $F \in \mathcal{E}_h$.

Step 2. The Kiwi Puzzle: Non-Archimedean Dynamics.

Computation of the Euler characteristic

Step 1. The Branner-Hubbard Puzzle for maps in $\mathcal{E}_h$.
In order to understand asymptotic behavior near ∞_h, we must first study the dynamics of maps $F \in \mathcal{E}_h$.

Step 2. The Kiwi Puzzle: Non-Archimedean Dynamics.
This will convert the Branner-Hubbard dynamic information into asymptotic information about the differences $a - F^{\circ j}(a)$.

Computation of the Euler characteristic

Step 1. The Branner-Hubbard Puzzle for maps in $\mathcal{E}_h$.
In order to understand asymptotic behavior near ∞_h, we must first study the dynamics of maps $F \in \mathcal{E}_h$.

Step 2. The Kiwi Puzzle: Non-Archimedean Dynamics.
This will convert the Branner-Hubbard dynamic information into asymptotic information about the differences $a - F^{\circ j}(a)$.

Step 3. Local Computation: The contribution of each ∞_h to $\chi(\overline{\mathcal{S}}_p)$.

Computation of the Euler characteristic

Step 1. The Branner-Hubbard Puzzle for maps in $\mathcal{E}_h$.
In order to understand asymptotic behavior near ∞_h, we must first study the dynamics of maps $F \in \mathcal{E}_h$.

Step 2. The Kiwi Puzzle: Non-Archimedean Dynamics.
This will convert the Branner-Hubbard dynamic information into asymptotic information about the differences $a - F^{\circ j}(a)$.

Step 3. Local Computation: The contribution of each ∞_h to $\chi(\overline{\mathcal{S}}_p)$.

Step 4. A Global Identity.

Computation of the Euler characteristic

Step 1. The Branner-Hubbard Puzzle for maps in $\mathcal{E}_h$. In order to understand asymptotic behavior near ∞_h, we must first study the dynamics of maps $F \in \mathcal{E}_h$.

Step 2. The Kiwi Puzzle: Non-Archimedean Dynamics. This will convert the Branner-Hubbard dynamic information into asymptotic information about the differences $a - F^{\circ j}(a)$.

Step 3. Local Computation: The contribution of each ∞_h to $\chi(\overline{\mathcal{S}}_p)$.

Step 4. A Global Identity. This will help piece the complicated local information together into a relatively simple formula.

The local computation and the global identity

Local computation

The local computation and the global identity

Local computation

For each $\mathcal{E}_h$ there exists $c_h > 0$ and $q_h \in \mathbb{Q}$ such that, we get the asymptotic formula,

$$|(a - a_1) \cdots (a - a_{p-1})| \sim c_h |a|^{q_h} \qquad \text{as} \quad |a| \to \infty$$

The local computation and the global identity

Local computation

For each $\mathcal{E}_h$ there exists $c_h > 0$ and $q_h \in \mathbb{Q}$ such that, we get the asymptotic formula,

$$|(a - a_1) \cdots (a - a_{p-1})| \sim c_h |a|^{q_h} \qquad \text{as} \quad |a| \to \infty$$

and prove that

$$w_n/\mu_h = p - 2 + q_h.$$

The local computation and the global identity

Local computation

For each $\mathcal{E}_h$ there exists $c_h > 0$ and $q_h \in \mathbb{Q}$ such that, we get the asymptotic formula,

$$|(a - a_1) \cdots (a - a_{p-1})| \sim c_h |a|^{q_h} \qquad \text{as} \quad |a| \to \infty$$

and prove that

$$w_n / \mu_h = p - 2 + q_h.$$

Global identity

$$\sum_h \mu_h q_h = 0.$$

The local computation and the global identity

Local computation

For each $\mathcal{E}_h$ there exists $c_h > 0$ and $q_h \in \mathbb{Q}$ such that, we get the asymptotic formula,

$$|(a-a_1)\cdots(a-a_{p-1})| \sim c_h|a|^{q_h} \qquad \text{as} \quad |a| \to \infty$$

and prove that

$$w_n/\mu_h = p - 2 + q_h.$$

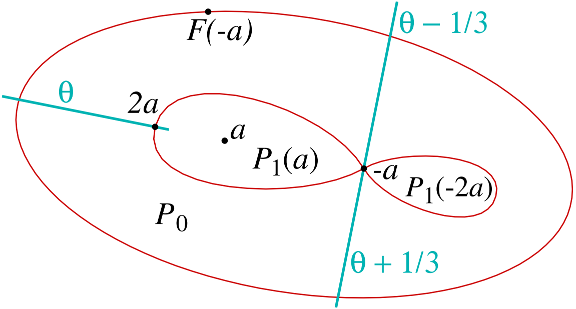
Global identity

$$\sum_h \mu_h q_h = 0.$$

Therefore

$$\sum_h w_h = (p-2)\sum_h \mu_h = (p-2)\, d_p\,.$$

Sketch of the dynamical plane



Here $\theta \in \mathbb{R}/\mathbb{Z}$ is the co-critical angle.

More on Escape Regions

Since

$$a = a_0 \mapsto a_1 = v \mapsto a_2 \mapsto \cdots \mapsto a_p = a$$

then each a_j, can be expressed as a meromorphic function of $\xi^{1/\mu}$ with a pole at the ideal point ∞_h.

More on Escape Regions

Since
$$a = a_0 \mapsto a_1 = v \mapsto a_2 \mapsto \cdots \mapsto a_p = a$$

then each a_j, can be expressed as a meromorphic function of $\xi^{1/\mu}$ with a pole at the ideal point ∞_h.

The periodic critical orbit elements are "asymptotic" to the periodic critical point $+a$ or to the cocritical point $-2a$.

More on Escape Regions

Since
$$a = a_0 \mapsto a_1 = v \mapsto a_2 \mapsto \cdots \mapsto a_p = a$$

then each a_j, can be expressed as a meromorphic function of $\xi^{1/\mu}$ with a pole at the ideal point ∞_h.

The periodic critical orbit elements are "asymptotic" to the periodic critical point $+a$ or to the cocritical point $-2a$. The cocritical point satisfies the property of also being a preimage of the marked critical value i.e.,

$$F_{a,v}(-2a) = v = F_{a,v}(+a).$$

More on Escape Regions

Since
$$a = a_0 \mapsto a_1 = v \mapsto a_2 \mapsto \cdots \mapsto a_p = a$$

then each a_j, can be expressed as a meromorphic function of $\xi^{1/\mu}$ with a pole at the ideal point ∞_h.

The periodic critical orbit elements are "asymptotic" to the periodic critical point $+a$ or to the cocritical point $-2a$. The cocritical point satisfies the property of also being a preimage of the marked critical value i.e.,

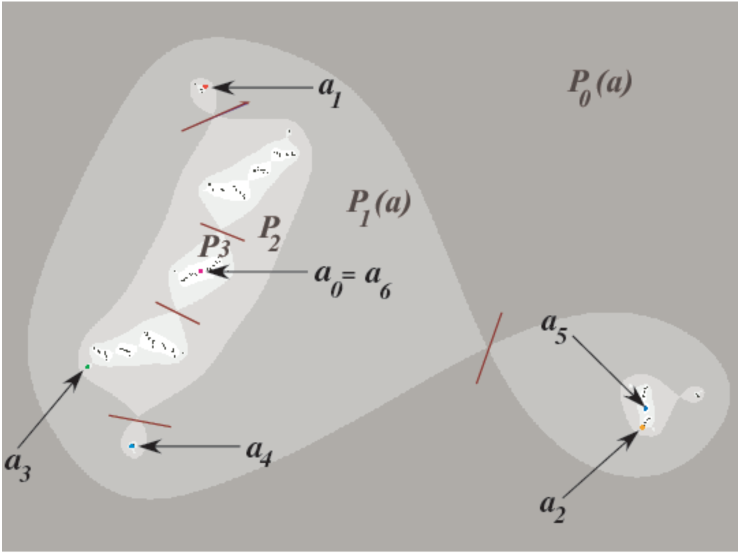
$$F_{a,v}(-2a) = v = F_{a,v}(+a).$$

We have

$$a_j = \begin{cases} a + O(1)\,, & \text{if} \quad a_j \in P_1(a) \quad (\sigma_j = 0) \\ -2a + O(1)\,, & \text{if} \quad a_j \in P_1(-2a) \quad (\sigma_j = 1)\,. \end{cases}$$

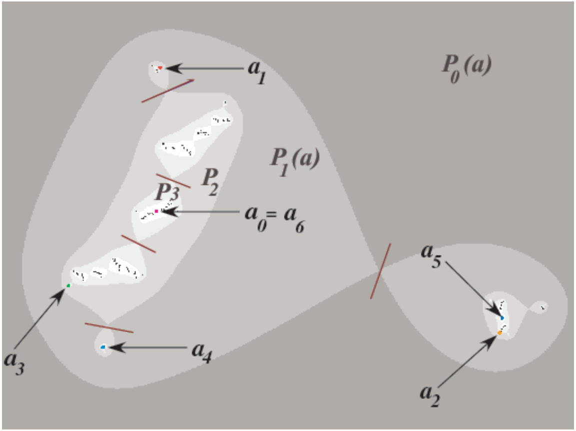
where each $O(1)$ term represents a holomorphic function of $\xi^{1/\mu}$ which is bounded for small $|\xi|$.

The Branner-Hubbard Puzzle. Let $a_j = F^{\circ j}(a)$

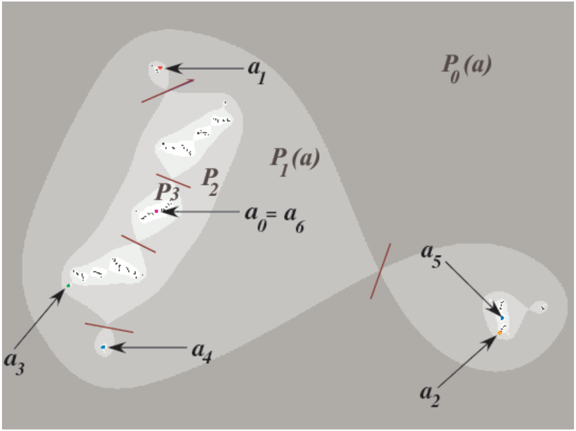


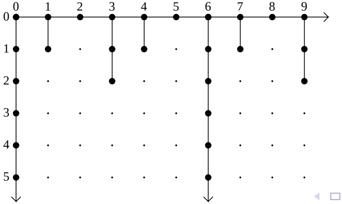
The Branner-Hubbard Puzzle for a polynomial with kneading sequence $\overline{010010}$.

The Branner-Hubbard Puzzle for a polynomial with kneading sequence $\overline{010010}$.



The Branner-Hubbard Puzzle for a polynomial with kneading sequence $\overline{010010}$.





Kneading sequence

If the critical point $+a \in \mathcal{E}_h$ then it determines a periodic sequence $\sigma(a) \in \{0, 1\}$, with $\sigma_{j+p}(a) = \sigma_j(a)$, and with $\sigma_0(a) = 0$.

Kneading sequence

If the critical point $+a \in \mathcal{E}_h$ then it determines a periodic sequence $\sigma(a) \in \{0, 1\}$, with $\sigma_{j+p}(a) = \sigma_j(a)$, and with $\sigma_0(a) = 0$.

The **kneading sequence** of an orbit $a_0 \mapsto a_1 \mapsto \cdots$ in K_F is the sequence

$$\sigma(a_0)\,\sigma(a_1)\,\sigma(a_2)\,\cdots$$

of zeros and ones, where

$$\sigma(a_j) = \begin{cases} 0 & \text{if} \quad P_1(a_o) = P_1(a_j), \\ 1 & \text{if} \quad P_1(a_o) \neq P_1(a_j)\,. \end{cases}$$

Kneading sequence

If the critical point $+a \in \mathcal{E}_h$ then it determines a periodic sequence $\sigma(a) \in \{0, 1\}$, with $\sigma_{j+p}(a) = \sigma_j(a)$, and with $\sigma_0(a) = 0$.

The **kneading sequence** of an orbit $a_0 \mapsto a_1 \mapsto \cdots$ in K_F is the sequence

$$\sigma(a_0)\,\sigma(a_1)\,\sigma(a_2)\,\cdots$$

of zeros and ones, where

$$\sigma(a_j) = \begin{cases} 0 & \text{if} \quad P_1(a_o) = P_1(a_j), \\ 1 & \text{if} \quad P_1(a_o) \neq P_1(a_j)\,. \end{cases}$$

The kneading sequence is briefly denoted as $\overline{\sigma_1 \ldots \sigma_{p-1} 0}$, where the overline indicates infinite repetition.

Kneading sequence

If the critical point $+a \in \mathcal{E}_h$ then it determines a periodic sequence $\sigma(a) \in \{0,1\}$, with $\sigma_{j+p}(a) = \sigma_j(a)$, and with $\sigma_0(a) = 0$.

The **kneading sequence** of an orbit $a_0 \mapsto a_1 \mapsto \cdots$ in K_F is the sequence

$$\sigma(a_0)\,\sigma(a_1)\,\sigma(a_2)\,\cdots$$

of zeros and ones, where

$$\sigma(a_j) \;=\; \begin{cases} 0 & \text{if} \quad P_1(a_o) \;=\; P_1(a_j), \\ 1 & \text{if} \quad P_1(a_o) \;\neq\; P_1(a_j)\,. \end{cases}$$

The kneading sequence is briefly denoted as $\overline{\sigma_1 \ldots \sigma_{p-1} 0}$, where the overline indicates infinite repetition.

The (minimal) period q of this kneading sequence is always a divisor of the period p of $+a$.

Escape Regions and Puiseux series

To replace the a_j by locally holomorphic functions on $\overline{\mathcal{S}}_p$, we introduce the new variables

$$u_j = \frac{a - a_j}{3\,a}.$$

Escape Regions and Puiseux series

To replace the a_j by locally holomorphic functions on $\overline{\mathcal{S}}_p$, we introduce the new variables

$$u_j = \frac{a - a_j}{3\,a}.$$

More precisely, each u_j has a power series of the form

$$u_j = \sigma_j + c_\mu\,\xi + c_{\mu+1}\,\xi^{1+1/\mu} + c_{\mu+2}\,\xi^{1+2/\mu} + \cdots$$

which converges for small $|\xi|$. Notice that $\sigma_j \in \{0, 1\}$.

Escape Regions and Puiseux series

To replace the a_j by locally holomorphic functions on $\overline{\mathcal{S}}_p$, we introduce the new variables

$$u_j \ = \ \frac{a - a_j}{3\,a} \, .$$

More precisely, each u_j has a power series of the form

$$u_j \ = \ \sigma_j \ + \ c_\mu \, \xi \ + \ c_{\mu+1} \, \xi^{1+1/\mu} \ + \ c_{\mu+2} \, \xi^{1+2/\mu} \ + \ \cdots$$

which converges for small $|\xi|$. Notice that $\sigma_j \ \in \ \{0, \, 1\}$.

We will refer to this as the ***Puiseux expansion*** of u_j .

Escape Regions and Puiseux series

To replace the a_j by locally holomorphic functions on $\overline{\mathcal{S}}_p$, we introduce the new variables

$$u_j = \frac{a - a_j}{3\,a} .$$

More precisely, each u_j has a power series of the form

$$u_j = \sigma_j + c_\mu\,\xi + c_{\mu+1}\,\xi^{1+1/\mu} + c_{\mu+2}\,\xi^{1+2/\mu} + \cdots$$

which converges for small $|\xi|$. Notice that $\sigma_j \in \{0, 1\}$.

We will refer to this as the ***Puiseux expansion*** of u_j.

The Puiseux series depends on the choice of μ-th. root of ξ, but different choices give series which are conjugate to each other by the Galois automorphism

$$\xi^{1/\mu} \mapsto \alpha\,\xi^{1/\mu}$$

where α is an arbitrary μ-th root of unity.

Characterization of Escape regions by Puiseux series

Characterization of Escape regions by Puiseux series

Theorem [BKM]. Each escape region $\mathcal{E}_h$ of $\mathcal{S}_p$ is characterized by an essentially unique Puiseux series.

Characterization of Escape regions by Puiseux series

Theorem [BKM]. Each escape region $\mathcal{E}_h$ of $\mathcal{S}_p$ is characterized by an essentially unique Puiseux series.

Passing to **formal Puiseux series** we can rewrite

$$\mathbf{u_j} = \sum_{k \geq k_0 \geq 0} c_k \, \xi^{k/\mu} \in \mathbb{C}[[\xi^{1/\mu}]], \qquad \text{with} \quad k_0 = 0 \quad \text{or} \quad k \geq \mu.$$

Characterization of Escape regions by Puiseux series

Theorem [BKM]. Each escape region $\mathcal{E}_h$ of $\mathcal{S}_p$ is characterized by an essentially unique Puiseux series.

Passing to **formal Puiseux series** we can rewrite

$$\mathbf{u_j} = \sum_{k \geq k_0 \geq 0} c_k\, \xi^{k/\mu} \in \mathbb{C}[[\xi^{1/\mu}]], \qquad \text{with} \quad k_0 = 0 \quad \text{or} \quad k \geq \mu.$$

Assuming that $c_{k_0} \neq 0$, define the **leading monomial** $\mathbf{m}(\mathbf{u_j})$ of $\mathbf{u_j}$ as $c_{k_0}\, \xi^{k_0/\mu}$.

Characterization of Escape regions by Puiseux series

Theorem [BKM]. Each escape region $\mathcal{E}_h$ of $\mathcal{S}_p$ is characterized by an essentially unique Puiseux series.

Passing to **formal Puiseux series** we can rewrite

$$\mathbf{u_j} = \sum_{k \geq k_0 \geq 0} c_k \, \xi^{k/\mu} \in \mathbb{C}[[\xi^{1/\mu}]], \qquad \text{with} \quad k_0 = 0 \quad \text{or} \quad k \geq \mu.$$

Assuming that $c_{k_0} \neq 0$, define the **leading monomial** $\mathbf{m}(\mathbf{u_j})$ of $\mathbf{u_j}$ as $c_{k_0} \, \xi^{k_0/\mu}$.

We say that $\vec{\mathbf{u}} = (\mathbf{u_1}, \ldots, \mathbf{u_{p-1}}, 0)$ is a vector of Puiseux series associated to the escape region $\mathcal{E}_h$.

Characterization of Escape regions by Puiseux series

Theorem [BKM]. Each escape region $\mathcal{E}_h$ of $\mathcal{S}_p$ is characterized by an essentially unique Puiseux series.

Passing to **formal Puiseux series** we can rewrite

$$\mathbf{u_j} = \sum_{k \geq k_0 \geq 0} c_k \, \xi^{k/\mu} \in \mathbb{C}[[\xi^{1/\mu}]], \qquad \text{with} \quad k_0 = 0 \quad \text{or} \quad k \geq \mu.$$

Assuming that $c_{k_0} \neq 0$, define the **leading monomial** $\mathbf{m}(\mathbf{u_j})$ of $\mathbf{u_j}$ as $c_{k_0} \, \xi^{k_0/\mu}$.

We say that $\vec{\mathbf{u}} = (\mathbf{u_1}, \ldots, \mathbf{u_{p-1}}, 0)$ is a vector of Puiseux series associated to the escape region $\mathcal{E}_h$.

The asymptotic behavior of $a_j - a$ is encoded by

$$\mathbf{m}(\vec{\mathbf{u}}) \; = \; \Big(\mathbf{m}(\vec{\mathbf{u}}_1), \; \ldots, \; \mathbf{m}(\vec{\mathbf{u}}_{p-1}), \; 0\Big).$$

Characterization of Escape regions by Puiseux series

Theorem [BKM]. Each escape region $\mathcal{E}_h$ of $\mathcal{S}_p$ is characterized by an essentially unique Puiseux series.

Passing to **formal Puiseux series** we can rewrite

$$\mathbf{u_j} = \sum_{k \geq k_0 \geq 0} c_k \, \xi^{k/\mu} \in \mathbb{C}[[\xi^{1/\mu}]], \qquad \text{with} \quad k_0 = 0 \quad \text{or} \quad k \geq \mu.$$

Assuming that $c_{k_0} \neq 0$, define the **leading monomial** $\mathbf{m}(\mathbf{u_j})$ of $\mathbf{u_j}$ as $c_{k_0} \, \xi^{k_0/\mu}$.

We say that $\vec{\mathbf{u}} = (\mathbf{u_1}, \ldots, \mathbf{u_{p-1}}, 0)$ is a vector of Puiseux series associated to the escape region $\mathcal{E}_h$.

The asymptotic behavior of $a_j - a$ is encoded by

$$\mathbf{m}(\vec{\mathbf{u}}) \; = \; \Big(\mathbf{m}(\vec{\mathbf{u}}_1), \; \ldots, \; \mathbf{m}(\vec{\mathbf{u}}_{p-1}), \; 0\Big).$$

Theorem [Kiwi]. Let $\vec{\mathbf{u}'}$ and $\vec{\mathbf{u}''}$ be vectors of **Puiseux series** associated to escape regions of $\mathcal{S}_p$.

If $\mathbf{m}(\vec{\mathbf{u}'}) = \mathbf{m}(\vec{\mathbf{u}''})$, then $\vec{\mathbf{u}'} = \vec{\mathbf{u}''}$.

Non-Archimedean fields, –the basics

Let $\mathbb{Q}^a$ be the algebraic closure of the set of rational numbers $\mathbb{Q}$, and

Non-Archimedean fields, –the basics

Let $\mathbb{Q}^a$ be the algebraic closure of the set of rational numbers $\mathbb{Q}$, and let $\mathbb{L} = \mathbb{L}(\xi)$ be the completion of the field of Puiseux series in ξ with coefficients in $\mathbb{Q}^a$.

Non-Archimedean fields, –the basics

Let $\mathbb{Q}^a$ be the algebraic closure of the set of rational numbers $\mathbb{Q}$, and let $\mathbb{L} = \mathbb{L}(\xi)$ be the completion of the field of Puiseux series in ξ with coefficients in $\mathbb{Q}^a$. By definition, this set consists of all finite or infinite series of the form

$$\mathbf{z} \;=\; c_0\,\xi^{q_0} + c_1\xi^{q_1} + \cdots \quad \text{with} \quad c_j \in \mathbb{Q}^a \diagdown \{0\}\,, \quad q_j \in \mathbb{Q}\,,$$

and with $q_0 < q_1 < q_2 < \ldots$, where $\lim_{j\to\infty} q_j = +\infty$ in the case of an infinite sum.

Non-Archimedean fields, –the basics

Let $\mathbb{Q}^a$ be the algebraic closure of the set of rational numbers $\mathbb{Q}$, and let $\mathbb{L} = \mathbb{L}(\xi)$ be the completion of the field of Puiseux series in ξ with coefficients in $\mathbb{Q}^a$. By definition, this set consists of all finite or infinite series of the form

$$\mathbf{z} \;=\; c_0\,\xi^{q_0} \;+\; c_1\xi^{q_1} \;+\; \cdots \quad \text{with} \quad c_j \in \mathbb{Q}^a \smallsetminus \{0\}\,,\;\; q_j \in \mathbb{Q}\,,$$

and with $q_0 < q_1 < q_2 < \ldots$, where $\lim_{j\to\infty} q_j = +\infty$ in the case of an infinite sum. This field is algebraically closed, and complete under the norm

$$\|z\| \;=\; e^{-q_0}\,, \qquad \text{with} \qquad \log\|z\| \;=\; -q_0\,,$$

for $\mathbf{z}$ as above, $\mathbf{z} \neq \mathbf{0}$; with $\|\mathbf{0}\| = 0$.

Non-Archimedean fields, –the basics

Let $\mathbb{Q}^a$ be the algebraic closure of the set of rational numbers $\mathbb{Q}$, and let $\mathbb{L} = \mathbb{L}(\xi)$ be the completion of the field of Puiseux series in ξ with coefficients in $\mathbb{Q}^a$. By definition, this set consists of all finite or infinite series of the form

$$\mathbf{z} \;=\; c_0\,\xi^{q_0} \;+\; c_1\xi^{q_1} \;+\;\cdots \quad \text{with} \quad c_j \in \mathbb{Q}^a\smallsetminus\{0\}\,, \;\; q_j \in \mathbb{Q}\,,$$

and with $q_0 < q_1 < q_2 < \ldots$, where $\lim_{j\to\infty} q_j = +\infty$ in the case of an infinite sum. This field is algebraically closed, and complete under the norm

$$\|z\| \;=\; e^{-q_0}\,, \qquad \text{with} \qquad \log\|z\| \;=\; -q_0\,,$$

for $\mathbf{z}$ as above, $\mathbf{z} \neq \mathbf{0}$; with $\|\mathbf{0}\| = 0$. Note the ultrametric inequality

$$\|\alpha + \beta\| \;\le\; \max(\,\|\alpha\|\,,\;\|\beta\|\,)\,,$$

with equality except possibly when $\|\alpha\| = \|\beta\|$.

Open balls and annulus

For any $r \in e^{\mathbb{Q}}$, a set of the form $\{\mathbf{z} \,;\, \|\mathbf{z} - \mathbf{z}_0\| < r\}$ is called an **open ball** of radius r.

Open balls and annulus

For any $r \in e^{\mathbb{Q}}$, a set of the form $\{\mathbf{z} \,;\, \|\mathbf{z}-\mathbf{z}_0\| < r\}$ is called an **open ball** of radius r. If $r_1 < r_2$ in $e^{\mathbb{Q}}$, then

$$\{\mathbf{z} \,;\, r_1 < \|\mathbf{z}-\mathbf{z}_0\| < r_2\}$$

is called an **annulus** of modulus $\log(r_2/r_1)$.

Open balls and annulus

For any $r \in e^{\mathbb{Q}}$, a set of the form $\{\mathbf{z}\,;\ \|\mathbf{z}-\mathbf{z}_0\| < r\}$ is called an **open ball** of radius r. If $r_1 < r_2$ in $e^{\mathbb{Q}}$, then

$$\{\mathbf{z}\,;\ r_1 < \|\mathbf{z}-\mathbf{z}_0\| < r_2\}$$

is called an **annulus** of modulus $\log(r_2/r_1)$. Thus all balls and annuli in $\mathbb{L}$ are round by definition, and all moduli are rational.

Dynamics setting in non-Archimedean field

Identify the marked critical point $a \in \mathbb{C}$ with the constant $\mathbf{a} = \xi^{-1} \in \mathbb{L}$, where $\log \|\mathbf{a}\| = +1$.

Dynamics setting in non-Archimedean field

Identify the marked critical point $a \in \mathbb{C}$ with the constant $\mathbf{a} = \xi^{-1} \in \mathbb{L}$, where $\log \|\mathbf{a}\| = +1$. For any $\mathbf{v} \in \mathbb{L}$ we have a polynomial map $\mathbf{f_v} : \mathbb{L} \to \mathbb{L}$ defined by

$$\mathbf{f_v}(\mathbf{z}) \;=\; \mathbf{z}^3 - 3\,\mathbf{a}^2\mathbf{z} + (2\,\mathbf{a}^3 + \mathbf{v})\,,$$

so that $\mathbf{f_v}(\mathbf{a}) \;=\; \mathbf{v}$.

Dynamics setting in non-Archimedean field

Identify the marked critical point $a \in \mathbb{C}$ with the constant $\mathbf{a} = \xi^{-1} \in \mathbb{L}$, where $\log \|\mathbf{a}\| = +1$. For any $\mathbf{v} \in \mathbb{L}$ we have a polynomial map $\mathbf{f_v} : \mathbb{L} \to \mathbb{L}$ defined by

$$\mathbf{f_v}(\mathbf{z}) = \mathbf{z}^3 - 3\,\mathbf{a}^2\mathbf{z} + (2\,\mathbf{a}^3 + \mathbf{v}),$$

so that $\mathbf{f_v}(\mathbf{a}) = \mathbf{v}$. We will assume that the Puiseux series $\mathbf{v}$ is chosen so that $\mathbf{f_v}^{\circ p}(\mathbf{a}) = \mathbf{a}$.

Dynamics setting in non-Archimedean field

Identify the marked critical point $a \in \mathbb{C}$ with the constant $\mathbf{a} = \xi^{-1} \in \mathbb{L}$, where $\log \|\mathbf{a}\| = +1$. For any $\mathbf{v} \in \mathbb{L}$ we have a polynomial map $\mathbf{f_v} : \mathbb{L} \to \mathbb{L}$ defined by

$$\mathbf{f_v}(\mathbf{z}) \;=\; \mathbf{z}^3 - 3\,\mathbf{a}^2\mathbf{z} + (2\,\mathbf{a}^3 + \mathbf{v})\,,$$

so that $\mathbf{f_v}(\mathbf{a}) \;=\; \mathbf{v}$. We will assume that the Puiseux series $\mathbf{v}$ is chosen so that $\mathbf{f_v}^{\circ p}(\mathbf{a}) \;=\; \mathbf{a}$.

The associated **Green's function** $G : \mathbb{L} \to [0,\ \infty)$ is defined by

$$G(\mathbf{z}) \;=\; \lim_{n\to\infty} \frac{1}{3^n} \log^+ \|\mathbf{f_v}^{\circ n}(\mathbf{z})\|\,,$$

so that $G\big(\mathbf{f_v}(\mathbf{z})\big) \;=\; 3\,G(\mathbf{z})$.

Dynamics setting in non-Archimedean field

Identify the marked critical point $a \in \mathbb{C}$ with the constant $\mathbf{a} = \xi^{-1} \in \mathbb{L}$, where $\log\|\mathbf{a}\| = +1$. For any $\mathbf{v} \in \mathbb{L}$ we have a polynomial map $\mathbf{f_v} : \mathbb{L} \to \mathbb{L}$ defined by

$$\mathbf{f_v}(\mathbf{z}) \;=\; \mathbf{z}^3 - 3\,\mathbf{a}^2\mathbf{z} + (2\,\mathbf{a}^3 + \mathbf{v})\;,$$

so that $\mathbf{f_v}(\mathbf{a}) \;=\; \mathbf{v}$. We will assume that the Puiseux series $\mathbf{v}$ is chosen so that $\mathbf{f_v}^{\circ p}(\mathbf{a}) \;=\; \mathbf{a}$.

The associated **Green's function** $G : \mathbb{L} \to [0,\ \infty)$ is defined by

$$G(\mathbf{z}) \;=\; \lim_{n\to\infty} \frac{1}{3^n} \log^+ \|\mathbf{f_v}^{\circ n}(\mathbf{z})\|\;,$$

so that $G\big(\mathbf{f_v}(\mathbf{z})\big) \;=\; 3\,G(\mathbf{z})$. For example $G(\mathbf{z}) = 0$ whenever $\mathbf{z}$ is periodic.

Dynamics setting in non-Archimedean field

Identify the marked critical point $a \in \mathbb{C}$ with the constant $\mathbf{a} = \xi^{-1} \in \mathbb{L}$, where $\log \|\mathbf{a}\| = +1$. For any $\mathbf{v} \in \mathbb{L}$ we have a polynomial map $\mathbf{f_v} : \mathbb{L} \to \mathbb{L}$ defined by

$$\mathbf{f_v}(\mathbf{z}) \;=\; \mathbf{z}^3 - 3\,\mathbf{a}^2\mathbf{z} + (2\,\mathbf{a}^3 + \mathbf{v})\,,$$

so that $\mathbf{f_v}(\mathbf{a}) \;=\; \mathbf{v}$. We will assume that the Puiseux series $\mathbf{v}$ is chosen so that $\mathbf{f_v}^{\circ p}(\mathbf{a}) \;=\; \mathbf{a}$.

The associated **Green's function** $G : \mathbb{L} \to [0,\ \infty)$ is defined by

$$G(\mathbf{z}) \;=\; \lim_{n\to\infty} \frac{1}{3^n} \log^+ \|\mathbf{f_v}^{\circ n}(\mathbf{z})\|\,,$$

so that $G\big(\mathbf{f_v}(\mathbf{z})\big) \;=\; 3\,G(\mathbf{z})$. For example $G(\mathbf{z}) = 0$ whenever $\mathbf{z}$ is periodic. Thus $G(\mathbf{a}) = G(\mathbf{v}) = 0$, but it is easy to check that $G(\mathbf{z}) = \log \|\mathbf{z}\|$ whenever $\log \|\mathbf{z}\| > 1$.

Dynamics setting in non-Archimedean field

Identify the marked critical point $a \in \mathbb{C}$ with the constant $\mathbf{a} = \xi^{-1} \in \mathbb{L}$, where $\log \|\mathbf{a}\| = +1$. For any $\mathbf{v} \in \mathbb{L}$ we have a polynomial map $\mathbf{f_v} : \mathbb{L} \to \mathbb{L}$ defined by

$$\mathbf{f_v}(\mathbf{z}) \;=\; \mathbf{z}^3 - 3\,\mathbf{a}^2\mathbf{z} + (2\,\mathbf{a}^3 + \mathbf{v})\;,$$

so that $\mathbf{f_v}(\mathbf{a}) \;=\; \mathbf{v}$. We will assume that the Puiseux series $\mathbf{v}$ is chosen so that $\mathbf{f_v}^{\circ p}(\mathbf{a}) \;=\; \mathbf{a}$.

The associated **Green's function** $G : \mathbb{L} \to [0,\ \infty)$ is defined by

$$G(\mathbf{z}) \;=\; \lim_{n\to\infty} \frac{1}{3^n} \log^+ \|\mathbf{f_v}^{\circ n}(\mathbf{z})\|\;,$$

so that $G\big(\mathbf{f_v}(\mathbf{z})\big) \;=\; 3\,G(\mathbf{z})$. For example $G(\mathbf{z}) = 0$ whenever $\mathbf{z}$ is periodic. Thus $G(\mathbf{a}) = G(\mathbf{v}) = 0$, but it is easy to check that $G(\mathbf{z}) = \log\|\mathbf{z}\|$ whenever $\log\|\mathbf{z}\| > 1$. For example $G\big(\mathbf{f_v}(-\mathbf{a})\big) = 3$, hence $G(-\mathbf{a}) = 1$.

The Kiwi Puzzle

The puzzle piece P_0 is the open ball consisting of all $\mathbf{z} \in \mathbb{L}$ with
$$G(\mathbf{z}) < 3 \quad \Longleftrightarrow \quad \log \|\mathbf{z}\| < 3 \ .$$

The Kiwi Puzzle

The puzzle piece P_0 is the open ball consisting of all $\mathbf{z} \in \mathbb{L}$ with

$$G(\mathbf{z}) < 3 \quad \Longleftrightarrow \quad \log \|\mathbf{z}\| < 3 \,.$$

Kiwi Lemma. For each $n > 0$, the set of all $\mathbf{z} \in \mathbb{L}$ such that

$$\mathbf{f}^{\circ n}(\mathbf{z}) \in P_0 \quad \Longleftrightarrow \quad G(\mathbf{z}) < 3^{1-n}$$

is a union of finitely many disjoint open balls.

The Kiwi Puzzle

The puzzle piece P_0 is the open ball consisting of all $\mathbf{z} \in \mathbb{L}$ with

$$G(\mathbf{z}) < 3 \quad \Longleftrightarrow \quad \log \|\mathbf{z}\| < 3 \; .$$

Kiwi Lemma. For each $n > 0$, the set of all $\mathbf{z} \in \mathbb{L}$ such that

$$\mathbf{f}^{\circ n}(\mathbf{z}) \in P_0 \quad \Longleftrightarrow \quad G(\mathbf{z}) < 3^{1-n}$$

is a union of finitely many disjoint open balls.

By definition, each of these balls is a **puzzle piece** P_n of level n.

The Kiwi Puzzle

The puzzle piece P_0 is the open ball consisting of all $\mathbf{z} \in \mathbb{L}$ with

$$G(\mathbf{z}) < 3 \quad \Longleftrightarrow \quad \log \|\mathbf{z}\| < 3 .$$

Kiwi Lemma. For each $n > 0$, the set of all $\mathbf{z} \in \mathbb{L}$ such that

$$\mathbf{f}^{\circ n}(\mathbf{z}) \in P_0 \quad \Longleftrightarrow \quad G(\mathbf{z}) < 3^{1-n}$$

is a union of finitely many disjoint open balls.

By definition, each of these balls is a **puzzle piece** P_n of level n.

Kiwi Theorem. If $\mathbf{v}$ is the Puiseux series associated with the escape region $\mathcal{E}_h$, then the marked grid for the corresponding Kiwi puzzle is identical with the marked grid for the Branner-Hubbard puzzle for any map $f \in \mathcal{E}_h$.

Corollary. The Branner-Hubbard puzzle determines the norm $\|\mathbf{a}_j - \mathbf{a}\|$ for each point in the periodic orbit $\mathbf{a} \mapsto \mathbf{a}_1 \mapsto \cdots$.

Corollary. The Branner-Hubbard puzzle determines the norm $\|\mathbf{a}_j - \mathbf{a}\|$ for each point in the periodic orbit $\mathbf{a} \mapsto \mathbf{a}_1 \mapsto \cdots$.

On the other hand, if

$$\log \|\mathbf{a}_j - \mathbf{a}\| = q \qquad \text{so that} \qquad \mathbf{a} - \mathbf{a}_j = c_j \, \xi^{-q} + (\text{higher order terms}) \, ,$$

then it follows easily that

$$a - a_j = c_j \, a^q + o(a^q) \qquad \text{as} \qquad |a| \to \infty \, .$$

Corollary. The Branner-Hubbard puzzle determines the norm $\|\mathbf{a}_j - \mathbf{a}\|$ for each point in the periodic orbit $\mathbf{a} \mapsto \mathbf{a}_1 \mapsto \cdots$.

On the other hand, if

$$\log \|\mathbf{a}_j - \mathbf{a}\| = q \qquad \text{so that} \qquad \mathbf{a} - \mathbf{a}_j = c_j\, \xi^{-q} + (\text{higher order terms}),$$

then it follows easily that

$$a - a_j = c_j\, a^q + o(a^q) \qquad \text{as} \qquad |a| \to \infty\,.$$

In particular, it follows that

$$\log |a_j - a| = q \log |a| + O(1)$$

as $(a,\, v) \in \mathcal{E}_h$ tends to the ideal point ∞_h.

References

B. Branner and J.H. Hubbard, *The iteration of cubic polynomials II, patterns and parapatterns*, Acta Math. **169** (1992) 229–325.

J. Kiwi,

- *Puiseux series polynomial dynamics and iteration of complex cubic polynomials*, Ann. Inst. Fourier (Grenoble) **56** (2006) 1337–1404.

- *Leading monomials of escape regions*. To appear in: "Frontiers in Complex Dynamics: a volume in honor of John Milnor's 80th birthday", (A. Bonifant, M. Lyubich, S. Sutherland, editors). In press, 2013, Princeton University Press.

Cubic Polynomial Maps with Periodic Critical Orbit:

J. Milnor, *Part I*. In "Complex Dynamics Families and Friends", ed. D. Schleicher, A. K. Peters 2009, pp. 333-411.

A. Bonifant, J. Kiwi and J. Milnor, *Part II: Escape Regions*, Journal of Conformal Geometry and Dynamics **14** (2010) 68–112 and 190–193.

A. Bonifant and J. Milnor, *Part III: External rays*. In preparation.