What are the Equations Defining Algebraic Varieties?

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Defining Equations of Projective Varieties

Chow's Theorem. Consider

$$X \subseteq \mathbf{P}^r = \mathbf{P}^r(\mathbf{C})$$

a complex submanifold. Then X is an algebraic variety, i.e. $\exists$ homogeneous polynomials

$$F_\alpha = F_\alpha(X_0, \ldots, X_r)$$

such that

$$X = \Big\{ F_1 = \ldots = F_N = 0 \Big\}.$$

Question: What can one say about the defining equations $\{F_\alpha\}$, e.g. their degrees?

Answer: In this generality, nothing.

(Choose any $\{F_\alpha\}$, use to define $X \subset \mathbf{P}^r$.)

Better Question: Study "nice" embeddings of given X.

Today: "nice" = "very positive"

• Embedding $X \subseteq \mathbf{P}^r$ defined by choosing holomorphic line bundle L, and basis

$$s_0 \ , \ \ldots \ , \ s_r \ \in \ \Gamma(X, L).$$

• Define

$$X \hookrightarrow \mathbf{P}^r \quad \text{via} \quad x \mapsto [s_0(x), \ldots s_r(x)].$$

• We will be interested in L where $c_1(L)$ is very positive.

<u>Example</u>. Take $C = \mathbf{P}^1$, $L = \mathcal{O}_{\mathbf{P}^1}(3)$, giving

$$\mathbf{P}^1 \hookrightarrow \mathbf{P}^3 \ , \ [s, t] \mapsto [s^3, s^2t, st^2, t^3].$$

Image is set

$$C \ = \ \left\{ \text{rank} \begin{bmatrix} T_0 & T_1 & T_2 \\ T_1 & T_2 & T_3 \end{bmatrix} \le \ 1 \right\} \ \subseteq \mathbf{P}^3.$$

So C cut out by three quadratic polynomials

$$\Delta_{01} = T_0 T_2 - T_1^2$$
$$\Delta_{02} = T_0 T_3 - T_1 T_2$$
$$\Delta_{12} = T_1 T_3 - T_2^2.$$

Example. Say $E = \mathbf{C}/\Lambda$ an elliptic curve.

- If $\deg L = 3$, get

$$E \subseteq \mathbf{P}^2, \quad E = \{G = 0\}$$

with $\deg(G) = 3$.

- If $\deg L = 4$, get

$$E \subseteq \mathbf{P}^3, \quad E = \{Q_1 = Q_2 = 0\},$$

with $\deg(Q_1) = \deg(Q_2) = 2$.

Theorem [Castelnuovo, Mumford, Kempf, ...]
If X is smooth variety, and

$$X \subseteq \mathbf{P}^r$$

is defined by L with

$$c_1(L) \gg 0,$$

then X cut out by polynomials of degree 2.

Example. (Castelnuovo) When X is curve of genus g, then conclusion holds when

$$\deg(L) \geq 2g + 2.$$

Sidman–Smith: When $c_1(L) \gg 0$, X is cut out in $\mathbf{P}^r$ by the 2×2 minors of matrix of linear forms.

Two Issues.

(I). The theorem guarantees that $X \subseteq \mathbf{P}^r$ is cut out by quadrics if $c_1(L)$ is sufficiently positive. What happens if we let L become even more positive?

Example. If $g(X) = g$, what can we say when

$$\deg(L) \ \geq \ 2g + 3?$$

(II). Can't easily read off invariants of X from number or form of quadrics defining it.

Green: Should study higher syzygies among defining equations.

Syzygies

Consider polynomial ring:

$$S \;=\; \mathbf{C}[T_0,\ldots,T_r],$$

ideal

$$I \;=\; \Big(Q_1,\ldots,Q_N\Big) \;\subseteq\; S$$

and to fix ideas say $\deg(Q_\alpha) = 2$.

Hilbert: Consider *syzygies* among the Q_α, i.e. relations of the form

$$\sum R_\alpha \cdot Q_\alpha \;\equiv\; 0 \qquad (*)$$

where the R_α are polynomials with $\deg R_\alpha = q$.

Say that (*) is a *second* syzygy of *weight* q.

Example: Return to

$$C \;=\; \mathbf{P}^1 \hookrightarrow \mathbf{P}^3\;,\;\; [s,t] \mapsto [s^3, s^2t, st^2, t^3].$$

Recall that

$$C \;=\; \left\{ \text{rank} \begin{bmatrix} T_0 & T_1 & T_2 \\ T_1 & T_2 & T_3 \end{bmatrix} \;\leq\; 1 \right\} \;\subseteq \mathbf{P}^3,$$

so C defined by

$$\begin{aligned}\Delta_{01} &= T_0T_2 - T_1^2\\ \Delta_{02} &= T_0T_3 - T_1T_2\\ \Delta_{12} &= T_1T_3 - T_2^2.\end{aligned}$$

Repeat row of matrix and expand resulting determinant:

$$\det\begin{bmatrix} T_0 & T_1 & T_2\\ T_1 & T_2 & T_3\\ T_0 & T_1 & T_2\end{bmatrix} \equiv 0,$$

so

$$T_2 \cdot \Delta_{01} - T_1 \cdot \Delta_{02} + T_0 \cdot \Delta_{12} = 0.$$

Similarly,

$$T_3 \cdot \Delta_{01} - T_2 \cdot \Delta_{02} + T_1 \cdot \Delta_{12} = 0.$$

No other syzygies.

So here all syzygies have minimal weight $q = 1$.

Example: Consider degree four elliptic curve

$$E = \mathbf{C}/\Lambda \ \subseteq \ \mathbf{P}^3.$$

Recall

$$E \ = \ \Big\{ Q_1 \ = \ Q_2 \ = \ 0. \Big\}.$$

Here only syzygy is

$$Q_2 \cdot Q_1 \ - \ Q_1 \cdot Q_2 \ = \ 0,$$

which has weight $q = 2$.

Returning to ideal

$$I = (Q_1, \ldots, Q_N) \subseteq S,$$

one considers next

$$\Big\{ \text{Third syzygies} \Big\} \ = \ \left\{ \begin{array}{l} \text{Relations among coeffi-} \\ \text{cients of second syzygies} \end{array} \right\},$$

and so on.

(Constructing *minimal free resolution* of I.)

Hibert's Theorem on Syzygies: Process stops after at most r steps.

Definition. Given L defining

$$X \subseteq \mathbf{P}^r$$

one says that L satisfies Property (N_p) if:

- X cut out by quadrics ($p = 1$);

- First p modules of syzygies of X generated by relations with minimal possible weight $q = 1$.

"Green's Principle": On any smooth X, Property (N_p) holds linearly in positivity of embedding line bundle.

Fix reference Kähler form ω_0, and suppose that L_d is line bundle such that

$$c_1(L_d) = d \cdot \omega_0 + \eta,$$

$\eta =$ fixed $(1,1)$-form.

Theorem. (Many people...) There exist constants $A, B > 0$ (depending on X, ω_0, η) such that L_d satisfies (N_p) when

$$d \geq A \cdot p + B.$$

Example (Green). Consider X a curve with $g(X) = g$, and suppose $\deg(L_d) = d$. Then (N_p) holds when

$$d \ \geq \ 2g + 1 + p.$$

Philosophy: As positivity of embedding grows, the algebraic properties of

$$X \ \subseteq \ \mathbf{P}^r$$

become simpler.

Note: Assume as above $c_1(L_d) = d \cdot \omega_0 + \eta$, and say

$$\dim X \ = \ n.$$

• Number of syzygy modules that occur is approximately

$$r(L_d) \ = \ C \cdot d^n + \text{LOT}$$

• Number of syzygy modules governed by results just stated grows linearly in d.

So: When $n = \dim X \geq 2$, Green's principle ignores most of syzygies that occur!

Ottaviani-Paoletti: For

$$X \ = \ \mathbf{P}^n \quad , \quad L_d \ = \ \mathcal{O}_{\mathbf{P}^n}(d),$$

(N_p) fails when $p = 3d - 2$.

Question: When $n \geq 2$, what can one say about the asymptotic shape of syzygies of embedding

$$X \ \subseteq \ \mathbf{P}^{r_d}$$

defined by L_d as $d \to \infty$?

Will initially focus on which weights of syzygies appear.

Asymptotic Non-Vanishing Thms (with L. Ein)

As before, consider X with $\dim X = n$, and L_d on X with

$$c_1(L_d) \ = \ d \cdot \omega_0 + \eta.$$

L_d defines

$$X \ \subseteq \ \mathbf{P}^{r_d} \ \text{ with } \ r_d = O(d^n).$$

Are interested in p^{th} syzygies of X for

$$1 \ \leq \ p \ \leq \ r_d$$

when $d \gg 0$.

General Facts. For $d \gg 0$:

(I). All syzygies of X have weights

$$1 \ \leq \ q \ \leq \ n + 1.$$

(II). [Green et al] L_d has syzygies of maximal weight $q = n + 1$ if and only if

$$\Gamma(X, \Omega_X^n) \neq 0,$$

in which case such syzygies appear only for a few large values of p.

Rmk. Follows from (I) and (II), that for curves, essentially only syzygies that appear are those of weight $q = 1$ ($\Longrightarrow$ Green's theorem.)

Problem: Fix $q \in [1, n]$. For which

$$p \in [1, r_d]$$

does L_d give rise to a p^{th} syzygy of weight q when $d \gg 0$?

Theorem A Fix $q \in [1, n]$. There exist constants $C_1, C_2 > 0$ with the property that if

$$d \gg 0,$$

then L_d determines p^{th} syzygy of weight q for every p with

$$C_1 \cdot d^{q-1} \leq p \leq r_d - C_2 \cdot d^{n-1}.$$

Rmk. For fixed $q \in [1, n]$, consider the ratio

$$\frac{\#\left\{ p \in [1, r_d] \mid \exists\ p^{\text{th}} \text{ syz. of weight } q \right\}}{\#\left\{ p \in [1, r_d] \right\}}$$

Since $r_d = O(d^n)$, Theorem implies:

$$\text{Ratio} \longrightarrow 1 \quad \text{as} \quad d \longrightarrow \infty.$$

(I.e. asymptotically in d, "essentially all" the syzygy modules that could have generator in weight q actually do have such generators.)

Conjecture. Fix $1 \leq q \leq n$. Then

$$K_{p,q}(L_d) \ = \ 0$$

for $p \leq O(d^{q-1})$.

Veronese Varieties

Take $X = \mathbf{P}^n$ and $L_d = \mathcal{O}_{\mathbf{P}^n}(d)$. Use all monomials of degree d to define embedding

$$\mathbf{P}^n \hookrightarrow \mathbf{P}^{r_d} \quad , \quad r_d = \binom{n+d}{d} - 1.$$

Image is d^{th} *Veronese embedding* of $\mathbf{P}^n$.

Syzygies of Veronese varieties studied eg by Ottaviani-Paoletti, Rubei, Bruns-Conca-Römer.

Theorem B. Fix $q \in [1,n]$. If $d \gg 0$ then the Veronese variety carries p^{th} syzygies of weight q for

$$\binom{d+q}{q} - \binom{d-1}{q} - q \;\leq\; p$$

$$p \;\leq\; \binom{d+n}{n} - \binom{d+n-q}{n-q} + \binom{n}{n-q} - q - 1.$$

Ex. Take $q = 2, n = 2$. Then $\exists$ syzygies of weight $q = 2$ for

$$3d - 2 \;\leq\; p \;\leq\; r_d - 2.$$

(Thm of Ottaviani-Paoletti.)

Conjecture. Bound is optimal for all $q \in [1,n]$, $d \geq q+1$.

Vector space of p^{th} syzygies of weight q are a representation of $\mathrm{SL}(n+1)$.

Ask: How many different irreducible representations appear?

Fulger-Zhou: For fixed p, as $d \to \infty$:

$$\left(\begin{array}{l}\text{\# of irreps in space of } p^{\text{th}} \text{ syzy-} \\ \text{gies of weight } \ q = 1\end{array}\right) \ = \ O(d^p).$$

Intuition for Proof of Thm A.

Fix a hypersurface $\overline{X} \subseteq X$, and consider composition

$$\overline{X} \ \subseteq \ X \ \subseteq \ \mathbf{P}^{r_d}.$$

Then $\overline{X}$ embeds in a linear space of very large codimension, and by induction on dim, one can see that syzygies of $\overline{X}$ in $\mathbf{P}^{r_d}$ have many different weights. Expect that these contribute to syzygies of X in $\mathbf{P}^{r_d}$.

Betti Numbers. (with Ein, Erman)

Consider X, L_d as before. Define

$$k_{p,q}(L_d) \ = \ \text{dim}\left\{p^{\text{th}} \text{ syzygies of weight } q\right\}.$$

Question. Fix $1 \leq q \leq n$. Can one say anything about the asymptotics of these betti numbers as $d \to \infty$?

Curves: Take

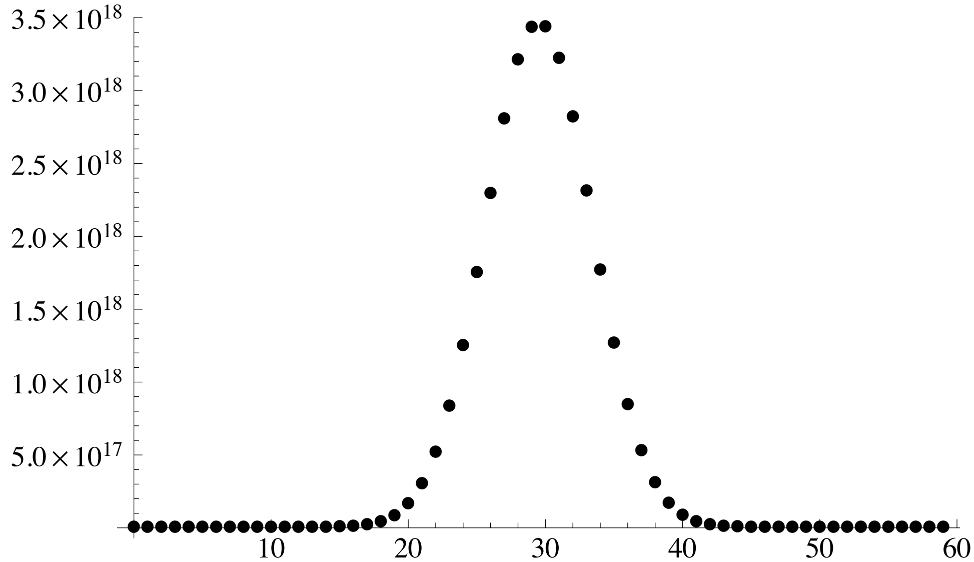
$$g(C) \ = \ g \quad , \quad \text{deg}(L_d) \ = \ d$$
$$r_d \ = \ d - g.$$

For large d, want to consider the behavior of the dimension

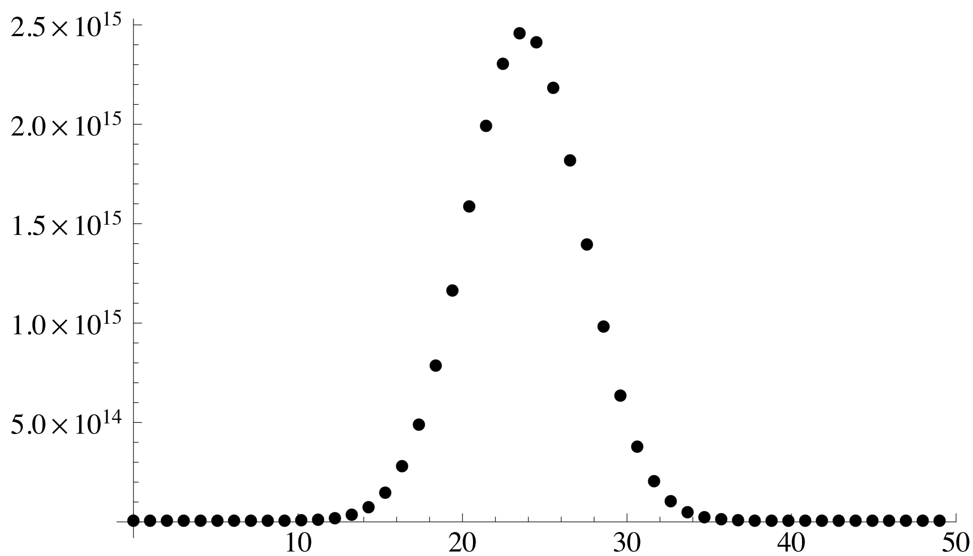
$$k_{p,1} \ =_{\text{def}} \ k_{p,1}(L_d)$$

as a function of p.

Ex. Plot of $k_{p,1}$ for $g = 0$, $d = 60$.



Ex. Plot of $k_{p,1}$ for $g = 10$, $d = 60$.



Prop. Fix C, L_d as above, and let $\{p_d\}$ be a sequence of integers such that

$$p_d \longrightarrow \frac{r_d}{2} + a \cdot \frac{\sqrt{r_d}}{2}$$

for some fixed number a (ie. $\lim \frac{2p_d - r_d}{\sqrt{r_d}} = a$). Then as $d \to \infty$,

$$\frac{1}{2^{r_d}} \cdot \sqrt{\frac{2\pi}{r_d}} \cdot k_{p_d,1} \longrightarrow e^{-a^2/2}.$$

What about general X, L_d?

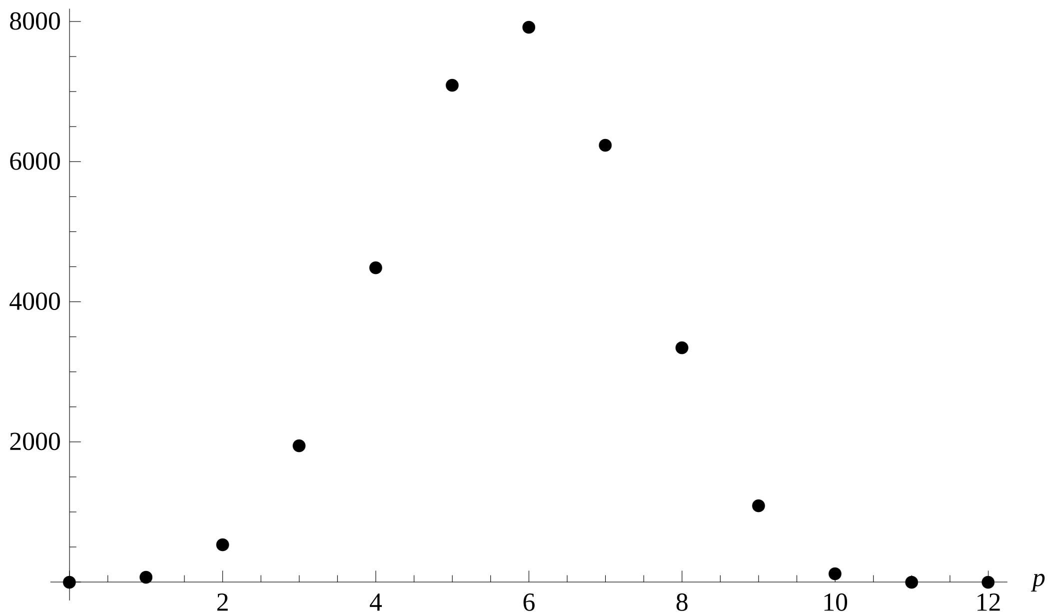
One can hope that similar picture holds for $k_{p,q}(L_d)$ for every $q \in [1, n]$.

Conjecture: For each $q \in [1, n]$ there is a function $F(d)$ (depending on X and geometric data) such that

$$F(d) \cdot k_{p_d,q}(L_d) \longrightarrow e^{-a^2/2}$$

as $d \to \infty$ and $p_d \to \frac{r_d}{2} + a \cdot \frac{\sqrt{r_d}}{2}$.

Example. Betti numbers $k_{p,1}$ of 4-fold Veronese embedding of $\mathbf{P}^2$.



(Biggest example we could work out exactly on computer.)

Confession: Don't know how to verify Conjecture for any X of dimension $n \geq 2$!

(Ex. What are asymptotics of betti numbers for Veronese embeddings of $\mathbf{P}^2$??)

Evidence for Conjecture comes from

Probabilistic Picture: For "random resolutions" having syzygies with fixed weights, betti numbers become normally distributed as length of resolution grows.

Ask: What does one mean by "random resolution?"

As model for syzygies of very positive embeddings of varieties of fixed dimension n, consider resolutions of modules M over polynomial rings in $r+1$ variables that have syzygies only in weights $1 \leq q \leq n$.

Eisenbud–Schreyer: Proved conjecture of Boij-Söderberg describing (up to scaling) all possible configurations of betti numbers $k_{p,q}(M)$ for M as above.

• Betti tables are (essentially) parametrized up to scaling by numerical parameters

$$x = \{x_I\} \in [0,1]^{\binom{r}{n-1}}$$

• So get functions

$$k_{p,q} : \Omega_r \ =_{\text{def}} \ [0,1]^{\binom{r}{n-1}} \longrightarrow \mathbf{R}$$

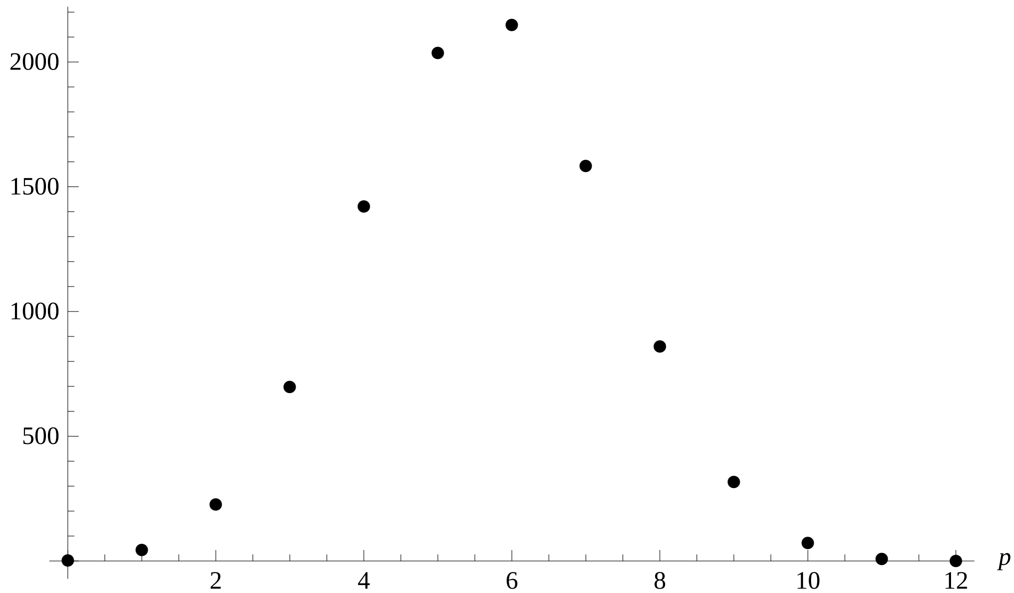
describing Betti numbers of formal resolution described by Boij-Söderberg coefficient vector $x \in \Omega_r$.

Plan: For fixed $1 \leq q \leq n$, choose

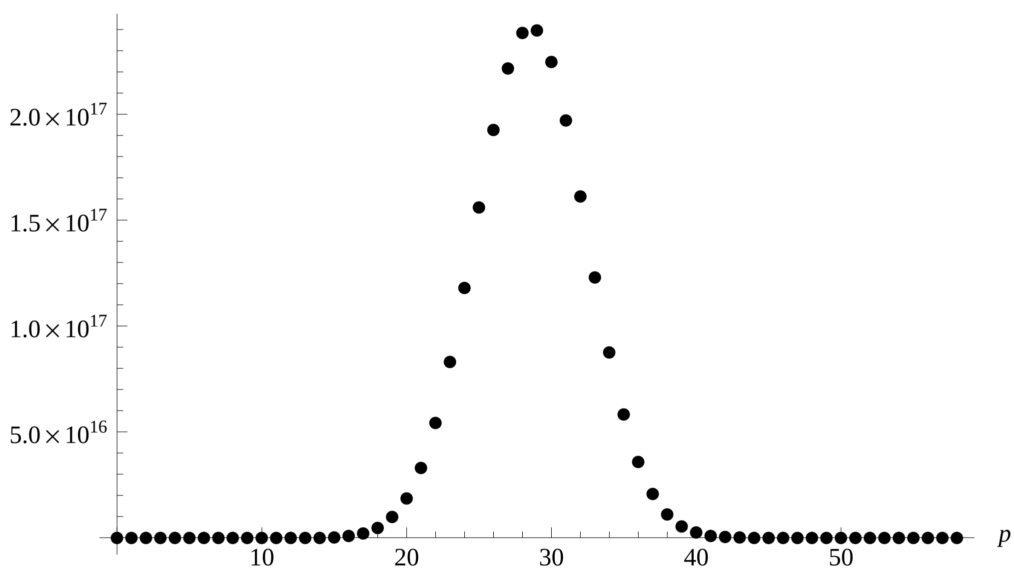
$$x \ \in \Omega_r \ \text{ uniformly at random,}$$

and study distribution in p of the formal betti numbers $k_{p,q}(x)$.

Example. Plot of $k_{p,1}$ for random x with $n = 2$ and $r = 14$.



Example. Plot of $k_{p,1}$ for random x with $n = 2$ and $r = 60$.



Theorem C. (Informal Statement). Fix $1 \leq q \leq n$. Then as $r \to \infty$, for "most" choices of

$$x \in \Omega_r$$

the formal betti numbers

$$k_{p,q}(x) \in \mathbf{R}$$

display the sort of normal distribution (in p) that is predicted by the conjecture.

So at least the Conjecture predicts that "real-life" Betti numbers have typical, rather than exceptional, behavior.