Shock Reflection and von Neumann conjectures

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Examples of shocks

Shock on aircraft:

Examples of shocks

Explosion:

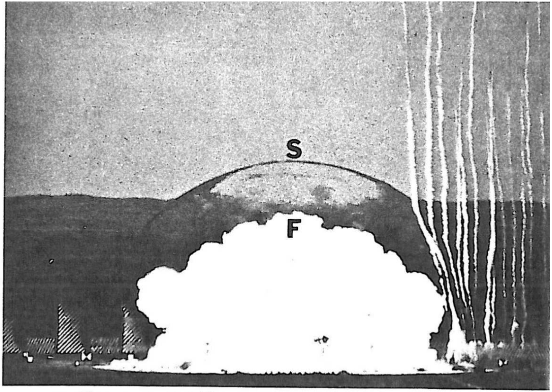
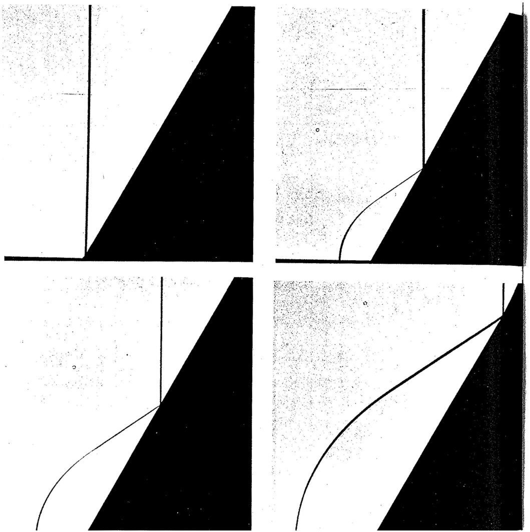
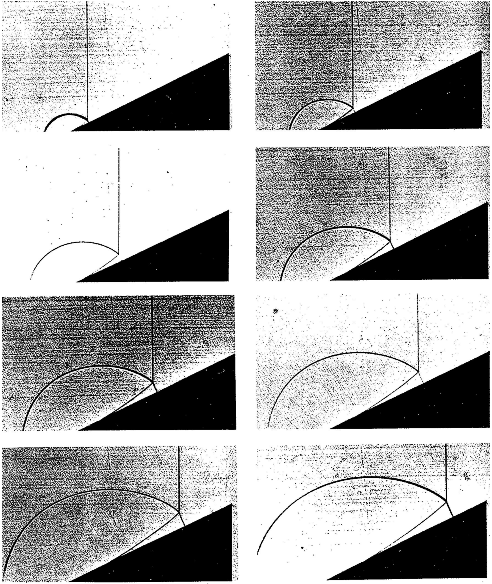


FIG. 22: EXPLOSION FROM A 20-TON HEMISPHERE OF TNT
The blast wave S, and fireball F, from a 20-ton TNT surface explosion are clearly shown. The backdrops are 50 feet by 30 feet and in conjunction with the rocket smoke trails, it is possible to distinguish shock waves and particle paths and to measure their velocities. Owing to unusual daylight conditions, the hemispherical shock wave became visible. (Courtesy: Defence Research Board of Canada).

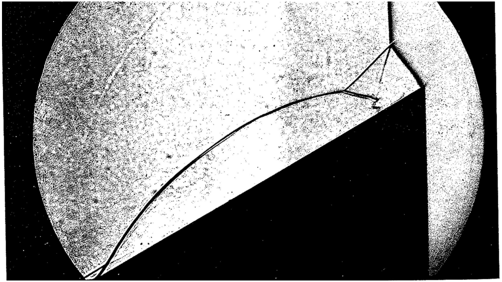
Shock reflection by a wedge: Regular reflection



Shock reflection by a wedge: Mach reflection



Shock reflection by a wedge: Irregular Mach reflection.



Self-similar flow: $(\vec{u}, p, \rho)(x, t) = (\vec{u}, p, \rho)(\frac{x}{t})$.

Systems of conservation laws

$$\partial_t u + \textbf{div} F(u) = 0 \quad \textbf{in} \quad \Omega \times \mathbb{R}^1,$$

$\Omega \subset \mathbb{R}^n$, $u : \Omega \times \mathbb{R}^1 \to \mathbb{R}^m$, $F : \mathbb{R}^m \to \mathbb{R}^{m \times n}$.
System: $m > 1$.

Systems in one dimensions, $n = 1$: Starting from 1950's, works by P. Lax, O.Oleinik, J. Glimm, and many other works. Long-time existence, uniqueness of weak solutions with small data (in BV), convergence of viscosity approximations, ...

Systems in multiple dimensions, $n > 1$: Very little is known about general time-dependent solutions. Thus study special solutions: Riemann problem.

Riemann Problem in Multi-D

System of conservation laws:

$$\partial_t u + \mathbf{div} F(u) = 0 \quad \textbf{in} \quad \Omega \times \mathbb{R}^1,$$

$\Omega \subset \mathbb{R}^n$, $u : \Omega \times \mathbb{R}^1 \to \mathbb{R}^m$, $F : \mathbb{R}^m \to \mathbb{R}^{m \times n}$, $m > 1$ **,** $n > 1$.

Riemann problem: piecewize-constant initial data

$\Omega = \Omega_1 \cup \Omega_2 \cdots \cup \Omega_s$ **,**

$$u_{|\Omega_k \times \{t=0\}} = u_k^0 - \textbf{constant vector,}\ k = 1, \ldots, s.$$

For appropriate Ω_k and boundary cond. (BC), expect self-similar solutions $u(x,t) = U(\xi)$, where $\xi = \frac{x}{t} \in \Omega$.

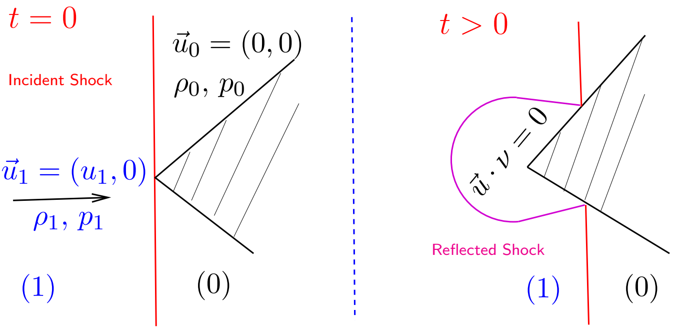
Self-similar system in Ω, for $U(\xi) = U(\frac{x}{t})$:

$$\mathbf{div} F(U) + (\xi \cdot \nabla) U = 0 \quad \textbf{in} \quad \Omega$$

plus BC, and conditions at infinity (from initial condition)

Solution may have some additional discontinuities – shocks, contact discontinuities, i.e. new subdomains: Free Boundary Problem

Shock reflection as a Riemann problem



Constant (uniform) states (0) and (1):
State (0): velocity $\vec{u}_0 = (0, 0)$, density ρ_0, pressure p_0.
State (1): velocity $\vec{u}_1 = (u_1, 0)$, density ρ_1, pressure p_1.

Self-similar solution: $(\vec{u}, \rho, p) = (\vec{u}, \rho, p)(\vec{\xi})$, where $\vec{\xi} = \dfrac{\vec{x}}{t}$.

Shock reflection

First described by E. Mach 1878. Reflection patterns: Regular reflection, Mach reflection.

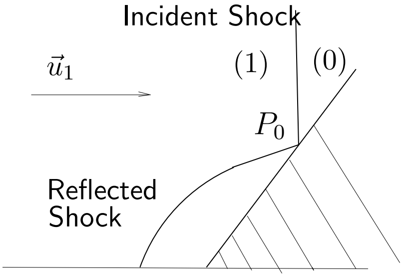
J. von Neumann, 1940s: on transition between patterns

Later works: experimental, computational. Asymptotic analysis: Lighthill, Keller, Blank, Hunter, Harabetian, Morawetz.
Reference: book by J. Glimm and A. Majda.

Analysis: Special models (Transonic small disturbance eq., pressure-gradient system): Gamba, Rosales, Tabak, Canic, Keyfitz, Kim, Lieberman, Zheng.
Local results: S.-X. Chen.

Recent works on global shock reflection solutions for potential flow: G.-Q.Chen-F., Elling-Liu, Elling, Bae-G.-Q.Chen-F.

Regular reflection in self-similar coordinates



Given:
State (0): velocity $\vec{u}_0 = (0, 0)$, density ρ_0, pressure p_0.
State (1): velocity $\vec{u}_1 = (u_1, 0)$, density ρ_1, pressure p_1.

Problem: Find self-similar solution: $(\vec{u}, \rho, p) = (\vec{u}, \rho, p)(\vec{\xi})$, where $\vec{\xi} = \dfrac{\vec{x}}{t}$, with asymptotic conditions at infinity determined by states (0) and (1), and satisfying $u \cdot \nu = 0$ on the boundary.

Compressible Euler system

Isentropic case:

$$\partial_t \rho + \textbf{div}(\rho\, \vec{u}) = 0,$$
$$\partial_t(\rho \vec{u}) + \textbf{div}(\rho\, \vec{u} \otimes \vec{u}) + \nabla p = 0$$

where:

$\vec{u} = (u_1, u_2)$ – velocity

ρ – density

$p = \rho^\gamma$ – pressure

$\gamma > 1$ – adiabatic exponent (it is a given constant)

Potential flow model: $\vec{u} = \nabla_x \Phi$.

$$\rho_t + \textbf{div}(\rho\, \nabla\Phi) = 0,$$
$$\Phi_t + \frac{1}{2}|\nabla\Phi|^2 + \frac{\rho^{\gamma-1} - 1}{\gamma - 1} = const$$

Potential flow: self-similar case

$\Phi(\vec{x}, t) = t\psi(\xi, \eta)$, $\rho(\vec{x}, t) = \rho(\xi, \eta)$ with $(\xi, \eta) = \frac{\vec{x}}{t} \in \mathbb{R}^2$.

Pseudo-potential: $\varphi = \psi - \frac{1}{2}(\xi^2 + \eta^2)$.

Equation for φ:

$$\operatorname{div}\left(\rho(|\nabla\varphi|^2, \varphi)\nabla\varphi\right) + 2\rho(|\nabla\varphi|^2, \varphi) = 0,$$

with $\quad \rho(|\nabla\varphi|^2, \varphi) = \left(K - (\gamma - 1)(\varphi + \frac{1}{2}|\nabla\varphi|^2)\right)^{\frac{1}{\gamma-1}}.$

Equation is of mixed type:

$$\begin{array}{rl} \textbf{elliptic} & |\nabla\varphi| < c(|\nabla\varphi|^2, \varphi, K), \\ \textbf{hyperbolic} & |\nabla\varphi| > c(|\nabla\varphi|^2, \varphi, K), \end{array}$$

where **sonic speed** c is:

$$c^2 = \rho^{\gamma-1} = K - (\gamma - 1)(\varphi + \frac{1}{2}|\nabla\varphi|^2).$$

Shocks, RH conditions, Entropy condition

Shocks are discontinuities in the pseudo-velocity $\nabla\varphi$:

if Ω^+ and $\Omega^- := \Omega \setminus \overline{\Omega^+}$ are nonempty and open, and $S := \partial\Omega^+ \cap \Omega$ is a C^1 curve where $\nabla\varphi$ has a jump, then $\varphi \in C^1(\Omega^\pm \cup S) \cap C^2(\Omega^\pm)$ is a global weak solution in Ω if and only if φ satisfies potential flow equation in $\Omega^\pm$ and the Rankine-Hugoniot (RH) condition on S:

$$
\begin{aligned}
&[\varphi]_S = 0, \\
&\left[\rho(|\nabla\varphi|^2, \varphi)\nabla\varphi \cdot \nu\right]_S = 0,
\end{aligned}
$$

where $[\cdot]_S$ is jump across S.

Entropy Condition on S: density increases across S in the flow direction.

Uniform states

Solutions with constant (physical) velocity (u, v):

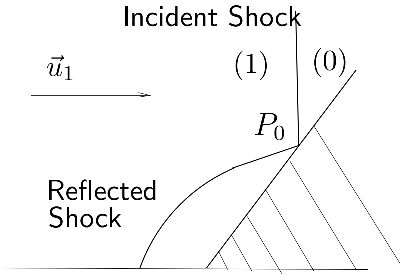
$$\varphi(\xi, \eta) = -\frac{\xi^2 + \eta^2}{2} + u\xi + v\eta + const.$$

Any such function is a solution.
Also (from formula) density $\rho(\nabla\varphi, \varphi) = const$, thus sonic speed $c = \rho^{\frac{\gamma-1}{2}} = const$. Then ellipticity region

$$|\nabla\varphi(\xi, \eta)| = |(u, v) - (\xi, \eta)| < c$$

is circle, centered at (u, v), radius c.

Regular reflection, state (2)



φ = pseudo-potential between the reflected shock and the wall
φ_1 = pseudo-potential of state (1)

Denote $\nabla\varphi(P_0) = (u_2, v_2)$. Since $\varphi_\nu = 0$ on wedge, then $v_2 = u_2 \tan\theta_w$.

Rankine-Hugoniot conditions at reflection point P_0, for φ and φ_1: algebraic equations for u_2, $\varphi(P_0)$

Regular reflection, state (2), detachment angle

If solution exists: Let

$$\varphi_2(\xi, \eta) = -(\xi^2 + \eta^2)/2 + u_2\xi + v_2\eta + C,$$

where C determined by $\varphi_2(P_0) = \varphi_1(P_0)$.

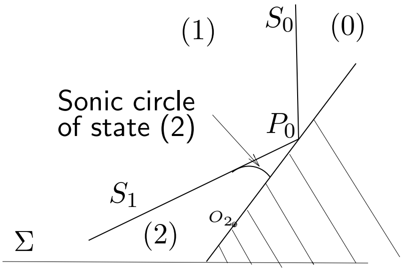
Existence of state (2) is necessary condition for existence of regular reflection

Given γ, ρ_0, ρ_1, there exists $\theta_{detach} \in (0, \frac{\pi}{2})$ such that:

state (2) exists for $\theta_w \in (\theta_{detach}, \frac{\pi}{2})$,
state (2) does not exist for $\theta_w \in (0, \theta_{detach})$.

If φ_2 exist, then RH is satisfied along the line
$S_1 := \{\varphi_1 = \varphi_2\}$.

Sonic angle



There exist $\theta_{sonic} \in (\theta_{detach}, \frac{\pi}{2})$ such that:
State 2 is supersonic at P_0 for $\theta_w \in (\theta_{sonic}, \frac{\pi}{2})$.
State 2 is subsonic at P_0 for $\theta_w \in (\theta_{detach}, \theta_{sonic})$.

Von Neumann's conjectures on transition between different reflection patterns

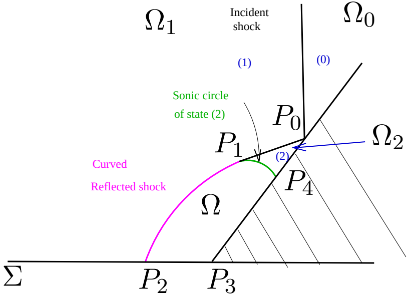
Recall: sonic angle θ_{sonic} and detachment angle θ_d satisfy $0 < \theta_d < \theta_{sonic} < \frac{\pi}{2}$.

Sonic conjecture:
Regular reflection for $\theta_w \in (\theta_{sonic}, \frac{\pi}{2})$

Von Neumann's detachment conjecture:
Regular reflection for $\theta_w \in (\theta_d, \frac{\pi}{2})$, Mach reflection for $\theta_w \in (0, \theta_d)$.

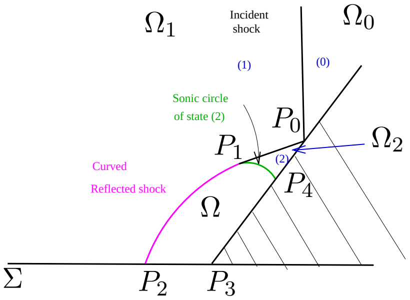
We discuss existence of regular reflection for $\theta_w \in (\theta_{sonic}, \frac{\pi}{2})$. Also a recent work indicates existence in case $\theta_w < \theta_{sonic}$ close to θ_{sonic}.
We use self-similar potential flow equation.

Existence for θ_w near $\pi/2$



Theorem 1. (G.-Q. Chen-F., PNAS 2005; Ann. of Math. 2010). There exist $\theta_c = \theta_c(\rho_0, \rho_1, \gamma) \in (0, \pi/2)$ and $\alpha = \alpha(\rho_0, \rho_1, \gamma) \in (0, 1/2)$ such that, when $\theta_w \in [\theta_c, \pi/2)$, there exists a global self-similar solution $\varphi \in C^{0,1}(\Lambda)$, satisfying:

Existence for θ_w near $\pi/2$



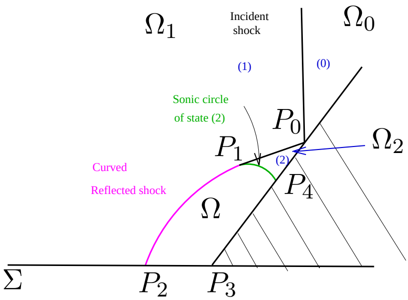
1) φ is a weak solution of Eq. in $\Lambda = \mathbb{R}^2_+ \setminus$ Obstacle,

2) $\partial_\nu \varphi = 0$ on ∂Wedge and Σ;

3) $\varphi = \varphi_i$ (state (i)) in Ω_i for $i = 0, 1, 2$;

4) Equation is elliptic in Ω, ellipticity degenerates near sonic arc $P_1 P_4$.

Existence for θ_w near $\pi/2$



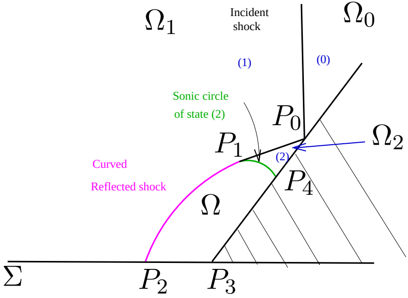
Moreover,

a) φ is $C^{1,1}$ near and across the sonic arc P_1P_4;

b) Reflected shock is $C^{2,\beta}$ curve for all $\beta \in (0, \frac{1}{2})$, a graph for a cone of directions between S_0 and S_1;

c) $\varphi_2 \leq \varphi \leq \varphi_1$ in Ω

Stability of normal reflection as $\theta_w \to \pi/2$



Furthermore,
the solutions φ converge in $W^{1,1}_{loc}$ to the solution of the normal refection as $\theta_w \to \pi/2$.

Further related work

V. Elling, T.P. Liu
- Ellipticity principle for potential flow (JHDU 2005)
- Supersonic flow onto wedge (Prandtl reflection): existence of weak solutions, (CPAM, 2008)

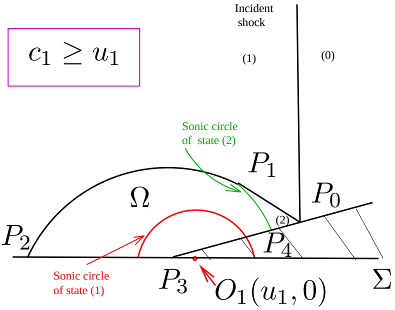
M. Bae, G.-Q. Chen, F. Regularity near sonic arc, (Invent. Math. 2009)

D. Serre. Shock interactions/reflection for Chaplygin gas, (ARMA 2009)

V. Elling Regular reflection: existence of weak solutions under condition of existence of a barrier for the shock (Commun. Math. Anal., 2010)

G.-Q. Chen, F. Regular reflection: existence of solutions with regularity as in Th. 1 for $\theta_w \in (\theta_{sonic}, \frac{\pi}{2})$, under condition $u_1 \leq c_1$ on parameters of states (1) and (2). In the case $u_1 > c_1$: possibility of "attached shock" (Preprint 2011)

Existence for $\theta_w \in (\theta_{sonic}, \frac{\pi}{2})$

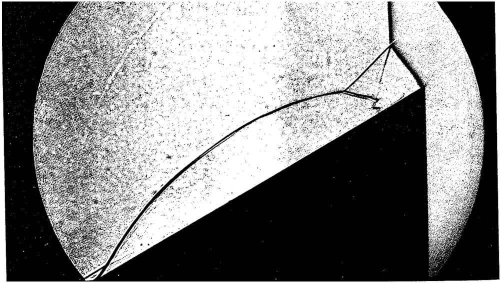


Theorem 2. (G.-Q. Chen-F., Preprint 2011). If $\rho_1 > \rho_0 > 0$, $\gamma > 1$ satisfy $u_1 \leq c_1$, then a regular reflection solution φ described in Th. 1 exists for all wedge angles $\theta_w \in (\theta_{sonic}, \frac{\pi}{2})$.

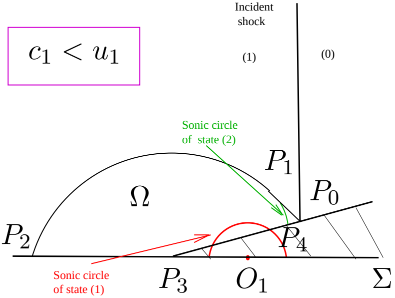
Solution satisfies all properties stated in Th. 1. In particular, φ is $C^{1,1}$ near and across the sonic arc $P_1 P_4$, and shock is $C^{2,\beta}$ curve for all $\beta \in (0, \frac{1}{2})$, and $\varphi_2 \leq \varphi \leq \varphi_1$ in Ω.

Attached reflected shock

For irregular Mach reflection attached case appears to be possible, see Fig. 238 (page 144) of
M. Van Dyke,*An Album of Fluid Motion*, The Parabolic Press: Stanford, 1982.



Existence for $\theta_w \in (\theta_{sonic}, \frac{\pi}{2})$

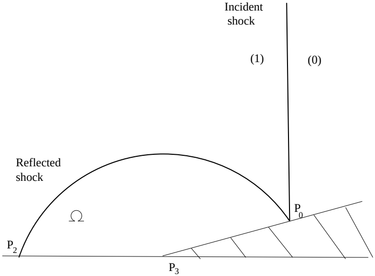


Theorem 3. (G.-Q. Chen-F., Preprint 2011). If $\rho_1 > \rho_0 > 0$, $\gamma > 1$ satisfy $u_1 > c_1$, then a regular reflection solution φ described in Th. 1 exists for all wedge angles $\theta_w \in (\theta_c, \frac{\pi}{2})$, where

-either $\theta_c = \theta_{sonic}$,

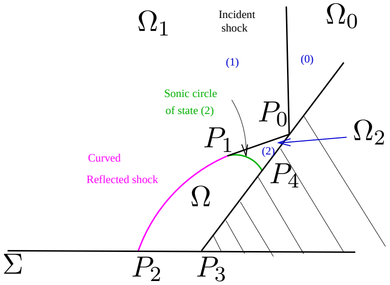
-or $\theta_c > \theta_{sonic}$ and for $\theta_w = \theta_c$ there exists an attached weak solution of regular reflection problem.

Subsonic regular reflection for $\theta_w < \theta_{sonic}$ near θ_{sonic}



Theorem 2'-3'. (G.-Q. Chen-F., 2012.) In the conditions of Theorem 2, or Theorem 3 with $\theta_c < \theta_{sonic}$, there exists $\theta_w^* \in [\theta_{detach}, \theta_{sonic})$ such that for all $\theta_w \in [\theta_w^*, \theta_{sonic}]$ there exists a subsonic regular reflection solution, i.e. with $P_0 = P_1 = P_4$.

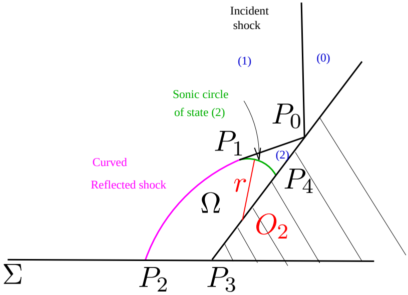
Regularity across sonic arc



Theorem 4. (Bae-Chen-F., Invent. Math. 2009) Let φ be a global regular reflection solution, supersonic at P_0, and $\varphi_2 \leq \varphi$ on curved reflected shock (and thus in Ω).
Then φ is *not* C^2 across the sonic arc P_1P_4.

Solutions constructed in Th. 1–3 satisfy condition of Th.4.
Thus $C^{1,1}$ regularity across P_1P_4 in Th. 1–3 is optimal.

Regularity in Ω near sonic arc



Theorem 5. (Bae-Chen-F., Invent. Math. 2009) Assume the solution φ satisfies:

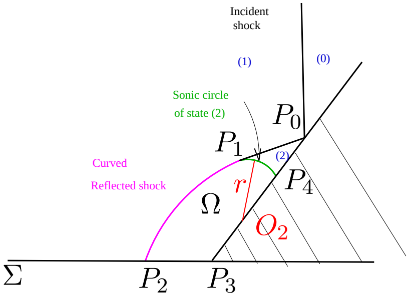
a) $\varphi_2 \leq \varphi$ on curved reflected shock (and thus in Ω)
b) $\varphi \in C^{1,1}$ near sonic arc P_1P_4.

Then:

1) For every P in sonic arc $(P_1P_4]$ (i.e. excluding P_1)

$\varphi \in C^{2,\alpha}(\overline{\Omega} \cap B_R(P))$, for some small $R > 0$, any $\alpha \in (0,1)$.

Regularity in Ω near sonic arc



2) $D^2\varphi$ has a jump across sonic arc P_1P_4:

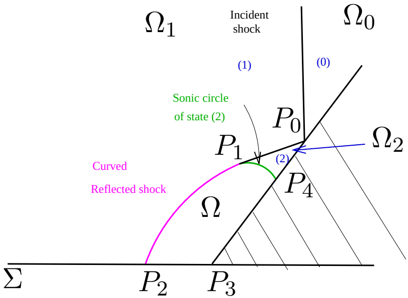
$$D_{rr}\varphi_{|\Omega} - D_{rr}\varphi_2 = \frac{1}{\gamma+1} \quad \text{on arc}(P_1P_4]$$

Thus φ is $C^{1,1}$ but *not* C^2 across sonic arc,

3) $D^2\varphi$ in Ω does *not* have a limit at P_1.

Remark: Solutions constructed in Th. 1–3 satisfy condition of Th.5.

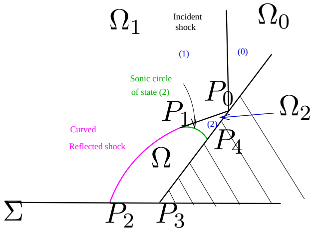
Approach: Free boundary problem



Unknowns: elliptic region Ω, its boundary part P_1P_2, and φ in Ω.

Free boundary problem for elliptic (?) equation:

Approach: Free boundary problem



$$\text{div}\left(\rho(|\nabla\varphi|^2,\varphi)\nabla\varphi\right) + 2\rho(|\nabla\varphi|^2,\varphi) = 0 \;\text{ in } \Omega,$$

$$\left.\begin{array}{l} \rho(|\nabla\varphi|^2,\varphi)\nabla\varphi\cdot\nu = \rho(|\nabla\varphi_1|^2,\varphi_1)\nabla\varphi_1\cdot\nu \\ \varphi = \varphi_1 \end{array}\right\} \text{ on } P_1P_2$$

$\varphi = \varphi_2$ on P_1P_4 (and prove $D_\nu\varphi = D_\nu\varphi_2$ on P_1P_4)

$\varphi_\nu = 0$ on Wedge P_3P_4, Symmetry line P_2P_3,

Solving FBP

Admissible solutions: satisfy ellipticity in Ω, nonstrict monotonicity $\partial_e(\varphi_1 - \varphi) \leq 0$ in Ω for any $e \in \text{cone}(e_\eta, e_{S_1})$.

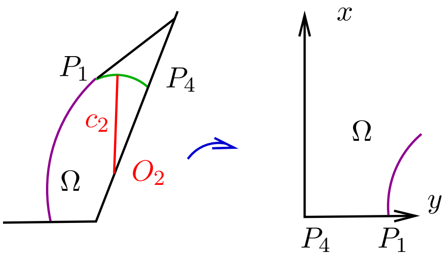
- Prove strict monotonicity of $\varphi_1 - \varphi$ for each $e \in \text{cone}(e_\eta, e_{S_1})$. $\Longrightarrow$ Γ_{shock} is a graph, Lip$[\Gamma_{shock}] \leq C$.
- Derive uniform basic estimates for admissible solutions: $\|\varphi\|_{C^{0,1}}(\Omega) \leq C$, diam$(\Omega) \leq C$, $0 < \rho_{min} \leq \rho(\nabla\varphi, \varphi) \leq \rho_{\max}$.
- Prove geometric properties of the free boundary Γ_{shock}: Uniform estimates on separation of shock with wedge and the symmetry line, uniform lower bound dist$(\Gamma_{shock}, B_{c_1}(O_1)) \geq \frac{1}{C}$.
- Prove "ellipticity"$(\xi, \eta) \geq \frac{1}{C}$dist$((\xi, \eta), \Gamma_{sonic})$.
- Derive estimates for φ in weighted/scaled $C^{2,\alpha}$ in $\overline{\Omega}$.
- Use method of continuity/degree theory to prove existence of admissible solutions for each wedge angle up to the sonic angle (if $u_1 \leq c_1$....)

Estimates near sonic arc

Flatten sonic arc: introduce coordinates

$$x = c_2 - r, \qquad y = \theta - \theta_w,$$

where (r, θ) are polar coordinates centered at $O_2 = (u_2, v_2)$.



Then $\Omega \cap \mathcal{N}(\Gamma_{sonic}) \subset \{x > 0\}$ and $\Gamma_{sonic} \subset \{x = 0\}$, where Γ_{sonic} is arc $P_1 P_4$.

Estimates near sonic arc

Let $\psi = \varphi - \varphi_2$. Self-similar potential flow equation rewritten for ψ in (x, y)-coordinates is:

$$[2x - (\gamma+1)\psi_x]\psi_{xx} + \frac{1}{c_2}\psi_{yy} - \psi_x = 0 \quad \text{in} \;\; \Omega \subset \{x > 0\}.$$

plus "small" terms. Full equation is homogeneous.

Also,

$$\psi > 0 \quad \text{in } \Omega, \qquad \leftarrow \text{recall: } \varphi \geq \varphi_2$$

$$\psi = 0 \quad \text{on } \Gamma_{sonic} = \partial\Omega \cap \{x = 0\}.$$

Equation is elliptic in $x > 0$ if

$$\psi_x < \frac{2}{\gamma+1}x.$$

Estimates near sonic arc

Thus expect ψ, ψ_x small: consider linearization near $\psi = 0$:

$$x\psi_{xx} + \frac{1}{2c_2}\psi_{yy} - \frac{1}{2}\psi_x = 0 \quad \text{in } \{x > 0\}.$$

Related works: P. Daskalopoulos - R. Hamilton, and F.-H. Lin - L. Wang

For ODE (in $x > 0$)

$$xu_{xx} - \alpha u_x = 0, \quad u(0) = 0,$$

solutions are $u = Cx^{1+\alpha}$.

Thus, in order to have $|u_x| \leq Cx$, need $\alpha \geq 1$.

But in our case $\alpha = \frac{1}{2}$, thus linearization near $\psi = 0$ does not work.

Estimates near sonic arc

Use nonlinear term: let

$$w_\varepsilon = \frac{1+\varepsilon}{2(\gamma+1)}x^2.$$

Then w_0 is a solution of the nonlinear (main terms) equation

$$[2x-(\gamma+1)\psi_x]\psi_{xx} + \frac{1}{c_2}\psi_{yy} - \psi_x = 0$$

and w_ε for $\varepsilon > 0$ is a strict supersolution of the full equation.
Note: we do *not* have a lower barrier, but we know $\psi \geq 0$.

Using this (and boundary conditions....), get

$$0 \leq \psi \leq Cx^2.$$

Estimates near sonic arc

Further estimates: use parabolic scaling.

From monotonicities (and other estimates) of $\psi = \varphi - \varphi_2$ near sonic arc, $0 \le \psi_x \le Cx$, $|\psi_y| \le Cx$. Allows to control coefficients of equation.

Then for simplicity, consider

$$x u_{xx} + u_{yy} - \alpha u_x = 0 \quad \text{in} \quad \{x > 0\}$$

and assume that

$$|u| \le Cx^2.$$

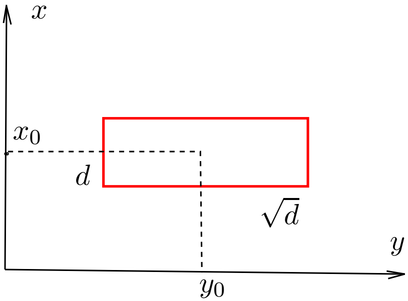
Note: this estimate has been proved in our "real" case.

Estimates near sonic arc

Let $x_0 = 2d > 0$, consider rectangle

$$Q_d(x_0, y_0) = \{(x, y) \mid |x - x_0| < d,\ |y - y_0| \le \sqrt{d}\}.$$

Note $Q_d(x_0, y_0) \subset \{x > 0\}$.



Estimates near sonic arc

Map $Q_d(x_0, y_0)$ to the unit square $Q_1(0,0)$ by

$$X = \frac{x - x_0}{d}, \quad Y = \frac{y - y_0}{\sqrt{d}}.$$

Define function $v(X,Y)$ on $Q_1(0,0)$ by

$$\frac{1}{d^2}u(x,y) = v(X,Y) \equiv v\left(\frac{x - x_0}{d}, \frac{y - y_0}{\sqrt{d}}\right).$$

Then $|u| \le Cx^2$ implies $|v| \le 9C$ in $Q_1(0,0)$.

Equation for u translates into:

$$(2 + X)v_{XX} + v_{YY} - \alpha v_X = 0,$$

which is uniformly elliptic in $Q_1(0,0)$.

Estimates near sonic arc

Thus

$$\|v\|_{C^{2,\alpha}(Q_{1/2})} \leq C\|v\|_{L^\infty(Q_1)} \leq \hat{C}.$$

Writing this in terms of $u(x,y)$ at $(x,y) = (x_0, y_0)$, get

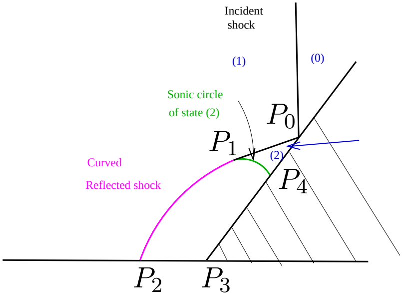
$$|u_x| \leq Cx, \quad |u_y| \leq Cx^{3/2},$$
$$|u_{xx}| \leq C, \quad |u_{xy}| \leq Cx^{1/2}, \quad |u_{yy}| \leq Cx.$$

Thus we get $C^{1,1}$ estimates up to $\{x = 0\}$.

In fact, we obtained more precise estimates. We call them estimates in parabolic norms (defined as supremum over (x_0, y_0) of $C^{2,\alpha}(Q_{1/2})$ norms of the rescaled functions $v(X,Y)$).

Estimates near sonic arc P_1P_4

Note: essentially, we used linearization near $w_0 = \frac{1}{2(\gamma+1)}x^2$ and showed that it controls nonlinear equation up to $C^{1,1}$. In the further regularity work (Theorem 5), we show that this control extends to $C^{2,\alpha}$ near P_1P_4 away from shock (i.e. away from point P_1), but cannot be better than $C^{1,1}$ near P_1.

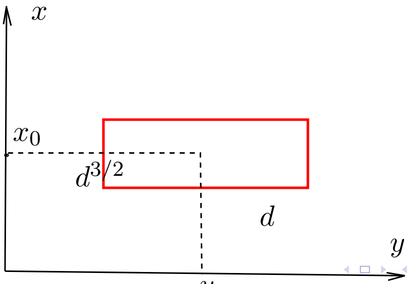


Subsonic reflection: $\theta_w < \theta_{sonic}$ near θ_{sonic}: Estimates near P_0

First: Uniform estimates for $\theta_w \geq \theta_{sonic}$ up to θ_{sonic}.
Let $x_0 = 2d > 0$. Can only have rectangle size Cd in y-direction (and note: $Cd << \sqrt{d}$ if d is small).
Then consider rectangle

$$Q_d(x_0, y_0) = \{(x, y) \mid |x - x_0| < d^{3/2},\ |y - y_0| \leq Cd\}.$$

Note $Q_d(x_0, y_0)$ fits into Ω if (x_0, y_0) is a sufficiently interior point.



Subsonic reflection: Estimates near P_0

Map $Q_d(x_0, y_0)$ to the unit square $Q_1(0,0)$ by

$$X = \frac{x - x_0}{d^{3/2}}, \quad Y = \frac{y - y_0}{d}.$$

Define function $v(X,Y)$ on $Q_1(0,0)$ by

$$\frac{1}{d^2}u(x,y) = v(X,Y) \equiv v\left(\frac{x - x_0}{d^{3/2}}, \frac{y - y_0}{d}\right).$$

Then $|u| \leq Cx^2$ implies $|v| \leq 9C$ in $Q_1(0,0)$.

Equation for u translates into:

$$(2 + Xd^{1/2})v_{XX} + v_{YY} - d^{1/2}\alpha v_X = 0,$$

which is uniformly elliptic in $Q_1(0,0)$. Get weighted/scaled $C^{1,\alpha}$ for u.

Negative solutions near sonic circle

Arise in the proof "shock does not hit sonic circle of state (1)". Let $\psi = \varphi - \varphi_1$. Then $\psi < 0$. In the (x, y)-coordinates related to sonic circle of state (1), equation as before:

$$[2x - (\gamma + 1)\psi_x]\psi_{xx} + \frac{1}{c_1}\psi_{yy} - \psi_x = 0 \quad \text{in} \quad \Omega \subset \{x > 0\}$$

plus "small" terms.

Define $u = -\psi$. Then $u > 0$ in $\{x > 0\}$, and satisfies:

$$[2x + (\gamma + 1)u_x]u_{xx} + \frac{1}{c_1}u_{yy} - u_x = 0 \quad \text{in} \quad \Omega \subset \{x > 0\}.$$

Negative solutions near sonic circle

$$[2x+(\gamma+1)u_x]u_{xx}+\frac{1}{c_1}u_{yy}-u_x=0 \quad \text{in} \;\; \Omega\subset\{x>0\},$$
$$u=0 \quad \text{on} \;\; \partial\Omega\cap\{x=0\}$$

Equation is uniformly elliptic on linear functions $u = kx$, where $k > 0$. Moreover, it has a positive subsolution U in $\{x > 0\}$ of linear growth and $U = 0$ on $\{x = 0\}$. (Compare with supersolution of quadratic growth in the previous case).

This implies that shock cannot hit sonic circle of state (1):

$$(\varphi_1-\varphi)\geq U \quad \text{in} \;\; \Omega\cap\{x>0\}, \quad \text{equality on} \;\; \partial\Omega\cap\{x=0\}$$

thus there would be a gradient jump at the touching point – contradiction to RH-conditions when one of sides is sonic.

Open problems

- Finalize the proof of existence of subsonic regular reflection for all wedge angles $\theta_w \in (\theta_{sonic}, \theta_{detach})$, i.e. prove von Neumann detachment conjecture;
- Uniqueness of regular reflection solution. Depends on the geometric properties of the shock: convexity would be sufficient.
- Prove all these results for Euler system. Difficulty: vorticity blowup near stagnation points, noticed by D. Serre for isentropic Euler system. It is possible that the full Euler system does not have this singularity;
- Mach reflection...