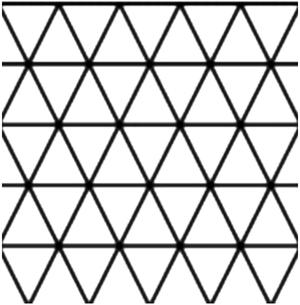
Percolation, Conformal welding, and SLE

Ilia Binder

University of Toronto

February 23, 2012

Site percolation on triangular lattice



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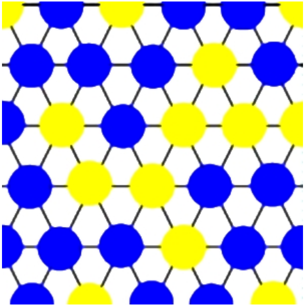

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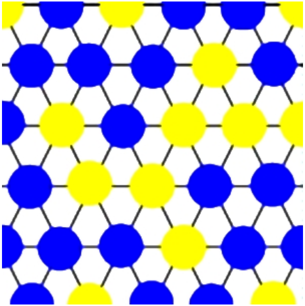

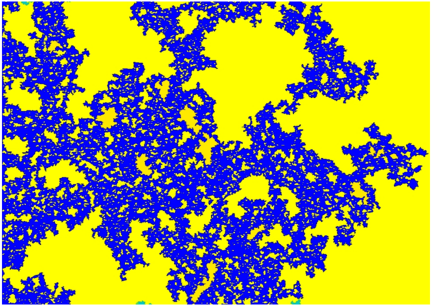

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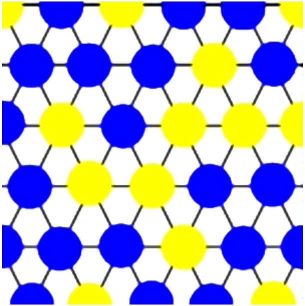
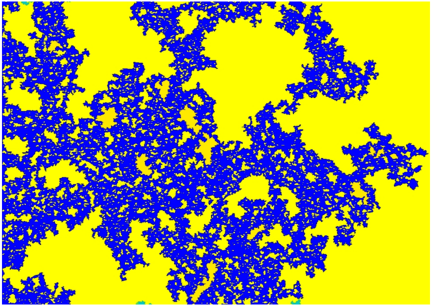

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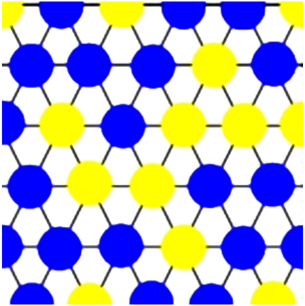

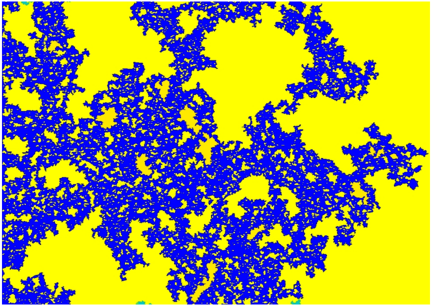

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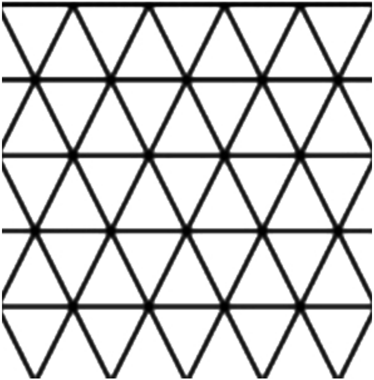
$p_c = 1/2$ - critical.

Bond percolation on triangular lattice



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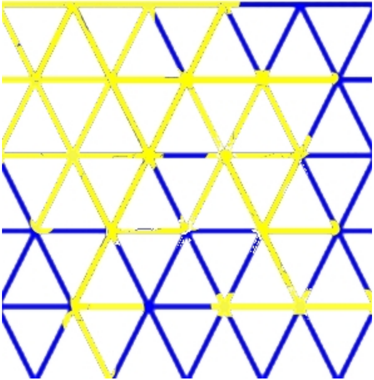

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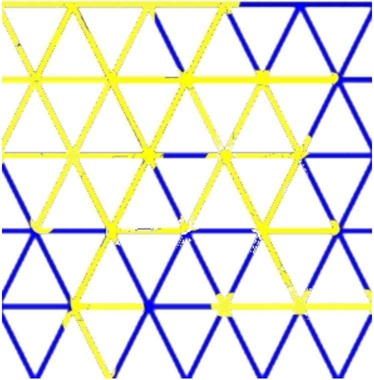
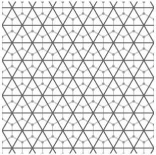


Here $p_c = 2\sin\pi/18 \approx 0.35$(because there is more connectivity!)

To hexagonal lattice

Hexagonal lattice is dual to triangular lattice:



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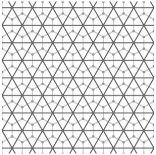
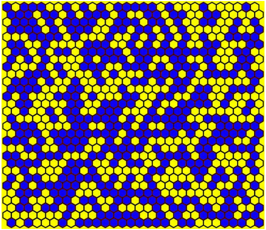
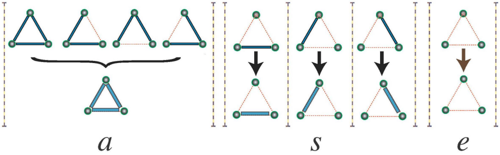


Site percolation: just color hexagons blue/yellow with probability 1/2.



To hexagonal lattice: bond percolation

Interested only in connectivity properties, so can group triangles with the same connectivity:



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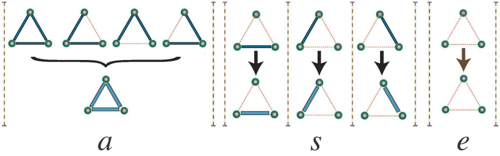

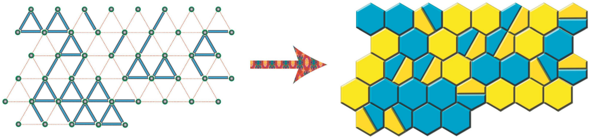

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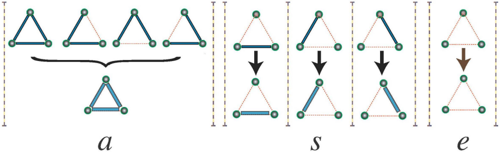
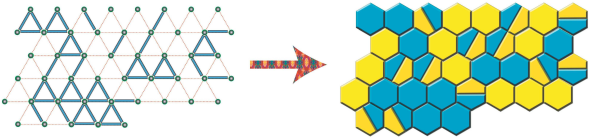
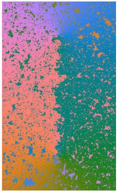




Not quite symmetric: 

Universality

It has been predicted by physicists that at criticality various lattice models, such as *Percolation* and



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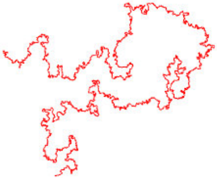

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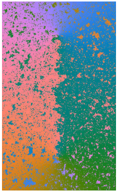

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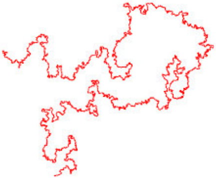
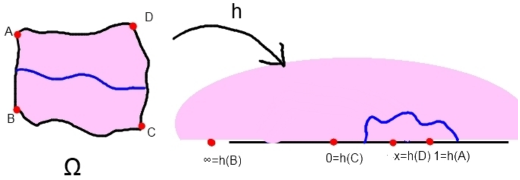


Self Avoiding Walk

have scaling limits which are conformally invariant and independent of the lattice selected.

Smirnov-Cardy observable: history

Cardy's observation (1992):



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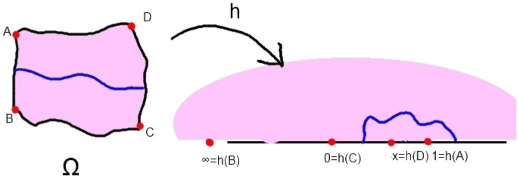

The formula is cumbersome: $F(\Omega; A, B, C, D) := \frac{\int_0^x (s(1-s))^{-2/3}\, ds}{\int_0^1 (s(1-s))^{-2/3}\, ds}$

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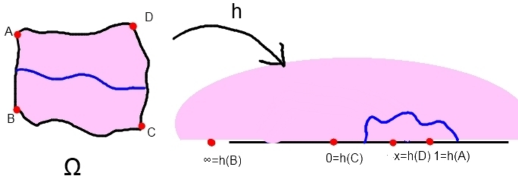

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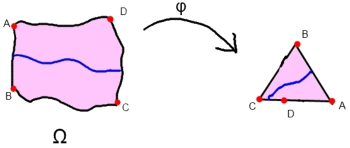

Carleson observation:
$F(\Omega; A, B, C, D) := \frac{|CD|}{|AC|}$

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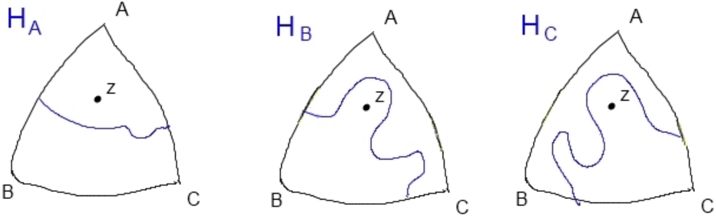

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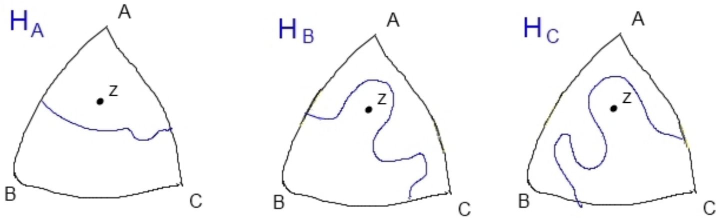

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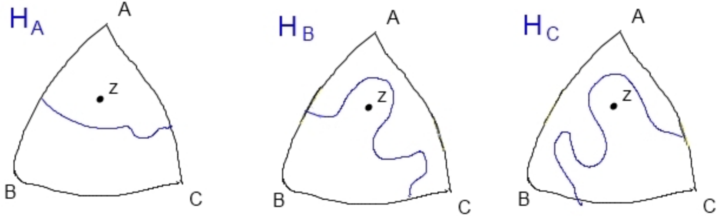
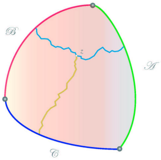
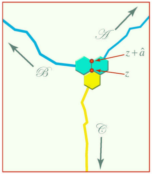


$H_A(z) + e^{2\pi i} H_B(z) + e^{-2\pi i} H_C(z)$ converge to ϕ. Coincides with Cardy's formula when $z \in \partial\Omega$!

Idea of proof





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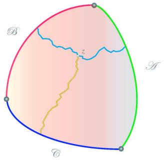

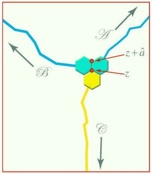

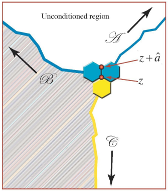

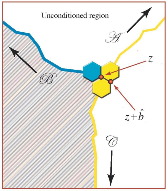

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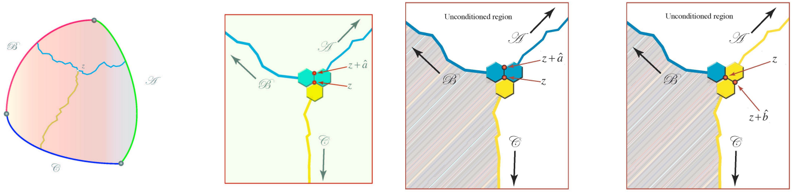

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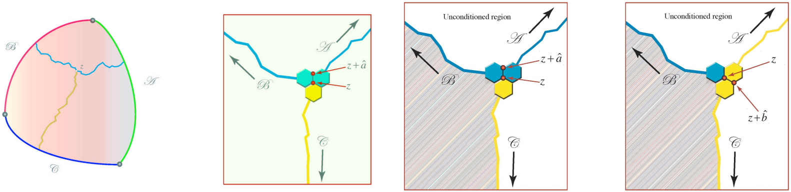
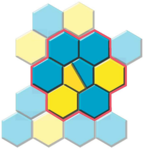


Does not work for bond percolation:

Modified bond percolation

Introduce flowers and irises. 

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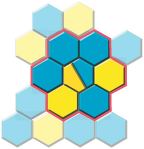

Introduce flowers and irises.

Rules

1. Flowers are disjoint
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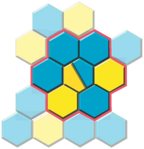

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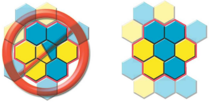

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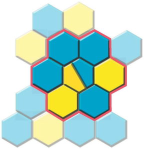

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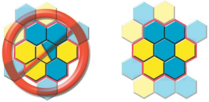

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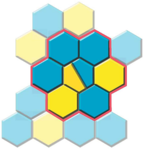

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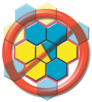
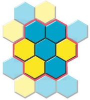
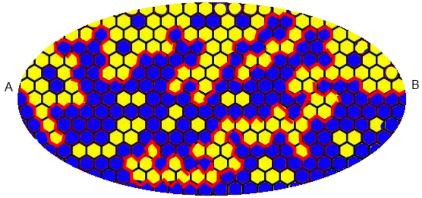




The last rule introduces local correlations. Cardy-Smirnov observable still works here!

Exploration process and metric on curves

In a simply connected domain Ω with two points A, B on the boundary, color all hexagons on $[AB]$ blue, on $[BA]$ yellow. Then there is unique interface between yellow and blue, a random curve from A to B in Ω. It is called the *exploration process*.



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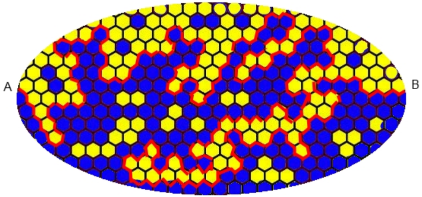

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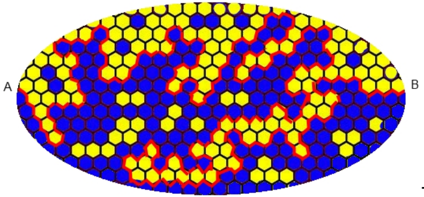

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For the convergence in law we need metric, which is

$$\mathrm{dist}_U(\gamma_1, \gamma_2) = \inf_{\text{parametrizations of } \gamma_1,\ \gamma_2} \|\gamma_1(t) - \gamma_2(t)\|_\infty$$

Löwner Evolution

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Parametrize the curve and consider conformal mapping g_t from $\mathbb{H} \setminus \gamma[0, t]$ back to $\mathbb{H}$ with hydrodynamic normalization at ∞:

$$g_t(z) = z + \frac{2a(t)}{z} + O(\frac{1}{|z|^2}).$$

Let us re-parametrize the curve so that $a(t) = t$. Let $\lambda(t) := g_t(\gamma(t)) \in C(\mathbb{R}_+)$.

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Schramm-Löwner Evolution and Schramm Theorem

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1. Conformal invariance.
2. *Domain Markov Property:* The law of $\gamma[t+T,\infty)$ is the same as the law of the exploration process in $\Omega \setminus \gamma[0,T]$.

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Strategy of the proof of convergence of exploration process to SLE_6

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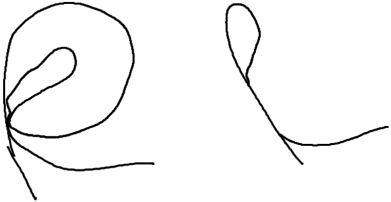

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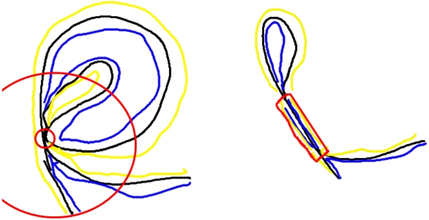
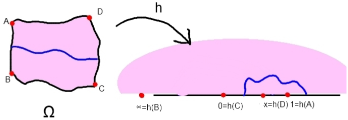


For percolation, both cases can be ruled out by five-arm exponent argument or Cardy's formula.

Smirnov-Cardy observable leads to convergence to SLE_6



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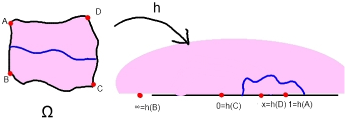

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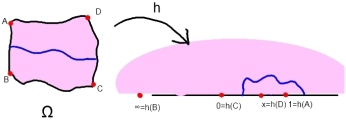

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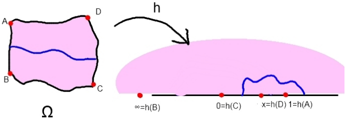

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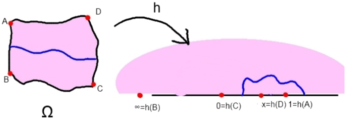

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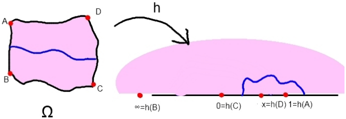

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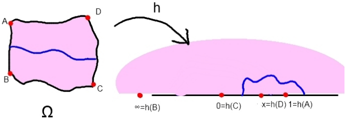

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Both restrictions in our theorem are essential.

Other models

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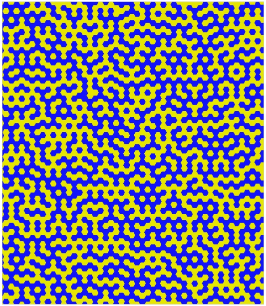
Sheffield-Schramm: Level lines of Discrete Gaussian Free Field on any regular lattice converge to SLE_4.

Lattice models: Cluster boundaries



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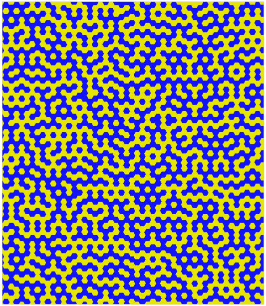

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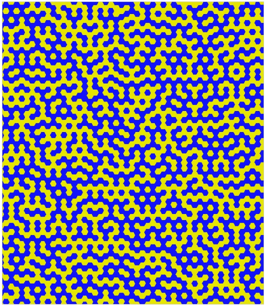
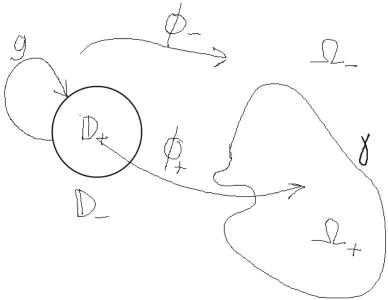


Another natural object related to lattice models: scaling limit of a random cluster boundary. A conformally invariant random loop. Should look like SLE locally.

Can be done through SLE, but there is another natural tool for description of random loops – *Conformal Welding*.

Conformal welding



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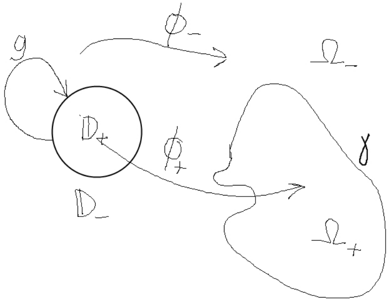

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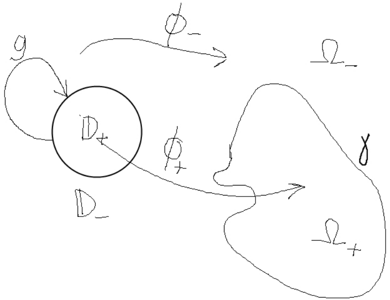

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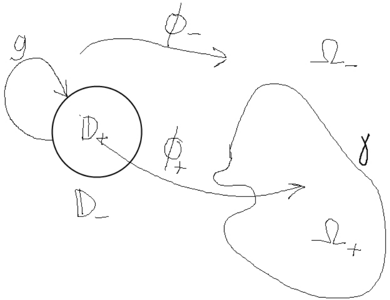

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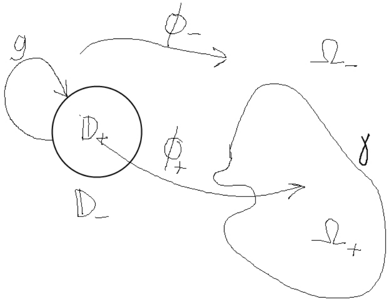

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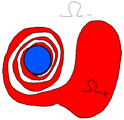
Not always. Example: 

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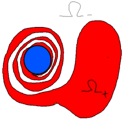
Not always. Example: 

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Theorem. (Ahlfors-Beurling)

Let g be a quasi-symmetric orientation-preserving homeomorphism of the circle, i.e. $\exists K : \forall z, w$,

$$\frac{|g(zw) - g(z)|}{|g(z) - g(zw^{-1})|} \leq K.$$

Then g is the conformal welding for a quasicircle γ (an image of the circle under a quasi-symmetric map).

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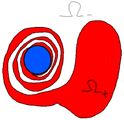

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Way too restrictive for SLE. It can be proven that for them one needs Lehto welding.

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Good enough for SLE curves: they are known to be Hölder.

Trace of GFF on unit circle

Let

$$X(e^{2\pi it}) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left(A_n \cos(2\pi nt) + B_n \sin(2\pi nt)\right),$$

where A_n, B_n are independent Standard Gaussians.

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$\frac{1}{2}$-derivative of Brownian motion.
Invariant under Möbius transformations of the circle (modulo constants).

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g_β is thus a.s. define a unique (up to a Möbius transformation) loop, which we denote by γ_β.

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Multifractal spectrum

μ – regular probability measure on a metric space.

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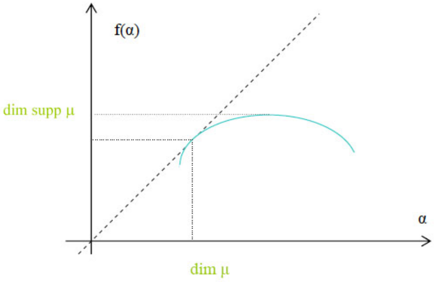
Dimension spectrum (a version of rigorous):

$$f(\alpha) = \lim_{\eta \to 0} \dim \left\{ x \mid \exists \delta_j \downarrow 0 : \delta_j^{\alpha+\eta} \le \mu(B(x,\delta_j)) \le \delta_j^{\alpha-\eta} \right\}$$

Dimension spectrum: some properties



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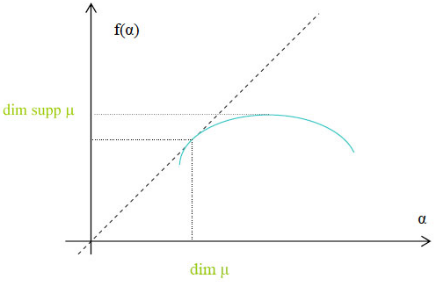

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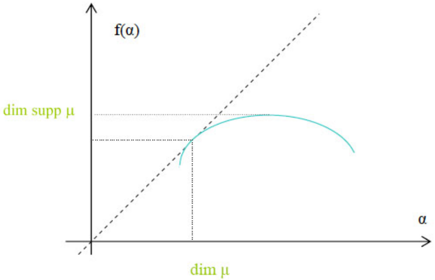

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- $\sup_\alpha f(\alpha) = \dim \operatorname{supp} \mu$

Harmonic measure and its spectrum

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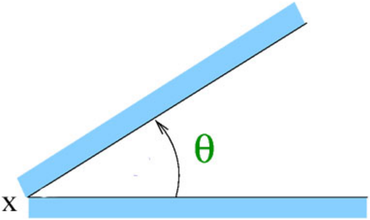

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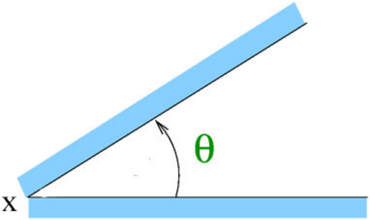

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$\pi/\dim_{loc} \omega(x)$ – generalized angle.

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In terms of g: ("$g(\omega_+) = \omega_-$")

$$F(\lambda) = \mathrm{Hdim} \left\{ t \in S^1 \mid \ \lim_{h \to 0} \frac{\log |g(e^{2\pi i(x+h)}) - g(e^{2\pi i x})|}{\log |h|} = \lambda \right\}$$

Two sided spectrum: AJKS welding

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$p(t) \neq q(t)$, so the AJKS welding is asymmetric.

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Physics prediction:

$$f(\alpha,\alpha') = \frac{25-c}{12} - \frac{1}{2(1-\gamma)\left(1-\frac{1}{2}\left(\frac{1}{\alpha}+\frac{1}{\alpha'}\right)\right)^{-1}} - \frac{1-c}{24}(\alpha+\alpha').$$

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