On the problem of local connectivity of the Mandelbrot set

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Stony Brook University
1 March 2018

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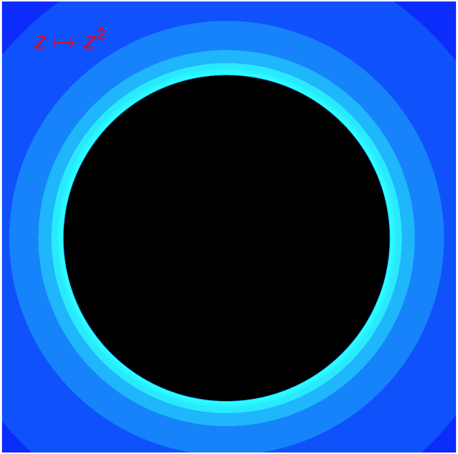
$\mathrm{orb}(z) = (z, f_c(z), f_c \circ f_c(z), f_c \circ f_c \circ f_c(z), \ldots)$

The Julia set $J_c = \partial\{z \mid \mathrm{orb}(z) \text{ is bounded}\}$



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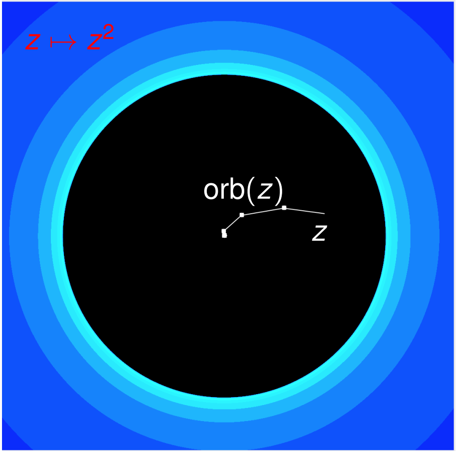

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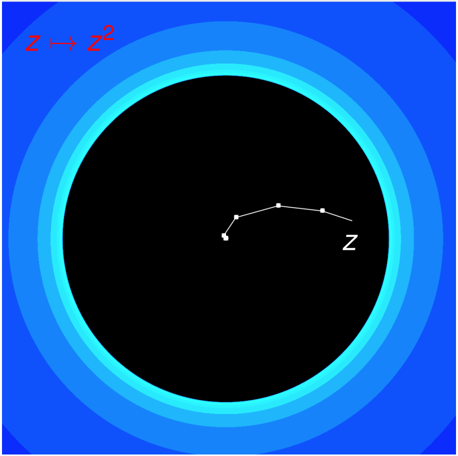

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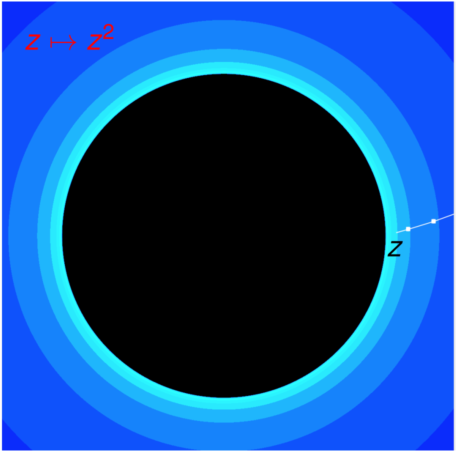

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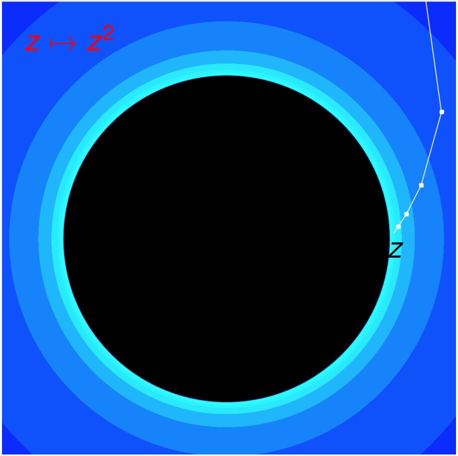

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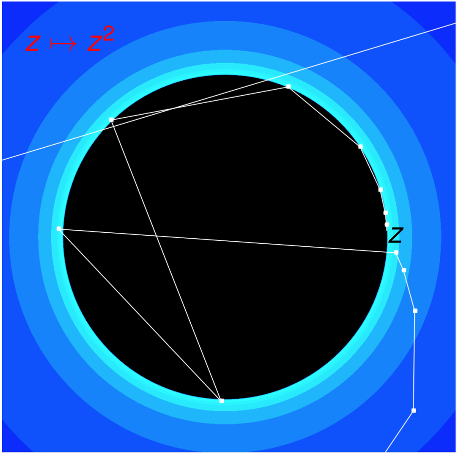

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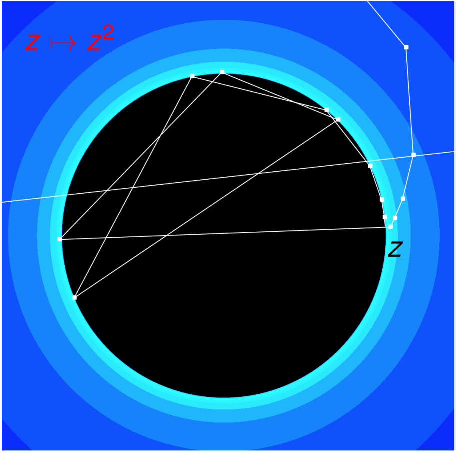

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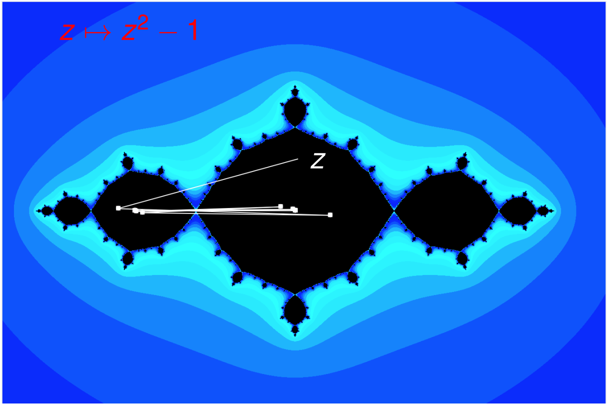

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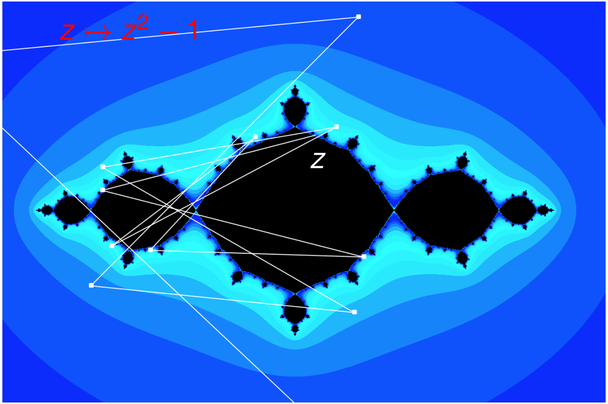

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The Julia set $J_c = \partial\{z \mid \text{orb}(z) \text{ is bounded}\}$ is either

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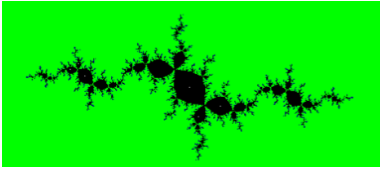
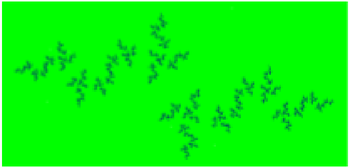
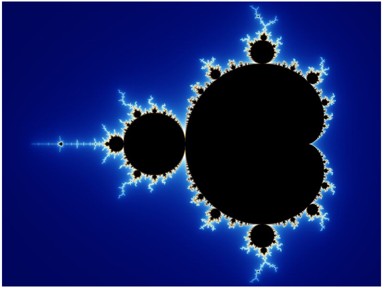


a Cantor set



The Mandelbrot set $\mathcal{M} = \{c \mid J_c \text{ is connected}\}$



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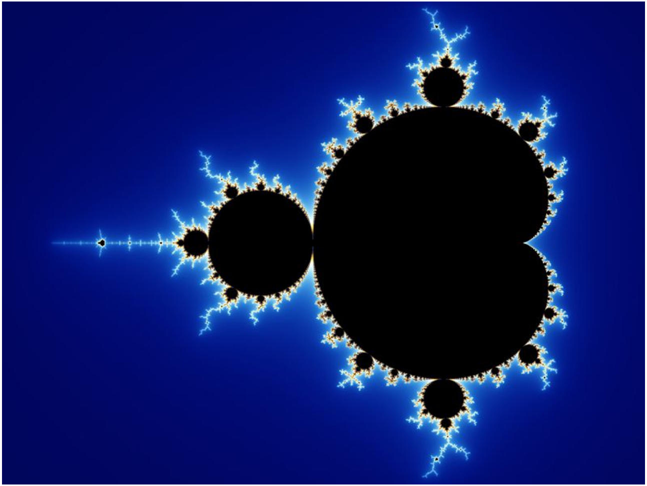

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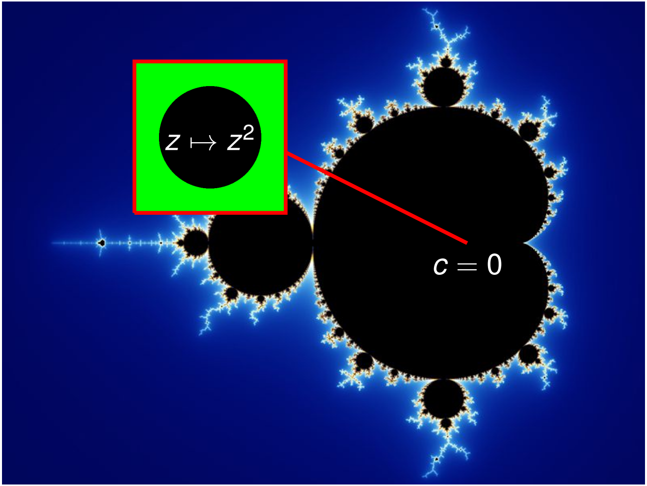

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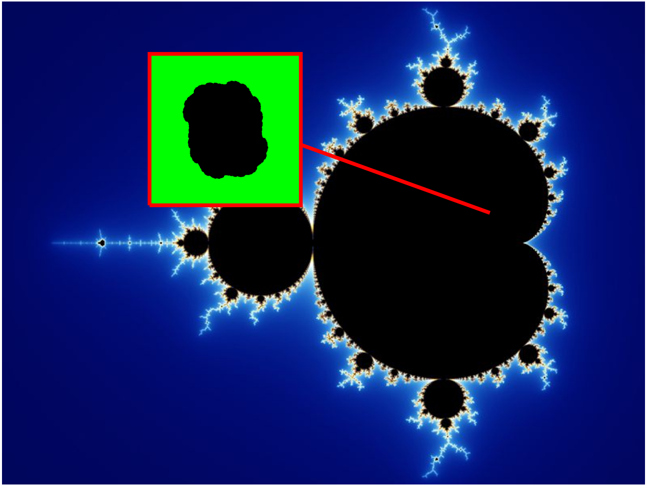

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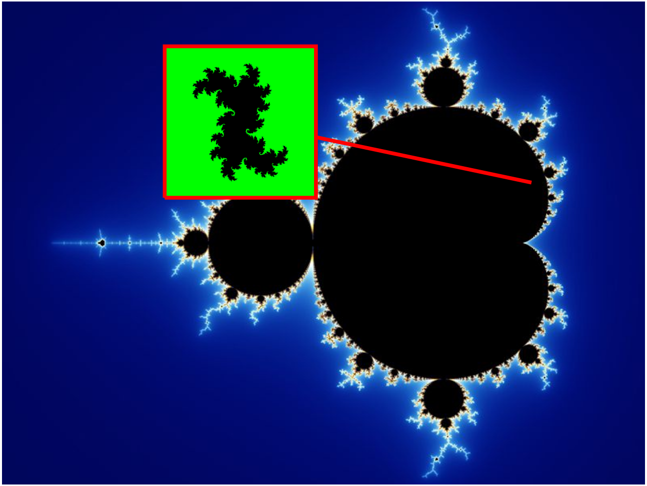

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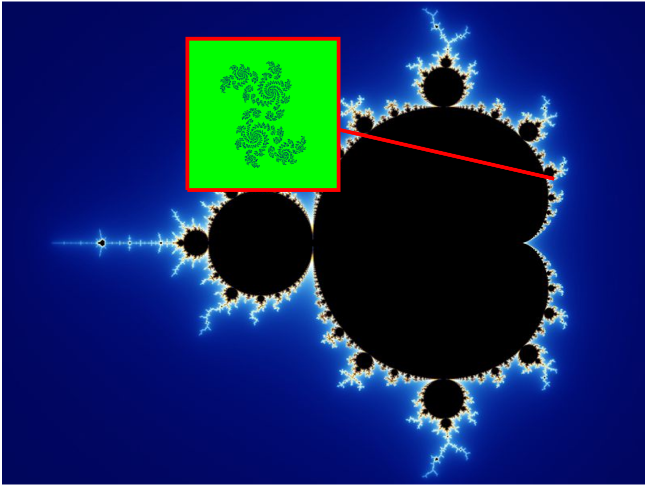

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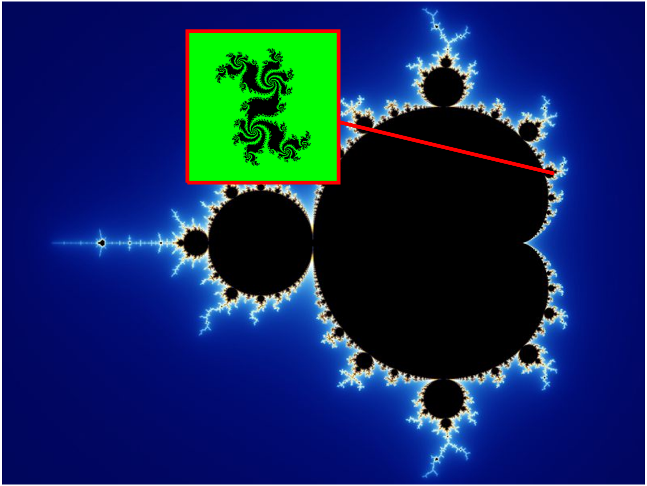

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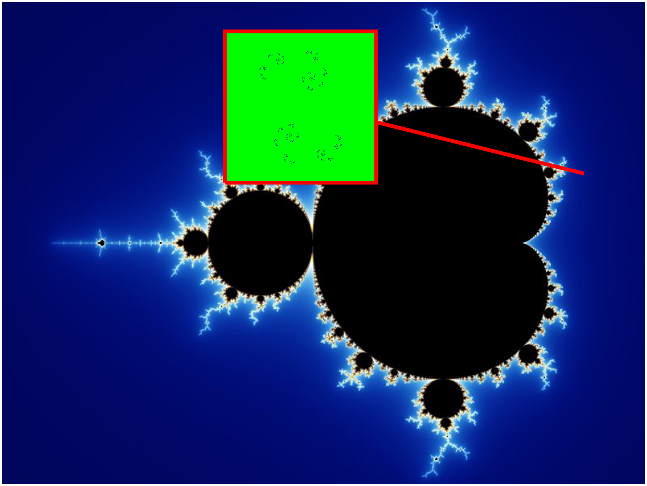

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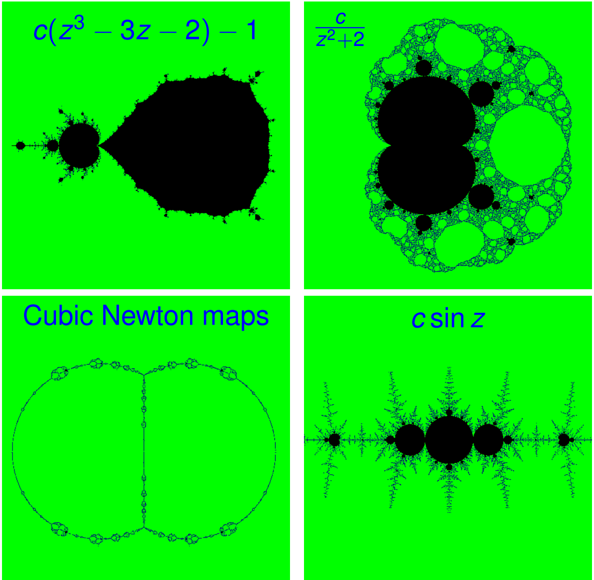
dim $=1$ parameter spaces



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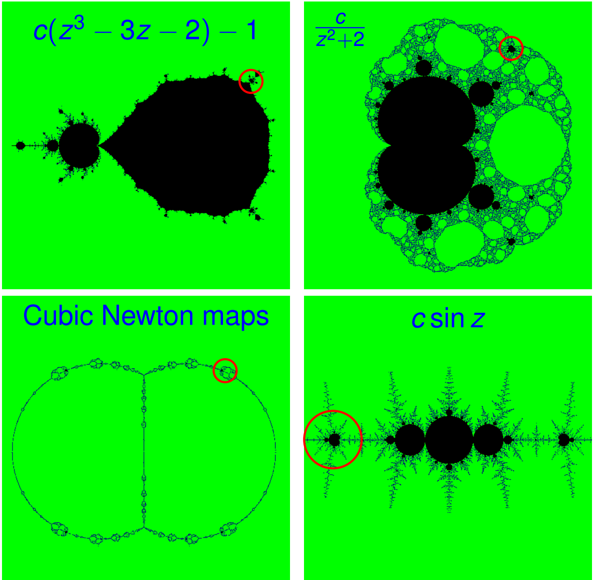

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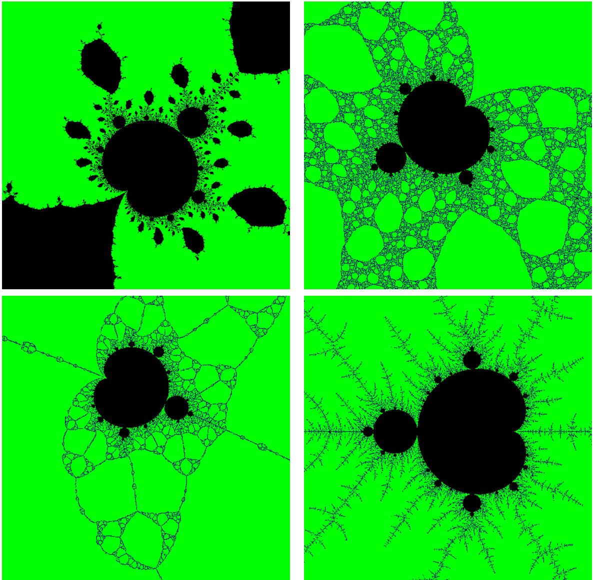
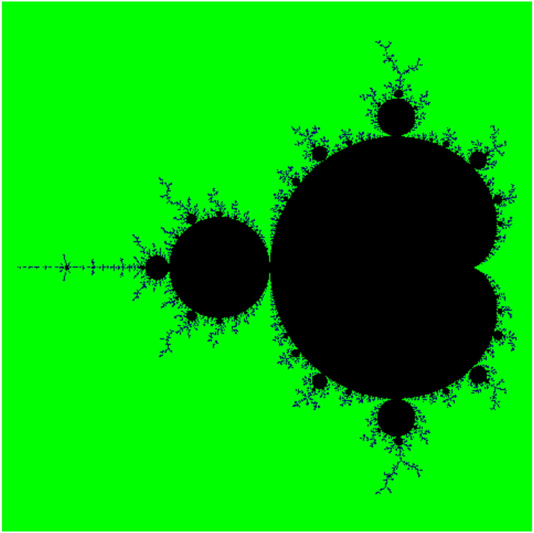


Douady, Hubbard: $\mathcal{M}$ has ∞-many copies of itself
every copy is canonically homeomorphic to $\mathcal{M}$



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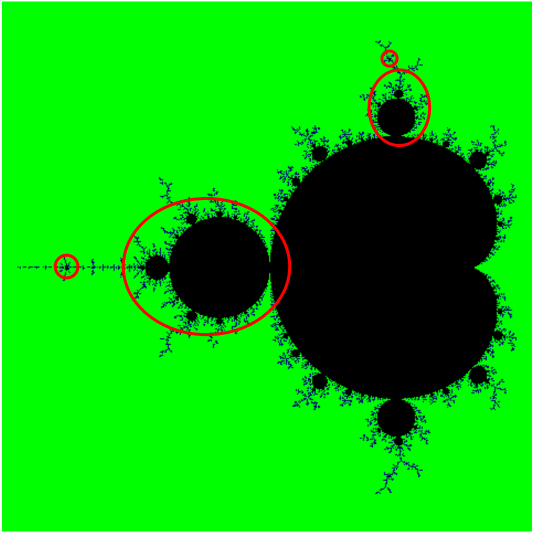

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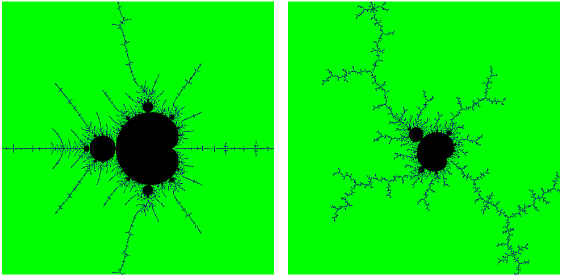
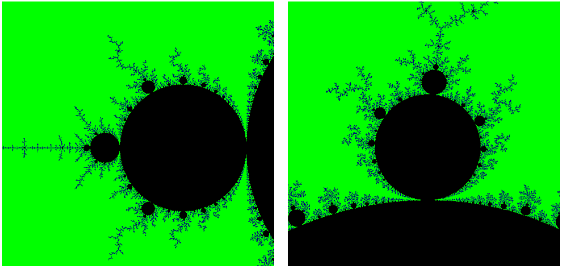


primitive copies

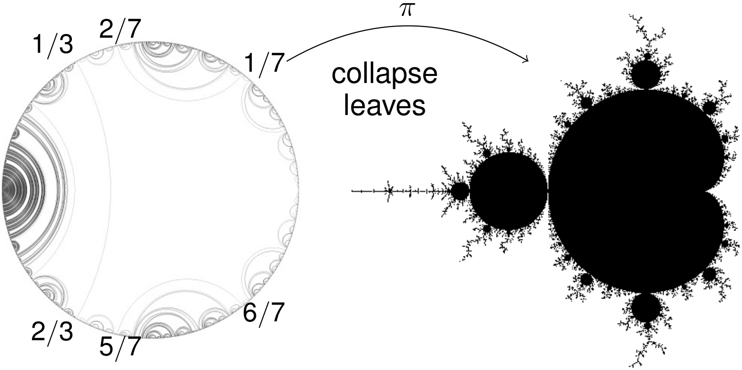


satellite copies

The MLC-conjecture: the Mandelbrot set is locally connected

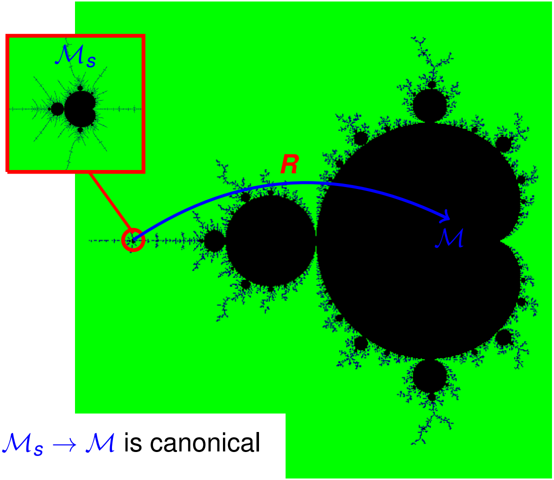
MLC iff $\exists \pi : \overline{\mathbb{D}^1} \to \mathcal{M}$ continuous

pinched disk model:



Yoccoz: MLC holds at “non-∞ renormalizable” parameters

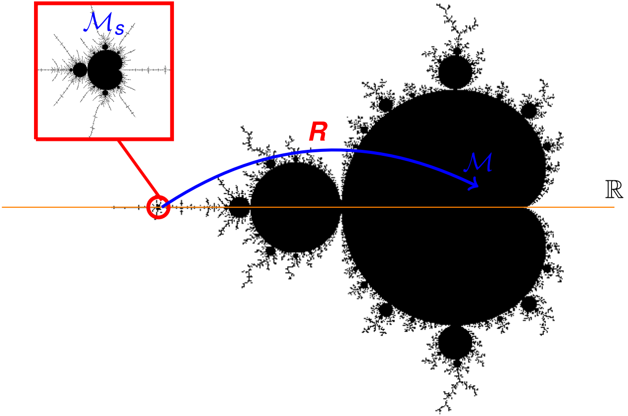
Cor: MLC iff canonical homeomorphisms are “expanding”; f.e.



$\boldsymbol{R}: \mathcal{M}_s \to \mathcal{M}$ is canonical

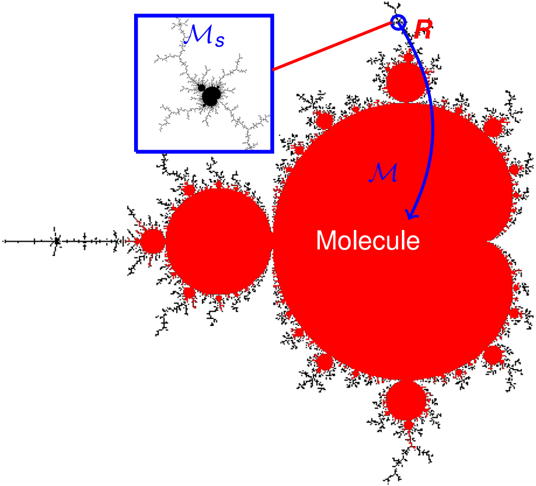
if $\bigcap_{n\geq 0} \boldsymbol{R}^{-n}(\mathcal{M}) = \{c_s\}$ is a singleton, then MLC holds at c_s

Lyubich; Graczyk and Świątek: $\mathbb{R}$-version of MLC:



$\bigcap_{n\geq 0} \boldsymbol{R}^{-n}(\mathcal{M}) \cap \mathbb{R} = \{c_s\}$ is a singleton if $\mathcal{M}_s \cap \mathbb{R} \neq \emptyset$

Kahn, Lyubich: $\forall \varepsilon > 0$, $\boldsymbol{R}: \mathcal{M}_s \to \mathcal{M}$ are simultaneously expanding if $\mathcal{M}_s$ are ε-away from the molecule (primitive case):



$\bigcap_{n>0} \boldsymbol{R}^{-n}(\mathcal{M}) = \{c_s\}$ is a singleton – MLC at c_s

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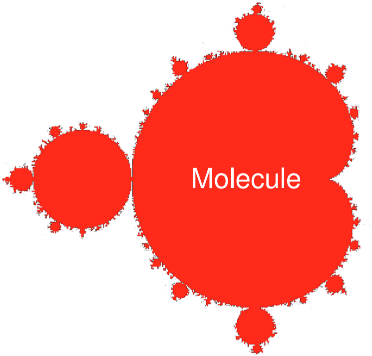

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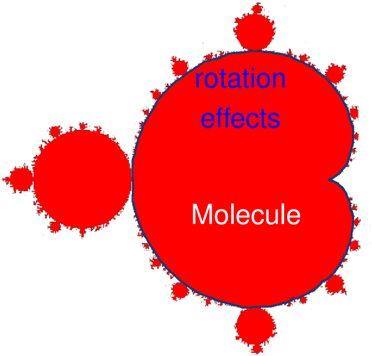
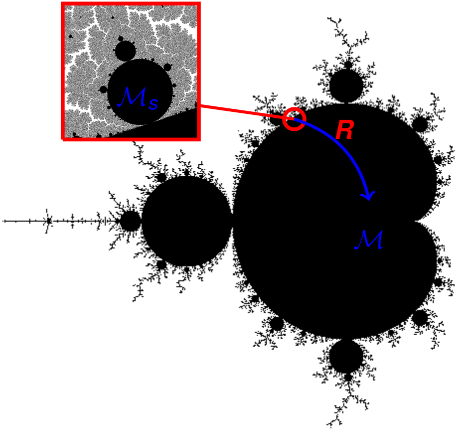


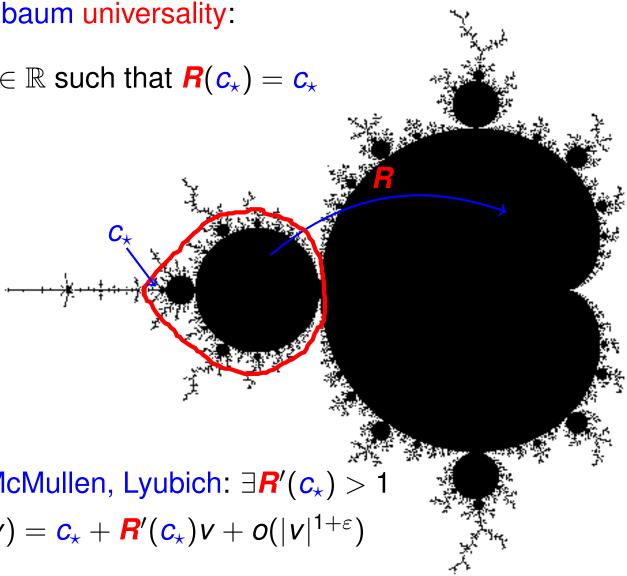
Thm (Lyubich and DD) ***R***: $\mathcal{M}_s \to \mathcal{M}$ is expanding for some satellite copies $\mathcal{M}_s$ on the molecule (first examples):



$\bigcap_{n\geq 0} \boldsymbol{R}^{-n}(\mathcal{M}) = \{c_s\}$ is a singleton – MLC at c_s

Feigenbaum universality:

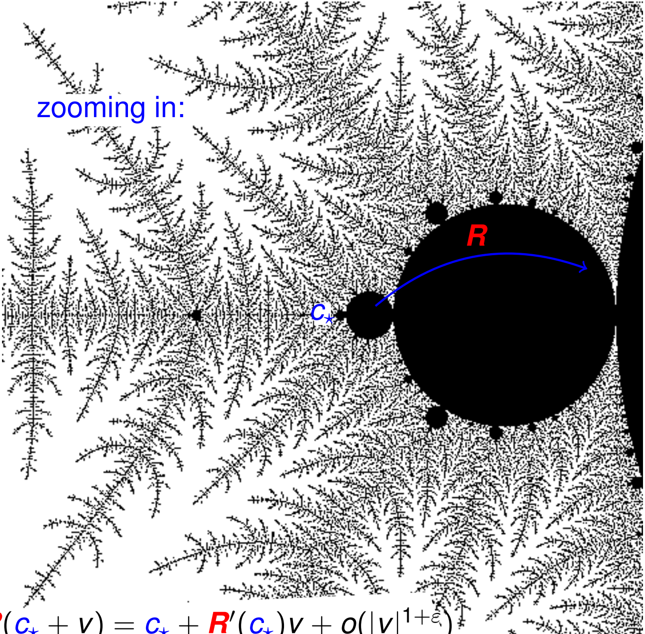
$$\exists! c_\star \in \mathbb{R} \text{ such that } \boldsymbol{R}(c_\star) = c_\star$$



Sullivan, McMullen, Lyubich: $\exists \boldsymbol{R}'(c_\star) > 1$

$$\boldsymbol{R}(c_\star + v) = c_\star + \boldsymbol{R}'(c_\star) v + o(|v|^{1+\varepsilon})$$

zooming in:



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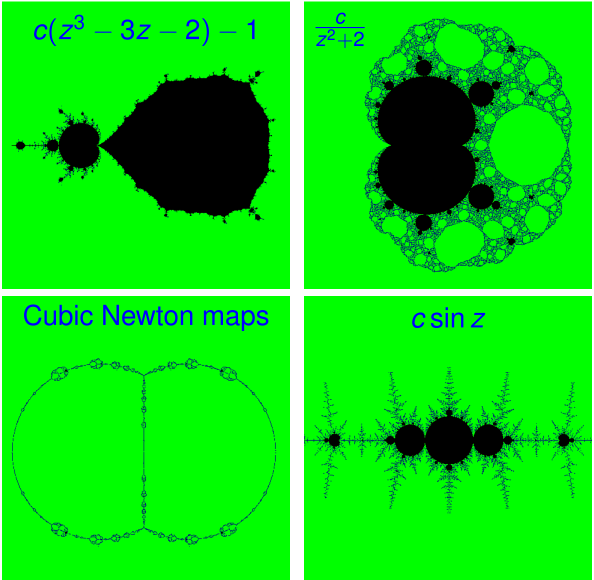
Feigenbaum scaling is universal:



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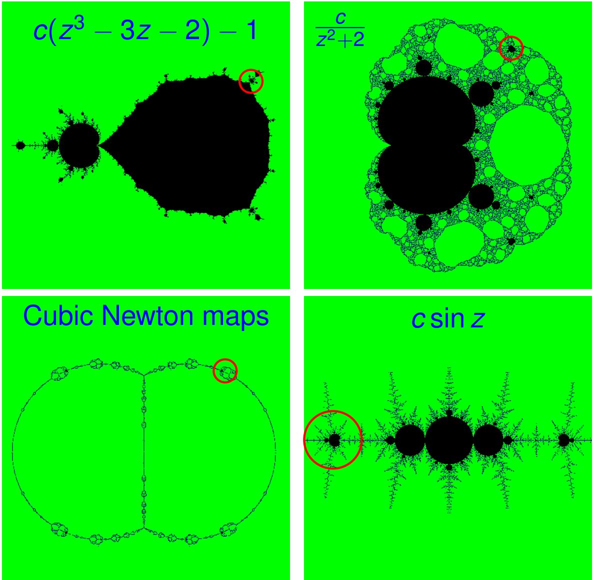

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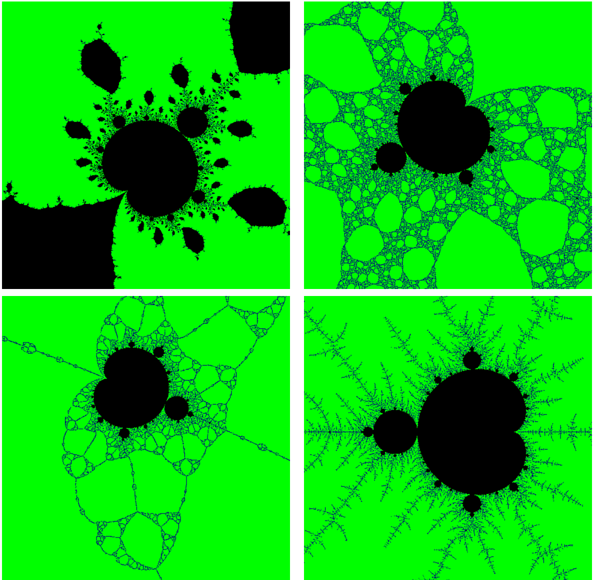

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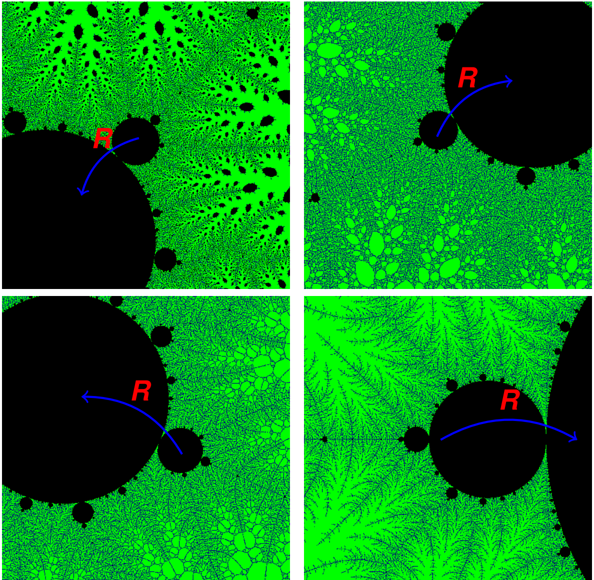

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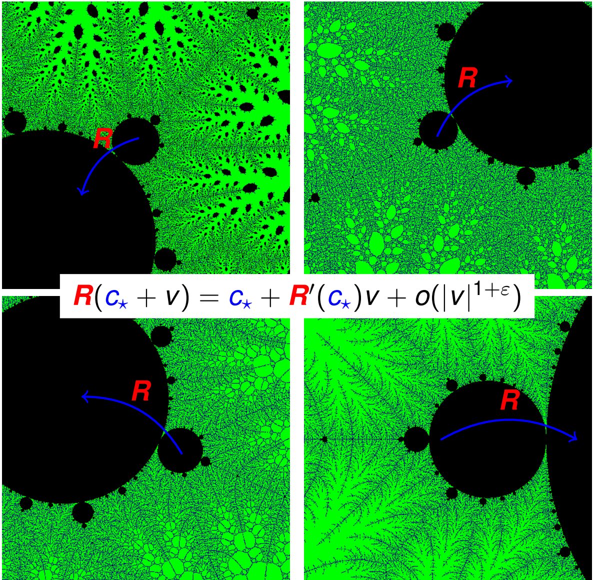

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Renormalization of f

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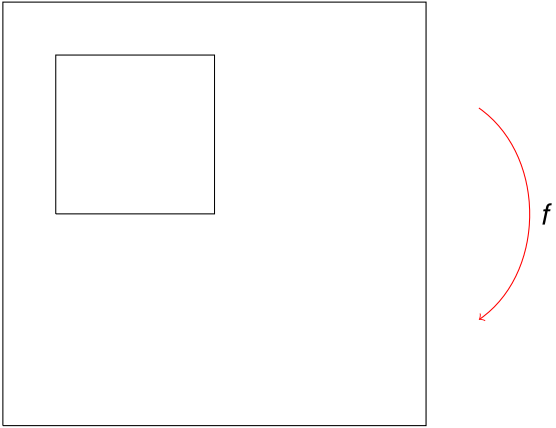

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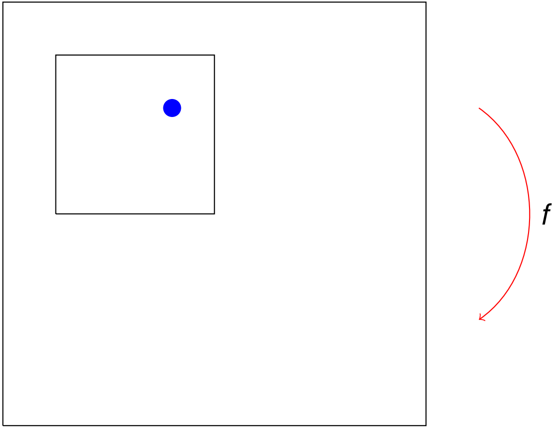

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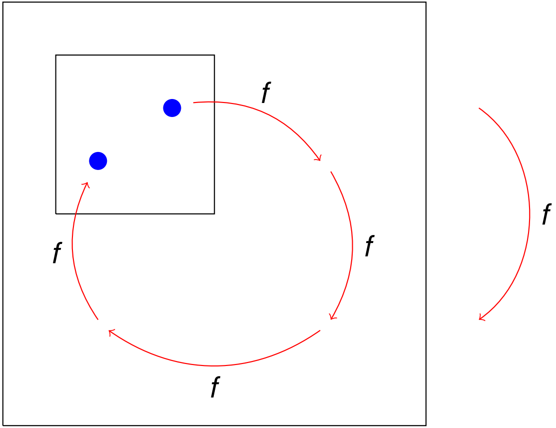

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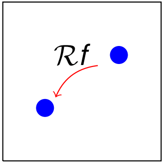
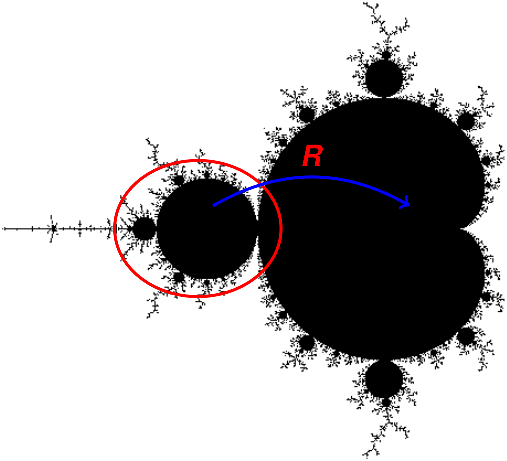
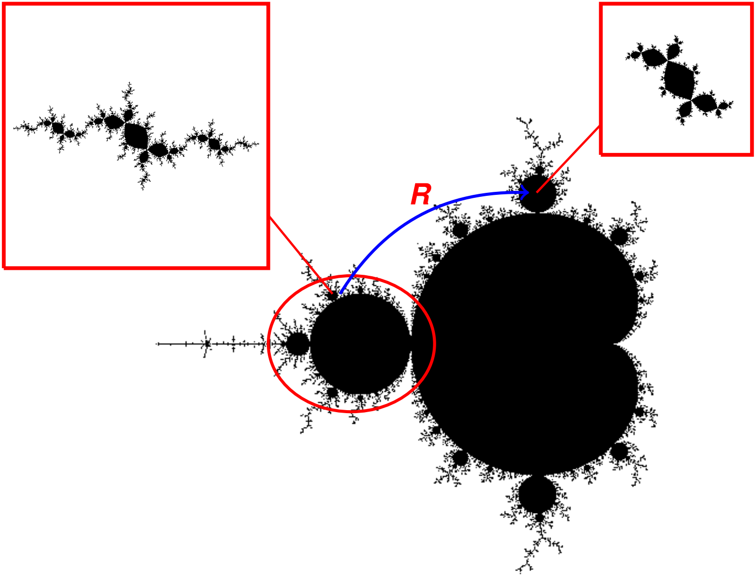


$\mathcal{R}f$ is the first return map

$$\mathcal{R}\colon \{\text{Maps}\}/_\sim \dashrightarrow \{\text{Maps}\}/_\sim$$

Canonical homeomorphism:

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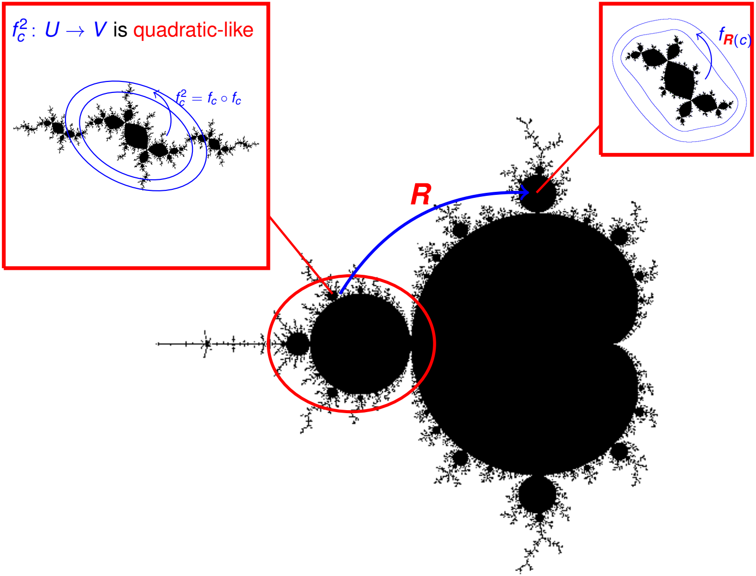
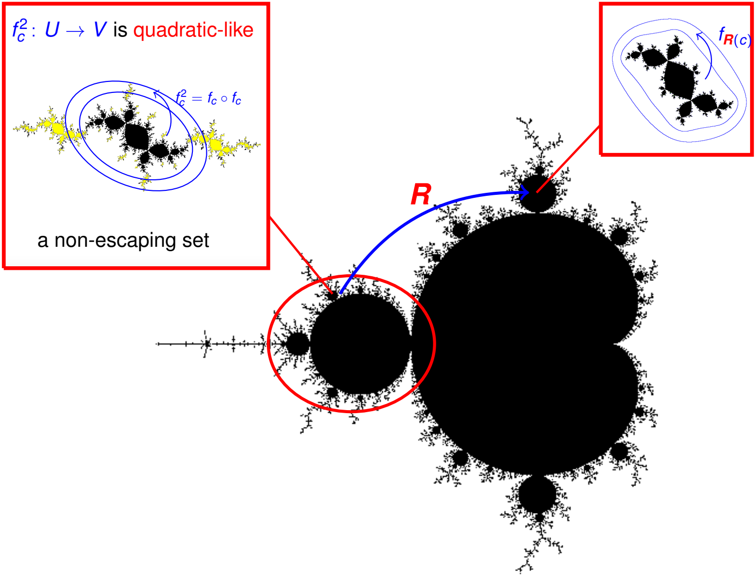


$f_c^2 : U \to V$ is quadratic-like
$f_c^2 = f_c \circ f_c$
R
$f_{R(c)}$



$f_c^2\colon U \to V$ is quadratic-like
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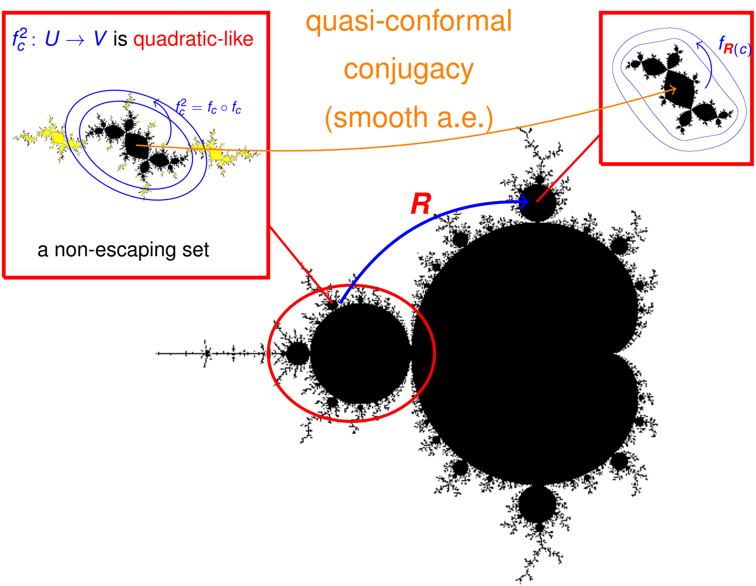
$f_c^2 : U \to V$ is quadratic-like
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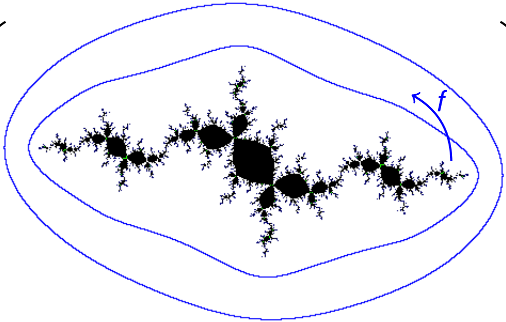
a non-escaping set
quasi-conformal
conjugacy
(smooth a.e.)
$f_{R(c)}$
R

Decomposition $\boldsymbol{R} = \text{holonomy} \circ \mathcal{R}$

QL := {



quadratic-like maps

(2 : 1 maps $f\colon U \to V$ with $U \Subset V$)

} / affine conjugacy

$\mathcal{R}\colon$ QL $\dashrightarrow$ QL is analytic (iteration+restriction)

dim(QL) $= \infty$, but qc-conjugate maps form leaves of

codim $= 1$ stable foliation

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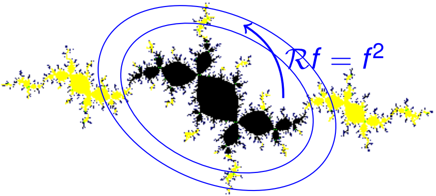

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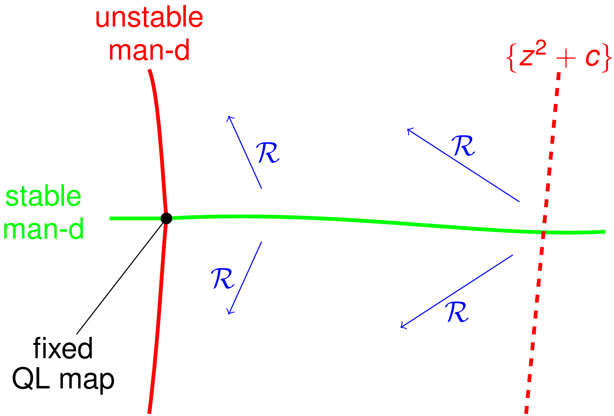
(2 : 1 maps $f : U \to V$ with $U \Subset V$)

} / affine conjugacy

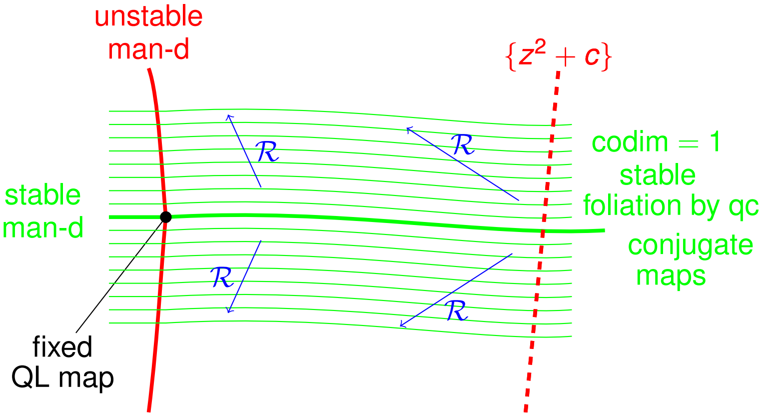
$\mathcal{R}$: QL $\dashrightarrow$ QL is analytic (iteration+restriction)

dim(QL) $= \infty$, but qc-conjugate maps form leaves of codim $= 1$ stable foliation

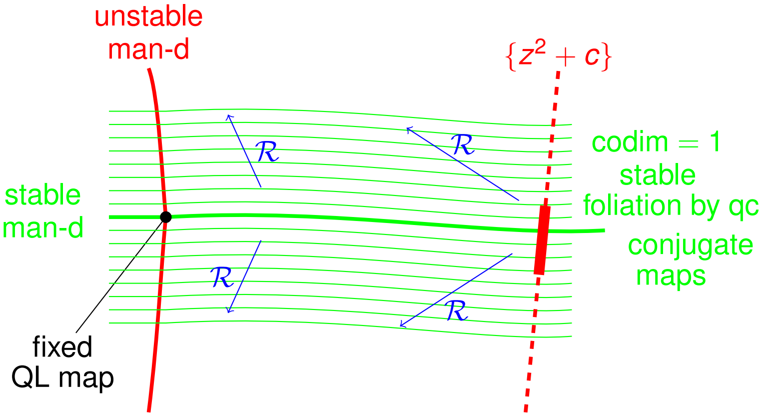
Hyperbolicity of $\mathcal{R}$: QL ⇢ QL



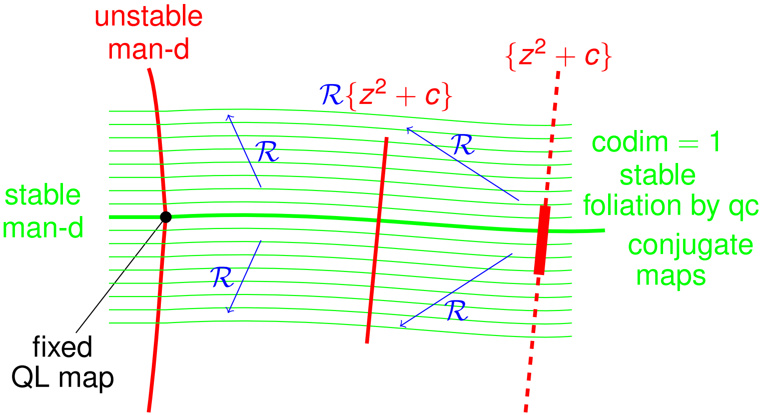
Hyperbolicity of $\mathcal{R}$: QL $\dashrightarrow$ QL



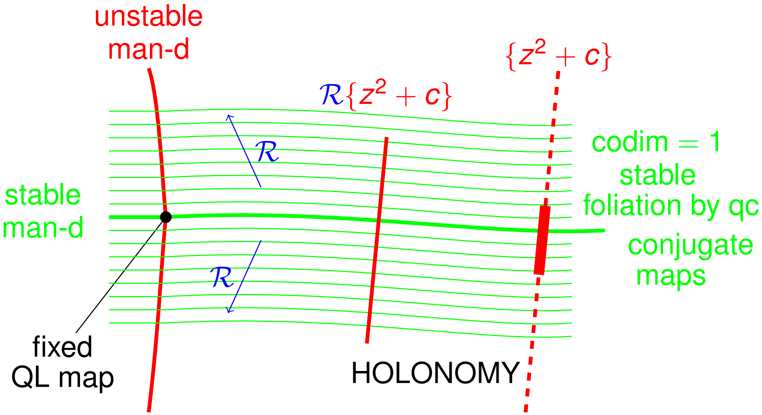
Hyperbolicity of $\mathcal{R}$: QL $\dashrightarrow$ QL



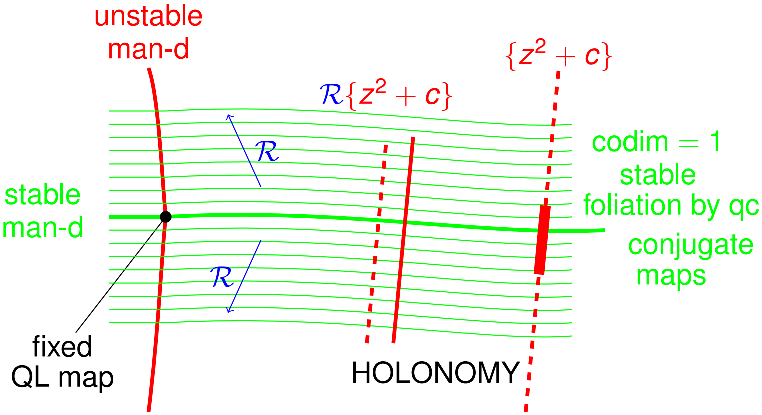
Hyperbolicity of $\mathcal{R}$: QL $\dashrightarrow$ QL



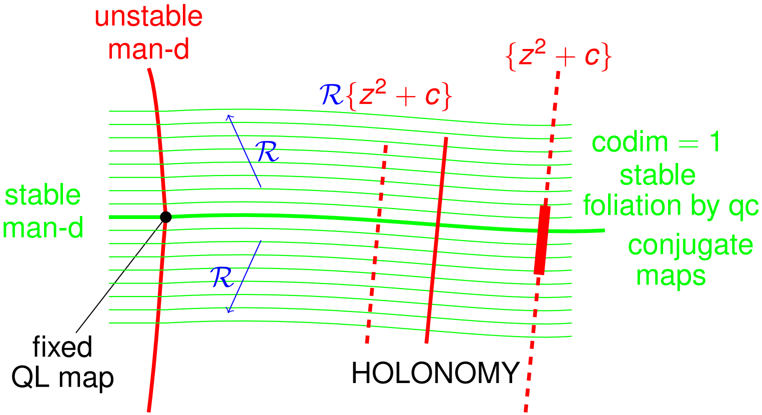
Hyperbolicity of $\mathcal{R}$: QL $\dashrightarrow$ QL



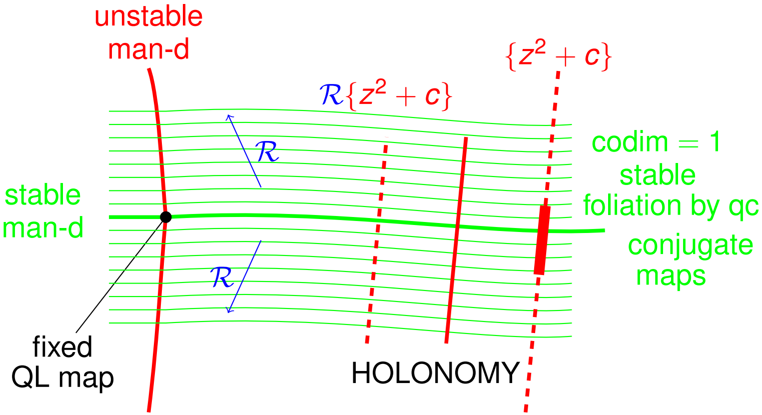
Hyperbolicity of $\mathcal{R}$: QL $\dashrightarrow$ QL



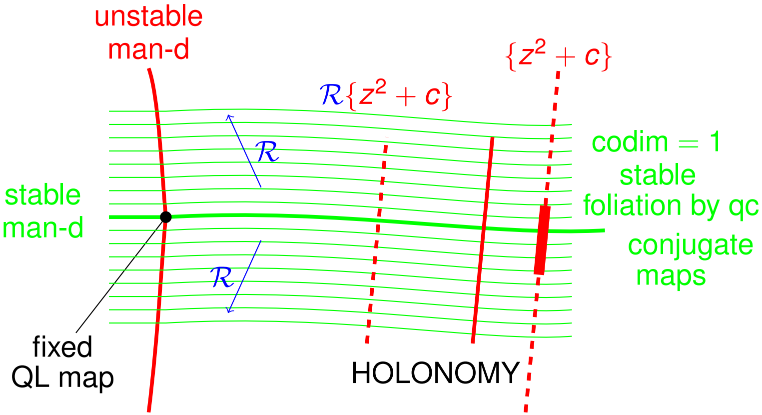
Hyperbolicity of $\mathcal{R}$: QL $\dashrightarrow$ QL



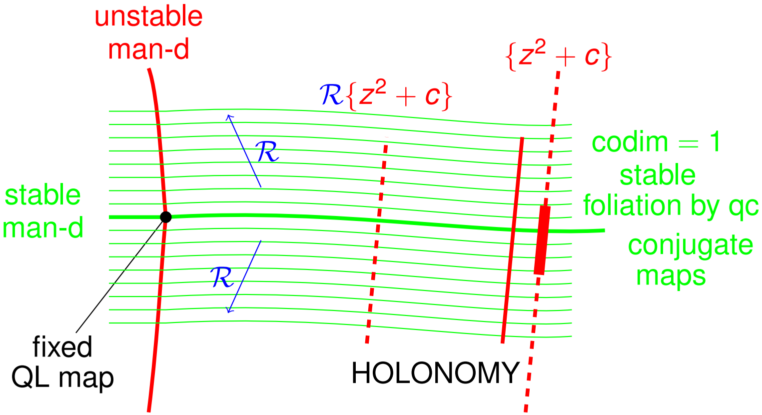
Hyperbolicity of $\mathcal{R}$: QL $\dashrightarrow$ QL



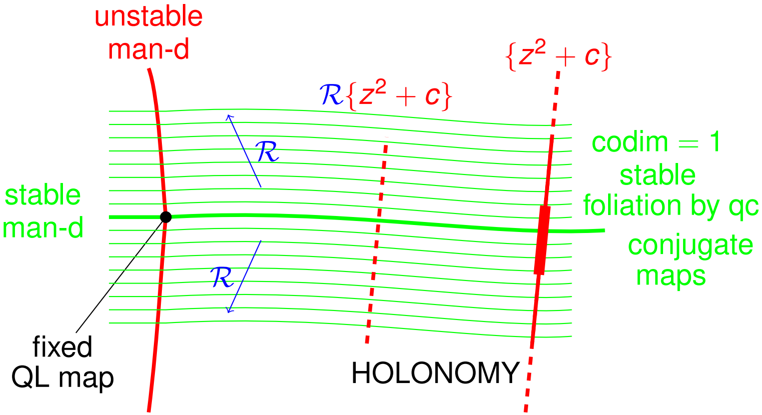
Hyperbolicity of $\mathcal{R}$: QL $\dashrightarrow$ QL



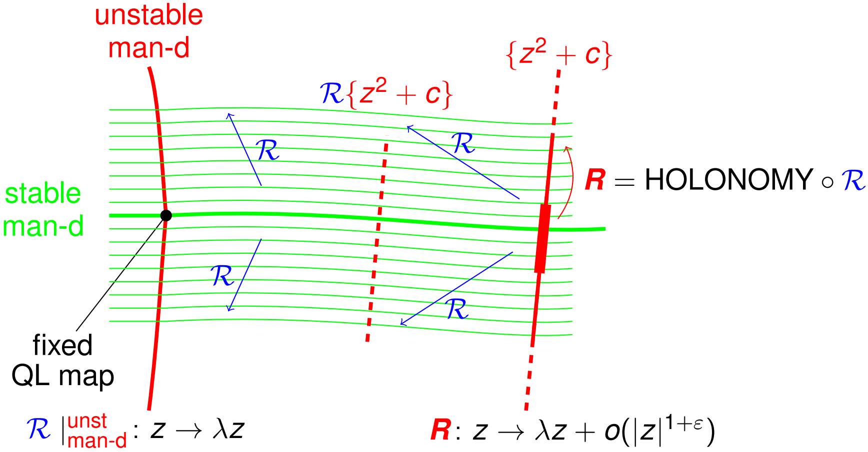
Hyperbolicity of $\mathcal{R}$: QL $\dashrightarrow$ QL



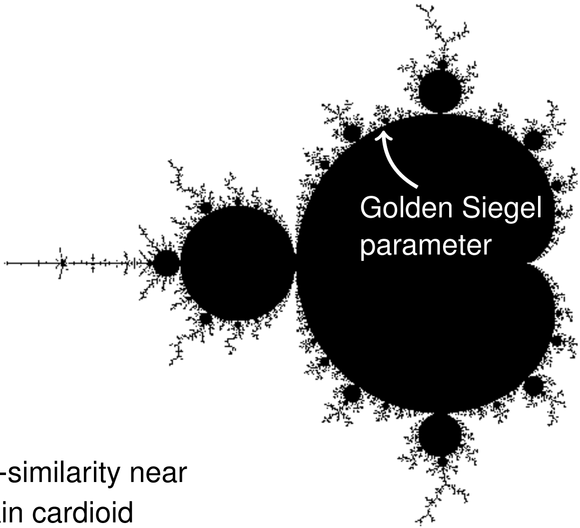
Hyperbolicity of $\mathcal{R}$: QL $\dashrightarrow$ QL



Hyperbolicity of $\mathcal{R}$: QL --→ QL

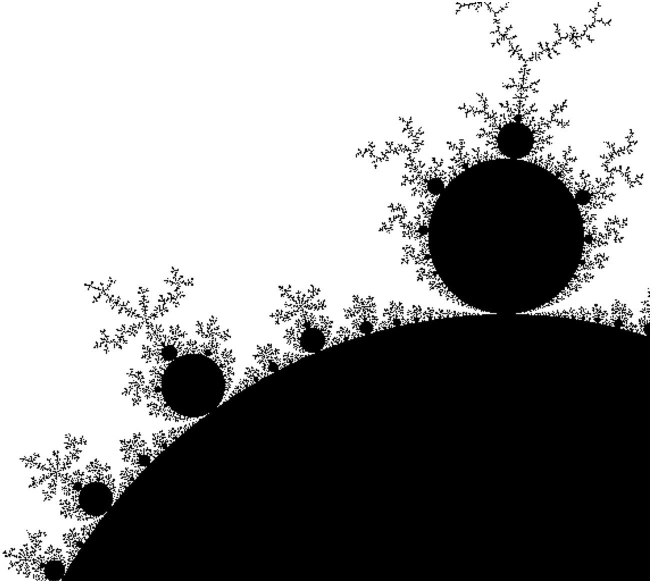


Sullivan, McMullen, Lyubich: hyperbolicity of $\mathcal{R}$ + holonomy prove universality

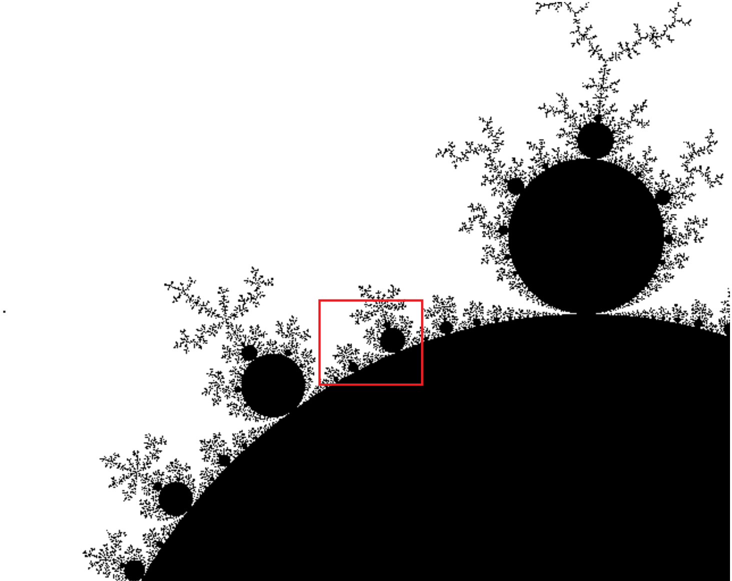


self-similarity near
main cardioid

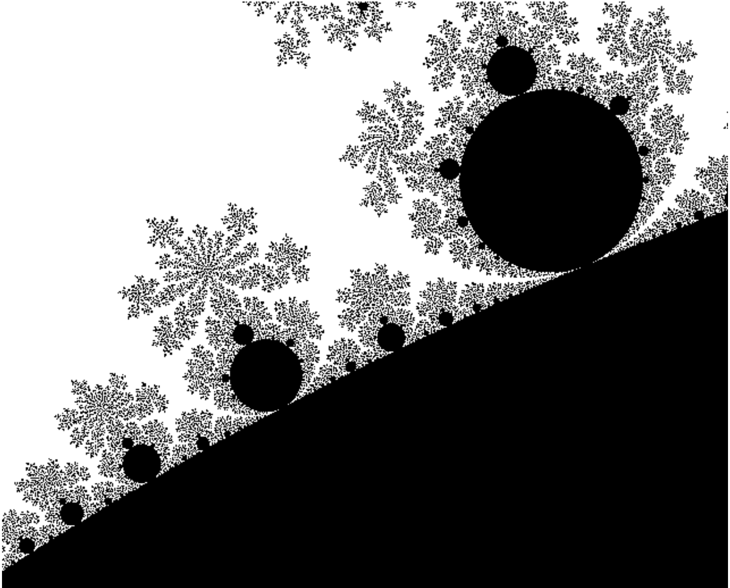
Scaling around the Golden Siegel parameter



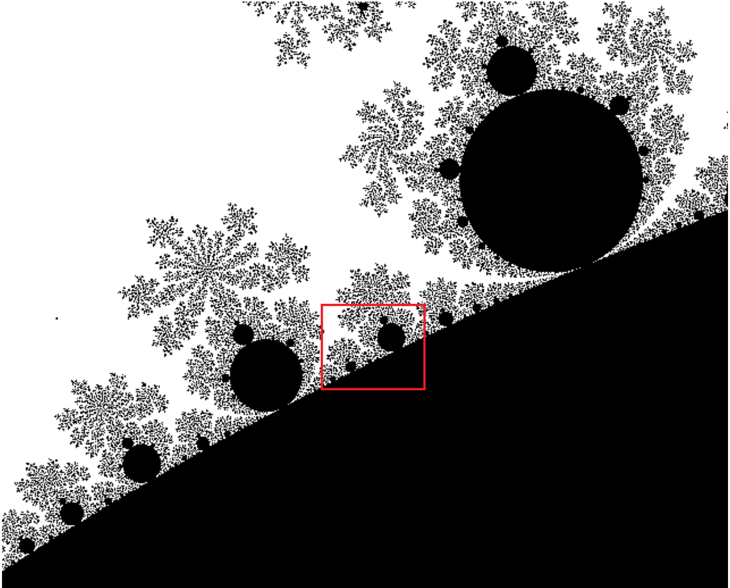
Scaling around the Golden Siegel parameter



Scaling around the Golden Siegel parameter



Scaling around the Golden Siegel parameter



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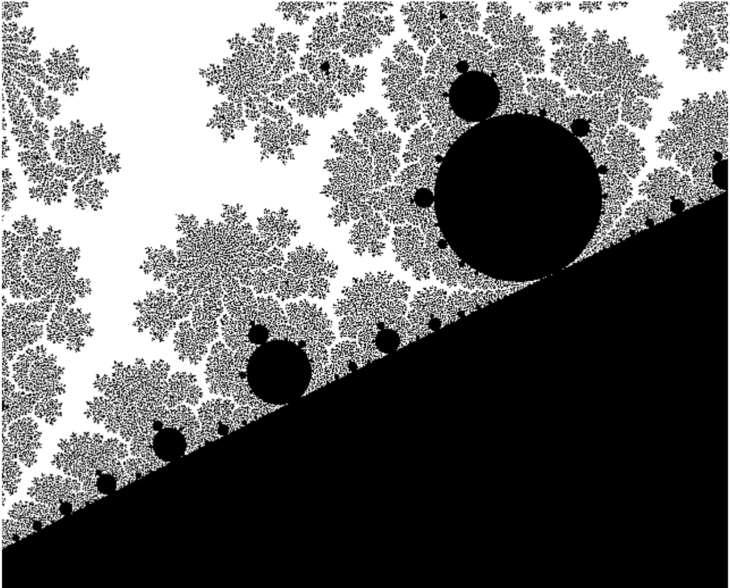

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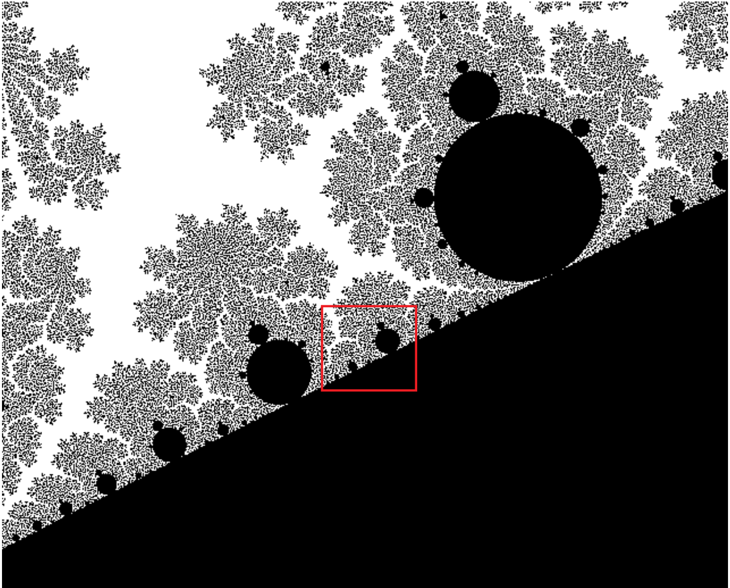
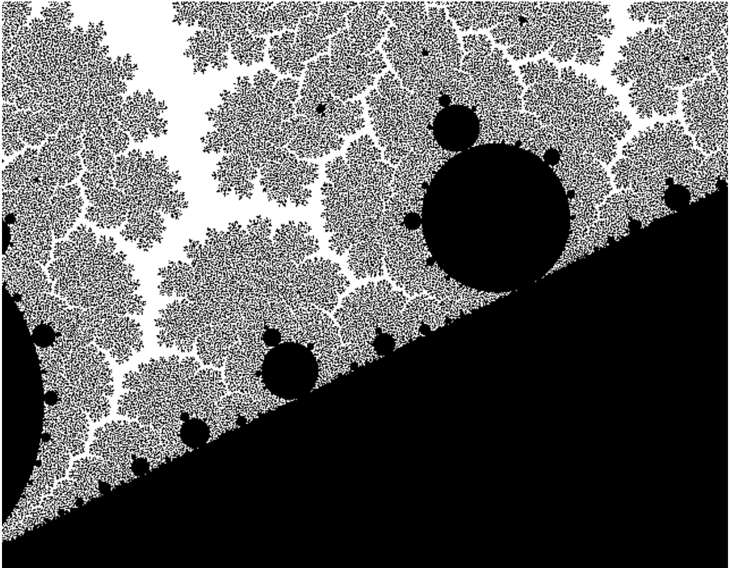
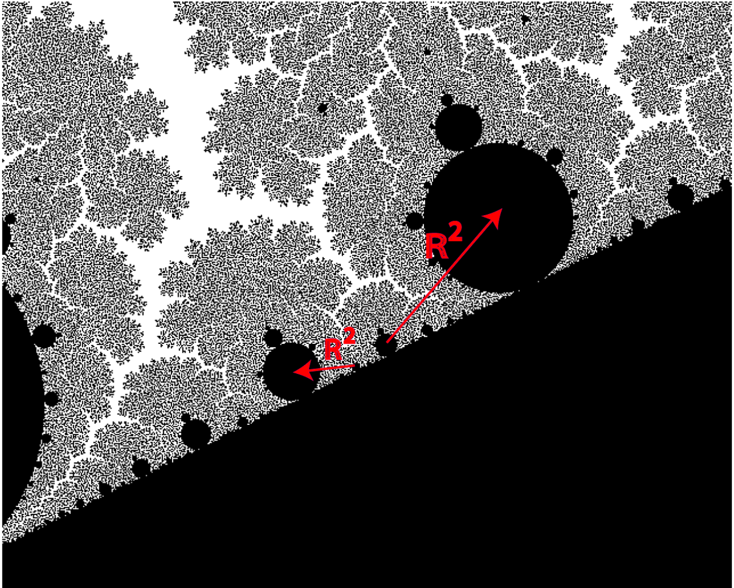
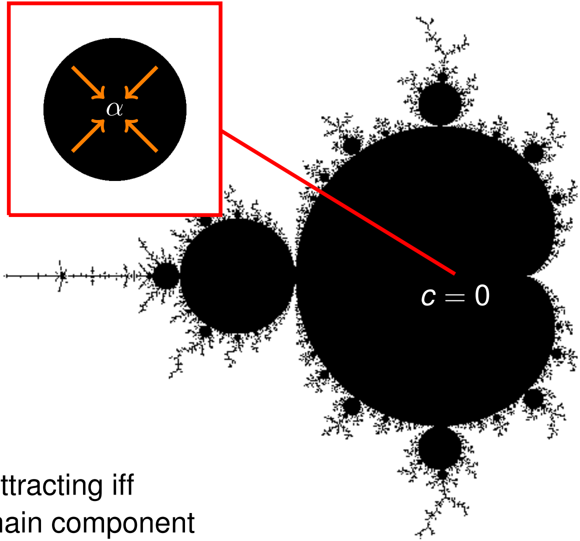


Scaling around the Golden Siegel parameter

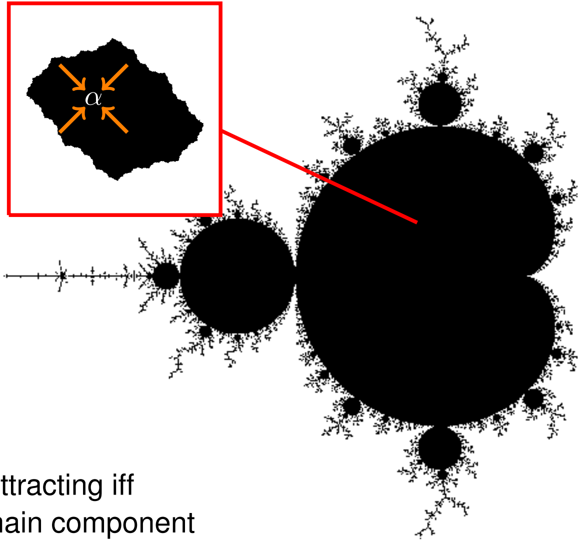


Scaling around the Golden Siegel parameter





α is attracting iff
$c \in$ main component



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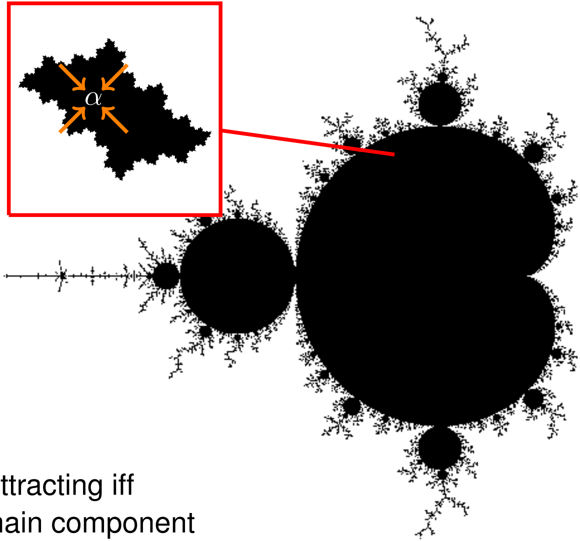

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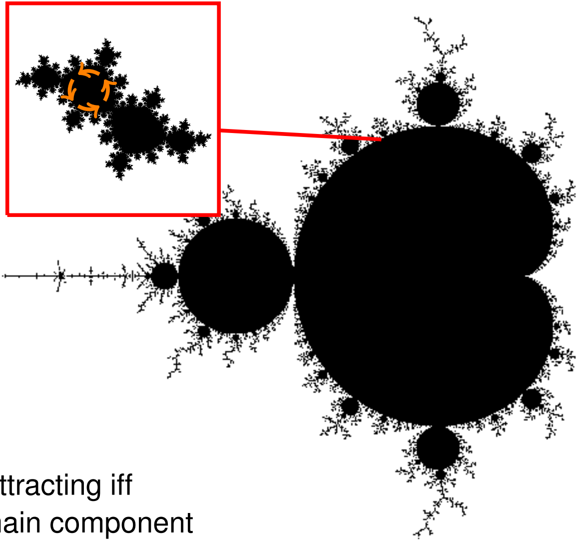

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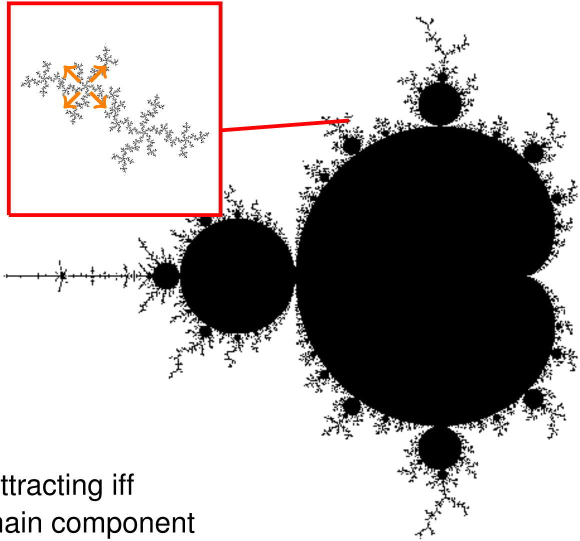

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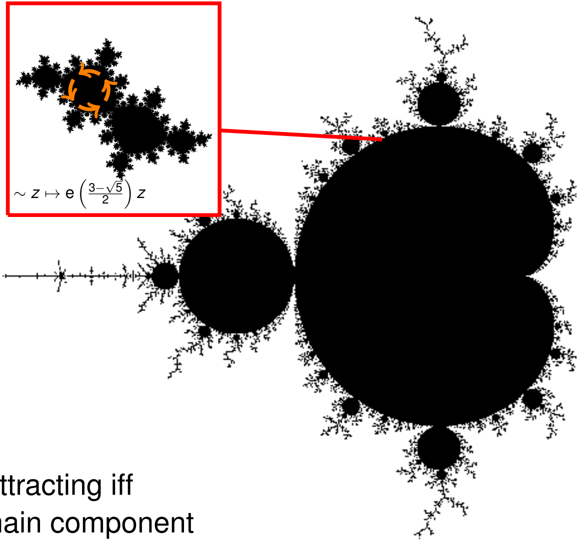

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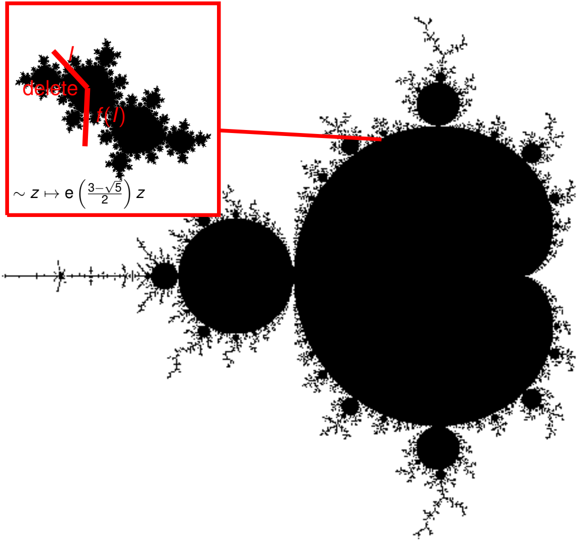
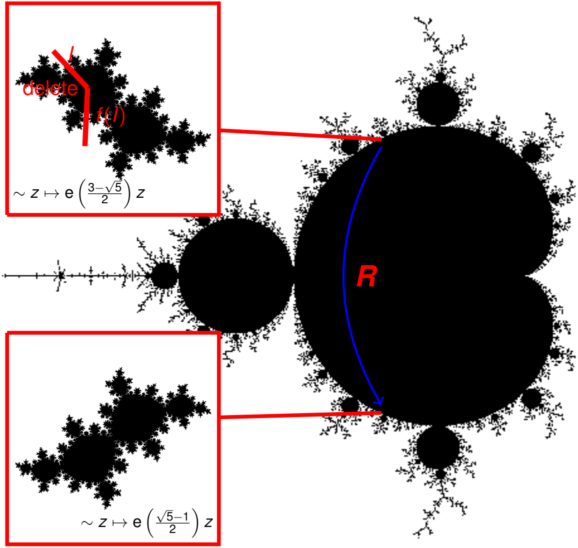
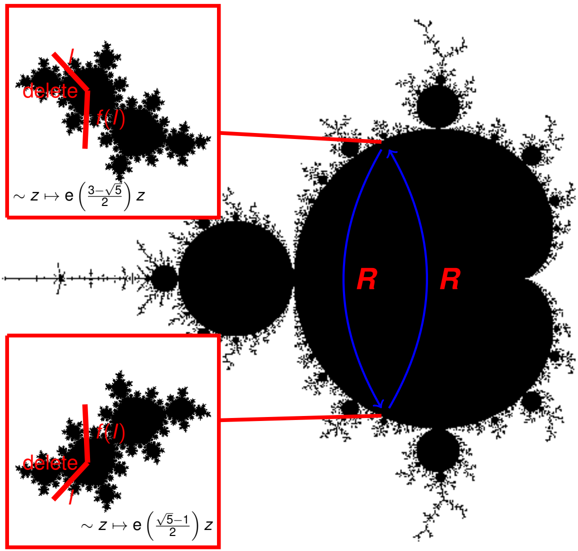


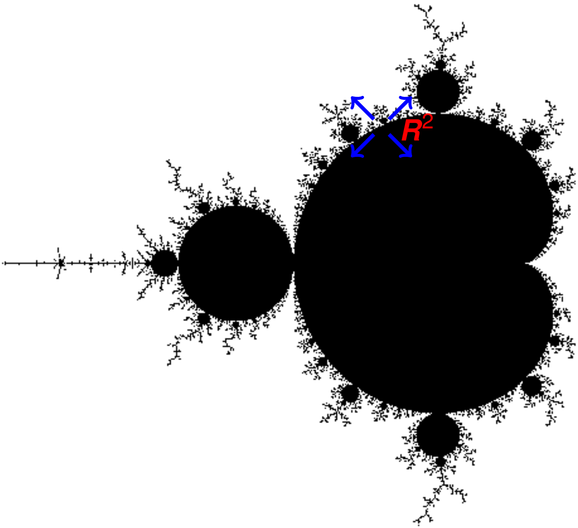
l
delete
f(l)
$\sim z \mapsto e\left(\frac{3-\sqrt{5}}{2}\right) z$

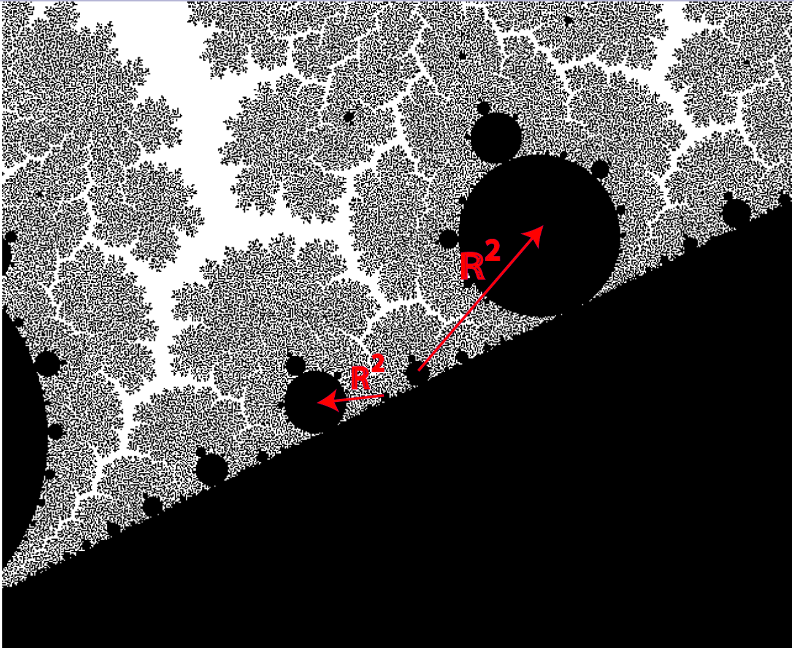


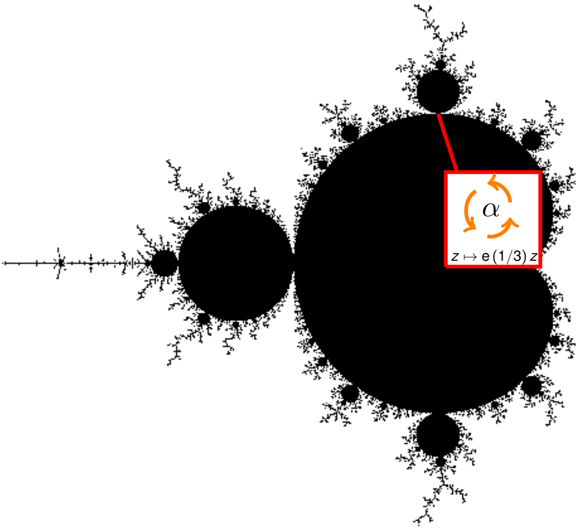
l
delete
$f(l)$
$\sim z \mapsto e\left(\frac{3-\sqrt{5}}{2}\right) z$
$\sim z \mapsto e\left(\frac{\sqrt{5}-1}{2}\right) z$
R

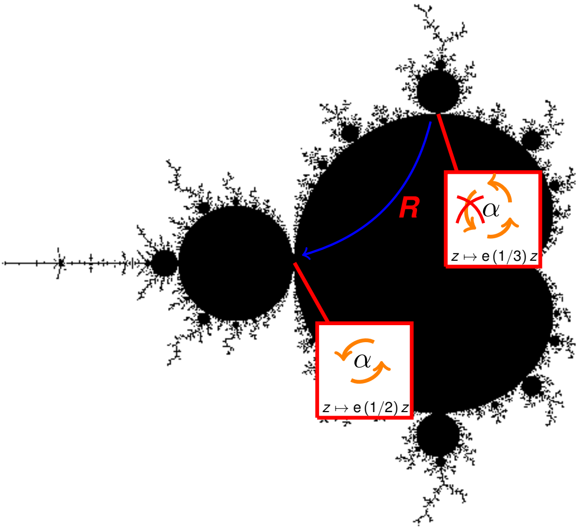


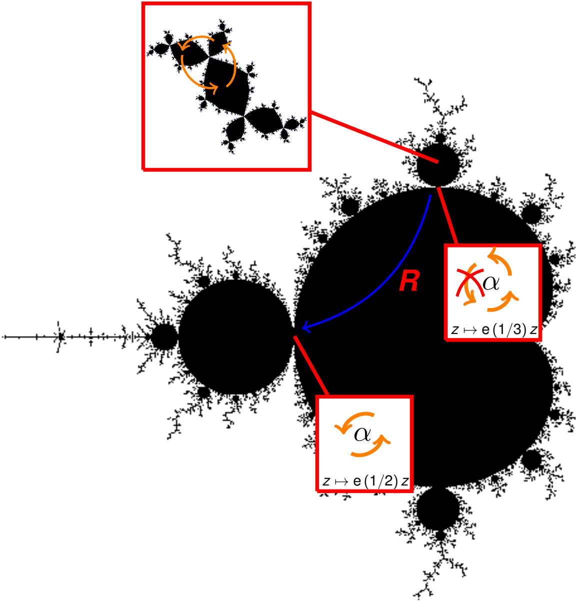
l
delete
f(l)
$\sim z \mapsto e\left(\frac{3-\sqrt{5}}{2}\right) z$
R
R
f(l)
delete
l
$\sim z \mapsto e\left(\frac{\sqrt{5}-1}{2}\right) z$


R^2


R^2
R^2

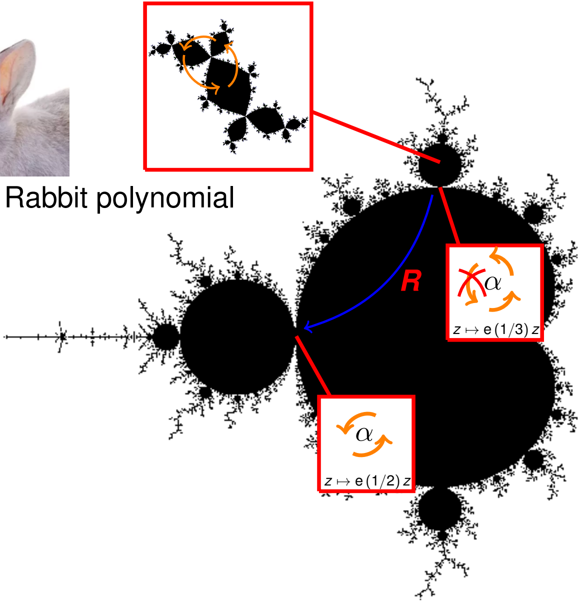

α
$z \mapsto e(1/3)\,z$

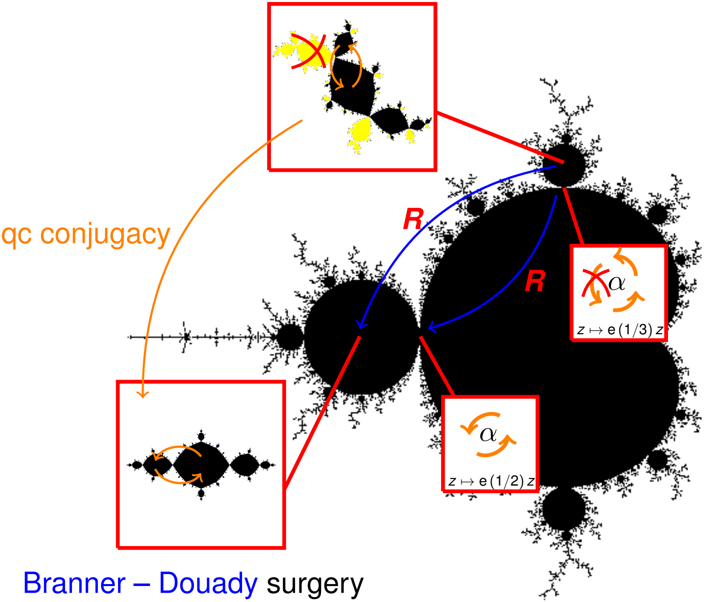

R
α
z ↦ e (1/3) z
α
z ↦ e (1/2) z



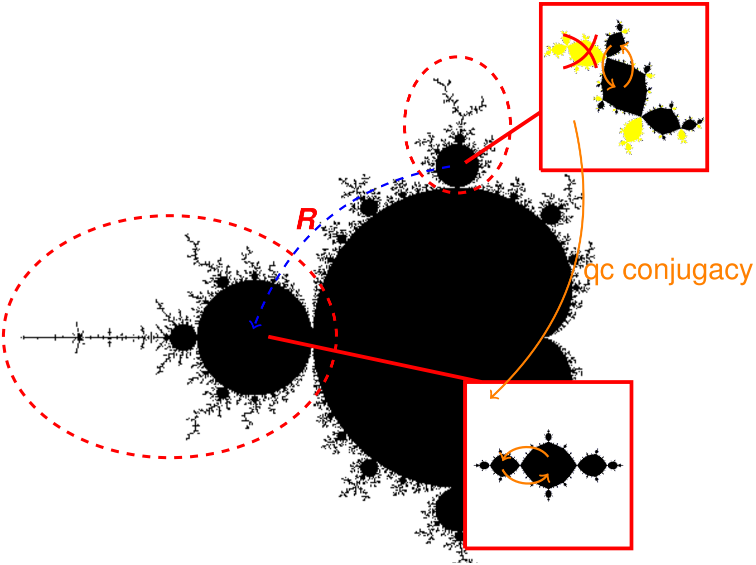
R
α
z ↦ e (1/3) z
α
z ↦ e (1/2) z

Rabbit polynomial

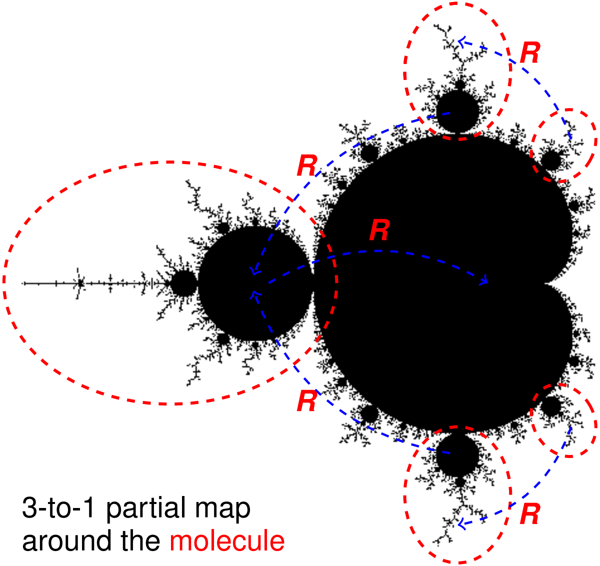




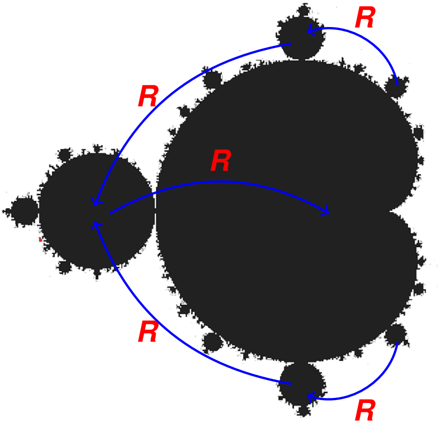
Branner – Douady surgery



Branner, Douady: ∃ partial surjective map



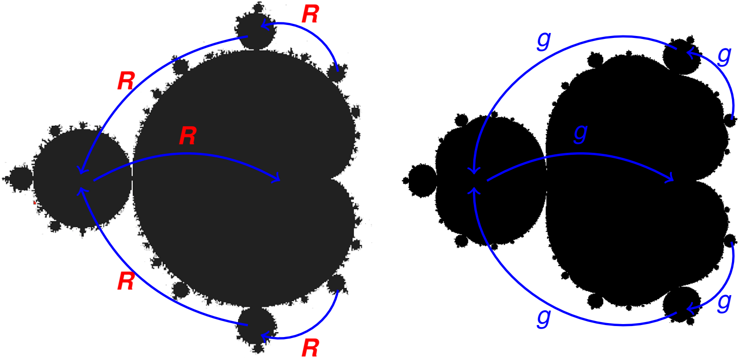
3-to-1 partial map
around the molecule



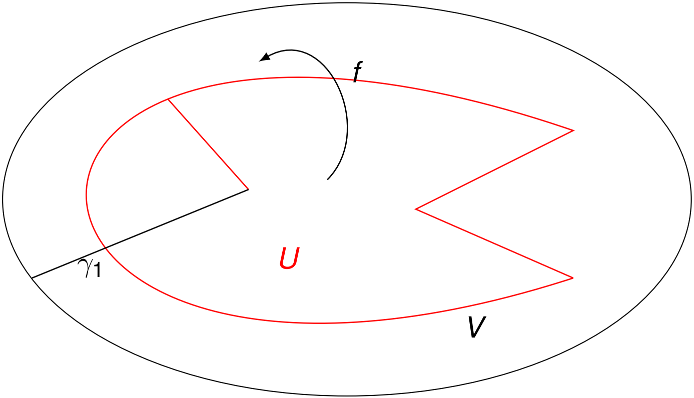
the molecule map
(3-to-1 continuous)

The molecule map and its model – conjugate if MLC

$$g(z) = z(z+1)^2$$



Pacman is a 2-to-1 map $f\colon U \to V$:



1UP
310
HIGH SCORE
310
www.gamesdbase.com

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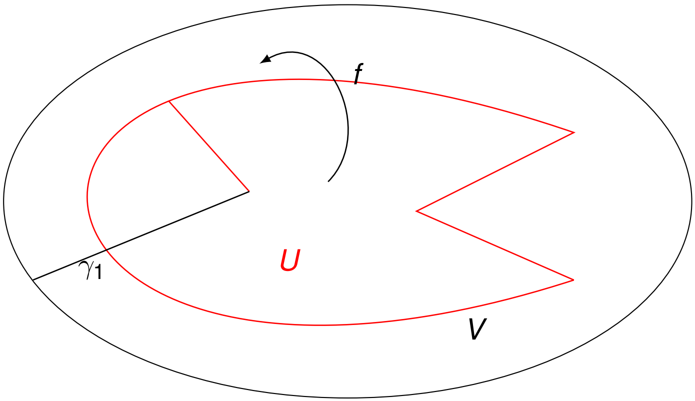

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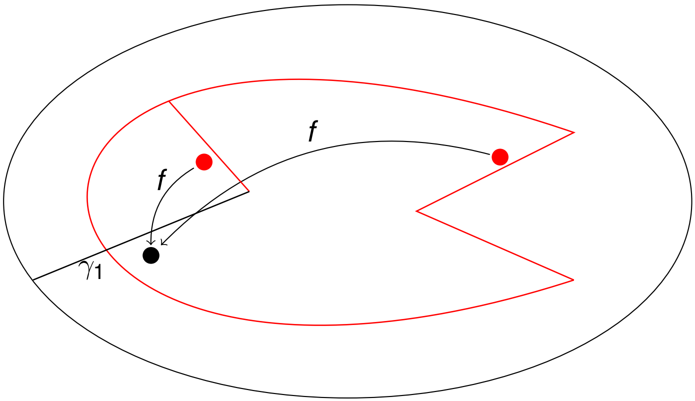
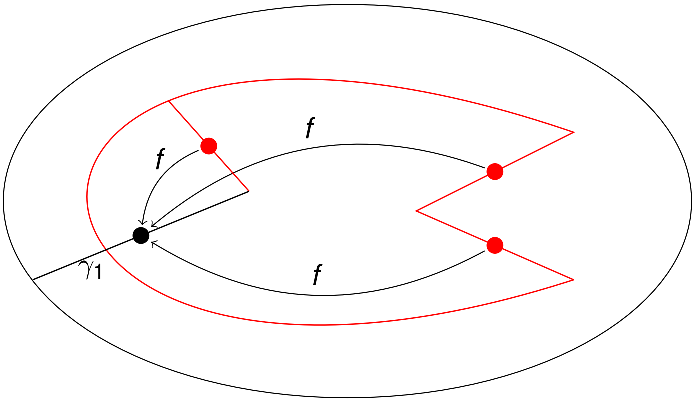


not a branched covering!

Pacman is a 2-to-1 map $f\colon U \to V$:



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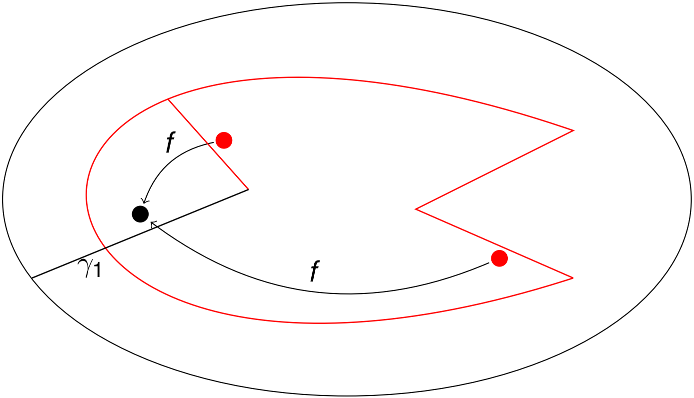
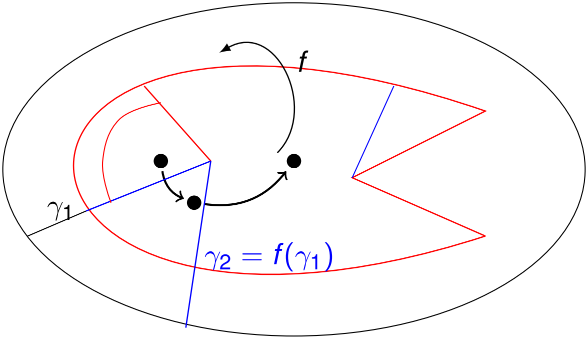
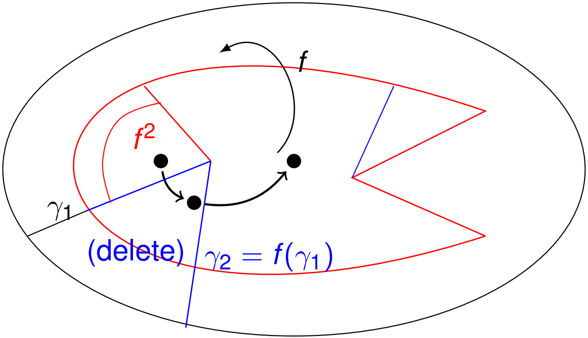


not a branched covering!

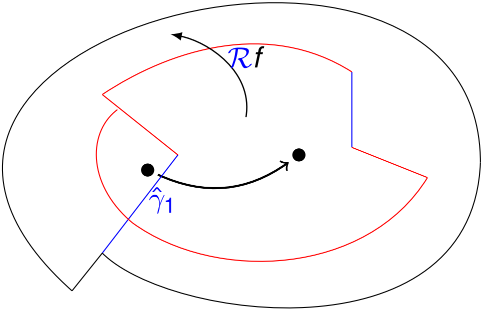
Renormalizable pacman



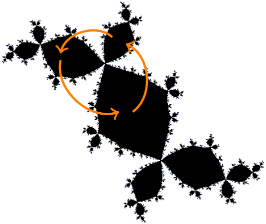
Renormalizable pacman:

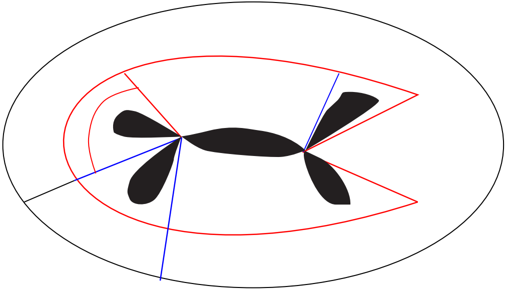


Pacman renormalization:

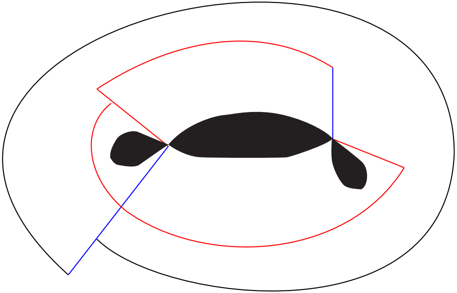


Renormalization of the Rabbit

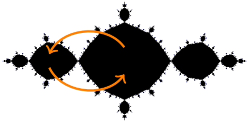


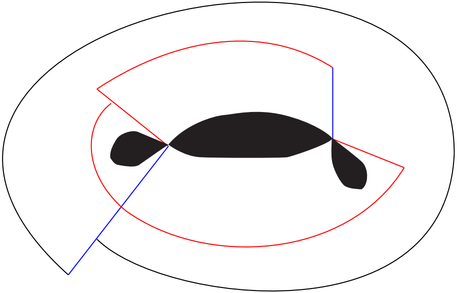


Renormalization of the Rabbit



Renormalization of the Rabbit

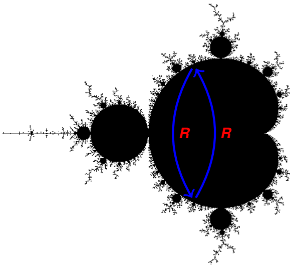


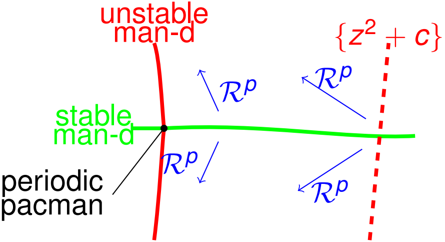


Branner – Douady surgery

analytic operator

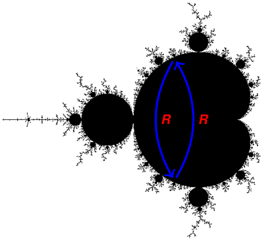
Thm (Lyubich, Selinger, and DD)
For periodic parameters we construct a hyperbolic analytic pacman renormalization operator $\mathcal{R}$ with $\dim = 1$ unstable man-d



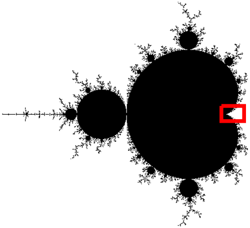


Rem. Periodic points were constructed in 1990s by McMullen for a "cylinder" renormalization

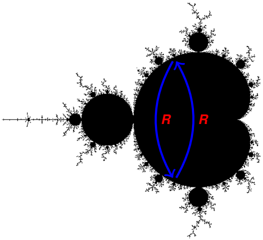
Thm (Lyubich, Selinger, and DD)
For periodic parameters we construct a hyperbolic analytic pacman renormalization operator $\mathcal{R}$ with dim $=$ 1 unstable man-d



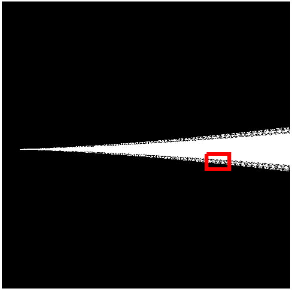
Inou, Shishikura: hyperbolicity for the cylinder renormalization for high type parameters (perturbative methods)



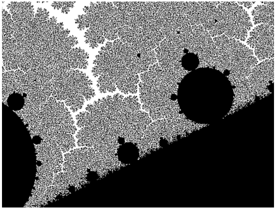
Thm (Lyubich, Selinger, and DD)
For periodic parameters we construct
a hyperbolic analytic pacman
renormalization operator $\mathcal{R}$
with dim $= 1$ unstable man-d



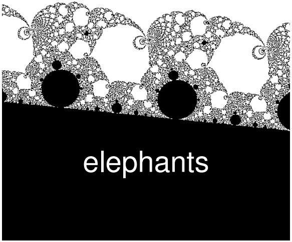
Inou, Shishikura: hyperbolicity
for the cylinder renormalization
for high type parameters
(perturbative methods)



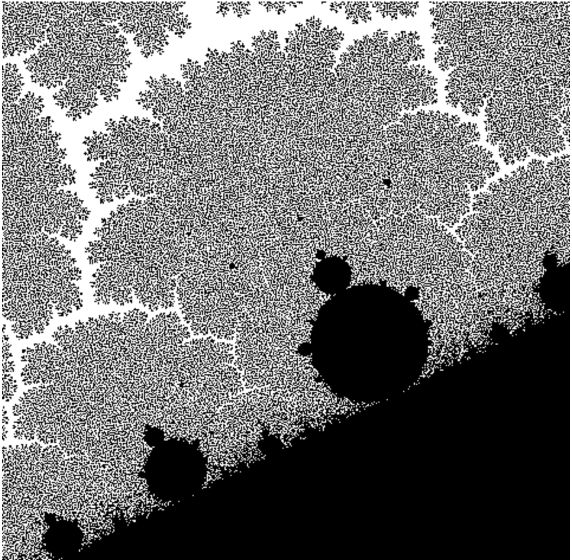
Thm (Lyubich, Selinger, and DD)
For periodic parameters we construct a hyperbolic analytic pacman renormalization operator $\mathcal{R}$ with dim $=1$ unstable man-d



Inou, Shishikura: hyperbolicity for the cylinder renormalization for high type parameters (perturbative methods)



Unstable manifold $\approx$ zoomed Mandelbrot set
can be studied as a transcendental family



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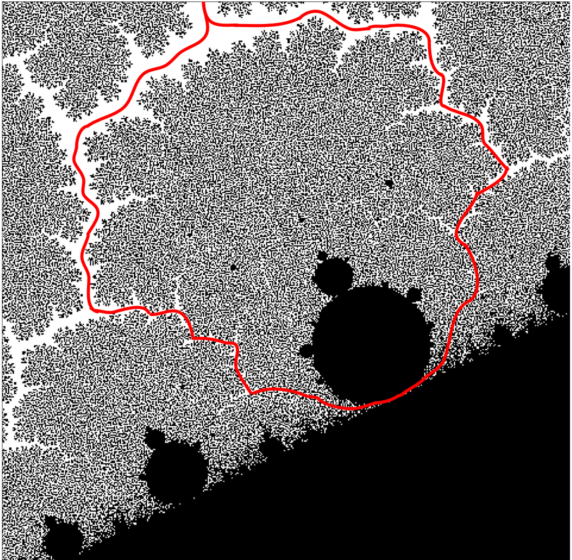
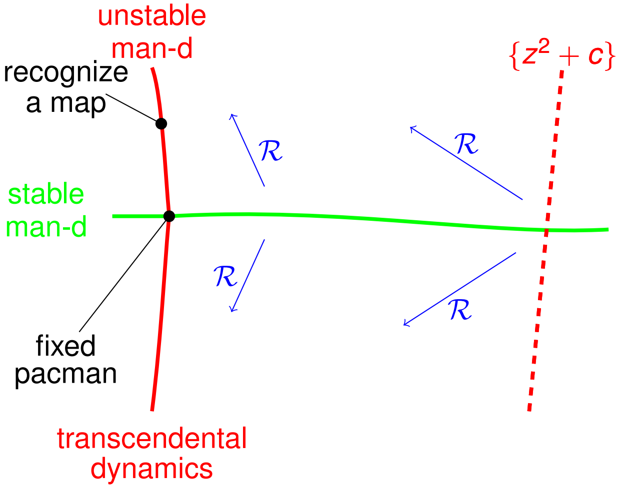
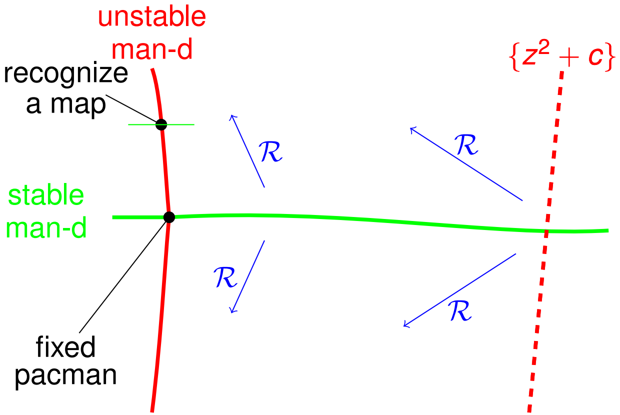


Lyubich and DD: there is a stable lamination

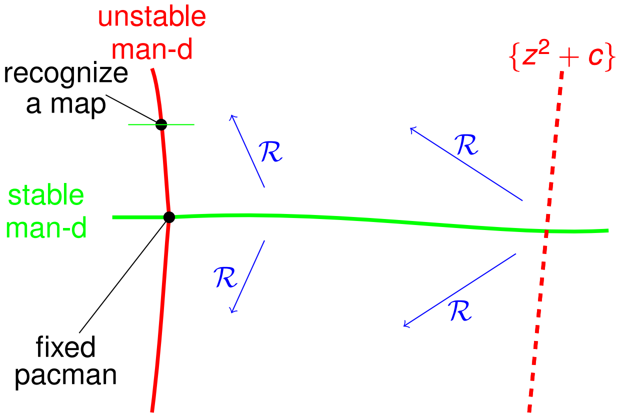


Lyubich and DD: there is a stable lamination



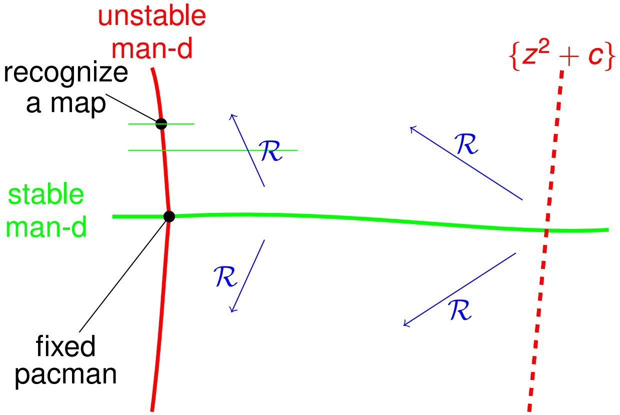
Construct a local leaf

Lyubich and DD: there is a stable lamination



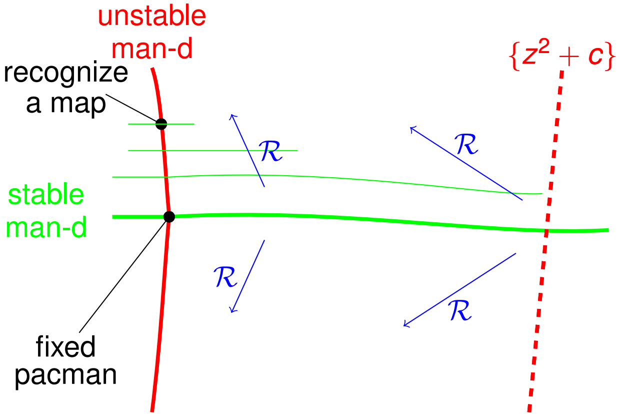
Construct a local leaf

Lyubich and DD: there is a stable lamination



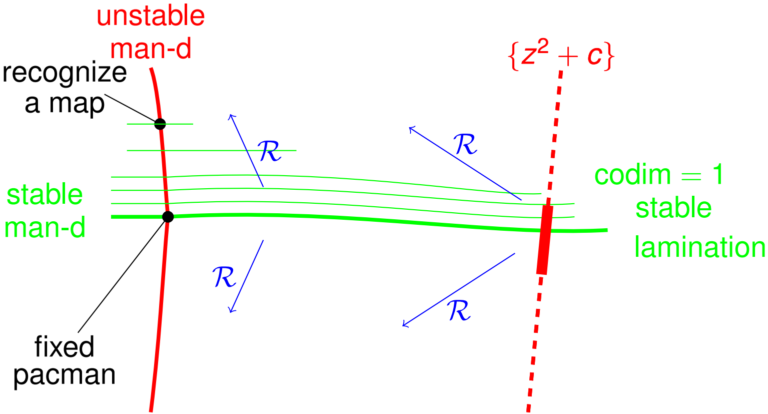
Construct a local leaf, run $\mathcal{R}$

Lyubich and DD: there is a stable lamination



Construct a local leaf, run $\mathcal{R}$

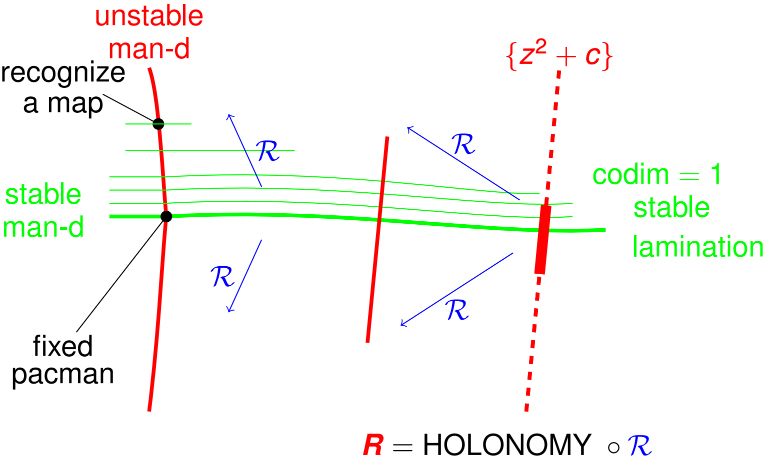
Lyubich and DD: there is a stable lamination



$\boldsymbol{R}$ = HOLONOMY $\circ\, \mathcal{R}$

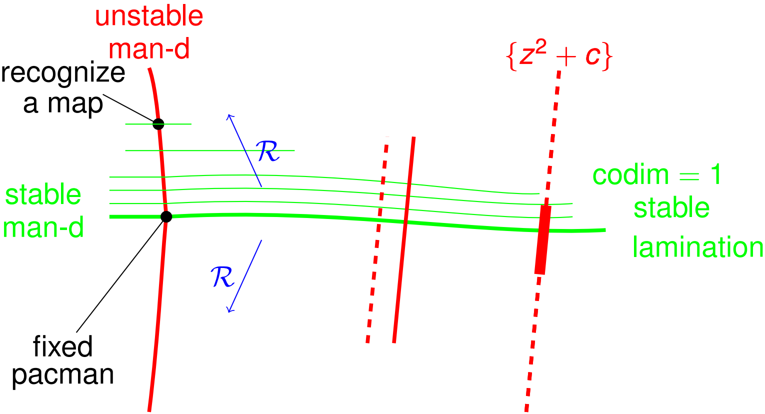
Construct a local leaf, run $\mathcal{R}$

Lyubich and DD: scaling



Construct a local leaf, run $\mathcal{R}$

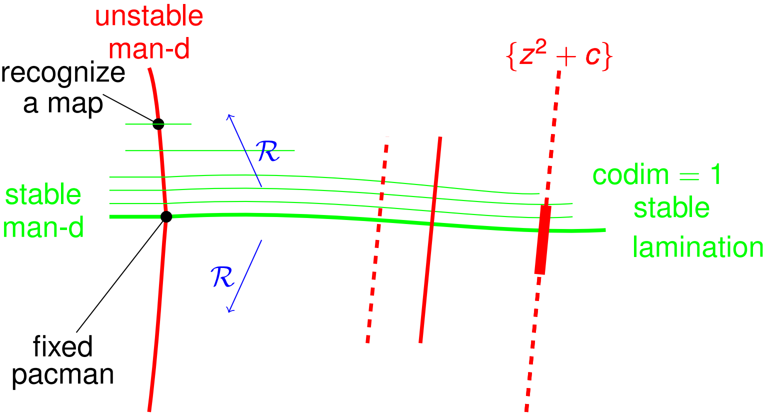
Lyubich and DD: scaling



$\boldsymbol{R} = \text{HOLONOMY} \circ \mathcal{R}$

Construct a local leaf, run $\mathcal{R}$

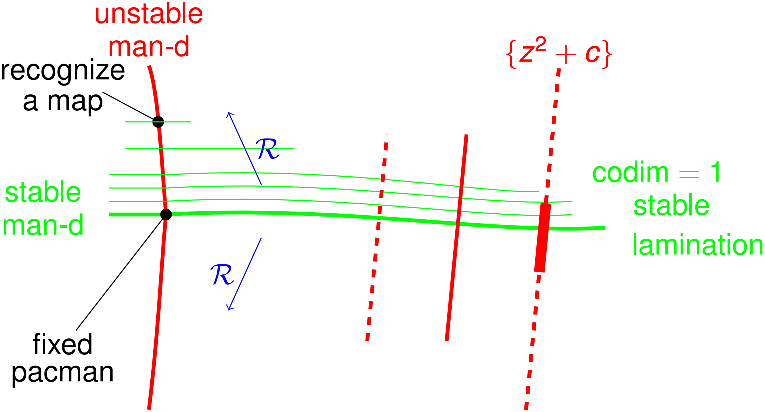
Lyubich and DD: scaling



$\boldsymbol{R} =$ HOLONOMY $\circ\, \mathcal{R}$

Construct a local leaf, run $\mathcal{R}$

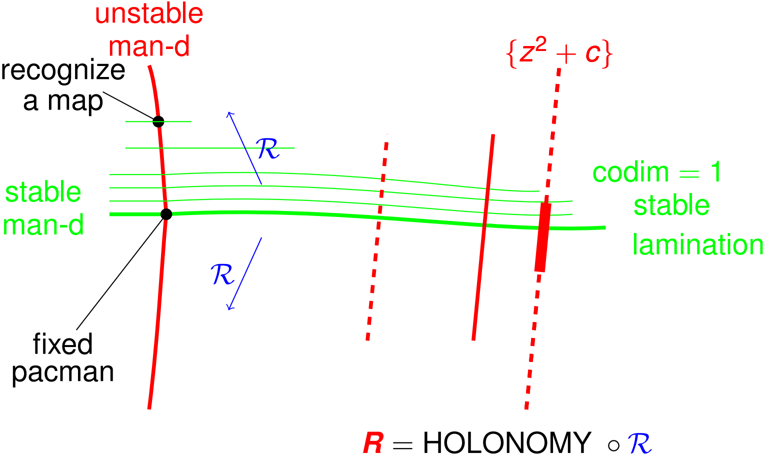
Lyubich and DD: scaling



$\boldsymbol{R}$ = HOLONOMY $\circ\,\mathcal{R}$

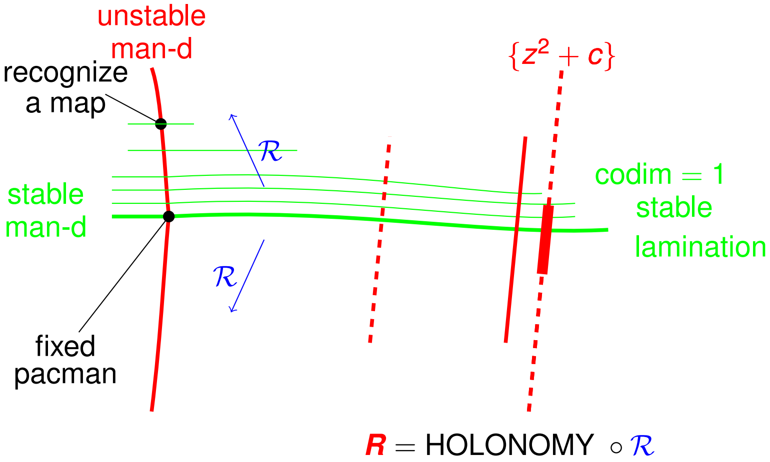
Construct a local leaf, run $\mathcal{R}$

Lyubich and DD: scaling



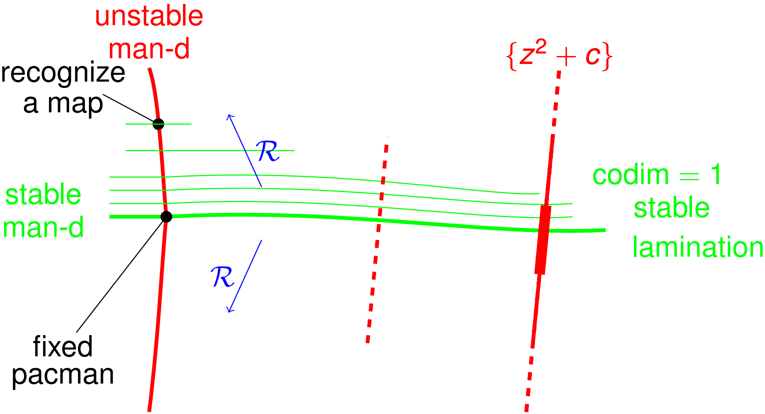
Construct a local leaf, run $\mathcal{R}$

Lyubich and DD: scaling



Construct a local leaf, run $\mathcal{R}$

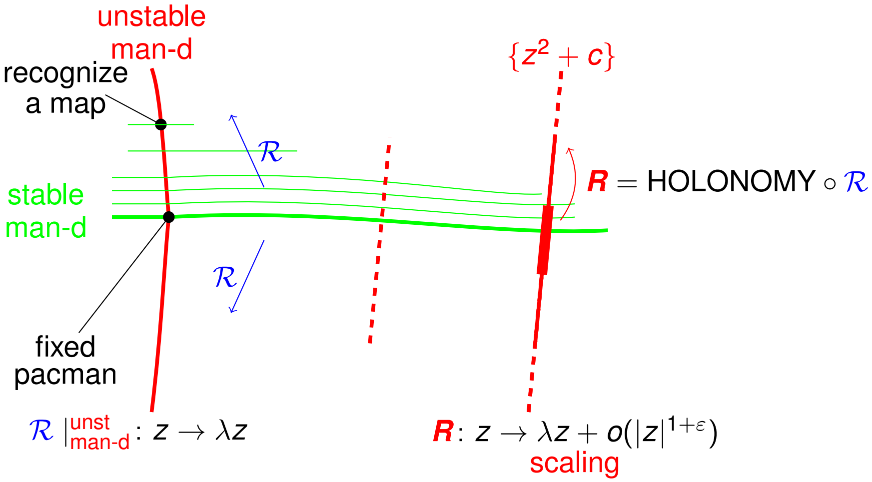
Lyubich and DD: scaling



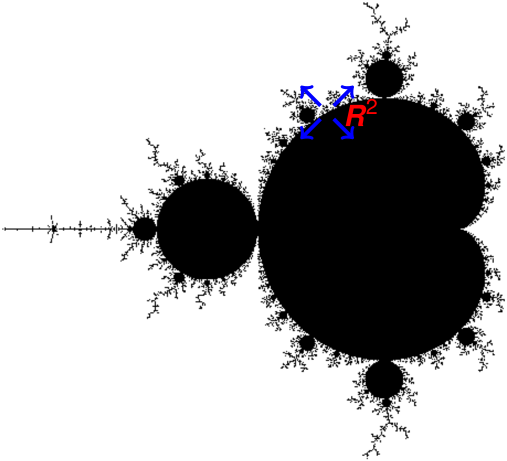
$\boldsymbol{R}$ = HOLONOMY $\circ\, \mathcal{R}$

Construct a local leaf, run $\mathcal{R}$

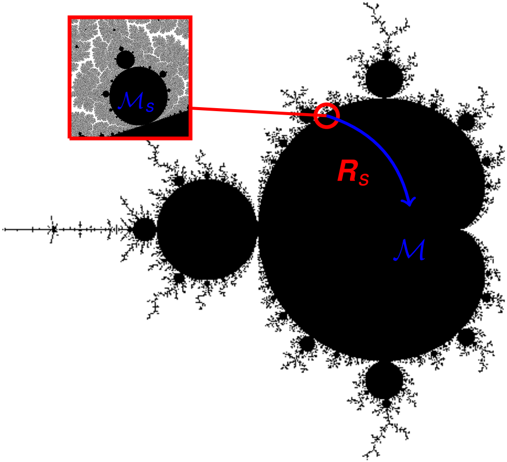
Lyubich and DD: scaling



Cor. Scaling: $\boldsymbol{R}^2(c_\star + c) = c_\star + \lambda c + o(|c|^{1+\varepsilon})$ with $\lambda > 1$ for “certain” c



Cor. Scaling: $\boldsymbol{R}^2(c_\star + c) = c_\star + \lambda c + o(|c|^{1+\varepsilon})$ with $\lambda > 1$ for “certain” c



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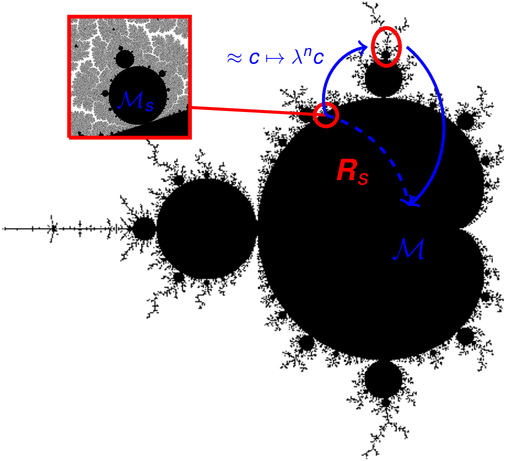
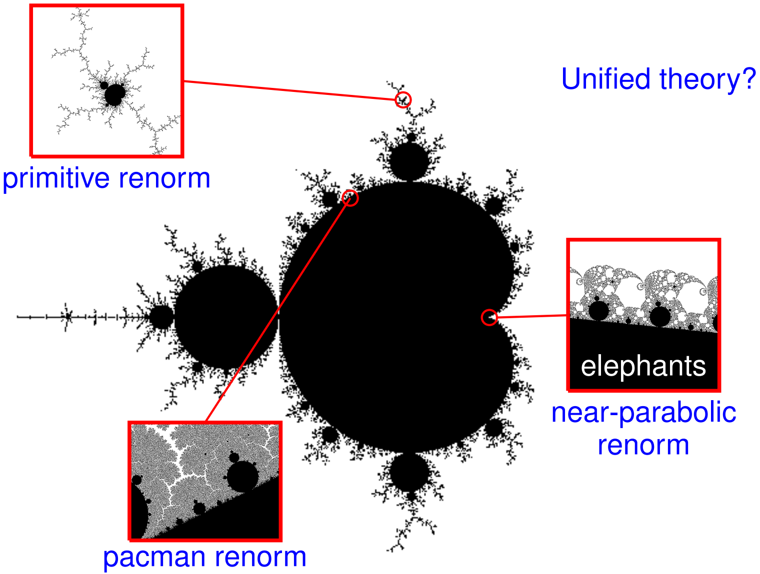
with $\lambda > 1$ for "certain" c



$\bigcap_{n\geq 0} \boldsymbol{R}_s^{-n}(\mathcal{M}) = \{c_s\}$ is a singleton – MLC at c_s


Unified theory?
primitive renorm
elephants
near-parabolic
renorm
pacman renorm