Variational theory of minimal surfaces and applications

Fernando Codá Marques
Princeton University

Stony Brook, October 2014

- Minimal submanifolds are solutions of the most basic variational problem of submanifold geometry, that of extremizing the area.

- Minimal submanifolds are solutions of the most basic variational problem of submanifold geometry, that of extremizing the area.

- Lagrange (1762): question of existence of surfaces of least area having a given closed curve in three-space as boundary.

- Minimal submanifolds are solutions of the most basic variational problem of submanifold geometry, that of extremizing the area.

- Lagrange (1762): question of existence of surfaces of least area having a given closed curve in three-space as boundary.

- The graph of a function $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ extremizes area if

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0.$$

- Minimal submanifolds are solutions of the most basic variational problem of submanifold geometry, that of extremizing the area.

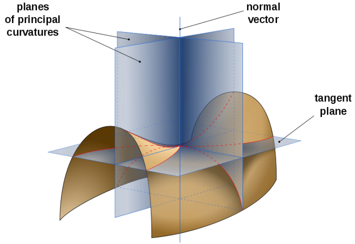
- Lagrange (1762): question of existence of surfaces of least area having a given closed curve in three-space as boundary.

- The graph of a function $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ extremizes area if

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0.$$

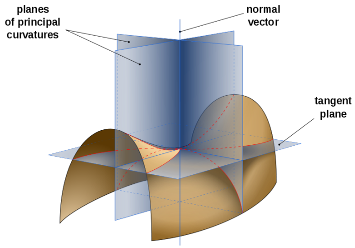
- This is equivalent to the vanishing of the mean curvature (Meusnier).

Let Σ be a two-dimensional oriented surface in $\mathbb{R}^3$, and let N denote a unit normal field.



- The local geometry at a point can be understood in terms of the principal curvatures k_1, k_2: the eigenvalues of the second fundamental form A.

Let Σ be a two-dimensional oriented surface in $\mathbb{R}^3$, and let N denote a unit normal field.



- The local geometry at a point can be understood in terms of the principal curvatures k_1, k_2: the eigenvalues of the second fundamental form A.

- The classical notions of curvature of a surface in three-space are:
  - the mean curvature $H = (k_1 + k_2)/2$,
  - the Gauss curvature $K = k_1 \cdot k_2$.

Let $F : (-\varepsilon, \varepsilon) \times \Sigma \to \mathbb{R}^3$ be a smooth variation of Σ, with $F(0, \cdot) = \mathrm{id}$ and initial velocity $X = \frac{\partial F}{\partial t}(0, \cdot)$.

- The First Variation Formula gives

$$\frac{d}{dt}_{|t=0} \mathrm{area}(\Sigma_t) = -\int_\Sigma \langle \vec{H}, X \rangle \, d\Sigma + \int_{\partial \Sigma} \langle \nu, X \rangle \, ds,$$

where $\Sigma_t = F_t(\Sigma)$, $\vec{H} = H \cdot N$ is the mean curvature vector of Σ in $\mathbb{R}^3$ and ν is the outward unit conormal vector of $\partial\Sigma$.

Let $F : (-\varepsilon, \varepsilon) \times \Sigma \to \mathbb{R}^3$ be a smooth variation of Σ, with $F(0, \cdot) = \mathrm{id}$ and initial velocity $X = \frac{\partial F}{\partial t}(0, \cdot)$.

- The First Variation Formula gives

$$\frac{d}{dt}_{|t=0} \mathrm{area}(\Sigma_t) = -\int_\Sigma \langle \vec{H}, X \rangle \, d\Sigma + \int_{\partial\Sigma} \langle \nu, X \rangle \, ds,$$

  where $\Sigma_t = F_t(\Sigma)$, $\vec{H} = H \cdot N$ is the mean curvature vector of Σ in $\mathbb{R}^3$ and ν is the outward unit conormal vector of $\partial\Sigma$.

- The formula applies to the more general setting of a k-dimensional submanifold Σ immersed in an n-dimensional Riemannian manifold M.

Let $F : (-\varepsilon, \varepsilon) \times \Sigma \to \mathbb{R}^3$ be a smooth variation of Σ, with $F(0, \cdot) = \mathrm{id}$ and initial velocity $X = \frac{\partial F}{\partial t}(0, \cdot)$.

- The First Variation Formula gives

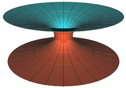
$$\frac{d}{dt}_{|t=0} \mathrm{area}(\Sigma_t) = - \int_\Sigma \langle \vec{H}, X \rangle \, d\Sigma + \int_{\partial\Sigma} \langle \nu, X \rangle \, ds,$$

  where $\Sigma_t = F_t(\Sigma)$, $\vec{H} = H \cdot N$ is the mean curvature vector of Σ in $\mathbb{R}^3$ and ν is the outward unit conormal vector of $\partial\Sigma$.

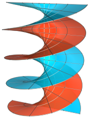
- The formula applies to the more general setting of a k-dimensional submanifold Σ immersed in an n-dimensional Riemannian manifold M.

- We say that $\Sigma^k \subset M^n$ is a *minimal submanifold* if its mean curvature vector vanishes ($\vec{H} = 0$) or, equivalently, if the first derivative of area is zero with respect to any variation that keeps the boundary fixed ($X = 0$ on $\partial\Sigma$).

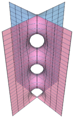
- Some examples in $\mathbb{R}^3$:



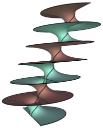
Catenoid



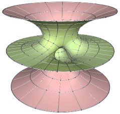
Helicoid



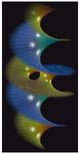
Singly-periodic Scherk



Riemann's example

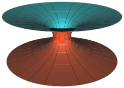


Costa surface

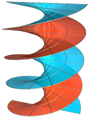


Genus one helicoid

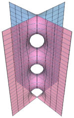
- Some examples in $\mathbb{R}^3$:



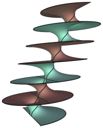
Catenoid



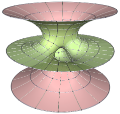
Helicoid



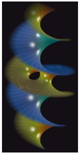
Singly-periodic Scherk



Riemann's example



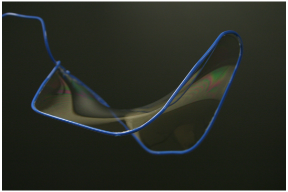
Costa surface



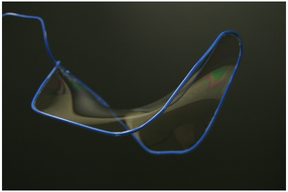
Genus one helicoid

- There are closed minimal surfaces of every genus in $\mathbb{S}^3$ (Lawson).

- Minimal surfaces can be physically represented as soap films. (Joseph Plateau, 19th century)



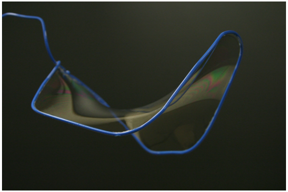
- Minimal surfaces can be physically represented as soap films. (Joseph Plateau, 19th century)



- The Plateau's Problem became a central question in the field, until it was independently solved in 1930 by Douglas and Radó.

  (two-dimensional surfaces as mappings of the unit disk)

- Minimal surfaces can be physically represented as soap films. (Joseph Plateau, 19th century)



- The Plateau's Problem became a central question in the field, until it was independently solved in 1930 by Douglas and Radó.

  (two-dimensional surfaces as mappings of the unit disk)

- Later Morrey extended this existence theory to two-dimensional surfaces in n-dimensional Riemannian manifolds.

- The search for solving the Plateau's problem in greater generality lead to the development of Geometric Measure Theory.

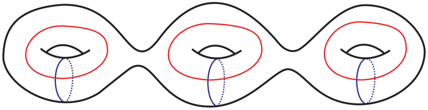
- The search for solving the Plateau's problem in greater generality lead to the development of Geometric Measure Theory.

  Federer and Fleming (1960): introduced integral currents to model k-dimensional domains of integration (good compactness properties).

- The search for solving the Plateau's problem in greater generality lead to the development of Geometric Measure Theory.

  Federer and Fleming (1960): introduced integral currents to model k-dimensional domains of integration (good compactness properties).

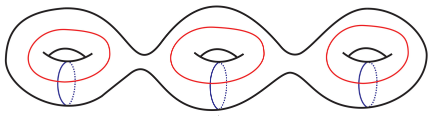
- There is an area-minimizing integral k-current in every nontrivial homology class $\alpha \in H_k(M^n, \mathbb{Z})$, M compact.



- The search for solving the Plateau's problem in greater generality lead to the development of Geometric Measure Theory.

  Federer and Fleming (1960): introduced integral currents to model k-dimensional domains of integration (good compactness properties).

- There is an area-minimizing integral k-current in every nontrivial homology class $\alpha \in H_k(M^n, \mathbb{Z})$, M compact.



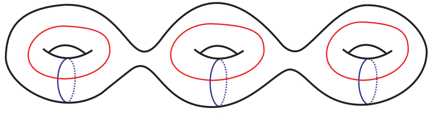
- Regularity (Almgren, De Giorgi, Federer, Fleming, Simons, Bombieri-De Giorgi-Giusti, De Lellis-Spadaro).

  In the case of codimension one, the area minimizing current is smooth outside a singular set of codimension 7.

- The search for solving the Plateau's problem in greater generality lead to the development of Geometric Measure Theory.

  Federer and Fleming (1960): introduced integral currents to model k-dimensional domains of integration (good compactness properties).

- There is an area-minimizing integral k-current in every nontrivial homology class $\alpha \in H_k(M^n, \mathbb{Z})$, M compact.



- Regularity (Almgren, De Giorgi, Federer, Fleming, Simons, Bombieri-De Giorgi-Giusti, De Lellis-Spadaro).

  In the case of codimension one, the area minimizing current is smooth outside a singular set of codimension 7.

- The Simons cone $C = \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x| = |y|\}$ in $\mathbb{R}^8$ is area-minimizing.

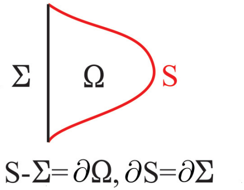
- An important source of area minimizing submanifolds comes from calibration theory (Harvey and Lawson).

- An important source of area minimizing submanifolds comes from calibration theory (Harvey and Lawson).
- A *calibration* of $\Sigma^k \subset M^n$ is a closed k-form ω ($d\omega = 0$) such that
  - $|\omega(e_1, \ldots, e_k)| \leq 1$ for any orthonormal frame $\{e_1, \ldots, e_k\}$ in M,
  - $\omega_\Sigma = \mathrm{vol}_\Sigma$.

- An important source of area minimizing submanifolds comes from calibration theory (Harvey and Lawson).
- A *calibration* of $\Sigma^k \subset M^n$ is a closed k-form ω ($d\omega = 0$) such that
  - $|\omega(e_1, \ldots, e_k)| \leq 1$ for any orthonormal frame $\{e_1, \ldots, e_k\}$ in M,
  - $\omega_\Sigma = \mathrm{vol}_\Sigma$.

If S is homologous to Σ, we have

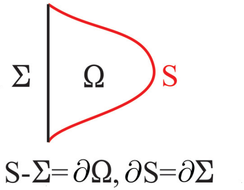
$$\mathrm{vol}(S) \geq \int_S \omega = \int_\Sigma \omega + \int_\Omega d\omega = \mathrm{vol}(\Sigma).$$



- An important source of area minimizing submanifolds comes from calibration theory (Harvey and Lawson).
- A *calibration* of $\Sigma^k \subset M^n$ is a closed k-form ω ($d\omega = 0$) such that
  - $|\omega(e_1, \dots, e_k)| \leq 1$ for any orthonormal frame $\{e_1, \dots, e_k\}$ in M,
  - $\omega_\Sigma = \mathrm{vol}_\Sigma$.

If S is homologous to Σ, we have

$$\mathrm{vol}(S) \geq \int_S \omega = \int_\Sigma \omega + \int_\Omega d\omega = \mathrm{vol}(\Sigma).$$



- Examples include:
  - minimal graphs
  - complex submanifolds in Kähler manifolds
  - special Lagrangian submanifolds in Calabi-Yau manifolds.

Closed minimal surfaces can be constructed by a variety of methods (besides minimizing in a homology class).

Closed minimal surfaces can be constructed by a variety of methods (besides minimizing in a homology class).

- Incompressible minimal surfaces in Riemannian manifolds. (Schoen and Yau, Sacks and Uhlenbeck)
  - minimize energy $E(f) = \int_{\Sigma_g} |df|^2 \, d\mu$ in a homotopy class. This produces a branched minimal immersion $h : \Sigma_g \to M$.

Closed minimal surfaces can be constructed by a variety of methods (besides minimizing in a homology class).

- Incompressible minimal surfaces in Riemannian manifolds. (Schoen and Yau, Sacks and Uhlenbeck)
  - minimize energy $E(f) = \int_{\Sigma_g} |df|^2 \, d\mu$ in a homotopy class. This produces a branched minimal immersion $h : \Sigma_g \to M$.

- Embedded minimal surfaces in three-dimensional manifolds. (Meeks, Simon and Yau)
  - minimize area in an isotopy class (three-manifold topology).

Closed minimal surfaces can be constructed by a variety of methods (besides minimizing in a homology class).

- Incompressible minimal surfaces in Riemannian manifolds. (Schoen and Yau, Sacks and Uhlenbeck)
  - minimize energy $E(f) = \int_{\Sigma_g} |df|^2 \, d\mu$ in a homotopy class. This produces a branched minimal immersion $h : \Sigma_g \to M$.
- Embedded minimal surfaces in three-dimensional manifolds. (Meeks, Simon and Yau)
  - minimize area in an isotopy class (three-manifold topology).
- Minimal two-spheres in compact Riemannian manifolds. (Sacks and Uhlenbeck)
  - the energy E is conformally invariant and the group of conformal transformations of S^2 is noncompact
  - renormalization (or blow-up) technique.
  - Siu-Yau (Frankel conjecture), Micallef-Moore (positive isotropic curvature)

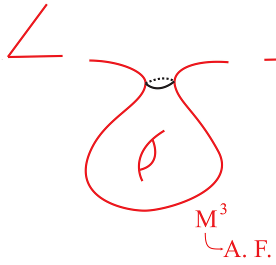
- A celebrated application of minimal hypersurfaces of minimizing type to mathematical physics is the proof of the Positive Mass Conjecture by Schoen and Yau (1979).

- A celebrated application of minimal hypersurfaces of minimizing type to mathematical physics is the proof of the Positive Mass Conjecture by Schoen and Yau (1979).
- Witten (1981) gave a different proof using harmonic spinors.

- A celebrated application of minimal hypersurfaces of minimizing type to mathematical physics is the proof of the Positive Mass Conjecture by Schoen and Yau (1979).
- Witten (1981) gave a different proof using harmonic spinors.
- **Theorem:** The total mass of an isolated gravitational system, modeled by an asymptotically flat spacetime obeying the dominant energy condition, must be positive unless the spacetime is the Minkowski space (of zero mass).

- A celebrated application of minimal hypersurfaces of minimizing type to mathematical physics is the proof of the Positive Mass Conjecture by Schoen and Yau (1979).
- Witten (1981) gave a different proof using harmonic spinors.

- **Theorem:** The total mass of an isolated gravitational system, modeled by an asymptotically flat spacetime obeying the dominant energy condition, must be positive unless the spacetime is the Minkowski space (of zero mass).



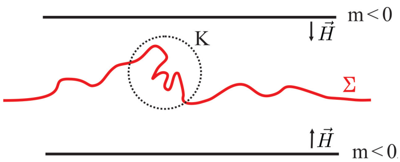
$\mathbb{R}^3$

$$g = \left(1 + \frac{m}{2|x|}\right)^4 \delta + O(|x|^{-2})$$

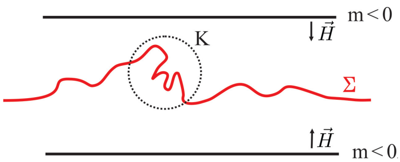
$$R_g \geq 0$$

Thm: m > 0,
unless $M \equiv \mathbb{R}^3$ (flat).

- The proof of Schoen and Yau is by contradiction. If the mass is negative, they construct a complete orientable area-minimizing minimal surface Σ in M.

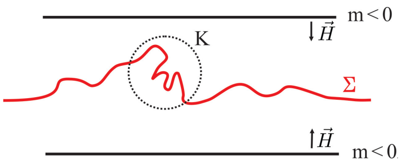


- The proof of Schoen and Yau is by contradiction. If the mass is negative, they construct a complete orientable area-minimizing minimal surface Σ in M.



- Curvature estimates for stable minimal submanifolds are needed in this process (Schoen, Schoen-Simon-Yau).

- The proof of Schoen and Yau is by contradiction. If the mass is negative, they construct a complete orientable area-minimizing minimal surface Σ in M.



- Curvature estimates for stable minimal submanifolds are needed in this process (Schoen, Schoen-Simon-Yau).

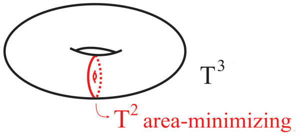
- By the Second Variation Formula, the stability condition gives that

$$\int_\Sigma K_\Sigma f^2 \, d\Sigma \geq \int_\Sigma \frac{1}{2}(R_M + |A|^2) f^2 \, d\Sigma - \int_\Sigma |\nabla f|^2 \, d\Sigma$$

for any smooth function f with compact support in Σ.

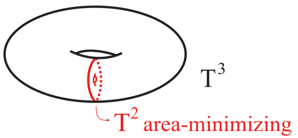
The idea is to exploit the stability inequality and arrive at a contradiction with the Gauss-Bonnet Theorem.

- Similarly, this argument shows that the three-dimensional torus T^3 does not admit a metric of positive scalar curvature.



(Gromov-Lawson, spinorial techniques for T^n)

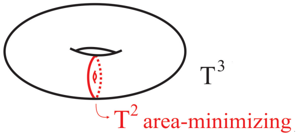
- Similarly, this argument shows that the three-dimensional torus T^3 does not admit a metric of positive scalar curvature.



(Gromov-Lawson, spinorial techniques for T^n)

- The Positive Mass Conjecture for nonspin manifolds in high dimensions remains an open problem.
  - Schoen and Yau: any dimension $3 \leq n \leq 7$
  - Witten: any dimension $n \geq 3$ if manifold is spin (topological condition).

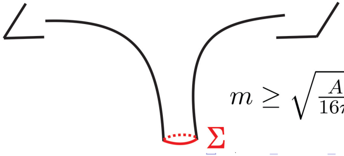
- Similarly, this argument shows that the three-dimensional torus T^3 does not admit a metric of positive scalar curvature.



(Gromov-Lawson, spinorial techniques for T^n)

- The Positive Mass Conjecture for nonspin manifolds in high dimensions remains an open problem.
  - Schoen and Yau: any dimension $3 \leq n \leq 7$
  - Witten: any dimension $n \geq 3$ if manifold is spin (topological condition).
- Minimal surfaces also play a very important role in general relativity by modeling apparent horizons of black holes.

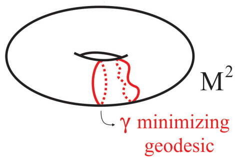
Penrose inequality:
Huisken-Ilmanen, Bray



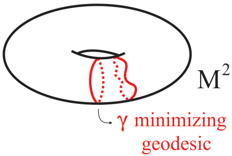
$$m \geq \sqrt{\frac{A}{16\pi}}$$

- A foundational question of Poincaré (1905) asks about the existence of closed geodesics in Riemannian two-spheres.

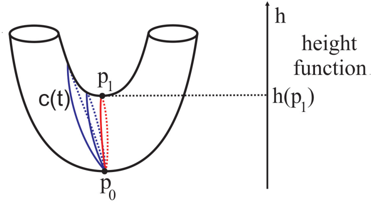
- A foundational question of Poincaré (1905) asks about the existence of closed geodesics in Riemannian two-spheres.
- If the surface has nontrivial genus, a closed geodesic can be found by minimization methods.



- A foundational question of Poincaré (1905) asks about the existence of closed geodesics in Riemannian two-spheres.
- If the surface has nontrivial genus, a closed geodesic can be found by minimization methods.

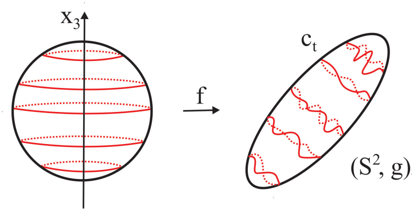


- In 1917, Birkhoff introduced the min-max method to this problem.



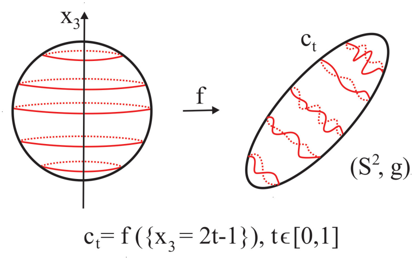
$$h(p_1) = \inf_{c} \sup_{t \in [0,1]} h(c(t))$$

- Birkhoff defined the notion of a sweepout.



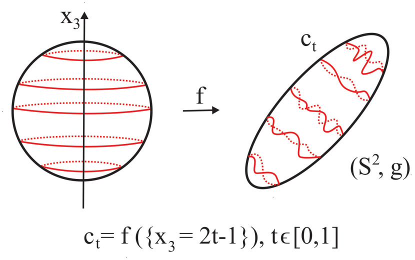
$$c_t = f(\{x_3 = 2t-1\}),\ t \in [0,1]$$

- Birkhoff defined the notion of a sweepout.



- These one-parameter families of curves are topologically nontrivial if we make the requirement that $\deg(f) = 1$.

- Birkhoff defined the notion of a sweepout.



- These one-parameter families of curves are topologically nontrivial if we make the requirement that $\deg(f) = 1$.

- Define
$$L = \min_{\{c_t\}} \max_{t\in[0,1]} L(c_t) \qquad (L\text{=length}).$$

Theorem (Birkhoff): Let (S^2, g) be any Riemannian sphere. Then $L > 0$, and $L = L(\gamma)$ for some smooth closed geodesic γ.

- The work of Birkhoff inspired the development of Morse theory and Lusternik-Schnirelman theory.

- The work of Birkhoff inspired the development of Morse theory and Lusternik-Schnirelman theory.

- In 1929, Lusternik and Schnirelmann proved:

  Three Closed Geodesics Theorem: Let (S^2, g) be any Riemannian sphere. Then there exist at least three distinct simple closed geodesics.

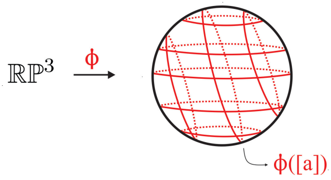
- The work of Birkhoff inspired the development of Morse theory and Lusternik-Schnirelman theory.

- In 1929, Lusternik and Schnirelmann proved:

  Three Closed Geodesics Theorem: Let (S^2, g) be any Riemannian sphere. Then there exist at least three distinct simple closed geodesics.

- The space of unoriented round circles in S^2 can be parametrized by $\mathbb{RP}^3$:

$$\Phi([a_0 : a_1 : a_2 : a_3]) = \{x \in S^2 : a_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 = 0\}.$$



- Finally, in the 1990s, by combining the works of Franks and Bangert (Hingston), the following theorem was proved:

  Theorem: Let (S^2, g) be any Riemannian sphere. Then there exist infinitely many geometrically distinct closed geodesics.

- Finally, in the 1990s, by combining the works of Franks and Bangert (Hingston), the following theorem was proved:

  Theorem: Let (S^2, g) be any Riemannian sphere. Then there exist infinitely many geometrically distinct closed geodesics.

- What about the area functional? How many minimal surfaces does a three-manifold have? This suggests looking for a Morse theory for minimal varieties.

- Finally, in the 1990s, by combining the works of Franks and Bangert (Hingston), the following theorem was proved:

  Theorem: Let (S^2, g) be any Riemannian sphere. Then there exist infinitely many geometrically distinct closed geodesics.

- What about the area functional? How many minimal surfaces does a three-manifold have? This suggests looking for a Morse theory for minimal varieties.

- Almgren computed the homotopy groups of the space $\mathcal{Z}_k(M, \mathbb{Z})$ of k-dimensional integral cycles (integral currents with boundary zero) of M:

$$\pi_l(\mathcal{Z}_k(M, \mathbb{Z}), \{0\}) = H_{k+l}(M, \mathbb{Z}).$$

- A similar result holds for the space $\mathcal{Z}_k(M, \mathbb{Z}_2)$ of modulo 2 flat cycles:

$$\pi_l(\mathcal{Z}_k(M, \mathbb{Z}_2), \{0\}) = H_{k+l}(M, \mathbb{Z}_2).$$

- A similar result holds for the space $\mathcal{Z}_k(M, \mathbb{Z}_2)$ of modulo 2 flat cycles:

$$\pi_l(\mathcal{Z}_k(M, \mathbb{Z}_2), \{0\}) = H_{k+l}(M, \mathbb{Z}_2).$$

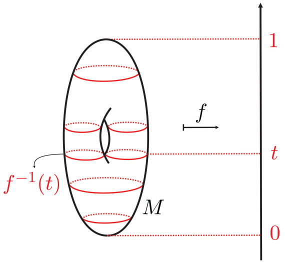
In particular, $\pi_1(\mathcal{Z}_{n-1}(M^n, \mathbb{Z}_2), \{0\}) = H_n(M^n, \mathbb{Z}_2) = \mathbb{Z}_2$.

- A similar result holds for the space $\mathcal{Z}_k(M, \mathbb{Z}_2)$ of modulo 2 flat cycles:

$$\pi_l(\mathcal{Z}_k(M, \mathbb{Z}_2), \{0\}) = H_{k+l}(M, \mathbb{Z}_2).$$

In particular, $\pi_1(\mathcal{Z}_{n-1}(M^n, \mathbb{Z}_2), \{0\}) = H_n(M^n, \mathbb{Z}_2) = \mathbb{Z}_2$.

- Let $f : M \to \mathbb{R}$ be a Morse function, with $f(M) = [0, 1]$.



The sweepout

$$t \in [0, 1] \mapsto \Phi(t) \quad = \quad \partial(\{x \in M : f(x) < t\})$$

generates the fundamental group of $\mathcal{Z}_{n-1}(M^n, \mathbb{Z}_2)$.

- In the 1960s, Almgren devised a very general min-max theory in the context of Geometric Measure Theory. It applied to families of cycles of any dimension and codimension, and any number of parameters.

- In the 1960s, Almgren devised a very general min-max theory in the context of Geometric Measure Theory. It applied to families of cycles of any dimension and codimension, and any number of parameters.
  - existence of a possibly singular minimal variety (stationary integral varifold)

- In the 1960s, Almgren devised a very general min-max theory in the context of Geometric Measure Theory. It applied to families of cycles of any dimension and codimension, and any number of parameters.
  - existence of a possibly singular minimal variety (stationary integral varifold)
- In 1980, Pitts improved the theory by showing that the minimal variety can be chosen to satisfy an additional variational property, the almost minimizing condition.

- In the 1960s, Almgren devised a very general min-max theory in the context of Geometric Measure Theory. It applied to families of cycles of any dimension and codimension, and any number of parameters.
  - existence of a possibly singular minimal variety (stationary integral varifold)
- In 1980, Pitts improved the theory by showing that the minimal variety can be chosen to satisfy an additional variational property, the almost minimizing condition.
  - in codimension one, Pitts used curvature estimates for stable minimal hypersurfaces (Schoen-Simon-Yau) to prove smoothness ($n \leq 6$)

- In the 1960s, Almgren devised a very general min-max theory in the context of Geometric Measure Theory. It applied to families of cycles of any dimension and codimension, and any number of parameters.
  - existence of a possibly singular minimal variety (stationary integral varifold)
- In 1980, Pitts improved the theory by showing that the minimal variety can be chosen to satisfy an additional variational property, the almost minimizing condition.
  - in codimension one, Pitts used curvature estimates for stable minimal hypersurfaces (Schoen-Simon-Yau) to prove smoothness ($n \leq 6$)
- Schoen and Simon (1981) extended regularity to higher dimensions, allowing singular sets of codimension 7.

- In the 1960s, Almgren devised a very general min-max theory in the context of Geometric Measure Theory. It applied to families of cycles of any dimension and codimension, and any number of parameters.
    - existence of a possibly singular minimal variety (stationary integral varifold)
- In 1980, Pitts improved the theory by showing that the minimal variety can be chosen to satisfy an additional variational property, the almost minimizing condition.
    - in codimension one, Pitts used curvature estimates for stable minimal hypersurfaces (Schoen-Simon-Yau) to prove smoothness ($n \leq 6$)
- Schoen and Simon (1981) extended regularity to higher dimensions, allowing singular sets of codimension 7.
- The main application of the Almgren-Pitts Min-Max Theory until very recently was:

  Theorem: Let (M^n, g) be a compact Riemannian manifold, with $3 \leq n \leq 7$. Then there exists a smooth closed embedded minimal hypersurface $\Sigma^{n-1} \subset M^n$.

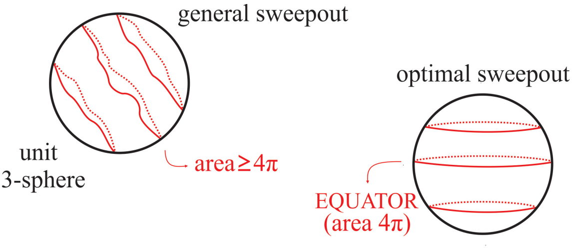
- In general the min-max minimal hypersurface comes as a disjoint collection of connected, embedded closed minimal hypersurfaces with positive integer multiplicities:

$$\Sigma = n_1 \Sigma_1 + \cdots + n_k \Sigma_k.$$

- In general the min-max minimal hypersurface comes as a disjoint collection of connected, embedded closed minimal hypersurfaces with positive integer multiplicities:

$$\Sigma = n_1\Sigma_1 + \cdots + n_k\Sigma_k.$$

- The minimal surface produced by Almgren and Pitts in the unit sphere S^3 is the equator.



The area is 4π and the Morse index is one.

- Min-max methods can also be used to produce branched minimal two-spheres, following the Sacks-Uhlenbeck approach.

- Min-max methods can also be used to produce branched minimal two-spheres, following the Sacks-Uhlenbeck approach.
- Colding and Minicozzi found an application to three-manifold topology and the theory of Ricci flow.

- Min-max methods can also be used to produce branched minimal two-spheres, following the Sacks-Uhlenbeck approach.
- Colding and Minicozzi found an application to three-manifold topology and the theory of Ricci flow.
- In 1982, R. Hamilton introduced the equation

$$\frac{\partial g}{\partial t} = -2Ric_g,$$

known as the Ricci flow, and proved the existence of short time solutions with arbitrary compact Riemannian manifolds as initial conditions.

Singularities should be expected in finite time.

- Min-max methods can also be used to produce branched minimal two-spheres, following the Sacks-Uhlenbeck approach.
- Colding and Minicozzi found an application to three-manifold topology and the theory of Ricci flow.
- In 1982, R. Hamilton introduced the equation

  $$\frac{\partial g}{\partial t} = -2Ric_g,$$

  known as the Ricci flow, and proved the existence of short time solutions with arbitrary compact Riemannian manifolds as initial conditions.

  Singularities should be expected in finite time.
- In three dimensions, the existence of a Ricci flow with surgeries and the study of its properties were accomplished by G. Perelman.

- Min-max methods can also be used to produce branched minimal two-spheres, following the Sacks-Uhlenbeck approach.
- Colding and Minicozzi found an application to three-manifold topology and the theory of Ricci flow.
- In 1982, R. Hamilton introduced the equation

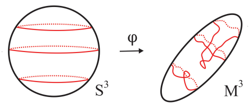
  $$\frac{\partial g}{\partial t} = -2Ric_g,$$

  known as the Ricci flow, and proved the existence of short time solutions with arbitrary compact Riemannian manifolds as initial conditions.

  Singularities should be expected in finite time.
- In three dimensions, the existence of a Ricci flow with surgeries and the study of its properties were accomplished by G. Perelman.

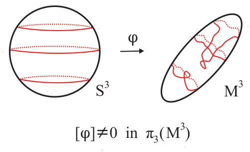
  Perelman proved that any Ricci flow on a homotopy three-sphere must become extinct in finite time (Poincaré conjecture).

- A homotopy three-sphere can be swept out by mappings from S^2, hence minimal spheres can be produced by min-max for the energy functional.



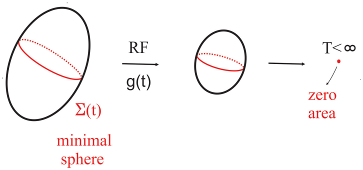
$[\varphi] \neq 0$ in $\pi_3(M^3)$

- A homotopy three-sphere can be swept out by mappings from S^2, hence minimal spheres can be produced by min-max for the energy functional.



- Colding and Minicozzi provided an alternative argument for the finite-time extinction by looking at the evolution equation of the area of these minimal spheres under Ricci flow:

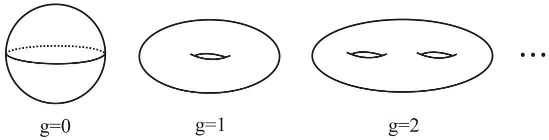
$$\frac{d}{dt}W(g(t)) \leq -4\pi + \frac{3}{4(t+C)}W(g(t)).$$



- Recently, the lecturer and A. Neves have been able to find a connection between the min-max theory of minimal surfaces and the Willmore conjecture (1965).

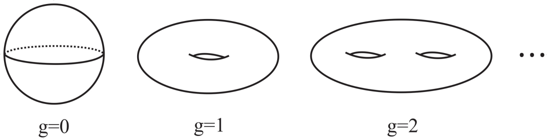
- Recently, the lecturer and A. Neves have been able to find a connection between the min-max theory of minimal surfaces and the Willmore conjecture (1965).

- Let Σ be an abstract closed surface of genus g.



What is the best realization (immersion) of Σ into Euclidean three-space?

- Recently, the lecturer and A. Neves have been able to find a connection between the min-max theory of minimal surfaces and the Willmore conjecture (1965).
- Let Σ be an abstract closed surface of genus g.



What is the best realization (immersion) of Σ into Euclidean three-space?

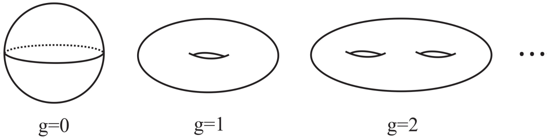
- The Willmore energy of a closed surface $\Sigma \subset \mathbb{R}^3$ is defined to be

$$\mathcal{W}(\Sigma) = \int_\Sigma H^2 \, d\Sigma,$$

where $H = \frac{1}{2}(k_1 + k_2)$ denotes the mean curvature (Germain, 1800s).

- Recently, the lecturer and A. Neves have been able to find a connection between the min-max theory of minimal surfaces and the Willmore conjecture (1965).

- Let Σ be an abstract closed surface of genus g.



  What is the best realization (immersion) of Σ into Euclidean three-space?

- The Willmore energy of a closed surface $\Sigma \subset \mathbb{R}^3$ is defined to be

$$\mathcal{W}(\Sigma) = \int_\Sigma H^2 \, d\Sigma,$$

  where $H = \frac{1}{2}(k_1 + k_2)$ denotes the mean curvature (Germain, 1800s).

  Remarkably, this functional is invariant under any conformal transformation of three-space (Blaschke, Thomsen, 1920s).

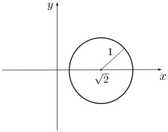
- The Willmore energy of any surface is at least 4π, and equality happens precisely when the surface is a round sphere.

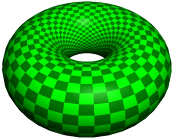
- The Willmore energy of any surface is at least 4π, and equality happens precisely when the surface is a round sphere.

- **Willmore Conjecture (1965)**: *The energy of any torus immersed in three-dimensional space must be at least* $2\pi^2$.

- The Willmore energy of any surface is at least 4π, and equality happens precisely when the surface is a round sphere.

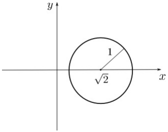
- **Willmore Conjecture (1965)**: *The energy of any torus immersed in three-dimensional space must be at least* $2\pi^2$.
- The conjectured minimizing torus (perfect doughnut) $\Sigma_{\sqrt{2}}$

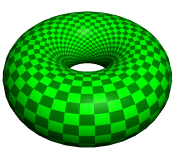




- The Willmore energy of any surface is at least 4π, and equality happens precisely when the surface is a round sphere.

- **Willmore Conjecture (1965)**: *The energy of any torus immersed in three-dimensional space must be at least* $2\pi^2$.
- The conjectured minimizing torus (perfect doughnut) $\Sigma_{\sqrt{2}}$



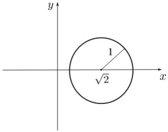


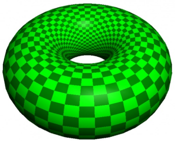
is the stereographic projection of the Clifford torus (minimal surface)

$$\hat{\Sigma} = S^1(\frac{1}{\sqrt{2}}) \times S^1(\frac{1}{\sqrt{2}}) \subset S^3 \subset \mathbb{R}^4.$$

- The Willmore energy of any surface is at least 4π, and equality happens precisely when the surface is a round sphere.

- **Willmore Conjecture (1965)**: *The energy of any torus immersed in three-dimensional space must be at least* $2\pi^2$.
- The conjectured minimizing torus (perfect doughnut) $\Sigma_{\sqrt{2}}$





is the stereographic projection of the Clifford torus (minimal surface)

$$\hat{\Sigma} = S^1(\frac{1}{\sqrt{2}}) \times S^1(\frac{1}{\sqrt{2}}) \subset S^3 \subset \mathbb{R}^4.$$

- By conformal invariance the problem can be phrased in terms of surfaces in the three-sphere S^3, where $W(\Sigma) \geq \mathrm{area}(\Sigma)$ with equality if and only if Σ is a minimal surface.

- There were many partial results towards the conjecture.

  In particular, P. Li and S. T. Yau (1982) proved that the energy of any closed surface with a self-intersection must be at least 8π.

- There were many partial results towards the conjecture.

  In particular, P. Li and S. T. Yau (1982) proved that the energy of any closed surface with a self-intersection must be at least 8π.

- **Theorem** (—, Neves) *Let* $\Sigma \subset S^3$ *be a closed embedded surface, with genus* $g \geq 1$. *Then*

$$\mathcal{W}(\Sigma) \geq 2\pi^2,$$

  and equality holds if and only if the surface Σ *is a conformal image of the Clifford torus.*

- There were many partial results towards the conjecture.

  In particular, P. Li and S. T. Yau (1982) proved that the energy of any closed surface with a self-intersection must be at least 8π.

- **Theorem** (—, Neves) *Let $\Sigma \subset S^3$ be a closed embedded surface, with genus $g \geq 1$. Then*

$$\mathcal{W}(\Sigma) \geq 2\pi^2,$$

  and equality holds if and only if the surface Σ is a conformal image of the Clifford torus.

  This implies the Willmore conjecture is true.

  It also implies that the area of any nonspherical minimal surface in S^3 is at least $2\pi^2$.

- There were many partial results towards the conjecture.

  In particular, P. Li and S. T. Yau (1982) proved that the energy of any closed surface with a self-intersection must be at least 8π.

- **Theorem** (—, Neves) *Let $\Sigma \subset S^3$ be a closed embedded surface, with genus $g \geq 1$. Then*

  $$\mathcal{W}(\Sigma) \geq 2\pi^2,$$

  and equality holds if and only if the surface Σ is a conformal image of the Clifford torus.

  This implies the Willmore conjecture is true.

  It also implies that the area of any nonspherical minimal surface in S^3 is at least $2\pi^2$.

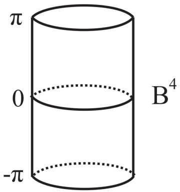
- We considered a new kind of sweepout, a five-parameter family of surfaces in S^3 that allowed us to produce the Clifford torus as a min-max minimal surface.

- For each closed embedded surface $\Sigma \subset S^3$, we construct a *canonical family* of surfaces $\Sigma_{(v,t)} \subset S^3$, where $(v, t) \in B^4 \times (-\pi, \pi)$,

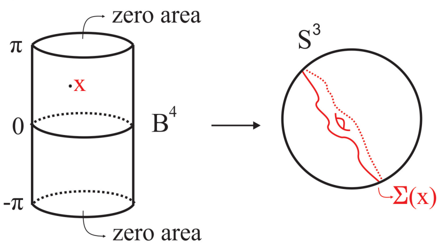
- For each closed embedded surface $\Sigma \subset S^3$, we construct a *canonical family* of surfaces $\Sigma_{(v,t)} \subset S^3$, where $(v, t) \in B^4 \times (-\pi, \pi)$, with the properties that:
  - $\Sigma_{(0,0)} = \Sigma$,
  - $\mathrm{area}(\Sigma_{(v,t)}) \leq \mathcal{W}(\Sigma)$ for every $(v, t) \in B^4 \times (-\pi, \pi)$.

- For each closed embedded surface $\Sigma \subset S^3$, we construct a *canonical family* of surfaces $\Sigma_{(v,t)} \subset S^3$, where $(v, t) \in B^4 \times (-\pi, \pi)$, with the properties that:
  - $\Sigma_{(0,0)} = \Sigma$,
  - $\mathrm{area}(\Sigma_{(v,t)}) \leq \mathcal{W}(\Sigma)$ for every $(v, t) \in B^4 \times (-\pi, \pi)$.
- The estimate follows from an inequality of Ros, Heintze-Karcher.

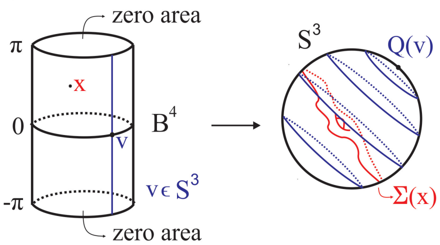
- For each closed embedded surface $\Sigma \subset S^3$, we construct a *canonical family* of surfaces $\Sigma_{(v,t)} \subset S^3$, where $(v, t) \in B^4 \times (-\pi, \pi)$, with the properties that:
  - $\Sigma_{(0,0)} = \Sigma$,
  - $\mathrm{area}(\Sigma_{(v,t)}) \leq \mathcal{W}(\Sigma)$ for every $(v, t) \in B^4 \times (-\pi, \pi)$.
- The estimate follows from an inequality of Ros, Heintze-Karcher.



- For each closed embedded surface $\Sigma \subset S^3$, we construct a *canonical family* of surfaces $\Sigma_{(v,t)} \subset S^3$, where $(v,t) \in B^4 \times (-\pi,\pi)$, with the properties that:
  - $\Sigma_{(0,0)} = \Sigma$,
  - $\mathrm{area}(\Sigma_{(v,t)}) \leq \mathcal{W}(\Sigma)$ for every $(v,t) \in B^4 \times (-\pi,\pi)$.
- The estimate follows from an inequality of Ros, Heintze-Karcher.



- For each closed embedded surface $\Sigma \subset S^3$, we construct a *canonical family* of surfaces $\Sigma_{(v,t)} \subset S^3$, where $(v,t) \in B^4 \times (-\pi,\pi)$, with the properties that:
  - $\Sigma_{(0,0)} = \Sigma$,
  - $\mathrm{area}(\Sigma_{(v,t)}) \leq \mathcal{W}(\Sigma)$ for every $(v,t) \in B^4 \times (-\pi,\pi)$.
- The estimate follows from an inequality of Ros, Heintze-Karcher.



- For each $v \in S^3 = \partial B^4$, $\{\Sigma(v,t)\}$ is the standard family of round spheres centered along the axis passing through some $Q(v) \in S^3$.

- We use the following theorem of Urbano (1990):

  Theorem. *Let* $\Sigma \subset S^3$ *be an immersed, closed minimal surface* $(H = 0)$*, with* $\mathrm{index}(\Sigma) \leq 5$ *and genus* $g \geq 0$*. Then* Σ *is either the Clifford torus (index 5) or the great sphere (index 1).*

- We use the following theorem of Urbano (1990):

  Theorem. *Let $\Sigma \subset S^3$ be an immersed, closed minimal surface ($H = 0$), with $\mathrm{index}(\Sigma) \leq 5$ and genus $g \geq 0$. Then Σ is either the Clifford torus (index 5) or the great sphere (index 1).*

- In order to rule out the possibility of producing great spheres, a new topological ingredient is needed.

- We use the following theorem of Urbano (1990):

  Theorem. *Let* $\Sigma \subset S^3$ *be an immersed, closed minimal surface* $(H = 0)$*, with* $\mathrm{index}(\Sigma) \leq 5$ *and genus* $g \geq 0$*. Then* Σ *is either the Clifford torus (index 5) or the great sphere (index 1).*

- In order to rule out the possibility of producing great spheres, a new topological ingredient is needed.

  The center map $Q : S^3 \to S^3$ has

$$\deg(Q) = \mathrm{genus}(\Sigma)\,!$$

- We use the following theorem of Urbano (1990):

  Theorem. *Let* $\Sigma \subset S^3$ *be an immersed, closed minimal surface* *(*$H = 0$*), with* $\mathrm{index}(\Sigma) \leq 5$ *and genus* $g \geq 0$*. Then* Σ *is either the Clifford torus (index 5) or the great sphere (index 1).*

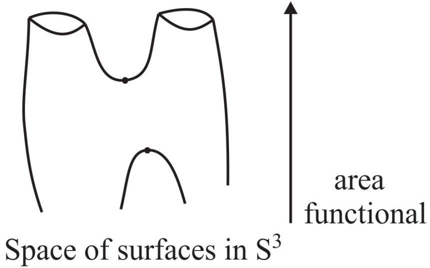
- In order to rule out the possibility of producing great spheres, a new topological ingredient is needed.

  The center map $Q : S^3 \to S^3$ has

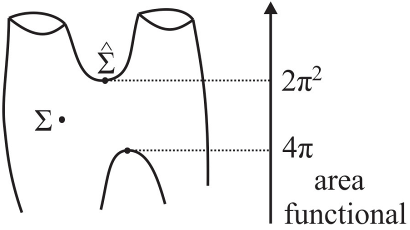
  $$\deg(Q) = \mathrm{genus}(\Sigma)\,!$$

  If $\mathrm{genus}(\Sigma) \geq 1$, the boundary of the cylinder is mapped onto the space of round spheres in a homotopically nontrivial way.

- An illustration of the proof:



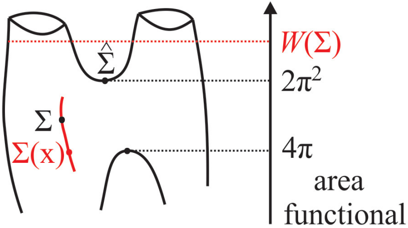
- An illustration of the proof:



Space of surfaces in S^3

$\hat{\Sigma}$ = Clifford torus

- An illustration of the proof:

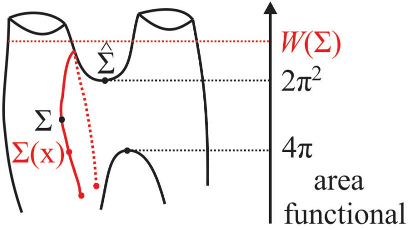


Space of surfaces in S^3

$\hat{\Sigma}$ = Clifford torus

area $\Sigma(x) \leq W(\Sigma)$, $x \in B^5$

- An illustration of the proof:



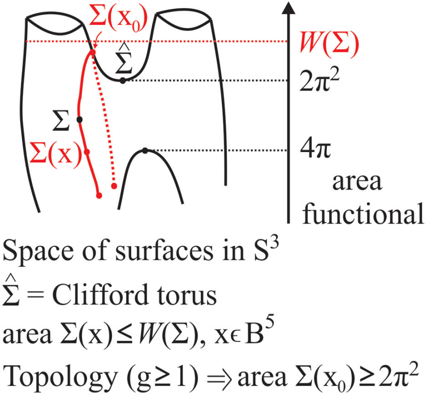
Space of surfaces in S^3

$\hat{\Sigma}$ = Clifford torus

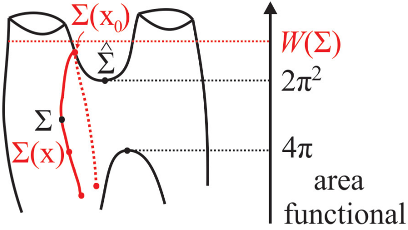
area $\Sigma(x) \leq W(\Sigma)$, $x \in B^5$

Topology ($g \geq 1$)

- An illustration of the proof:



- An illustration of the proof:



Space of surfaces in S^3

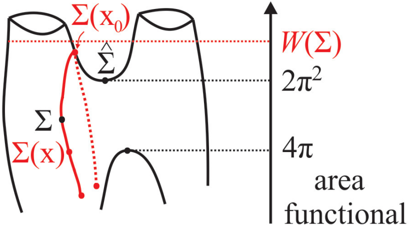
$\hat{\Sigma}$ = Clifford torus

area $\Sigma(x) \leq W(\Sigma)$, $x \in B^5$

Topology ($g \geq 1$) $\Rightarrow$ area $\Sigma(x_0) \geq 2\pi^2$

$W(\Sigma) \geq \text{area}\,\Sigma(x_0) \geq 2\pi^2$

- An illustration of the proof:



Space of surfaces in S^3

$\hat{\Sigma}$ = Clifford torus

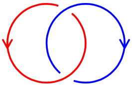
area $\Sigma(x) \leq W(\Sigma)$, $x \in B^5$

Topology $(g \geq 1) \Rightarrow$ area $\Sigma(x_0) \geq 2\pi^2$

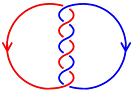
$W(\Sigma) \geq \text{area}\,\Sigma(x_0) \geq 2\pi^2$

(Willmore Conjecture)

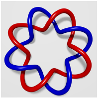
Let γ_1 and γ_2 be two linked curves in $\mathbb{R}^3$ with linking number $\mathrm{lk}(\gamma_1, \gamma_2)$.



$\mathrm{lk} = 1$

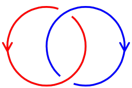


$\mathrm{lk} = 3$

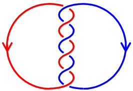


$\mathrm{lk} = 4$

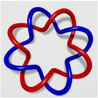
Let γ_1 and γ_2 be two linked curves in $\mathbb{R}^3$ with linking number $\mathrm{lk}(\gamma_1, \gamma_2)$.



$\mathrm{lk}\ = 1$



$\mathrm{lk}\ = 3$



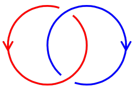
$\mathrm{lk}\ = 4$

- The Möbius cross energy of the link (γ_1, γ_2) is defined to be

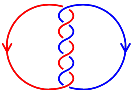
$$E(\gamma_1, \gamma_2) = \int_{S^1 \times S^1} \frac{|\gamma_1'(s)||\gamma_2'(t)|}{|\gamma_1(s) - \gamma_2(t)|^2}\, ds\, dt.$$

The energy $E(\gamma_1, \gamma_2)$ is conformally invariant.

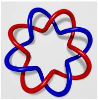
Let γ_1 and γ_2 be two linked curves in $\mathbb{R}^3$ with linking number $\mathrm{lk}(\gamma_1, \gamma_2)$.



$\mathrm{lk} = 1$



$\mathrm{lk} = 3$



$\mathrm{lk} = 4$

- The Möbius cross energy of the link (γ_1, γ_2) is defined to be

$$E(\gamma_1, \gamma_2) = \int_{S^1 \times S^1} \frac{|\gamma_1'(s)||\gamma_2'(t)|}{|\gamma_1(s) - \gamma_2(t)|^2} \, ds \, dt.$$

  The energy $E(\gamma_1, \gamma_2)$ is conformally invariant.

- We have that $E(\gamma_1, \gamma_2) \geq 4\pi |\mathrm{lk}\,(\gamma_1, \gamma_2)|$, by the Gauss formula:

$$\mathrm{lk}(\gamma_1, \gamma_2) = \frac{1}{4\pi} \int_{S^1 \times S^1} \frac{\det(\gamma_1'(s), \gamma_2'(t), \gamma_1(s) - \gamma_2(t))}{|\gamma_1(s) - \gamma_2(t)|^3} \, ds \, dt.$$

  It is natural to search for the optimal configuration in the case of nontrivial links.

- **Freedman-He-Wang Conjecture (1994):** *The energy of any 2-component link in* $\mathbb{R}^3$ *with linking number equal to* ± 1 *is at least* $2\pi^2$*.*

- **Freedman-He-Wang Conjecture (1994):** *The energy of any 2-component link in* $\mathbb{R}^3$ *with linking number equal to* ± 1 *is at least* $2\pi^2$*.*

  If true, every non-trivial link has $E(\gamma_1, \gamma_2) \geq 2\pi^2$ (He).

- **Freedman-He-Wang Conjecture (1994):** *The energy of any 2-component link in* $\mathbb{R}^3$ *with linking number equal to* ± 1 *is at least* $2\pi^2$*.*

  If true, every non-trivial link has $E(\gamma_1, \gamma_2) \geq 2\pi^2$ (He).

- **Theorem** (Agol, — , Neves): *The conjecture is true.*

  If equality holds, then (γ_1, γ_2) *is conformal to the standard Hopf link in* S^3*:*

$$\beta_1(t) = (\cos t, \sin t, 0, 0) \quad \textit{and} \quad \beta_2(s) = (0, 0, \cos s, \sin s).$$

- **Freedman-He-Wang Conjecture (1994):** *The energy of any 2-component link in $\mathbb{R}^3$ with linking number equal to ± 1 is at least $2\pi^2$.*

  If true, every non-trivial link has $E(\gamma_1, \gamma_2) \geq 2\pi^2$ (He).

- **Theorem** (Agol, — , Neves): *The conjecture is true.*

  If equality holds, then (γ_1, γ_2) is conformal to the standard Hopf link in S^3:

$$\beta_1(t) = (\cos t, \sin t, 0, 0) \quad \textit{and} \quad \beta_2(s) = (0, 0, \cos s, \sin s).$$

- The Gauss map $g : S^1 \times S^1 \to S^3$ of a link $(\gamma_1, \gamma_2) \subset S^3$, given by

$$g(s,t) = \frac{\gamma_1(s) - \gamma_2(t)}{|\gamma_1(s) - \gamma_2(t)|},$$

  satisfies $\mathrm{area}(g(S^1 \times S^1)) \leq E(\gamma_1, \gamma_2)$.

- **Freedman-He-Wang Conjecture (1994):** *The energy of any 2-component link in $\mathbb{R}^3$ with linking number equal to ± 1 is at least $2\pi^2$.*

  If true, every non-trivial link has $E(\gamma_1, \gamma_2) \geq 2\pi^2$ (He).

- **Theorem** (Agol, — , Neves): *The conjecture is true.*

  If equality holds, then (γ_1, γ_2) is conformal to the standard Hopf link in S^3:

$$\beta_1(t) = (\cos t, \sin t, 0, 0) \quad \textit{and} \quad \beta_2(s) = (0, 0, \cos s, \sin s).$$

- The Gauss map $g : S^1 \times S^1 \to S^3$ of a link $(\gamma_1, \gamma_2) \subset S^3$, given by

$$g(s,t) = \frac{\gamma_1(s) - \gamma_2(t)}{|\gamma_1(s) - \gamma_2(t)|},$$

  satisfies $\mathrm{area}(g(S^1 \times S^1)) \leq E(\gamma_1, \gamma_2)$.

- We construct a 5-parameter family of surfaces in S^3 with area bounded above by $E(\gamma_1, \gamma_2)$, and such that the associated center map $Q : S^3 \to S^3$ satisfies $|\deg(Q)| = 1$ if $|\mathrm{lk}(\gamma_1, \gamma_2)| = 1$.

- **Yau's Conjecture** (1982): Every compact Riemannian three-manifold admits an infinite number of smooth, closed, immersed minimal surfaces.

- **Yau's Conjecture** (1982): Every compact Riemannian three-manifold admits an infinite number of smooth, closed, immersed minimal surfaces.
- **Theorem** (—, Neves) *Let M^{n+1} be a compact Riemannian manifold with positive Ricci curvature, $3 \leq (n+1) \leq 7$. Then there exist infinitely many distinct closed embedded minimal hypersurfaces in M.*

- **Yau's Conjecture** (1982): Every compact Riemannian three-manifold admits an infinite number of smooth, closed, immersed minimal surfaces.
- **Theorem** (—, Neves) *Let M^{n+1} be a compact Riemannian manifold with positive Ricci curvature, $3 \leq (n+1) \leq 7$. Then there exist infinitely many distinct closed embedded minimal hypersurfaces in M.*

  Kahn and Markovic (hyperbolic three-manifolds)
  Kapouleas (desingularizing approach)

- **Yau's Conjecture** (1982): Every compact Riemannian three-manifold admits an infinite number of smooth, closed, immersed minimal surfaces.
- **Theorem** (—, Neves) *Let M^{n+1} be a compact Riemannian manifold with positive Ricci curvature, $3 \leq (n+1) \leq 7$. Then there exist infinitely many distinct closed embedded minimal hypersurfaces in M.*

  Kahn and Markovic (hyperbolic three-manifolds)
  Kapouleas (desingularizing approach)
- By Almgren,

$$\pi_k(\mathcal{Z}_n(M, \mathbb{Z}_2), \{0\}) = H_{k+n}(M^{n+1}, \mathbb{Z}_2).$$

- **Yau's Conjecture** (1982): Every compact Riemannian three-manifold admits an infinite number of smooth, closed, immersed minimal surfaces.
- **Theorem** (—, Neves) *Let M^{n+1} be a compact Riemannian manifold with positive Ricci curvature, $3 \leq (n+1) \leq 7$. Then there exist infinitely many distinct closed embedded minimal hypersurfaces in M.*

  Kahn and Markovic (hyperbolic three-manifolds)
  Kapouleas (desingularizing approach)
- By Almgren,

$$\pi_k(\mathcal{Z}_n(M, \mathbb{Z}_2), \{0\}) = H_{k+n}(M^{n+1}, \mathbb{Z}_2).$$

  All higher homotopy groups vanish but the first one:

$$\pi_1(\mathcal{Z}_n(M, \mathbb{Z}_2)) = \mathbb{Z}_2.$$

- **Yau's Conjecture** (1982): Every compact Riemannian three-manifold admits an infinite number of smooth, closed, immersed minimal surfaces.
- **Theorem** (—, Neves) *Let M^{n+1} be a compact Riemannian manifold with positive Ricci curvature, $3 \leq (n+1) \leq 7$. Then there exist infinitely many distinct closed embedded minimal hypersurfaces in M.*

  Kahn and Markovic (hyperbolic three-manifolds)
  Kapouleas (desingularizing approach)
- By Almgren,

$$\pi_k(\mathcal{Z}_n(M, \mathbb{Z}_2), \{0\}) = H_{k+n}(M^{n+1}, \mathbb{Z}_2).$$

  All higher homotopy groups vanish but the first one:

$$\pi_1(\mathcal{Z}_n(M, \mathbb{Z}_2)) = \mathbb{Z}_2.$$

- There are nontrivial families parametrized by projective spaces.

- Let $f : M^{n+1} \to \mathbb{R}$ be a Morse function. Define

$$\psi : \mathbb{RP}^k \to \mathcal{Z}_n(M, \mathbb{Z}_2)$$

by

$$\psi([a_0 : \cdots : a_k]) = \{x \in M : p_a(f(x)) = 0\}.$$

where $p_a(t) = a_0 + a_1 t + \cdots + a_k t^k$.

- Let $f : M^{n+1} \to \mathbb{R}$ be a Morse function. Define

  $$\psi : \mathbb{RP}^k \to \mathcal{Z}_n(M, \mathbb{Z}_2)$$

  by

  $$\psi([a_0 : \cdots : a_k]) = \{x \in M : p_a(f(x)) = 0\}.$$

  where $p_a(t) = a_0 + a_1 t + \cdots + a_k t^k$.
- Note that $\psi([a_0 : a_1 : 0 : \cdots : 0]) = \{x \in M : f(x) = -a_0/a_1\}$ reproduces a one-parameter sweepout of M.

- Let $f : M^{n+1} \to \mathbb{R}$ be a Morse function. Define

$$\psi : \mathbb{RP}^k \to \mathcal{Z}_n(M, \mathbb{Z}_2)$$

by

$$\psi([a_0 : \cdots : a_k]) = \{x \in M : p_a(f(x)) = 0\}.$$

where $p_a(t) = a_0 + a_1 t + \cdots + a_k t^k$.

- Note that $\psi([a_0 : a_1 : 0 : \cdots : 0]) = \{x \in M : f(x) = -a_0/a_1\}$ reproduces a one-parameter sweepout of M.
- (Gromov) Let $\bar{\lambda} \in H^1(\mathcal{Z}_n(M, \mathbb{Z}_2), \mathbb{Z}_2) = \mathbb{Z}_2$ be the generator. Then

$$\lambda = \psi^*(\bar{\lambda})$$

is the generator of $H^1(\mathbb{RP}^k, \mathbb{Z}_2) = \mathbb{Z}_2$.

- Let $f : M^{n+1} \to \mathbb{R}$ be a Morse function. Define

$$\psi : \mathbb{RP}^k \to \mathcal{Z}_n(M, \mathbb{Z}_2)$$

by

$$\psi([a_0 : \cdots : a_k]) = \{x \in M : p_a(f(x)) = 0\}.$$

where $p_a(t) = a_0 + a_1 t + \cdots + a_k t^k$.

- Note that $\psi([a_0 : a_1 : 0 : \cdots : 0]) = \{x \in M : f(x) = -a_0/a_1\}$ reproduces a one-parameter sweepout of M.
- (Gromov) Let $\bar{\lambda} \in H^1(\mathcal{Z}_n(M, \mathbb{Z}_2), \mathbb{Z}_2) = \mathbb{Z}_2$ be the generator. Then

$$\lambda = \psi^*(\bar{\lambda})$$

is the generator of $H^1(\mathbb{RP}^k, \mathbb{Z}_2) = \mathbb{Z}_2$. Hence

$$\lambda^k \neq 0 \in H^k(\mathbb{RP}^k, \mathbb{Z}_2) = \mathbb{Z}_2,$$

where $\lambda^k = \lambda \smile \cdots \smile \lambda$ (k-th cup power). Such maps are called *k-sweepouts.*

- We denote by $\mathcal{P}_k$ the space of all k-sweepouts, and define the min-max invariant:

$$\omega_k(M) := \inf_{\Phi \in \mathcal{P}_k} \sup_{x \in \mathrm{dmn}(\Phi)} \mathrm{area}(\Phi(x)),$$

where $\mathrm{dmn}(\Phi)$ stands for the domain of Φ.

- We denote by $\mathcal{P}_k$ the space of all k-sweepouts, and define the min-max invariant:

$$\omega_k(M) := \inf_{\Phi \in \mathcal{P}_k} \sup_{x \in \mathrm{dmn}(\Phi)} \mathrm{area}(\Phi(x)),$$

  where $\mathrm{dmn}(\Phi)$ stands for the domain of Φ.
- The numbers $\omega_k(M)$ satisfy (Gromov, Guth):

$$C_1 k^{\frac{1}{n+1}} \leq \omega_k(M) \leq C_2 k^{\frac{1}{n+1}},$$

  for constants $C_1, C_2 > 0$ that depend only on M.

- We denote by $\mathcal{P}_k$ the space of all k-sweepouts, and define the min-max invariant:

$$\omega_k(M) := \inf_{\Phi \in \mathcal{P}_k} \sup_{x \in \mathrm{dmn}(\Phi)} \mathrm{area}(\Phi(x)),$$

  where $\mathrm{dmn}(\Phi)$ stands for the domain of Φ.
- The numbers $\omega_k(M)$ satisfy (Gromov, Guth):

$$C_1 k^{\frac{1}{n+1}} \leq \omega_k(M) \leq C_2 k^{\frac{1}{n+1}},$$

  for constants $C_1, C_2 > 0$ that depend only on M.
- The condition of positive Ricci curvature implies a "Frankel property": any two embedded minimal hypersurfaces must intersect.

- We denote by $\mathcal{P}_k$ the space of all k-sweepouts, and define the min-max invariant:

$$\omega_k(M) := \inf_{\Phi \in \mathcal{P}_k} \sup_{x \in \mathrm{dmn}(\Phi)} \mathrm{area}(\Phi(x)),$$

  where $\mathrm{dmn}(\Phi)$ stands for the domain of Φ.
- The numbers $\omega_k(M)$ satisfy (Gromov, Guth):

$$C_1 k^{\frac{1}{n+1}} \leq \omega_k(M) \leq C_2 k^{\frac{1}{n+1}},$$

  for constants $C_1, C_2 > 0$ that depend only on M.
- The condition of positive Ricci curvature implies a "Frankel property": any two embedded minimal hypersurfaces must intersect.

  Hence $\omega_k(M) = n_k \cdot \mathrm{area}(\Sigma_k)$, where $n_k \in \mathbb{N}$ and Σ_k is a minimal hypersurface.

- We denote by $\mathcal{P}_k$ the space of all k-sweepouts, and define the min-max invariant:

$$\omega_k(M) := \inf_{\Phi \in \mathcal{P}_k} \sup_{x \in \mathrm{dmn}(\Phi)} \mathrm{area}(\Phi(x)),$$

where $\mathrm{dmn}(\Phi)$ stands for the domain of Φ.

- The numbers $\omega_k(M)$ satisfy (Gromov, Guth):

$$C_1 k^{\frac{1}{n+1}} \leq \omega_k(M) \leq C_2 k^{\frac{1}{n+1}},$$

for constants $C_1, C_2 > 0$ that depend only on M.

- The condition of positive Ricci curvature implies a "Frankel property": any two embedded minimal hypersurfaces must intersect.

Hence $\omega_k(M) = n_k \cdot \mathrm{area}(\Sigma_k)$, where $n_k \in \mathbb{N}$ and Σ_k is a minimal hypersurface.

- We combine Lusternik-Schnirelmann ideas with counting arguments to derive a contradiction with the sublinear growth of $\omega_k(M)$ if there are only finitely many closed minimal hypersurfaces.

- An important open problem in this min-max theory consists in relating the Morse index of the min-max minimal surface to the number of parameters. This is a subtle question because of the phenomenon of multiplicity. (X. Zhou: when $k = 1$, $Ric > 0$, $3 \leq n \leq 7$).

- An important open problem in this min-max theory consists in relating the Morse index of the min-max minimal surface to the number of parameters. This is a subtle question because of the phenomenon of multiplicity. (X. Zhou: when $k = 1$, $Ric > 0$, $3 \leq n \leq 7$).
- The problem of controlling the topology of the min-max minimal surface has been studied in the three-dimensional case (Simon-Smith, Colding and De Lellis, De Lellis and Pellandini, Ketover).

- An important open problem in this min-max theory consists in relating the Morse index of the min-max minimal surface to the number of parameters. This is a subtle question because of the phenomenon of multiplicity. (X. Zhou: when $k = 1$, $Ric > 0$, $3 \leq n \leq 7$).
- The problem of controlling the topology of the min-max minimal surface has been studied in the three-dimensional case (Simon-Smith, Colding and De Lellis, De Lellis and Pellandini, Ketover).
- Gromov has proposed to consider the sequence $\{\omega_k(M)\}_{k\in\mathbb{N}}$ as a nonlinear analogue of the Laplace spectrum of M. The asymptotic behavior of the area of nodal sets of eigenfunctions has been conjectured by Yau.

- An important open problem in this min-max theory consists in relating the Morse index of the min-max minimal surface to the number of parameters. This is a subtle question because of the phenomenon of multiplicity. (X. Zhou: when $k = 1$, $Ric > 0$, $3 \leq n \leq 7$).
- The problem of controlling the topology of the min-max minimal surface has been studied in the three-dimensional case (Simon-Smith, Colding and De Lellis, De Lellis and Pellandini, Ketover).
- Gromov has proposed to consider the sequence $\{\omega_k(M)\}_{k \in \mathbb{N}}$ as a nonlinear analogue of the Laplace spectrum of M. The asymptotic behavior of the area of nodal sets of eigenfunctions has been conjectured by Yau.
- We conjecture that under generic conditions the minimal hypersurfaces Σ_k we have produced should have index k, multiplicity one and should become equidistributed in space.