K3 surfaces and their punctual Hilbert schemes

(joint with Brendan Hassett)

Examples:

- $S_2 : w^2 = f_6(x, y, z) \subset \mathbb{P}(1, 1, 1, 3)$
- $S_4 \subset \mathbb{P}^3$, quartic
- $S_8 \subset \mathbb{P}^5$, intersection of 3 quadrics
- $S = \widetilde{A/\pm}$, a Kummer surface

Main questions (for us)

We work mostly over nonclosed fields:

$$k = \mathbb{F}_p, \quad \mathbb{Q}_p, \quad \mathbb{Q} \text{ or } \mathbb{C}((t)), \quad \mathbb{C}(t)$$

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- Zariski density of k-rational points
- Existence of rational curves and their interaction with rational points

Geometric invariants

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The cone of pseudo-effective divisors is generated by

$$\{D \in \mathrm{Pic}(S) \,|\, (D, D) \geq -2, \quad (D, f) > 0\}.$$

This cone encodes the geometry of S: fibrations, automorphisms.

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- If $\rho = 1$ then $\mathrm{Aut}(S)$ is finite.
- If $\rho = 2$ and there exists a class of square zero or square -2 then $\mathrm{Aut}(S)$ is finite.

Torelli

Let (S, f) be a polarized K3 surface.

- Pick an $\sigma \in H^0(S, \Omega_S^2)$, it is unique up to scalars.
- Choose a basis $\gamma_1, \ldots, \gamma_{22} \in H_2(S, \mathbb{Z})$.

The period is given by

$$(\int_{\gamma_1} \sigma, \ldots, \int_{\gamma_{22}} \sigma).$$

A K3 surface is **uniquely** determined by its period, i.e., given polarized K3 surfaces $(S, f), (S', f')$ and an isomorphism of lattices

$$\phi : H_2(S, \mathbb{Z}) \to H_2(S', \mathbb{Z}), \quad \text{with} \quad \phi(f) = f',$$

and inducing equality of periods (up to scalars), there exists a unique isomorphism $S \to S'$ of surfaces inducing ϕ.

Applications

- Description of automorphisms
- Lattice-polarized K3 surfaces
- Relation to abelian varieties (Kuga-Satake construction)

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Over $\bar{\mathbb{Q}}$, there are *ad hoc* methods, e.g.

van Luijk

Let X be a K3 surface over $\mathbb{Q}$. If

$$\rho(\bar{X}_{p_1}) = \rho(\bar{X}_{p_2}) = 2, \quad \operatorname{disc}(\operatorname{Pic}(\bar{X}_{p_1})) \neq \operatorname{disc}(\operatorname{Pic}(\bar{X}_{p_2})),$$

for primes $p_1 \neq p_2$, then

$$\operatorname{Pic}(\bar{X}) \simeq \mathbb{Z}.$$

Computing Pic(X)

Hassett-Kresch-T. 2012

Let X be a degree 2 K3 surface over a number field k. There exists an algorithm, with *a priori* bounded running time, to compute

- $\mathrm{Pic}(X_{\bar{k}})$ and its generators,
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Main Issue: The moduli space of degree $2d$-polarized K3s is $\mathcal{D}_{2d}/\Gamma_{2d}$, where $\mathcal{D}_{2d}$ is a bounded symmetric domain and Γ_{2d} a discrete group. It is quasi-projective, by Bailey-Borel.

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Jumping Picard ranks

Let $E = E_X$ be the endomorphism algebra of $T(X) := \mathrm{Pic}(X)^{\perp}$, it is either a totally real field or a CM-field.

F. Charles 2011

We have

$$\rho(\bar{X}_p) \geq \rho(\bar{X}) + \eta(\bar{X}) := \begin{cases} 0 & \text{if } E \text{ is CM or } \dim_E(T(X)) \text{ even} \\ [E:\mathbb{Q}] & \text{otherwise} \end{cases}$$

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Is this set infinite? Our main concern is the case $\rho(\bar{X}) = 2$.

Jumping Picard ranks: Kummer surfaces

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If $A = C_1 \times C_2$, a product of two elliptic curves, then

- $\rho(\bar{X}) \geq 18$
- $\rho(\bar{X}) \geq 19$, if $C_1 \sim C_2$,
- $\rho(\bar{X}) \geq 20$, if in addition, C_1 has complex multiplication

Jumping Picard ranks: Kummer surfaces

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- Even when $C_1 \not\sim C_2$, $\Pi_{\text{jump}}(X)$ is infinite (F. Charles 2014).

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- Even when $C_1 \not\sim C_2$, $\Pi_{\text{jump}}(X)$ is infinite (F. Charles 2014).
- Not known for A absolutely simple.

Numerical experiments (E. Costa 2014)

What can be said about

$$\gamma(X,B) := \frac{\#\{p \le B \mid p \in \Pi_{\text{jump}}(X)\}}{\#\{p \le B\}}, \quad \text{as} \quad B \to \infty?$$

When $\rho(\bar{X}) = 1$ and $E_X = \mathbb{Q}$ we observe $\gamma(X,B) \sim c/\sqrt{B}$.

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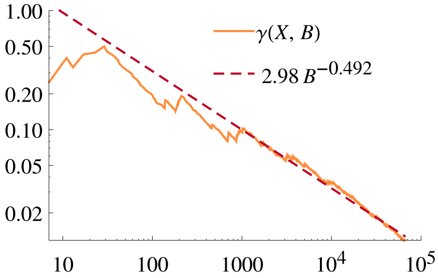
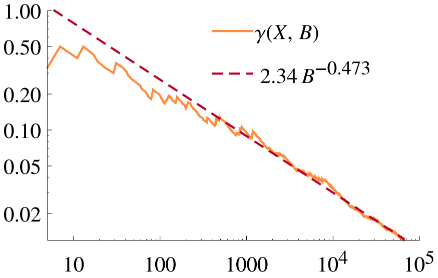
Example 1



Example 2



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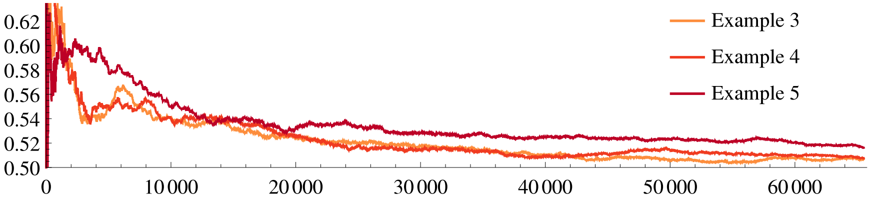
When $\rho(X) = \rho(\bar{X}) = 2$ we observe slow convergence $\gamma(X, B) \to 1/2$.



Jumping Picard ranks: applications

Combining Bogomolov-Hassett-T. (2009) and Li-Liedtke (2010):

Assume that $\Pi_{\mathrm{jump}}(X)$ is infinite, for K3 surfaces X over a number field with $\rho(\bar{X}) = 2, 4$. Then every K3 surface over any algebraically closed field contains infinitely many rational curves.

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- by duality, have

$$H_2(F, \mathbb{Z}) = H_2(S, \mathbb{Z}) \oplus_\perp \mathbb{Z}\delta^\vee, \quad (\delta^\vee, \delta^\vee) = -1/2.$$

Effective curves conjecture

Hassett-T. (1999)

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- A square-zero class on the boundary gives rise to an abelian fibration $F \to \mathbb{P}^2$ (Markman 2013, under some assumptions, Bayer-Macri 2013 for Bridgeland moduli spaces).

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- A square-zero class on the boundary gives rise to an abelian fibration $F \to \mathbb{P}^2$ (Markman 2013, under some assumptions, Bayer-Macri 2013 for Bridgeland moduli spaces).
- A (-2)-class corresponds to a family of rational curves parametrized by a K3 surface, which gets blown down to a rational double point
- A (-10)-class corresponds to a family of lines contained in a plane which gets contracted to a point.

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- The ample cone is at most as large as predicted (2009). This relies on the Torelli theorem by Verbitsky, on results of Markman concerning monodromy, and on Boucksom, Druel.

Example

Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold.

Beauville-Donagi 1985

The Fano variety of lines $F = F(X)$ is a holomorphic symplectic variety deformation equivalent to $S^{[2]}$, for some K3 surface S.

Assume that X contains a cubic scroll, or equivalently, a cubic hyperplane section $Y \subset X$ with 6 double points, in general position. Then $F(Y)$ has 3 components: planes Π, Π' and a cubic surface.

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The Beauville-Bogomolov form on $\mathrm{Pic}(F)$ restricts to

$$J_{12} := \begin{array}{c|cc} & g & \tau \\ \hline g & 6 & 6 \\ \tau & 6 & 2 \end{array}$$

of discriminant -24.

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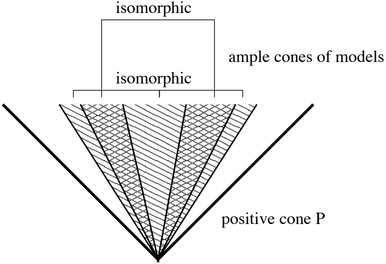
- α_1 (resp. α_1') induces a contraction $F \to \hat{F}$ taking the Lagrangian plane Π (resp. Π') to a point.

The effective cone



The partition of the effective cone into ample cones for isomorphism classes of minimal models.

Variety of lines of a cubic fourfold

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Let $\ell \in F$ be a general line on X. There exists a unique plane $\Pi = \Pi_\ell \subset \mathbb{P}^5$ which is tangent to X along ℓ. Then $\Pi \not\subset X$. The residual to ℓ in Π is a line $\ell' \in F$.

Applications to potential density

Amerik-Campana (2005)

Let F be a Fano variety of lines of a very general cubic fourfold. Then the orbit $\{\phi^n(x)\}_{n\in\mathbb{N}}$ of a very general $x \in F$ is Zariski dense.

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There exist cubic fourfolds X over some number field k such that the corresponding Fano variety of lines F has geometric Picard group $\mathrm{Pic}(F_{\bar{k}}) = \mathbb{Z}$ and satisfies potential density over k.

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If the Picard group of F has rank two then $\mathrm{Aut}(F) = 0$ and F is not birational to an abelian fibration.

Applications to HP and WA

Given a K3 surface (S, f) with $\rho(X) = 1$, and $\alpha \in \mathrm{Br}(S)[2]$, one can construct a cubic fourfold $X \subset \mathbb{P}^5$ containing a plane such that the corresponding variety of lines $F = F(X)$ is birational to $S^{[2]}$ (Voisin 86).

The lattices $H^4(X, \mathbb{Z})$ and $H^2(F, \mathbb{Z})$ take the form

	h^2	T
h^2	3	1
T	1	3

	g	τ
g	6	2
τ	2	-2

The (-2)-class gives rise to a divisorial contraction. We have

$$\begin{array}{ccc} W & \xrightarrow{\iota} & F \\ {\scriptstyle\pi}\downarrow & & \\ S & & \end{array}$$

where π is a conic bundle (Azumaya algebra) over S and ι is an inclusion (W parametrizes lines incident to the plane Π).

Applications to HP and WA

Hassett–Várilly-Alvarado 2011

Let S be a degree 2 K3 surface given by

$$w^2 = -\frac{1}{2}\det\begin{pmatrix} 2A & B & C \\ B & 2D & E \\ C & E & 2G \end{pmatrix}.$$

There exist quadratic polynomials $A, B, C, D, E, G \in \mathbb{Z}[x_0, x_1, x_2]$ such that $\rho(S) = 1$ and S fails the Hasse principle (or weak approximation).

Applications to unirational parametrizations

Hassett-T. 2001

Assume F contains a smooth rational curve of degree n and that the corresponding scroll T is not a cone. Then there exists a rational map

$$\mathbb{P}^4 \dashrightarrow X$$

of degree

$$\frac{(n-2)^2}{4} + \frac{(R,R)}{2} + 1.$$

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In particular, cubic fourfolds with odd degree unirational parametrizations are dense in moduli.

Higher dimensions

Let F be a holomorphic symplectic variety of $K3^{[n]}$-type. Then

$$H^2(F,\mathbb{Z}) \simeq H^2(S,\mathbb{Z}) \oplus_\perp \mathbb{Z}\delta, \quad (\delta,\delta) = -2(n-1),$$

for the Beauville-Bogomolov quadratic form (2δ is the class of the diagonal). Thus

$$H_2(F,\mathbb{Z}) \simeq H_2(S,\mathbb{Z}) \oplus_\perp \mathbb{Z}\delta^\vee, \quad (\delta^\vee,\delta^\vee) = -1/2(n-1),$$

Huybrechts 1999

If there is a class of positive square then F is projective.

Higher dimensions

Initial idea (Hassett-T. 2008)

The quadratic form on $\mathrm{Pic}(F) \subset H^2(F, \mathbb{Z})$ determines the ample and effective cone.

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Conjecture (Hassett-T. 2008)

Let (F, g) be a polarized holomorphic symplectic of $\mathrm{K3}^{[n]}$-type. Then

$$\overline{NE}_1(F) = \langle C \mid (C, g) > 0, \quad (C, C) \geq -(n+3)/2 \rangle.$$

Higher dimensions

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Major breakthrough in birational geometry of $S^{[n]}$, for S a Del Pezzo surface, came in the work of Huizenga (2012), with Arcara, Bertram, Coscun (2013), who applied Bridgeland stability conditions to determine the structure of the effective cone.

Higher dimensions

Building on

- Bayer–Macri (2013) in the $S^{[n]}$-case,
- Markman's monodromy theorem, and
- Torelli theorem, due to Verbitsky,

we have:

Theorem (Bayer-Hassett-T. 2013)

Complete description of the cone of ample divisors of holomorphic symplectic varieties of $K3^{[n]}$-type, for all n.

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- Mongardi 2013

$$\overline{NE}_1(F) \subseteq ...$$

Higher dimensions

New insight

To understand ample and effective cones in $\mathrm{Pic}(F)$ one needs additional structure.

There is a natural extension

$$H^2(F,\mathbb{Z}) \subset \tilde{\Lambda} \simeq (-E_8)^2 \oplus U^4$$

with orthogonal complement generated by a primitive vector v. We have $(v,v) = 2n-2$.

Example

For $F = S^{[n]}$, we have

$$\tilde{\Lambda} = H^2(S, \mathbb{Z}) \oplus U.$$

We saw that

$$H^2(S^{[n]}, \mathbb{Z}) = H^2(S, \mathbb{Z}) \oplus \mathbb{Z}\delta,$$

here δ generates $v^\perp$ in U and satisfies $(\delta, \delta) = -2(n-1)$.

Torelli

F and F' are birational iff there exists a Hodge isometry

$$\tilde{\Lambda} \to \tilde{\Lambda}'$$

inducing an isomorphism

$$H^2(F, \mathbb{Z}) \to H^2(F', \mathbb{Z}).$$

There is a canonical homomorphism

$$\theta^\vee : \tilde{\Lambda} \longrightarrow H_2(F, \mathbb{Z})$$

inducing an inclusion of finite index

$$H^2(F, \mathbb{Z}) \hookrightarrow H_2(F, \mathbb{Z})$$

Higher dimensions

Bayer-Hassett-T. 2014

Let (F, g) be a polarized holomorphic symplectic variety of K3-type. The Mori cone $\overline{NE}_1(F)$ is generated by

$$\{D \in H^2(F, \mathbb{R}) \mid (D, D) > 0\}$$

and

$$\{\theta^\vee(a) \mid a \in \tilde{\Lambda}_{\mathrm{alg}}, \quad a^2 \geq -2, \quad |(a, v)| \leq v^2/2, \quad (g, a) > 0\}.$$

Understanding extremal rays

Hassett-T. 2015

Given $R \in H_2(F, \mathbb{Z})$ with

$$-\frac{n+3}{2} \leq (R, R) < 0$$

there exists a K3 surface S with $\mathrm{Pic}(S) = \mathbb{Z}f$ and extremal rational curve $\mathbb{P}^1 \subset S^{[n]}$ with $\mathbb{R}_{\geq 0}[\mathbb{P}^1]$ monodromy-equivalent to $\mathbb{R}_{\geq 0}R$. In particular, the Mori cone on $S^{[n]}$ is generated by $\delta^\vee$ and $[\mathbb{P}^1]$.

Characterizing Lagrangian $\mathbb{P}^n$

The goal is to characterize numerically the class of a line in a Lagrangian $\mathbb{P}^n \subset F$, for F of K3-type.

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The goal is to characterize numerically the class of a line in a Lagrangian $\mathbb{P}^n \subset F$, for F of K3-type. A long computation in cohomology ring reduces the problem to the need to show that:

- $n = 3$: (Harvey-Hassett-T. 2010) The only solution of

$$y^2 = -\frac{5^2}{2^{16} \cdot 3 \cdot 11} x^3 - \frac{3}{2^{11} \cdot 11} x^2 - \frac{5}{2^8 \cdot 3 \cdot 11} x - 1$$

  with $x \in \mathbb{Z}$ and $y \in \mathbb{Q}$ is $y = 0$ and $x = -48$.
- $n = 4$: (Bakker-Jorza 2011) The only solution of

$$y^2 = \frac{5^2}{2^{12} \cdot 3^4 \cdot 7} x^4 + \frac{5^2}{2^9 \cdot 3^4} x^3 + \frac{13 \cdot 31}{2^9 \cdot 3^2 \cdot 7} x^2 + \frac{3^2}{2^7} x - \frac{3^2 \cdot 5 \cdot 7^2 \cdot 197}{2^8}$$

  with $y \in \mathbb{Q}$ and $x \in \mathbb{Z}$ is $y = 0$ and $x = -126$.

Characterizing Lagrangian $\mathbb{P}^n$

Bakker 2013

- Let $R \in H_2(F, \mathbb{Z})$ be the class of a line in a Lagrangian $\mathbb{P}^n \subset F$. Then

$$(R, R) = -\frac{n+3}{2} \quad \text{and} \quad 2R \in H^2(F, \mathbb{Z}) \quad (*)$$

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- A primitive class R generating an extremal ray of the Mori cone is the class of a line in a Lagrangian $\mathbb{P}^n$ iff R satisfies $(*)$. These lie in a single monodromy orbit.

Geometry of exceptional loci, $\dim(F) = 4$

(a,a)	(a,v)	$(v-a, v-a)$	**Interpretation**
-2	0	0	$\mathbb{P}^1$-bundle over S
-2	1	-2	$\mathbb{P}^2$
0	1	0	$\mathbb{P}^1$-bundle over S

Geometry of exceptional loci, $\dim(F) = 6$

(a,a)	(a,v)	$(v-a, v-a)$	**Interpretation**
-2	0	2	$\mathbb{P}^1$-bundle over $S^{[2]}$
-2	1	0	$\mathbb{P}^2$-bundle over S
-2	2	-2	$\mathbb{P}^3$
0	1	2	$\mathbb{P}^1$-bundle over $S \times S$
0	2	0	$\mathbb{P}^1$-bundle over $S \times S'$, S and S' are isogenous

Geometry of exceptional loci, $\dim(F) = 8$

(a,a)	(a,v)	$(v-a,v-a)$	**Interpretation**
-2	0	4	$\mathbb{P}^1$-bundle over $S^{[3]}$
-2	1	2	$\mathbb{P}^2$-bundle over $S^{[2]}$
-2	2	0	$\mathbb{P}^3$-bundle over S
-2	3	-2	$\mathbb{P}^4$
0	1	4	$\mathbb{P}^1$-bundle over $S \times S^{[2]}$
0	2	2	$\mathbb{P}^1$-bundle over $S' \times S^{[2]}$ S, S' are isogenous
0	3	0	$\mathbb{P}^2$-bundle over $S \times S'$ S, S' are isogenous

Geometry of exceptional loci, $\dim(F) = 10$

(a, a)	(a, v)	$(v-a, v-a)$	**Interpretation**
-2	0	6	$\mathbb{P}^1$-bundle over $S^{[4]}$
-2	1	4	$\mathbb{P}^2$-bundle over $S^{[3]}$
-2	2	2	$\mathbb{P}^3$-bundle over $S^{[2]}$
-2	3	0	$\mathbb{P}^4$-bundle over S
-2	4	-2	$\mathbb{P}^5$
0	1	6	$\mathbb{P}^1$-bundle over $S \times S^{[3]}$
0	2	4	$\mathbb{P}^1$-bundle over $S' \times S^{[3]}$ S, S' are isogenous
0	3	2	$\mathbb{P}^2$-bundle over $S' \times S^{[2]}$ S, S' are isogenous
0	4	0	$\mathbb{P}^3$-bundle over $S \times S'$ S, S' are isogeneous

Hassett-T. 2015

- There exist S (e.g., of degree 114) such that $S^{[3]}$ admits an automorphism not arising from an automorphism of any K3 surface T with $T^{[3]} \simeq S^{[3]}$.

Applications

Hassett-T. 2015

- There exist S (e.g., of degree 114) such that $S^{[3]}$ admits an automorphism not arising from an automorphism of any K3 surface T with $T^{[3]} \simeq S^{[3]}$.
- There exist polarized holomorphic symplectic varieties $(F, g), (F', g')$ admitting an isomorphism of Hodge structures

$$\phi : H^2(F, \mathbb{Z}) \to H^2(F', \mathbb{Z}), \quad \phi(g) = g',$$

not preserving ample cones. (We can take $F = S^{[7]}$).

Further applications

- Constructing explicit Azumaya algebras realizing transcendental Brauer-Manin obstructions to weak approximation and the Hasse principle (Hassett–Varilly Alvarado)
- Modular constructions of isogenies between K3 surfaces and interpretation of moduli spaces of K3 surface with level structure (McKinnie–Sawon–Tanimoto–Varilly-Alvarado 2014)
- Explicit descriptions of derived equivalences among K3 surfaces and varieties of K3 type
- Analysis of birational and biregular automorphisms of holomorphic symplectic varieties (Hassett–T. 2011, Boissiére–Cattaneo–Nieper-Wisskirchen–Sarti 2014).

Derived equivalent K3 surfaces

Let X be a K3 surface over $\mathbb{C}$ and

$$T(X) := \mathrm{Pic}(X)^{\perp} \subset H^2(X, \mathbb{Z})$$

its transcendental lattice.

Derived equivalent K3 surfaces

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its transcendental lattice.

X, Y are derived equivalent if

$$T(X) \simeq T(Y),$$

as Hodge structures, or equivalently, there exists an

$$\mathcal{E} \subset D^b(X \times Y)$$

such that

$$\Phi_{\mathcal{E}} : D^b(X) \to D^b(Y)$$

is an equivalence of triangulated categories.

In high Picard rank, derived equivalence implies isomorphism, e.g., if

- $\rho(X) \geq 12$
- X admits an elliptic fibration with a section
- $\rho(X) \geq 3$ and the discriminant group of $\mathrm{Pic}(X)$ is cyclic.

Derived equivalent K3 surfaces: first examples

$$\Lambda_X = \begin{array}{c|cc} & C & f \\ \hline C & 2 & 13 \\ f & 13 & 12 \end{array} \quad \Lambda_Y = \begin{array}{c|cc} & D & g \\ \hline D & 8 & 15 \\ g & 15 & 10 \end{array}$$

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Let X and Y be K3 surfaces with split Picard groups Λ_X and Λ_Y over a field k.

Hope: One can relate arithmetic properties of X and Y.

Derived equivalent K3 surfaces

We have:

- X and Y admit decomposable zero-cycles of degree one
- $X(k) \neq \emptyset$: the rational points arise from the smooth rational curves with classes $2f - C$ and $25C - 2f$, both of which admit zero-cycles of odd degree and thus are $\simeq \mathbb{P}^1$ over k
- $Y(k')$ is dense for some finite k'/k, since $|\mathrm{Aut}(Y_{\bar{k}})| = \infty$

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- $Y(k')$ is dense for some finite k'/k, since $|\mathrm{Aut}(Y_{\bar{k}})| = \infty$

We do not know whether

- $X(k')$ is dense for any finite k'/k
- $Y(k) \neq \emptyset$

Derived equivalent K3 surfaces

Hassett-T. 2014

Assume that X and Y are derived equivalent over a field k of characteristic $\neq 2$. Then

- $\mathrm{Pic}(X) \sim \mathrm{Pic}(Y)$, stably isomorphic as $\mathrm{Gal}(\bar{k}/k)$-modules

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- $\mathrm{Pic}(X) \sim \mathrm{Pic}(Y)$, stably isomorphic as $\mathrm{Gal}(\bar{k}/k)$-modules
- $\mathrm{Br}(X)[n] \simeq \mathrm{Br}(Y)[n]$, if $\mathrm{char}(k) \nmid n$,
- if X has a zero-cycle of degree 1 over k then Y also has a zero-cycle of degree 1 over k.

Derived equivalent K3 surfaces

Lieblich-Olsson 2011

Let X and Y be derived equivalent K3 surfaces over $k = \mathbb{F}_q$. Then

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Over $\mathbb{R}$

Let X and Y be derived equivalent K3 surfaces over $\mathbb{R}$. Then $X(\mathbb{R})$ and $Y(\mathbb{R})$ are diffeomorphic and in particular

$$X(\mathbb{R}) \neq \emptyset \Leftrightarrow Y(\mathbb{R}) \neq \emptyset.$$

What about $k = \mathbb{Q}_p$? What about $\mathbb{Q}$?

Derived equivalent K3 surfaces: the p-adic case

Hassett-T. 2014

Assume that X and Y are derived equivalent over a p-adic field k of residue characteristic ≥ 7 and that both admit ADE-reduction. Then

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Potentially good reduction is a derived invariant (Y. Matsumoto 2014).

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Potentially good reduction is a derived invariant (Y. Matsumoto 2014). Is ADE-reduction a derived invariant? There are some results in this direction (Liedtke-Matsumoto 2014).

Derived equivalent K3 surfaces: the geometric case

Let X be a K3 surface over $k = \mathbb{C}((t))$. Fix an integral model over

$$\mathcal{X} \to \Delta := \mathrm{Spec}(\mathbb{C}[[t]]).$$

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Goal: Show that having or not having a section is a derived invariant.

Derived equivalent K3 surfaces: the geometric case

After a finite base change $\Delta_2 \to \Delta$, there exists a Kulikov model, whose central fiber is one of the following

- a K3 surface
- a chain of surfaces glued along elliptic curves, with rational surfaces at the ends and elliptic ruled surfaces in between
- a union of rational surfaces (combinatorially, a triangulation of a sphere)

Derived equivalent K3 surfaces: the geometric case

Hassett-T. 2014

Assume that X and Y are derived equivalent over $k = \mathbb{C}((t))$ and that X admits a Kulikov model. Then Y also admits a Kulikov model, and both $X(k)$ and $Y(k)$ are nonempty.

Isotrivial families

Let $G = \mathbb{Z}/N$ act on a K3 surface X_0 via $G \twoheadrightarrow H \subset \mathrm{Aut}(X_0)$ and on Δ_2, the unit disc, via $z \mapsto \zeta_N z$. We get an isotrivial family

$$\pi : \mathcal{X} \to (X_0 \times \Delta_2)/G \to \Delta_1 := \Delta_2/G.$$

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$$\pi : \mathcal{X} \to (X_0 \times \Delta_2)/G \to \Delta_1 := \Delta_2/G.$$

These structures are preserved by derived equivalence.

Lemma

π admits a section if and only if H-action on X_0 has a fixed point.

Suppose that a finite cyclic group H acts on X_0 and Y_0 effectively and that there exists an isomorphism of Hodge structures

$$T(X_0) \simeq T(Y_0),$$

compatible with the H-actions.

Isotrivial families

Suppose that a finite cyclic group H acts on X_0 and Y_0 effectively and that there exists an isomorphism of Hodge structures

$$T(X_0) \simeq T(Y_0),$$

compatible with the H-actions. Does X_0 admit an H-fixed point if and only if Y_0 admits an H-fixed point?

Isotrivial families

Let
$$H = \langle \sigma \rangle \simeq \mathbb{Z}/N \subset \mathrm{Aut}(X)$$
and X^σ be the fixed point locus for σ.

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Cyclic automorphisms

- $\phi(N) \leq 20$ and all such N arise.

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Cyclic automorphisms

- $\phi(N) \leq 20$ and all such N arise.
- If σ is symplectic then $1 \leq N \leq 8$ and $X^\sigma \neq \emptyset$.

$\phi(N) \leq 20$

	20	18	16	12	10	8	6	4	2	1
	66	54	60	42	22	30	18	12	6	2
	50	38	48	36	11	24	14	10	4	1
	44	27	40	28		20	9	8	3	
	33	19	34	26		16	7	5		
	25		32	21		15				
			17	13						

$\phi(N) \leq 20$

	20	18	16	12	10	8	6	4	2	1
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	44	27	40	28		20	9	8	3	
	33	19	34	26		16	7	5		
	25		32	21		15				
			17	13						

There is a very detailed analysis of such actions in the literature (Artebani, Keum, Kondo, Machida, Nikulin, Oguiso, Sarti, Taki). **Hopefully**, this contains the information we need.

To do

- Develop a mixed-characteristic version of Kulikov's models

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- Develop a mixed-characteristic version of Kulikov's models
- Develop a mixed-characteristic version of Bridgeland-Maciocia's Fourier-Mukai transforms for K3 fibrations

Conclusion

- Birational geometry of punctual Hilbert schemes of K3 surfaces is very rich.

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- Birational geometry of punctual Hilbert schemes of K3 surfaces is very rich.
- Recent advances
    - Torelli theorem (Verbitsky)
    - Computation of Monodromy (Markman)
    - Analysis of Bridgeland stability (Bayer–Macri)

  opened the door to interesting applications in arithmetic geometry.