Some New Results on Ricci Flow

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My Ricci Flow History

My First Encounter with Ricci Flow:

Richard Hamilton's 1982 PNGS Talk in Vancouver

Theorem

If (M^3, g_0) has $Rc(g_0) \geq 0$, then $M^3 = S^3$

Proof via Ricci Flow Convergence

My Second Encounter with Ricci Flow:

Mauro Carfora's 1985 GRG Society Talk in Florence

Conjecture

The Einstein Evolution of "Smoothed" initial data accurately models general cosmological solutions of Einstein's equations

Never proven

My Third Encounter with Ricci Flow:

Collaboration with Mauro Carfora and Martin Jackson on JDG Paper

Theorem:

The Ricci flow of "Gowdy-type metrics" on T^3 always converges to a flat metric.

Significance:

The first Ricci flow convergence result for metrics of positive and negative Ricci curvature.

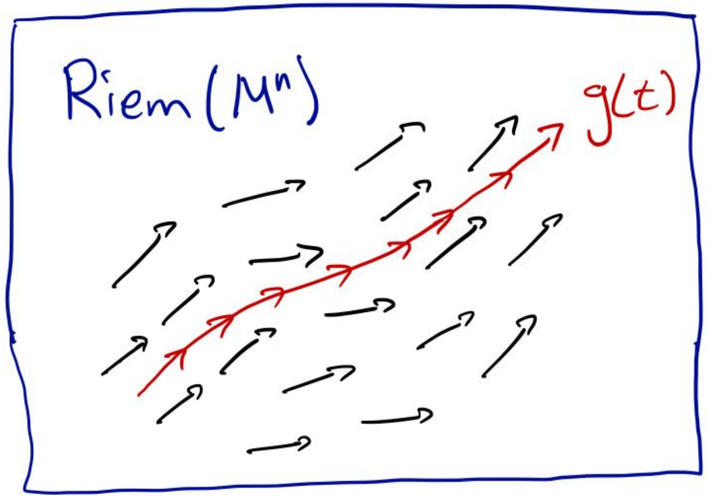
Ricci flow

Two ways to think about Ricci flow:

–Initial value problem for a (weakly) parabolic PDE system

$$\frac{\partial}{\partial t}g = -2\,\mathrm{Rc} + \frac{2}{n}\left(\fint R\,d\mu\right)g, \qquad g(0) = g_0.$$

– Dynamical System



We know a lot about Ricci flow for:

– 2 and 3 Dimensional Riemannian Manifolds

– Riemannian Metrics on Compact Manifolds with Large Isometry Groups

Outline of the Talk

Part A of the talk: Convergence Stability of Ricci flow for Flat Geometries on the Torus and for the Hyperbolic Geometry in Asymptotically Hyperbolic Metrics

(with Chris Guenther and Eric Bahuaud)

Application A 1: Convergence of Ricci flow for initial metrics near warped product geometries (with no isometry) to a flat metric

Application A 2: Convergence of Ricci flow for initial metrics near rotationally symmetric asymptotically hyperbolic geometries (with no symmetry) to the hyperbolic metric

Part B of the talk: Singularity Formation for Ricci flow for Multi-Warped Complete Geometries on Non-Compact manifolds

(with Tim Carson, Dan Knopf, and Natasa Sesum)

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Application B 1: Unexpected behavior of blowup sequences in Ricci flow solutions with Type I singularities at spatial infinity

– some blowup subsequences form gradient shrinking Ricci solitons

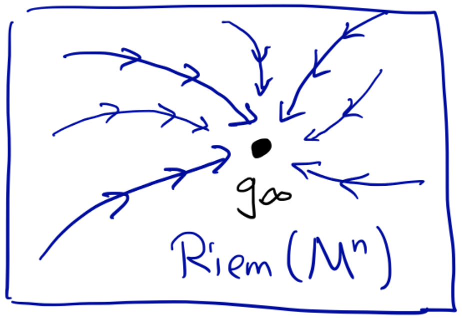
– other blowup subsequences form ancient solutions which are not solitons

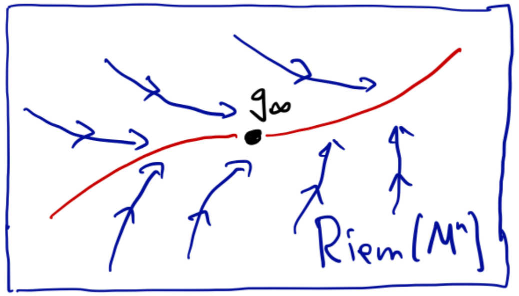
Application B 2: Weak stability of generalized cylinders under Ricci flow.

Convergence Stability for Ricci Flow

The Idea:

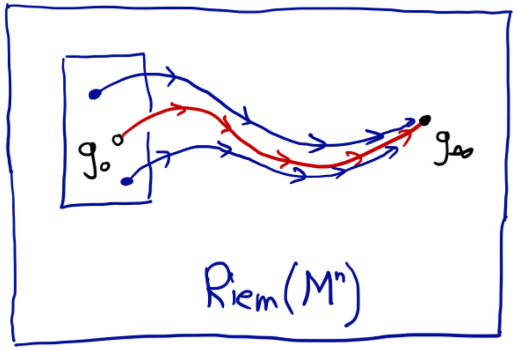
–Let g_∞ be a stable point for Ricci flow: either asymptotic stable fixed point or stable fixed point.

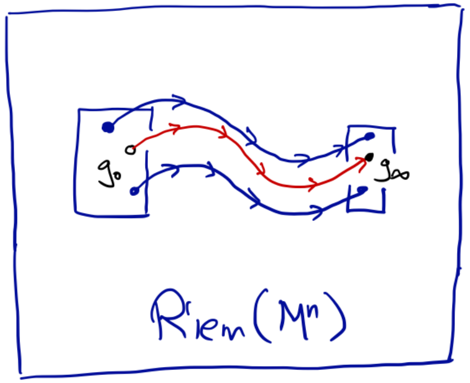




–Let g_0 be a metric such that the Ricci flow $g_0(t)$ with initial metric g_0 converges to g_∞.

–Ricci flow is **convergent stable** at g_0 if there exists a neighborhood N of g_0 such that for every metric g_1 in N, the Ricci flow starting at g_1 converges to g_∞, or to a metric in the corresponding center manifold.





Example Application:
Since Lott and Sesum show that the Ricci flow of warped product geometries of the circle over the torus always converge to flat geometries, and since one can prove convergence stability for Ricci flow on T^3, it follows that the Ricci flow for metrics near warped product geometries must also converge to flat geometries.

Example Application:

Since Bahuaud and Woolgar show that the Ricci flow for rotationally symmetric asymptotically hyperbolic geometries always converges to the hyperbolic geometry, and since one can prove convergence stability for Ricci flow for asymptotically hyperbolic geometries, it follows that the Ricci flow for asymptotically hyperbolic metrics which are nearly rotationally symmetric must converge to the hyperbolic geometry.

Ingredients for Proving Convergence Stability for Ricci flow

– Prove that Ricci flow for a certain class of metrics is stable or asymptotically stable at g_∞ with respect to some chosen topology (in terms of some function space)

– Prove that Ricci flow for that same class of metrics is well-posed in the sense of continuous dependence on initial data.

Proof of Convergence Stability for Ricci Flow on T^n at Flat Metrics

We need:

– Stability Theorem at the Flat Metric

– Continuous Dependence of Ricci Flow on Compact Manifolds

Stability Theorem for flat metrics on T^n

Theorem (Guenther-I-Knopf)

Let $\mathrm{h}^{\{\mathrm{k},\alpha\}}$ denote the little Holder space of symmetric 2-tensor fields whose k derivatives are α $-$ Holder smooth.

Let g_∞ be a flat Riemannian metric on T^n.

There exists a neighborhood N of g_∞ (defined via the norm of $h^{\{2,\alpha\}}$) such that the Ricci flow $g_1(t)$ for any initial metric g_1 in N must converge exponentially to a flat metric (generally not g_∞).

Key ideas for proving the Stability Theorem:

1) Semi-Group Approach (DaPrato, Lunardi)

2) Use of DeTurck Flow (as "shadow" to Ricci flow)

$$\frac{d}{dt}g = -2\mathrm{Rc}(g) + 2\mathcal{L}_W g,$$

3) Linearization of the DeTurck Flow

$$\frac{d}{dt}h = \Delta_L h,$$

4) Verify "sectoriality" of the Lichnerowicz Laplacian

(i.e., the spectrum forms a wedge in a left complex 1/2-plane)

5) Verify boundedness of the "resolvent" of the Lichnerowicz Laplacian

6) Verify that the RHS of the DeTurck flow is continuous, and has a Frechet derivative which is Lipschitz.

7) Verify that the diffeomorphism relating Ricci flow and DeTurck flow is bounded. (Involves analysis of a harmonic map flow generating this diffeomorphism.)

Continuous Dependence Theorem for Ricci Flow on Compact Manifolds

Theorem 1.2 (Bahuaud-Guenther-I-'19). *Let (M, g_0) be a compact Riemannian manifold. Let $g_0(t)$ be the maximal solution of the normalized Ricci flow for $t \in [0, \tau(g_0))$, $\tau(g_0) \leq \infty$, where $\tau(g_0)$ denotes the maximal time of existence. Choose $0 < \tau < \tau(g_0)$. There exist positive constants r and C depending only on g_0 and τ such that if*

$$||g_1 - g_0||_{h^{k,\alpha}} \leq r,$$

then for the unique solution $g_1(t)$ of the Ricci flow starting at g_1, the maximal existence time $\tau(g_1)$ satisfies

$$\tau(g_1) \geq \tau,$$

and

$$||g_1(t) - g_0(t)||_{h^{k,\alpha}} \leq C||g_1 - g_0||_{h^{k,\alpha}}$$

for all $t \in [0, \tau]$.

Tools Needed to Prove Continuous Dependence:

– Short Time Uniqueness and Continuous Dependence (Bamler-Brendle)

[Proven using similar tools as for Stability Theorem]

– Long Time Continuous Dependence (Bahuaud-Guenther-I ‘19)

[Proven using Iteration of Uniform short time continuous dependence]

–The Convergence Stability Theorem on Closed Manifolds

Theorem (Bahuaud, Guenther, I; '19)

- *If* $(\mathcal{M}, g_0)$ *is a smooth closed torus and* $g_0(t)$ *converges to a flat metric under Ricci flow, then solutions starting at nearby metrics converge to flat metrics.*
- *If* $(\mathcal{M}, g_0)$ *is a smooth closed Riemmanian manifold and* $g_0(t)$ *converges to a hyperbolic metric* g_∞ *under a normalized Ricci flow, then solutions starting at nearby metrics converge to* g_∞

Note: This last result is a bonus.

Proof of Convergence Stability

– Let g_0(t) be the normalized Ricci flow of a metric g_0 which converges exponentially to a stable fixed point flat metric g_∞

– It follows from the definition of convergence that within a finite time, the flow g_0(t) enters any specified neighborhood of g_∞ , including the neighborhood N in which all normalized Ricci flows must converge exponentially to g_∞ or a nearby flat metric.

– If g_1 is a metric sufficiently close to g_0 , then it follows from continuous dependence that for any specified finite time, the flow g_1(t) remains close to the flow g_0(t) and therefore enters N.

– It follows that g_1(t) converges exponentially to a flat metric in the center manifold containing g_∞.

Convergence Stability of Ricci Flow for Asymptotically Hyperbolic Geometries

Theorem (Bahuaud-Guenther-I ‘21)

Let (M, g_0) be a rotationally symmetric asymptotically hyperbolic metric with the same conformal infinity as the hyperbolic metric. There exists an open neighborhood N of g_0 (with respect to a specified little Holder space) for asymptotically hyperbolic metrics such that the normalized Ricci flow with initial metric $g_1 \in$ N converges exponentially to the hyperbolic metric.

Sketch of the Proof of Convergence Stability of Ricci Flow for AH Geometries

[Same idea as for Proof of Convergence Stability on Compact Manifolds;
But using Weighted Holder Spaces adapted to the AH geometries]

– Stability Theorem for the Hyperbolic Metric
(Bamler-Brendle)

– Long Time Continuous Dependence Theorem of Ricci Flow for AH Geometries
(Bahuaud-Guenther-I ‘21)

– Convergence of Rotationally Symmetric AH Geometries to the Hyperbolic Geometry
(Bahuaud-Woolgar)

Singularity Formation for Ricci flow for Multi-Warped Complete Geometries on Non-Compact Manifolds

(Work with Carson, Knopf, and Sesum)

Motivation:

–Exploring Ricci flow Solutions with Singularities Forming at Spatial Infinity

– Obtaining a Weak Stability Result for Generalized Cylinders under Ricci flow

The Setup for Multi-Warped Products with Einstein Fibers

1.2. **Manifolds.** Let $(\mathcal{B}^n, g_{\mathcal{B}})$ be a complete noncompact Riemannian manifold. For $\alpha \in \{1, \ldots, A < \infty\}$, let $(\mathcal{F}_\alpha^{n_\alpha}, g_{\mathcal{F}_\alpha})$ be a collection of space forms, and let μ_α be constants such that $\mu_\alpha g_{\mathcal{F}_\alpha} = 2\,\mathrm{Rc}[g_{\mathcal{F}_\alpha}]$. Given functions $u_\alpha : \mathcal{B}^n \to \mathbb{R}_+$, there is a warped product metric g on the manifold $\mathcal{M}^{\mathcal{N}} = \mathcal{B}^n \times \mathcal{F}_1^{n_1} \times \cdots \times \mathcal{F}_A^{n_A}$, where $\mathcal{N} = n + \sum_{\alpha=1}^{A} n_\alpha$, given by

$$g = g_{\mathcal{B}} + \sum_{\alpha=1}^{A} u_\alpha g_{\mathcal{F}_\alpha}. \tag{1}$$

The Ricci Flow Equations for These Metrics

$$(2a) \qquad \partial_t \, g_{\mathrm{B}} + 2\,\mathrm{Rc}[g_{\mathrm{B}}] = -2 \sum_{\alpha=1} n_\alpha u_\alpha^{-1/2} \nabla^2 (u_\alpha^{1/2}),$$

$$(2b) \qquad (\partial_t - \Delta)\, u_\alpha = -\mu_\alpha - u_\alpha^{-1} |\nabla u_\alpha|^2, \qquad (\alpha \in \{1, \ldots, A\}).$$

Derivation of These Equations:
Based on calculation of relation between curvature on the base, the fibers, and the full manifold
– Straightforward
– Tedious
– "Where the dead horses are buried" (Cliff Taubes)

A Model Direct Product Solution and a New Choice of Variables

If some $u_\alpha(x,0)$ is a constant a_α, then $u_\alpha(x,t) = a_\alpha - \mu_\alpha t$ is an explicit solution of (2b) for as long as the flow remains smooth. Since we are interested in studying perturbations of spatially homogeneous solutions, we set $a_\alpha = \inf_{x\in\mathcal{B}} u_\alpha(x,0)$ and define $v_\alpha(\cdot,0) : \mathcal{B} \to \mathbb{R}_+$ by

$$v_\alpha(x,0) = u_\alpha(x,0) - a_\alpha, \tag{4}$$

The new form of the metric:

$$g(x,t) = g_{\mathcal{B}}(x,t) + \sum_{\alpha=1}^{A} \{(a_\alpha - \mu_\alpha t) + v_\alpha(x,t)\} g_{\mathcal{F}_\alpha}. \tag{5}$$

Controlling Functions for the Curvature

$$\gamma_\alpha(x,t) = |\nabla v_\alpha(x,t)|^2, \tag{6a}$$

$$\chi_\alpha(x,t) = |\nabla^2 v_\alpha(x,t)|^2_{g_{\mathcal{B}}}, \tag{6b}$$

$$\rho(x,t) = \left|\,\mathrm{Rm}[g_{\mathcal{B}}](x,t)\right|^2_{g_{\mathcal{B}}}, \tag{6c}$$

With the Estimate:

$$\left|\mathrm{Rm}[g] - \sum_{\alpha=1}^{A} u_\alpha^{-1}\,\mathrm{Rm}[g_{\mathcal{T}_\alpha}]\right|_g \leq C\left\{\rho^{1/2} + \sum_{\alpha=1}^{A}\left(u_\alpha^{-2}\gamma_\alpha + u_\alpha^{-1}\chi_\alpha^{1/2}\right)\right\}. \tag{7}$$

The Main Assumptions

Assumption Parameterization Functions:

–G_α contained in

$$\mathcal{G} = \left\{ G : \mathbb{R}_+ \to \mathbb{R}_+ : \|G\|_{\mathcal{G}} := \|G\|_{2,\mathrm{mon}} + \sup_{s \in \mathbb{R}_+} \frac{G(s)}{s^2} < \infty \right\}. \tag{9}$$

Given a smooth function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, we define

$$\|\varphi\|_{2,\mathrm{mon}} = \sup_{s \in \mathbb{R}_+} \left(1 + \frac{s|\varphi'(s)|}{\varphi(s)} + \frac{s^2|\varphi''(s)|}{\varphi(s)} \right).$$

This is a quasi-norm!

–H_α defined by

$$H_\alpha[s_1, \ldots, s_A](s_\alpha) = \left(\sum_{\beta=1}^{A} \frac{G_\beta(s_\beta)}{s_\beta^2} \right) G_\alpha(s_\alpha). \tag{10}$$

– We impose these conditions on the initial data:

Main Assumptions. *There exist a constant* C_{init} *and functions* $G_\alpha \in \mathcal{G}$ *such that for* $\alpha \in \{1, \ldots, A\}$,

$$\|G_\alpha\|_{\mathcal{G}} \le C_{\text{init}}, \tag{11a}$$

$$\gamma_\alpha(x,0) \le C_{\text{init}}\, G_\alpha\big(v_\alpha(x,0)\big) \qquad \textit{for all } x \in \mathcal{B}, \tag{11b}$$

$$\chi_\alpha(x,0) \le C_{\text{init}}\, H_\alpha\big(v_\alpha(x,0)\big) \qquad \textit{for all } x \in \mathcal{B}, \tag{11c}$$

$$\rho(x,0) \le C_{\text{init}} \qquad \textit{for all } x \in \mathcal{B}. \tag{11d}$$

We further assume that $|\nabla \operatorname{Rm}[g(\cdot,0)]|_{g(\cdot,0)}$ *is bounded and that at least one* $\mu_\alpha > 0$, i.e., *that at least one fiber is a space form of positive Ricci curvature.*

Our Main Theorem (Non-Technical Version)

3. Theorem. *Let* $(\mathcal{M}, g(t))$ *be a solution of the Ricci flow system* (2) *that originates from initial data satisfying our Main Assumptions and exists for* $t \in [0, T_{\text{small}}]$.

There exists a constant $C_* = C_*(n, n_\alpha, C_{\text{init}})$ *such that for* $t \in [0, \min\{T_{\text{small}}, C_*^{-1}\})$ *the metric can be written as*

:

$$g(x,t) = \big(1 + \mathcal{O}(1)\big) g_{\mathcal{B}}(x,0) + \sum_{\alpha=1}^{A} \left\{ (a_\alpha - \mu_\alpha t) + \left(1 + \mathcal{O}\left(\frac{G_\alpha\left(v_\alpha(x,0)\right)}{v_\alpha^2(x,0)}\right)\right) v_\alpha(x,0) \right\} g_{\mathcal{F}_\alpha}.$$

Our Main Theorem (Technical Version)

4. **Theorem.** *Let* $(\mathcal{M}, g_{\text{init}})$ *satisfy the Main Assumptions stated in Section 2.1. Then there exists a constant* $C_* = C_*(n, n_\alpha, C_{\text{init}})$ *such that the following are true:*

A solution

$$g(x,t) = g_{\mathcal{B}}(x,t) + \sum_{\alpha=1}^{A} \{a_\alpha - \mu_\alpha t + v_\alpha(x,t)\} g_{\mathcal{F}_\alpha}$$

of the Ricci flow initial value problem with $g(x,0) = g_{\text{init}}(x)$ *exists with curvatures bounded in space at all times* $t \in [0, T_*)$, *where* $T_* := \min\{T_{\text{sing}}, C_*^{-1}\}$, *and* T_{sing} *is the (finite) singularity time,* i.e., *the maximal existence time of a smooth solution.*

The v_α *are uniformly equivalent for* $t \in [0, T_*)$. *Specifically, one has*

$$\frac{1}{C_*} v_\alpha(x,t) \le v_\alpha(x,0) \le C_* v_\alpha(x,t).$$

Moreover, for each $x \in \mathcal{B}$ *and* $t \in [0, T_*)$, *one has*

$$\rho(x,t) \leq C_{\text{init}}\,(1 + C_* t), \tag{8a}$$

and for $\alpha \in \{1, \ldots, A\}$,

$$\gamma_\alpha(x,t) \leq C_{\text{init}} \left(1 + C_* t \, \frac{G_\alpha\big(v_\alpha(x,t)\big)}{v_\alpha^2(x,t)}\right) G_\alpha(v_\alpha(x,t)), \tag{8b}$$

$$\chi_\alpha(x,t) \leq C_{\text{init}}\,(1 + C_* t)\, H_\alpha\big(v_\alpha(x,t)\big), \tag{8c}$$

where G_α *and* H_α *are functions specified in the Main Assumptions.*

Corollary:
Equivalent Initial Data with The Singularity at Spatial Infinity

5. **Corollary.** *There exist initial data* $(\mathcal{M}, g'_{\text{init}})$ *satisfying the Main Assumptions stated in Section 2.1 with the same constant* C_{init}*, the same initial values* v_α*, the same real-valued functions* G_α *and* H_α*, but with changed constants* a_α*, such that the conclusions of Theorem 4 hold for the Ricci flow evolution of* $(\mathcal{M}, g'_{\text{init}})$ *at all times* $[0, T_{\text{sing}})$*. Moreover,* $T_{\text{sing}} = T_{\text{form}}$*; there are no finite singular points in space; and the singularity is Type-I and occurs at spatial infinity.*

Here T_{form} takes the form

$$a_\varsigma/\mu_\varsigma = \min\{a_\alpha/\mu_\alpha : \mu_\alpha > 0\}.$$

Key Ideas Used To Prove Main Results

I] Evolution Estimates for the Curvature Controlling Quantities:

There exists a constant C_N depending only on dimensions such that

$$(\partial_t - \Delta)\gamma_\alpha \le -\frac{1}{2}\frac{|\nabla \gamma_\alpha|^2}{\gamma_\alpha} + 6\left(\frac{\gamma_\alpha}{u_\alpha^2}\right)\gamma_\alpha, \tag{13}$$

$$(\partial_t - \Delta)\chi_\alpha \le -\frac{1}{2}\frac{|\nabla \chi_\alpha|^2}{\chi_\alpha} + C_N L \chi_\alpha + C_N L \sum_{\beta=1}^{A} \frac{\gamma_\beta}{u_\beta^2}\gamma_\alpha, \tag{14}$$

and

$$(\partial_t - \Delta)\rho \le -\frac{|\nabla \rho|^2}{\rho} + C_N L^3, \tag{15}$$

where

$$L := \rho^{1/2} + \sum_{\beta=1}^{A} \frac{\gamma_\beta}{u_\beta^2} + \sum_{\beta=1}^{A} \frac{\chi_\beta^{1/2}}{u_\beta}.$$

II] Standard Short Time Existence Results for Ricci Flow:

Solutions exist for short time with curvature bounds

III] Lemma: Curvature Bound on Short Interval

&

Evolution Estimates for Curvature Control Functions

Produces

Linear growth in time for the Curvature Control Functions

(on a possibly shorter time interval)

(with the estimate constants C depending on the curvature)

IV) Linear In Time Growth of Curvature Control Functions

Produces

Estimates on Curvature with Uniform Constants

(Possibly on Reduced Time Interval)

These improved estimates with uniform constants again bound the curvature linearly in time.

V) Use "Open-Closed Argument" to Extend the Time Interval of Curvature Estimates to the Time of the Singularity T_{sing}

Proof of the Corollary (Singularities are Type-I, and at Spatial Infinity)

VI) Bounds on the Curvature Control Functions on $[0, T_{sing})$ Implies that for t near T_{sing}, there exist constants such that

$$|\,\mathrm{Rm}\,| = \sum_{\alpha=1}^{A} \frac{c_\alpha}{u_\alpha} + C', \tag{43}$$

Since on any compact set,

$$u_\alpha(x,t) = (a_\alpha - \mu_\alpha t) + v_\alpha(x,t)$$

and since $v_a(x,t)$ Is positive everywhere on the manifold, but has infimum zero, It follows that

$$\sup_{x \in \mathcal{B}} u_{\varsigma}^{-1}(x,t) = (a_{\varsigma} - \mu_{\varsigma} t)^{-1},$$

which tells us that the singularity is Type-I, and occurs at Spatial Infinity.

A Workhorse Lemma for Proving a Number of These Steps

13. **Lemma.** *Let* $(\mathcal{M}, g(t))$ *be a smooth solution of Ricci flow for* $t \in [0, T]$, *and let* $U, V, W : \mathcal{M} \times [0, T] \to \mathbb{R}_+$ *be smooth functions. Suppose that there exist constants* $0 < c < 1 < C$ *such that*

$$
\begin{aligned}
(\partial_t - \Delta)U &\leq C(UW + VW) - c\frac{|\nabla U|^2}{U}, \\
\frac{|(\partial_t - \Delta)V|}{V} + \frac{|\nabla V|^2}{V^2} &\leq CW, \\
|(\partial_t - \Delta)W| + |\nabla W|^2 &\leq CW, \\
W &\leq C,
\end{aligned}
$$

where the Laplacian and norms above are computed with respect to the solution $g(t)$ *of Ricci flow.*

Then there exist $\lambda = \lambda(c, C)$ *and* $T' = T'(c, C, T) \in (0, T]$ *such that for all* $t \in [0, T']$,

$$(\partial_t - \Delta)\left\{\frac{U}{V} - \lambda t\left(1 + \frac{U}{V}\right)W\right\} \leq 0.$$

Moreover, if there exist a point $x' \in \mathcal{B}$ *and a constant* C' *such that one has* $U(x,t) \leq C' e^{C' d^2_{g(t)}(x',x)} V(x,t)$ *on* $[0, T']$*, then for* $t \in [0, T']$,

$$\frac{U(x,t)}{V(x,t)} \leq \sup_{y \in \mathcal{M}} \frac{U(y,0)}{V(y,0)} + 2\lambda t W(x,t)\left(1 + \sup_{y \in \mathcal{M}} \frac{U(y,0)}{V(y,0)}\right).$$

This "Horse Cemetery" of a lemma allows us to estimate a function U
(say a geometric quantity on a short time interval)
in terms of a comparison function V
(say a geometric quantity at the initial time)
and a controlling function W
(say a large constant depending on curvatures).

Application B1: Unexpected Behavior of Blowup Sequences in Solutions of Ricci Flow with Singularities at Spatial Infinity

Some Useful Preliminary Definitions:

Define $\mathcal{R}(t) = \sup_{\mathcal{M}} |\mathrm{Rm}[g(t)]|$

We have a singularity at time T if $\limsup_{t \to T} \mathcal{R}(t) = \infty$

A singularity is Type I if $\mathcal{R}(t) \leq \dfrac{C}{T-t}$

An Essential Blowup Sequence in a singular Ricci flow solution is a sequence $(p_i, t_i) \in M\ x\ [0, T)$ such that for some constant c

$$|\operatorname{Rm}[g(t_i)]|_{g(t_i)}(p_i, t_i) \geq \frac{c}{T - t_i}.$$

If a Ricci flow solution is Type-I and if a point p contained in M is the limit point of an essential blowup sequence, then p is a Type-I singular point for that flow.

Enders, Moeller and Topping show that if a Type-I Ricci flow solution (M, g(t)) contains a Type-I singular point p$\in M$, then for every infinite sequence λ_j, the corresponding rescaled Ricci flows

$$g_j(t) = \lambda_j g(T + \lambda_j^{-1} t)$$

on the time interval [-$\lambda_j T$,0) subconverge to a normalized nontrivial gradient shrinking soliton; i.e., a solution (g,f) of

$$\mathrm{Rc} + \nabla^2 f = \frac{1}{2(T-t)} g \qquad \text{and} \qquad \frac{\partial}{\partial t} f = |\nabla f|^2.$$

We show in this example that a Type-I singular Ricci flow solution with the singularity developing at spatial infinity– and nowhere on the Ricci flow solution manifold M –may have essential blowup sequences with some subsequences that have a gradient soliton limit and other subsequences that do not have a gradient soliton limit.

This theorem and its corollary (work done with Carson, Knopf, and Sesum) illustrate this.

6. **Theorem.** *There exist complete, noncompact, κ-noncollapsed Ricci flow solutions $(\mathcal{M}, g(t))$, with $\mathcal{M} := \mathbb{R} \times \mathcal{S}^p \times \mathcal{S}^p$, that develop Type-I singularities at spatial infinity.*

On each of these solutions, there exist essential blowup sequences along which a blowup limit yields a nontrivial gradient shrinking Ricci soliton, and there exist essential blowup sequences along which no blowup limit can be a gradient shrinking Ricci soliton.

7. **Corollary.** *Under the conditions of Theorem 6, a blowup limit of the flow along a sequence (x_j, t_j) with $|x_j| \to \infty$ and $t_j \to a_*$ is a nontrivial gradient soliton if and only if*

$$\lim_{j\to\infty} \frac{|\operatorname{Rm}(x_j, t_j)|}{\sup_{\mathcal{M}} |\operatorname{Rm}(\cdot, t_j)|} = 1.$$

The examples that we consider in proving this theorem and corollary have metrics of the double warped product form

$$g(x,t) = \mathrm{d}x^2 + \big(a_* - t + v_1(x,t)\big)\, g_{S^p} + \big(a_* - t + v_2(x,t)\big)\, g_{S^p}. \qquad (47)$$

We choose initial conditions on the functions v_1 and v_2 so that the metric satisfies the Main Assumptions, and so that the singularity occurs at time T= a_* and at spatial infinity.

Results from Angenent and Knopf show that for metrics of this form, we obtain a gradient shrinking soliton if and only if the curvature of the Ricci flow solution satisfies the condition stated in the corollary.

By construction, we find that there are some subsequences of the essential blowup sequences of these Ricci flow solutions which satisfy the curvature condition in the corollary, and there are other subsequences which should not satisfy this curvature condition.

Application B2: Weak Stability of the Generalized Cylinder Under Ricci Flow

Colding and Minicozzi have proven a strong stability theorem for the mean curvature flow of embedded cylinders.

As a corollary of our main results, Carson, Knopf, Sessum and I can prove the following weak stability theorem for Ricci flow solutions near generalized cylindrical metrics.

Theorem:

Let M=$R^k x\ S^p$ for positive integers k and p≥2. The cylinder metric on M is

$$g_{\mathrm{cyl}}(0) = g_{\mathrm{Eucl}} + a_* g_{S^p}.$$

for some constant a_*.

The Ricci flow for this initial metric $g_{cyl}(0)$ is

$$g_{\mathrm{cyl}}(t) = g_{\mathrm{Eucl}} + (a_* - t)\, g_{\mathbb{S}^p}$$

which becomes singular at time a_*.

If we choose a real-valued function δ(x) such that the perturbed initial metric

$$g(x,0) = g_{\mathrm{Eucl}} + u(x,0) g_{\mathbb{S}^p},$$

with u(x,0)=a_*+ δ(x)

satisfies the Main assumptions and decays appropriately at spatial infinity, Then the Ricci flow of this perturbed initial data g(x,0) develops a Type-I singularity at spatial infinity at time a_*, and satisfies the stability condition

$$\sup_{t\in[0,a_*)} |g(x,t) - g_{\mathrm{cyl}}(t)|_{g_{\mathrm{cyl}}(0)} = \big(1 + o(1;\delta(x) \searrow 0)\big)\, |g(x,0) - g_{\mathrm{cyl}}(0)|_{g_{\mathrm{cyl}}(0)}$$

Addendum: Instability of Generalized Cylinders Under Rescaled Ricci Flow

For some studies of stability of Ricci flow, one focuses on rescaled solutions.

It is easy to see that the Ricci flow of generalized cylinders is not stable under small perturbations of the initial data:

Consider a generalized cylinder metric on $R^1 x\ S^p$

$$g_{\mathrm{cyl}} = dx^2 + g_{S^p}$$

and consider a small perturbation of such a metric

$$g_\epsilon = dx^2 + (1 - \epsilon) g_{S^p}$$

If we rescale the Ricci flow of the perturbed cylinder metric g_{ϵ} by the factor $(T-t)^{-1}$ where T is the collapse time for the spherical metrics, then for ϵ positive, the rescaled Ricci flow becomes infinitely large with no singularity, while for ϵ negative, the rescaled Ricci flow becomes singular before the time T.

It follows that in this sense, the rescaled Ricci flow for cylindrical metrics is unstable.