On null singularities for the Einstein vacuum equations

and

the strong cosmic censorship conjecture in general relativity

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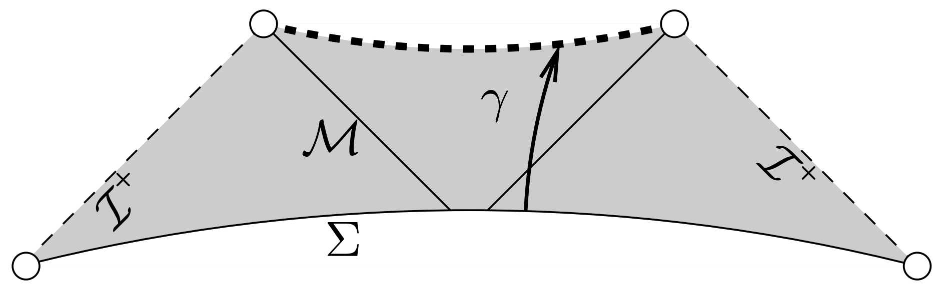
1. Schwarzschild, Reissner–Nordström/Kerr and the strong cosmic censorship conjecture

Schwarzschild

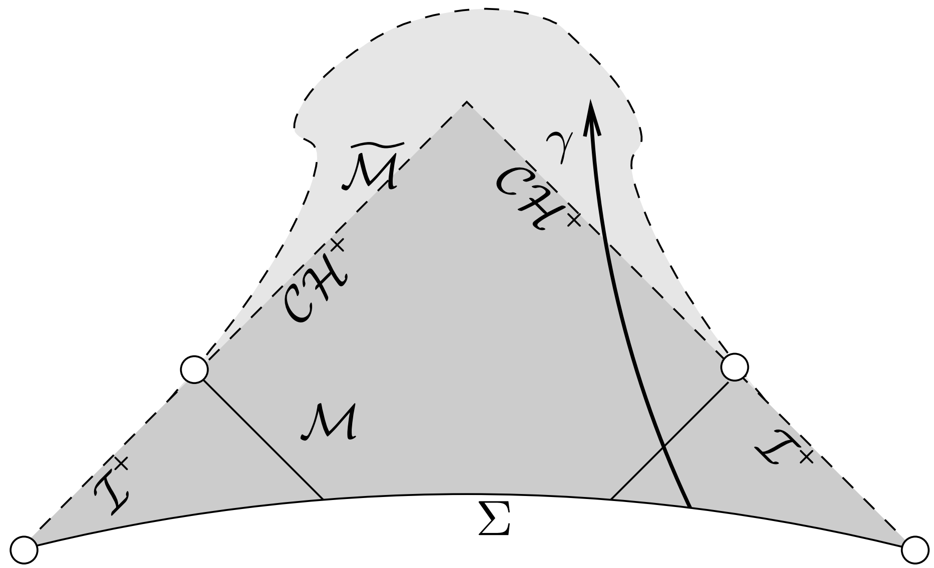


The Schwarzschild spacetime $(\mathcal{M}^{3+1}, g)$ is geodesically <u>in</u>complete: There are observers–like poor γ–who live only for finite proper time.

It turns out that **all** such observers are **torn apart** by infinite tidal forces. The spacetime is *inextendible* as a Lorentzian manifold with C^0 metric.

Is this latter prediction stable to arbitrary perturbation of initial data?

Reissner–Nordström $0 < Q < M$ **or Kerr** $0 < |a| < M$



The part of spacetime determined by initial data is extendible C^∞ into a larger spacetime into which observers γ enter in finite time.

These extensions are severly non-unique. *What happens to the observers?*

Strong cosmic censorship

Conjecture (Strong cosmic censorship, PENROSE 1972)**.** *For generic asymptotically flat initial data for the Einstein vacuum equations* $\mathrm{Ric} = 0$*, the maximal Cauchy development is future* <u>*inextendible*</u> *as a suitably regular Lorentzian manifold.*

One should think of this conjecture as a statement of *global uniqueness*, or, in more colloquial language:

"The future is uniquely determined by the present".

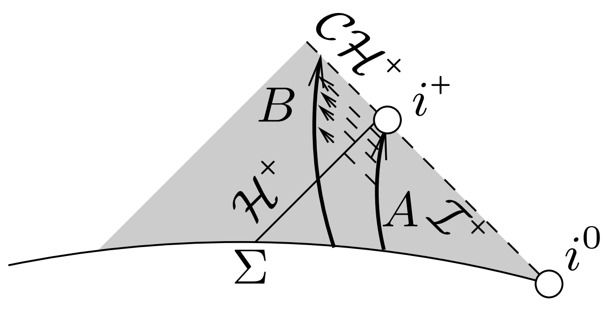
The **in**extendibility requirement of the conjecture is *true* then in Schwarzschild, but *false* in Reissner–Nordström and Kerr for $Q \neq 0$, $a \neq 0$ respectively.

Thus, within the class of explicit stationary solutions, it is *extendibility* that is generic, not *inextendibility*, which only holds with $a = Q = 0$!

Why would one ever conjecture then that strong cosmic censorship holds?

Blue-shift instability (PENROSE, 1968)

A possible mechanism for instability is the celebrated blue-shift effect, first pointed out by PENROSE:

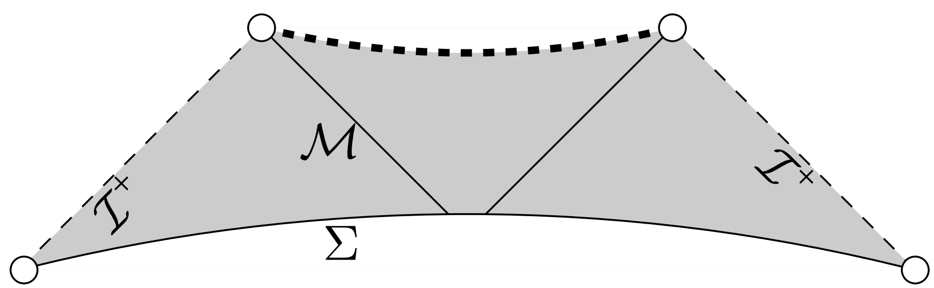


PENROSE argued that this would cause linear perturbations to blow-up in some way on a Reissner–Nordström background. Subsequent numerical study by SIMPSON–PENROSE on Maxwell fields (1972).

This suggests Cauchy horizon formation is an unstable phenomenon *once a wave-like dynamic degree of freedom is allowed*.

While linear perturbations as a matter of principle can at worst blow up *at the Cauchy horizon* $\mathcal{CH}^+$, in the full non-linear theory governed by the Einstein vacuum equations, one might expect that the non-linearities would kick in so as for blow-up to occur *before the Cauchy horizon has the chance to form.*

The conclusion which was drawn from the Simpson–Penrose analysis was that for generic dynamic solutions of the Einstein equations, the picture would revert to Schwarzschild:



The blue-shift effect in linear theory

The simplest mathematical realisation of the PENROSE heuristic account of the blue shift instability can be given as a corollary of a general recent result on the Gaussian beam approximation on Lorentzian manifolds, due to SBIERSKI. This gives:

Theorem 1 (SBIERSKI, 2012)**.** *In subextremal Reissner–Nordström or Kerr, let Σ be a two-ended asymptotically flat Cauchy surface and choose a spacelike hypersurface $\widetilde{\Sigma}$ transverse to $\mathcal{CH}^+$, let $E_\Sigma[\psi]$, $E_{\widetilde{\Sigma}}[\psi]$ denote the energy measured with respect to the normal of Σ, $\widetilde{\Sigma}$, respectively.*

Then

$$\sup_{\psi \in C^\infty : E_\Sigma[\psi]=1} E_{\widetilde{\Sigma}}[\psi] = \infty.$$

On the other hand, the radiation emitted to the black hole from initially *localised data* should in fact decay and a priori this decay could compete with the blue-shift effect. We have, however:

Theorem 2 (M.D. 2003)**.** *In subextremal Reissner–Nordström, for sufficiently regular solutions of* $\Box\psi = 0$ *of intially compact support, then if the spherical mean* ψ_0 *satisfies*

$$|\partial_v \psi_0| \geq c v^{-4} \tag{1}$$

<u>*along the event horizon* $\mathcal{H}^+$</u>*, for some constant* $c > 0$ *and all sufficiently large* v*, then* $E_{\widetilde{\Sigma}}[\psi] = \infty$*.*

The lower bound (1) is conjecturally true for generic initial data of compact support, cf. Bicak, Gundlach–Price–Pullin, . . .

The blow-up given by the above theorem, if it indeed occurs is, however, in a sense weak!

In particular, the L^∞ norm of the solution remains bounded.

Theorem 3 (A. Franzen, 2013)**.** *In subextremal Reissner–Nordström or Kerr with $M > Q \neq 0$ or $M > a \neq 0$, respectively, let ψ be a sufficiently regular solution of the wave equation. Then*

$$|\psi| \leq C$$

globally in the black hole interior up to and including $\mathcal{CH}^+$.

The above result generalised a previous result (M.D. 2003) concerning spherically symmetric solutions in the Reissner–Nordström case.

See upcoming results of Gajic for the extremal case.

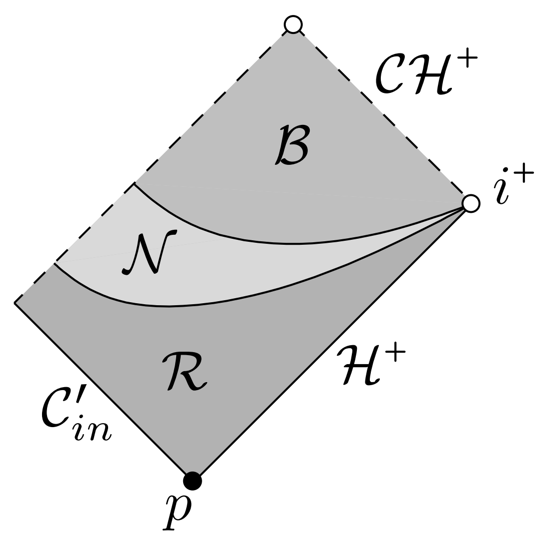
The first input into the proof is an upper bound for the decay rate of a scalar field along the event horizon $\mathcal{H}^+$ of a general subextremal Kerr metric ($0 < |a| < M$) which follows from work of M.D.–Rodnianski–Shlapentokh- Rothman on the wave equation on exterior Kerr:

$$\int_v^\infty |\partial_v \psi|^2 \leq v^{-1-\delta}$$

(A similar estimate holds in the much easier Reissner–Nordström case (cf. Blue.))

Having decay on the event horizon for $\partial_v \psi$, one now needs to propagate estimates in the black hole interior.

The interior can be partitioned into a red-shift region $\mathcal{R}$, a no-shift region $\mathcal{N}$, and a blue-shift region $\mathcal{B}$, separated by constant-r curves where $r = r_+ - \epsilon$, $r = r_- + \epsilon$, respectively.



In the blue-shift region $\mathcal{B}$, one applies the energy identity corresponding to a vector field

$$v^p \partial_v + u^p \partial_u$$

in Eddington–Finkelstein coordinates, with $p > 1$. In a regular coordinate V with $V = 0$ at the Cauchy horizon, this is $(\log V)^{-p} V \partial_V + u^p \partial_u$.

One can derive an energy estimate yielding the boundedness of the flux

$$\int_{\mathbb{S}^2} \int v^p (\partial_v \phi)^2 + (r^2 - 2Mr + Q^2) u^p |\nabla\!\!\!/ \psi|^2 r^2 dv d\sigma_{\mathbb{S}^2}$$

The uniform boundedness of ϕ then follows from

$$\phi \leq \int \partial_v \phi dv + data \leq \int v^p (\partial_v \phi)^2 dv + \int v^{-p} dv + data,$$

commutation with angular momentum operators Ω_i, and Sobolev.

$\square$

If one "naively" extrapolates the linear behaviour of $\Box\psi = 0$ to the non-linear $\mathrm{Ric}(g) = 0$, where we think of ψ representing the metric itself in perturbation theory, whereas derivatives of ψ representing the Christoffel symbols, this suggests that the metric may extend continuously to the Cauchy horizon whereas the Christoffel symbols blow up, failing to be square integrable.

On the other hand, if one believes the original intuition, then the non-linearities of the Einstein equations should induce blow-up earlier.

Which of the two scenario holds?

Fully non-linear toy-models under spherical symmetry

The programme of studying this problem with spherically symmetric *toy-models* was initiated by Hiscock 1983, Poisson–Israel 1989, and Ori 1990.

The Einstein–Maxwell–(real) scalar field model under spherical symmetry

The simplest toy model which allows for the study of this problem in spherical symmetry with a true wave-like degree of freedom is that of a self-gravitating *real-valued* scalar field in the presence of a self-gravitating electromagnetic field.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi (T^{\phi}_{\mu\nu} + T^{F}_{\mu\nu})$$

$$T^{\phi}_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial^\alpha \phi \partial_\alpha \phi$$

$$T^{F}_{\mu\nu} = \frac{1}{4\pi} (g^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta})$$

$$\Box_g \psi = 0, \qquad \nabla^\mu F_{\mu\nu} = 0, \qquad dF = 0$$

$$(\mathcal{M}, g, \phi), \qquad g = -2\Omega^2 dudv + r^2 d\sigma_{\mathbb{S}^2}$$

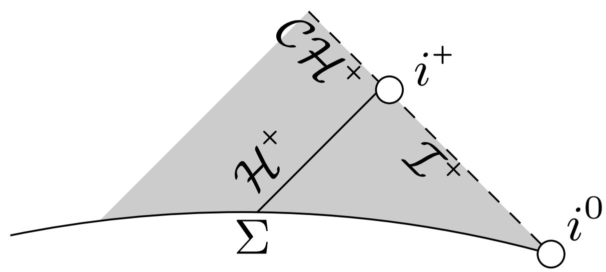
$$\begin{gathered}
\partial_u \partial_v r = -\frac{\Omega^2}{4r} - \frac{1}{r}\partial_v r \partial_u r + \frac{1}{4}\Omega^2 r^{-3} Q^2, \\
\partial_u \partial_v \log \Omega^2 = -4\pi \partial_u \phi \partial_v \phi + \frac{\Omega^2}{4r^2} + \frac{1}{r^2}\partial_v r \partial_u r - \frac{\Omega^2 Q^2}{2r^4}, \\
\partial_u (r \partial_v \phi) = -\partial_u \phi \partial_v r, \\
\partial_u (\Omega^{-2} \partial_u r) = -4\pi r \Omega^{-2} (\partial_u \phi)^2, \\
\partial_v (\Omega^{-2} \partial_v r) = -4\pi r \Omega^{-2} (\partial_v \phi)^2.
\end{gathered}$$

The system was studied numerically, originally with conflicting results

Gnedin–Gnedin 1993, Gundlach–Price–Pulin 1994, Bonano–Droz–Israel–Morsink 1995, Brady–Smith 1995, Burko 1997

It turns out, however, that one can in fact mathematically *prove* that solutions indeed exhibit all the features first discussed by Poisson–Israel and Ori.

Theorem 4 (M.D. 2001, 2003)**.** *For arbitrary asymptotically flat spherically symmetric data for the Einstein–Maxwell–real scalar field system for which the scalar field decays suitably at spatial infinity i^0, then if the charge is non-vanishing and the event horizon $\mathcal{H}^+$ is asymptotically subextremal, it follows that the Penrose diagramme contains a subset which is as below*



where $\mathcal{CH}^+$ is a <u>*non-empty*</u> *piece of* <u>*null*</u> *boundary. Moreover, the spacetime can be continued beyond $\mathcal{CH}^+$ to a strictly larger manifold with C^0 Lorentzian metric, to which the scalar field also extends continuously.*

Like for the recent result of FRANZEN about the linear problem without symmetry, the proof of the above theorem requires in particular as an input the fact that suitable decay bounds are known for the scalar field along the horizon $\mathcal{H}^+$ under the assumptions of the theorem, a statement which in turn was proven in joint work with RODNIANSKI, 2003.

Assuming such a decay statement on $\mathcal{H}^+$ for the more complicated Einstein–Maxwell–*charged* scalar field system–this is yet to be proven!–a version of Theorem 4 has recently been obtained by KOMMEMI.

Theorem 5 (M.D. 2001, 2003)**.** *If a suitable* **lower** *bound on the decay rate of the scalar field on the event horizon $\mathcal{H}^+$ is assumed (c.f. the discussion of formula* (1)*), then the non-empty piece of null boundary $\mathcal{CH}^+$ of Theorem 4 is in fact a <u>weak null singularity</u> along which the Hawking mass blows up identically, in particular, the metric cannot be continued beyond $\mathcal{CH}^+$ as a C^2 metric, in fact, as a continuous metric with square-integrable Christoffel symbols. The scalar field cannot be extended beyond $\mathcal{CH}^+$ as a H^1_{loc} function.*

The above results suggest that **"inextendible as a Lorentzian manifold with continuous metric and with Christoffel symbols in L^2_{loc}"** may be the correct formulation of *"inextendible as a suitably regular Lorentzian metric"* in the statement of strong cosmic censorship. This formulation is due to CHRISTODOULOU.

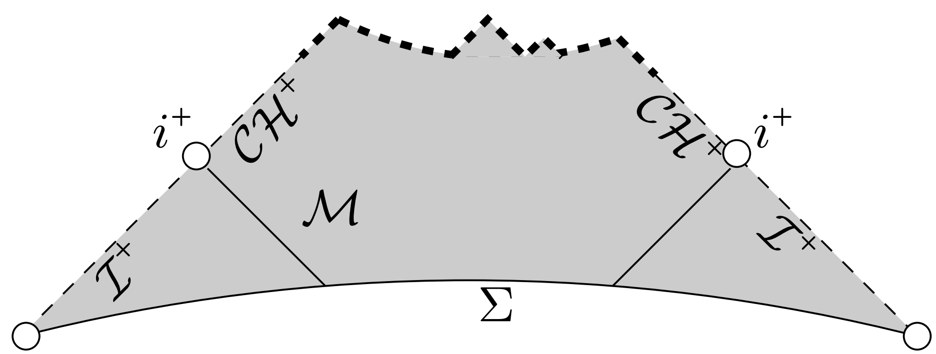
This notion of inextendibility, though not sufficient to show that macroscopic observers are torn apart in the sense of a naive Jacobi field calculation, ensures that the boundary of spacetime is singular enough so that one cannot extend the spacetime as a *weak solution* to a suitable Einstein–matter system. In this sense, it is sufficient to ensure a version of the "determinism" which SCC tries to enforce.

The Einstein–Maxwell–real scalar field system is such that for the Maxwell tensor to be non-trivial, complete initial data necessarily will have two asymptotically flat ends–just like Schwarzschild and Reissner–Nordström.

The theorems of the previous section only probed the structure of the boundary of spacetime in a neighbourhood of i^+.

What about the remaining boundary?

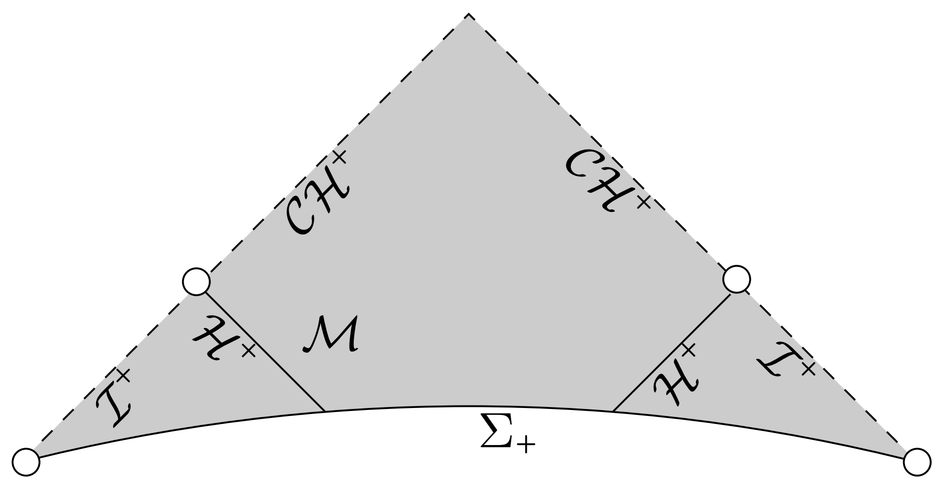
A preliminary result, using the fact that the matter model is, in language due to KOMMEMI, "strongly tame", implies that, if the initial data hypersurface Σ is moreover assumed to be "future admissible", this boundary in general is as below:



where in addition to the null boundary components $\mathcal{CH}^+$ emanating from i^+, on which r is bounded below (at this level of generality, *these components are possibly empty*, but are indeed non-empty if Theorem 4 applies), there is an (again, *possibly empty!*) achronal boundary on which r extends continuously to 0, depicted above as the thicker-shaded dotted line.

Theorem 6 (M.D. 2011). *Let $(\mathcal{M}, g, \phi, F)$ be the maximal Cauchy development of sufficiently small spherically symmetric perturbations of asymptotically flat two-ended data corresponding to subextremal Reissner–Nordström with parameters $0 < Q_{RN} < M_{RN}$, under the evolution of the Einstein–Maxwell–real scalar field system.*

Then there exists a later Cauchy surface Σ_+ which is future-admissible and such that to the future of Σ_+, the Penrose diagramme of $(\mathcal{M}, g)$ is given by:



The global bound

$$r \geq M_{RN} - \sqrt{M_{RN}^2 - Q_{RN}^2} - \epsilon$$

holds for the area-radius r *of the spherically symmetric spheres, where* $\epsilon \to 0$ *as the 'size' of the perturbation tends to* 0*. Moreover, the metric extends continuously beyond* $\mathcal{CH}^+$ *to a strictly larger Lorentzian manifold* $(\widetilde{\mathcal{M}}, \widetilde{g})$*, making* $\mathcal{CH}^+$ *a bifurcate null hypersurface in* $\widetilde{\mathcal{M}}$*. The scalar field* ϕ *extends to a continuous function on* $\widetilde{\mathcal{M}}$*. All future-incomplete causal geodesics in* $\mathcal{M}$ *extend to enter* $\widetilde{\mathcal{M}}$*.*

Finally, if ϕ *satisfies the assumption of Theorem 5 on both components of the horizon* $\mathcal{H}^+$*, then the Hawking mass extends "continuously" to* ∞ *on all of* $\mathcal{CH}^+$*. In particular,* $(\mathcal{M}, g)$ *is future inextendible as a spacetime with square integrable Christoffel symbols.*

Beyond toys:

Null singularities for the vacuum Einstein equations without symmetry

The first question one might ask is, can one construct weak null singularities for the vacuum and are they "stable" to perturbation?

This has recently been resolved in a remarkable new result of LUK

Theorem 7 (LUK)**.** *Let us be given characteristic data for the Einstein vacuum equations* $\mathrm{Ric}(g) = 0$ *defined on a bifurcate null hypersurface* $\mathcal{N}^{\mathrm{out}} \cup \mathcal{N}^{\mathrm{in}}$*, where* $\mathcal{N}^{\mathrm{out}}$ *is parameterised by affine parameter* $\underline{u} \in [0, \underline{u}^*)$*, and the data are regular on* $\mathcal{N}^{\mathrm{in}}$ *while singular on* $\mathcal{N}^{\mathrm{out}}$*, according to*

$$|\hat{\chi}| \sim |\log(\underline{u}^* - \underline{u})|^{-p} |\underline{u}^* - \underline{u}|^{-1}, \tag{2}$$

for appropriate $p > 1$*. Then the solution exists in a region foliated by a double null foliation with level sets* u, $\bar{u}$ *covering the region* $0 \le u < u^*$, $0 \le \underline{u} < \underline{u}^*$ *for* $\underline{u}^*$ *as above and sufficiently small* u^**, and the bound* (2) *propagates. The spacetime is continuously extendible beyond* $\underline{u} = \underline{u}^*$*, but the Christoffel symbols fail to be square integrable in this extension.*

The above theorem can be thought of as an extension of a recent result of LUK–RODNIANSKI on the propagation of impulsive gravitational waves, generalising explicit plane-wave solutions of PENROSE.

The setup was similar, but for impulsive gravitational waves, the shear $\hat{\chi}$ was *bounded* (but discontinuous at $\underline{u} = \underline{u}^*$), enducing on the curvature component α a delta function singularity at $\underline{u} = \underline{u}^*$.

In the new result of LUK, in contrast, $\hat{\chi}$ fails to be even in L^2 (in fact any L^p, $p > 1$), and thus the solutions cannot be extended beyond the singular front at $\underline{u} = \underline{u}^*$ as weak solutions to the vacuum Einstein equations.

Thus the situation for the new result is considerably more singular!

In their theory of impulsive waves, LUK–RODNIANSKI also had a result on the interaction of *two* impulsive wave fronts (generalising the interacting impulsive plane waves of KHAN–PENROSE).

An analogue of this result for weak null singularities is:

Theorem 8 (LUK). *Now suppose both $\mathcal{N}^{\text{in}} \cup \mathcal{N}^{\text{out}}$ are parameterised by $u \in [0, u^*)$, $\underline{u} \in [0, \underline{u}^*)$, with u^*, $\underline{u}^*$ sufficiently small, and suppose initially that both*

$$|\hat{\chi}| \sim |\log(\underline{u}^* - \underline{u})|^{-p} |\underline{u}^* - \underline{u}|^{-1}, \qquad |\underline{\hat{\chi}}| \sim |\log(u^* - u)|^{-p} |u^* - u|^{-1}, \quad (3)$$

Then the solution exists in $[0, u^*) \times [0, \underline{u}^*)$ and both bounds (3) propagate.

Theorem 9 (M.D.–Luk, to appear)**.** *Suppose we are given characteristic initial data for the Einstein vacuum equations on two intersecting null hypersurfaces* $\mathcal{H}^+_A \cup \mathcal{H}^+_B$*, such that, along each, the data are near to and in fact asymptote to (at a sufficiently fast rate) event-horizon data of a subextremal Kerr with* $a \neq 0$*.*

Then there exists a future extension $(\widetilde{\mathcal{M}}, \widetilde{g})$ *of the solution* $(\mathcal{M}, g)$ *with* C^0 *metric* $\widetilde{g}$ *such that* $\partial\mathcal{M}$ *is a bifurcate null cone in* $\widetilde{\mathcal{M}}$ *and* **all** *future incomplete geodesics in* γ *pass into* $\widetilde{\mathcal{M}} \setminus \mathcal{M}$*.*

Thus, the (conjectural) stability of the Kerr black hole exterior (up to and including the event horizon) would imply by the above theorem the global C^0-stability of the Kerr Cauchy horizon!

More generally, the above theorem implies that any spacetime settling down to Kerr in its exterior region will have a non-empty Cauchy horizon in its interior across which the metric extends C^0.

What is left to be done?

Conjecture 1. *Small perturbations of Kerr initial data on a Cauchy hypersurface indeed form an event horizon outside of which the solution settles down to a nearby Kerr solution at a sufficiently fast polynomial rate.*

If the above conjecture is true, then the statement of our theorem applies to arbitrary small perturbations of Kerr initial data on a spacelike hypersurface. It would follow that for arbitrary small perturbations on a spacelike Cauchy hypersurface, the metric can be extended as a continuous Lorentzian metric across a bifurcate Cauchy horizon. **This would then disprove the strongest formulations of strong cosmic censorship.**

Conjecture 2. *For* generic *such initial data, the resulting Cauchy horizon is indeed (globally) singular in the sense that any C^0 extension $\widetilde{\mathcal{M}}$ as above will fail to have L^2 Christoffel symbols in a neighbourhood of any point of $\partial\mathcal{M}$.*

If the above conjecture is also true, then the statement of our theorem **proves** CHRISTODOULOU's formulation of strong cosmic censorship *in a neighbourhood of the Kerr family*.

In the case of small perturbations of two-ended Kerr, the above would imply that there is no spacelike singularity period.

What happens more generally, in particular, in the one ended case?

Is there also a spacelike portion of the singularity?

Or does this null piece close up before such a singularity can occur?