MAT 539 Algebraic Topology Spring 2004

GENERAL INFORMATION

Description and goals: This is a first course in algebraic topology and will cover basic material, i.e. the elements of homotopy theory, homology and cohomology. The subject is vast, and it will be impossible to do everything. I have decided to use the book by Aguilar, Gitler and Prieto that defines the homology and cohomology groups of a space X in terms of the homotopy groups of an associated space SP(X), the infinite symmetric product. This means that the first half of the course will be almost exclusively about homotopy theory: we shall have to delve quite deeply into this, in particular proving the Dold--Thom theorem about quasifibrations, to have enough tools to prove the basic results on (co)homology. Thus students should get a good understand of homotopy theory. There should also be time to establish the main properties of homology and cohomology, including the multiplication on cohomology. If there is more time, students will decide with me what direction to go in: we could do more homological algebra (eg universal coefficient theorem), we could discuss spectral sequences, or K-theory. Here is a preliminary syllabus.

Class Assignments: There will be weekly homework assignments, posted on this page. You are expected to hand in one or two problems each week to be graded. I will be out of town during the last two weeks in March and so will miss four classes. Students will be divided into four groups, each reponsible for preparing one of these classes. I will give you topics and work with you on this assignment, and will arrange for a faculty member to attend the classes. Each student will write up a portion of the lecture material and hand it in so that I can see it as well. There will be no final exam. Grades will be based on the assignments and class participation.

Prerequisites: I will assume that students know the material in MAT 530/531. Of special relevance are the following concepts: homotopy of maps, the fundamental group and covering spaces. An excellent preparation for the class would be to read Chapters 0 and 1 in Hatcher or the introduction and chapter 1 from Aguilar, Gitler and Prieto.

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HOMEWORKS

All documents posted in this section are in PDF format.

Homework 1(updated)	Solutions
Homework 2(updated)	Solutions
Homework 3	Solutions
Homework 4 (updated, corrected and with notes)	Solutions
Homework 5 (revised, with notes)	Solutions
Homework 6	Solutions
Homework 7 (preliminary version with notes)	Solutions

Homework 8	Solutions
Homework 9 (preliminary version)	Solutions

ANNOUNCEMENTS

- Question 2 on Homework 1 was correct as stated originally; one needs to use the first part of the question to prove (ii). So I rewrote it again.
- You should choose topics for your presentations by March 5 at the very latest. For outlines of possible topics, click here.

FOR PEOPLE WITH DISABILITIES

If you have a physical, psychological, medical or learning disability that may impact your course work, please contact Disability Support Services, ECC (Educational Communications Center) Building, room 128, (631) 632-6748. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential. Students requiring emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information, go to the following web site: http://www.ehs.stonybrook.edu/fire/disabilities.asp

Last modified: Feb 3, 2004

At the moment I do not know how much we will be able to cover. Initial topics include:

the Seifert–van Kampen theorem,

H-spaces and H co-spaces (loop spaces and suspensions),

homotopy groups and associated exact sequences,

cofibrations and the homotopy extension property,

fibrations and the homotopy lifting property,

locally trivial bundles are Serre fibrations,

the Whitehead theorems on CW complexes,

infinite symmetric products and quasifibrations, the Dold–Thom theorem,

definition and main properties of homology groups, (Eilenberg–Steenrod axioms)

the Blakers–Massey theorem (excision for homotopy groups),

Hurewicz theorem on relation between homotopy and homology,

Moore spaces and Eilenberg–McLane spaces,

definition and main properties of cohomology groups,

cellular (co)homology.

Math 539 Homework 1

January 30, 2004

I recorrected Q 2, which can be proved in its original form if you use (i).

Problem 1. (i) Show that if X is a Hausdorff topological space then every compact subset of X is closed.

(ii) Show that if X is locally compact and Hausdorff (i.e. X is the union of open subsets with compact closures) then for every open U and point $x \in U$ there is an open set V with compact closure such that $x \in V \subset \overline{V} \subset U$.

Problem 2. (i) Suppose that X is Hausdorff, let $K \subset X$ be compact and suppose that $K \subset U_1 \cup \cdots \cup U_n$ where the U_i are open. Show that K is the union of a finite number of compact sets $K_j, j = 1, \ldots, M$, such that each K_j is contained in some U_i.

(ii) Consider the iterated mapping space $M(X, M(Y, Z))$ where X is Hausdorff and Y is locally compact and Hausdorff. Let $K \subset X$ be compact, $L_1, L_2 \subset Y$ be compact and $U_1, U_2 \subset Z$ be open. Denote by U^L the set of maps $f : Y \to Z$ such that $f(L) \subset U$. Show that

$$\left(U_1^{L_1} \cup U_2^{L_2}\right)^K$$

is a finite intersection of sets of the form $(U^L)^K$. Hence deduce that the sets $(U^L)^K$ (with U open and K, L compact) form a subbasis for the compact–open topology in $M(X, M(Y, Z))$. (We assume that $M(Y, Z)$ is also given the compact-open topology.)

(iii) Show that the sets $U^{K\times L}$ form a subbasis for the topology on $M(X \times Y, Z)$ where U (resp. K, L) ranges over all open (resp. compact) subsets of Z (resp. X, Y).

(iv) Deduce that the map $\phi : M(X \times Y, Z) \to M(X, M(Y, Z))$ is a homeomorphism, where for each $g \in M(X \times Y, Z)$

$$\phi(g)(x) : y \mapsto g(x, y).$$

Problem 3 (i) If A is a subspace of X and B is a subspace of Y we denote by $M(X, A; Y, B)$ the subset of $M(X, Y)$ consisting of maps $f : X \to Y$ such that $f(A) \subset B$. We give it the subspace topology. Suppose that B is a single point $*$ in Y. Then there is an obvious bijection

$$\phi : M(X, A; Y, *) \to M(X/A, *; Y, *)$$

where X/A denotes the quotient space with base point $*$ equal to the image of A in X/A. Show that if A is compact this is a homeomorphism.

(ii) Let X be the unit ball B in $\mathbf{R}^n$ and A be its boundary $\partial B = S^{n-1}$. Show that the quotient space X/A is homeomorphic to the sphere S^n (where you define S^n as the unit sphere in $\mathbf{R}^{n+1}$.)

Math 539 Homework 2

February 5, 2004

Homework is due on Tuesdays; give it to me or Yasha.

Problem 1. Let $X = \cup_n C_n \subset \mathbf{R}^2$, where C_n is the circle of radius $1/n$ centered at $(1/n, 0)$. Give X the subspace topology and put the base point at $(0,0)$.
(i) Describe the elements in $\pi_1(X, x_0)$.

(ii) Describe the group structure on $\pi_1(X, x_0)$. eg Can you give generators and relations? Can you describe it as an amalgamated free product?

Problem 2: on the direct limit. Consider a sequence $X_i, i \geq 1$, of topological spaces and, for each i let

$$j^i_{i+1} : X_i \to X_{i+1}$$

be a closed imbedding. Thus j^i_{i+1} is an injective map such that a subset $A \subset X_i$ is closed in X_i iff $j^i_{i+1}(A)$ is closed in X_{i+1}. As a set, the direct limit $X := \lim_{i\to\infty} X_i$ is the quotient of the disjoint union $\coprod_i X_i$ by the equivalence relation generated by setting $x \sim j^i_{i+1}(x)$ for all $x \in X_i$. X is given the quotient topology.

Note: *direct limits* are also called *colimits*.

Denote the composite of j^i_{i+1} with the quotient map by $j_i : X_i \to X$. (Thus we can also identify X with the union of its subsets $X'_i := j_i(X_i)$.)

(i) Show that a subset $U \subset X$ is open in X iff $j_i^{-1}(U)$ is open in X_i for all i, iff $U \cap X'_i$ is open in X'_i for all i.

(ii) Show that X has the following universal property: Suppose given maps $g_i : X_i \to Y, i \geq 1$, such that $g_{i+1} \circ j^i_{i+1} = g_i$ for all i, then there is a unique continuous map $g : X \to Y$ such that $g_i = g \circ j_i$.

(iii) Suppose that the X_i are Hausdorff. Show that every compact subset of X lies in X'_n for some n.

Problem 3. Consider the torus

$$T^2 = [0,1] \times [0,1], \quad (x,0) = (x,1)\ \forall x,\ (0,y) = (1,y)\ \forall y.$$

Write it as $X_1 \cup X_2$ where

$$X_1 = \big\{(x,y) : 1/6 < x, y < 5/6\big\} \cup L_\epsilon, \quad X_2 = \big\{(x,y) : |x - 1/2| > 1/4\big\} \cup X_2 = \big\{(x,y) : |y - 1/2| > 1/4\big\},$$

where L_ϵ is an ϵ-nhbd of the line segment $\{(x,x) : 0 \leq x \leq 1/6\}$ in T^2 (with $\epsilon < 1/6$). Put the base point x_0 at $(0,0)$.

(i) Show that X_2 is homotopy equivalent to $S^1 \vee S^1$ so that $\pi_1(X_2, x_0) = \mathbb{Z} * \mathbb{Z}$, the free group on two generators α, β where $\alpha = [f]$ is represented by $f(t) = (t, 0)$ and $\beta = [g]$ is represented by $g(t) = (0, t)$.

(ii) Calculate $\pi_1(X_1, x_0), \pi_1(X_1 \cap X_2, x_0)$. Using this decomposition of T^2 show that $\pi_1(T^2, x_0) = \mathbb{Z} \oplus \mathbb{Z}$.

(iii) (an exercise meant to illustrate the proof of van Kampen's theorem.)

Define a based homotopy

$$F : \big([0,1] \times [0,1], [0,1] \times \{0,1\}\big) \to \big(T^2, x_0\big)$$

so that $F(t,0) = f * g(t), F(t,1) = g * f(t)$ and so that the domain I^2 can be divided by the lines $t = 1/2$ and $\lambda = 1/3, 2/3$ into 6 rectangles R_{ij} such that $F(R_{ij})$ belongs either to X_1 or to X_2. Describe the paths $F(t,\lambda_j)$ for $\lambda_j = 0, 1/3, 2/3, 1$ as words in $\pi_1(X_1) *_{\pi_1(X_1 \cap X_2} \pi_1(X_2)$. Explain where the relation $(j_1)_*(h) = (j_2)_*(h)$ appears in the transition from the word corresponding to $F(t,0)$ to that given by $F(t,1)$. **Hint:** Instead of writing a formula for F it might be easiest to draw a picture of F as a map $I^2 \to I^2$. The most natural way to divide the path $F(t,0)$ into two is so that it corresponds to the word $[f][g]$, but since $[f],[g]$ both belong to $\pi_1(X_2)$ one could make different divisions, eg into $[h_1][h_2]$, where $h_1 \sim f * g$ and h_2 is nullhomotopic.

Math 539 Homework 3

February 11, 2004

Problem 1. (for those of you who know de Rahm cohomology.) Let M be a compact smooth manifold, and denote by $\mathrm{Map}_*^{sm}(M, S^1)$ the space of smooth based maps $M \to S^1$ with the topology of uniform C^1-convergence. Let $[M, S^1]_*^{sm}$ be the set of smooth homotopy classes of such maps. (Such a homotopy class corresponds to a *smooth* map $M \times I \to S^1$.) Since S^1 is a group with smooth multiplication $S^1 \times S^1 \to S^1$ (and hence an H-space) this is a group. Show that this group is isomorphic to the first de Rham cohomology group $H^1_{\mathrm{deR}}(M)$, the quotient of the closed 1-forms on M by the exact 1-forms. Under this isomorphism the map $f : M \to S^1$ corresponds to the 1-form $f^*(d\theta)$ where θ is the angular coordinate on S^1.

Question 2: (Do this after question 1) Prove that $[M, S^1]_*^{sm} = [M, S^1]_*$, in other words every continuous map $f : M \to S^1$ is homotopic to a smooth map and every continuous homotopy is homotopic (rel end points) to a smooth homotopy. **Hint:** The proof uses a smoothing procedure which I outline below. You should fill in the important details and then apply it to the case of maps $M \to S^1$.

Using partitions of unity on M one can reduce to considering a cts map $f : B \to \mathbf{R}^n$ where $B = B_1(0)$ is the closed unit ball in $\mathbf{R}^k$. Choose a smooth bump function $\beta : B \to \mathbf{R}$ such that $0 \le \beta(x) \le 1, \beta(0) = 1, \beta(x) = 0$ near ∂B; define β_ϵ by $\beta_\epsilon(x) := \beta(x/\epsilon)$ – this has support in ϵB – and then define

$$f_\epsilon(x) = c_\epsilon \int \beta_\epsilon(x - y) f(y)\, dy.$$

Here the constant c_ϵ is chosen so that $c_\epsilon \int \beta_\epsilon(y) dy = 1$.

Problem 3: on the product. (i) Show that the product $X \times Y$ of two topological spaces (given the product topology), together with the projection maps p_X, p_Y to X and Y, has the following universal property.

Consider triples (Z, f, g) in the category of topological spaces where $f \in \mathrm{Mor}(Z, X), g \in \mathrm{Mor}(Z, Y)$. These form the objects of a category $\mathcal{C}$ whose morphisms are:

$$\mathrm{Mor}\big((Z, f, g), (Z', f', g')\big) = \big\{\alpha \in \mathrm{Mor}(Z, Z') : f = f' \circ \alpha, g = g' \circ \alpha\big\}.$$

Then $(X \times Y, p_X, p_Y)$ is a *final object* in $\mathcal{C}$, i.e. for each object (Z, f, g) in $\mathcal{C}$ there is a morphism $(Z, f, g) \to (X \times Y, p_X, p_Y)$. Moreover this morphism is unique, and any other final object is isomorphic (in $\mathcal{C}$) to $(X \times Y, p_X, p_Y)$.

(ii) Formulate the equivalent universal property for the wedge $X \vee Y$. of based (or pointed) topological spaces. (This is an exercise in reversing arrows!)

(iii) What is the equivalent of the wedge in the category of all (unpointed) spaces?

(iv) What are the equivalents to the product and the wedge in the category of groups, of finitely generated commutative rings with unit?

Problem 4. Suppose that the contravariant functor

$$F : \Big(\text{based top. spaces}\Big) \to \Big(\text{groups}\Big), \quad F(X) := [X, W]_*,$$

defined by the space W has the property that the constant map $e : X \to W$ represents the identity element in $F(X)$ for all X. Show that W is a homotopy associative H-space with homotopy inverses.

Again, you want to take the same statement for the covariant functor $X \rightsquigarrow [Q, X]_*$ and reverse all arrows (making appropriate substitutions for the wedge etc.).

Math 539 Homework 4 plus Notes

February 17, 2004

Problem 0: I did not state the first problem on last week's homework with enough precision. The correct statement is that for connected manifolds $H^1(M;\mathbb{Z}) \cong [M, S^1]_*$. ie one starts with integral cohomology. The idea: each class in $H^1(M;\mathbb{Z})$ can be represented by a closed 1-form α whose integral round each closed loop in M is an integer. Given such α define $f_\alpha : M \to S^1 = \mathbf{R}/\mathbb{Z}$ that takes the base point $x_0 \in M$ to $0 \in \mathbf{R}/\mathbb{Z}$ by defining $f_\alpha(x) = \int_\gamma \alpha$ where γ is any smooth path from x_0 to x. Now check that $f_\alpha^*(d\theta) = \alpha$. **Hint:** You must calculate the value of $f_\alpha^*(d\theta)$ in the direction $v \in T_x(M)$. Since $f_\alpha(x)$ is indep of the choice of path γ you may assume that γ is tangent to v at its endpoint x.

Problem 1: I promised a homework problem about**basepoints**, good and bad, to help answer the question of when X is homeomorphic to the quotient X/A (for some closed subset $A \subset X$.) There does not seem to be a good general answer to this question. In the cases we are interested in (eg $I^n/\partial I^n \cong S^n$) the quotient X/A is an n-manifold. In this case the base point x_0 in X/A has a neighborhood that is homeomorphic to an open n ball and so X is homeomorphic to X/A iff A has a neighborhood N such that $N \setminus A$ is homeomorphic to the annulus $S^{n-1} \times (0,1)$.

There is a notion of a *nondegenerate* base point x_0 in a space X. Here the condition is that the inclusion $x_0 \hookrightarrow X$ has the HEP. This condition has its uses, but it does not help in the homeomorphism problem. Here are some questions.

(i) Suppose that A is a closed subset of X with the HEP. Show that the base point $x_0 \in X/A$ also has the HEP.

(ii) Find an example of a closed contractible subset $A \subset X$ such that the base point in X/A is nondegenerate but X is not homeomorphic to X/A.

NOTE: the next defn is slightly changed from what I said in lecture: it is probably better to stick to the language in the text book.

A **map** $j : A \to X$ is called a **cofibration** if given any homotopy $F : A \times I \to Z$ and any map $f : X \to Z$ that extends $F(\cdot, 0) : A \to Z$ in the sense that $f \circ j = F(\cdot, 0)$, the map f is the time 0 map of a homotopy $\widehat{F} : X \times I \to Z$ such that $\widehat{F}(j(a), t) = F(a, t)$ for all $a \in A, t \in [0,1]$. (If we are in the based category all maps are assumed to preserve the base point.)

A **pair** (X, A) is said to **have the HEP (Homotopy Extension Property)** if the inclusion $j : A \to X$ is a cofibration. It follows from Ex 3 below that a map $j : A \to X$ is a cofibration iff j is a homeomorphism onto its image and the pair $(X, j(A))$ has the HEP.

Problem 2 (The Hopf map) Think of S^3 as the unit sphere in $\mathbb{C}^2$:

$$S^3 = \{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1\}.$$

Define $f : S^3 \to S^2 = \mathbb{C} \cup \{\infty\}$ by $f(z_1, z_2) = z_1/z_2$.
(i) Show that the inverse image of each point in S^2 is a circle in S^3.

(ii) Show that the complex projective plane may be decomposed as $S^2 \cup_f B^4$ where $f : \partial B^4 = S^3 \to S^2$ is the Hopf map.

Hint: Think of $\mathbf{C}P^2$ as the space of all complex lines through 0 in $\mathbb{C}^3$. The subset of lines that intersect the plane $z_3 = 1$ forms an open subset U of $\mathbf{C}P^2$ whose complement can be identified with $S^2 := \mathbf{C}P^1$, "the line at infinity". Identify U with the interior of the (real) 4-ball $B^4 \subset \mathbf{R}^4$ in such a way that you see the attaching map is f. As a warmup, it is probably a good idea to do the real projective plane $\mathbf{R}P^2$ and the complex projective line $S^2 = \mathbf{C}P^1 = pt \cup B^2$.

Problem 3 (More on the HEP and cofibrations) (i) Show that if $j : A \to X$ is a cofibration then $j : A \to j(A)$ is a homeomorphism (where $j(A)$ is given the subspace topology). ie we can think of j as the inclusion of a subset A of X into X. (Hint: take Z to be the mapping cylinder or the mapping cone of j.)

(ii) Let $A \subset X$ be closed. Show that (X, A) has the HEP iff $W := X \times \{0\} \cup A \times I$ is a retract of $X \times I$, ie there is a map $r : X \times I \to W$ that is the identity on $W \subset X \times I$.

(iii) Show that if X is normal and A is closed, the pair (X, A) has the HEP iff there is a neighborhood V of A in X such that (V, A) has the HEP. (ie in this case having the HEP is a local property for A, depending only on a neighborhood of A in X.) *Recall:* X is **normal** if any two closed sets can be separated bu disjoint open sets. The relevant property is given by Urysohn's lemma.

NOTE There is an interesting class of metric spaces called ANRs (ANR= **Absolute Neighborhood Retract**) with the property that if X and A are ANRs such that A is a closed subset of X then (X, A) has the HEP. Every finite dimensional manifold and every paracompact manifold modelled on a Banach space is an ANR. I won't have time to go into this, but this is often a useful technical condition.

Problem 4 (More on mapping cones) The first part spells out what it means for the mapping cone to be a "natural" construction; the second part shows that its homotopy type only depends on the "homotopy class" of $f : X \to Y$.

Let $\mathcal{C}$ be the category whose objects are morphisms $f : X \to Y$ in the category $\mathcal{T}_*$ of based top spaces, and whose morphisms are commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow g_X & & \downarrow g_Y \\ X' & \xrightarrow{f'} & Y'. \end{array}$$

More formally, $\mathrm{Mor}_{\mathcal{C}}((X, Y, f), (X', Y', f'))$ is the set of pairs (g_X, g_Y) that make the diagram commute, where $g_X \in \mathrm{Mor}_{\mathcal{T}_*}(X, X')$, $g_Y \in \mathrm{Mor}_{\mathcal{T}_*}(Y, Y')$.

(i) Show that the mapping cone $f \rightsquigarrow C_f$ is a functor from $\mathcal{C}$ to $\mathcal{T}_*$.

(ii) Show that if the morphisms g_X, g_Y are homotopy equivalences then C_f is homotopy equivalent to $C_{f'}$.

Math 539 Homework 5

February 25, 2004

Given a map $f : X \to Y$ the **mapping path space** E_f is

$$E_f = \big\{(x, \widetilde{y}) \in X \times M(I, Y) \ : \ \widetilde{y}(1) = x\big\}.$$

There is a map $\pi : E_f \to Y$ where $(x, \widetilde{y}) \mapsto \widetilde{y}(0)$ and the **homotopy fiber** $P_f := \pi^{-1}(y_0)$.

A map $p : E \to B$ has the HLP if for every $f : Z \to E$ and $F : Z \times I \to B$ such that $p \circ f = F(\cdot, 0)$ there is a lift $\widehat{F} : Z \times I \to E$ such that $p \circ \widehat{F} = F$ and $\widehat{F}(\cdot, 0) = f$.

Problem 1: Comparing fiber and homotopy fiber (i) Suppose that Y is path connected. Show that the homotopy type of the homotopy fiber P_f is independent of the choice of point y_0.

(ii) Suppose that B is path connected and $p : E \to B$ has the HLP. Fix $b_0 \in B$ and set $F := p^{-1}(b_0)$, $P_p := \pi^{-1}(b_0)$, where $\pi : E_p \to B$ as above. Define a map $F \to P_p$ and show that it is a homotopy equivalence.

(iii) (i) and (ii) imply that all the fibers $F_b := p^{-1}(b)$ of p are homotopy equivalent. Prove this directly using the HLP.

Problem 2 Examples (i) Show that any projection map $p : F \times B \to B$ has the HLP.

(ii) Let $E = \big\{(a, b) \in [0, 2] \times (0, 2) \ : b \le 1 \text{ if } a \le 1\big\}$, and let $p : E \to [0, 2]$ be the projection $(a, b) \mapsto a$. Show that $p : E \to A$ has the HLP.

(iii) Let $E' := E \cup (\{1\} \times (0, 2))$. Show that projection $p : E' \to A$ does NOT have the HLP, even though its fibers are all homotopy equivalent.

Problem 3 Pullbacks Let $p' : g^*(E) \to A$ be the pullback of $p : E \to B$ by the map $g : A \to B$. So

$$g^*(E) = \{(a, e) : g(a) = p(e)\}.$$

Show that $p' : g^*(E) \to A$ has the HLP if $p : E \to B$ does.

Problem 4 Barrett–Puppe fibration sequence Suppose that X and Y are path connected. In class I sketched a proof that for any based map $f : X \to Y$ with homotopy fiber P_f the homotopy fiber P_q of the map $q : P_f \to X$ is homotopy equivalent to the based loop space ΩY. (i) gives another proof.

(i) Show that $q : P_f \to X$ has the HLP. Use Problem 1(ii) to conclude that there is a homotopy equivalence $\Omega Y \to P_q$. Describe the induced map $r : \Omega Y \to P_f$

It follows that any W there is a long exact sequence

$$\cdots \to [W, \Omega X]_* \to [W, \Omega Y]_* \to [W, P_f]_* \to [W, X]_* \to [W, Y]_*.$$

Therefore, taking $W = S^n, n \ge 0$, we get the long exact sequence

$$\ldots \pi_{n+1}(X) \to \pi_{n+1}(Y) \to \pi_n(P_f) \to \pi_n(X) \to \pi_n(Y) \to \ldots.$$

Here all the maps except for $\pi_{n+1}(Y) \to \pi_n(F)$ are are the obvious ones, induced by the maps $q : P_f \to X$ and $f : X \to Y$.

(ii) Suppose that $f : X \to Y$ has the HLP. Then by Problem 1 we can replace P_f by the fiber F, so that there is a map $\delta : \pi_{n+1}(Y) \to \pi_n(F)$. Work out a nice description of this map using the commutative diagram

$$\begin{array}{ccc} \pi_{n+1}(Y) & \xrightarrow{\delta} & \pi_n(F) \\ \downarrow & & \downarrow \\ \pi_n(\Omega Y) & \xrightarrow{r} & \pi_n(P_f), \end{array}$$

where r is as in (i).

Math 539 Homework 6

March 3, 2004

Problem 1 Let $E = S^1 \times \mathbf{R} \setminus A \times (-\infty, 0]$ where $A \neq S^1$ is an open arc in S^1. The projection $p : E \to S^1$ does not have HLP (as in question 5.2(iii).)
(i) Check that $p : E \to S^1$ is a quasifibration.
(ii) Give a direct proof that the homotopy $F : E \times I \to S^1$, $F(\theta, \lambda, t) \to \theta + t$ can be lifted after a preliminary homotopy to a map $H : E \times I \to E$ such that $H(e, 0) = e$. ie there is $D : E \times I \times I \to S^1$ and H as above such that

$$D|_{E\times I\times 0} = F, \;\; D|_{E\times I\times 1} = p\circ H, \;\; D(e, 0, u) = p(e) \; \forall e \in E, u \in I.$$

Note You should think of D as a 1-parameter family of homotopies that all start at the same map (in this case p); i.e. for each u $D(\cdot, \cdot, u)$ is a homotopy starting at p.

Problem 2 The five lemma Suppose given a commutative diagram of abelian groups

$$\begin{array}{ccccccccc} A & \stackrel{i_A}{\to} & B & \stackrel{i_B}{\to} & C & \stackrel{i_C}{\to} & D & \stackrel{i_D}{\to} & E \\ \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \delta\downarrow & & \epsilon\downarrow \\ A' & \stackrel{i_{A'}}{\to} & B' & \stackrel{i_{B'}}{\to} & C' & \stackrel{i_{C'}}{\to} & D' & \stackrel{i_{D'}}{\to} & E' \end{array}$$

where the rows are exact, α is surjective, β and δ are isomorphisms and ϵ is injective. Show that γ is an isomorphism. The proof is by "diagram chasing". eg suppose that $c \in C$ is in $\ker \gamma$. If the image $d := i_C(c)$ of c in D is nonzero, then $\delta(d)$ is nonzero. But $\delta(d) = \delta \circ i_C(c) = i_{C'} \circ \gamma(c) = 0$, a contradiction. Therefore $i_C(c) = 0$ and so...

Problem 3 (Paracompactness) A space X is called **paracompact** if every open covering $\{U_\alpha\}_{\alpha\in A}$ of X has a locally finite subcovering. (ie each point in X has a neighborhood that intersects only finitely many elements in the subcovering.)

Show that any CW complex is paracompact.

Note: paracompactness is a very useful property; paracompact spaces are normal; also any covering on a paracompact space has a subordinate partition of unity. See Munkres, for example.)

Problem 4 Define $\pi_n(X, A) := [(I^n, \partial I^n, p_0); (X, A, x_0)]$ for $n \geq 1$ where $p_0 \in \partial I^n, x_0 \in A \subset X$. Show that $\pi_2(X, A)$ is a group and that $\pi_k(X, A)$ is an abelian group for $k > 2$. Find an example where $\pi_2(X, A)$ is nonabelian.
Note; This deals with some unfinished business from class. Note that $\pi_1(X, A)$ is just a pointed set and that $\pi_0(X, A)$ is not defined.

Problem 5 Write down a careful proof that that the inclusion $X \hookrightarrow Y := X \cup e^{n+1}$ is an n-equivalence. Here Y is obtained from X by attaching an $(n+1)$-cell. You may use the fact that an inclusion $A \hookrightarrow Y$ is an k equivalence iff the relative homotopy groups $\pi_i(Y, A) = 0$ for $1 \leq i \leq k$ and $\pi_0(A) \to \pi_0(X)$ is onto.

Math 539 Homework 7

April 2, 2004, due Thursday April 15

Definition of homology groups

Let (X, A) be a CW pair, i.e. X is a CW complex with subcomplex A. Then

$$H_n(X,A) := \widetilde{H}_n(X \cup CA) \cong \widetilde{H}_{n+1}\big(\Sigma(X \cup CA)\big) = \pi_{n+1}\big(SP(\Sigma(X \cup CA))\big).$$

If $X \cup CA$ is connected one can use the equivalent definition $\widetilde{H}_n(X \cup CA) = \pi_n\big(SP(X \cup CA)\big)$.

Properties of homology groups

1. Functoriality in category of pairs;
2. Excision: if X is a CW complex that is the union of subcomplexes A and B, then $H_n(A, A\cap B) = H_n(X, B)$.
3. Long exact sequence for pairs: if X is a CW complex with subcomplex A then the sequence

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \ldots$$

is exact, where i_*, j_* are induced by the inclusions. It is also functorial. One way to get this sequence: consider the sequence

$$\Sigma(A^+) \to \Sigma(X^+) \to \Sigma(X^+)/\Sigma(A^+) \simeq \Sigma(X \cup CA)$$

and the associated quasifibration

$$SP(\Sigma(A^+)) \to SP(\Sigma(X^+)) \to SP(\Sigma(X \cup CA)),$$

and take its long exact sequence in homotopy. This defines ∂ as the composite

$$\pi_{n+1}\big(SP(\Sigma(X \cup CA))\big) \cong \pi_{n+1}\big(SP(\Sigma(X^+)), SP(\Sigma(A^+))\big) \to \pi_n\big(SP(\Sigma(A^+))\big),$$

where the last map is the boundary in the long exact homotopy sequence.

Problem 1: (Picky details about basepoints) (i) Check that (for unpointed X, A) $\Sigma(X^+/A^+) = \Sigma(X^+)/\Sigma(A^+) \simeq \Sigma(X \cup CA)$. Here (since A is unpointed) CA denotes the unreduced cone $CA = A \times [0,1]/A \times \{0\}$, with base point the image of $A \times \{0\}$.

(ii) If B is pointed, then CB denotes the reduced cone $B \times [0,1]/(B \times \{0\} \cup \{b_0\} \times I$. Show that for an unpointed set A the spaces $C(A^+)$ and CA are homeomorphic.
(iii) Give an example to show that if A is pointed then $\Sigma(A^+) \not\simeq \Sigma(A)$.

Another way to define a boundary map in the l.e.s. for homology: define $\partial_1 : H_n(X,A) \to H_{n-1}(A)$ as the composite

$$H_n(X,A) = \pi_{n+1}\big(SP(\Sigma(X \cup CA))\big) \xrightarrow{q_*} \pi_{n+1}\big(SP(\Sigma(\Sigma(A^+)))\big) \overset{S}{\cong} \pi_n\big(SP(\Sigma(A^+))\big) = H_{n-1}(A),$$

where q_* is induced by the quotient map $q : X \cup CA \to \Sigma(A^+)$ and S is the desuspension isomorphism.

Problem 2: Show that $\delta = \delta_1$ by considering the diagram:

$$\begin{array}{ccc} A^+ & \to & A^+ \\ \downarrow & & \downarrow \\ X^+ & \xrightarrow{\phi} & CA^+ \\ \downarrow & & \downarrow \\ X \cup CA & \xrightarrow{q} & \Sigma(A^+) \end{array}$$

where ϕ is the constant map that takes X to the base point in CA. **Hint:** First establish that the above diagram is homotopy commutative. Then apply the functor $SP \circ \Sigma$ to get a map between two quasifibrations. Therefore there is a map between the two corresponding long exact sequences in homotopy, i.e. you get a *commutative* diagram of the form

$$\begin{array}{ccccccccc}
\pi_n(A) & \to & \pi_n(B) & \to & \pi_n(C) & \to & \pi_{n-1}(A) & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\pi_n(A') & \to & \pi_n(B') & \to & \pi_n(C') & \to & \pi_{n-1}(A') & \ldots
\end{array}$$

Now relate the boundary maps in these long exact seqeunces to δ, δ_1.

Problem 3 Unions (i) Use the functoriality of reduced homology to show that $\widetilde{H}_n(A \vee B) = \widetilde{H}_n(A) \oplus \widetilde{H}_n(B)$. (Hint: use the properties of the obvious maps $A \to A \vee B, B \to A \vee B$ and $A \vee B \to A, A \vee B \to B$.)

(ii) Show by a similar argument that $H_n(A \coprod B) = H_n(A) \oplus H_n(B)$, where $\coprod$ denotes the disjoint union.

Problem 4 Direct Limits (i) Let $X_1 \subset X_2 \subset \ldots$ be subcomplexes of X with union X. Then (essentially by definition) X is the direct limit (or colimit) $\lim_i X_i$. Show that $H_n(X) = \lim_i H_n(X_i)$, where $\lim_i H_n(X_i)$ should be interpreted as the direct limit in the category of groups.
(ii) Deduce (using Exercise 3) that if X is the disjoint union of an arbitrary number of sets X_i then $H_n(X) = \oplus_i H_n(X_i)$.

Problem 5 Show that

$$SP(\{0,1\}) \to SP(I) \to SP(S^1)$$

is not a quasifibration. Conclude that A must be path connected in the Dold–Thom theorem. Also do the example: $A = S^1 \times \{0,1\}, X = S^1 \times I$, where $a_0 = (0,0)$.

Math 539 Homework 8

April 15, 2004, due Thursday April 22

Problem 1: (Excision for π_1) Suppose that $X = A \cup B$ and set $C := A \cap B$. Find the best conditions you can under which the inclusion $\pi_1(A, C) \to \pi_1(X, B)$ is an isomorphism (of pointed sets). Prove your claim.
Hint: Adapt the proof of the Seifert–van Kampen theorem.

Problem 2: (Step 2 in Blakers–Massey theorem.) Let X be a CW complex that is the union of the subcomplexes A, B. Set $C := A \cap B$. Suppose that (A, C) is $(m-1)$-connected and that (B, C) is $(n-1)$-connected. The Blakers–Massey thm says that the map

$$i_* : \pi_q(A, C) \to \pi_q(X, B)$$

is an isomorphism for $2 \le q < m + n - 2$ and an epi for $q = m + n - 2$. We proved this in class when $A = C \cup e^m, B = C \cup e^n$ by showing that the triad homotopy group $\pi_q(X; A, B) = 0$ for $2 \le q \le m + n - 2$.

Prove this when $A = C \cup$ (cells of dim $\ge m$) and $B = C \cup$ (cells of dim $\ge n$).

Hint: You only need to prove this when A, B are obtained by adding a finite number of cells. (Why?) Therefore you can argue by induction on the numbers of added cells. Suppose you obtain A by adding a single cell to $A' \supset C$. Let $X' = A' \cup B$. Then consider the relation of the triads $(X; A, B), (X'; A', B)$ and $(X; A, X')$. The argt is easier when you add cells to B.

Problem 3: (Calculating $\pi_n(S^n)$.) There is a homomorphism $\phi : \pi_n(S^n) \to \mathbb{Z}$ given by taking the degree of any smooth map homotopic to $f : S^n \to S^n$.
(i) Define ϕ precisely, show it is well defined.
(ii) Show that ϕ is injective.
Hint: Assume f is smooth, pick a regular value x and then homotop f so that it is "linear" (ie has standard form) in a finite set of disjoint discs centered on the points in $f^{-1}(x)$. Then homotop f so it takes the interiors of these discs onto $S^n \setminus y$ (where y is the antipode of x) and takes the rest of S^n to y. Then f is a composite

$$S^n \to S^n \vee \cdots \vee S^n \xrightarrow{g} S^n,$$

where the middle space is the one point union of k copies of S^n, $k := \#\{f^{-1}(x)\}$. If f has degree 0 then $k = 2\ell$ and you can construct g to be the identity on ℓ of the spheres and a reflection on the other ℓ spheres. Now show how to homotop such a pair of maps $S^n \vee S^n \to S^n$ to zero.

Go through the above steps first for $n = 1$ and then for $n = 2$. It would be okay to write out the above proof in the case $n = 2$.
Note: In general I am rather lax in my treatment of base points. But this is permissible. eg if X is simply connected then there is a bijective correspondence between the homotopy classes of based maps $(X, x_0) \to (X, x_0)$ and the homotopy classes of arbitrary (unbased) maps $X \to X$. So when $n > 1$ we need not worry about keeping the base point fixed when calculating $\pi_n(S^n)$.

Math 539 Homework 9

April 27, 2004, due Thursday May 6

Problem 1: (Calculating $\pi_n(S^n \vee S^n)$).
(i) In class we calculated $\pi_n(S^n \vee S^n), n \geq 2$, by the following argument: because $S^n \times S^n = (S^n \vee S^n) \cup e^{2n}$,

$$\pi_n(S^n \vee S^n) = \pi_n(S^n \times S^n) = \mathbb{Z} \oplus \mathbb{Z}, \qquad n \geq 2.$$

Check all details. You may use the fact that $\pi_n(S^n) \cong \mathbb{Z}$.

(ii) Here is another argument, that also works for $n \geq 2$. Consider the pair $(X, A) = (S^n \vee S^n, S^n)$ where the sphere A is one of the obvious factors. Use the Blakers–Massey theorem to show that $\pi_n(S^n \vee S^n, S^n) \cong \pi_n(S^n)$. Now argue using the exact sequence of the pair $(S^n \vee S^n, S^n)$. (Remember there are maps from $S^n \vee S^n$ to both of its factors.)

(iii) Deduce from (i) or (ii) that $\pi_n(\vee_{j\in A} S^n_j \cong \oplus_{j\in A}\mathbb{Z}$.

Problem 2: (Symmetric products of wedges of spheres.)
(i) Show that $SP(X \vee Y) \cong SPX \times SPY$. (Here $\cong$ means that these spaces are homeomorphic.) You can show this by direct construction.
(ii) For our purposes it is enough to know that the spaces $SP(X \vee Y)$ and $SPX \times SPY$ are homotopy equivalent. Prove this by considering the fibration coming from the sequence $X \to X \vee Y \to Y$ etc.
(iii) Deduce from (ii) and Problem 1 that if X is the wedge product of spheres $\vee_{j\in A} S^n_j$ then $\pi_n(X) \to \pi_n(SPX)$ is an $(n+1)$-equivalence.

Problem 3: What is $SP(S^1 \times S^1)$? Can you work it out using the fact that $S^1 \times S^1 = (S^1 \vee S^1) \cup e^2$ and you know $SP(S^1 \vee S^1)$?

Problem 4: (Maps to $K(\pi, n)$s) (i) In class we saw that if X is $(n-1)$-connected and

$$f : \pi_n(X) \to G = \pi_n(K(G, n))$$

is any homomorphism then there is a map $\widehat{f} : X \to K(G, n)$ such that

$$\widehat{f}_* : \pi_n(X) \to \pi_n(K(G, n)$$

is f. Show that $\widehat{f}$ is unique up to homotopy.
Hint: You have a map $X \times \{0, 1\} \to K(G, n)$. Show that it can be extended to $X \times I$.

(ii) Deduce that if X is $(n-1)$-connected then $H^n(X; G) \cong \mathrm{Hom}(\pi_n(X), G)$.

Math 539 Possible topics for Presentations

February 26, 2004

Here are some suggestions. I would like you to have formed into four groups of about three each and settled on topics by the end of next week (March 5) at the very latest. Each person in the group should talk for about 20–25 mins, so you will need to divide up the topic. For these presentations to be successful they will have to be very well planned and focussed. You should pick one or two main results to concentrate on. You won't be able to go into all details, but you should try to explain all concepts and definitions and state theorems clearly, then do one or two proofs or do some examples.

I will be around through March 10 to help you plan. Tony Phillips and Jack Milnor have agreed to act as consultants and one of them will come to the lectures.

1. Homological algebra Explain the basic concepts: chain complex and its homology, maps between chain complexes, what is the equivalent of a homotopy of complexes? the five lemma; short exact sequence of chain complexes gives rise to long exact sequence in homology (and cohomology.) (Thm 7.4.10 in Aguilar, Gitler and Prieto). For reference look at almost any book on algebraic topology *except* AGP.

2. Framed cobordism and the homotopy groups of spheres You could try to explain Pontriagin's proof that $\pi_3(S^2) = \mathbb{Z}$ while $\pi_4(S^3) = \mathbb{Z}/2\mathbb{Z}$. Pontriagin showed that if M is a smooth manifold of dimension k without boundary then the set of homotopy classes of maps $M \to S^p$ are in 1-1 correspondence with the set of framed codimension $k - p$ submanifolds of M. (Proofs are given in §7 of Milnor: *Topology from a differentiable viewpoint.* Also see problems 16,17 (p.54) for how to make framed cobordisms classes into a group. This kind of geometric interpretation for homotopy classes is important now, for example in the recent proof of the Mumford conjecture about the stable homology of the mapping class groups.) To calculate $\pi_{p+1}(S^p)$ Pontriagin then classified framed 1-dimensional submanifolds of S^{p+1}. These are circles, but you have to understand the framing. It would be hard to give a complete proof here, but at least you could explain why the standard framed circle has infinite order in S^3 but order 2 in S^4.

You could well find other possible subjects in Milnor's book, such as Brouwer degree and the Poincaré–Hopf theorem on the index of vector fields. These may be too close to topics covered by the last semester's Differential Topology course. Oh the other hand, you might be interested in looking again at those topics and interpreting them in a more topological way. eg you could put the Poincaré–Hopf theorem in the context of the Euler class of the tangent bundle. (See also Topic 4 below.)

Topic 3: Hopf invariant You could do a lecture on different ways of understanding the Hopf invariant for a map $f : S^{2p-1} \to S^p$ (in particular, for the Hopf map $\pi : S^3 \to S^2$.) Problems 14 and 15 in Milnor outline its construction as a linking number. Ch 9.3 of Spanier (*Algebraic Topology*) describes it in terms of CW complexes and long exact homotopy sequences. (It is in a chapter called applications of the homology spectral sequence, but I don't think he uses either homology or spectral sequences.) There is a discussion in

AGP Ch 10.6, but it looks rather too advanced for the present. We might do it at the end of the semester.

Topic 4: Vector bundles and Characteristic classes

(a) **First Chern class** You can define the first Chern class of a complex line bundle over a Riemann surface by counting the zeros of a generic section. Using this you can define the first Chern for any complex vector bundle. Relate this to the Euler number of a vector field as mentioned in 2. (This approach is outlined for example in McDuff–Salamon: *Introduction to Symplectic Topology* Chapter 2 (see Thm 2.69, Remark 2.70) but in the context of symplectic vector bundles.)

(b) **Classifying spaces** You could discuss other approaches to characteristic classes, or talk about the classifying space for (complex) vector (or line) bundles. This is a space, usually called $BU(n)$ that carries a universal rank n vector bundle $E \to BU(n)$ and has the property that the set of isomorphism classes of vector bundle over a paracompact space X is in bijective correspondence with the homotopy classes of maps $X \to BU(n)$, i.e.

$$\mathrm{Vect}(X) = [X, BU(n)].$$

Reference: Milnor and Stasheff: *Characteristic classes* or AGP Ch 8.