MAT 542: Complex Analysis I Spring 2014

Lectures:

Lecture	Time	Location	Instructor	Office hours
L01	MW 11:00am-12:20pm,	Physics 122	Tanya Firsova `tanya AT math.sunysb.edu`	Mon, Wed 1:00-2:00pm, 3-121

Grader: Zhongshan An, zhongshan.an AT stonybrook.edu

Textbook: Lars Ahlfors, Complex Analysis, third edition.

We will also use Lecture notes by Donald Marshall

Syllabus We will follow the basic outline from the graduate core course requirements (see below), not necessarily in the same order.

Midterm: There will be one midterm, date TBA.

Grading Scheme: homeworks 30%, midterm 30% final 40%

Homeworks: There will be a homework assigned each week and collected during the Wednesday lecture. The list of problems will be posted on the course webpage.

Homework 1 is due Wednesday, February 12 on the lecture. Homework 1

Homework 2 is due Wednesday, February 19 on the lecture. Homework 2

Homework 3 is due Wednesday, February 26 on the lecture. Homework 3

Homework 4 is due Wednesday, March 5 on the lecture. Homework 4

Homework 5 is due Wednesday, March 12 on the lecture. Homework 5

Homework 6 is due Monday, March 24 on the lecture. Homework 6

Homework 7 is due Wednesday, April 9 on the lecture. Homework 7

Homework 8 is due Wednesday, April 16 on the lecture. Homework 8

Homework 9 is due Wednesday, April 30 on the lecture. Homework 9

Homework 10 is due Wednesday, May 7 on the lecture. Homework 10

Final: The final will be Thursday, 5/15/14 11:15 AM - 1:45 PM.

The list of topics for the final with references.

Notes on hyperbolic metric. The list of facts and definitions about the hyperbolic metrics that were covered in class and/or homeworks.

Special Needs

If you have a physical, psychological, medical, or learning disability that may impact your course work, please contact Disability Support Services at (631) 632-6748 or online. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential. Students who require assistance during emergency evacuation are encouraged to discuss their needs with their instructors and Disability Support Services. For procedures and information go to the following website.

Academic Integrity

Each student must pursue his or her academic goals honestly and be personally accountable for all submitted work. Representing another person's work as your own is always wrong. Faculty are required to report any suspected instances of academic dishonesty to the Academic Judiciary. Faculty in the Health Sciences Center (School of Health Technology & Management, Nursing, Social Welfare, Dental Medicine) and School of Medicine are required to follow their school-specific procedures. For more comprehensive information on academic integrity, including categories of academic dishonesty, please refer to the academic judiciary website .

Homework 1.

1. Find all complex solutions of the equation $z^5 = 1$. Draw them on the complex plane.

2. Consider a map $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}$, $f(z) = \frac{1}{z}$. Find the image of (a) the interior of the unit circle, (b) the exterior of the unit circle. Prove that the map sends the circles into circles or lines. Which circles are mapped into lines?

3. Prove the Fundamental Theorem of Algebra: Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0 = 0$ be a polynomial. Show that there exists $z_0 \in \mathbb{C}$, such that $p(z_0) = 0$.

   (a) Show that there exist M and R such that $|p(z)| > M$ whenever $|z| > R$. (Hint: Show that the term z^n dominates.)

   (b) Show that there exists z_0, such that $p(z_0) = \min_{z\in\mathbb{C}} p(z)$.

   (c) Assume that $p(z_0) \neq 0$. Let $q(z) = \frac{p(z-z_0)}{p(z_0)}$. Then

   $$q(z) = 1 + b_k z^k + \ldots b_n z^n.$$

   Obtain a contradiction by showing that $\min_{z\in\mathbb{C}} |q(z)| < 1$. (Hint: take $z = \left(-\frac{1}{b_k}\right)^{\frac{1}{k}} \epsilon$.)

4. Verify the Cauchy-Riemann equations for the function $f(z) = z^3$ by splitting f into its real and imaginary parts.

5. Let $x = r\cos\phi$, $y = r\sin\phi$. Show that the Cauchy-Riemann equations for the function $F = U + iV$ in polar coordinates are

$$r\frac{\partial U}{\partial r} = \frac{\partial V}{\partial \theta}$$
$$r\frac{\partial V}{\partial r} = -\frac{\partial U}{\partial \theta}$$

6. Determine whether each of the following functions can be a real part of a holomorphic function

   (a) $f(x) = e^x$;

(b) $f(x) = e^x \sin y$.

If yes, exhibit the holomorphic function. If not, give a proof.

Homework 2.

- *We say that U is a region if it is an open connected set.*
- *We say that a function $f(z)$ is analytic in a region U if for every point $z_0 \in U$ there exists r such that $f(z)$ can be expressed as a convergent power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ in the disk $D_r(z_0) \subset U$ (where $D_r(z_0)$ is a disk of radius r around a point z_0).*

1. Let $f(z)$ and $g(z)$ be analytic functions in a region U. Check that $f(z)g(z)$ is an analytic function on U.

2. Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence 1 and suppose $a_n \geq 0$ for all n. Prove that $z = 1$ is a singular point of f. That is, there is no function g analytic in a neighborhood U of $z = 1$ such that $f = g$ on $U \cap D$, where D is a unit disk.

3. Let $g(z)$ be an analytic function in a region U. Assume that $\forall z \in U$, $g(z) \neq 0$. Prove that $\frac{1}{g(z)}$ is an analytic function on U. Assume that there exists $z_0 \in U$ such that $|g(z_0)| = \inf_{z \in U} |g(z)|$. Conclude that $g(z)$ is a constant map.

4. Define $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.

   (a) Prove that this series converges for all $z \in \mathbb{C}$.

   (b) Prove that $e^z e^w = e^{z+w}$.

   (c) Define $\cos z = \frac{e^{iz}+e^{-iz}}{2}$ and $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$. Let $\theta \in \mathbb{R}$. Show that the series expansion for $\sin\theta$ and $\cos\theta$ are the same you learned in a calculus course.

   (d) Check that $\sin^2 z + \cos^2 z = 1$. In particular, for $\theta \in \mathbb{R}$, $e^{i\theta} = \cos\theta + i\sin\theta$ is a point on a unit circle.

   (e) Show that $|e^z| = e^{Re z}$.

   (f) Find all solutions of the equation $e^z = 1$.

5. Assume $f(z)$ is analytic in a region U and one the following conditions is satisfied:

   (a) $|f(z)| = \text{const}$ in U;

(b) $Ref(z) = \text{const}$ in U.

Show that $f(z) = \text{const}$ in U.

Homework 3.

- *We say that a function $f(z)$ defined in a region U is analytic in a point z_0 if there exists r such that $f(z)$ can be expressed as a convergent power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ in the disk $D_r(z_0) \subset U$ (where $D_r(z_0)$ is a disk of radius r around a point z_0).*
- *Recall that the upper half plane is $\mathbb{H} = \{z : \ Im\, z > 0\}$.*

1. Prove that $f(z) = z^{1/n}$ is an analytic function in a point $z = 1$.
2. Let $\mathbb{D} = \{z : |z| < 1\}$. Find a map $f : \mathbb{D} \to \mathbb{H}$ that is analytic and bijective.
3. Let $S = \{z : \ 0 < \mathrm{Im} z < 2\pi\}$. Find a map $f : S \to \mathbb{H}$ that is analytic and bijective.
4. Let $T = \{z : \mathrm{Im}\, z > 0, \mathrm{Re}\, z > 0\}$, $S = \{z : 0 < \mathrm{Im}\, z < 2\pi\}$. Find a map $f : T \to S$ that is analytic and bijective.
5. Prove that if f is non-constant and analytic on all of $\mathbb{C}$ then $f(C)$ is dense in $\mathbb{C}$.
6. Suppose f and g are analytic in $\mathbb{C}$ and $|f(z)| \leq |g(z)|$ for all z. Prove that there exists a constant c so that $f(z) = cg(z)$ for all z.

Homework 4.

- *Unit disk* $\mathbb{D} = \{|z| < 1\}$.
- *Upper half plane* $\mathbb{H} = \{Im\, z > 0\}$.

1. (a) Let $a \in \mathbb{D}$, $\phi \in [0, 2\pi)$. Let $T_{a,\phi} = e^{i\phi}\frac{z-a}{1-\bar{a}z}$. Check that each $T_{a,\phi}$ is an analytic bijective map from $\mathbb{D}$ to $\mathbb{D}$.

   (b) Let $T : \mathbb{D} \to \mathbb{D}$ be an analytic bijective map, such that $T(0) = 0$, then $T = e^{i\phi}z$, where $\phi \in [0, 2\pi)$.

   (c) Let $T : \mathbb{D} \to \mathbb{D}$ be an analytic bijective map. Show that $T = T_{a,\phi}$ for some $a \in \mathbb{C}$, $\phi \in [0, 2\pi)$.

   We say that maps $T_{a,\phi}$ form a group of conformal automorphisms of a disk $\mathbb{D}$.

2. Let $p(z)$ and $q(z)$ be polynomials. Let $f(z) = \frac{p(z)}{q(z)}$. Prove that there is a polynomial $s(z)$ and constants $c_{k,j}$ so that

$$f(z) = s(z) + \sum_{j=1}^{N} \sum_{k=1}^{n_j} \frac{c_{k,j}}{(z - z_j)^k}$$

3. (a) Suppose p is a polynomial with all its zeroes in the upper half plane $\mathbb{H}$. Prove that all of the zeroes of p' are contained in $\mathbb{H}$.

   (b) Use (a) to prove that if p is a polynomial then the zeroes of p' are contained in the (closed) convex hull of the zeroes of p. (The closed convex hull is the intersection of all half planes containing the zeroes.)

4. We say that a function $f(z)$ is analytic at infinity, if $f(\frac{1}{z})$ is analytic in a neighborhood of 0.

   Check that $\frac{p(z)}{q(z)}$ is analytic at infinity if $\deg q(z) \geq \deg p(z)$.

5. Prove that for the function $f(z) = \sum_{n=0}^{\infty} z^{2^n}$ each point on the unit circle is singular.

6. Let $f(z)$ be an analytic function in a region U. Assume it satisfies $|f^2(z) - 1| < 1$ for all points $z \in U$. Prove that either $\mathrm{Re}\, f(z) > 0$ or $\mathrm{Re}\, f(z) < 0$ throughout U.

Homework 5.

- *Let $\delta > 0$. Assume that $f(z)$ is an analytic function in $U_{z_0} := \{z : 0 < |z - z_0| < \delta\}$. We say that z_0 is a removable singularity if there exists $g(z)$, analytic in* $\{z : |z - z_0| < \delta\}$ such that $g(z) = f(z)$ for all $z \in U_{z_0}$.

- *You can use the following theorem: Let $\delta > 0$ and let $f(z)$ be an analytic function in $U_{z_0} := \{z : 0 < |z - z_0| < \delta\}$. Assume that $|f(z)| < M$ for all $z \in U_{z_0}$, then z_0 is a removable singularity.*

1. Let $f(z)$ be an analytic function in a simply connected region U. Fix a point $z_0 \in U$. Let $F(z) = \int_{\gamma_z} f(z)dz$, where γ_z is a piece-wise smooth curve that connects z_0 and z. Show that $F(z)$ is a well-defined analytic function and $F' = f$.

2. Let $\log z = \log |z| + i \arg z$. Notice that $\log z$ is not a well-defined function unless we specify the values of $\arg z$ (*it is a multifunction*).

   (a) Let $f(z) = \log z$, where $\arg z \in [0, 2\pi)$. Show that $e^{f(z)} = z$ for $z \in U_0 := \{z : z \in \mathbb{C} \backslash \mathrm{Re}_+\}$, where Re_+ is a positive real axis. Conclude that $f(z)$ is an analytic function on U_0. Is it possible to extend $f(z)$ as an analytic function to a point $z \in U_0$? What is the image of $f(z)$?

   (b) Find all analytic functions $g(z) : U_0 \to \mathbb{C}$ such that $e^{g(z)} = z$.

   (c) Let $U_\theta = \{z : z \in \mathbb{C} \backslash R_\theta\}$, where R_θ is the ray, that forms an angle θ with the positive real axis. Find all analytic functions $g(z) : U_\theta \to \mathbb{C}$ such that $e^{g(z)} = z$. Describe their images.

3. Let $\delta > 0$. Assume that $f(z)$ is analytic in $\{z : 0 < |z - z_0| < \delta\}$, $\lim_{z \to z_0} f(z) = \infty$. Prove that there exists a positive integer n such that $f(z) = \frac{g(z)}{(z - z_0)^n}$, where $g(z)$ is analytic in $\{z : |z - z_0| < \delta\}$, $g(z_0) \neq 0$.

4. (a) Let B be a base of the standard topology on $\mathbb{C}$. Consider a space $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$. We say that $U_R := \{|z| > R\}$ are base neighborhoods of ∞. The topology on $\hat{\mathbb{C}}$ is generated by $B \cup_R U_R$. Check that the space $\hat{\mathbb{C}}$ is compact and is homeomorphic to a sphere.

(b) Let $\delta > 0$, $U_\delta = \{z : |z-a| < \delta\}$. Let $g : U_\delta \backslash \{a\} \to \mathbb{C}$. Assume that $g(z) = \frac{f(z)}{(z-a)^n}$, where $f(z)$ is analytic in U_δ, $f(a) \neq 0$. Let us define $g(a) = \infty$. Then $g : U_\delta \to \hat{\mathbb{C}}$. Check that g is continuous and open. Moreover, there exists δ' such that $g : U_{\delta'} \backslash \{a\} \to \mathbb{C}$ is n-to-1 to its image.

5. Compute the following integrals

   (a) $\int_{|z|=2} \frac{dz}{z^2+1}$;

   (b) $\int_{|z|=1} e^z z^{-n} dz$.

6. Assume that f is analytic in Ω, $D_r(z_0) \subset \Omega$. Prove the Mean-Value Theorem:
$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta.$$

7. We say that a metric defined on $U \subset \mathbb{C}$ is conformal if it has the form $E(x,y)(dx^2 + dy^2)$, where E is positive function on U. We can write it in the form $(\phi(z))^2 |dz|^2$, where $\phi(z)$ is a positive function on U. Then $\phi(z)|dz|$ measures infinitesimal length of vectors, we will refer to it as a metric as well. Check that
$$\rho_D = \frac{2|dz|}{1-|z|^2}$$
is invariant with respect to conformal automorphisms (bijective analytic self-maps) of the unit disk. Moreover, if a conformal metric on D is invariant with respect to conformal autmorphisms of D, then it is a multiple of ρ_D.

Homework 6.

- *We define the singularity of $f(z)$ at ∞ to be the singularity at 0 of $g(z) = f(\frac{1}{z})$.*

1. How many roots does the equation $3z^5 + 21z^4 + 5z^3 + 6z + 7 = 0$ have in $\overline{\mathbb{D}}$? How many in $\{z : 1 < |z| < 2\}$? (*Hint: use Rouché's Theorem*)

2. (a) Let f be a meromorphic function in a region U. Let $\gamma \subset U$ be a piece-wise smooth curve. Assume that $f|_\gamma$ doesn't have zeroes or poles. Then $\frac{1}{2\pi i}\int_\gamma \frac{f'}{f} dz$ is the winding number of the curve $f(\gamma)$ around 0.

   (b) How many roots of the equation $z^4 + 8z^3 + 3z^2 + 8z + 3 = 0$ lie in the right half plane? *Hint: use Argument principle.*

3. Evaluate $\int_{|z|=1} \frac{dz}{z^2+2az+1}$.

4. Show that the function $\frac{1}{z^2}$ can not be uniformly approximated by polynomials on the annulus $A = \{z : \frac{1}{2} \leq |z| \leq 2\}$.

5. Find Laurent series expansion of the function $f(z) = e^{\frac{1}{z}}$ in $\mathbb{C}\backslash\{0\}$. Show that 0 is an essential singularity. Let n be a positive integer, find the image of $\{z : 0 < |z| < \frac{1}{n}\}$ under the map f.

6. Find the singularity at ∞ of the following functions. If the singularity is removable, give the value, if the singularity is a zero or pole, give the order.

   (a) $\frac{z^2+1}{e^z}$;
   (b) $\frac{1}{e^{\frac{1}{z}}-1} - z$;
   (c) $e^{\frac{z}{1-z}}$;
   (d) $z^2 - z$;

7. Find the expansion in powers of z for
$$\frac{z}{(z^2+4)(z-3)^2(z-4)}$$
which converges in $\{z : 3 < |z| < 4\}$.

8. Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a meromorphic function that has a pole or a removable singularity at ∞. Prove that f is rational.

Homework 7.

- *A holomorphic bijective self-map $f : U \to U$ is called a conformal automorphism.*
- *Let $U \subset \mathbb{C}$. We say that $\phi(z)|dz|$, where $\phi(z) > 0$, is a conformal metric.*
- *Let $z_1, z_2 \in U$. The curve $\gamma : [0,1] \to U$, $g(0) = z_1$, $g(1) = z_2$ is called a geodesic of the metric ϕ if $\int_\gamma \phi(z)|dz| = \inf_{\alpha:[0,1]\to U, \alpha(0)=z_1, \alpha(1)=z_2} \int_\alpha \phi(z)|dz|$.*

1. Evaluate the following integrals by method of residues:
   (a) $\int_0^\infty \frac{x^2}{x^4+5x^2+6} dx$;
   (b) $\int_0^\infty \frac{x^{3/2}}{1+x^3} dx$
   (c) $\int_0^\infty (1+x^2)^{-1} \log x dx$;
   (d) $\int_0^\infty \frac{\sin^2 x}{x^2} dx$;
2. Find a holomorphic bijective map between $\mathbb{H}$ and
   (a) $\mathbb{H}\backslash\{z : \mathrm{Im} z \geq 1, \mathrm{Re} z = 0\}$;
   (b) $\mathbb{D}\backslash\{z : \mathrm{Re} z = 0, \mathrm{Im} z \geq 0\}$.
3. Show that $f(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$ defines a conformal automorphism $f : \mathbb{H} \to \mathbb{H}$. Prove that each conformal automorphism of H has such form.
4. (a) The metric $\rho_{\mathbb{H}} = \frac{dz}{y}$ ($y = \mathrm{Im} z$) on $\mathbb{H}$ is called the hyperbolic metric. Check that $\rho_{\mathbb{H}}$ is invariant under conformal automorphisms of $\mathbb{H}$. Prove that any invariant conformal metric on $\mathbb{H}$ is a multiple of $\rho_{\mathbb{H}}$.
   (b) Describe geodesics of $\rho_{\mathbb{H}}$ on $\mathbb{H}$.
   (c) Describe geodesics of the hyperbolic metric $\rho_{\mathbb{D}} = \frac{2|dz|}{1-|z|^2}$ on $\mathbb{D}$.
   (d) Calculate the distance between $z_1, z_2 \in \mathbb{D}$ in the hyperbolic metric.

Homework 8.

- *Theorem: A family of locally uniformly bounded functions is normal.*

1. Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic map. Let $d_{\mathbb{D}}$ be the hyperbolic distance on $\mathbb{D}$. Prove that

$$d_{\mathbb{D}}(f(z_1), f(z_2)) \le d_{\mathbb{D}}(z_1, z_2).$$

Hint: Use Schwarz lemma.

2. (a) We say that $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix of a Möbius transformation $f(z) = \frac{az+b}{cz+d}$. Let $M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$, $M_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ be matrices of f_1 and f_2, show that $M_2 M_1$ is a matrix of $f_2 \circ f_1$.

   (b) Show that each Möbius transformation $f(z) \not\equiv z$ is conjugate to one of the following transformations:

   i. $z \mapsto z + 1$;

   ii. $z \mapsto \lambda z$, $\lambda \in \mathbb{C} \backslash \{0, 1\}$

   How many fixed points does the Möbius transformation have in case (1) and case (2)?

3. Assume that $f(z)$ is an automorphism of $\mathbb{H}$ that does not have fixed points in $\mathbb{H}$. Show that f is conjugate to $z \mapsto z + 1$, or $z \mapsto \lambda z$, where $\lambda \in \mathbb{R}_+$.

4. Give an example of an equicontinuous family of functions $f : \mathbb{D} \to \mathbb{R}$ that is not normal.

5. Show that the family of holomorphic functions $f : \mathbb{H} \to \mathbb{H}$ is normal.

Homework 9.

- Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function. Let $\chi(z, w)$ be chordal distance on the Riemann sphere. We say that a point z belongs to *Fatou set* $F(f)$ if for $\forall \epsilon$ there exists δ such that $\chi(w, z) < \delta$ implies

  $$\chi(f^{\circ n}(w), f^{\circ n}(z)) < \epsilon \text{ for all } n.$$

  The compliment of the Fatou set is called the *Julia set* $J(f)$.

- **Big Montel's Theorem** Let $\Omega \subset C$ be a region. The family of functions

  $$f : \Omega \to \mathbb{C} \backslash \{0, 1, \infty\}$$

  is normal.

1. Let $\Omega \subset \mathbb{C}$ be a region. Let $f_n : \Omega \to \mathbb{C}$ be holomorphic injective functions. Assume that f_n converges to f uniformly on compact sets. Prove that f is injective.

2. Show that the Fatou set $F(f)$ of a rational function f is open and the Julia set $J(f)$ is closed.

3. Describe the Fatou and Julia set for the function $z \mapsto z^n$.

4. Let $z \in J(f)$. Show that for every neighborhood U of a point z, open sets $f^{\circ n}(U)$ cover all of $\hat{\mathbb{C}}$ except for at most 2 exceptional points.

5. Prove that the Riemann surface of $\ln z$ is isomorphic to $\mathbb{C}$.

6. Prove Zalcman lemma: Let F be a family of meromorphic functions on the unit disk $\mathbb{D}$ which are not normal at 0. Then there exist sequences f_n in F, z_n, ρ_n and a nonconstant function f meromorphic in the plane such that

   $f_n(z_n + \rho_n z) \to f(z)$, locally and uniformly (in the spherical sense) in the complex plane $\mathbb{C}$, where $z_n \to 0$ and $\rho_n \to 0$. (Attempt to solve the problem without consulting Marshal's book).

Homework 10.

1. Let $f : U \to \mathbb{C}$ be a holomorphic function. Let R_f be its Riemann surface. Let $\pi : R_f \to \mathbb{C}$ be the natural projection. Construct an example of a function f (or show that it exists) such that

   (a) $\mathrm{Im}\pi = \mathbb{D}$ and π is a bijection to its image;

   (b) $\mathrm{Im}\pi = \mathbb{C}$ and π is a bijection to its image;

   (c) $\mathrm{Im}\pi = \{z : 1 < |z| < 2\}$ and π is a bijection to its image;

   (d) $\mathrm{Im}\pi = \{z : 1 < |z| < 2\}$ and π is a two-to-one covering map to its image;

   (e) π is not a covering from R_f to its image (for example, different points have different number of preimages).

2. (a) *Monodromy Theorem.* Let $f : \Omega \to \mathbb{C}$ be a holomorphic map. Let us fix a point z. Let $\gamma : [0,1] \times [0,1] \to \mathbb{C}$ be a continuous map. Let $\gamma_t(s) = \gamma(t,s)$. Assume $\gamma_t(0) = z$, $\gamma_t(1) = w$. Assume that f can be analytically continued along all γ_t for all $t \in [0,1]$. Let us denote the germ of analytic continuation along γ_t in the point w by f_t. Show that $f_t = f_0$ for all $t \in [0,1]$.

   (b) Let S_1, S_2 be 1-dimensional complex manifolds, $\pi : S_1 \to S_2$ be a holomorphic covering map. Let $f : \mathbb{D} \to S_2$ be a holomorphic map. Show that there exists a holomorphic map $g : \mathbb{D} \to S_1$ such that $\pi \circ g = f$.

3. (a) Let $h_\omega : \mathbb{C} \to \mathbb{C}$, $h_\omega(z) = z + \omega$. Take $\omega_1, \omega_2 \subset \mathbb{C}$ such that $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$. Let us consider a relation on $\mathbb{C}$, $z \sim w$, if $z = h^n_{\omega_1} \circ h^m_{\omega_2}(w)$, $m, n \in \mathbb{Z}$. Show that $\sim$ is an equivalence relation, and the factor space $T_{\omega_1,\omega_2} := \mathbb{C}/\sim$ is a complex manifold, homeomorphic to a torus.

   (b) Let $\tau = \frac{\omega_1}{\omega_2}$, then the torus $T_{1,\tau}$ is biholomorphic to T_{ω_1,ω_2}. Later we will use notation T_τ for $T_{1,\tau}$.

   (c) Let $f : T_\tau \to T_{\tau'}$ be a conformal bijective map. Let $\pi_1 : \mathbb{C} \to T_\tau$, $\pi_2 : \mathbb{C} \to T_{\tau'}$ be analytic covering maps. Show that there exists $g : \mathbb{C} \to \mathbb{C}$, a bijective analytic map, $g(0) = 0$ such that $f \circ \pi_1 = \pi_2 \circ g$.

(d) Show that T_τ is biholomorphic to $T_{\tau'}$, if and only if $\tau' = \frac{a\tau+b}{c\tau+d}$, where $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$.

4. (a) Let $\omega_1, \omega_2 \in \mathbb{C}$. Assume $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$. The Weirstrass function

$$\rho(z) := \frac{1}{z^2} + \sum_{\omega \neq 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right),$$

where summation goes over $\omega = n\omega_1 + m\omega_2$, $n, m \in \mathbb{Z}$.

Show that the series for $\rho(z)$ is convergent.

(b) Show that every double periodic even function with periods ω_1 and ω_2 can be expressed in the form
$C \prod_{k=1}^{n} \frac{\rho(z)-\rho(a_k)}{\rho(z)-\rho(b_k)}$.

(c) Show that $\rho'(z) = 4(\rho(z) - e_1)(\rho(z) - e_2)(\rho(z) - e_3)$, where $e_1 = \rho\left(\frac{\omega_1}{2}\right)$, $e_2 = \rho\left(\frac{\omega_2}{2}\right)$, $e_3 = \rho\left(\frac{\omega_1+\omega_2}{2}\right)$. (In class we have shown that the equality holds up to a constant, very the constant.)

(d) Let $\lambda(\tau) = \frac{e_3-e_2}{e_1-e_2}$, where $\tau = \frac{\omega_1}{\omega_2}$. Show that $\lambda(\frac{a\tau+b}{c\tau+d}) = \lambda(\tau)$, if $a, b, c, d \in \mathbb{Z}$, $a \equiv d \equiv 1(\mathrm{mod}2)$, $b \equiv c \equiv 0(\mathrm{mod}2)$.

List of topics for the final.

1. Geometry of complex numbers. (Section I.1, Marshal's notes)

2. Analytic functions. Local properties. (Chapter II, Marshal's notes).

3. Maximum modulus principle, Schwartz lemma, Liouville's Theorem. (Chapter III, Marshal's notes).

4. Integration. Cauchy integral formula. Equivalence of analytic and holomorphic (Chapter IV, Marshal's notes).

5. Removable singularities. Laurent series, Argument principle. Rouche's Theorem. (Chapter V, Marshal's notes).

6. Calculus of residues. Calculation of Definite Integrals (Section 4.5, Ahlfors).

7. Elementary functions (Section 2.3, Ahlfors; Chapter VI, Marshal's notes).

8. Groups of conformal automorphisms of the Riemann sphere, complex plane, unit disk (upper half plane) (Section 1.1, Marshal's notes; homework).

9. The hyperbolic metric on the unit disk and upper half plane. Invariant form of the Schwartz lemma (Lecture notes, homework).

10. Normal families. Montel's Theorem. Big Montel's Theorem (Section 5.5, Ahlfors; lecture notes; Freitag, Complex Analysis 2, Section III.3).

11. Analytic continuation (Section 8.1, Ahlfors), Riemann surfaces.

12. Picard's Theorems (Section 8.3, Ahlfors; lecture notes; Freitag, Complex Analysis 2, Section III.3).

Hyperbolic geometry

1 Conformal Metric, Geodesics

Definition 1.1. *Let $U \subset \mathbb{C}$ be a region, $\phi : U \to \mathbb{R}$, $\phi(z) > 0$ for all $z \in U$. Then $\phi(z)|dz|$ is called a conformal metric.*

Definition 1.2. *Let $z_1, z_2 \in U$. The curve $\gamma : [0,1] \to U$, $g(0) = z_1$, $g(1) = z_2$ is called a geodesic of the metric ϕ if*

$$\int_\gamma \phi(z)|dz| = \inf_{\alpha:[0,1]\to U, \alpha(0)=z_1, \alpha(1)=z_2} \int_\alpha \phi(z)|dz|.$$

2 Unit Disk

$$\mathbb{D} = \{z : |z| < 1\}$$

is a unit disk.

Lemma 2.1. *Conformal automorphisms of the unit disk have the form*

$$T_{\phi,a}(z) = e^{\imath\phi}\frac{z-a}{1-\bar{a}z},$$

where $a \in \mathbb{D}$.

Lemma 2.2. *The metric $\rho_{\mathbb{D}}(z) = \frac{2|dz|}{1-|z|^2}$ on $\mathbb{D}$ is invariant under the conformal automorphisms of the unit disk. If $\phi(z)|dz|$ is invariant under automorphisms of the unit disk, then $\phi = c\rho_{\mathbb{D}}$, where $c \in \mathbb{R}$, $c > 0$.*

The metric $\rho_{\mathbb{D}}$ is called the hyperbolic metric on $\mathbb{D}$.

Lemma 2.3. *(Invariant form of the Schwarz lemma) Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic map, then*

$$\rho_{\mathbb{D}}(f(z), f(w)) \le \rho_{\mathbb{D}}(z, w).$$

3 Upper Half Plane

$$\mathbb{H} = \{z : \mathrm{Im}(z) > 0\}$$

is upper half plane

Lemma 3.1. *Cayley transform* $C : \mathbb{H} \to \mathbb{D}$,

$$C(z) = \frac{z - i}{z + i}$$

is a bijective holomorphic map.

Lemma 3.2. *Conformal automorphisms of* $\mathbb{H}$ *have the form*

$$T(z) = \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$.

Lemma 3.3. *The metric* $\rho_{\mathbb{H}} = \frac{|dz|}{y}$ *on* $\mathbb{H}$ *invariant with respect to conformal automorphisms. Any conformal metric on* $\mathbb{H}$ *invariant with respect to conformal auromorphisms is propotional to* ρ.

The metric $\rho_{\mathbb{H}}$ is called the hyperbolic metric on $\mathbb{H}$.