MAT 550 (Real Analysis II), Spring 2015

Instructor

Dr. Malik Younsi

Contact

malik.younsi@stonybrook.edu

Lecture

TuTh 10:00am-11:20am, Math Tower 4-130.

Office hours

Tuesdays 8:30 - 10:00, Thursdays 8:30 - 10:00, and by appointment. Math Tower 4-118.

Special Announcements

- There will be a make up class on Friday April 17, 11:00-12:20 in 4-130.

Syllabus

Homeworks

- Homework 1 (due in class February 5, 2015). Solution
- Homework 2 (due in class February 12, 2015). Solution
- Homework 3 (due in class February 19, 2015). Solution
- Homework 4 (due in class February 26, 2015). Solution
- Homework 5 (due in class March 5, 2015). Solution
- Homework 6 : Read the first three sections of these notes (courtesy of Dror Varolin).
- Homework 7 (due in class April 16, 2015). Solution
- Homework 8 (due in class April 23, 2015). Solution
- Homework 9 (due in class April 30, 2015). Solution
- Homework 10. Solution

Exams

- Midterm Exam. Solution
- Final Exam. Solution

Final Grades

- Final Grades.

Link to previous course : Real Analysis I

Math Tower 4-130 Tuesday, Thursday 10:00 - 11:20

This syllabus contains the policies and expectations that the instructor has established for this course. Please read the entire syllabus carefully before continuing in this course. These policies and expectations are intended to create a productive learning atmosphere for all students. Unless you are prepared to abide by these policies and expectations, you risk losing the opportunity to participate further in the course.

Instructor: Dr. Malik Younsi (malik.younsi@stonybrook.edu)
Office: Math Tower 4-118
Office Hours: Tuesdays 8:30 - 10:00, Thursdays 8:30 - 10:00, and by appointment.

Course Description

This course is the second half of a two course introduction to real analysis. Topics to be covered include: Hilbert spaces, L^p spaces, Existence and uniqueness of ODEs, Radon measures, Fourier analysis, Distribution theory. Additional topics will be chosen as time permits.

Important Dates

Observed Holidays and Breaks (Spring 2015)

- March 16–22: no classes (Spring recess).

Exam Dates

- Midterm Exam : To be announced on the Course Website at least one week in advance.
- Final Exam : Friday May 15, 2015 11:15-1:45 pm.

Required Resources

- **Course Webpage:** www.math.sunysb.edu/∼myounsi/teaching/mat550/
- **Lecture Notes:** Attend lectures regularly and take your own notes.
- **Textbook:** Gerald B. Folland, *Real Analysis: Modern Techniques and Their Applications*, Second Edition, Wiley, John & Sons, 1999. (Reading and homework assignments will be assigned out of this textbook. Make sure you can access a copy of the textbook.)

About Attendance

Attendance is highly encouraged. **Lectures may include material not in the textbook!** The lectures may also present material in a different order than in the textbook.

Graded Components

- **Homework** – 40% of course average
  There will be a homework assignment due in class on most Thursdays. Homework assignments will be posted on the course webpage.
- **Midterm Exam** – 30% of course average
  There will be one closed book, closed notes midterm exam in class. The exam will be scheduled during the course, with the exam date posted at least one week in advance on the course webpage.
- **Final Exam** – 30% of course average
  There will be one closed book, closed notes final exam as scheduled by the university on Friday May 15, 2015 11:15-1:45 pm.

Your *course average* will be determined by a weighted average of the graded components above. Your *final grade* for the class will be based on your course average *and* on your participation.

Late Homework Policy

A student's homework assignment shall be considered late if it is not turned in to the instructor by the end of lecture on the due date. Late homework assignments may be turned in to the instructor by the end of the first lecture after the due date, but will be penalized 40%. Late homework assignments will not be accepted more than one lecture past the due date.

Missed Exam Policy

No make-up exams will be given. If a student misses a midterm exam, then the student's final exam grade will be substituted for the missed midterm. A student must sit the final exam at the scheduled time in order to receive a passing grade in the class.

Classroom Policies

Students are expected to arrive to lecture on time and remain until the lecture is concluded. (Leaving early creates distraction and is disrespectful to the instructor and your fellow students.) Cell phones should be silenced for the duration of the lecture. Tablet and laptop computers should not be used during lecture, except for taking notes.

Disability Support Services

If you have a physical, psychological, medical, or learning disability that may impact your course work, please contact Disability Support Services (631) 632-6748 or

`studentaffairs.stonybrook.edu/dss/`

They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential. Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to the following website:

`www.sunysb.edu/facilities/ehs/fire/disabilities`

Academic Integrity

Each student must pursue his or her academic goals honestly and be personally accountable for all submitted work. Representing another person's work as your own is always wrong. Faculty are required to report any suspected instance of academic dishonesty to the Academic Judiciary. For more comprehensive information on academic integrity, including categories of academic dishonesty, please refer to the academic judiciary website at

`www.stonybrook.edu/uaa/academicjudiciary/`

Critical Incident Management

Stony Brook University expects students to respect the rights, privileges, and property of other people. Faculty are required to report to the Office of Judicial Affairs any disruptive behavior that interrupts their ability to teach, compromises the safety of the learning environment, and/or inhibits students' ability to learn.

Syllabus Revision

The standards and requirements set forth in this syllabus may be modified at any time by the course instructor. Notice of such changes will be by announcement in class and changes to this syllabus will be posted on the course website.

This syllabus was last revised on January 21, 2015.

MAT 550, Real Analysis II, Spring 2015 — Homework 1

Due in Class : February 5, 2015.

Reading : Read Chapter 5, Sect. 5.5.

Turn in the following exercices. Exercise a.b refers to Exercise b in Chapter a in the Textbook.

Problem 1. Exercise 5.2. You can take for granted that $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a vector space.

Problem 2. Exercise 5.4.

Problem 3. Exercise 5.7.

Problem 4. Exercise 5.25.

Problem 5. Exercise 5.27. Note : The term *Meager* means that the set is a countable union of nowhere dense sets.

Problem 6. Exercise 5.32.

Problem 7. Exercise 5.38.

Problem 8. Show that if (α_j) is a sequence of complex numbers such that $\sum_j \alpha_j \xi_j$ converges for every sequence (ξ_j) of complex numbers with $\xi_j \to 0$ as $j \to \infty$, then $\sum_j |\alpha_j| < \infty$.

Problem 9. Does there exist a sequence of continuous positive functions f_n on $\mathbb{R}$ such that the sequence $(f_n(x))$ is unbounded if and only if x is rational? *Hint: Is* $\mathbb{Q}$ *a* G_δ?

Problem 1. Exercise 5.2.

Denote by $\alpha_1, \alpha_2, \alpha_3$ the three expressions of (5.3). Let us first prove that $\alpha_1 = \alpha_2 = \alpha_3$. If $x \in X, x \neq 0$, then $x/\|x\|$ has norm 1 and thus $T(x)/\|x\| = T(x/\|x\|) \leq \alpha_1$. Taking the supremum shows that $\alpha_2 \leq \alpha_1$. If $x \in X$, then $\|Tx\| \leq \alpha_2\|x\|$ and therefore $\alpha_3 \leq \alpha_2$. Finally, if $x \in X, \|x\| = 1$, then $\|Tx\| \leq \alpha_3$ and taking the supremum over such x's yield $\alpha_1 \leq \alpha_3$.

Let us now prove that $\|\cdot\|$ defines a norm on $\mathcal{L}(X, Y)$. If $S, T \in \mathcal{L}(X, Y)$ and $x \in X$, then

$$\|(S + T)x\| = \|Sx + Tx\| \leq \|Sx\| + \|Tx\| \leq (\|S\| + \|T\|)\|x\|$$

and hence $\|S + T\| \leq \|S\| + \|T\|$. If $\lambda \in \mathbb{C}$, then $\|\lambda Tx\| = |\lambda|\|Tx\| \leq |\lambda|\|T\|\|x\|$ which shows that $\|\lambda T\| \leq |\lambda|\|T\|$. For the reverse inequality, replace λ by λ^{-1} and T by λT. Lastly, if $\|T\| = 0$ then $\|Tx\| = 0$ for all x and $T = 0$.

Problem 2. Exercise 5.4.

Suppose that $T_n \to T$ and $x_n \to x$. Then we have

$$\|T_n x_n - Tx\| = \|T_n(x_n - x) + (T_n - T)x\| \leq \|T_n\|\|x_n - x\| + \|T_n - T\|\|x\|$$

and the right-hand side tends to 0 as $n \to \infty$.

Problem 3. Exercise 5.7.

a. Note that if S is any bounded linear operator and $n \in \mathbb{N}$, then $\|S^n\| \leq \|S\|^n$. Hence $\|(I - T)^n\| \leq \|(I - T)\|^n$ and since $\|I - T\| < 1$, we have that the partial sums of $\sum(I - T)^n$ form a Cauchy sequence in $\mathcal{L}(X, X)$, which converges to some element S by completeness. If S_N denotes the N-th partial sum, then we have

$$TS_N = S_N T = I - (I - T)^{N+1}$$

and letting $N \to \infty$ shows that $S = T^{-1}$.

b. Note that $\|I - T^{-1}S\| = \|T^{-1}(T - S)\| \leq \|T^{-1}\|\|T - S\| < 1$, so by **a.** the operator $T^{-1}S$ is invertible. Clearly, this implies that S is invertible.

Problem 4. Exercise 5.25.

We follow the suggestion and let $x_n \in X$ such that $\|x_n\| = 1$ and $|f_n(x_n)| \geq \frac{1}{2}\|f_n\|$, where (f_n) is a countable dense subset of X^*. We claim that $\operatorname{span}\{x_n\}$

is dense in X. This implies separability of X, since one can obtain a countable dense subset by considering linear combinations of the x_n's with coefficients in $\mathbb{Q} + i\mathbb{Q}$. To prove the claim, suppose the contrary and let $M := \overline{\operatorname{span}\{x_n\}}$, so that $M \neq X$. By the first consequence of the Hahn-Banach theorem, there is a nonzero $f \in X^*$ which vanishes on M Let (f_{n_j}) be a subsequence of the dense sequence (f_n) with $f_{n_j} \to f$. Then we have

$$\frac{1}{2}\|f_{n_j}\| \leq |f_{n_j}(x_{n_j})| = |f_{n_j}(x_{n_j}) - f(x_{n_j})| \leq \|f_{n_j} - f\|.$$

Letting $j \to \infty$ gives $\|f\| = 0$, a contradiction.

Problem 5. Exercise 5.27. Note : The term *Meager* means that the set is a countable union of nowhere dense sets.

Let $\{r_j\}$ be an enumeration of the rationals and for $j, n \in \mathbb{N}$, let

$$V_{j,n} := \left\{ x \in \mathbb{R} : |x - r_j| < \frac{1}{n2^{j+1}} \right\}.$$

Define $A := \cup_n \cap_j V_{j,n}^c$. Then $m(A^c) \leq 1/n$ for all n, so $m(A^c) = 0$. Moreover, it is easy to see that for each n, $\cap_j V_{j,n}^c$ has empty interior. Therefore A is meagre.

Problem 6. Exercise 5.32.

It suffices to apply the Corollary to the Open Mapping Theorem to the identity map from $(X, \|\cdot\|_2)$ to $(X, \|\cdot\|_1)$.

Problem 7. Exercise 5.38.

Clearly T is linear. Since $\sup_n \|T_n x\| < \infty$ for all $x \in X$, we are in the first case of the Uniform Boundedness Principle and $M := \sup_n \|T_n\| < \infty$. For all n and all x, we have $\|T_n x\| \leq \|T_n\|\|x\| \leq M\|x\|$. Letting $n \to \infty$ shows that $\|Tx\| \leq M\|x\|$ for all x, so that T is bounded and $\|T\| \leq M$.

Problem 8. Show that if (α_j) is a sequence of complex numbers such that $\sum_j \alpha_j \xi_j$ converges for every sequence (ξ_j) of complex numbers with $\xi_j \to 0$ as $j \to \infty$, then $\sum_j |\alpha_j| < \infty$.

Let c_0 denote the space of all sequences of complex numbers which tend to zero, equipped with the uniform norm

$$\|x\| := \sup_j |\xi_j| \qquad (x = (\xi_1, \xi_2, \dots)).$$

It is easy to see that c_0 is a Banach space (it is closed in the set of all bounded sequences). For $n \in \mathbb{N}$ and $x = (\xi_1, \xi_2, \dots) \in c_0$, let $T_n(x) := \sum_{j=1}^n \alpha_j \xi_j$. Clearly T_n is linear. Also, $|T_n(x)| \leq (\sum_{j=1}^n |\alpha_j|)\|x\|$, so that T_n is bounded and $\|T_n\| \leq \sum_{j=1}^{n} |\alpha_j|$. By taking $x \in c_0$ defined by $\xi_j = \overline{\alpha_j}/|\alpha_j|$ if $1 \leq j \leq n$ and $\alpha_j \neq 0$ and zero otherwise, we see that in fact $\|T_n\| = \sum_{j=1}^{n} |\alpha_j|$. Since $\lim_{n\to\infty} T_n(x)$ converges for every $x \in c_0$, we have that $\sup_n \|T_n\| < \infty$ by the Uniform Boundedness Principle. In other words, $\sum_{j=1}^{\infty} |\alpha_j| < \infty$.

Problem 9. Does there exist a sequence of continuous positive functions f_n on $\mathbb{R}$ such that the sequence $(f_n(x))$ is unbounded if and only if x is rational? *Hint: Is $\mathbb{Q}$ a G_δ?*

No, such a sequence does not exist. First note that $\mathbb{Q}$ is not a G_δ. Indeed, if $\mathbb{Q} := \{r_n\}_n$ were a G_δ, say $\mathbb{Q} = \cap_n V_n$ where each V_n is open, then

$$\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q}) = \{r_n\}_n \cup (\cup_n V_n^c)$$

would be a countable union of nowhere dense sets, contradicting the Baire Category theorem. However, the set of points A at which a sequence of positive continuous functions is unbounded is a G_δ :

$$A = \cap_m \cup_n \{x \in \mathbb{R} : f_n(x) > m\}.$$

MAT 550, Real Analysis II, Spring 2015 — Homework 2

Due in Class : February 12, 2015.

Reading : Finish reading Chapter 5 and start reading Chapter 6.

Turn in the following exercices. Exercise a.b refers to Exercise b in Chapter a in the Textbook.

In the following, $\mathcal{H}$ denotes a Hilbert space.

Problem 1. Exercise 5.56.

Problem 2. Exercise 5.57.

Problem 3. Exercise 5.66.

Problem 4. Show that $\mathcal{H}$ is reflexive by proving that the linear isometry $x \mapsto \hat{x}$ from $\mathcal{H}$ to $\mathcal{H}^{**}$ is surjective.

Problem 5. If $M := \{x \in \mathcal{H} : f(x) = 0\}$, where $f \in \mathcal{H}^*$ is non identically zero, prove that $M^{\perp}$ is a vector space of dimension one.

Problem 1. Exercise 5.56.

We already saw in class that $M^\perp$ is a closed vector subspace of $\mathcal{H}$ for any subset M of $\mathcal{H}$. In particular, $(E^\perp)^\perp$ is a closed subspace. If $x \in E$, then $x \perp y$ for all $y \in E^\perp$, which shows that $E \subset (E^\perp)^\perp$. Finally, if F is a closed subspace of $\mathcal{H}$ containing E, then clearly $E^\perp \supset F^\perp$ and $(E^\perp)^\perp \subset (F^\perp)^\perp$, so it suffices to show that $(F^\perp)^\perp \subset F$. If $x \in (F^\perp)^\perp$, write $x = Px + Qx$ where $Px \in F$ and $Qx \in F^\perp$. Since x is orthogonal to $F^\perp$, we obtain that $x - Px = Qx$ is orthogonal to $F^\perp$, which implies that $Qx = 0$, so $x = Px \in F$.

Problem 2. Exercise 5.57.

a. Note that for $y \in \mathcal{H}$, the map $x \mapsto \langle Tx, y\rangle$ is a bounded linear functional on $\mathcal{H}$, so there exists a unique element of $\mathcal{H}$, call it T^*y, such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in \mathcal{H}$. This defines a map $T^* : \mathcal{H} \to \mathcal{H}$.

To prove uniqueness, note that if $\langle x, Sy\rangle = \langle x, T^*y\rangle$ for all $x, y \in \mathcal{H}$, then $Sy - T^*y$ is orthogonal to every element of $\mathcal{H}$, in particular to itself, so $Sy = T^*y$ and this holds for every $y \in \mathcal{H}$.

It is easy to see by construction that T^* is linear. Moreover, T^* is bounded since by construction the norm of T^*y is equal to the norm of the linear functional $x \mapsto \langle Tx, y\rangle$, which is less than $\|T\|\|y\|$ by the Schwarz inequality. This also shows that $\|T^*\| \le \|T\|$.

b. To prove that $\|T\| = \|T^*\|$, recall that if $v \in \mathcal{H}$, then the norm of the linear functional $u \mapsto \langle u, v\rangle$ is exactly the norm of v. Hence by construction of T^*, we have

$$\|T^*\| = \sup_{\|y\|=1} \|T^*y\| = \sup_{\|y\|=1} \sup_{\|x\|=1} |\langle Tx, y\rangle| = \sup_{\|x\|=1} \|Tx\| = \|T\|.$$

To prove that $\|T^*T\| = \|T\|^2$, note first that

$$\|T^*T\| \le \|T^*\|\|T\| = \|T\|^2.$$

Also, as in above, we have

$$\|T^*T\| = \sup_{\|x\|=\|y\|=1} |\langle y, T^*Tx\rangle| \ge \sup_{\|x\|=1} |\langle x, T^*Tx\rangle| = \sup_{\|x\|=1} \|Tx\|^2 = \|T\|^2.$$

Finally, the remaining identities follow easily from uniqueness of the adjoint.

c. We have that $x \in \mathcal{N}(T^*)$ if and only if $\langle T^*x, y\rangle = 0$ for all $y \in \mathcal{H}$, i.e. $\langle x, Ty\rangle = 0$ for all $y \in \mathcal{H}$, which happens if and only if $x \in \mathcal{R}(T)^\perp$. This shows that $\mathcal{N}(T^*) = \mathcal{R}(T)^\perp$. Replacing T by T^* and taking orthogonal complement on both sides, we get $\mathcal{N}(T)^\perp = (\mathcal{R}(T^*)^\perp)^\perp = \overline{\mathcal{R}(T^*)}$, where we used Exercise 5.56. Here $\overline{\mathcal{R}(T^*)}$ means the smallest closed subspace of $\mathcal{H}$ containing $\mathcal{R}(T^*)$.

d. If T is unitary, then T is invertible by definition. Moreover, if $x, y \in \mathcal{H}$, we have

$$\langle x, T^{-1}y\rangle = \langle Tx, TT^{-1}y\rangle = \langle Tx, y\rangle$$

so that $T^{-1} = T^*$ by uniqueness of the adjoint. On the other hand, if T is invertible and $T^{-1} = T^*$, then for all $x, y \in \mathcal{H}$ we have

$$\langle Tx, Ty\rangle = \langle x, T^*Ty\rangle = \langle x, y\rangle,$$

so that T is unitary.

Problem 3. Exercise 5.66.

a. Let I be the identity map from $(M, \|\cdot\|_2)$ to $(M, \|\cdot\|_u)$. Note that these are both Banach spaces since M is closed (we take for granted that $L^2([0,1], m)$ is complete, which we will prove in Chapter 6). Let $f_n, f, g \in M \subset C([0,1])$ such that $\|f_n - f\|_2 \to 0$ and $\|f_n - g\|_u \to 0$ as $n \to \infty$. It is easy to show (either by using the Dominated Convergence Theorem or the continuity of the functions) that $f = g$ everywhere on $[0,1]$. By the Closed Graph Theorem, I is bounded, i.e. there is a $C > 0$ such that

$$\|f\|_u \le C\|f\|_2 \qquad (f \in M).$$

b. For $x \in [0,1]$, let $T_x : (M, \|\cdot\|_2) \to \mathbb{C}$ be the evaluation functional $T_x(f) = f(x)$. Then clearly T_x is linear. Moreover, T_x is bounded by Part a. Since $(M, \|\cdot\|_2)$ is a Hilbert space, there is a $g_x \in M$ such that $f(x) = \langle f, g_x\rangle$ for all $f \in M$. Moreover, the norm of g_x is equal to the norm of T_x which is less than C by Part a.

c. If (f_j) is an orthonormal sequence in M and $x \in [0,1]$, then by Bessel's inequality

$$\sum_j |f_j(x)|^2 = \sum_j |\langle f_j, g_x\rangle|^2 \le \|g_x\|^2 \le C^2.$$

Integrating between 0 and 1 with respect to x yields the desired conclusion.

Problem 4. Show that $\mathcal{H}$ is reflexive by proving that the linear isometry $x \mapsto \hat{x}$ from $\mathcal{H}$ to $\mathcal{H}^{**}$ is surjective.

Let $\psi \in \mathcal{H}^{**}$. We want to show that there exists an element $x \in \mathcal{H}$ such that $\hat{x} = \psi$, i.e. $f(x) = \psi(f)$ for all $f \in \mathcal{H}^*$. First note that the map

$$y \mapsto \psi(\langle \cdot, y \rangle)$$

is a bounded conjugate-linear functional on $\mathcal{H}$, so there exists an element $x \in \mathcal{H}$ such that

$$\psi(\langle \cdot, y \rangle) = \langle x, y \rangle \qquad (y \in \mathcal{H}).$$

Now, if $f \in \mathcal{H}^*$, then $f = \langle \cdot, y \rangle$ for some $y \in \mathcal{H}$, so that

$$\psi(f) = \psi(\langle \cdot, y \rangle) = \langle x, y \rangle = f(x).$$

This shows that the map $x \to \hat{x}$ is surjective.

Problem 5. If $M := \{x \in \mathcal{H} : f(x) = 0\}$, where $f \in \mathcal{H}^*$ is non identically zero, prove that $M^\perp$ is a vector space of dimension one.

We already saw in class that $M^\perp$ is a vector space. Since $f \in \mathcal{H}^*$, there is an element $y \in \mathcal{H}, y \neq 0$ such that

$$f(x) = \langle x, y \rangle \qquad (x \in \mathcal{H}).$$

It follows that $M = \{y\}^\perp$, thus $M^\perp = (\{y\}^\perp)^\perp$ is the smallest closed subspace of $\mathcal{H}$ containing $\{y\}$, by Exercise 5.56. This closed subspace is the span of $\{y\}$ which has dimension one.

MAT 550, Real Analysis II, Spring 2015 — Homework 3

Due in Class : February 19, 2015.

Reading : Finish reading Chapter 6.

Turn in the following exercices. Exercise a.b refers to Exercise b in Chapter a in the Textbook.

Problem 1. Let $C([0,1])$ be the vector space of all (complex-valued) continuous functions on $[0,1]$, with the supremum norm. Let M consists of all $f \in C([0,1])$ such that

$$\int_0^{1/2} f(t)\,dt - \int_{1/2}^1 f(t)\,dt = 1.$$

Show that M is a nonempty closed convex subset of $C([0,1])$ which contains no element of minimal norm.

Problem 2. Let M be the set of all $f \in L^1([0,1])$ relative to Lebesgue measure such that

$$\int_0^1 f(t)\,dt = 1.$$

Show that M is a closed convex subset of $L^1([0,1])$ which contains infinitely many elements of minimal norm.

Problem 3. Exercise 6.5.

Problem 4. Exercise 6.7

Problem 5. Exercise 6.8

Problem 6. Exercise 6.9.

Problem 7. Exercise 6.10.

Problem 8. Exercise 6.13.

Problem 1. Let $C([0,1])$ be the vector space of all (complex-valued) continuous functions on $[0,1]$, with the supremum norm. Let M consists of all $f \in C([0,1])$ such that

$$\int_0^{1/2} f(t)\,dt - \int_{1/2}^1 f(t)\,dt = 1.$$

Show that M is a nonempty closed convex subset of $C([0,1])$ which contains no element of minimal norm.

The fact that M is nonempty, closed and convex is elementary. To show that M contains no element of minimal norm, first observe that if $f \in M$, then

$$1 = \left| \int_0^{1/2} f(t)\,dt - \int_{1/2}^1 f(t)\,dt \right| \le \frac{1}{2}\|f\|_\infty + \frac{1}{2}\|f\|_\infty = \|f\|_\infty$$

by the triangle inequality, hence $\|f\|_\infty \ge 1$. It is easy to see that there exist functions in M whose supremum norms are arbitrarily close to one. Suppose that M contains an element of minimal norm f with $\|f\|_\infty = 1$. Then we must have

$$1 = \operatorname{Re}\left(\int_0^{1/2} f(t)\,dt - \int_{1/2}^1 f(t)\,dt \right) = \int_0^{1/2} \operatorname{Re} f(t)\,dt - \int_{1/2}^1 \operatorname{Re} f(t)\,dt.$$

Rewrite this as

$$\int_0^{1/2} (\operatorname{Re} f(t) - 1)\,dt + \int_{1/2}^1 (-1 - \operatorname{Re} f(t))\,dt = 0.$$

Observe that each of the functions inside the integrals are negative, since $\|f\|_\infty = 1$. It follows easily that $\operatorname{Re} f(t) = 1$ for $t \in [0,1/2)$ and $\operatorname{Re} f(t) = -1$ for $t \in (1/2,1]$, so f is discontinuous at $1/2$, a contradiction.

Problem 2. Let M be the set of all $f \in L^1([0,1])$ relative to Lebesgue measure such that

$$\int_0^1 f(t)\,dt = 1.$$

Show that M is a closed convex subset of $L^1([0,1])$ which contains infinitely many elements of minimal norm.

Again, it is straightforward to show that M is a closed convex subset. If $f \in M$, then

$$1 = \left| \int_0^1 f(t)\,dt \right| \le \|f\|_1.$$

The functions $f_n := n\chi_{[0,1/n]}$ all have minimal norms.

Problem 3. Exercise 6.5.

We only prove the first statement, the proof of the second one being very similar. Suppose that there is an $\epsilon > 0$ such that every measurable set in X of positive measure has measure bigger than ϵ. Let us prove that $L^p \subset L^q$. Let $f \in L^p$, and assume that f is not identically zero. For $n \in \mathbb{N}$, define $E_n := \{|f| > n\}$. It is easy to show that

$$\mu(E_n) \leq \frac{\|f\|_p^p}{n^p}$$

(this is usually referred to as Chebyshev's inequality). It follows that $\mu(E_n) = 0$ for n large enough. But then we have

$$\int |f|^q = \int_{E_n^c} |f|^q = \int_{E_n^c} n^q \left(\frac{|f|}{n}\right)^q \leq \int_{E_n^c} n^q \left(\frac{|f|}{n}\right)^p \leq n^{q-p}\|f\|_p^p < \infty$$

so that $f \in L^q$.

Conversely, assume that X contains sets of arbitrarily small positive measure. By a standard procedure, it also contains disjoint such sets, so there is a disjoint sequence $\{E_n\}$ of measurable sets such that $0 < \mu(E_n) < 2^{-n}$. Define

$$f = \sum_{n=1}^{\infty} a_n \chi_{E_n},$$

where $a_n := \mu(E_n)^{-1/q}$. Then one easily verifies that $f \in L^p$ but $f \notin L^q$.

The case $q = \infty$ works similarly.

Problem 4. Exercise 6.7

First note that

$$|f|^q = |f|^{q-p}|f|^p \leq \|f\|_\infty^{q-p}|f|^p$$

almost everywhere. Integrating and taking q-th root gives

$$\|f\|_q \leq \|f\|_\infty^{1-p/q}\|f\|_p^{p/q},$$

hence

$$\limsup_{q\to\infty} \|f\|_q \leq \|f\|_\infty.$$

On the other hand, assume that $\|f\|_\infty > 0$ and for $0 < a < \|f\|_\infty$, define $E_a := \{|f| \geq a\}$. Then

$$\|f\|_q^q \geq \int_{E_a} |f|^q \geq a^q \mu(E_a).$$

Taking q-th root and the liminf as $q \to \infty$, we obtain

$$a \leq \liminf_{q\to\infty} \|f\|_q.$$

Letting $a \to \|f\|_\infty$ gives the result.

Problem 5. Exercise 6.8

We can assume that $\|f\|_p > 0$.
a. Let M be such that $x > M$ implies $\log x \leq x^p$. Then by splitting the integral over $\{|f| < M\}$ and its complement, we see that $\log |f|$ is integrable. The desired inequality then follows directly from Jensen's inequality.

b. Note that $\log x \leq x-1$ if $x \in [0, \infty)$. Applying this inequality to $x = \|f\|_q^q$ gives the first result.

For the second statement, note that for $a > 0$, $(a^x - 1)/x$ is increasing on $(0, \infty)$ and converges to $\log a$ as $x \to 0^+$. Moreover, we have $(\int |f|^q - 1)/q = \int(|f|^q - 1)/q$. Use the fact that $(1 - |f|^q)/q$ increases to $-\log |f|$ as q decreases to zero and apply the monotone convergence theorem.

c. By a. and b., we have

$$\exp\left(\int \log|f|\right) \leq \|f\|_q \leq \exp\left(\left(\int |f|^q - 1\right)/q\right).$$

Since the right-hand side converges to $\exp(\int \log |f|)$ as $q \to 0$, this gives the result.

Problem 6. Exercise 6.9.

If $\epsilon > 0$, then by Chebyshev's inequality (see 6.17) et have

$$\mu(\{|f_n - f| > \epsilon\}) \leq \frac{\|f_n - f\|_p^p}{\epsilon^p},$$

so that $f_n \to f$ in measure if $\|f_n - f\|_p \to 0$ as $n \to \infty$. By Theorem 2.30, there is a subsequence that converges to f almost everywhere.

On the other hand, assume that $f_n \to f$ in measure and $|f_n| \leq g \in L^p$ for all n. If $\|f_n - f\|_p \nrightarrow 0$, then there is a subsequence (f_{n_j}) and some $\epsilon > 0$ such that $\|f_{n_j} - f\|_p \geq \epsilon$ for all j. Again by Theorem 2.30, f_{n_j} has a subsequence which converges to f almost everywhere. Note that $|f| \leq g$, so we can apply the dominated convergence theorem to deduce that $\|f_{n_{j_k}} - f\|_p \to 0$, a contradiction.

Problem 7. Exercise 6.10.

If $\|f_n - f\|_p \to 0$, then $\|f_n\|_p \to \|f\|_p$ since by the triangle inequality we have

$$|\|f_n\|_p - \|f\|_p| \leq \|f_n - f\|_p.$$

Conversely, assume that $\|f_n\|_p \to \|f\|_p$. First note that for arbitrary complex numbers α and β, we have

$$|\alpha - \beta|^p \leq 2^{p-1}(|\alpha|^p + |\beta|^p)$$

(this can be proved using the triangle inequality and the convexity of the function $t \mapsto t^p$ on $[0, \infty)$). Put

$$h_n := 2^{p-1}(|f|^p + |f_n|^p) - |f - f_n|^p.$$

By Fatou's lemma, we get

$$2^p \|f\|_p^p \leq \liminf \int h_n = 2^p \|f\|_p^p - \limsup \int |f - f_n|^p$$

which implies that $\|f - f_n\|_p \to 0$ as $n \to \infty$.

Problem 8. Exercise 6.13.

There are several ways to prove that $L^p(\mathbb{R}^n, m)$ is separable. One way is the following. Let $\mathcal{R}$ denote the countable collection of sets in $\mathbb{R}^n$ of the form $R = \prod_{j=1}^N (a_j, b_j)$ where $a_j, b_j \in \mathbb{Q}$, $j = 1, \ldots, N$. Let S be the set of all finite linear combinations with rational coefficients of the functions $\{\chi_R\}_{R \in \mathcal{R}}$. Then it follows from Proposition 6.7 and Theorem 2.40c that S is a countable dense subset of $L^p(\mathbb{R}^n, m)$.

To show that $L^\infty(\mathbb{R}^n, m)$ is not separable, consider the uncountable family of functions $\{\chi_{[0,a)^n}\}_{a>0}$. Note that if $a_1 \neq a_2$, then $\|\chi_{[0,a_1)^n} - \chi_{[0,a_2)^n}\|_\infty = 1$. For $a > 0$, let B_a denote the open ball in $L^\infty(\mathbb{R}^n, m)$ centered at $\chi_{[0,a)^n}$ of radius $1/2$. Then all of the B_a's are disjoint open sets, so if S is any dense subset in $L^\infty(\mathbb{R}^n, m)$ then each B_a must contain a distinct element of S. This shows that S must be uncountable.

Due in Class : February 26, 2015.

Turn in the following exercices. Exercise a.b refers to Exercise b in Chapter a in the Textbook.

Problem 1. Let $1 \leq p < \infty$ and q be the exponent conjugate to p. Suppose that μ is a positive σ-finite measure and g is a measurable function such that $fg \in L^1(\mu)$ for every $f \in L^p(\mu)$. Prove that then $g \in L^q(\mu)$.

Problem 2. Show that there exists a nontrivial bounded linear functional ψ on $L^\infty([0,1], m)$ that is zero on $C([0,1])$. Deduce that there is no $g \in L^1([0,1], m)$ that satisfies

$$\psi(f) = \int_{[0,1]} fg\, dm \qquad (f \in L^\infty([0,1], m)).$$

Here m is the Lebesgue measure.

Problem 3. Show that $L^p(\mu)$ is reflexive if $1 < p < \infty$ and μ is σ-finite.

Problem 4. Exercise 6.43.

Problem 5. Let $0 < p < \infty$. Suppose that f is in weak L^p and that $\mu(\{f \neq 0\}) < \infty$. Show that $f \in L^q$ for all $0 < q < p$.

Problem 1. Let $1 \le p < \infty$ and q be the exponent conjugate to p. Suppose that μ is a positive σ-finite measure and g is a measurable function such that $fg \in L^1(\mu)$ for every $f \in L^p(\mu)$. Prove that then $g \in L^q(\mu)$.

Let $X_1 \subset X_2 \subset \dots$ be measurable sets of finite measure such that $X = \cup_n X_n$. For $n \in \mathbb{N}$, set $g_n := g\chi_{X_n}\chi_{\{|g|\le n\}}$, so that each g_n is in $L^q(\mu)$. Define $T_n : L^p(\mu) \to \mathbb{C}$ by

$$T_n(f) := \int f g_n \qquad (f \in L^p(\mu)).$$

Then T_n is a bounded linear functional with $\|T_n\| = \|g_n\|_q$, by a theorem that we proved in class. Clearly we are in the first case of the Uniform Boundedness Principle, so $\sup_n \|T_n\| < \infty$. By Fatou's lemma (or just pointwise convergence of the g_n's to g if $p = 1$), we get that $g \in L^q(\mu)$.

Problem 2. Show that there exists a nontrivial bounded linear functional ψ on $L^\infty([0,1], m)$ that is zero on $C([0,1])$. Deduce that there is no $g \in L^1([0,1], m)$ that satisfies

$$\psi(f) = \int_{[0,1]} fg\, dm \qquad (f \in L^\infty([0,1], m)).$$

Here m is the Lebesgue measure.

Note that $C([0,1])$ as a subset of $L^\infty([0,1], m)$ is not dense (any noncontinuous function in $L^\infty([0,1], m)$ does not belong to the closure of $C([0,1])$). The first statement is then a consequence of the Hahn-Banach Theorem (see Theorem 5.8 a.). Suppose that there is a $g \in L^1([0,1], m)$ such that

$$\psi(f) = \int_{[0,1]} fg\, dm \qquad (f \in L^\infty([0,1], m)).$$

In particular,

$$0 = \int_{[0,1]} fg\, dm \qquad (f \in C([0,1])).$$

By Lusin's theorem (or Theorem 2.41), the above identity holds with f any characteristic function of a measurable set. It follows easily that g is zero, contradicting the fact that ψ is not trivial.

Problem 3. Show that $L^p(\mu)$ is reflexive if $1 < p < \infty$ and μ is σ-finite.

This follows easily from the representation theorem of bounded linear functionals on $L^p(\mu)$ that we proved in class, as in Solution 2 Problem 4.

Problem 4. Exercise 6.43.

An annoying but simple calculation shows that

$$H\chi_{(0,1)}(x) = \begin{cases} \frac{1}{2(1-x)} & \text{if } x \leq 0 \\ 1 & \text{if } 0 < x < 1. \\ \frac{1}{2x} & \text{if } x \geq 1. \end{cases}$$

It follows easily that $H\chi_{(0,1)} \in L^p$ if $p > 1$, but that $H\chi_{(0,1)} \notin L^1$. Furthermore, using the explicit formula for $H\chi_{(0,1)}$ it is easy to see that

$$m(\{x : |H\chi_{(0,1)}(x)| > t\}) = O\left(\frac{1}{t}\right)$$

as $t \to 0$, so that $H\chi_{(0,1)}$ is in weak L^1. Finally, a calculation shows that

$$\|H\chi_{(0,1)}\|_p^p = 1 + \frac{2^{1-p}}{p-1}$$

if $p > 1$, so that $\|H\chi_{(0,1)}\|_p$ tends to ∞ like $(p-1)^{-1}$ as $p \to 1$.

Problem 5. Let $0 < p < \infty$. Suppose that f is in weak L^p and that $\mu(\{f \neq 0\}) < \infty$. Show that $f \in L^q$ for all $0 < q < p$.

By a Corollary proved in class, we have

$$\int |f|^q \, d\mu = q \int_0^\infty t^{q-1} \lambda_f(t) \, dt.$$

Set $M_1 := \mu(\{f \neq 0\})$ and $M_2 := \sup_{t>0} t^p \lambda_f(t)$. Then the right-hand side equals

$$q \int_0^1 t^{q-1} \lambda_f(t) \, dt + q \int_1^\infty t^{q-1-p} t^p \lambda_f(t) \, dt \leq M_1 q \int_0^1 t^{q-1} \, dt + M_2 q \int_1^\infty t^{q-1-p} \, dt < \infty$$

since $0 < q < p$.

MAT 550, Real Analysis II, Spring 2015 — Homework 5

Due in Class : March 5, 2015.

Turn in the following exercices. Exercise a.b refers to Exercise b in Chapter a in the Textbook.

Problem 1. Show that a subset E of $\mathbb{R}$ is the support of a continuous function $f : \mathbb{R} \to \mathbb{C}$ if and only if it is the closure of an open set.

Problem 2. Exercise 7.2.

Problem 3. Let X be a locally compact Hausdorff topological space and let μ be a positive regular measure on X. Suppose that $f : X \to \mathbb{C}$ is measurable and vanishes outside a set of finite measure. Show that if $|f| \leq 1$, then there is a sequence $\{g_n\}_{n\geq 1}$ such that $g_n \in C_c(X)$, $|g_n| \leq 1$ for all n and

$$f(x) = \lim_{n\to\infty} g_n(x) \qquad a.e.$$

Problem 1. Show that a subset E of $\mathbb{R}$ is the support of a continuous function $f : \mathbb{R} \to \mathbb{C}$ if and only if it is the closure of an open set.

If $E \subset \mathbb{R}$ is the support of a continuous function $f : \mathbb{R} \to \mathbb{C}$, then E is the closure of the open set $\{x \in \mathbb{R} : f(x) \neq 0\}$.

Conversely, assume that $E = \overline{V}$ for some open set $V \subset \mathbb{R}$. If E is empty, then E is the support of the zero function. Otherwise, write V as the countable union of disjoint open intervals (a_j, b_j). For each j, let $f_j : \mathbb{R} \to \mathbb{C}$ be a positive continuous function which is nonzero on (a_j, b_j) and vanishes identically on the complement of (a_j, b_j). For instance, one can take $f_j(x) := \mathrm{dist}(x, (a_j, b_j)^c)$. Define $f : \mathbb{R} \to \mathbb{C}$ by

$$f(x) = \sum_{j=1}^{\infty} \frac{f_j(x)}{2^j \|f_j\|_u}.$$

Then the series converges uniformly by the Weierstrass M-test, hence f is continuous. Finally, it is easy to see that $V = \{x \in \mathbb{R} : f(x) \neq 0\}$. Therefore, E is the support of f.

Problem 2. Exercise 7.2.

a. Let N be the union of all open sets $U \subset X$ such that $\mu(U) = 0$. Then N is open since it is a union of open sets. If K is any compact subset of N, then by compactness and the definition of N we get that K is contained in a finite union of open sets of measure zero, so $\mu(K) = 0$. By inner regularity of μ on N, it follows that $\mu(N) = 0$.

b. If $x \in N$, then there is an open set U with $x \in U$ and $\mu(U) = 0$. Let $f \in C_c(X)$ such that $\{x\} \prec f \prec U$. Then $0 \leq f \leq 1$, $f(x) = 1 > 0$ and $\int f \, d\mu \leq \int_U d\mu = 0$.

Conversely, assume that $x \in \mathrm{supp}(\mu)$. Let $f \in C_c(X)$, $0 \leq f \leq 1$ with $f(x) > 0$. Then $V := \{f > 0\}$ is open and contains x. Since x belongs to the support of μ, we must have $\mu(V) > 0$. It follows that $\int f d\mu \geq \int_V f d\mu > 0$.

Problem 3. Let X be a locally compact Hausdorff topological space and let μ be a positive regular measure on X. Suppose that $f : X \to \mathbb{C}$ is measurable and vanishes outside a set of finite measure. Show that if $|f| \leq 1$, then there is a sequence $\{g_n\}_{n\geq 1}$ such that $g_n \in C_c(X)$, $|g_n| \leq 1$ for all n and

$$f(x) = \lim_{n\to\infty} g_n(x) \qquad a.e.$$

By Lusin's theorem, for each n, there is a function $g_n \in C_c(X)$ with $|g_n| \leq 1$ such that $\mu(E_n) \leq 2^{-n}$, where

$$E_n := \{x \in X : f(x) \neq g_n(x)\}.$$

This implies that for almost every $x \in X$, we have $f(x) = g_n(x)$ for all n sufficiently large. Indeed, let A be the set of all $x \in X$ which lie in infinitely many E_n's. Put

$$g(x) := \sum_{n=1}^{\infty} \chi_{E_n}(x) \qquad (x \in X).$$

Then clearly $g(x) = \infty$ if and only if $x \in A$. By the monotone convergence theorem, we have

$$\int_X |g| \, d\mu = \sum_{n=1}^{\infty} \mu(E_n) \leq \sum_{n=1}^{\infty} 2^{-n} < \infty,$$

so that $g \in L^1(\mu)$ and therefore $g(x) < \infty$ almost everywhere. In other words, $\mu(A) = 0$. It follows that

$$f(x) = \lim_{n \to \infty} g_n(x) \qquad a.e.$$

ORDINARY DIFFERENTIAL EQUATIONS
NOTES FOR MAT 550 SPRING 2013

1. Definition of ODE

Definition 1.1. Let $D \subset \mathbb{R}^n$ be a domain, i.e., an open connected set.

(i) A time-dependent vector field on D is a pair consisting of a domain $V \subset D \times \mathbb{R}$ together with a continuous map $F : V \to \mathbb{R}^n$.

(ii) The time-dependent vector field F is said to be autonomous (or one simply omits the adjective *time-dependent*) if $V = D \times \mathbb{R}$ and for each $x \in D$, $F(x, \cdot)$ is constant. That is to say, a vector field on D is a continuous map $\xi : D \to \mathbb{R}^n$. (In particular, $\xi(x) = F(x,t)$ for all $(x,t) \in D \times \mathbb{R}$.) $\diamond$

Remark 1.2. In principal, the above definition can be made for more general measurable vector fields. However, since for a given vector field we will be seeking a measurable function that we will compose with the vector field in question, continuity is a natural assumption. $\diamond$

Vector fields are the data of ordinary differential equations (ODE). From this data, we want to extract so-called *integral curves*, which we now define.

Definition 1.3. Let F be a time-dependent vector field on a domain $V \subset D \times \mathbb{R}$. An integral curve through $x \in D$ with initial time s is an open set $I_{(x,s)} \subset \mathbb{R}$ containing s, together with a differentiable curve $\gamma_{(x,s)} : I_{(x,s)} \to D$, such that

(i) $\gamma_{(x,s)}(s) = x$,

(ii) $(\gamma_{(x,s)}(t), t) \in V$ for all $t \in I_{(x,s)}$, and

(iii) one has

$$\frac{d\gamma_{(x,s)}(t)}{dt} = F(\gamma_{(x,s)}(t), t).$$

The central question of ODE is whether, for a given vector field, integral curves exist and, if so, are unique. This question is partially answered in Section 3, after we establish, in Section 2, a simple fact about contraction mappings.

2. Contraction mappings

In the proof of the existence and uniqueness theorem to be stated in the next section, we will need to make use of an iteration scheme due to Picard. The convergence of this iteration scheme depends on the concept of contraction mapping, which we now define.

Definition 2.1. Let $A \subset X$ be a subset of a metric space. A mapping $S : A \to A$ is said to be a *contraction mapping* if there exists some $r \in (0,1)$ such that

$$d(Sx, Sy) \le r \cdot d(x,y)$$

for all $x, y \in X$. $\diamond$

The basic fact about contraction mappings is the following result.

PROPOSITION 2.2. *Let X be a complete metric space and let $A \subset X$ be a closed subset. Let $S : A \to A$ be a contraction mapping. Then S has a unique fixed point.*

Proof. Let $x \in A$ be any point. Consider the sequence $\{x_j\}$ defined by

$$x_j := S^{(j)}x, \quad j = 0, 1, 2, ...$$

where $S^{(0)} = \mathrm{Id}$ is the identity map and $S^{(j)} := S \circ S^{(j-1)}$ for all $j \in \mathbb{N}$. Then for all $j < k$ we have

$$d(x_j, x_k) \leq \sum_{\ell=j}^{k-1} d(x_\ell, x_{\ell+1}) \leq \sum_{\ell=j}^{k-1} r^\ell d(x, Sx) = \frac{r^j(1 - r^{k-j-1})}{1-r} d(x, Sx) \leq \frac{r^j}{1-r} d(x, Sx).$$

It follows that $\{x_j\}$ is a Cauchy sequence, and since A is closed (hence complete), the limit

$$x_* := \lim x_j$$

exists and lies in A. Since a contraction mapping is continuous,

$$x_* = \lim S^{(j)}x_* = \lim S \circ S^{(j-1)}x_* = S(\lim S^{(j)}x_*) = Sx_*.$$

Thus x_* is a fixed point of S.

Finally, if y is another fixed point of S, then

$$0 \leq (1-r)d(x_*, y) = d(Sx_*, Sy) - rd(x_*, y) \leq (r - r)d(x_*, y) = 0.$$

Thus $y = x_*$, and the proof is complete. □

3. THE EXISTENCE AND UNIQUENESS THEOREM FOR FIRST ORDER ODE

DEFINITION 3.1. Let $f : U \to \mathbb{R}^n$ be a function defined on a domain $U \subset \mathbb{R}^m$. We say that f is *locally Lipschitz* if for each $p \in U$ and each $\varepsilon \in (0, \mathrm{dist}(p, U^c))$ there exists a constant $K = K_{\varepsilon,p}$ such that

$$|f(x) - f(y)| \leq K|x - y|$$

for all $x, y \in B(p, \varepsilon) := \{z \in \mathbb{R}^m \ ; \ |z - p| < \varepsilon\}$. ◇

REMARK 3.2. Note that any differentiable function is locally lipschitz, but that the converse is not true, as is shown by the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$. ◇

Before stating our next definition and result, we introduce the following notation for the sake of convenience. Let $D \subset \mathbb{R}^n$ and $V \subset D \times \mathbb{R}$ be domains. For each $t \in \mathbb{R}$, we write

$$V_t = \{x \in D \ ; \ (x, t) \in V\}.$$

(Note that this set may be empty.)

DEFINITION 3.3. Let $D \subset \mathbb{R}^n$ and $V \subset D \times \mathbb{R}$ be domains, let $F : V \to \mathbb{R}^n$ be a continuous time-dependent vector field. We say that the function $F_t : V_t \to \mathbb{R}^n$ defined by $F_t(x) := F(x, t)$ is *locally uniformly Lipschitz* on V_t if F_t is locally Lipschitz and moreover the Lipschitz constant can be taken locally uniform with respect to t. ◇

The main theorem of these notes is the following result.

THEOREM 3.4. *Let $D \subset \mathbb{R}^n$ and $V \subset D \times \mathbb{R}$ be domains, let $F : V \to \mathbb{R}^n$ be a continuous time-dependent vector field. Assume that for each $t \in \mathbb{R}$ the function $F_t : V_t \to \mathbb{R}^n$ defined by $F_t(x) := F(x,t)$ is locally uniformly Lipschitz on V_t. Then for each $(x,s) \in V$ there exists an integral curve $\gamma_{(x,s)} : I_{(x,s)} \to D$ for F. Moreover, the set of integral curves possesses the following uniqueness property: if $\gamma_{(x,s)} : I_{(x,s)} \to D$ and $\tilde{\gamma}_{(x,s)} : \tilde{I}_{(x,s)} \to D$ are two integral curves through x at time s, then $\gamma_{(x,s)}(t) = \tilde{\gamma}_{(x,s)}(t)$ for all $t \in I_{(x,s)} \cap \tilde{I}_{(x,s)}$.*

Proof. Let $(x_o, t_o) \in V$ and choose $\varepsilon > 0$ such that F is continuous in $B(x_o, \varepsilon) \times (-\varepsilon, \varepsilon)$ and Lipschitz in the first variable with Lipschitz constant K, i.e.,

$$|F(x,t) - F(y,t)| \leq K|x-y|$$

for all $(x,t), (y,t) \in B(x_o, \varepsilon) \times (-\varepsilon, \varepsilon)$. By continuity there exists a constant $M > 0$ such that

$$|F(x,t)| \leq M$$

for all $(x,t) \in B(x_o, \varepsilon) \times (-\varepsilon, \varepsilon)$.

Choose positive constants α and β such that

(i) with $I_\alpha := \{t \in \mathbb{R} \; ; \; |t - t_o| \leq \alpha\}$ and $B_\beta := \{x \in \mathbb{R}^n \; ; \; |x - x_o| \leq \beta\}$,

$$B_\beta \times I_\alpha \subset B(x_o, \varepsilon) \times (-\varepsilon, \varepsilon),$$

(ii) $\alpha M < \beta$, and

(iii) $\alpha K < 1$.

Let $\mathscr{A}$ denote the set of continuous maps $\phi : I_\alpha \to \mathbb{R}^n$ such that

$$|\phi(t) - x_o| \leq \beta \quad \text{for all } t \in I_\alpha.$$

Equipping $\mathscr{A}$ with the uniform norm

$$||\phi||_u := \sup_{I_\alpha} |\phi|$$

makes $\mathscr{A}$ into a closed bounded subset of a Banach (and hence complete metric) space, as we have discussed earlier in the course. Thus $\mathscr{A}$ is itself a complete metric space with respect to the metric

$$d(\phi, \tilde{\phi}) := ||\phi - \tilde{\phi}||_u.$$

Consider the operator T defined by

$$T\phi(t) := x_o + \int_{t_o}^{t} F(\phi(s), s) ds.$$

Observe first that if $\phi \in \mathscr{A}$ then clearly $T\phi$ is continuous and defined on all of I_α. Moreover, for $t \in I_\alpha$ one has

$$|T\phi(t) - x_o| \leq M|t - t_o| \leq M\alpha < \beta,$$

and thus $T\phi \in \mathscr{A}$. That is to say,

$$T : \mathscr{A} \to \mathscr{A}.$$

Next, observe that for $\phi_1, \phi_2 \in \mathscr{A}$ one has

$$\begin{aligned} |T\phi_1(t) - T\phi_2(t)| &= \left| \int_{t_o}^{t} \left(F(\phi_1(s), s) - F(\phi_2(s), s)\right) ds \right| \\ &\leq \int_{t_o}^{t} K \left|\phi_1(s) - \phi_2(s)\right| ds \\ &\leq K\alpha \sup_{I_\alpha} |\phi_1 - \phi_2| . \end{aligned}$$

It follows that for some $r \in (0, 1)$,

$$||T\phi_1 - T\phi_2||_u \leq r||\phi_1 - \phi_2||_u.$$

Thus $T : \mathscr{A} \to \mathscr{A}$ is a contraction mapping, and therefore by Proposition 2.2 it has a unique fixed point $\phi_* \in \mathscr{A}$.

Being a fixed point of T, ϕ_* satisfies the equation

$$\phi_*(t) = x_o + \int_{t_o}^{t} F(\phi_*(s), s)ds, \tag{1}$$

and therefore

$$\frac{\phi_*(t+h) - \phi_*(t)}{h} = \frac{1}{h}\int_{t}^{t+h} F(\phi_*(s), s)ds \xrightarrow{h\to 0} F(\phi_*(t), t).$$

Since $\phi_* \in \mathscr{A}$, the latter limit is continuous, and thus the fixed point ϕ_* of T is differentiable. Differentiation of the integral equation (1) with respect to t shows that

$$\phi_*'(t) = F(\phi_*(t), t).$$

Since $\phi_*(t_o) = x_o$, we see that $\gamma_{(x_o,t_o)}(t) := \phi_*(t)$ is an integral curve of F through x_o at time t_o.

Conversely, any integral curve of F satisfies the equation (1), and is therefore a fixed point of T. Since contraction mappings have a unique fixed point, we see that any two integral curves must agree on I_α. By carrying out the same proof in small intervals centered at all points of the intersection of the open set $I_{(x,s)} \cap \tilde{I}_{(x,s)}$, we obtain the uniqueness statement claimed in the theorem. The proof is therefore complete. □

4. Maximal Integral Curves, Fundamental Domains, and Flows

Our next goal is to 'glue together' the integral curves of a time-dependent vector fields. The first task is to maximally extend integral curves.

Let $D \subset \mathbb{R}^n$ and $V \subset D \times \mathbb{R}$ be domains, and let $F : V \to \mathbb{R}^n$ be a continuous, time-dependent vector field such that for each $s \in \mathbb{R}$, $F_s : V_s \to \mathbb{R}^n$ is locally uniformly Lipschitz. Fix an initial condition $(x, s) \in V$. By Theorem 3.4, F has an integral curve through x with initial time s.

Proposition 4.1. *With the notation above, there exists a unique integral curve $\gamma_{(x,s)} : I_{(x,s)} \to D$ for F passing through x with initial time s such that if $\phi : I \to D$ is another integral curve for F through (x, s) then $I \subset I_{(x,s)}$.*

Proof. With respect to inclusion of domains, the set $\mathscr{I}_{(x,s)}$ of all integral curves for F passing through x with initial time s is partially ordered. Moreover, given two such integral curves $\phi_i : I_i \to D$, $i = 1, 2$, Theorem 3.4 implies that the function

$$\phi(t) := \begin{cases} \phi_1(t)\ , & t \in I_1 \\ \phi_2(t)\ , & t \in I_2 \end{cases}$$

is well-defined, and therefore $\phi : I_1 \cup I_2 \to D$ is also an integral curve for F passing through x with initial time s. It follows that $\mathscr{I}_{(x,s)}$ is a directed set. We have to show that it has a maximal element, which is then of course unique.

To this end, let $\{\phi_i : I_i \to D\}_{i\in I}$ be a maximal linearly ordered subset of $\mathscr{I}_{(x,s)}$. Then the set $I := \bigcup_{i\in I} I_i$ is open, and the curve $\phi : I \to D$ defined by

$$\phi(t) = \phi_i(t), \quad t \in I_i$$

is well-defined by the uniqueness part of Theorem 3.4, and therefore in $\mathscr{I}_{(x,s)}$. Thus $\mathscr{I}_{(x,s)}$ has a unique maximal element in $\mathscr{I}_{(x,s)}$. □

DEFINITION 4.2. The unique maximal element of the set $\mathscr{I}_{(x,s)}$ defined in the proof of the previous proposition is called the *maximal integral curve for F through (x,s)*. We shall denote the maximal integral curve for F through (x,s) by

$$\Gamma_{(x,s)} : \mathcal{I}_{(x,s)} \to D.$$

One can also consider the unions of the graphs of the maximal integral curves.

DEFINITION 4.3. The set

$$\mathscr{U}_F := \{(x,s,t)\ ;\ (x,s) \in V, t \in \mathcal{I}_{(x,s)}\} \subset V \times \mathbb{R}$$

is called the fundamental domain of the time-dependent vector field F, and the map

$$\Phi_F : \mathscr{U}_F \to V$$

defined by $\Phi_F(x,s,t) := (\Gamma_{(x,s)}(t), t)$ is called the time-dependent flow of F. ◇

Let us denote by $\pi_D : V \to D$ the restriction to V of the natural projection $\pi : D \times \mathbb{R} \to D$.

DEFINITION 4.4. The map $\Phi_s^t : D \to D$

$$\Phi_s^t(x) := \Gamma_{(x,s)}(t) = \pi_D \circ \Phi_F(x,s,t) \tag{2}$$

is called the time-t map for the initial time s. ◇

The uniqueness part of Theorem 3.4 implies a symmetry appearing in the composition law for the maps (2), stated in the following result.

PROPOSITION 4.5. *For each $s \in \mathbb{R}$ one has*

$$\Phi_s^s(x) = x \quad \textit{for all } x \in V_s.$$

Moreover, if $(x,s,t) \in \mathscr{U}_F$ and $(\Phi_s^t(x), t, r) \in \mathscr{U}_F$, we have the pseudo-group law

$$\Phi_t^r \circ \Phi_s^t(x) = \Phi_s^r(x).$$

5. AUTONOMOUS VECTOR FIELDS

From the point of view of classical mechanics, the general setting of time-dependent vector fields corresponds to physical systems in which the laws of physics change with time. Such situations can happen, but in nature we mostly find them when the particular physical system we are studying is not *closed*, i.e., it is part of a larger physical system.

By definition, the vector field representing a closed physical system is autonomous. That is to say, for each $x \in D$

$$t \mapsto F(x,t)$$

is constant. In this case, we choose the convention of always taking initial value problems to start at time $s = 0$.

The fundamental domain and the flow are defined just slightly differently, so as to eliminate the initial time. Let us make the definitions precise.

DEFINITION 5.1. Let $\xi : D \to \mathbb{R}^n$ be a vector field on a domain $D \subset \mathbb{R}^n$.

(i) The *maximal integral curve* for ξ through $x \in D$ is the maximal integral curve

$$\Gamma_x : \mathcal{I}_x \to D$$

where $\Gamma_x := \Gamma_{(x,0)}$ and $\mathcal{I}_x := \mathcal{I}_{(x,0)}$.

(ii) The *fundamental domain* of ξ is the domain

$$\mathscr{U}_\xi^0 := \{(\Gamma_x(t), t) \, ; x \in D\} \subset D \times \mathbb{R}.$$

(iii) The flow of ξ is the map $\Phi_\xi : \mathscr{U}_\xi^0 \to D$ defined by

$$\Phi_\xi(x, t) = \Gamma_x(t).$$

The time-t map is the map Φ_ξ^t defined by

$$\Phi_\xi^t(x) = \Phi_\xi(x, t).$$

$\diamond$

Note that $\mathscr{U}_\xi^0$ always contains $D \times \{0\}$. Note as well that the time-t maps define the pseudo-group law

$$\Phi_\xi^t \circ \Phi_\xi^s = \Phi_\xi^{t+s}. \tag{3}$$

The link between the autonomous and time-dependent scenarios is the identity

$$\Phi_s^t = \Phi_\xi^{t-s}.$$

The pseudo-group law (3) is not a group law only because integral curves are not defined for a long enough time, i.e., even if t and s both lie in the domains of their respective integral curves, $t+s$ may not. The situation in which this failure does not happen is therefore particularly important, and we study it in more detail now.

DEFINITION 5.2. A vector field $\xi : D \to \mathbb{R}^n$ is said to be *complete* (sometimes also called *completely integrable*) if every maximal integral curve is defined on the entire real line. $\diamond$

We have the following simple Proposition.

PROPOSITION 5.3. *Let $\xi : D \to \mathbb{R}^n$ be a locally Lipschitz vector field defined on a domain $D \subset \mathbb{R}^n$. Then the following are equivalent.*

(i) *ξ is complete.*
(ii) *There exists a positive number ε such that for each $x \in D$, $\mathcal{I}_x \supset (-\varepsilon, \varepsilon)$.*
(iii) *For each $t \in \mathbb{R}$, the map Φ_ξ^t is a $\mathscr{C}^1$-diffeomorphism of D: $\Phi_\xi^t \in \mathrm{Diff}^1(D)$.*
(iv) *For some $t \in \mathbb{R} - \{0\}$, $\Phi_\xi^t \in \mathrm{Diff}^1(D)$.*
(v) *The set of maps $\{\Phi_\xi^t\}_{t \in \mathbb{R}}$ is a 1-parameter subgroup of $\mathrm{Diff}^1(D)$.*
(vi) *The fundamental domain of ξ is $D \times \mathbb{R}$.*

The proof is left to the reader as an exercise.

6. APPROXIMATION

In this section we study a technique, initiated by Euler, for the approximation of integral curves and more generally flows. We confine ourselves to autonomous vector fields for the time being.

DEFINITION 6.1. Let $\xi : D \to \mathbb{R}^n$ be a vector field on a domain $D \subset \mathbb{R}^n$ and let $I \subset \mathbb{R}$ be an open interval containing 0. An *algorithm for* ξ is a map $H : D \times I \to D$ such that, with $H_t(x) := H(x,t)$,

(i) $H_0 = \mathrm{Id}$,
(ii) $H(x, \cdot)$ is $\mathscr{C}^1$ and its derivative is continuous in $D \times I$, and
(iii) $\frac{\partial H}{\partial t}\Big|_{t=0} = \xi$.

The basic approximation theorem is the following result.

THEOREM 6.2. *Let H be an algorithm for a Lipschitz vector field ξ. If $(t,x) \in \mathscr{U}^0_\xi$ then for all $N >> 0$, $H^{(N)}_{t/N}(x)$ is defined, and converges to $\Phi^t_\xi(x)$. Conversely, if $H^{(N)}_{t/N}(x)$ is defined and converges for $t \in [0,T]$ then $(T,x) \in \mathscr{U}^0_\xi$ and*

$$\lim_{N\to\infty} H^{(N)}_{t/N}(x) = \Phi^t_\xi(x).$$

In both statements, the converges is locally uniform on $D \times I$.

Before proving Theorem 6.2, we establish the following lemma which we shall need.

LEMMA 6.3. *Fix a Lipschitz vector field $\xi : D \to \mathbb{R}^n$ defined on a domain $D \subset \mathbb{R}^n$, a point $x_o \in D$ and a number $\varepsilon \in (0, \mathrm{dist}(x, D^c))$. Fix a constant $K > 0$ such that*

$$||\xi(x) - \xi(y)|| \le K||x - y||, \quad x, y \in B(x_o, \varepsilon).$$

Then for any interval $I \subset \mathbb{R}$ containing 0 such that $\Phi^t_\xi(x)$ is defined for all $t \in I$ and $x \in B(x_o,\varepsilon)$, we have the estimate

$$||\Phi^t_\xi(x) - \Phi^t_\xi(y)|| \le e^{K|t|}||x - y||, \quad x, y \in B(x_o, \varepsilon),\ t \in I.$$

Proof. Observe that with $f(t) := ||\Phi^t_\xi(x) - \Phi^t_\xi(y)||$ we have

$$f(t) = \left|\left| x - y + \int_0^t (\xi(\Phi^s_\xi(x)) - \xi(\Phi^s_\xi(y)))ds + x - y \right|\right| \le ||x-y|| + K\int_0^t f(s)ds =: g(t).$$

Now, $g'(t) = Kf(t) \le Kg(t)$, and we have

$$\frac{d}{dt}\left(e^{-Kt}g(t)\right) \le 0.$$

Thus

$$g(t) \le g(0)e^{Kt} \le g(0)e^{K|t|}$$

which is what we want. ∎

Proof of Theorem 6.2. We begin by showing that the convergence holds locally. To this end, let $x_o \in D$. Then

$$H_t(x) = x + O(t) \quad \text{and} \quad \Phi^t_\xi(x) - H_t(x) = o(t). \tag{4}$$

If $H^{(j)}_{t/j}(x)$ is well-defined for x in a small neighborhood of x_o, for $j = 1, 2, ..., N-1$, then the semi-group law for time-t maps and the first estimate in (4) shows that

$$\begin{aligned} H^{(N)}_{t/N}(x) - x &= H^{(N)}_{t/N}(x) - H^{(N-1)}_{t/N}(x) + H^{(N-1)}_{t/N}(x) - H^{(N-2)}_{t/N}(x) \\ &\quad + ... + H_{t/N}(x) - x \\ &= NO(t/N) = O(t), \end{aligned}$$

which is small independently of N, for t sufficiently small. Thus for x sufficiently close to x_o and t sufficiently small, $H^{(N)}_{t/N}(x)$ remains close to x_o for all N. In other words, with

$$x_j = H^{(j)}_{t/j}(x),$$

$||x_j - x_o|| < \varepsilon$ for x sufficiently close to x_o and t sufficiently small. From the semi-group law for Φ^t_ξ, we also have

$$\begin{aligned} \Phi^t_\xi(x) - H^{(N)}_{t/N}(x) &= (\Phi^{t/N}_\xi)^{(N)}(x) - H^{(N)}_{t/N}(x) \\ &= (\Phi^{t/N}_\xi)^{(N-1)}(\Phi^{t/N}_\xi(x)) - (\Phi^{t/N}_\xi)^{(N-1)}(H_{t/N}(x)) \\ &\quad + \sum_{j=2}^{N} (\Phi^{t/N}_\xi)^{(N-j)}(\Phi^{t/N}_\xi(x_j)) - (\Phi^{t/N}_\xi)^{(N-j)}(H_{t/N}(x_j)), \end{aligned}$$

Thus, by repeated application of Lemma 6.3 we find the estimate

$$\begin{aligned} ||\Phi^t_\xi(x) - H^{(N)}_{t/N}(x)|| &\leq \sum_{k=1}^{N} e^{K|t|(N-k)/N} ||\Phi^{t/N}_\xi(x_{N-k-1}) - H_{t/N}(x_{N-k-1})|| \\ &\leq N e^{K|t|} o(t/N), \end{aligned}$$

and the last quantity converges, as $N \to \infty$, to 0 uniformly on a small ball centered at x_o and for all sufficiently small t. The final estimate uses the second estimate of (4).

Having handled the case of short times, we now proceed to longer times. To this end, suppose first that $\Phi^t_\xi(x)$ is defined for all $t \in [0, T]$. By what we have just done, if k is sufficiently large then

$$\Phi^{t/k}_\xi(y) = \lim_{k\to\infty} H^{(k)}_{t/k}(y)$$

holds uniformly for $t \in [0, T]$ and y in a bounded neighborhood of the curve $\{\Phi^t_\xi(x)\ ;\ t \in [0, T]\}$. Thus

$$\Phi^t_\xi(x) = (\Phi^{t/k}_\xi)^{(k)}(x) = \lim_{N\to\infty} (H^{(N)}_{t/(kN)})^{(k)}(x) = \lim_{N\to\infty} H^{(Nk)}_{t/(kN)}(x) = \lim_{N\to\infty} H^{(N)}_{t/N}(x).$$

Conversely, suppose $t \mapsto H^{(N)}_{t/N}(x)$ converges to a curve $c : [o, T] \to D$. Let

$$S = \{t \in [0, T]\ ;\ \Phi^t_\xi(x) \text{ is defined and equal to } c(t)\}.$$

Clearly $0 \in S$, and from the local result S is relatively open. Let $\{t_k\} \subset S$ and suppose $t_k \to t$. Then $\Phi^{t_k}_\xi(x) \to c(t)$ so by Theorem 3.4 $\Phi^t_\xi(x)$ is defined, and by continuity, $\Phi^t_\xi(x) = c(t)$. Thus S is closed, and hence $S = [0, T]$.

Finally, observe that by existence and uniqueness, $\Phi^{-t}_\xi = \Phi^t_{-\xi}$, so the above proof applies to negative times as well. $\square$

7. SUSPENSION: A REMARK

Autonomous vector fields are special cases of time-dependent vector fields. In this section, we note that the converse is true. To this end, let $D \subset \mathbb{R}^n$ and $V \subset D \times \mathbb{R}$ be domains and let $F : V \to \mathbb{R}^n$ be a time-dependent vector field. Define $\xi_F : V \to \mathbb{R}^n \times \mathbb{R}$ by the formula

$$\xi_F(x, s) := (F(x, s), 1)$$

The vector field ξ_F is then autonomous, and its flow is given by the time-t maps

$$\Phi^t_{\xi_F}(x, s) = (\Phi^{s+t}_s(x), s + t).$$

It is therefore possible to extract the flow of F from that of ξ_F. If one can find the latter flow, this is of course possible. In fact, the hypotheses of Theorem 3.4 apply to ξ_F as soon as F is Lipschitz on V.

MAT 550, Real Analysis II, Spring 2015 — Homework 7

Due in Class : April 16, 2015.

Turn in the following exercices. Exercise a.b refers to Exercise b in Chapter a in the Textbook.

Problem 1. Show that the sum of two closed subsets of $\mathbb{R}^n$ is not necessarily closed.

Problem 2. Let A, B be two measurable subsets of $\mathbb{R}$ with positive and finite measure. Show that $A + B$ contains a segment.
(*Hint* : Consider $\chi_A * \chi_B$).

Problem 3. Exercise 8.15. (Note : In **c.**, omit the part about uniform convergence).

Problem 4. Let $h := \chi_{[-1,1]}$ and for $n \in \mathbb{N}$, let $g_n := \chi_{[-n,n]}$.
a. Compute $g_n * h$ explicitly.
b. Show that $g_n * h$ is the Fourier transform of a function $f_n \in L^1(\mathbb{R})$ defined by

$$f_n(x) := \frac{\sin 2\pi x \sin 2\pi n x}{\pi^2 x^2}.$$

c. Show that $\|f_n\|_1 \to \infty$ as $n \to \infty$.
d. Deduce that $f \mapsto \hat{f}$ maps $L^1(\mathbb{R})$ onto a proper subset of $C_0(\mathbb{R})$.

Problem 5. Use Exercise 8.15a to deduce the Fourier transform of $(\sin x/x)^2$.

Problem 6.
a. Compute the Fourier transform of $e^{-|x|}$ on $\mathbb{R}$.
b. Deduce the value of the integral

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}.$$

Problem 7. Does there exist a function $u \in L^1(\mathbb{R}^n)$ such that $f * u = f$ for all $f \in L^1(\mathbb{R}^n)$?

Problem 8.
a. Let $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ be a linear map such that

$$T(\partial_j \phi) = \partial_j T(\phi) \quad \text{and} \quad T(x_j \phi) = x_j T(\phi) \qquad (\phi \in \mathcal{S}(\mathbb{R}^n), j = 1, \ldots, n).$$

Show that T is a multiple of the identity.

(*Hint* : You can take for granted the fact that if $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$ is such that $\phi(y) = 0$, then there exist $\phi_1, \ldots, \phi_n \in \mathcal{S}(\mathbb{R}^n)$ such that $\phi(x) = \sum_{j=1}^{n}(x_j - y_j)\phi_j(x)$ for all $x \in \mathbb{R}^n$).

b. Use part **a.** to give another proof of the Fourier Inversion Theorem.

Problem 9. Prove that if ϕ is a complex homomorphism on a Banach algebra A, then ϕ is a bounded linear functional of norm at most one.

Problem 10. Is $L^2(\mathbb{R})$ closed under convolution?

Problem 1. Show that the sum of two closed subsets of $\mathbb{R}^n$ is not necessarily closed.

Sol. Let $F_1 := \{(x, 0) : x \in \mathbb{R}\}$ and $F_2 := \{(x, e^x) : x \in \mathbb{R}\}$. Then F_1 and F_2 are two closed subsets of $\mathbb{R}^2$, but $F_1 + F_2 = \{(x, y) : x \in \mathbb{R}, y > 0\}$ is not closed.

Problem 2. Let A, B be two measurable subsets of $\mathbb{R}$ with positive and finite measure. Show that $A + B$ contains a segment.
(*Hint* : Consider $\chi_A * \chi_B$).

Sol. First note that $\chi_A * \chi_B$ is uniformly continuous on $\mathbb{R}$, by a proposition proved in class. Furthermore, this function is not identically zero since by Fubini's theorem and translation invariance of Lebesgue measure, we have

$$\int \chi_A * \chi_B(x)\, dx = m(A)m(B) > 0.$$

Let $x_0 \in \mathbb{R}$ such that $\chi_A * \chi_B(x_0) > 0$, and let $\delta > 0$ such that $|x - x_0| < \delta$ implies $\chi_A * \chi_B(x) > 0$. We claim that $A + B$ contains the interval $(x_0 - \delta, x_0 + \delta)$. Indeed, if x belongs to that interval, then $\chi_A * \chi_B(x) > 0$ so there exists $y \in B$ such that $x - y \in A$. Therefore $x = x - y + y \in A + B$.

Problem 3. Exercise 8.15. (Note : In **c.**, omit the part about uniform convergence).

a. Sol. A simple calculation shows that the Fourier transform of $\chi_{[-a,a]}$ at $\xi \in \mathbb{R}$ is $2a \operatorname{sinc} 2a\xi$.

b. Sol. Clearly $\mathcal{H}_a$ is a vector subspace of L^2. To show that it is a Hilbert space, it suffices to prove that it is closed in L^2. If $(f_n) \subset \mathcal{H}_a$, $f_n \to f$ in L^2, then $\hat{f_n} \to \hat{f}$ in L^2 by Plancherel, hence there is a subsequence (f_{n_j}) such that $\hat{f_{n_j}} \to \hat{f}$ almost everywhere. It easily follows that $\hat{f}(\xi) = 0$ for almost every $|\xi| > a$, i.e. $f \in \mathcal{H}_a$. For the second part, set $g_k(\xi) := \sqrt{2a} \operatorname{sinc}(2a\xi - k)$ and $h_a := \chi_{[-a,a]}$. Then for $j, k \in \mathbb{Z}$, we have

$$\begin{aligned}
\langle g_j, g_k \rangle &= \frac{1}{2a} \langle \hat{h_a}(\xi - j/2a), \hat{h_a}(\xi - k/2a) \rangle \\
&= \frac{1}{2a} \langle (e^{2\pi i \frac{j}{2a} x} h_a)\hat{}(\xi), (e^{2\pi i \frac{k}{2a} x} h_a)\hat{}(\xi) \rangle \\
&= \frac{1}{2a} \langle e^{2\pi i \frac{j}{2a} x} h_a, e^{2\pi i \frac{k}{2a} x} h_a \rangle = \delta_{jk},
\end{aligned}$$

by Plancherel and the properties of the Fourier transform. This shows orthonormality.

To prove that $\{g_k\}_{k\in\mathbb{Z}}$ is a basis, assume that $g \in \mathcal{H}_a$ is orthogonal to every g_k. Then by a calculation similar to the preceding one, we get that

$$\int_{-a}^{a} \hat{g}(x)e^{i\pi kx/a}\,dx = 0 \qquad (k \in \mathbb{Z})$$

i.e.

$$\int_{-\pi}^{\pi} \hat{g}(at/\pi)e^{ikt}\,dt = 0 \qquad (k \in \mathbb{Z}).$$

It follows from the density of $\{e^{ikt} : k \in \mathbb{Z}\}$ in $L^2([-\pi,\pi])$, which is an easy consequence of Stone-Weierstrass, that $\hat{g} = 0$ a.e. on $[-a,a]$. Thus $\hat{g} = 0$ a.e. on $\mathbb{R}$, since $g \in \mathcal{H}_a$. Hence $g = 0$ a.e. and $\{g_k\}_{k\in\mathbb{Z}}$ is a basis.

c. Sol. If $f \in \mathcal{H}_a$, then $\hat{f} \in L^1$ so by a Corollary of Plancherel's theorem proved in class, we have that $f(\xi) = \hat{\hat{f}}(-\xi)$ almost everywhere, hence f is the Fourier transform of a function in L^1, which implies that $f \in C_0$ (Riemann-Lebesgue lemma). By part **b.**, we get that

$$f(x) = \sum_{k=-\infty}^{\infty} \langle f, g_k\rangle g_k(x)$$

where the series converges in L^2. But for $k \in \mathbb{Z}$, we have

$$\begin{aligned}
\langle f, g_k\rangle &= \frac{1}{\sqrt{2a}}\langle f, \hat{h_a}(\xi - k/2a)\rangle \\
&= \frac{1}{\sqrt{2a}}\langle \hat{\hat{f}}(-\xi), (e^{2\pi i \frac{k}{2a}x}h_a)\hat{\ }(\xi)\rangle \\
&= \frac{1}{\sqrt{2a}}\langle \hat{f}(-\xi), e^{2\pi i \frac{k}{2a}\xi}h_a\rangle \\
&= \frac{1}{\sqrt{2a}}\int_{-a}^{a} \hat{f}(-\xi)e^{-2\pi i \frac{k}{2a}\xi}\,d\xi \\
&= \frac{1}{\sqrt{2a}}\hat{\hat{f}}\left(-\frac{k}{2a}\right) = \frac{1}{\sqrt{2a}}f\left(\frac{k}{2a}\right),
\end{aligned}$$

where we used Plancherel and the fact that $\hat{f}(\xi) = 0$ for all $|\xi| > a$. Therefore, we obtain

$$f(x) = \sum_{k=-\infty}^{\infty} f(k/2a)\,\mathrm{sinc}(2ax - k),$$

which is what we wanted to prove.

Problem 4. Let $h := \chi_{[-1,1]}$ and for $n \in \mathbb{N}$, let $g_n := \chi_{[-n,n]}$.

a. Compute $g_n * h$ explicitly.

Sol. A simple calculation shows that

$$g_n * h(x) = \begin{cases} 0 & \text{if } x \leq -n-1 \\ x+1+n & \text{if } -n-1 < x \leq -n+1. \\ 2 & \text{if } -n+1 < x \leq n-1. \\ n-x+1 & \text{if } n-1 < x \leq n+1. \\ 0 & \text{if } x > n+1. \end{cases}$$

b. Show that $g_n * h$ is the Fourier transform of a function $f_n \in L^1(\mathbb{R})$ defined by

$$f_n(x) := \frac{\sin 2\pi x \sin 2\pi n x}{\pi^2 x^2}.$$

Sol. Note that $g_n * h \in L^1$ since $g_n, h \in L^1$. By the inversion theorem, $g_n * h$ is the Fourier transform of $f_n(x) := (\hat{g_n}\hat{h})(-x)$. A simple calculation using Problem 3 shows that f_n has the desired form.

c. Show that $\|f_n\|_1 \to \infty$ as $n \to \infty$.

Sol. Setting $y = 2\pi n x$ in the integral for $\|f_n\|_1$, we get

$$\|f_n\|_1 = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{|\sin y/n||\sin y|}{|y/n||y|}\, dy$$

hence by Fatou's lemma,

$$\liminf_{n\to\infty} \|f_n\|_1 \geq \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{|\sin y|}{|y|}\, dy = \infty.$$

d. Deduce that $f \mapsto \hat{f}$ maps $L^1(\mathbb{R})$ onto a proper subset of $C_0(\mathbb{R})$.

Sol. Suppose for a contradiction that $f \mapsto \hat{f}$ maps $L^1(\mathbb{R})$ onto $C_0(\mathbb{R})$. Since $\|\hat{f}\|_u \leq \|f\|_1$ for all $f \in L^1$ and the Fourier transform is one-to-one on L^1 by the inversion theorem, this would mean that $f \mapsto \hat{f}$ is a bounded linear bijection between the Banach spaces $L^1(\mathbb{R})$ and $C_0(\mathbb{R})$. By the open mapping theorem, there is a constant $C > 0$ such that

$$\|\hat{f}\|_u \geq C\|f\|_1 \qquad (f \in L^1(\mathbb{R})).$$

However, note that the functions f_n as above belong to $L^1(\mathbb{R})$ and satisfy $\|\hat{f}_n\|_u = \|g_n * h\|_u = 2$ for all n, yet $\|f_n\|_1 \to \infty$ as $n \to \infty$. This is a contradiction.

Problem 5. Use Exercise 8.15a to deduce the Fourier transform of $(\sin x/x)^2$.

Sol. Let $f(x) = (\sin 2\pi x/x)^2$ and let $h = \chi_{[-1,1]}$. By Problem 3, $\hat{h}(\xi) = \sin 2\pi\xi/(\pi\xi)$, so that

$$\hat{f}(\xi) = \pi^2(\hat{h}\hat{h})\hat{\ }(\xi) = \pi^2(h * h)\hat{\hat{\ }}(\xi) = \pi^2(h * h)(-\xi).$$

Let $g(x) = (\sin x/x)^2 = (2\pi)^{-2} f(x/2\pi)$. Then

$$\hat{g}(\xi) = (2\pi)^{-1}\hat{f}(2\pi\xi) = \frac{\pi}{2}(h * h)(-2\pi\xi).$$

Problem 4 part a then gives an explicit expression for $\hat{g}(\xi)$.

Problem 6.
a. Compute the Fourier transform of $e^{-|x|}$ on $\mathbb{R}$.

Sol. A simple calculation shows that the Fourier transform of $f(x) := e^{-|x|}$ is $\hat{f}(\xi) = 2/(1 + 4\pi^2\xi^2)$.

b. Deduce the value of the integral

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}.$$

Sol. By a simple change of variable and Pancherel, we get

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{2}\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\, d\xi = \frac{\pi}{2}\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{\pi}{2}.$$

Problem 7. Does there exist a function $u \in L^1(\mathbb{R}^n)$ such that $f * u = f$ for all $f \in L^1(\mathbb{R}^n)$?

Sol. If such a function $u \in L^1(\mathbb{R}^n)$ exists, then we must have $\hat{f}\hat{u} = \hat{f}$ for all $f \in L^1(\mathbb{R}^n)$. Apply this to some $f \in L^1(\mathbb{R}^n)$ such that $\hat{f}$ is nowhere vanishing, for instance $f(x) = e^{-|x|^2}$. We obtain that $\hat{u}$ is identically equal to one, which contradicts the fact that $\hat{u} \in C_0(\mathbb{R}^n)$.

Problem 8.
a. Let $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ be a linear map such that

$$T(\partial_j\phi) = \partial_j T(\phi) \quad \text{and} \quad T(x_j\phi) = x_j T(\phi) \qquad (\phi \in \mathcal{S}(\mathbb{R}^n), j = 1, \ldots, n).$$

Show that T is a multiple of the identity.
(*Hint* : You can take for granted the fact that if $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$ is such that $\phi(y) = 0$, then there exist $\phi_1, \ldots, \phi_n \in \mathcal{S}(\mathbb{R}^n)$ such that $\phi(x) = \sum_{j=1}^n (x_j - y_j)\phi_j(x)$ for all $x \in \mathbb{R}^n$).

Sol. First we show that if $\phi \in \mathcal{S}$ and $y \in \mathbb{R}^n$ is such that $\phi(y) = 0$, then $T\phi(y) = 0$. By the hint, there exist $\phi_1, \ldots, \phi_n \in \mathcal{S}(\mathbb{R}^n)$ such that

$\phi(x) = \sum_{j=1}^{n}(x_j - y_j)\phi_j(x)$ for all $x \in \mathbb{R}^n$. Thus

$$T\phi(x) = \sum_{j=1}^{n}(x_j - y_j)T\phi_j(x) \qquad (x \in \mathbb{R}^n)$$

so $T\phi(y) = 0$. Now, fix $x_0 \in \mathbb{R}^n$ and let $\phi \in \mathcal{S}$. Let $\psi \in C_c^\infty(\mathbb{R}^n)$ with $\psi(x_0) = 1$. Then we have

$$T\phi(x_0) = T(\phi - \phi(x_0)\psi)(x_0) + T(\phi(x_0)\psi)(x_0) = \phi(x_0)T(\psi)(x_0),$$

since the function $\phi - \phi(x_0)\psi$ vanishes at x_0. But x_0 was arbitrary, so $T(\phi) = \phi f$ for some function f on $\mathbb{R}^n$. Applying this to a function $\phi \in \mathcal{S}$ which is nonvanishing on a given ball shows that f is C^∞ on that ball, hence $f \in C^\infty(\mathbb{R}^n)$. Finally, fix $j \in \{1, \dots, n\}$. Since T commutes with ∂_j, we get

$$f(\partial_j\phi) = \partial_j(f\phi) = \phi\partial_j f + f\partial_j\phi \qquad (\phi \in \mathcal{S}),$$

hence $\phi\partial_j f \equiv 0$ for all $\phi \in \mathcal{S}$, which implies that $\partial_j f \equiv 0$. Since this holds for each j, we get that f is constant. Therefore, $T\phi = c\phi$ is a multiple of the identity.
b. Use part **a.** to give another proof of the Fourier Inversion Theorem.

Sol. Let $T : \mathcal{S} \to \mathcal{S}$ defined by $T\phi(\xi) = \hat{\hat{\phi}}(-\xi)$. Simple manipulations show that T satisfies the hypotheses of **a.**, so $T\phi = c\phi$ for some constant c. Applying T to $\phi(x) = e^{-\pi|x|^2}$, we get that $c = 1$.

Problem 9. Prove that if ϕ is a complex homomorphism on a Banach algebra A, then ϕ is a bounded linear functional of norm at most one.

Sol. Assume, to get a contradiction, that $|\phi(x_0)| > \|x_0\|$ for some $x_0 \in A$. Put $\lambda = \phi(x_0)$ and set $x = x_0/\lambda$. Then $\|x\| < 1$ and $\phi(x) = 1$. Since $\|x^n\| \leq \|x\|^n$ and $\|x\| < 1$, the elements

$$s_n = -x - x^2 - \cdots - x^n$$

form a Cauchy sequence in A. By completeness, there exists $y \in A$ such that $s_n \to y$ in A. Clearly $x + s_n = xs_{n-1}$ for all n, so that

$$x + y = xy$$

and thus

$$1 + \phi(y) = \phi(x) + \phi(y) = \phi(x + y) = \phi(xy) = \phi(x)\phi(y) = \phi(y),$$

a contradiction.

Problem 10. Is $L^2(\mathbb{R})$ closed under convolution?

First Sol. Suppose that $L^2(\mathbb{R})$ is closed under convolution. First, we prove that this implies that there is a constant $C > 0$ such that

$$\|f * g\|_2 \leq C\|f\|_2\|g\|_2 \qquad (f, g \in L^2(\mathbb{R})).$$

For $g \in L^2(\mathbb{R})$, define $T_g : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$T_g(f) = f * g \qquad (f \in L^2(\mathbb{R})).$$

Clearly, for each g, T_g is a linear operator. Let us show that T_g is bounded with the Closed Graph Theorem. Assume that $f_n \to f$ in $L^2(\mathbb{R})$ and $T_g(f_n) = f_n * g \to h$ in $L^2(\mathbb{R})$. Then we have

$$\|f_n * g - f * g\|_u \leq \|f_n - f\|_2 \|g\|_2 \to 0$$

as $n \to \infty$. Taking a subsequence such that $f_{n_j} * g \to h$ almost everywhere shows that $T_g(f) = f * g = h$ almost everywhere. This proves that each T_g is bounded, i.e. $\|f * g\|_2 \leq \|T_g\| \|f\|_2$ for all $f \in L^2(\mathbb{R})$. Consider now the family of bounded linear operators $\{T_g\}$ where $g \in L^2(\mathbb{R})$, $\|g\|_2 \leq 1$. Fix $f \in L^2(\mathbb{R})$. Then

$$\|T_g(f)\| \leq \|T_f\| \|g\|_2 \leq \|T_f\| \qquad (g \in L^2(\mathbb{R}), \|g\|_2 \leq 1).$$

Thus we are in the first case of the Uniform Boundedness Principle, so that $\|T_g\| \leq C < \infty$ for all $g \in L^2(\mathbb{R}), \|g\|_2 \leq 1$. Normalizing, we get

$$\|f * g\|_2 \leq C \|f\|_2 \|g\|_2 \qquad (f \in L^2(\mathbb{R}), g \in L^2(\mathbb{R})).$$

Now, let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}), f \neq 0$ and for $t > 0$, define $f_t(x) = f(x/t)$. By the above inequality with $f = g = f_t$ and using the fact that $(f_t * f_t)\hat{} = \hat{f_t}^2$, we get

$$\|\hat{f_t}^2\|_2 \leq C \|f_t\|_2^2.$$

But $\hat{f}_t(\xi) = t\hat{f}(t\xi)$, so a simple change of variable yields

$$\sqrt{t}\|\hat{f}^2\|_2 \leq C\|f\|_2^2.$$

Letting $t \to \infty$ gives a contradiction.

Second Sol. First, we show that if $f, g \in L^2(\mathbb{R})$ and $f * g \in L^2(\mathbb{R})$, then $(f * g)\hat{} = \hat{f}\hat{g}$. To prove this, first note that $\hat{f}\hat{g} \in L^1(\mathbb{R})$, so its Fourier transform at $\xi \in \mathbb{R}$ is

$$\begin{aligned}
\int \hat{f}(x)\hat{g}(x)e^{-2\pi i \xi x}\, dx &= \langle \hat{f}, \overline{\hat{g}} e^{2\pi i \xi x} \rangle \\
&= \langle \hat{f}, \check{\overline{g}} e^{2\pi i \xi x} \rangle \\
&= \langle \hat{f}, (\tau_{-\xi} \overline{\widetilde{g}})\hat{} \rangle \\
&= \langle f, \tau_{-\xi} \overline{\widetilde{g}} \rangle \\
&= \int f(x) g(-\xi - x)\, dx = (f * g)(-\xi),
\end{aligned}$$

where $\widetilde{g}(x) := g(-x)$. We used Plancherel and also the fact that $\overline{\hat{g}} = \check{\overline{g}}$. In particular, the Fourier transform of $\hat{f}\hat{g}$ belongs to $L^2(\mathbb{R})$, which implies that $\hat{f}\hat{g} \in L^2(\mathbb{R})$, by Plancherel. Let $h \in L^2(\mathbb{R})$ be such that $\hat{h} = \hat{f}\hat{g}$. Since $\hat{f}\hat{g} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we have, by a Corollary proved in class and the above calculation,

$$h(x) = \int \hat{f}(\xi)\hat{g}(\xi)e^{2\pi i\xi x}\, d\xi = (f * g)(x) \qquad (a.e.\, x \in \mathbb{R}),$$

and thus $\hat{f}\hat{g} = \hat{h} = (f * g)\hat{}$.

Now, consider a function $F \in L^2(\mathbb{R})$ such that $F^2 \notin L^2(\mathbb{R})$, for instance $F(x) := x^{-1/3}e^{-x^2}$. Let $f \in L^2(\mathbb{R})$ such that $\hat{f} = F$. We claim that $f * f \notin L^2(\mathbb{R})$. Indeed, if $f * f \in L^2(\mathbb{R})$, then by the above formula with $g = f$, we get

$$F^2 = \hat{f}\hat{f} = (f * f)\hat{} \in L^2(\mathbb{R}),$$

a contradiction.

Due in Class : April 23, 2015.

Turn in the following exercices. Exercise a.b refers to Exercise b in Chapter a in the Textbook.

Problem 1.

a. Let $f(t) = t$ for $t \in (-\pi, \pi]$, extended to be 2π-periodic on $\mathbb{R}$. Compute the Fourier coefficients of f.

b. Deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Problem 2. Let $u : C_c^\infty(\mathbb{R}^n) \to \mathbb{C}$ be a linear functional which is positive in the sense that $\langle u, \phi \rangle \geq 0$ whenever $\phi \in C_c^\infty(\mathbb{R}^n)$ is positive. Show that u is a distribution of order zero.

Problem 3. Define $u : C_c^\infty((0,\infty)) \to \mathbb{C}$ by

$$\langle u, \phi \rangle := \sum_{n=1}^{\infty} \phi^{(n)}(1/n) \qquad (\phi \in C_c^\infty((0,\infty)))\,.$$

a. Show that u is a distribution on $(0, \infty)$ of infinite order.

b. Does there exist a distribution v on $\mathbb{R}$ such that $v = u$ on $(0, \infty)$?

Problem 4. Let $U \subset \mathbb{R}^n$ be open, let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of U and, for each $\alpha \in I$, let $u_\alpha \in \mathcal{D}'(U_\alpha)$. Suppose that $u_\alpha = u_\beta$ on $U_\alpha \cap U_\beta$, for all $\alpha, \beta \in I$. Show that there is a unique $u \in \mathcal{D}'(U)$ such that $u = u_\alpha$ on U_α for each $\alpha \in I$. (*Hint* : Partition of unity.)

Problem 5.

a. For $n \in \mathbb{N}$, define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) := \int_{-n}^{nx} \frac{\sin t}{t} dt \qquad (x \in \mathbb{R}).$$

Show that $f_n \to \pi H$ in $\mathcal{D}'(\mathbb{R})$ as $n \to \infty$, where $H = \chi_{(0,\infty)}$ is the Heaviside function. (*Hint* : You can take for granted the fact that $\int_{-\infty}^{\infty} \frac{\sin t}{t} dt = \pi$.)

b. Deduce that $\frac{\sin nx}{x} \to \pi\delta$ in $\mathcal{D}'(\mathbb{R})$ as $n \to \infty$.

c. Deduce that if $\phi \in C_c^\infty(\mathbb{R})$, then

$$\int_{-n}^{n} \left(\int_{-\infty}^{\infty} \phi(x) e^{-itx} \, dx \right) dt \to 2\pi\phi(0)$$

as $n \to \infty$.

Problem 6. What is the order of the distribution p.v. $\frac{1}{x}$?

Problem 7. Let $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\phi \in C_c^\infty(\mathbb{R}^n)$. True or false?

a. If $\langle u, \phi \rangle = 0$, then $\phi u = 0$.

b. If $\phi u = 0$, then $\langle u, \phi \rangle = 0$.

Problem 1.

a. Let $f(t) = t$ for $t \in (-\pi, \pi]$, extended to be 2π-periodic on $\mathbb{R}$. Compute the Fourier coefficients of f.

Sol. A simple calculation shows that $\hat{f}(0) = 0$ and $\hat{f}(n) = i(-1)^n/n$ for $n \neq 0$.

b. Deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Sol. By Parseval's identity, we have

$$2\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2\, dt = \frac{\pi^2}{3},$$

from which the result follows.

Problem 2. Let $u : C_c^\infty(\mathbb{R}^n) \to \mathbb{C}$ be a linear functional which is positive in the sense that $\langle u, \phi\rangle \geq 0$ whenever $\phi \in C_c^\infty(\mathbb{R}^n)$ is positive. Show that u is a distribution of order zero.

Sol. Let $K \subset \mathbb{R}^n$ be compact, and let $\psi \in C_c^\infty(\mathbb{R}^n)$ with $0 \leq \psi \leq 1$ and $\psi = 1$ on K. Let $\phi \in C_c^\infty(\mathbb{R}^n), \operatorname{supp}\phi \subset K$ and set $M := \sup|\phi|$. Then we have

$$-\psi M \leq \operatorname{Re}\phi \leq \psi M$$

on $\mathbb{R}^n$. By positivity of u, this implies that

$$-M\langle u, \psi\rangle \leq \langle u, \operatorname{Re}\phi\rangle \leq M\langle u, \psi\rangle$$

i.e.

$$|\langle u, \operatorname{Re}\phi\rangle| \leq M\langle u, \psi\rangle.$$

Similarly for $\operatorname{Im}\phi$. Therefore, setting $C := \langle u, \psi\rangle$, we get

$$|\langle u, \phi\rangle| \leq |\langle u, \operatorname{Re}\phi\rangle| + |i\langle u, \operatorname{Im}\phi\rangle| \leq 2CM = 2C\sup|\phi|,$$

which shows that u is a distribution of order zero.

Problem 3. Define $u : C_c^\infty((0,\infty)) \to \mathbb{C}$ by

$$\langle u, \phi\rangle := \sum_{n=1}^{\infty} \phi^{(n)}(1/n) \qquad (\phi \in C_c^\infty((0,\infty)))\,.$$

a. Show that u is a distribution on $(0, \infty)$ of infinite order.

Sol. Let $K \subset (0,\infty)$ be compact. Let $N \in \mathbb{N}$ such that $K \subset [1/N, \infty)$. Then for every $\phi \in C_c^\infty((0,\infty))$, $\operatorname{supp}\phi \subset K$, we have

$$|\langle u, \phi\rangle| \leq \sum_{n=1}^{N} \sup|\phi^n|,$$

so that u is a distribution by a proposition proved in class.

Assume that u is of finite order, say N. Let $\psi \in C_c^\infty(\mathbb{R})$ supported on $[-1,1]$ with $\psi^{(N+1)}(0) \neq 0$ (for instance, one can take ψ to be x^{N+1} times a smooth bump function which is one at zero). For $\epsilon > 0$ small, set

$$\phi(x) = \psi\left(\frac{x - \frac{1}{N+1}}{\epsilon}\right).$$

Then ϕ is supported on a small interval around $1/(N+1)$, so that

$$\langle u, \phi\rangle = \phi^{(N+1)}(1/(N+1)) = \epsilon^{-(N+1)}\psi^{(N+1)}(0).$$

Since u is of order N, there must exist a constant $C > 0$ independent of ϵ such that

$$|\langle u, \phi\rangle| \leq C\sum_{n=0}^{N} \sup|\phi^{(n)}| \leq C\sum_{n=0}^{N} \epsilon^{-n}\sup|\psi^{(n)}|,$$

which implies that

$$|\psi^{(N+1)}(0)| \leq C\sum_{n=0}^{N} \epsilon^{N+1-n}\sup|\psi^{(n)}|.$$

Letting $\epsilon \to 0$ gives a contradiction.

b. Does there exist a distribution v on $\mathbb{R}$ such that $v = u$ on $(0,\infty)$?

Sol. No. Indeed, assume that such a distribution v exists. Then there exist C, N such that

$$|\langle v, \psi\rangle| \leq C\sum_{n=0}^{N} \sup|\psi^{(n)}| \qquad (\psi \in C_c^\infty(\mathbb{R}), \operatorname{supp}\psi \subset [-2,2]).$$

Since the support of u is contained in $[0,1]$, the above inequality would imply that u is of order N, a contradiction.

Problem 4. Let $U \subset \mathbb{R}^n$ be open, let $\{U_\alpha\}_{\alpha\in I}$ be an open cover of U and, for each $\alpha \in I$, let $u_\alpha \in \mathcal{D}'(U_\alpha)$. Suppose that $u_\alpha = u_\beta$ on $U_\alpha \cap U_\beta$, for all $\alpha, \beta \in I$. Show that there is a unique $u \in \mathcal{D}'(U)$ such that $u = u_\alpha$ on U_α for each $\alpha \in I$. (*Hint* : Partition of unity.)

Sol. First, we prove existence. Let $\phi \in C_c^\infty(U)$, and let $U_{\alpha_1}, \ldots, U_{\alpha_m}$ be a finite subcover of $\operatorname{supp}\phi$. Let $\psi_1, \ldots, \psi_n$ be a partition of unity for $\operatorname{supp}\phi$ subordinate to $U_{\alpha_1}, \ldots, U_{\alpha_m}$. Then we set

$$\langle u, \phi\rangle := \sum_{j=1}^{m} \langle u_{\alpha_j}, \phi\psi_j\rangle.$$

If $U_{\beta_1}, \ldots, U_{\beta_l}$ is another subcover of $\operatorname{supp}\phi$ with corresponding partition of unity $\widetilde{\psi}_1, \ldots, \widetilde{\psi}_l$, we have

$$\sum_{j=1}^{m} \langle u_{\alpha_j}, \phi\psi_j\rangle = \sum_{j=1}^{m}\sum_{k=1}^{l} \langle u_{\alpha_j}, \phi\psi_j\widetilde{\psi}_k\rangle = \sum_{j=1}^{m}\sum_{k=1}^{l} \langle u_{\beta_k}, \phi\psi_j\widetilde{\psi}_k\rangle.$$

By a similar calculation, we get

$$\sum_{k=1}^{l} \langle u_{\beta_k}, \phi\widetilde{\psi}_k\rangle = \sum_{j=1}^{m} \langle u_{\alpha_j}, \phi\psi_j\rangle,$$

so that $\langle u, \phi\rangle$ is well-defined. Clearly $u = u_\alpha$ on each U_α. Also, it is easy to see that u defined in this way is a distribution, for if K is any compact subset of U, then one can use the same cover for all $\phi \in C_c^\infty(U)$, $\operatorname{supp}\phi \subset K$. Then u satisfies the semi-norm estimate, by Leibniz's theorem.

Finally, the uniqueness of u follows from the fact proved in class that if two distributions are equal on some open sets, then they are equal on the union of all these open sets.

Problem 5.

a. For $n \in \mathbb{N}$, define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) := \int_{-n}^{nx} \frac{\sin t}{t} dt \qquad (x \in \mathbb{R}).$$

Show that $f_n \to \pi H$ in $\mathcal{D}'(\mathbb{R})$ as $n \to \infty$, where $H = \chi_{(0,\infty)}$ is the Heaviside function. (*Hint* : You can take for granted the fact that $\int_{-\infty}^{\infty} \frac{\sin t}{t} dt = \pi$.)

Sol. Clearly, $f_n(x) \to \pi H(x)$ as $n \to \infty$, for all $x \in \mathbb{R} \setminus \{0\}$. Let us show now that there is a $g \in L^1_{loc}(\mathbb{R})$ such that $|f_n| \leq g$ for all n. To do this, note that the function

$$y \mapsto \left| \int_0^y \frac{\sin t}{t} dt \right|$$

is continuous on $\mathbb{R}$ and tends to a limit as $y \to \pm\infty$, therefore it is bounded, say by M. It follows that $|f_n| \leq 2M \in L^1_{loc}(\mathbb{R})$, for all n. By a remark made in class, we get that $f_n \to \pi H$ in $\mathcal{D}'(\mathbb{R})$ as $n \to \infty$.

b. Deduce that $\frac{\sin nx}{x} \to \pi\delta$ in $\mathcal{D}'(\mathbb{R})$ as $n \to \infty$.

Sol. Since $f_n \to \pi H$ in $\mathcal{D}'(\mathbb{R})$, it follows that $f_n' \to (\pi H)'$ in $\mathcal{D}'(\mathbb{R})$, i.e. $\frac{\sin nx}{x} \to \pi\delta$ as $n \to \infty$.

c. Deduce that if $\phi \in C_c^\infty(\mathbb{R})$, then

$$\int_{-n}^{n} \left(\int_{-\infty}^{\infty} \phi(x) e^{-itx}\, dx \right) dt \to 2\pi\phi(0)$$

as $n \to \infty$.

Sol. A simple calculation using Fubini's theorem shows that the integral is equal to

$$2\int_{-\infty}^{\infty} \frac{\sin nx}{x}\phi(x)\, dx = 2\langle \frac{\sin nx}{x}, \phi\rangle$$

which tends to $2\pi\langle\delta, \phi\rangle = 2\pi\phi(0)$ as $n \to \infty$, by part b.

Problem 6. What is the order of the distribution p.v. $\frac{1}{x}$?

The order is one. Indeed, let $K \subset [-a, a]$ be a compact subset of $\mathbb{R}$. Then for all $\phi \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}\phi \subset K$, we have

$$|\langle \text{p.v.}\,\frac{1}{x}, \phi\rangle| = |\langle \partial \log|x|, \phi\rangle| = \left| \int_{-a}^{a} \log|x|\phi'(x)dx \right| \leq \sup|\phi'| \int_{-a}^{a} |\log|x||dx,$$

which shows that the order of p.v. $\frac{1}{x}$ is less than or equal to one.

If the order is zero, then there would exist a constant $C > 0$ such that

$$|\langle \text{p.v.}\,\frac{1}{x}, \phi\rangle| \leq C \sup|\phi| \qquad (\phi \in C_c^\infty(\mathbb{R}), \operatorname{supp}\phi \subset [0, 2]).$$

Let $\delta > 0$ and let $\phi \in C_c^\infty(\mathbb{R}), 0 \leq \phi \leq 1$ such that $\phi = 1$ on $[\delta, 1]$ and $\operatorname{supp}\phi \subset [0, 2]$. The above inequality gives

$$|\langle \text{p.v.}\,\frac{1}{x}, \phi\rangle| \leq C,$$

while on the other hand

$$\langle \text{p.v.}\,\frac{1}{x}, \phi\rangle = \int_0^2 \frac{\phi(x)}{x} dx \geq \int_\delta^1 \frac{1}{x} dx = \log(1/\delta).$$

Letting $\delta \to 0$ gives a contradiction.

Problem 7. Let $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\phi \in C_c^\infty(\mathbb{R}^n)$. True or false?

a. If $\langle u, \phi\rangle = 0$, then $\phi u = 0$.

Sol. False. Indeed, take $u = \delta' \in \mathcal{D}'(\mathbb{R})$ and $\phi \in C_c^\infty(\mathbb{R})$ with $\phi = 1$ near 0. Then $\langle u, \phi \rangle = -\phi'(0) = 0$, but $\phi u = \delta' \neq 0$.

b. If $\phi u = 0$, then $\langle u, \phi \rangle = 0$.

True. Indeed, take $\psi \in C_c^\infty(\mathbb{R}^n)$ with $\psi = 1$ on $\operatorname{supp} \phi$. Then we have

$$\langle u, \phi \rangle = \langle u, \phi\psi \rangle = \langle \phi u, \psi \rangle = 0.$$

MAT 550, Real Analysis II, Spring 2015 — Homework 9

Due in Class : April 30, 2015.

Turn in the following exercices. Exercise a.b refers to Exercise b in Chapter a in the Textbook.

Problem 1. Define $u : C_c^\infty(\mathbb{R}) \to \mathbb{C}$ by

$$\langle u, \phi \rangle := \sum_{n=1}^{\infty} \frac{\phi(1/n) - \phi(0)}{n} \qquad (\phi \in C_c^\infty(\mathbb{R})).$$

a. Show that u is a distribution on $\mathbb{R}$.

b. Show that the support of u is the compact set $K := \{1/n : n \geq 1\} \cup \{0\}$.

c. Show that there are no constants C, N such that

$$|\langle u, \phi \rangle| \leq C \sum_{j \leq N} \sup_K |\phi^{(j)}|$$

for all $\phi \in C_c^\infty(\mathbb{R})$.

Problem 2. Let $U \subset \mathbb{R}^n$ be open and let $u \in \mathcal{D}'(U)$. Prove that

a. if $v \in \mathcal{D}'(U)$, then

$$\operatorname{supp}(u + v) \subset \operatorname{supp} u \cup \operatorname{supp} v.$$

b. if α is any multi-index, then

$$\operatorname{supp}(\partial^\alpha u) \subset \operatorname{supp} u.$$

c. if $f \in C^\infty(U)$, then

$$\operatorname{supp}(fu) \subset \operatorname{supp} f \cap \operatorname{supp} u.$$

d. If $V \subset \mathbb{R}^n$ is open and $f : V \to U$ is a smooth diffeomomorphism, then

$$\operatorname{supp}(u \circ f) = f^{-1}(\operatorname{supp} u).$$

Problem 3. Find all solutions $u \in \mathcal{D}'(\mathbb{R})$ of the following equations.

a. $u'' = 0$.

b. $xu' = \delta$.

c. $xu' + u = 0$.

Problem 4.

a. Show that if $u \in \mathcal{D}'(\mathbb{R}^n)$ is homogeneous of degree λ, then it satisfies Euler's equation:

$$\sum_{j=1}^{n} x_j \partial_j u = \lambda u.$$

b. Use this to find all distributions on $\mathbb{R}$ which are homogeneous of degree -1.

Problem 1. Define $u : C_c^\infty(\mathbb{R}) \to \mathbb{C}$ by

$$\langle u, \phi \rangle := \sum_{n=1}^{\infty} \frac{\phi(1/n) - \phi(0)}{n} \qquad (\phi \in C_c^\infty(\mathbb{R})).$$

a. Show that u is a distribution on $\mathbb{R}$.

Sol. Linearity is clear. By the mean value theorem, we have $|\phi(1/n) - \phi(0)| \leq (1/n) \sup |\phi'|$, so that

$$|\langle u, \phi \rangle| \leq \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right) \sup |\phi'|,$$

which shows that u satisfies the semi-norm estimate and thus is a distribution.

b. Show that the support of u is the compact set $K := \{1/n : n \geq 1\} \cup \{0\}$.

Sol. Clearly, $u = 0$ on $\mathbb{R} \setminus K$, so that $\operatorname{supp} u \subset K$. On the other hand, let $n \geq 1$. We claim that $1/n \in \operatorname{supp} u$. Indeed, assume the contrary. Then there is a small interval I around $1/n$ which does not contain any other $1/m$ such that $u = 0$ on I. But then consider $\phi \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} \phi \subset I$ and $\phi(1/n) = 1$, so that $\langle u, \phi \rangle = 1/n \neq 0$, contradicting the fact that $u = 0$ on I.
This shows that $1/n \in \operatorname{supp} u$ for all $n \geq 1$. Since $\operatorname{supp} u$ is closed, it must also contain 0. Hence $\operatorname{supp} u = K$.

c. Show that there are no constants C, N such that

$$|\langle u, \phi \rangle| \leq C \sum_{j \leq N} \sup_K |\phi^{(j)}|$$

for all $\phi \in C_c^\infty(\mathbb{R})$.

Sol. Let $k \geq 1$. Let $\phi \in C_c^\infty(\mathbb{R})$ with $\phi = 1$ on a neighborhood of $[1/k, 1]$ and $\operatorname{supp} \phi \subset (1/(k+1), 2)$. Then $\sup_K |\phi| = 1$ and $\sup_K |\phi^{(j)}| = 0$ for all $j \geq 1$. Moreover,

$$\langle u, \phi \rangle = \sum_{n=1}^{k} 1/n.$$

If the inequality is true for some C, N, then this would imply that

$$\sum_{n=1}^{k} \frac{1}{n} \leq C$$

for all k, which is impossible.

Problem 2. Let $U \subset \mathbb{R}^n$ be open and let $u \in \mathcal{D}'(U)$. Prove that

a. if $v \in \mathcal{D}'(U)$, then

$$\operatorname{supp}(u+v) \subset \operatorname{supp} u \cup \operatorname{supp} v.$$

Sol. Let $\phi \in C_c^\infty(U)$ with $\operatorname{supp}\phi \subset U \setminus (\operatorname{supp} u \cup \operatorname{supp} v)$. Since $u = 0$ on $U \setminus \operatorname{supp} u$ and $\operatorname{supp}\phi \subset U \setminus \operatorname{supp} u$, we get $\langle u, \phi\rangle = 0$. Similarly, $\langle v, \phi\rangle = 0$. Hence $\langle u+v, \phi\rangle = 0$ and $u+v$ vanishes on the complement of $(\operatorname{supp} u \cup \operatorname{supp} v)$, which implies that

$$\operatorname{supp}(u+v) \subset \operatorname{supp} u \cup \operatorname{supp} v.$$

b. if α is any multi-index, then

$$\operatorname{supp}(\partial^\alpha u) \subset \operatorname{supp} u.$$

Sol. Let $\phi \in C_c^\infty(U)$ with $\operatorname{supp}\phi \subset U \setminus \operatorname{supp} u$. Then $\operatorname{supp}\partial^\alpha\phi \subset \operatorname{supp}\phi \subset U \setminus \operatorname{supp} u$, so that $\langle u, \partial^\alpha\phi\rangle = 0$. Hence $\langle \partial^\alpha u, \phi\rangle = 0$, so that $\partial^\alpha u$ vanishes on the complement of $\operatorname{supp} u$, which implies that

$$\operatorname{supp}(\partial^\alpha u) \subset \operatorname{supp} u.$$

c. if $f \in C^\infty(U)$, then

$$\operatorname{supp}(fu) \subset \operatorname{supp} f \cap \operatorname{supp} u.$$

Sol. First, we prove that $\operatorname{supp}(fu) \subset \operatorname{supp} f$. Let $\phi \in C_c^\infty(U)$ with $\operatorname{supp}\phi \subset U \setminus \operatorname{supp} f$. Then $f\phi = 0$, so that $\langle fu, \phi\rangle = 0$. This shows that fu vanishes on the complement of $\operatorname{supp} f$, so that $\operatorname{supp}(fu) \subset \operatorname{supp} f$.

Now, we prove that $\operatorname{supp}(fu) \subset \operatorname{supp} u$. Let $\phi \in C_c^\infty(U)$ with $\operatorname{supp}\phi \subset U \setminus \operatorname{supp} u$. Since $\operatorname{supp}(f\phi) \subset \operatorname{supp}\phi$ and u vanishes on the complement of $\operatorname{supp} u$, we get that $\langle u, f\phi\rangle = 0$, i.e. $\langle fu, \phi\rangle = 0$. Hence fu vanishes on the complement of $\operatorname{supp} u$, so that $\operatorname{supp}(fu) \subset \operatorname{supp} u$.

Combining these two inclusions, we get $\operatorname{supp}(fu) \subset \operatorname{supp} f \cap \operatorname{supp} u$.

d. If $V \subset \mathbb{R}^n$ is open and $f : V \to U$ is a smooth diffeomomorphism, then

$$\operatorname{supp}(u \circ f) = f^{-1}(\operatorname{supp} u).$$

Sol. Let $\phi \in C_c^\infty(V)$ with $\operatorname{supp} \phi \subset V \setminus f^{-1}(\operatorname{supp} u)$. Then the support of $(\phi \circ f^{-1})|\det(f^{-1})'|$ is contained in $U \setminus \operatorname{supp} u$, from which it follows that $\langle u \circ f, \phi \rangle = 0$. Therefore, $u \circ f$ vanishes outside $f^{-1}(\operatorname{supp} u)$, which implies that $\operatorname{supp}(u \circ f) \subset f^{-1}(\operatorname{supp} u)$.

Replacing u by $u \circ f$ and f by f^{-1}, we get $\operatorname{supp} u \subset f(\operatorname{supp}(u \circ f))$, i.e. $\operatorname{supp}(u \circ f) \supset f^{-1}(\operatorname{supp} u)$.

Combining both inclusions, we get $\operatorname{supp}(u \circ f) = f^{-1}(\operatorname{supp} u)$.

Problem 3. Find all solutions $u \in \mathcal{D}'(\mathbb{R})$ of the following equations.

a. $u'' = 0$.

Sol. By a theorem proved in class, u' is constant, say $u' = a \in \mathbb{C}$. Then $(u - ax)' = 0$, so that $u = ax + b$ for some constant $b \in \mathbb{C}$. Clearly, all of such u's are solutions.

b. $xu' = \delta$.

Sol. Assume that $xu' = \delta$. A simple calculation shows that $x\delta' = -\delta$, so that $x(u + \delta)' = 0$. By the solution to the division problem, we get that $(u+\delta)' = a\delta$ for some $a \in \mathbb{C}$. Hence $(u+\delta-aH)' = 0$, so that $u = -\delta+aH+b$ for some $b \in \mathbb{C}$. Clearly, all of such u's are solutions.

c. $xu' + u = 0$.

Assume that $xu' + u = 0$. Then $(xu)' = 0$, so that $xu = a$ for some constant $a \in \mathbb{C}$. A solution to this equation is $a\,\mathrm{p.v.}\,1/x$. Hence $x(u - a\,\mathrm{p.v.}\,1/x) = 0$, so that $u = a\,\mathrm{p.v.}\,1/x + b\delta$ for some $b \in \mathbb{C}$. Again, such u's are all solutions.

Problem 4.

a. Show that if $u \in \mathcal{D}'(\mathbb{R}^n)$ is homogeneous of degree λ, then it satisfies Euler's equation:

$$\sum_{j=1}^{n} x_j \partial_j u = \lambda u.$$

Sol. Let $\phi \in C_c^\infty(\mathbb{R}^n)$. By the theorem on test functions depending on a parameter, we have

$$\frac{d}{dt}\langle u_t, \phi\rangle = \frac{d}{dt}\langle u, t^{-n}\phi_{1/t}\rangle = \left\langle u(x), \frac{\partial}{\partial t}\left(t^{-n}\phi(x/t)\right)\right\rangle.$$

But

$$\frac{\partial}{\partial t}\left(t^{-n}\phi(x/t)\right) = -nt^{-n-1}\phi(x/t) - t^{-n-2}\sum_{j=1}^{n} x_j\partial_j\phi(x/t).$$

Set $t = 1$ to get

$$\frac{d}{dt}\langle u_t, \phi\rangle\big|_{t=1} = -n\langle u, \phi\rangle - \Big\langle u, \sum_{j=1}^{n} x_j\partial_j\phi\Big\rangle = \Big\langle \sum_{j=1}^{n} x_j\partial_j u, \phi\Big\rangle.$$

On the other hand, $\langle u_t, \phi\rangle = t^\lambda\langle u, \phi\rangle$ since u is homogeneous of degree λ, so that

$$\frac{d}{dt}\langle u_t, \phi\rangle\big|_{t=1} = \lambda\langle u, \phi\rangle.$$

Comparing these formulas gives the desired formula.

b. Use this to find all distributions on $\mathbb{R}$ which are homogeneous of degree -1.

Applying part a. with $n = 1$ and $\lambda = -1$, we obtain the equation $xu' + u = 0$, whose solutions are $u = a\,\text{p.v.}\,1/x + b\delta$ for $a, b \in \mathbb{C}$, by Problem 3 part c. All of these distributions are indeed homogeneous of degree -1.

N.B. : This homework will not be graded. However, I will ask one of the following problems in the Final Exam for the course.

Problem 1. Let $V \subset \mathbb{R}^n$ be open and let $u \in \mathcal{D}'(\mathbb{R} \times V)$. Show that $x_1 u = 0$ if and only if $u = \delta \otimes v$, where $v \in \mathcal{D}'(V)$ and δ is the Dirac distribution on $\mathbb{R}$.

Problem 2.
a. Let $u \in \mathcal{D}'(\mathbb{R}^n)$ and define $T : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ by $T\phi := u * \phi$. Show that

(1) $\tau_h(T\phi) = T(\tau_h\phi)$ for all $h \in \mathbb{R}^n$ and all $\phi \in C_c^\infty(\mathbb{R}^n)$,
(2) $(T\phi_n)(0) \to (T\phi)(0)$ whenever $\phi_n \to \phi$ in $C_c^\infty(\mathbb{R}^n)$.

b. Conversely, show that if $T : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is a linear map satisfying (1) and (2), then there exists $u \in \mathcal{D}'(\mathbb{R}^n)$ such that $T\phi = u * \phi$ for all $\phi \in C_c^\infty(\mathbb{R}^n)$.

Problem 3.
a. Show that
$$E(z) := \frac{1}{\pi z}$$
is a fundamental solution of the Cauchy-Riemann operator $\partial/\partial\overline{z}$ on $\mathbb{C} = \mathbb{R}^2$.

b. Find a fundamental solution of the differential operator ∂^α on $\mathbb{R}^n$.

Problem 4. Show that $e^x \cos(e^x)$ is a tempered distribution on $\mathbb{R}$, while e^x is not.

Problem 5. Let $u \in \mathcal{S}'(\mathbb{R}^n)$ be a tempered distribution that is homogeneous of degree λ. Show that its Fourier transform $\hat{u}$ is also homogeneous, and find its degree.

Problem 6. Find all distributions $u \in S'(\mathbb{R}^n)$ such that $\Delta u = u$. Is the answer different if we replace $\mathcal{S}'$ by $\mathcal{D}'$?

Problem 1. Let $V \subset \mathbb{R}^n$ be open and let $u \in \mathcal{D}'(\mathbb{R}\times V)$. Show that $x_1 u = 0$ if and only if $u = \delta \otimes v$, where $v \in \mathcal{D}'(V)$ and δ is the Dirac distribution on $\mathbb{R}$.

Sol. If $u = \delta\otimes v$ for some $v \in \mathcal{D}'(V)$, then clearly $x_1 u = (x_1\delta)\otimes v = 0\otimes v = 0$. Conversely, assume that $x_1 u = 0$. For $\chi \in C_c^\infty(V)$, define $v_\chi : C_c^\infty(\mathbb{R}) \to \mathbb{C}$ by

$$\langle v_\chi, \phi\rangle = \langle u, \phi \otimes \chi\rangle \qquad (\phi \in C_c^\infty(\mathbb{R})).$$

Clearly, $v_\chi \in \mathcal{D}'(\mathbb{R})$. Moreover, we have

$$\langle x_1 v_\chi, \phi\rangle = \langle u, (x_1\phi) \otimes \chi\rangle = \langle x_1 u, \phi \otimes \chi\rangle = 0$$

since $x_1 u = 0$. Thus $v_\chi = C_\chi \delta$ for some $C_\chi \in \mathbb{C}$, by the solution to the division problem. Define $v : C_c^\infty(V) \to \mathbb{C}$ by $\langle v, \chi\rangle := C_\chi$. Then for $\phi \in C_c^\infty(\mathbb{R})$ and $\chi \in C_c^\infty(V)$, we have

$$\langle u, \phi \otimes \chi\rangle = \langle v_\chi, \phi\rangle = \langle \delta, \phi\rangle\langle v, \chi\rangle.$$

Taking $\phi_0 \in C_c^\infty(\mathbb{R})$ with $\phi_0(0) = 1$, we get $\langle v, \chi\rangle = \langle u, \phi_0\otimes\chi\rangle$, which shows that $v \in \mathcal{D}'(V)$. Finally, the above equality shows that $u = \delta \otimes v$.

Problem 2.
a. Let $u \in \mathcal{D}'(\mathbb{R}^n)$ and define $T : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ by $T\phi := u * \phi$. Show that

(1) $\tau_h(T\phi) = T(\tau_h\phi)$ for all $h \in \mathbb{R}^n$ and all $\phi \in C_c^\infty(\mathbb{R}^n)$,
(2) $(T\phi_n)(0) \to (T\phi)(0)$ whenever $\phi_n \to \phi$ in $C_c^\infty(\mathbb{R}^n)$.

Sol. Both statements follow directly from the regularization theorem.

b. Conversely, show that if $T : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is a linear map satisfying (1) and (2), then there exists $u \in \mathcal{D}'(\mathbb{R}^n)$ such that $T\phi = u * \phi$ for all $\phi \in C_c^\infty(\mathbb{R}^n)$.

Sol. Define $u : C_c^\infty(\mathbb{R}^n) \to \mathbb{C}$ by

$$\langle u, \phi\rangle := T\widetilde{\phi}(0) \qquad (\phi \in C_c^\infty(\mathbb{R}^n)).$$

Then u is linear and sequentially continuous by (2). Moreover, for $\phi \in C_c^\infty(\mathbb{R}^n)$, we have, by the regularization theorem

$$(u*\phi)(x) = \langle u(y), \phi(x-y)\rangle = T(\tau_{-x}\phi)(0) = \tau_{-x}(T\phi)(0) = T\phi(x) \qquad (x \in \mathbb{R}^n),$$

by (1). Hence $T\phi = u * \phi$.

Problem 3.

a. Show that
$$E(z) := \frac{1}{\pi z}$$
is a fundamental solution of the Cauchy-Riemann operator $\partial/\partial\overline{z}$ on $\mathbb{C} = \mathbb{R}^2$.

Sol. Proceeding as in the example of the Laplacian, we get that $\partial E/\partial\overline{z} = c_0\delta$ for some $c_0 \in \mathbb{C}$. To find the value of c_0, fix $\phi \in C_c^\infty(\mathbb{C})$ radial. Then $\partial\phi/\partial\overline{z} = (e^{i\theta}/2)\partial\phi/\partial r$, so that
$$\left\langle \frac{\partial E}{\partial \overline{z}}, \phi \right\rangle = -\lim_{\epsilon\to 0}\int_\epsilon^\infty \int_0^{2\pi} \frac{1}{\pi r e^{i\theta}} \frac{e^{i\theta}}{2}\frac{\partial\phi}{\partial r} r\,d\theta dr = \phi(0).$$
Thus $c_0 = 1$ and $\partial E/\partial\overline{z} = \delta$.

b. Find a fundamental solution of the differential operator ∂^α on $\mathbb{R}^n$.

Sol. Write $\alpha = (\alpha_1, \ldots, \alpha_n)$. Let us first find a fundamental solution u_j of the differential operator $\partial_j^{\alpha_j}$ on $\mathbb{R}$. If $\alpha_j = 0$, then $u_j = \delta$ is trivially a fundamental solution. If $\alpha_j \geq 1$, then a simple calculation shows that
$$\partial_j^{\alpha_j}(x_j^{\alpha_j - 1} H_{x_j}) = (\alpha_j - 1)!\,\delta,$$
so that $u_j := 1/(\alpha_j - 1)!\, x_j^{\alpha_j-1} H_{x_j}$ is a fundamental solution. Now, for a fundamental solution of ∂^α on $\mathbb{R}^n$, it suffices to consider $u := u_1 \otimes u_2 \otimes \cdots \otimes u_n$.

Problem 4. Show that $e^x\cos(e^x)$ is a tempered distribution on $\mathbb{R}$, while e^x is not.

Sol. Since $e^x\cos(e^x) = (\sin(e^x))'$ and $\sin(e^x) \in L^\infty(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$, it follows that $e^x\cos(e^x) \in \mathcal{S}'(\mathbb{R})$.

For the second part, set $u := e^x$ and assume that $u \in \mathcal{S}'(\mathbb{R})$. Since $u' - u = 0$, we can take the Fourier transform on both sides to deduce that $(2\pi i\xi - 1)u' = 0$. Since $2\pi i \xi - 1 \neq 0$ for all $\xi \in \mathbb{R}$, it easily follows that $u' = 0$, so that $u = 0$, which is a contradiction.

Problem 5. Let $u \in \mathcal{S}'(\mathbb{R}^n)$ be a tempered distribution that is homogeneous of degree λ. Show that its Fourier transform $\hat{u}$ is also homogeneous, and find its degree.

Sol. For $t > 0$, let $f(x) = tx$ and $u_t := u \circ f$. Then $t^{-\lambda}\hat{u} = (u_{1/t})\hat{\ } = t^n(\hat{u})_t$, so that $(\hat{u})_t = t^{-\lambda-n}\hat{u}$ and $\hat{u}$ is homogeneous of degree $-\lambda - n$.

Problem 6. Find all distributions $u \in S'(\mathbb{R}^n)$ such that $\Delta u = u$. Is the answer different if we replace $\mathcal{S}'$ by $\mathcal{D}'$?

Sol. If $u \in S'(\mathbb{R}^n)$ and $\Delta u = u$, then by taking the Fourier transform we get $-4\pi^2|\xi|^2\hat{u} = \hat{u}$, i.e. $(1 + 4\pi^2|\xi|^2)\hat{u} = 0$. It easily follows that $\hat{u} = 0$ and so $u = 0$.

The answer is different if we replace $\mathcal{S}'$ by $\mathcal{D}'$. Indeed, $u = e^x$ on $\mathbb{R}$ satisfies $\Delta u = u$.

MAT 550, Real Analysis II, Spring 2015 — Midterm Exam

Name :

Student ID :

Problem 1 (20 pts).

Let H be a Hilbert space.

a. State Riesz's theorem describing the dual of H.

b. Let M be a subspace of H and let f be a bounded linear functional on M. We know by Hahn-Banach that f has an extension to a bounded linear functional F on H such that $\|F\| = \|f\|$. Prove that this extension F is unique and that it vanishes on $M^{\perp}$.

Problem 2 (20 pts).

a. State Hölder's inequality and describe exactly when equality occurs.

b. Suppose μ is a positive measure on a set X with $\mu(X) < \infty$. Let $f \in L^\infty(\mu)$ such that $\|f\|_\infty > 0$, and define

$$\alpha_n := \int_X |f|^n d\mu \qquad (n \in \mathbb{N}).$$

Show that

$$\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty.$$

(*Hint* : You can take for granted the fact that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.)

Problem 3 (20 pts).

Let X be a locally compact Hausdorff topological space.

a. Define what is a Radon measure on X.

b. Suppose that μ is a positive Radon measure on X with $\mu(X) = \infty$. Show that there is a positive function $f \in C_0(X)$ such that $\int f d\mu = \infty$.
(*Hint* : Use the fact that for all $n \in \mathbb{N}$, there exists $f_n \in C_c(X)$ such that $0 \leq f_n \leq 1$ and $\int f_n \, d\mu > 2^n$).

Problem 4 (20 pts).

a. State the theorem about the dual of L^p for $1 \leq p < \infty$.

b. Let $1 < p < \infty$ and let q be the exponent conjugate to p. Suppose that (ξ_j) is a sequence of complex numbers such that $\sum_j |\xi_j \eta_j| < \infty$ for every sequence (η_j) of complex numbers such that $\sum_j |\eta_j|^p < \infty$. Prove that then $\sum_j |\xi_j|^q < \infty$.

Problem 5 (20 pts).

True or False? Give a short justification.

a. If X is a locally compact Hausdorff space, then every positive linear functional on $(C_c(X), \|\cdot\|_u)$ is bounded.

b. The Hardy-Littlewood maximal operator H on $L^1_{loc}(\mathbb{R}^n)$ is strong type (p,p) for all $1 < p \leq \infty$.

c. If μ is a σ-finite Radon measure on a locally compact Hausdorff topological space X and if $E \subset X$ is Borel, then $E = A \cup N$ for an F_σ set A and a Borel set N with $\mu(N) = 0$.

d. Define $F : \mathbb{R}^2 \to \mathbb{R}$ by

$$F(x,t) := x + \sin x \sin e^t.$$

Then the differential equation

$$x'(t) = F(x,t)$$

has a unique solution $x(t)$ near $t = 0$ such that $x(0) = 0$.

Solution

Problem 1 (20 pts).

Let H be a Hilbert space.

a. State Riesz's theorem describing the dual of H.

Sol. The theorem says that if $f \in H^*$, then there is a unique $y \in H$ such that

$$f(x) = \langle x, y \rangle \qquad (x \in H).$$

Moreover, we have $\|f\| = \|y\|$.

b. Let M be a subspace of H and let f be a bounded linear functional on M. We know by Hahn-Banach that f has an extension to a bounded linear functional F on H such that $\|F\| = \|f\|$. Prove that this extension F is unique and that it vanishes on $M^\perp$.

Sol. The existence of a norm-preserving extension F of f is given by Hahn-Banach. By Riesz's theorem, we have $F(x) = \langle x, y \rangle$ for all $x \in H$, for some $y \in H$. Write $y = Py + Qy$ where $Py \in \overline{M}$ and $Qy \in \overline{M}^\perp$. Then we have

$$f(x) = F(x) = \langle x, Py + Qy \rangle = \langle x, Py \rangle \qquad (x \in M),$$

from which it follows that $\|Py\|^2 = \|f\|^2 = \|F\|^2 = \|y\|^2 = \|Py\|^2 + \|Qy\|^2$ and thus $Qy = 0$, so that $F(x) = \langle x, Py \rangle$ vanishes on $M^\perp$. If G is any other norm-preserving extension of f, then by the same argument G must vanish on $M^\perp$ and

$$G(x) = G(Px) = f(Px) = F(Px) = F(x) \qquad (x \in H),$$

which proves uniqueness.

Problem 2 (20 pts).

a. State Hölder's inequality and describe exactly when equality occurs.

Sol. Let $1 \leq p \leq \infty$ and let q be the conjugate exponent to p. If $f, g : X \to \mathbb{C}$ are measurable functions, then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

with equality if and only if $\alpha|f|^p = \beta|g|^q$ almost everywhere for some constants α, β not both zero.

b. Suppose μ is a positive measure on a set X with $\mu(X) < \infty$. Let $f \in L^\infty(\mu)$ such that $\|f\|_\infty > 0$, and define

$$\alpha_n := \int_X |f|^n d\mu \qquad (n \in \mathbb{N}).$$

Show that

$$\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty.$$

(*Hint* : You can take for granted the fact that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.)

Sol. First, note that we have

$$\alpha_{n+1} \leq \alpha_n \|f\|_\infty \qquad (n \in \mathbb{N}),$$

which implies that

$$\limsup_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} \leq \|f\|_\infty.$$

For the reverse inequality, we apply Hölder's inequality to the pair of exponents $p = (n+1)/n$ and $q = n+1$ to obtain

$$\alpha_n \leq \left(\int_X |f|^{n+1} \right)^{n/(n+1)} \mu(X)^{1/(n+1)},$$

so that

$$\frac{\alpha_{n+1}}{\alpha_n} \geq \left(\int_X |f|^{n+1} \right)^{1/(n+1)} \mu(X)^{-1/(n+1)},$$

hence

$$\liminf_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} \geq \liminf_{n\to\infty} \left(\int_X |f|^{n+1} \right)^{1/(n+1)} = \|f\|_\infty,$$

by the hint.

Problem 3 (20 pts).

Let X be a locally compact Hausdorff topological space.

a. Define what is a Radon measure on X.

Sol. A Radon measure is a positive Borel measure on X which is finite on compact sets, outer regular on Borel sets and inner regular on open sets.

b. Suppose that μ is a positive Radon measure on X with $\mu(X) = \infty$. Show that there is a positive function $f \in C_0(X)$ such that $\int f \, d\mu = \infty$.
(*Hint* : Use the fact that for all $n \in \mathbb{N}$, there exists $f_n \in C_c(X)$ such that $0 \leq f_n \leq 1$ and $\int_X f_n \, d\mu > 2^n$).

Sol. Note that $Tf := \int f\, d\mu$ defines a positive linear functional on $C_c(X)$, so by the Riesz representation theorem, we have

$$\mu(X) = \sup\{T(f) : f \in C_c(X), 0 \le f \le 1\}.$$

If $\mu(X) = \infty$, we get that there are functions $f_n \in C_c(X)$ such that $0 \le f_n \le 1$ and $T(f_n) = \int f_n\, d\mu > 2^n$. Consider $f := \sum_{n=1}^{\infty} 2^{-n} f_n$. Then f is a uniform limit of functions in $C_c(X)$, so it belongs to $C_0(X)$. Moreover, we have for every $N \in \mathbb{N}$,

$$\int_X f\, d\mu \ge \sum_{n=1}^{N} \int_X 2^{-n} f_n\, d\mu \ge \sum_{n=1}^{N} 2^{-n} 2^n = N.$$

Hence $\int_X f\, d\mu = \infty$.

Problem 4 (20 pts).

a. State the theorem about the dual of L^p for $1 \le p < \infty$.

Sol. Suppose $1 \le p < \infty$, μ is a positive σ-finite measure on X and ψ is a bounded linear functional on $L^p(\mu)$. Then there is a unique $g \in L^q(\mu)$ where q is the conjugate exponent to p such that

$$\psi(f) = \int fg\, d\mu \qquad (f \in L^p(\mu)).$$

Moreover, we have $\|\psi\| = \|g\|_q$.

b. Let $1 < p < \infty$ and let q be the exponent conjugate to p. Suppose that (ξ_j) is a sequence of complex numbers such that $\sum_j |\xi_j \eta_j| < \infty$ for every sequence (η_j) of complex numbers such that $\sum_j |\eta_j|^p < \infty$. Prove that then $\sum_j |\xi_j|^q < \infty$.

Sol. Note that this is a special case of Problem 1, Homework 4. For $n \in \mathbb{N}$, define a linear functional T_n on ℓ^p by

$$T_n((\eta_j)_{j\in\mathbb{N}}) := \sum_{j=1}^{n} \xi_j \eta_j \qquad ((\eta_j)_{j\in\mathbb{N}} \in \ell^p).$$

Then T_n is bounded and $\|T_n\| = \left(\sum_{j=1}^{n} |\xi_j|^q\right)^{1/q}$, by **a.**. By the uniform boundedness principle, we have that $\sup_n \|T_n\| < \infty$, from which it directly follows that $\sum_j |\xi_j|^q < \infty$.

Problem 5 (20 pts).

True or False? Give a short justification.

a. If X is a locally compact Hausdorff space, then every positive linear functional on $(C_c(X), \|\cdot\|_u)$ is bounded.

Sol. False. An example is given by $X = \mathbb{R}$ and $T(f) = \int_{\mathbb{R}} f\,dm$ for $f \in C_c(\mathbb{R})$, where m is the Lebesgue measure on $\mathbb{R}$.

b. The Hardy-Littlewood maximal operator H on $L^1_{loc}(\mathbb{R}^n)$ is strong type (p,p) for all $1 < p \leq \infty$.

Sol. True. This follows from the Marcinkiewicz Interpolation Theorem with $p_0 = q_0 = 1$ and $p_1 = q_1 = \infty$.

c. If μ is a σ-finite Radon measure on a locally compact Hausdorff topological space X and if $E \subset X$ is Borel, then $E = A \cup N$ for an F_σ set A and a Borel set N with $\mu(N) = 0$.

Sol. True. By a proposition proved in class, there is an F_σ set A and a G_δ set B such that $A \subset E \subset B$ and $\mu(B \setminus A) = 0$. Then set $N := E \setminus A$.

d. Define $F : \mathbb{R}^2 \to \mathbb{R}$ by

$$F(x,t) := x + \sin x \sin e^t.$$

Then the differential equation

$$x'(t) = F(x,t)$$

has a unique local solution $x(t)$ near $t = 0$ such that $x(0) = 0$.

Sol. True. Indeed, we have

$$|F(x,t) - F(y,t)| \leq |x-y| + |\sin e^t||\sin x - \sin y| \leq 2|x-y|$$

by the mean value theorem. In particular, $F_t(x) = F(x,t)$ is locally uniformly Lipschitz on $\mathbb{R}$. The statement now follows from the Existence and Uniqueness Theorem for first order ODE's.

MAT 550, Real Analysis II, Spring 2015 — **Final Exam**

Name :

Student ID :

Problem 1 (20 pts).

a. Show that if $1 \le p < \infty$ and $f \in L^p(\mathbb{R}^n)$, then $\|\tau_y f - f\|_p \to 0$ as $y \to 0$ in $\mathbb{R}^n$, where $\tau_y f(x) = f(x-y)$ for $x \in \mathbb{R}^n$.

(*Hint:* You can take for granted the fact that $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.)

b. Use this to prove that if $f, \phi \in L^1(\mathbb{R}^n)$ and $\int \phi(x)\,dx = 1$, then $f * \phi_t \to f$ in $L^1(\mathbb{R}^n)$ as $t \to 0$, where

$$\phi_t(x) := t^{-n}\phi(x/t) \qquad (x \in \mathbb{R}^n).$$

Problem 2 (20 pts).

a. State the Plancherel Theorem on the Fourier transform on $L^2(\mathbb{R}^n)$.

b. Let $f, g \in L^2(\mathbb{R}^n)$ and assume that the convolution $f * g$ belongs to $L^2(\mathbb{R}^n)$. Show that $(f * g)\hat{} = \hat{f}\hat{g}$.

c. Deduce that there exist $f, g \in L^2(\mathbb{R})$ such that $f * g \notin L^2(\mathbb{R})$.

Problem 3 (20 pts).

a. Give the definition of the convolution of two distributions on $\mathbb{R}^n$, one of which is compactly supported, and state the Regularization Theorem.

b. Let $u \in \mathcal{D}'(\mathbb{R}^n)$ and define $T : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ by $T\phi := u * \phi$. Show that

(1) $\tau_h(T\phi) = T(\tau_h\phi)$ for all $h \in \mathbb{R}^n$ and all $\phi \in C_c^\infty(\mathbb{R}^n)$,
(2) $(T\phi_n)(0) \to (T\phi)(0)$ whenever $\phi_n \to \phi$ in $C_c^\infty(\mathbb{R}^n)$.

c. Conversely, show that if $T : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is a linear map satisfying (1) and (2), then there exists $u \in \mathcal{D}'(\mathbb{R}^n)$ such that $T\phi = u * \phi$ for all $\phi \in C_c^\infty(\mathbb{R}^n)$.

Problem 4 (20 pts).

a. Give the definition of the support of a distribution and state the theorem on the characterization of distributions u on $\mathbb{R}^n$ such that $\operatorname{supp} u = \{0\}$.

b. Let $P(\partial) := \sum_{|\alpha| \leq k} c_\alpha \partial^\alpha$ be a linear differential operator with constant coefficients. Show that if the polynomial $P(\xi) = \sum_{|\alpha| \leq k} c_\alpha \xi^\alpha$ satisfies $P(\xi) \neq 0$ for all $\xi \neq 0$, then every tempered distribution u on $\mathbb{R}^n$ such that $P(\partial)u = 0$ is a polynomial.

Problem 5 (20 pts).

True or False? Give a short justification.

a. The map $f \to \hat{f}$ is an isomorphism of $L^1(\mathbb{T})$ onto c_0.

b. If $f \in C(\mathbb{T})$ and $S_n f(x) := \sum_{k=-n}^{n} \hat{f}(k) e^{ikx}$ for $n \in \mathbb{N}$, then the sequence $(S_n f(x))_n$ converges for every $x \in \mathbb{R}$, but not necessarily to $f(x)$.

c. If u is a distribution on $\mathbb{R}^n$ that is homogeneous of degree λ and $\alpha = (1, 2, \ldots, n)$, then $\partial^\alpha u$ is homogeneous of degree $\lambda - n(n+1)/2$.

d. A distribution u on an open set $U \subset \mathbb{R}^n$ is said to have *finite order* if there exist $C > 0$ and $N \in \mathbb{N} \cup \{0\}$ such that for every compact set $K \subset U$, we have

$$|\langle u, \phi \rangle| \leq C \sum_{|\alpha| \leq N} \sup |\partial^\alpha \phi| \qquad (\phi \in C_c^\infty(U), \operatorname{supp} \phi \subset K).$$

MAT 550, Real Analysis II, Spring 2015 — Final Exam

Solution

Problem 1 (20 pts).

a. Show that if $1 \leq p < \infty$ and $f \in L^p(\mathbb{R}^n)$, then $\|\tau_y f - f\|_p \to 0$ as $y \to 0$ in $\mathbb{R}^n$, where $\tau_y f(x) = f(x-y)$ for $x \in \mathbb{R}^n$.

(*Hint:* You can take for granted the fact that $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.)

Sol. Let $\epsilon > 0$ and let $g \in C_c(\mathbb{R}^n)$ such that $\|f - g\|_p < \epsilon/3$. Then for every $y \in \mathbb{R}^n$ small enough, we have $\|\tau_y g - g\|_p < \epsilon/3$. Indeed, let $K \subset \mathbb{R}^n$ be compact such that $\operatorname{supp} \tau_y g \subset K$ for all $y \in \mathbb{R}^n, |y| \leq 1$. Then for such y's we have

$$\int |\tau_y g - g|^p \, dm \leq \sup_{x \in K} |g(x-y) - g(x)|^p m(K)$$

which goes to zero as $y \to 0$ in $\mathbb{R}^n$.

It follows that for all $y \in \mathbb{R}^n$ small enough, we have

$$\|\tau_y f - f\|_p \leq \|\tau_y(f-g)\|_p + \|\tau_y g - g\|_p + \|g - f\|_p < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

b. Use this to prove that if $f, \phi \in L^1(\mathbb{R}^n)$ and $\int \phi(x)\, dx = 1$, then $f * \phi_t \to f$ in $L^1(\mathbb{R}^n)$ as $t \to 0$, where

$$\phi_t(x) := t^{-n} \phi(x/t) \qquad (x \in \mathbb{R}^n).$$

Sol. We have

$$f * \phi_t(x) - f(x) = \int (f(x-y) - f(x))\phi_t(y)\, dy = \int (\tau_{tz} f(x) - f(x))\phi(z)\, dz,$$

where we made the change of variable $y = tz$. Hence

$$|f * \phi_t(x) - f(x)| \leq \int |(\tau_{tz} f(x) - f(x))||\phi(z)|\, dz.$$

Integrating and using Fubini's theorem, we get

$$\|f * \phi_t - f\|_1 \leq \int \|\tau_{tz} f - f\|_1 |\phi(z)|\, dz$$

and the result follows from part a. and the Dominated Convergence Theorem.

Problem 2 (20 pts).

a. State the Plancherel Theorem on the Fourier transform on $L^2(\mathbb{R}^n)$.

Sol. The Plancherel theorem states that if $f \in L^1 \cap L^2$, then $\hat{f} \in L^2$; and the restriction of the Fourier transform operator to $L^1 \cap L^2$ extends uniquely to a unitary isomorphism on L^2.

b. Let $f, g \in L^2(\mathbb{R}^n)$ and assume that the convolution $f * g$ belongs to $L^2(\mathbb{R}^n)$. Show that $(f * g)\hat{} = \hat{f}\hat{g}$.

Sol. First note that $\hat{f}\hat{g} \in L^1(\mathbb{R}^n)$, so its Fourier transform at $\xi \in \mathbb{R}^n$ is

$$\begin{aligned} \int \hat{f}(x)\hat{g}(x)e^{-2\pi i\xi\cdot x}\,dx &= \langle \hat{f}, \overline{\hat{g}}e^{2\pi i\xi\cdot x}\rangle \\ &= \langle \hat{f}, \overline{\check{g}}e^{2\pi i\xi\cdot x}\rangle \\ &= \langle \hat{f}, (\tau_{-\xi}\overline{\widetilde{g}})\hat{}\,\rangle \\ &= \langle f, \tau_{-\xi}\overline{\widetilde{g}}\,\rangle \\ &= \int f(x)g(-\xi - x)\,dx = (f * g)(-\xi), \end{aligned}$$

where $\widetilde{g}(x) := g(-x)$. In particular, the Fourier transform of $\hat{f}\hat{g}$ belongs to $L^2(\mathbb{R}^n)$, which implies that $\hat{f}\hat{g} \in L^2(\mathbb{R}^n)$, by Plancherel. The result then follows from the Fourier inversion theorem for $L^2(\mathbb{R}^n)$ proved in class.

c. Deduce that there exist $f, g \in L^2(\mathbb{R})$ such that $f * g \notin L^2(\mathbb{R})$.

Sol. Consider a function $F \in L^2(\mathbb{R})$ such that $F^2 \notin L^2(\mathbb{R})$, for instance $F(x) := x^{-1/3}e^{-x^2}$. Let $f \in L^2(\mathbb{R})$ such that $\hat{f} = F$. We claim that $f * f \notin L^2(\mathbb{R})$. Indeed, if $f * f \in L^2(\mathbb{R})$, then by part b. with $g = f$, we would have

$$F^2 = \hat{f}\hat{f} = (f * f)\hat{} \in L^2(\mathbb{R}),$$

a contradiction.

Problem 3 (20 pts).

a. Give the definition of the convolution of two distributions on $\mathbb{R}^n$, one of which is compactly supported, and state the Regularization Theorem.

Sol. If $u \in \mathcal{E}'(\mathbb{R}^n)$ and $v \in \mathcal{D}'(\mathbb{R}^n)$, then the convolution $u * v$ is the distribution on $\mathbb{R}^n$ defined by

$$\langle u * v, \phi\rangle = \langle u(x) \otimes v(y), \rho(x)\phi(x+y)\rangle \qquad (\phi \in C_c^\infty(\mathbb{R}^n)),$$

where $\rho \in C_c^\infty(\mathbb{R}^n)$, $\rho = 1$ on a neighborhood of the support of u.

The Regularization Theorem states that if $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\psi \in C_c^\infty(\mathbb{R}^n)$, then $u * \psi \in C^\infty(\mathbb{R}^n)$ and

$$u * \psi(x) = \langle u(y), \psi(x-y)\rangle \qquad (x \in \mathbb{R}^n).$$

b. Let $u \in \mathcal{D}'(\mathbb{R}^n)$ and define $T : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ by $T\phi := u * \phi$. Show that

(1) $\tau_h(T\phi) = T(\tau_h\phi)$ for all $h \in \mathbb{R}^n$ and all $\phi \in C_c^\infty(\mathbb{R}^n)$,
(2) $(T\phi_n)(0) \to (T\phi)(0)$ whenever $\phi_n \to \phi$ in $C_c^\infty(\mathbb{R}^n)$.

Sol. Both statements follow directly from the regularization theorem.

c. Conversely, show that if $T : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is a linear map satisfying (1) and (2), then there exists $u \in \mathcal{D}'(\mathbb{R}^n)$ such that $T\phi = u * \phi$ for all $\phi \in C_c^\infty(\mathbb{R}^n)$.

Sol. Define $u : C_c^\infty(\mathbb{R}^n) \to \mathbb{C}$ by

$$\langle u, \phi\rangle := T\widetilde{\phi}(0) \qquad (\phi \in C_c^\infty(\mathbb{R}^n)).$$

Then u is linear and sequentially continuous by (2). Moreover, for $\phi \in C_c^\infty(\mathbb{R}^n)$, we have, by the regularization theorem

$$(u*\phi)(x) = \langle u(y), \phi(x-y)\rangle = T(\tau_{-x}\phi)(0) = \tau_{-x}(T\phi)(0) = T\phi(x) \qquad (x \in \mathbb{R}^n),$$

by (1). Hence $T\phi = u * \phi$.

Problem 4 (20 pts).

a. Give the definition of the support of a distribution and state the theorem on the characterization of distributions u on $\mathbb{R}^n$ such that $\operatorname{supp} u = \{0\}$.

Sol. The support of a distribution u on an open set $U \subset \mathbb{R}^n$ is the complement in U of the set of all points having an open neighborhood on which $u = 0$. Equivalently, it is also the smallest closed subset F of U such that $u = 0$ on $U \setminus F$.

The theorem states that $\operatorname{supp} u = \{0\}$ if and only if $u = \sum_{|\alpha| \le k} c_\alpha \partial^\alpha \delta$ for some $k \in \mathbb{N} \cup \{0\}$ and some constants $c_\alpha \in \mathbb{C}$.

b. Let $P(\partial) := \sum_{|\alpha| \le k} c_\alpha \partial^\alpha$ be a linear differential operator with constant coefficients. Show that if the polynomial $P(x) = \sum_{|\alpha| \le k} c_\alpha \xi^\alpha$ satisfies $P(\xi) \neq 0$ for all $\xi \neq 0$, then every tempered distribution u on $\mathbb{R}^n$ such that $P(\partial)u = 0$ is a polynomial.

Sol. If u is a tempered distribution with $P(\partial)u = 0$, then by taking the Fourier transform on both sides we get $(2\pi i)^{|\alpha|} P(\xi)\hat{u} = 0$. Since $P(\xi)$ is nonvanishing for $\xi \neq 0$, it easily follows that $\hat{u}$ is supported at the origin, so that

$$\hat{u} = \sum_{|\alpha| \le k} c_\alpha \partial^\alpha \delta$$

for some $k \in \mathbb{N} \cup \{0\}$ and some constants $c_\alpha \in \mathbb{C}$. The result then follows from the inversion theorem and from the fact that $\hat{\delta} = 1$.

Problem 5 (20 pts).

True or False? Give a short justification.

a. The map $f \to \hat{f}$ is an isomorphism of $L^1(\mathbb{T})$ onto c_0.

Sol. False. We proved in class that $f \to \hat{f}$ is a bounded one-to-one linear operator of $L^1(\mathbb{T})$ into (but not onto) c_0.

b. If $f \in C(\mathbb{T})$ and $S_n f(x) := \sum_{k=-n}^{n} \hat{f}(k) e^{ikx}$ for $n \in \mathbb{N}$, then the sequence $(S_n f(x))_n$ converges for every $x \in \mathbb{R}$, but not necessarily to $f(x)$.

Sol. False. For most continuous functions on the circle, the sequence $(S_n f(x))_n$ diverges at a dense set of points.

c. If u is a distribution on $\mathbb{R}^n$ that is homogeneous of degree λ and $\alpha = (1, 2, \ldots, n)$, then $\partial^\alpha u$ is homogeneous of degree $\lambda - n(n+1)/2$.

Sol. True. We proved in class that $\partial^\alpha u$ is homogeneous of degree $\lambda - |\alpha|$. In this case, $|\alpha| = n(n+1)/2$.

d. A distribution u on an open set $U \subset \mathbb{R}^n$ is said to have *finite order* if there exist $C > 0$ and $N \in \mathbb{N} \cup \{0\}$ such that for every compact set $K \subset U$, we have

$$|\langle u, \phi \rangle| \leq C \sum_{|\alpha| \leq N} \sup |\partial^\alpha \phi| \qquad (\phi \in C_c^\infty(U), \operatorname{supp} \phi \subset K).$$

Sol. False. In the definition of *finite order*, the constant C is allowed to depend on the compact set K.

Student ID	HW1 (90)	HW2 (50)	HW3 (80)	HW4 (50)	HW5 (30)	Exam1 (100)	HW7 (100)	HW8 (70)	HW9 (40)	Final Exam (100)	Grade (100)	Grade
47	62	33	66	31	30	57	69	58	34	73	69	B
85	78	42	77	47	26	80	89	51	33	74	81	A-
28	0	0	0	0	13	41	21	31	26	47	34	F
19	33	16	0	33	22	62	49	0	40	67	54	C-
39	38	29	58	38	20	70	0	0	32			I
3	89	42	78	49	30	62	100	65	35	67	77	A-
40	73	39	58	45	30	61	74	49	40	77	73	B+
86	8	27	59	38	27	44	71	27	28	67	56	C-
69	87	49	78	50	30	75	78	61	40	72	81	A-
14	69	37	75	48	30	76	89	60	38	70	79	A-
42	76	31	63	45	27	56	78	60	25	54	65	B-
95	76	48	78	45	30	48	95	67	40	58	69	B
34	87	45	80	45	28	69	97	67	40	94	87	A
82	85	42	77	49	30	80	90	62	40	85	87	A

MAT 544, Real Analysis I

fall 2014

Christopher Bishop

Professor, Mathematics
SUNY Stony Brook

Office: 4-112 Mathematics Building
Phone: (631)-632-8274
Dept. Phone: (631)-632-8290
FAX: (631)-632-7631

Time and place: TuTh 1:00-2:20, Physics P-129

We will use the text `Real Analysis' by by Gerald Folland, secondedition, published by Wiley. I hope to cover Chapters 1-3, and parts of 4 and 5. Chapter 0 is prerequisite material but I may discuss it briefly if needed.

My office hours will be Tu-Th 11:10-12:00 in my office, 4-112 in the Math Building.

This is an introductory course on measure theory, with a bit of point set topology and functional analysis thrown in.

Homework problems will be handed in at class each Tuesday.

The following is a tentative lecture and homework schedule .

problem set for Chapter 0.

Although it is not required, you may wish to consider writing up your solution in TeX, since eventually you will probably use this to write your thesis and papers. Here are a sample LaTex file and what the resulting output looks like . You can use the first file as a template to create your own TeX files. Numerous guides exist online that give the basic rules and commands.

The first lecture is Tuesday, Feb 1. The last class meeting is Thursday May 12. There is no class during spring recess: April 18-24 There will be a midterm exam on Thursday March 31 and a final at 11:15am-1:45pm on Thursday, May 19 (in Physics P-124, the usual room). Homework, midterm and final will count for 30%, 30% and 40% of your grade respectively.

The final will be in our usual room.

Here are the midgterms and finals for a 2-semester course from Rudin's 'Principles of Mathematical Analysis'. These should give you an idea of what would be good to know entering this course: midterm 1 , final 1 , midterm 2 , final 2 ,

Disability Support Services (DSS) Statement:

If you have a physical, psychological, medical or learning disability that may impact your course work, please contact Disability Support Services, ECC (Educational Communications Center) Building, room 128, (631) 632-6748. They will determine with you what accommodations, if any, are necessary and appropriate. All information and documentation is confidential. Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to the following website: http://www.stonybrook.edu/ehs/fire/disabilities]

Academic Integrity Statement:

Each student must pursue his or her academic goals honestly and be personally accountable for all submitted work. Representing another person's work as your own is always wrong. Faculty are required to report any suspected instances of academic dishonesty to the Academic Judiciary. Faculty in the Health Sciences Center (School of Health Technology & Management, Nursing, Social Welfare, Dental Medicine) and School of Medicine are required to follow their school-specific procedures. For more comprehensive information on academic integrity, including categories of academic dishonesty, please refer to the academic judiciary website at http://www.stonybrook.edu/commcms/academic_integrity/index.html

Critical Incident Management Statement:

Stony Brook University expects students to respect the rights, privileges, and property of other people. Faculty are required to report to the Office of Judicial Affairs any disruptive behavior that interrupts their ability to teach, compromises the safety of the learning environment, or inhibits students' ability to learn. Faculty in the HSC Schools and the School of Medicine are required to follow their school-specific procedures.

Send me email at: bishop at math.sunysb.edu

Link to history of mathematics