MAT 515: Geometry for Teachers, Fall 2012

- **Instructor:** Nikita Selinger, office 4-115 Math,
- **Office hours:** Tuesdays 4pm-5pm in the Math Learning Center, Tuesdays 5pm-6pm and Wednesdays 2pm-3pm in my office, or by appointment.
- **Class meetings:** MoWe, 5:30-6:50pm, Math 4-130.

Homework is a compulsory part of the course. Homework assignments are due each week at the beginning of the Wednesday's class. Under no circumstances will late homework be accepted.

Grading system: The final grade is the weighted average according the following weights: homework 10%, in-class tests 20%, Midterm 30%, Final 40%.

Textbook: *Kiselev's Geometry (Book I, Planimetry) Sumizdat, El Cerrito, Calif., 2006.* You can find the first 33 pages of the textbook here.

CORRECTION: The final is still in P-131.

CORRECTION: The final is in our usual classroom 4-130.

Sample solutions of HW10.

Here is the list of topics to review for the final exam.

Sample solutions of HW9.

We are going to have additional office hours on Saturday, December 8, 3pm.

Here are sample solutions for the make-up midterm. The final is at 5.30pm, December 12 in room P-131. That is the same room in which we had the make-up midterm.

We are going to have additional office hours on Sunday, November 18 at 10.00am.

Here are sample solutions for the midterm. The make up midterm is on November 28. We plan to move the final to 5.30pm, December 12. Let me know ASAP if you have a conflict.

We will have a **midterm exam** on Wednesday, October 31. You can also look at an exam from previous years, but please don't be mislead into thinking that there will be some "similar" problems, problems of particular "types", etc.

Week 11 (11/26– 11/30). Read the following lecture notes on isometries by Olga Plamenevskaya and by Oleg Viro. **Homework 10, due Dec 5, before class.** Solve the following problems .

Week 10 (11/12 – 10/17). Read pages 117-126 of the textbook and the following lecture notes on isometries by Olga Plamenevskaya and by Oleg Viro. **Homework 9, due Nov 26, before class.** Solve the following excercises from the textbook: 332, 334, 336, 342, 343, 344.

Week 9 (10/22 – 10/26). Prepare for the midterm. Here is the list of topics to review. Here are sample solutions of for HW problems 261, 274, 261, and Quiz 5.

Week 8 (10/15 – 10/19). Rewiev the material of the previous week and start preparing for the midterm. **Homework 8, due Oct 24, before class.** Solve the following excercises from the textbook: 276, 277, 295, 300, 303, 304.

Week 7 (10/08 – 10/12). Read pages 89-108 of the textbook. **Homework 7, due Oct 17, before class.** Solve the following excercises from the textbook: 261, 266, 267, 274, 275, 302.

Week 6 (10/01 – 10/05). Read pages 78-89 of the textbook. **Homework 6, due Oct 10, before class.** Solve the following excercises from the textbook: 217, 220, 222, 224, 231, 236.

Week 5 (09/24 – 09/28). Read pages 55-78 of the textbook. **Homework 5, due Oct 3, before class.** Solve the following excercises from the textbook: 146, 150, 153, 157, 163, 171, 172, 198, 205, 206.

Week 4 (09/17 – 09/21). Read pages 45-55 of the textbook. **Homework 4, due Sep 26, before class.** Solve the following excercises from the textbook: 109, 114, 117, 119, 121, 133, 136, 138.

Week 3 (09/10 – 09/14). Read pages 30-44 of the textbook. **Homework 3, due Sep 19** Solve the following excercises from the textbook: 92, 95, 99, 102, 103.

Week 2 (09/03 – 09/07). Read pages 22-30 of the textbook. **Homework 2, due Sep 12** Solve the following excercises from the textbook: 53, 54, 58, 67, 72.

Week 1 (08/27 – 08/31). Read the book up to page 22. Most of the material we have covered in class, some of it we *have not*. The reading assignments are important because they teach you how to follow the proof carefully, and how to build your own proofs and to use correct notation. Recall the following properites of our plane that we agreed upon:
(i) One can superimpose a plane on itself or any other plane in a way that takes one given point to any other given point.
(ii) One can superimpose a plane on itself or any other plane in a way that takes one given ray to any other given ray.
(iii) A plane can be superimposed on itself keeping all the points of a given straight line fixed. This "flip" can be done in a unique way.
Homework 1, due Sep 5 Solve the following excercises from the textbook: 28, 33, 36, 39, 50.

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Academic integrity statement: Each student must pursue his or her academic goals honestly and be personally accountable for all submitted work. Representing another person's work as your own is always wrong. Faculty are required to report any suspected instance of academic dishonesty to the Academic Judiciary. For more comprehensive information on academic integrity, including categories of academic dishonesty, please refer to the academic judiciary website at http://www.stonybrook.edu/uaa/academicjudiciary/.

Critical incident management: Stony Brook University expects students to respect the rights, privileges, and property of other people. Faculty are required to report to the Office of Judicial Affairs any disruptive behavior that interrupts their ability to teach, compromises the safety of the learning environment, and/or inhibits students' ability to learn.

Nikita Selinger

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I'm a postdoc at the Institute for Mathematical Sciences, Stony Brook University

Office hours Fall 2013: MoWe 4.00-5.30pm.

Teaching:

MAT 200 Language, Logic and Proof

Past Teaching

Research:

My research area is complex dynamics. One of the most appealing aspects of this field is that it has deep connections to many areas of mathematics, such as combinatorics, number theory, group theory (in particular, the recently developed theory of self-similar groups), and ergodic theory, to name a few. My own research lies in the intersection of complex analysis, hyperbolic and differential geometry, and Teichmüller theory.

PDF's:

My papers on arxiv.org

My PhD thesis

Slides for my talk at ICERM

Embedded Secure Document

The file *http://www.sumizdat.org/pl_1_33.pdf* is a secure document that has been embedded in this document. Double click the pushpin to view.

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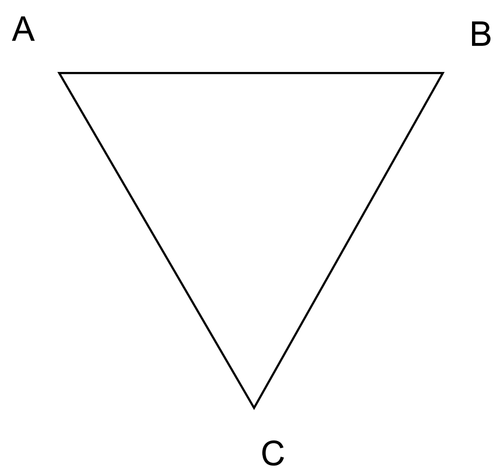
Fall 2012

Sample Solutions

1. Find the composition of two rotations: the rotation in the counterclockwise direction about a point A by angle $120°$ followed by the rotation in the counterclockwise direction about a point B by angle $120°$: prove that this is a rotation, find the center and angle of this rotation.

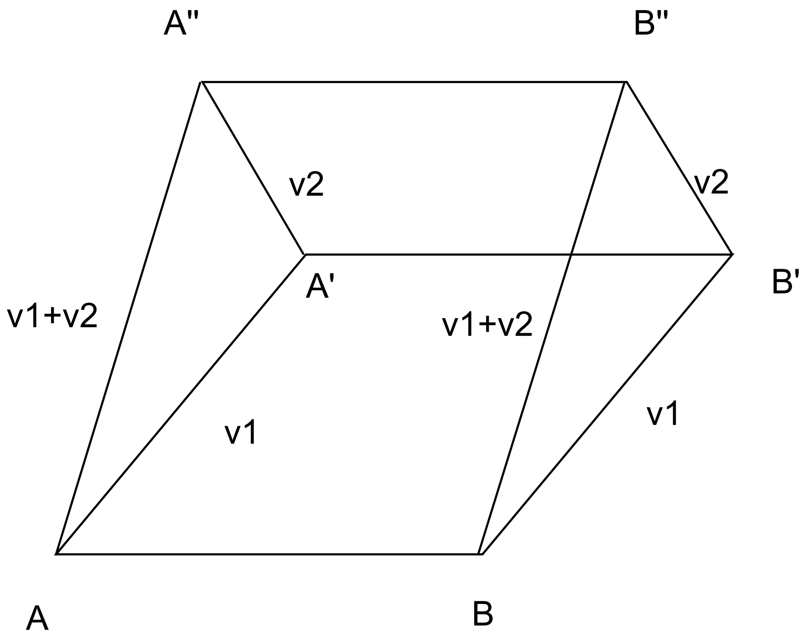
Solution



Consider an equilateral triangle $\triangle ABC$ such that A, B and C are oriented clockwise (see the picture). Then, by Theorem 8 from the lecture notes, $Rot_{A,120^\circ} = R_{AB} \circ R_{AC}$ and $Rot_{B,120^\circ} = R_{BC} \circ R_{AB}$. Thus the composition is $Rot_{B,120^\circ} \circ Rot_{A,120^\circ} = (R_{BC} \circ R_{AB}) \circ (R_{AB} \circ R_{AC}) = R_{BC} \circ (R_{AB} \circ R_{AB}) \circ R_{AC} = R_{BC} \circ R_{AC} = Rot_{C,-120^\circ}$. We see that the composition is the rotation around C in the clockwise direction by angle $120°$.

2. Prove that a composition of two translations is a translation. What is the vector of translation for the composition?

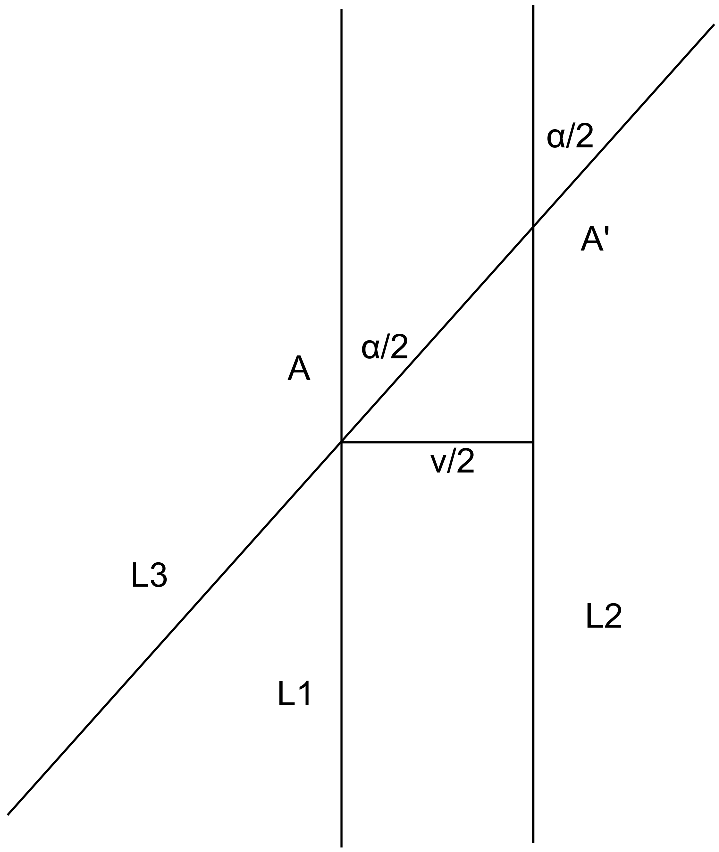
Solution



Denote the two translations by T_{v_1} and T_{v_2}. Pick any two distinct points A and B. Let $A' = T_{v_1}(A), B' = T_{v_1}(B)$ and $A'' = T_{v_2}(A'), B'' = T_{v_2}(B')$. By definition of translations, $ABB'A'$ and $A'B'B''A''$ are parallelograms. Therefore $AB = A'B' = A''B''$ and all these segments are parallel. Thus $ABB''A''$ is also a parallelogram and $T_{v_2} \circ T_{v_1}$ is a translation. The translation vector is the sum $v_1 + v_2$.

4. Prove that a composition of a rotation and a translation is a rotation. (This is true for any order of the rotation/translation; you can assume that the rotation is performed first.)

Solution



Let T_v and $Rot_{A,\alpha}$ be given isometries. We can write the translation $T_v = R_{l_2} \circ R_{l_1}$ as a composition of reflections with respect to two lines l_1 and l_2 that are perpendicular to v and at distance $|v|/2$ apart. We may assume that l_1 passes through the origin A of the rotation $Rot_{A,\alpha}$. We can also write $Rot_{A,\alpha} = R_{l_1} \circ R_{l_3}$ as a composition of reflections with respect to two lines l_3 and l_1 which intersect at A and form an angle $\alpha/2$. Then the composition is $T_v \circ Rot_{A,\alpha} = (R_{l_2} \circ R_{l_1}) \circ (R_{l_1} \circ R_{l_3}) = R_{l_2} \circ R_{l_3}$. We know that the latter is the rotation with angle α about the intersection point A' of l_2 and l_3.

MAT515. Final topics

Here is the list of topics recommended for review.

1. Theorem about vertical angles, section 26.
2. Existence and uniqueness of perpendicular to a line from a point, sections 24, 65 and 66.
3. Theorems about isosceles triangles and their properties, sections 35, 36.
4. Congruence tests for triangles section 40.
5. Inequality between exterior and interior angles in a triangle, sections 41 - 43.
6. Relations between sides and opposite angles sections 44 - 45.
7. Triangle inequality and its corollaries, sections 48 and 49.
8. Perpendicular and slants, sections 51 - 53.
9. Segment and angle bisectors, sections 56 and 57.
10. Basic construction problems, sections 61 - 69.
11. Tests for parallel lines, section section 73.
12. The parallel postulate, sections 75 and 76.
13. Angles formed by parallel lines and a transversal, sections 77 and 78.
14. Angles with respectively parallel sides, section 78.
15. Angles with respectively perpendicular sides, section 79.
16. The sum of interior angles in a triangle, section 81.
17. The sum of interior angles in a convex polygon, section 82.
18. Properties of a parallelogram, sections 85 - 87.
19. Special types of parallelograms and their properties, sections 90 - 92.
20. The midline theorem, sections 93, 94, 95.
21. The midline of trapezoid, sections 96, 97.
22. Existence and uniqueness of a circle passing through three points, sections 103, 104.
23. Constructions that use isometries, sections 98 - 101
24. Theorems about inscribed angles, section 123.
25. Corollaries of the theorem about inscribed angles, sections 125,126.
26. Constructions using theorems about inscribed angles, sections 127 - 130, 133.
27. Inscribed and circumscribed circles, sections 136, 137.
28. Concurrency points in a triangle, sections 140 - 142.
29. Mensurability, sections 143-153.
30. Lectures on isometries by Olga Plamenevskaya.

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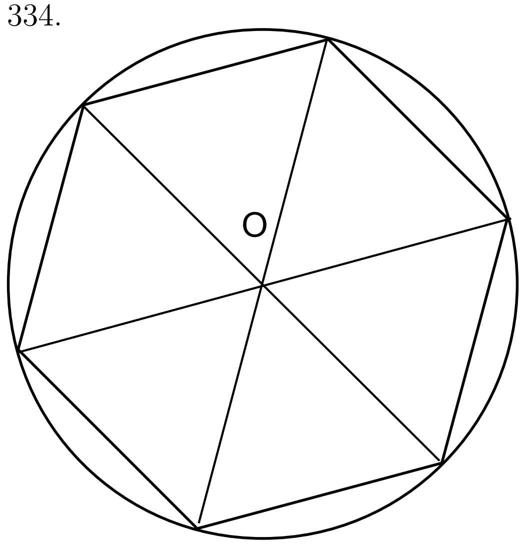
Sample Solutions

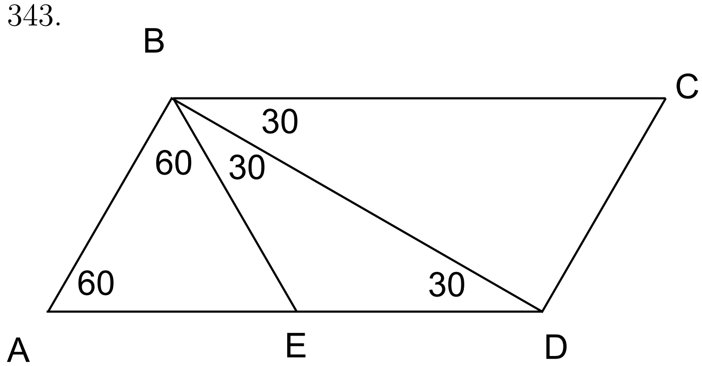
332. If one side, say a, is a common measure of the other two sides, then they can be written as na and ma for some positive integers m, n. If $n < m$, then $n + 1 \leq m$ and $a + na = (n + 1)a \leq ma$ which contradicts the triangle inequality. Similarly $n > m$ is impossible as well. We conclude that $m = n$ and $ma = na$ so the triangle is isosceles.

334.



Connect the vertices of the hexagon to the center O of the circumscribed circle. We get 6 triangles which are all isosceles with lateral sides congruent to the radius of the circle. Since all bases are also congruent by assumption, all triangles are congruent by SSS-test. In particular, the 6 angles at vertex O are all congruent and, hence, equal to $360°/6 = 60°$. Then the lateral angles in each triangle are congruent to $(180° - 60°)/2 = 60°$. We see that all sides of the hexagon are congruent to the radius of the circle and the perimeter of the hexagon is 3 time larger then the diameter of the circle.

343.



By assumptions $\angle BAD = 60°$. Then $\angle ABC = 180° - \angle BAD = 120°$ and $\angle ABD = 3/4\angle ABC = 90°$. Construct a line BE with E on AD such that $\angle ABE = 60°$. Then $\angle EBD = \angle ABD - \angle ABE = 30°$ and $\angle ADB = 180° - \angle DAB - \angle ABD = 30°$. We see that $\triangle ABE$ is equilateral and $\triangle BED$ is isosceles and $AB = AE = BE = ED$. Thus $AD = AE + ED = 2AB$.

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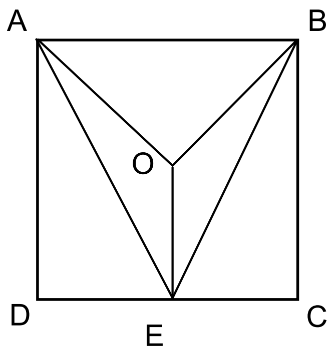
Makeup Midterm Solutions

1. Prove that in a triangle, the angle opposite to a greater side is greater than the angle opposite to a smaller side.

Solution. See Section 44 in the textbook.

2. Let O be a point in the interior of a square $ABCD$. Prove that $AO + BO \leq 3AB$.



Let E be a point on CD such that O is in the interior of $\triangle AEB$. For example, we can take E such that $OE||BC$. Then we can prove that $AO + BO < AE + BE$ (see the solution of Question 2 from the previous midterm). Applying the triangle inequality to $\triangle BCE$ and $\triangle ADE$ we get $AE < AD + DE$ and $BE < BC + CE$. We conclude

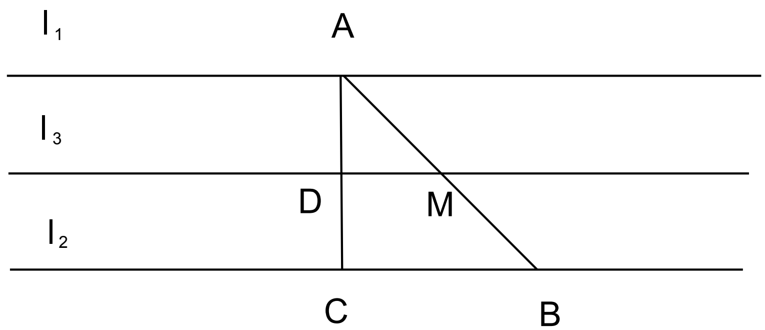
$$AO + BO < AE + BE < AD + BC + DE + CE = AD + BC + CD = 3AB,$$

as all sides of the square are congruent.

Remark. Please note that while we add inequalities in the way $a < b$ and $c < d$ implies $a + c < b + d$, you can not subtract them the same way! These two inequalities imply $a - d < b - c$ and **do not imply** $a - c < a - d$.

3. Given two parallel lines l_1 and l_2, find a geometric locus of midpoints of all segments AB such that A is on the line l_1 and B is on the line l_2.

Solution. The geometric locus is the line l_3 that is parallel and equidistant to l_1 and l_2.

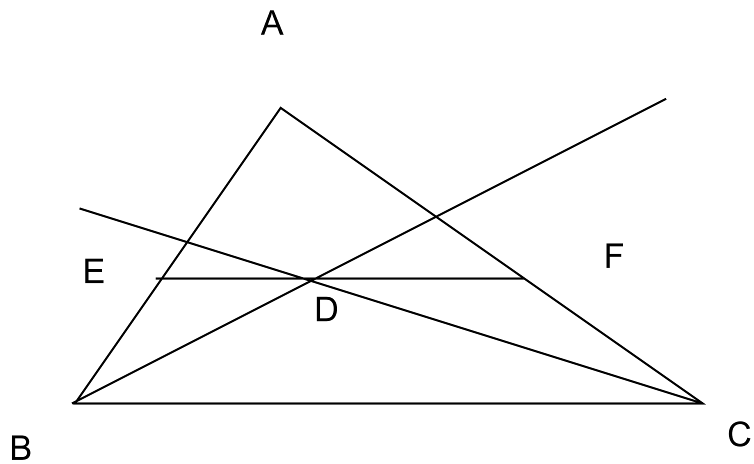


I. Suppose M is a midpoint of some AB with $A \in l_1$ and $B \in l_2$. Drop a perpendicular AC from a to l_2. Then l_3, by its definition, passes through the middle D of AC and is parallel to $BC = l_2$. Therefore l_3 intersects AB in the midpoint M and thus M is on l_3.

II. On the other hand, every point point M on l_3 is the midpoint of the segment with endpoints on l_1 and l_2 which is perpendicular to both lines and passes through M.

4. In a triangle, draw a line parallel to its base and such that the line segment contained between the lateral sides is congruent to the sum of the segments cut out on the lateral sides and adjacent to the base.

Solution.



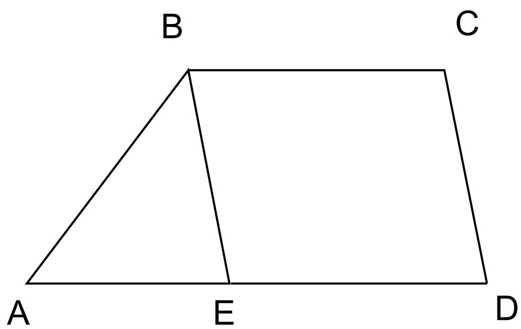
I. **Construction**. Construct angle bisectors of $\angle B$ and $\angle C$. Denote by D their intersection point and construct the line that is parallel to BC and passes through D.

II. **Justification**. Denote by E and F the intersection points of the constructed line with sides AB and AC respectively. By construction, $EF||BC$. Angles $\angle CBD$ and $\angle BDE$ are alternating, hence congruent. Since BD is the bisector of $\angle B$ we have $\angle EBD = \angle CBD = \angle BDE$. Therefore $\triangle EBD$ is isosceles: $EB = ED$. Similar reasoning yields $DF = FC$. We conclude $EF = ED + DF = BE + FC$ as required.

III. **Research.** Since two bisectors always intersect in a unique point in the interior of the triangle, for any triangle there is a unique solution.

5. Construct a trapezoid given all of its sides.

Solution. Let a, b, c, d be the given sides, where a, c are the two bases and b, d are lateral sides. Without loss of generality assume a is longer than c.



I. **Construction**.

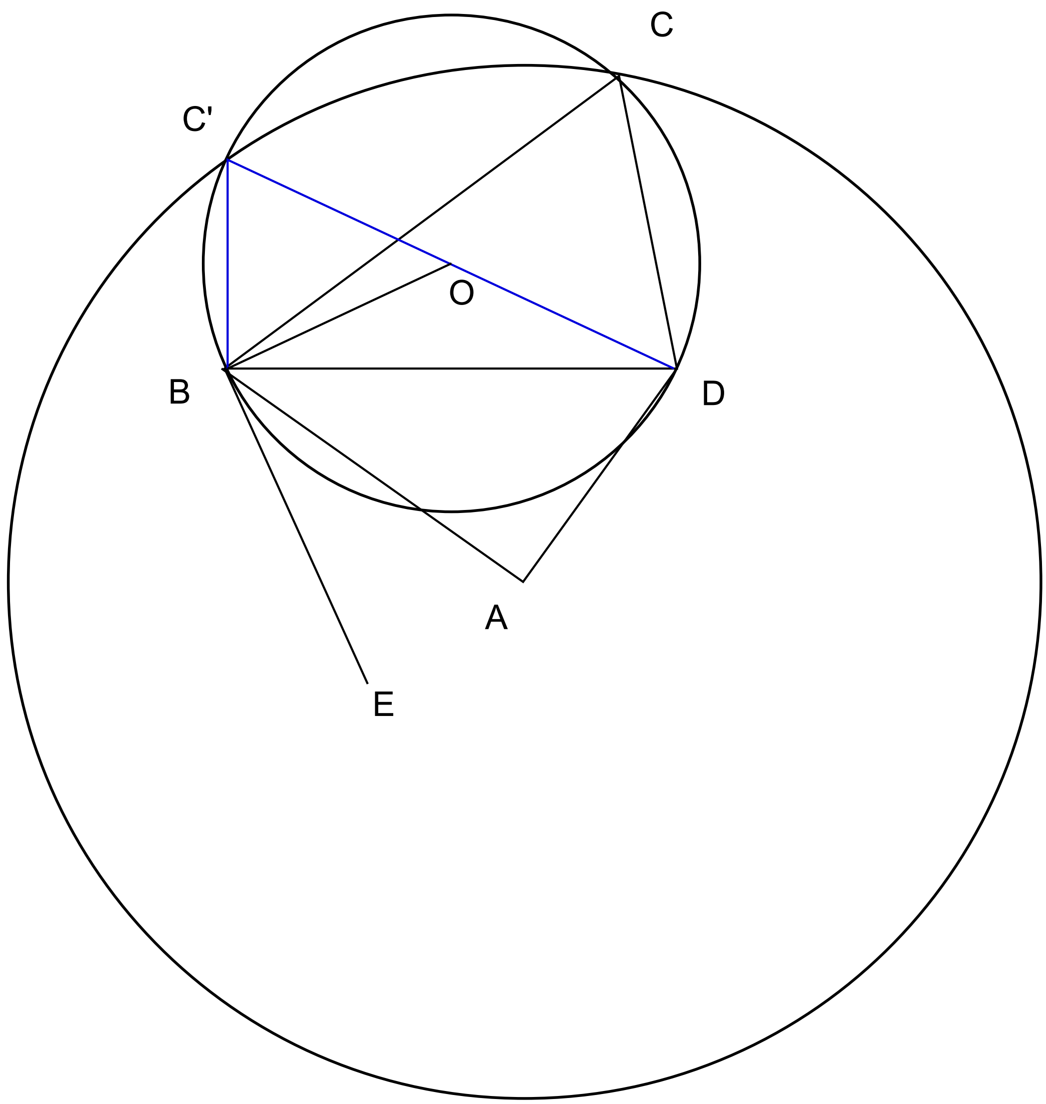
1. Construct a triangle ABE with $AB = b$, $BE = d$ and $AE = a - c$. 2. Extend AE past E and mark a point D on AE such that $ED = c$. 3. Construct a line parallel to AD through B and a line parallel to BE trough D. Denote the intersection of the two lines by C. 4. $ABCD$ is the required trapezoid.

II. **Justification**. By construction $BCED$ is a parallelogram. Therefore $CD = BE = d$ and $BC = ED = c$. Also $AD = AE + ED = (a - c) + c = a$ and $AB = b$. Since BC is parallel to AD, $ABCD$ is a trapezoid.

III. **Research.** The exists a unique solution if and only if b, d and $a - c$ form a triangle, that is satisfy the triangle inequalities ($b+d > a-c$, $d+a-c > b$ and $b + a - c > d$).

6. Construct a quadrilateral, given its diagonals, two adjacent sides, and the angle between the two remaining sides.

Solution. We want to construct a quadrilateral $ABCD$ such that $AB = a, AD = b, AC = c, BD = d$ and $\angle C = \alpha$.



I. **Construction**.

1. Construct a triangle ABD with $AB = a$, $AD = b$ and $BD = d$. 2. Construct an angle DBE congruent to α in the same halfplane as A with

respect to BD. 3. Construct a line perpendicular to BE and passing through B and denote by O the point where it intersects the median bisector of BD. 4. Draw a circle C_1 about O with radius BO and a circle C_2 about A with radius c. 4. Denote by C an intersection point of the two circles. $ABCD$ satisfies the conditions of the problem.

II. **Justification**. By construction $AB = a$, $AD = b$, $AC = c$ and $BD = d$. Since O is on the median bisector of BD, $BO = BD$ and D lies on C_1. Then $\angle BCD$ is inscribed in C_1 and subtends the same arc as the angle $\angle EBD$ between a chord and a tangent line. Therefore $\angle BCD = \angle BED = \alpha$.

III. **Research.** Any intersection point of circles C_1 and C_2 will work. The problem can have 0,1,2 or infinitely many different solutions depending on whether these circles are disjoint, tangent, intersecting or coincide.

MAT 515, Geometry for Teachers
Fall 2012

Midterm

Name:____________________________________

This is a closed book, closed notes test. No consultations with others. Calculators are not allowed.

Please turn off and take off the desk cell phones, pagers, etc. Only the exam and pens/pencils should be on your desk.

Please explain all your answers, show all work, and give careful proofs. Answers without explanation will receive little credit.

The problems are not in the order of difficulty. You may want to look through the exam and do the easier questions first.

DO NOT TURN THIS PAGE UNTIL INSTRUCTED TO DO SO

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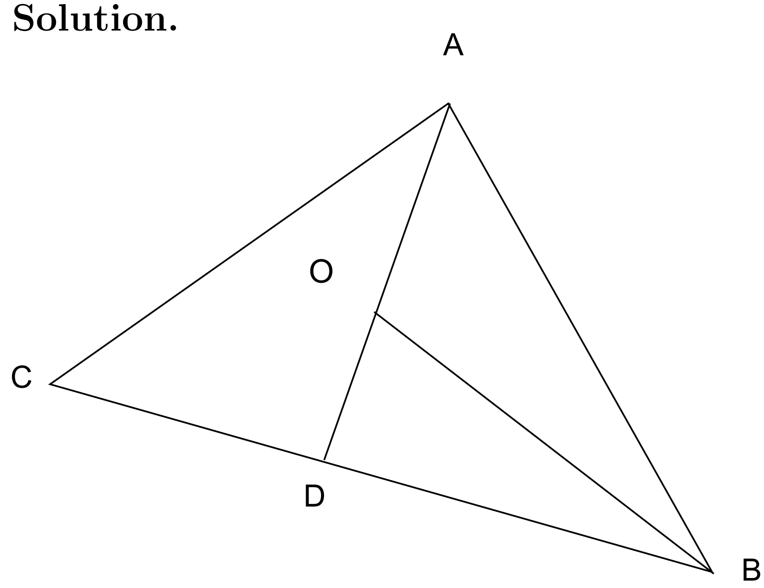
1	2	3	4	5	Total

1. Formulate and prove the midline theorem.
Solution. See section 95 of the book.

2. Let O be a point in the interior of a triangle ABC. Prove that $AO + BO \leq AC + BC$.

Solution.



Extend AO beyond O and mark the intersection point D of AO and BC. Using the triangle inequality in $\triangle BOD$ and $\triangle ACD$ we get

$$BO \leq DO + BD$$

and

$$AD \leq AC + CD.$$

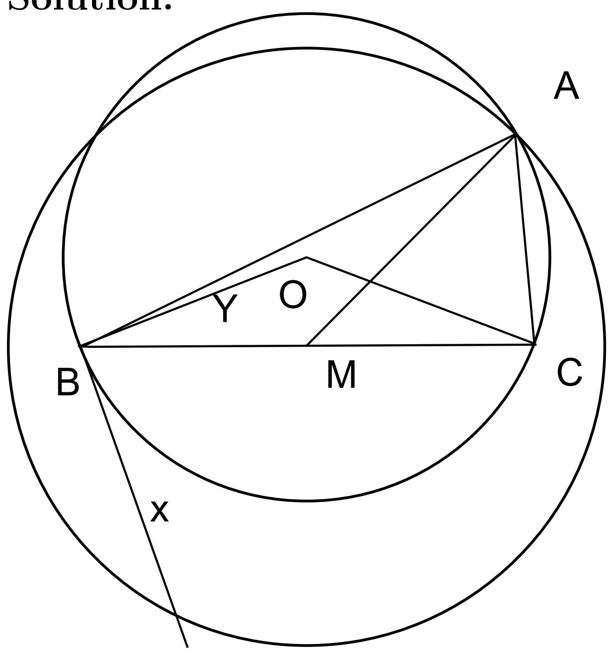
Noticing that $AD = AO + DO$ and $BC = BD + CD$ we take the sum of the previous inequalities:

$$AO+BO+DO = BO+AD \leq DO+BD+AC+CD = DO+BC+AC.$$

Subtracting DO from both sides we get the desired result.

3. Construct a triangle ABC, given an angle congruent to its interior angle at vertex A, a segment congruent to the median AM, and a segment congruent to the side BC. (You may use, without a detailed description, the following elementary constructions: segment and angle bisection, raising a perpendicular at a point on the line, dropping a perpendicular from a point not on the line, constructing segments and angles congruent to given ones.)

Solution.



I. Construction.

1. Construct a segment BC congruent to a given one.

2. Construct a ray BX such that $\angle XBC$ is congruent to the given angle.

3. Construct a line BY which is perpendicular to BX.

4. Construct the median bisector of BC and find its intersection M with BC (the middle of BC) and O with BY.

5. Construct circle C_1 using M as a center and the given

length of the median as a radius.

6. Construct circle C_2 using O as a center and OB as a radius.

7. Mark A one of the intersection points of the two circles. Connect A to B and C to get the triangle $\triangle ABC$.

II. Proof.

By construction BC and AM are congruent to the given segments. Since BO is a radius of C_2 and $BO \perp BX$ by construction, the line BX is tangent to C_2 and $\angle XBC$ is an angle between a tangent and a chord, so it is congruent to the angle $\angle A$ which is inscribed in C_2 and subtends the same arc. Therefore, $\angle A = \angle XBC$ and the latter was constructed congruent to the given angle.

III. Research. When C_1 and C_2 intersect in two points, the two intersection points are symmetric with respect to the median bisector of BC and will produce congruent triangles, so in this case the solution is unique. When C_1 and C_2 are tangent the solution is also unique. If C_1 and C_2 do not intersect there are no solutions. When $\angle A = 90°$ and $AM = 1/2BC$, the two circles coincide and we get infinitely many solutions: in this case any right triangle with hypotenuse BC is a solution.

4. Find a geometric locus of midpoints of all segments AB such that A and B belong to a given line l.

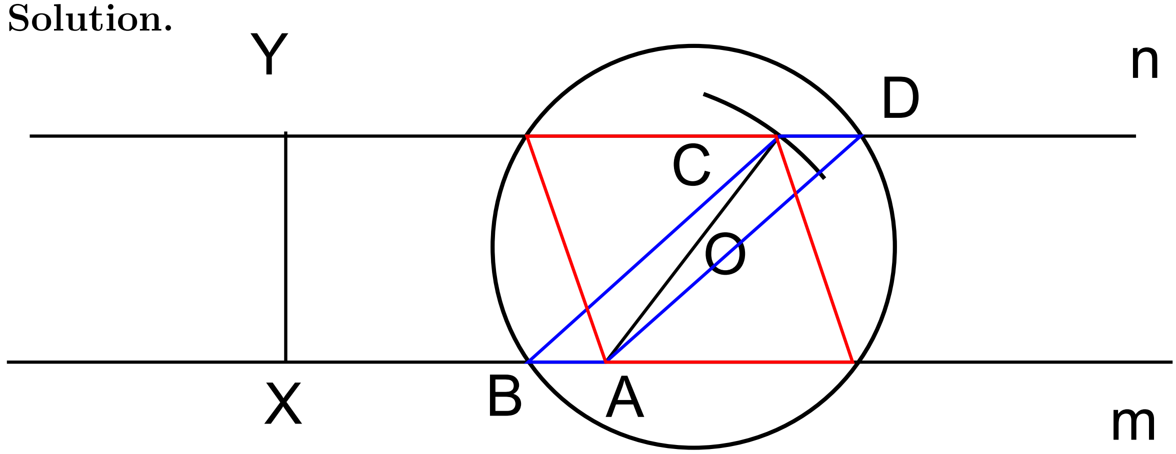
Solution. The geometric locus is the line l itself.

1. If M is a midpoint of a segment AB with A and B on l, then, in particular, M is on the line $AB = l$. Therefore the geometric locus is a subset of l.

2. If M is on l, take any point A on l different from M and construct a segment BM congruent to AM on the other side of line l (in other words reflect A with respect to M). Then by construction M is the middle of AB and both A and B are on l. We conclude that every point on l is in the geometric locus.

5. Construct a parallelogram, given its diagonals and an altitude.

Solution.



I. Construction.

1. Take an arbitrary line m and a point X on m.

2. Construct a perpendicular to m at X and mark a point Y on that perpendicular so that XY is congruent to the given altitude.

3. Construct a line n perpendicular to the line XY and going trough Y.

4. Take a point A on m and swing an arc of radius congruent to the first given diagonal. Mark the intersection point C of that arc with the line n.

5. Construct the midpoint O of the segment AC.

6. Bisect the second given diagonal. Construct a circle about O using half of this diagonal as a radius.

7. Mark one of the intersection points of the circle with m as B. Mark as D the point that is diametrically opposite to B on the circle. Connect $ABCD$ to get the desired parallelogram.

II. Proof.

If CH is an altitude of $ABCD$ then H must lie on line AB which is the same as m. Hence, $CHXY$ is a rectangle yielding

$CH = XY$. Thus, the altitude CH of $ABCD$ is congruent to the given segment. The diagonals AC and BD are congruent to the given segments by construction. Since the diagonals of $ABCD$ bisect each other, $ABCD$ is indeed a parallelogram.

III. Research.

If either of the diagonals is shorter than the altitude, there are clearly no solutions because in a right triangle a hypotenuse is always longer than a leg. If the constructed circle intersects the line m in two points that are both distinct from A, we get two different solutions (in the picture one is red and the other one is blue). When the two diagonals are congruent, one of the intersection points will coincide with A and the solution will be a unique rectangle. The circle will be tangent to m if and only if the second diagonal is congruent to the altitude; in that case we also get a unique solution.

MAT 360, Geometry

Spring 2011

Midterm I

Name:________________________________

This is a closed book, closed notes test. No consultations with others. Calculators are not allowed.

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1. Consider an isosceles triangle ABC (where AC is the base), and let CD be the bisector of the angle C. Suppose that the angle ADC is 150 degrees. Find angles of the triangle ABC.

2. Four houses A, B, C, D form vertices of a square. The residents would like to dig a well at a point W such that the sum of distances $AW + BW + CW + DW$ from all the houses to the well is the smallest possible. Where should they dig the well?

3. Using a straightedge and a compass, construct a rhombus whose diagonals are congruent to the two given segments. Justify your construction.

(You can use, without a detailed description, the following elementary constructions: segment and angle bisection, raising a perpendicular at a point on the line, dropping a perpendicular from a point not on the line, constructing segments and angles congruent to given ones.)

4. Let the chords AB and CD of the given circle be congruent. Extend the chord AB past the point B, the chord CD past C, and suppose that the resulting two rays intersect at the point P. Prove that the triangles APD and BPC are isosceles.

Isometries.

Congruence mappings as isometries. The notion of *isometry* is a general notion commonly accepted in mathematics. It means a mapping which preserves distances. The word *metric* is a synonym to the word *distance*. We will study isometries of the plane. In fact, we have already encountered them, when we superimposed a plane onto itself in various ways (eg by reflections or rotations) to prove congruence of triangles and such. We now show that each isometry is a "congruence mapping" like that.

Theorem 1. *An isometry maps*
(i) straight lines to straight lines;
(ii) segments to congruent segments;
(iii) triangles to congruent triangles;
(iv) angles to congruent angles.

Proof. Let's show that an isometry S maps a segment AB to segment $S(A)S(B)$ which is congruent to AB. It is clear (from the definition of isometry) that the distance between $S(A)$ and $S(B)$ is the same as the distance between A and B. However, we need to check that the image of AB will indeed be a straight line segment. To do so, pick an arbitrary point X on AB. Then $S(A)S(B) = AB = AX + XB = S(A)S(X)+S(X)S(B)$, and by the triangle inequality the point $S(X)$ must be on the segment $S(A)S(B)$ (otherwise we would have $S(A)S(X) + S(X)S(B) > S(A)S(B)$. So image of the segment AB lies in the segment $S(A)S(B)$, and indeed, covers the whole of $S(A)S(B)$ without leaving any holes: if X' is a point on $S(A)S(B)$, find X on AB such that $XA = X'S(A)$, $XB = X'S(B)$, then $S(X) = X'$. □

Examples of isometries. We have encountered quite a few examples before: reflections, rotations, and translations are all isometries. (It is pretty easy to see that the distances are preserved in each case: for instance, a reflection R_l through the line l maps any segment AB to a symmetric, and thus congruent, segment $A'B'$.) Let's look at some examples more closely.

Translations and central symmetries. A map of the plane to itself is called a *translation* if, for some fixed points A and B, it maps a point X to a point $T(X)$ such that $ABT(X)X$ is a parallelogram. (Note the order of points!)

Here we have to be careful with the notion of parallelogram, because a parallelogram may degenerate to a figure in a line. Not any degenerate quadrilateral fitting in a line deserves to be called a parallelogram, although any two sides of such a degenerate quadrilateral are parallel. By a parallelogram we mean a sequence of four segments KL, LM, MN and MK such that KL is congruent and parallel to MN and LM is congruent and parallel to MK. This definition describes the usual parallelograms, for which congruence can be deduced from parallelness and vice versa, and the degenerate parallelograms.

Theorem 2. *For any points A and B there exists a translation mapping A to B. A translation is an isometry.*

Proof. Any three points A, B and X can be completed in a unique way to a parallelogram $ABX'X$. Define $T(X) = X'$. For any points X, Y the quadrilateral $XYT(Y)T(X)$ is a parallelogram, since $XT(X)||AB||YT(Y)$. Therefore, $XY = T(X)T(Y)$, so T is an isometry. □

Denote by T_{AB} the translation which maps A to B.

Theorem 3. *The composition of any two translations is a translation.*

Proof. Exercise. □

Theorem *3* means that $T_{BC} \circ T_{AB} = T_{AC}$.

Fix a point O. A map of the plane to itself which maps a point A to a point B such that O is a midpoint of the segment AB is called the *symmetry about a point* O.

Theorem 4. *A symmetry about a point is an isometry.*

Proof. SAS-test for congruent triangles (extended appropriately to degenerate triangles.) □

Theorem 5. *The composition of any two symmetries in a point is a translation. More precisely, $S_B \circ S_A = T_{2\overrightarrow{AB}}$, where S_X denotes the symmetry about point X.*

Proof. Exercise. □

Remark. The equality

$$S_B \circ S_A = T_{2\overrightarrow{AB}}$$

implies a couple of other useful equalities. Namely, compose both sides of this equality with S_B from the left:

$$S_B \circ S_B \circ S_A = S_B \circ T_{2\overrightarrow{AB}}$$

Since $S_B \circ S_B$ is the identity, it can be rewritten as

$$S_A = S_B \circ T_{2\overrightarrow{AB}}.$$

Similarly, but multiplying by S_A from the right, we get

$$S_B = T_{2\overrightarrow{AB}} \circ S_A.$$

Corollary. *The composition of an even number of symmetries in points is a translation; the composition of an odd number of symmetries in points is a symmetry in a point.*

Remark. In general, it is clear that a composition of isometries is an isometry: if each mapping keeps distances the same, their composition also will. It is trickier, however, to see the resulting isometry explicitly; we will prove a few theorems related to compositions of isometries. To practice with compositions, consider, for example, a reflection about a line l and a rotation by $90°$ counterclockwise about a point

$O \in l$. When composed in different order (rotation followed by reflection vs reflection followed by rotation), these yield reflections about **different** lines. The proof that the composition is a reflection can be obtained by an explicit examination of which points go where; by Theorem *6*, it suffice to examine 3 non-collinear points.

Recovering an isometry from the image of three points.

Theorem 6. *An isometry of the plane can be recovered from its restriction to any triple of non-collinear points.*

Proof. Given images A', B' and C' of non-collinear points A, B, C under and isometry, let us find the image of an arbitrary point X. Using a compass, draw circles c_A and c_B centered at A' and B' of radii congruent to AX and BX, respectively. They intersect in at least one point, because segments AB and $A'B'$ are congruent and the circles centered at A and B with the same radii intersect at X. There may be two intersection point. The image of X must be one of them. In order to choose the right one, measure the distance between C and S and choose the intersection point X' of the circles c_A and c_B such that $C'X'$ is congruent to CX. □

In fact, there are exactly two isometries with the same restriction to a pair of distinct points. They can be obtained from each other by composing with the reflection about the line connecting these points.

Isometries as compositions of reflections.

Theorem 7. *Any isometry of the plane is a composition of at most three reflections.*

Proof. Choose three non-collinear points A, B, C. By theorem *6*, it would suffice to find a composition of at most three reflections which maps A, B and C to their images under a given isometry S.

First, find a reflection R_1 which maps A to $S(A)$. The axis of such a reflection is a perpendicular bisector of the segment $AS(A)$. It is uniquely defined, unless $S(A) = A$. If $S(A) = A$, one can take either a reflection about any line passing through A, or take, instead of reflection, an identity map for R_1 .

Second, find a reflection R_2 which maps segment $S(A)R_1(B)$ to $S(A)S(B)$. The axis of such a reflection is the bisector of angle $\angle R_1(B)S(A)S(B)$.

The reflection R_2 maps $R_1(B)$ to $S(B)$. Indeed, the segment $S(A)R_1(B) = R_1(AB)$ is congruent to AB (because R_1 is an isometry), AB is congruent to $S(A)S(B) = S(AB)$ (because S is an isometry), therefore $S(A)R_1(B)$ is congruent to $S(A)S(B)$. Reflection R_2 maps the ray $S(A)R_1(B)$ to the ray $S(A)S(B)$, preserving the point $S(A)$ and distances. Therefore it maps $R_1(B)$ to $S(B)$.

Triangles $R_2 \circ R_1(\triangle ABC)$ and $S(\triangle ABC)$ are congruent via an isometry $S \circ (R_2 \circ R_1)^{-1} = S \circ R_1 \circ R_2$, and the isometry is identity on the side $S(AB) = R_2 \circ R_1(AB)$. Now either $R_2(R_1(C)) = C$ and then $S = R_2 \circ R_1$, or the triangles $R_2 \circ R_1(\triangle ABC)$ and $S(\triangle ABC)$ are symmetric about their common side $S(AB)$. In the former case

$S = R_2 \circ R_1$, in the latter case denote by R_3 the reflection about $S(AB)$ and observe that $S = R_3 \circ R_2 \circ R_1$. $\square$

Compositions of two reflections.

Theorem 8. *The composition of two reflections in non-parallel lines is a rotation about the intersection point of the lines by the angle equal to doubled angle between the lines. In formula:*

$$R_{AC} \circ R_{AB} = Rot_{A,2\angle BAC},$$

where R_{XY} denotes the reflection in line XY, and $Rot_{X,\alpha}$ denotes the rotation about point X by angle α.

Proof. Pick some points whose images under reflections are easy to track. From symmetries/congruent triangles in the picture, it is clear that effect of two refections is that of a rotation. Since we know that an isometry is determined by the image of 3 non-collinear points, the ir no need to consider all possible positions of the points. $\square$

Theorem 9. *The composition of two reflections in parallel lines is a translation in a direction perpendicular to the lines by a distance twice larger than the distance between the lines.*

More precisely, if lines AB and CD are parallel, and the line AC is perpendicular to the lines AB and CD, then

$$R_{CD} \circ R_{AB} = T_{2\overrightarrow{AC}}.$$

Proof. Similar to the above. $\square$

Application: finding triangles with minimal perimeters. We have considered the following problem:

Problem 1. *Given a line l and points A, B on the same side of l, find a point $C \in l$ such that the broken line ACB would be the shortest.*

Recall that a solution of this problem is based on reflection. Namely, let $B' = R_l(B)$. Then the desired C is the intersection point of l and AB'.

Notice that this problem can be reformulated as finding $C \in l$ such that the perimeter of the triangle ABC is minimal.

Problem 2. *Given lines l, m and a point A, find points $B \in l$ and $C \in m$ such that the perimeter of the triangle ABC is the smallest possible.*

Idea that solves Problem 2. Reflect point A through lines l and m, that is, consider points $B' = R_l(A)$ and $C' = R_m(A)$. Use these points to find B and C (how?), and prove that the resulting triangle indeed has the smallest perimeter.

Problem 3. *Given lines l, m and n, no two of which are parallel to each other. Find points $A \in l$, $B \in m$ and $C \in n$ such that triangle ABC has minimal perimeter.*

If we knew a point $A \in l$, the problem would be solved like Problem 2: we would connect points $R_m(A)$ and $R_n(A)$ and take B and C to be the intersection points of this line with m and n. So, we have to find a point $A \in l$ such that the segment $R_m(A)R_n(A)$ would be minimal.

The endpoints $R_m(A)$, $R_n(A)$ of this segment belong to the lines $R_m(l)$ and $R_n(l)$ and are obtained from the same point $A \in l$. Therefore

$$R_n(A) = R_n(R_m(R_m(A))) = R_n \circ R_m(B),$$

where $B \in R_m(l)$. So, one endpoint is obtained from another by $R_n \circ R_m$.

By Theorem *9*, $R_n \circ R_m$ is a rotation about the point $m \cap n$. We look for a point B on $R_m(l)$ such that the segment $BR_n \circ R_m(B)$ is minimal.

The closer a point to the center of rotation, the closer this point to its image under the rotation. Therefore the desired B is the base of the perpendicular dropped from $m \cap n$ to $R_m(l)$. Hence, the desired A is the base of perpendicular dropped from $m \cap n$ to l.

Since all three lines are involved in the conditions of the problem in the same way, the desired points B and C are also the endpoints of altitudes of the triangle formed by lines l, m, n.

Composition of rotations.

Theorem 10. *The composition of rotations (about points which may be different) is either a rotation or a translation.*

Prove this theorem by representing each rotation as a composition of two reflections about a line. Choose the lines in such a way that the second line in the representation of the first rotation would coincide with the first line in the representation of the second rotation. Then in the representation of the composition of two rotations as a composition of four reflections the two middle reflections would cancel and the whole composition would be represented as a composition of two reflections. The angle between the axes of these reflections would be the sum of of the angles in the decompositions of the original rotations. If this angle is zero, and the lines are parallel, then the composition of rotations is a translation by Theorem *9*. If the angle is not zero, the axes intersect, then the composition of the rotations is a rotations around the intersection point by the angle which is the sum of angles of the original rotations.

Similar tricks with reflections allows to simplify other compositions.

Glide reflections. A reflection about a line l followed by a translation along l is called a *glide reflection*. In this definition, the order of reflection and translation does not matter, because they commute: $R_l \circ T_{AB} = T_{AB} \circ R_l$ if $l \parallel AB$.

Theorem 11. *The composition of a central symmetry and a reflection is a glide reflection.*

Use the same tricks as for Theorem *10*

Classification of plane isometries.

Theorem 12. *Any isometry of the plane is either a reflection about a line, a rotation, a translation, or a gliding reflection.*

This theorem can be deduced from Theorem *7* by taking into account relations between reflections in lines. By Theorem *7*, any isometry of the plane is a composition of at most 3 reflections about lines. By Theorems *8* and *9*, a composition of two reflections is either a rotation about a point or a translation.

Lemma. A composition of three reflections is either a reflection or a gliding reflection.

Proof. We will consider two cases: 1) all three lines are parallel, 2) not all lines are parallel (although two of the three may be parallel to one another).

The first one is easier; it is pretty straightforward to see (at least in some examples) that the composition is a translation. However, since the order of reflections matters, for a precise proof we wouold have to check different cases (if the lines are all vertical, the first reflection may be done about the leftmost, the rightmost, or the middle lien, etc.) To avoid this, we proceed as follows. Notice that $R_{l_3} \circ R_{l_2} \circ R_{l_1} = R_{l_3} \circ (R_{l_2} \circ R_{l_1})$, and the composition $R_{l_2} \circ R_{l_1}$ of two reflections in parallel lines is a translation. This translation depends only on the direction of the lines and the distance between them, ie $R_{l_2} \circ R_{l_1} = R_{l'_2} \circ R_{l'_1}$ for any two lines l'_1, l'_2 that are parallel to l_1, l_2 and have the same distance between them. Thus, we translate the first two lines to make the second line coincide with the third, ie choose l'_1, l'_2 so that $l'_2 = l_3$. Then

$$R_{l_3} \circ R_{l_2} \circ R_{l_1} = R_{l_3} \circ R_{l'_2} \circ R_{l'_1} = R_{l_3} \circ R_{l_3} \circ R_{l'_1} = R_{l'_1}$$

since two reflections about the same line l_3 cancel. Therefore, the result is a reflection (about the line l'_1).

If the three lines are not all parallel, then the second line l_2 is not parallel to l_1 or l_3. Let's suppose l_1 and l_2 are not parallel (the other case is very similar). Then the composition $R_{l_2} \circ R_{l_1}$ of reflections about intersecting lines is a rotation (that depends only on the point where the lines intersect, and the angle at which they intersect). So the lines l_1, l_2 can be rotated simultaneously about their intersection point by the same angle without changing the composition.

By an appropriate rotation, make the second line l_2 perpendicular to the third line l_3 (which is not rotated), ie replace l_1, l_2 by l'_1, l'_2 so that $R_{l_2} \circ R_{l_1} = R_{l'_2} \circ R_{l'_1}$, and $l'_2 \perp l_3$.

Then by rotating these two perpendicular lines l'_2, l_3 about their intersection point, make the middle line l_2 parallel to the line l_1. That is, we replace the lines l'_2, l_3 by lines l''_2, l''_3 so that

$$R_{l_3} \circ R_{l_2} \circ R_{l_1} = R_{l_3} \circ R_{l'_2} \circ R_{l'_1} = R_{l''_3} \circ R_{l''_2} \circ R_{l'_1}.$$

Now, the configuration of lines consists of two parallel lines and a line perpendicular to them: l_1', l_2'' are parallel, l_3'' is perpendicular to them both. The composition of reflections $R_{l_2''} \circ R_{l_1'}$ is a translation by a vector perpendicular to these two lines (and thus parallel to the third); so $R_{l_3''} \circ (R_{l_2''} \circ R_{l_1'})$ is a glide symmetry. But the composition of these three reflections is the same as the composition of reflections about the original three lines.

Properties of the four types of isometries. We have just seen that any isometry of the plane belongs to one of the four types. How do we detect to which type it belongs? In particular, it may seem a bit mysterious that while composition of 3 reflections is a reflection or glide reflection, a composition of two isometries can never be a reflection, but only a rotation or translation. This can be explained as follows. Suppose our plane lies in the 3-space (as a horizontal xy-plane), and its top is painted black, its bottom white. Suppose that the reflections are done by rotating the plane around the line (axis of reflection) in the 3-space. Then after a reflection, the white side will be on top, the black side on the bottom. Notice that the colors will flip this way if we perform any odd number of reflections, but after an even number of reflections the colors do not flip. (Eg after two reflections, the top will be black again, the bottom white.) By contrast, rotations and translations do not flip the colors. This explains why the composition of two reflections can be a rotation or translation, but never a reflection.

Another fundamental characteristic of an isometry is the points that it leaves fixed. For instance, a rotation doesn't move the center (but moves any other point); a reflection fixes every point of its axis. We summarize these properties in the chart below.

type of isometry	points that stay fixed	flips colors?
rotation	the center	no
reflection	every point on axis	yes
translation	none	no
glide reflection	none	yes

These properties help detect the type of isometry. In particular, the chart shows that a glied reflection cannot belong to any of the other three types.

□

Isometries of the plane. Draft

Oleg Viro

Below formulations of the main statements (theorems and problems) that are to be proved are separated from the rest of the text. The proofs are postponed to the end of the text. The reader is encouraged to invent proofs on her/his own. Nonetheless, the reader has to read the proofs, no matter, if you found a proof or not. The reader is encouraged also to draw missing pictures. Each theorem should be illustrated with a picture!

1. Relocations as isometries

The notion of *isometry* is a general notion commonly used in mathematics. It means a mapping which preserves distances. The word *metric* is a synonym to the word *distance*.

In the context of this course, an isometry is a mapping of the plane that maps each segment s to a segment s' congruent to s. Therefore each relocation is an isometry. In fact, each isometry of the plane is a relocation.

2. Recovery of an isometry from its restriction to three points

Theorem A. *An isometry of the plane can be recovered from its restriction to any triple of non-collinear points.*

Recall that a restriction of a mapping f to a subset is the mapping from this subset which maps each point exactly as f. Theorem A claims that an isometry can be restored if one forgets how it moves all the points besides some three points that are not contained in a line.

In fact, an isometry can be almost recovered from its restriction to a pair of points: there are exactly two isometries with the same restriction to a pair of distinct points. They can be obtained from each other by composing with the reflection in the line connecting these points.

3. Isometries as compositions of reflections

Theorem B. *Any isometry of the plane is a composition of at most three reflections.*

4. Translations and central symmetries

A map of the plane to itself is called a *translation* if, for some fixed points A and B, it maps a point X to a point $Y = T(X)$ such that $XYBA$ is a parallelogram.

Here we have to be careful with the notion of parallelogram, because a parallelogram may degenerate to a figure in a line. Not any quadrilateral squeezed to a figure in a line deserves to be called a parallelogram, although any two sides of such a degenerate quadrilateral are parallel. By a parallelogram we mean a sequence of four segments KL, LM, MN and MK such that KL is congruent and parallel to MN and LM is congruent and parallel to MK. This definition describes both usual parallelograms, for which congruence of opposite sides can be deduced from parallelness and vice versa, and the degenerate parallelograms.

Theorem C. *For any points A and B there exists a translation which maps A to B. Any translation is an isometry.*

Denote by $T_{\overrightarrow{AB}}$ the translation which maps A to B.

Theorem D. *The composition of any two translations is a translation.*

Theorem D implies that $T_{\overrightarrow{BC}} \circ T_{\overrightarrow{AB}} = T_{\overrightarrow{AC}}$.

Fix a point O. A map of the plane to itself which maps a point A to a point B such that O is the midpoint of the segment AB is called the *symmetry about a point O*.

Theorem E. *A symmetry about a point is an isometry.*

Theorem F. *The composition of any two symmetries in a point is a translation. In details, $S_B \circ S_A = T_{2\overrightarrow{AB}}$, where S_X denotes the symmetry about point X.*

Corollary G. *A composition of a translation and a symmetry about a point is a symmatry in a point.*

Corollary H. *The composition of an even number of symmetries in points is a translation; the composition of an odd number of symmetries in points is a symmetry in a point.*

Problem 1. *Given centers of sides of a pentagon, find the vertices of the pentagon.*

Problem 2. *Which sets of $2n$ points are centers of sides of $2n$-gon? Hown many $2n$-gons have the same centers of sides?*

Problem 3. *Given a circle c, a line l and a point A, find points $B \in l$ and $C \in c$ such that A is the midpoint of segment BC.*

Problem 4. *Given circles c_1 and c_2 meeting at point A, find points $X_1 \in c_1$ and $X_2 \in c_2$ such that A is the midpoint of segment C_1C_2.*

Problem 5. *Given circles c_1 and c_2 and a segment s, find points $X_1 \in c_1$ and $X_2 \in c_2$ such that the segment is congruent and parallel to s.*

5. Compositions of two reflections

Theorem I. *The composition of two reflections in non-parallel lines is a rotation about the intersection point of the lines by the angle equal to doubled angle between the lines. In formula:*

$$R_{AC} \circ R_{AB} = Rot_{A,2\angle BAC},$$

where R_{XY} denotes the reflection in line XY, and $Rot_{X,\alpha}$ denotes the rotation about point X by angle α.

Theorem J. *The composition of two reflections in parallel lines is a translation in a direction perpendicular to the lines by a distance twice larger than the distance between the lines.*

More precisely, if lines AB and CD are parallel, and the line AC is perpendicular to the lines AB and CD, then

$$R_{CD} \circ R_{AB} = T_{2\overrightarrow{AC}}.$$

6. Application: finding triangles with minimal perimeters

Problem 6. *Given a line l and points A, B on the same side of l, find a point $C \in l$ such that the broken line ACB would be the shortest.*

Recall that a solution of this problem relies on reflection. Namely, let $B' = R_l(B)$. Then the desired C is the intersection point of l and AB'.

Notice that this problem can be reformulated as finding $C \in l$ such that the perimeter of the triangle ABC is minimal.

Problem 7. *Given lines l, m and a point A, find points $B \in l$ and $C \in m$ such that the perimeter of the triangle ABC is minimal.*

Problem 8. *Given lines l, m and n, no two of which are parallel to each other. Find points $A \in l$, $B \in m$ and $C \in n$ such that triangle ABC has minimal perimeter.*

7. Composition of rotations

Theorem K. *The composition of rotations (about points which may be different) is either a rotation or translation.*

9 (Napolean Theorem). *For any triangle $\triangle ABC$ and equilateral triangles $\triangle BCU$, $\triangle CAV$ and $\triangle ABW$ having no common interior points with $\triangle ABC$, points X, Y and Z that are centers of $\triangle BCU$, $\triangle CAV$ and $\triangle ABW$, respectively, are vertices of an equilateral triangle.*

8. Glide reflections

A reflection about a line l followed by a translation along l is called a *glide reflection.* In this definition, the order of reflection and translation does not matter, because they commute: $R_l \circ T_{AB} = T_{AB} \circ R_l$ if $l \parallel AB$.

Theorem L. *The composition of a central symmetry and a reflection is a glide reflection.*

9. Classification of plane isometries

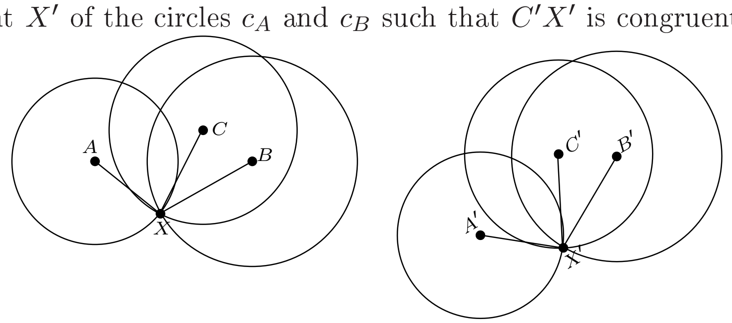
Theorem M. *Any isometry of the plane is either a reflection about a line, or rotation, or translation, or gliding reflection.*

Lemma N. *A composition of three reflections is either a reflection, or a gliding reflection.*

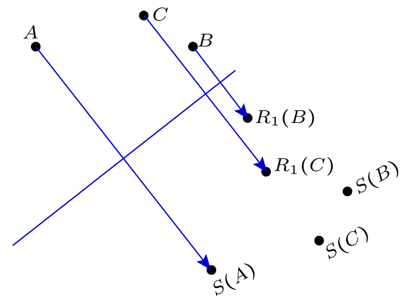
Exercise. Generalize everything that follows into the setup of the 3-space.

Proofs and Comments

A Given images A', B' and C' of non-collinear points A, B, C under and isometry, let us find the image of an arbitrary point X. Using a compass, draw circles c_A and c_B centered at A' and B' of radii congruent to AX and BX, respectively. They intersect in at least one point, because segments AB and $A'B'$ are congruent and the circles centered at A and B with the same radii intersect at X. There may be two intersection point. The image of X must be one of them. In order to choose the right one, measure the distance between C and S and choose the intersection point X' of the circles c_A and c_B such that $C'X'$ is congruent to CX.



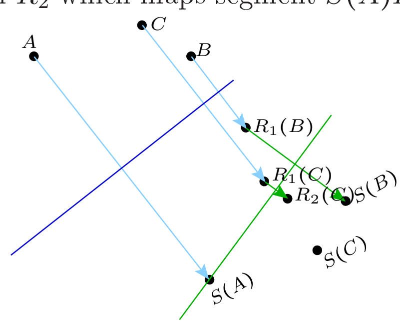
B Choose three non-collinear points A, B, C. By theorem A, it would suffice to find a composition of at most three reflections which maps A, B and C to their images under a given isometry S.
First, find a reflection R_1 which maps A to $S(A)$.



The axis of such a reflection is a perpendicular bisector of the segment

$AS(A)$. It is uniquely defined, unless $S(A) = A$. If $S(A) = A$, one can take either a reflection about any line passing through A, or take, instead of reflection, an identity map for R_1.

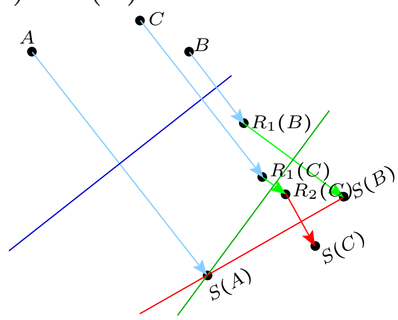
Second, find a reflection R_2 which maps segment $S(A)R_1(B)$ to $S(A)S(B)$.



The axis of such a reflection is the bisector of angle $\angle R_1(B)S(A)S(B)$.

The reflection R_2 maps $R_1(B)$ to $S(B)$. Indeed, the segment $S(A)R_1(B) = R_1(AB)$ is congruent to AB (because R_1 is an isometry), AB is congruent to $S(A)S(B) = S(AB)$ (because S is an isometry), therefore $S(A)R_1(B)$ is congruent to $S(A)S(B)$. Reflection R_2 maps the ray $S(A)R_1(B)$ to the ray $S(A)S(B)$, preserving the point $S(A)$ and distances.

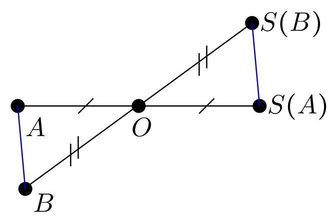
Therefore it maps $R_1(B)$ to $S(B)$.



Triangles $R_2 \circ R_1(\triangle ABC)$ and $S(\triangle ABC)$ are congruent via an isometry $S \circ (R_2 \circ R_1)^{-1} = S \circ R_1 \circ R_2$, and the isometry is identity on the side $S(AB) = R_2 \circ R_1(AB)$. Now either $R_2(R_1(C)) = C$ and then $S = R_2 \circ R_1$, or the triangles $R_2 \circ R_1(\triangle ABC)$ and $S(\triangle ABC)$ are symmetric about their common side $S(AB)$. In the former case $S = R_2 \circ R_1$, in the latter case denote by R_3 the reflection about $S(AB)$ and observe that $S = R_3 \circ R_2 \circ R_1$.

C Any points A, B and X can be completed in a unique way to a parallelogram $ABYX$ (maybe degenerated, that is all four points are collinear and $AB = XY$, $BY = AX$). Define $T(X) = Y$. For any points X, Y the quadrilateral $XYT(Y)T(X)$ is a parallelogram (maybe, degenerated). Therefore, T is an isometry.

E SAS-test for congruent triangles (extended appropriately to degenerate triangles.)



F Let X be an arbitrary point. Its image $Y = S_A(X)$ can be obtain from it by the translation $T_{\overrightarrow{XY}} = T_{\overrightarrow{AY}} \circ T_{\overrightarrow{XA}} = T_{2\overrightarrow{AY}}$. The image Z of Y under S_B can be obtained from Y by the translation $T_{\overrightarrow{YZ}} = T_{\overrightarrow{BZ}} \circ T_{\overrightarrow{YB}} = T_{2\overrightarrow{YB}}$. Hence

$$Z = T_{2\overrightarrow{YB}}(T_{2\overrightarrow{AY}}(X)) = T_{2(\overrightarrow{AY}+\overrightarrow{YB})}(X) = T_{2\overrightarrow{AB}}(X).$$

Draw the picture!

G The equality

$$S_B \circ S_A = T_{2\overrightarrow{AB}}$$

implies a couple of other useful equalities. Namely, compose both sides of this equality with S_B from the left:

$$S_B \circ S_B \circ S_A = S_B \circ T_{2\overrightarrow{AB}}$$

Since $S_B \circ S_B$ is the identity, it can be rewritten as

$$S_A = S_B \circ T_{2\overrightarrow{AB}}.$$

Similarly, but multiplying by S_A from right, we get

$$S_B = T_{2\overrightarrow{AB}} \circ S_A.$$

6 **Construction** that solves Problem 2. Reflect point A in l and m, that is find $B' = R_l(A)$ and $C' = R_m(A)$. Then $B = l \cap B'C'$ and $C = m \cap B'C'$. Exercise: provide a proof and research.

8 If we knew a point $A \in l$, the problem would be solved as Problem 2: we would connect points $R_m(A)$ and $R_n(A)$ and take for B and C the intersection points of this line with m and n. So, we have to find a point $A \in l$ such that the segment $R_m(A)R_n(A)$ would be minimal.

The end points $R_m(A)$, $R_n(A)$ of this segment belong to the lines $R_m(l)$ and $R_n(l)$ and are obtained from the same point $A \in l$. Therefore

$$R_n(A) = R_n(R_m(R_m(A))) = R_n \circ R_m(B),$$

where $B \in R_m(l)$. So, one end point is obtained from another by $R_n \circ R_m$.

By Theorem J, $R_n \circ R_m$ is a rotation about the point $m \cap n$. We look for a point B on $R_m(l)$ such that the segment $BR_n \circ R_m(B)$ is minimal.

The closer a point to the center of rotation, the closer this point to its image under the rotation. Therefore the desired B is the base of the perpendicular dropped from $m \cap n$ to $R_m(l)$. Hence, the desired A is the base of perpendicular dropped from $m \cap n$ to l.

Since all three lines are involved in the conditions of the problem in the same way, the desired points B and C are also the end points of altitudes of the triangle formed by lines l, m, n.

K Prove this theorem by representing each rotation as a composition of two reflections about a line. Choose the lines in such a way that the second line in the representation of the first rotation would coincide with the first line in the representation of the second rotation. Then in the representation of the composition of two rotations as a composition of four reflections the two middle reflections would cancel and the whole composition would be represented as a composition of two reflections. The angle between the axes of these reflections would be the sum of of the angles in the decompositions of the original rotations. If this angle is zero, and the lines are parallel, then the composition of rotations is a translation by Theorem J. If the angle is not zero, the axes intersect, then the composition of the rotations is a rotations around the intersection point by the angle which is the sum of angles of the original rotations.

Similar tricks with reflections allows to simplify other compositions.

L Use the same tricks as for Theorem K

M This theorem can be deduced from Theorem B by taking into account relations between reflections in lines. By Theorem B, any isometry of the plane is a composition of at most 3 reflections about lines. By Theorems I and J, a composition of two reflections is either rotation about a point or translation.

N If all three axes of the reflections are parallel, then the firs two can be translated without changing of their composition (the composition of reflections about two parallel lines depends only on the direction of lines and the distance between them). By translating the first two lines, make the second of them coinciding with the third line. Then in the total composition they cancel, and the composition is just the reflection in the first line.

If not all three lines are parallel, then the second is not parallel to one of the rest. The composition of reflections about these two non-parallel lines is a rotation, and the lines can be rotated simultaneously about their intersection point by the same angle without changing of the composition.

By an appropriate rotation, make the middle line perpendicular to the line which was not rotated. Then by rotating of these two perpendicular lines about their intersection point, make the middle one parallel to the other line. Now the configuration of lines consists of two parallel lines and a line perpendicular to them. The composition of reflections about them (the order does not matter any more, because they commute) is a gliding symmetry.

MAT 515, Geometry for Teachers
Fall 2012

Homework 10

1. Find the composition of two rotations: the rotation in the counterclockwise direction about a point A by angle $120°$ followed by the rotation in the counterclockwise direction about a point B by angle $120°$: prove that this is a rotation, find the center and angle of this rotation.

2. Prove that a composition of two translations is a translation. What is the vector of translation for the composition?

3. Prove that a composition of two gliding reflections along non-parallel lines is a rotation. How to find the angle of the rotation? How to find its center?

4. Prove that a composition of a rotation and a translation is a rotation. (This is true for any order of the rotation/translation; you can assume that the rotation is performed first.)

5. Prove that any isometry maps a circle into a circle of the same radius.

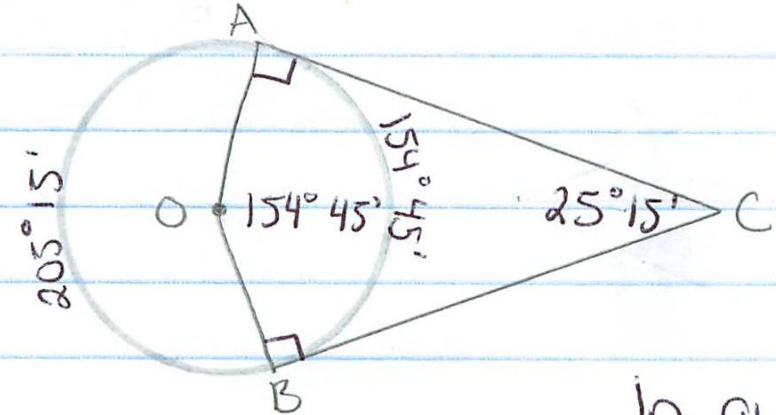
MAT515. Midterm topics

Here is the list of topics recommended for review.

1. Theorem about vertical angles, section 26.
2. Existence and uniqueness of perpendicular to a line from a point, sections 24, 65 and 66.
3. Theorems about isosceles triangles and their properties, sections 35, 36.
4. Congruence tests for triangles section 40.
5. Inequality between exterior and interior angles in a triangle, sections 41 - 43.
6. Relations between sides and opposite angles sections 44 - 45.
7. Triangle inequality and its corollaries, sections 48 and 49.
8. Perpendicular and slants, sections 51 - 53.
9. Segment and angle bisectors, sections 56 and 57.
10. Basic construction problems, sections 61 - 69.
11. Tests for parallel lines, section section 73.
12. The parallel postulate, sections 75 and 76.
13. Angles formed by parallel lines and a transversal, sections 77 and 78.
14. Angles with respectively parallel sides, section 78.
15. Angles with respectively perpendicular sides, section 79.
16. The sum of interior angles in a triangle, section 81.
17. The sum of interior angles in a convex polygon, section 82.
18. Properties of a parallelogram, sections 85 - 87.
19. Special types of parallelograms and their properties, sections 90 - 92.
20. The midline theorem, sections 93, 94, 95.
21. The midline of trapezoid, sections 96, 97.
22. Existence and uniqueness of a circle passing through three points, sections 103, 104.
23. Constructions that use isometries, sections 98 - 101
24. Theorems about inscribed angles, section 123.
25. Corollaries of the theorem about inscribed angles, sections 125,126.
26. Constructions using theorems about inscribed angles, sections 127 - 130, 133.
27. Inscribed and circumscribed circles, sections 136, 137.
28. Concurrency points in a triangle, sections 140 - 142.

10

261) The angle between two tangents drawn from the same point to a circle is 25°15'. Compute the arcs contained between the tangency points.



In circle O, AO and BO are radii and CA and CB are tangents. Then, $AO \perp CA$ and $BO \perp CB$.

In quadrilateral AOBC, the sum of the interior angles is 360°. So,

$90° + 90° + 25°15' + \angle AOB = 360° \Rightarrow \angle AOB = 154°45'$

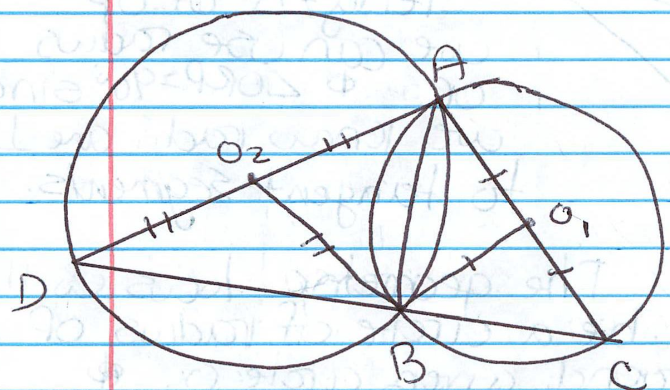
Since $\angle AOB$ is a central angle, it equals $\overset{\frown}{AB}$. (small)

Then big $\overset{\frown}{AB} = 360° - (\text{small } \overset{\frown}{AB})$

small $\overset{\frown}{AB} = 154°45'$ big $\overset{\frown}{AB} = 205°15'$

274. Through one of the two intersection points of two circles, a diameter in each of these circles is drawn. Prove that the line connecting the endpoints of these diameters passes through the other intersection point.

10



Given circles O_1, O_2 with intersections A, B. Through point A, diameter AC is drawn through O_1. Diameter AD is drawn through O_2. Connect D, C. Connect O_1 and B, O_2 and B to form radii.

$O_1A = O_1C = O_1B$ > they are all radii.
$O_2A = O_2D = O_2B$

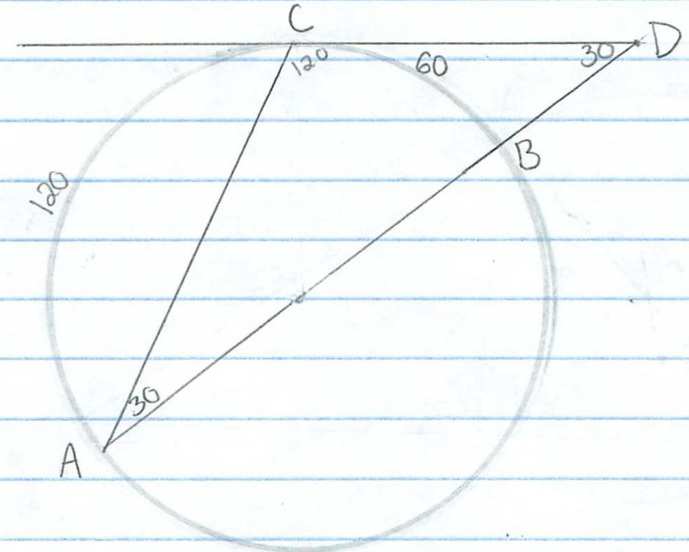
Connect AB. $\triangle CO_1B$, $\triangle BO_1A$, $\triangle BO_2A$, $\triangle BO_2D$ are all isosceles.

$\angle BCA + \angle CAB + \angle BAD + \angle ADB = 180°$ because the angles of a triangle add up to 180° ($\triangle ADC$).
$\angle ADB = \angle O_2BD$, $\angle BCA = \angle CBO_1$, $\angle CAB = \angle ABO_1$, $\angle BAD = \angle ABO_2$ ~~by definition of isosceles triangle.~~
by Theorem about isosceles triangles

$\angle CBO_1 + \angle O_1BA + \angle ABO_2 + \angle O_2BD = 180°$ by substitution so $\angle DBC = 180°$. $\therefore$ DBC is a straight line so C, B, and D are collinear.

10

275) A diameter AB and a chord AC form an angle of 30°. Through C, the tangent is drawn intersecting the extension of AB at the point D. Prove that $\triangle ACD$ is isosceles.



Pf:

We know $\overset{\frown}{BC} = 2(\angle CAB)$, so $\overset{\frown}{BC} = 2(30°) = 60°$. Since AB is a diameter, then $\overset{\frown}{ACB} = 180°$ and $\overset{\frown}{AC} + \overset{\frown}{BC} = 180°$, so $\overset{\frown}{AC} = 120°$. We know $\angle D = \frac{1}{2}(\overset{\frown}{AC} - \overset{\frown}{BC}) = \frac{1}{2}(120 - 60) = 30°$. Since the interior angles of a triangle sum to 180°, $\angle ACD = 120°$. In $\triangle ACD$, $\angle A = \angle D$, so it is an isosceles triangle. ■

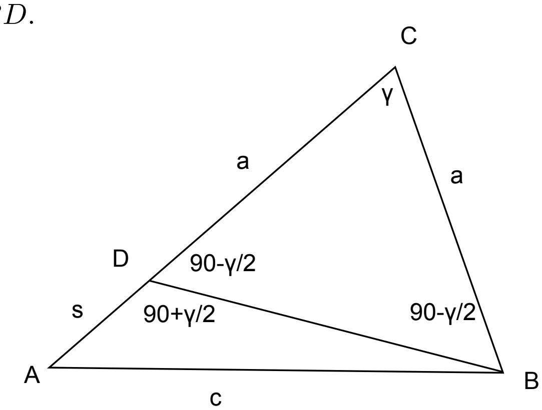
MAT515. Quiz 5. Name: Nikita Selinger

Construct a triangle with given side, angle opposite to that side, and the difference of the other two sides.

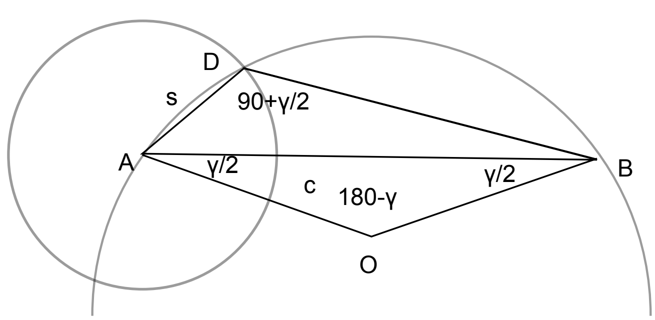
Solution.

I. Analysis.

Assume we have a triangle $\triangle ABC$ such that $\angle C = \gamma$, $AB = c$, and $AC - BC = s$, where γ, c, s are given. Mark a point D on AC such that $CD = BC = a$. Then $AD = AC - CD = AC - BC = s$. Since $CD = BC$, triangle $\triangle DCB$ is isosceles and, hence, $\angle DBC = \angle BDC$. As the sum of angles in $\triangle DCB$ is equal to $180°$, we see that $180° = \angle DBC + \angle BDC + \angle C = 2\angle BDC + \gamma$. Therefore $\angle BDC = 90° - \gamma/2$ and $\angle ADB$, which is supplementary to $\angle BDC$, is congruent to $\angle ADB = 180° - \angle BDC = 90 + \gamma/2$. Now it is enough to be able to construct $\triangle ADB$, after that we can recover C as the intersection of the line AD and the median bisector of BD.



In $\triangle ADB$ we know two sides and an angle. The point D must lie on a circle of radius s about A since $AD = s$. Also D must be on the geometric locus of points, from which AB is seen at angle $90° + \gamma/2$. We know that the latter is a union of two arcs and we may assume, by symmetry that D is on either of these arcs. To reconstruct this arc, we construct an isosceles triangle $\triangle AOB$ with base AB such that $\angle AOB = 180° - \gamma$. Then we draw the arc using O as a center and OA as a radius.



II. Construction.

The analysis yields the following construction.

1. Bisect γ to get an angle congruent to $\gamma/2$.
2. Construct a segment AB congruent to the given segment c and construct rays AX and BY in the same half-plane with respect to AB such that $\angle XAB = \angle YBA = \gamma/2$. Denote by O the intersection point of AX and BY.
3. Draw a circle S_1 using O as a center and AO as a radius.
4. Draw a circle S_2 using A as a center and the given segment s as a radius. Denote by D the intersection point of S_1 and S_2 that lies on the smaller arc $\overset{\frown}{\mathrm{AB}}$ of S_1.
5. Extend AD past D and find the intersection point C of AD and the median bisector of BD.

III. Synthesis.

Let us prove now that $\triangle ABC$ satisfies the required conditions. Indeed, $AB = c$ by construction. Since C is on the median bisector of BD, $BC = CD$ and $AC - BC = AC - CD = AD = s$ because D lies on the circle about A of radius s. In $\triangle AOB$, $\angle AOB = 180° - \angle OAB - \angle OBA = 180° - \gamma/2 - \gamma/2 = 180° - \gamma$. Therefore the angular measure of $\overset{\frown}{\mathrm{ADB}}$ is $180° - \gamma$ and the angular measure of the complementary arc $\overset{\frown}{\mathrm{AB}}$ is $360° - (180° - \gamma) = 180° + \gamma$. The measure of the angle $\angle ADB$, which is inscribed in S_1 and subtends $\overset{\frown}{\mathrm{AB}}$, is then equal to $90° + \gamma/2$. We already know that $BC = CD$ so that $\triangle BCD$ is isosceles and $\angle DBC = \angle BDC = 180° - \angle ADB = 180° - (90° + \gamma/2) = 90° - \gamma/2$. Then $\angle C = 180° - \angle BDC - \angle DBC = 180° - (90° - \gamma/2) - (90° - \gamma/2) = \gamma$ as required.

IV. Research.

We naturally assume that $\gamma < 180°$. If $s > c$, then the triangle inequality would be violated for any triangle satisfying $AB = c$ and $AC - BC = s > c = AB$. If $s < c$, then the two constructed circles intersect at two points and exactly one of them is on the smaller arc between A and B. In this case, every step of the construction can be done yielding a unique solution $\triangle ABC$.