MAT 513: Analysis for teachers

Spring 2011

Schedule	TuTh 5:20pm-6:40pm
Office hours	M-W 5-6 pm or by appointment
Office	Math Tower 3-102
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Text

William C. Bauldry, *INTRODUCTION TO REAL ANALYSIS An Educational Approach*

Course Content

Topics in differential calculus, its foundations, and its applications. This course is designed for teachers and prospective teachers of advanced placement calculus.

Homework

The homework are assigned during class, and posted on the web page. You will have about a week to complete the homework assignment. Late homework will not be accepted.

Here is the list of assigmnents (click on the links to open the pdf file):

Homework 1 Due date: **Tuesday, Feb 8**

Homework 2 Due date: **Tuesday, Feb 15**

Homework 3 Due date: **Tuesday, Feb 22**

Homework 4 Due date: **Thursday, March 10**

Homework 5 Due date: **Tuesday, March 22**

Homework 6 Due date: **Tuesday, March 29**

Homework 7 Due date: **Tuesday, April 5**

Homework 8 Due date: **Thursday, April 28**

Homework 9 Due date: **Thursday, May 5**

Grading

The grading will be weighted as follows: homework 25%, midterm I 20%, midterm II 20%, final 35%. The grades are available on blackboard .

Exams schedule

Midterm 1: Tuesday, March 8th (class time)

Midterm 2: Tuesday, April 12 (class time)

Final: May 19, 5:15 pm-7:45 pm

Midterm 1: suggested problems

Click here . I will write some more problems soon.

Midterm I: some comments and solutios

Click here.

Midterm 2: suggested problems

Click here.

Final Exam: practice problems

Click here

Stony Brook University Syllabus Statement

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Homework 1

Homework 1 is due **Tuesday, February 8**.

(1) Page 117: number 2.4
(2) Page 117: number 2.6
(3) Consider the closed interval $[0, 1]$. Prove that $\varepsilon = \frac{1}{2}$ is the maximum among the numbers such that the neighborhood
$$\mathrm{N}_\varepsilon\left(\frac{1}{2}\right) = \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right)$$
is contained in $[0, 1]$.
(4) Find the accumulation point(s) of the following subset of $\mathbb{R}$:
$$S = \{\frac{1}{n} \quad | \quad n = positive \quad integer\}$$
(5) Let $S = \{x \quad | \quad x \in \mathbb{Q} \quad and \quad 0 < x < 1\}$. Determine the set $\overline{S}$, the closure of S in $\mathbb{R}$ (hint: use Theorem 2.5).
(6) Let S be a subset of $\mathbb{R}$, and assume that $s = \mathrm{supS}$ exists. Prove that if $s \notin S$, then s is an accumulation point for S. Show with examples that if $s \in S$, then s can be either isolated or an accumulation point.
(7) (**optional**) Let $\mathbb{R}^2$ be the plane, with coordinates (x, y). We say that a subset S of $\mathbb{R}^2$ is open if for any point $P = (x_0, y_0) \in S$, there is a $\varepsilon > 0$ such that the set $N_\varepsilon(P) = \{(x, y) \quad | \quad \sqrt{(x - x_0)^2 + (y - y_0)^2} < \varepsilon\}$ is contained in S. Show that the intersection of a finite collection of open subsets is open, and that the union of any collection of open subsets is open.

Homework 2

Homework 2 is due **Tuesday, February 15**

(1) Page 117-118: numbers 2.14, 2.16, 2.21
(2) Using the **definition** of limit, prove that
$$\lim_{x\to 1} x^2 + 1 = 2$$
(3) Write down carefully what it means that: "the limit of f as x approoeaches a does not exists ". Then, consider the following function $f : \mathbb{R} \to \mathbb{R}$:
$$f(x) = \begin{cases} 1 & if \quad x \leq 0 \\ x & if \quad x > 0 \end{cases}$$
Prove that the limit of f as x approaches 0 does not exist.
(4) Prove Theorem 2.11 part 3 (using the definition of continuity)
(5) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Use the Intermediate Value Theorem (2.17) and the Extreme Value Theorem (2.19) to prove that the image (or range) of f is a closed bounded interval: that is, $f([a, b]) = [c, d]$ for some real numbers c, d.

Homework 3

Homework 3 is due **Thursday, February 24**

(1) Page 118: numbers 2.18, 2.29, 2.32, 2.33, 2.34
(2) For n positive integer, let $f(x) = x^n$. Prove, from the definition of derivative, that $f'(x) = nx^{n-1}$
(3) Let $f : D \to \mathbb{R}$ be a continous function. Suppose that there exits $c \in D$ such that $f(c) > 0$. Prove that there exits an $\varepsilon > 0$ such that $f(x) > 0$ for any $x \in D \cap (c - \varepsilon, c + \varepsilon)$.
(4) (**optional**) Find the derivative of the function

$$f(x) = \frac{\tan x + e^{x^2}}{\sqrt{x+1} - \ cosx}$$

(5) (**optional**) Find the absolute maximum and minimun value of $f(x) = x + \frac{1}{x}$ on the interval $[\frac{1}{2}, 2]$.
(6) Prove using a geometrical argument that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Homework 4

Homework 4 is due **Thursday, March 11**.

Definition. A function $f : D \to \mathbb{R}$ is called **increasing** (respectively, **decreasing**) if $f(x_1) \leq f(x_2)$ (respectively, $f(x_1) \geq f(x_2)$) for all x_1, x_2 in D such that $x_1 < x_2$.

A function $f : D \to \mathbb{R}$ is called **strictly increasing** (respectively, **strictly decreasing**) if $f(x_1) < f(x_2)$ (respectively, $f(x_1) > f(x_2)$) for all x_1, x_2 in D such that $x_1 < x_2$.

(1) Let $f : (a, b) \to \mathbb{R}$ be differentiable on (a, b). Prove that f is decreasing on (a, b) if and only if $f'(x) \leq 0$ for all $x \in (a, b)$.
(2) Let $f : (a, b) \to \mathbb{R}$ be differentiable on (a, b). Prove that if $f'(x) < 0$, then f is stricly decreasing on (a, b).
(3) Give an example of a striclty increasing differentiable function f on $\mathbb{R}$ such that $f'(x) = 0$ for some $x \in \mathbb{R}$
(4) The proof of Theorem 2.33 has a small mistake: find it and fix it.
(5) Page 120: numbers 2.50, 2.51, 2.52

Homework 5

Homework 5 is due **Thursday, March 17**.
Reference for Homework 5: Walter Rudin, *Principles of Mathematical Analysis*

(1) Repeat the proof of Theorem 1.37 to prove that there is a unique positive real number Y such that $Y^2 = 2$.)
(2) Study the definition of sum of two cuts (see Theorem 1.12 and Definition 1.13).
(3) Let α be a cut. Theorem 1.16 states that there is one and only one cut β such that $\alpha+\beta = 0^*$ ($0*$ is the cut defined by 0). In the proof, β is defined as the set of all rational numbers p such that $-p$ is an upper bound of α but not the smallest upper bound. What goes wrong if instead we define $\beta = \{-p \quad | \quad p \in \alpha\}$?
(4) Study the definition of products of two cuts. Prove that $\alpha\beta = \beta\alpha$ for all cuts α and β.
(5) Prove that a convergent sequence is a Cauchy sequence.
(6) Prove that a Cauchy sequence is bounded.

Homework 6

Homework 6 is due **Tuesday, March 29**

(1) Given a convergent sequence a_n, consider the set of its values:

$$S = \{a_n \quad | \quad n \in N\}$$

Prove that if S is finite, then the sequence is eventually constant: $\exists N$ such that $a_n = a_N \; \forall n \geq N$

(2) Given a convergent sequence a_n, consider the set of its values:

$$S = \{a_n \quad | \quad n \in N\}$$

Prove that if S is infinite, then $a = \lim a_n$ is the unique accumulation point of S.

(3) Give an example of a sequence a_n, such that the set

$$S = \{a_n \quad | \quad n \in N\}$$

has at two (or, if you prefer, more than two) accumulation points.

(4) Let a_n be a sequence of real numbers. Consider a strictly increasing sequence

$$n_1 < n_2 < n_3 < ...$$

of positive integers. Then a_{n_k} is a sequence, and it is said to be a **subsequence** of a_n.

a) Let

$$a_n = \frac{(-1)^n n + 1}{n}$$

Prove that a_n is not convergent.

b) Find two subsequences of a_n that are convergent.

(5) Use Bolzano-Weierstrass Theorem (2.54) to prove that if a_n is a bounded sequence, there exists a convergent subsequence of a_n

(6) Determine if the series is convergent or not

$$\sum_{n=0}^{\infty} \frac{n^3 - n}{n^4 + n^2 + 7}$$

(7) Determine the values of p such that the series is convergent:

$$\sum_{n=0}^{\infty} n^p$$

(8) Find the sum of the series:

$$\sum_{n=0}^{\infty} \left(\frac{2^{n-1}}{3^n} + \frac{1}{n!} \right)$$

Homework 7

Homework 7 is due **Tuesday, April 5**

(1) Page 121: numbers 2.63, 2.64, 2.65, 2.66, 2.68

Homework 8

Homework 8 is due **Thursday, April 24**

(1) Compute the following limits

a)

$$\lim_{x\to\infty} \frac{3x+5}{x-4}$$

b)

$$\lim_{x\to -3^+} \frac{x+2}{x+3}$$

c)

$$\lim_{x\to \frac{\pi}{2}^+} e^{\tan x}$$

(2) Graph the function

a)

$$f(x) = \frac{x^2}{x^2-1}$$

b)

$$f(x) = e^{\frac{-1}{x+1}}$$

c)

$$f(x) = \frac{e^x}{1+e^x}$$

d)

$$f(x) = \sqrt{x^2+1} - x$$

(3) Compute the integrals

a)

$$\int_0^1 10^x dx$$

b)

$$\int_0^{\frac{\pi}{3}} \frac{\sin x + \sin x(\tan x)^2}{(\sec x)^2} dx$$

c)

$$\int_1^e \frac{dx}{x\sqrt{\ln x}}$$

d)

$$\int \frac{dx}{x^2+6x+8}$$

Homework 9

Homework 8 is due **Thursday, May 5**

Page 121: from 2.69 to 2.73

Exam 1, suggested problems

(1) Prove, from the definition of limit, that the following limit does not exists:

$$\lim_{x\to 0} \sin\left(\frac{1}{x^2}\right)$$

(2) Prove, using the definition of limit of a sequence, that

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

(3) Suppose $f : (a, b) \to \mathbb{R}$ is a increasing and bounded above on the interval (a, b). Prove that

$$\mathrm{sup\{f(x) \quad | \quad x \in (a, b)\} = \lim_{x\to b^-} f(x)}$$

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be a function continuous at a. Suppose that $f(a) = 1$. Prove that there exists a neighborhood $N = (a - \varepsilon, a + \varepsilon)$ of a such that $f(x) > 0$ $\forall x \in N$.

(5) Let $f(x)$ be a polynomial function of degree 3 (that is $f(x) = ax^3+bx^2+cx+d$). Prove that the equation $f(x) = 0$ can have at most 3 real solutions.

(6) Now use induction to prove the general statement: if $f(x)$ is a polynomial function of degree n, then the equation $f(x) = 0$ has at most n real solutions.

(7) Let $f : [a, b] \to \mathbb{R}$ be continous on $[a, b]$ and differentiable on (a, b). Suppose that there exists a constant $M > 0$ such that $|f'(x)| \leq M$ for all $x \in (a, b)$. Prove that $|f(x) - f(y)| \leq M|x - y|$, for all x y in $[a, b]$.

(8) Let S and T be two subsets of $\mathbb{R}$. Suppose that for all $s \in S$ and $t \in T$ we have $s < t$. Prove that supS $\leq$ infT (note: first prove that supS and infT exist!)

(9) Use the definition of derivative to compute $f'(0)$:

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & if \quad x \neq 0 \\ x \quad if & x = 0 \end{cases}$$

(10) Prove that the equation $\cos x = x$ has exactly one solution on the interval $[0, 1]$

(11) Calculate

$$\lim_{n\to\infty} \sqrt{n^2+n} - n$$

MATH 513-EXAM 1

PLEASE PRINT YOUR NAME:

(1) For the sequence $a_n = \sqrt{4n^2 + n} - 2n$

$$\lim_{x \to \infty} a_n$$

Note: If, at some point of your calculations, you use de L'Hospital rule, and you want full credits, you have to give a reason why it can be applied to a sequence.

(2) Using the definition of derivative, calculate the derivative function $f'(x)$ for

$$f(x) = \frac{1}{x^2}$$

(3) Compute the derivative of $f(x) = xe^{(\cos x)^2} + 5x + 1$.

(4) Prove that a convergent sequence is bounded.

Remark Problems (1) (2) and (3) are of basic calculus type. If you missed them, you have to refresh your basic calculus knowledge. The statement of Problem (4) is proved in the book (Theorem 2.49)

(5) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Let $a \in \mathbb{R}$. Prove that if $-1 < f(a) < 1$, there exists a $\delta > 0$ such that $-1 < f(x) < 1$ for all $x \in (a-\delta, a+\delta)$.

Solution

The function f is continous at a. Let $\varepsilon < \min\{1-\mathrm{f(a)}, 1+\mathrm{f(a)}\}$, $\varepsilon > 0$. Then there exists a $\delta > 0$ such that for all $x \in (a-\delta, a+\delta)$,

$$|f(x) - f(a)| < \varepsilon$$

The above estimate is equivalent to

$$f(a) - \varepsilon < f(x) < f(a) + \varepsilon$$

and our choice of ε implies that

$$-1 = f(a) - f(a) - 1 < f(x) < f(a) + 1 - f(a) = 1$$

(indeed, $-\varepsilon > \max\{\mathrm{f(a)} - 1, -1 - \mathrm{f(a)}\} > -1 - \mathrm{f(a)}$)

Remarks

- We suggest to draw a graph in order to understand the choice of ε we made.
- This problem is similar to Problem 4, Exam 1 SUGGESTED PROBLEMS and Problem 3, Homework 3.

(6) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable funtion. Suppose that $|f'(x)| < 1$ for all $x \in \mathbb{R}$. Prove that there exists at most one value a such that $f(a) = a$.

Solution

Suppose that there are two distinct values x and y, such that $f(x) = x$ and $f(y) = y$. We can assume that $x < y$. Since f is differentiable on $\mathbb{R}$, it is differentiable on (x, y) and continous on $[x, y]$. Therefore according to the Mean Value Theorem there is a number c such that

$$\frac{f(y) - f(x)}{y - x} = f'(c)$$

This yields the following contradiction:

$$\frac{|f(y) - f(x)|}{|y - x|} = \frac{|y - x|}{|y - x|} = 1 = |f'(c)| < 1.$$

Remark For other applications of the Mean Value Theorem, see for example Homework 5, problems 1 and 2.

Exam 2, suggested problems

Note

The following is a list of suggested problems, complimentary to Homework 5 (problems 5 and 6 only) Homework 6 and Homework 7.

(1) Construct a sequence f_n of continous functions on $[0,1]$ that satisfies both the following conditions: f_n converges the function $f = 0$ pointwise but not uniformly and

$$\lim \int_0^1 f_n(x)dx = 0$$

.

HINT: modify example 2.24...

(2) Prove that the sequence $f_n = e^{\frac{-x^2}{n}}$ converges uniformly on $[-1,1]$

(3) For a continous function g on $[0,1]$ (or any other closed interval) one can prove the following inequality

$$|\int_0^1 f(x)dx| \leq \int_0^1 |f(x)|dx$$

Let f_n be a sequence of continous functions on $[0,1]$, converging uniformly to a function f. Using the above inequality, prove that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$$

(4) In class we define the following length function on the space $C([0,1])$ of continous functions on the **closed** interval $[0,1]$:

$$||f|| = \sup_{x\in[0,1]} |f(x)|$$

Consider now the space $C((0,1))$ of continous functions on the **open** interval $(0,1)$. What goes wrong if we try to define the length as we did for $C([0,1])$?

(5) Suppose that $\sum a_n$ is convergent, $a_n > 0$. Prove that $\sum \frac{a_n}{n}$ is convergent. If $\sum a_n$ is divergent, is it true that $\sum \frac{a_n}{n}$ is divergent?

(6) Determine whether the series is convergent or divergent:

$$a) \quad \sum_{n=1457}^{\infty} \frac{1}{n}$$

$$b) \quad \sum_{n=1}^{\infty} \frac{n^2+4}{n^4+5}$$

$$c) \quad \sum_{n=0}^{\infty} e^{-n}$$

$$d) \quad \sum_{n=0}^{\infty} e^{n}$$

$$e) \quad \sum_{n=1}^{\infty} ne^{-n^2}$$

(7) Calculate the sum of the following series:

$$\sum_{n=0}^{\infty} \frac{1+3^n}{4^n}$$

$$\sum_{n=0}^{\infty} \left(\frac{3^n}{n!} + \frac{1}{2^n} \right)$$

EXAM 2, SUGGESTED PROBLEMS

Note

The final exam will be open book/notes. No calculators and no computers are allowed. The final will consists of some calculus type questions, and some proving questions. Here is a list of some new practice problem.

(1) Let A and B be non-empty and bounded from above subsets of $\mathbb{R}$. Define the set $A + B$ as

$$A + B = \{a + b \quad | \quad a \in A, b \in B\}$$

Prove that $\sup A + \sup B = \sup(A + B)$

(2) Let f be a continous function on $[0, 1]$, and suppose that $f(x) > 0$ for all $x \in [0, 1]$. Prove that there exists $\varepsilon > 0$ such that $f(x) \geq \varepsilon$ for all $x \in [0, 1]$.

(HINT: suppose not, then $\min f = ...$)

(3) Let f be a continous function on $\mathbb{R}$. Prove that the set

$f^{-1}(0, 1) = \{x \quad | \quad 0 < f(x) < 1\}$ is open.

(4) Let $f_n : D \to \mathbb{R}$ be a sequence of **bounded** functions converging uniformly to $f : D \to \mathbb{R}$. Prove that f is bounded.

(5) Let $S \subset \mathbb{R}$, and $a_n \in S$. Prove that if $\lim_{n\to\infty} a_n = a$, then $a \in \overline{S}$.

(6) Prove the alternating series test (Theorem 1.39)