MAT 362 Differential Geometry, Spring 2011

Instructors' contact information

Course information

Take-home exam

Take-home final exam, due Thursday, May 19, at 2:15 PM. Please read the directions carefully.

Handouts

Overview of final projects pdf
Notes on differentials of C^1 maps pdf tex
Notes on dual spaces and the spectral theorem pdf tex
Notes on solutions to initial value problems pdf tex

Topics and homework assignments

Assigned homework problems may change up until a week before their due date.
Assignments are taken from texts by Banchoff and Lovett (B&L) and Shifrin (S), unless otherwise noted.

Topics and assignments through spring break (April 24)
Solutions to first exam
Solutions to second exam
Solutions to third exam

April 26-28: Parallel transport, geodesics. Read B&L 8.1-8.2; S2.4.
Homework due Tuesday May 3:

- B&L 8.1.4, 8.2.10
- S2.4: 1, 2, 4, 6, 11, 15, 20
- **Bonus:** Figure out what map projection is used in the graphic here. (A Facebook account is not needed.)

May 3-5: Local and global Gauss-Bonnet theorem. Read B&L 8.4; S3.1.
Homework due Tuesday May 10:

- B&L: 8.1.8, 8.4.5, 8.4.6
- S3.1: 2, 4, 5, 8, 9
- Project assignment: Submit final version of paper electronically to me BY FRIDAY MAY 13.

May 10: Hyperbolic geometry. Read B&L 8.5; S3.2.
No homework this week.

Third exam: May 12 (in class)

Take-home exam: due May 19 (at presentation of final projects)

Instructors for MAT 362 Differential Geometry, Spring 2011

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back to MAT 362 main page

MAT 362 Differential Geometry, Spring 2011

Syllabus in pdf format

Introduction to the course

This course is an introduction to the theory of curves and surfaces in Euclidean space, from the differentiable viewpoint. Our main goal is to cover "the local and global geometry of surfaces: geodesics, parallel transport, curvature, isometries, the Gauss map, the Gauss-Bonnet theorem." We will first spend some time (about 3-4 weeks) studying local and global properties of curves; these give insight into analogous results about surfaces, as well as tools for analyzing surfaces via the curves they contain.

The main prerequisites for this material are linear algebra, calculus in several variables, and the topology of R^n (such as one can get in an analysis course). These topics will be reviewed as needed, according to the students' background.

This is one of the most advanced courses offered by the math department at the undergraduate level. You are expected to spend about 10-15 hours each week outside of class working on the material. Grading will be based on homework, exams, and a final project.

Grading

- 30% Weekly homework (due each Tuesday, except following an exam)
- 15% First exam: **Thursday, February 24**
- 15% Second exam: **Thursday, March 24**
- 15% Third exam: **Thursday, May 12** (last day of class)
- 15% Final project: papers due **Tuesday, May 10**, presentations on **Thursday, May 19** (scheduled exam period)
- 10% Take-home final exam (distributed last day of class, collected at presentations on May 19)

Textbooks

We will use two texts as references for this class:

- *Differential Geometry of Curves and Surfaces*, by Thomas Banchoff and Stephen Lovett, available in the bookstore.
  Here is the site containing the authors' applets.
- *Differential Geometry: A First Course in Curves and Surfaces*, by Theodore Shifrin, available for (free) download here.

Here are a few other books about classical differential geometry, which I will be using:

- *Differential Geometry of Curves and Surfaces*, by Manfredo P. do Carmo
- *Differential Geometry: Curves -- Surfaces -- Manifolds* 2nd ed., by Wolfgang Kühnel
- *Elementary Differential Geometry*, by Andrew Pressley
- *Lectures on Classical Differential Geometry* 2nd ed., by Dirk J. Struik

Many other resources are available, both in the library and online.

Disability Support Services

If you have a physical, psychological, medical, or learning disability that may impact your course work, please contact Disability Support Services at (631) 632-6748 or http://studentaffairs.stonybrook.edu/dss/. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential.

Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to the following website: http://www.sunysb.edu/facilities/ehs/fire/disabilities.

Academic Integrity

Each student must pursue his or her academic goals honestly and be personally accountable for all submitted work. Representing another person's work as your own is always wrong. Faculty are required to report any suspected instances of academic dishonesty to the Academic Judiciary. For more comprehensive information on academic integrity, including categories of academic dishonesty, please refer to the academic judiciary website at http://www.stonybrook.edu/uaa/academicjudiciary/

Critical Incident Management

Stony Brook University expects students to respect the rights, privileges, and property of other people. Faculty are required to report to the Office of Judicial Affairs any disruptive behavior that interrupts their ability to teach, compromises the safety of the learning environment, or inhibits students' ability to learn.

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back to MAT 362 main page

MAT 362 Differential Geometry
Take-home exam
Due 19 May 2011

Instructor: Joshua Bowman

Name (please print): ____

Instructions:

- Don't panic. Try to have fun. ☺
- Keep in mind that most of the problems can be solved in several different ways.
- Your solutions should be submitted directly to me by 2:15 PM on Thursday, May 19. They may be typed or handwritten; they should be stapled together, and a copy of this page should be included as a cover sheet.
- This exam consists of three multipart problems. Read each question carefully, and answer the same way. Partial credit will be given where appropriate.
- You may use any books, notes, or other resources you find, but you may not discuss the exam with anyone except me until after 5:00 PM on May 19.

Question	Points	Score
1	10	
2	10	
3	10	
Total	30	

Each student must pursue his or her academic goals honestly and be personally accountable for all submitted work.

Understanding this, I declare I shall not give, use, or receive unauthorized aid in this examination.

Signature: ____

1. Let (x, y, z) be coordinates in $\mathbb{R}^3$ and (x, y, z, w) be coordinates on $\mathbb{R}^4$. Let $S^3 \subset \mathbb{R}^4$ be the three-dimensional unit sphere. Stereographic projection P from $S^3 \setminus \{(0,0,0,1)\}$ to $\mathbb{R}^3$ is defined by

$$P(x, y, z, w) = \left(\frac{x}{1-w}, \frac{y}{1-w}, \frac{z}{1-w} \right).$$

   (a) Show that

   $$\vec{X}(u, v) = (\cos u \cos v, \sin u \cos v, \cos u \sin v, \sin u \sin v).$$

   parametrizes a compact surface S in S^3. What surface is S homeomorphic to?

   (b) Show that $P \circ \vec{X}$ is a conformal parametrization of a regular surface in $\mathbb{R}^3$. (*Hint:* show that $\vec{X}$ is conformal, then use the fact that P is conformal.) Give a natural domain for this parametrization.

   (c) Show that the u- and v-coordinate curves of $P \circ \vec{X}$ are circles in $\mathbb{R}^3$.

   (d) Two circles in $\mathbb{R}^3$ form a *Hopf link* if one of them is the boundary of a topological disk that is intersected by the other exactly once. Show that at least one pair of v-coordinate curves for $P \circ \vec{X}$ form a Hopf link. (In fact, this is the case for any such pair, but you only need to prove it for one.)

Let $\tanh x = \sinh x / \cosh x = (e^x - e^{-x})/(e^x + e^{-x})$ be the hyperbolic tangent. For problems 2 and 3, you will need the fact that $\frac{d}{dx} \tanh x = 1/\cosh^2 x$.

2. (a) Let $\mathbb{D}$ be the unit disk $\{(x, y) \mid x^2 + y^2 < 1\}$ with the hyperbolic metric, given by

   $$g_{11} = g_{22} = \frac{4}{(1 - x^2 - y^2)^2}, \qquad g_{12} = 0.$$

   Show that the curve

   $$\gamma(t) = (\tanh t, 0), \qquad t \in \mathbb{R},$$

   is a geodesic in the hyperbolic metric.

   (b) Use the result of part (a) to compute the area of a disk $D_r \subset \mathbb{D}$, centered at $(0, 0)$, with hyperbolic radius r (that is, Euclidean radius $\tanh r$).

   (c) Use the result of part (b), the Taylor expansion for e^x, and Theorem 8.3.15 in Banchoff and Lovett's text to show that the curvature of $\mathbb{D}$ with the hyperbolic metric is -1 at the origin. (You do not need to prove the theorem.)

   (d) Now let $\mathbb{H}$ be the upper half-plane $\{(x, y) \mid y > 0\}$ with the hyperbolic metric, given by

   $$g_{11} = g_{22} = \frac{1}{y^2}, \qquad g_{12} = 0.$$

   Show that any line through the origin, parametrized by $t \mapsto (ae^t, e^t)$, has constant geodesic curvature with respect to the hyperbolic metric, and express this curvature as a function of a. Also show that any horizontal line in $\mathbb{H}$ has constant geodesic curvature.

3. The surface C_1 obtained by rotating the curve $y = \cosh x$ around the x-axis is called the *catenoid*.

   (a) Find the Gaussian curvature of the catenoid.

   (b) Now find the area element dA and compute the total Gaussian curvature

   $$\iint_{C_1} K\,dA.$$

   (Note that this is an improper integral.)

   (c) Prove (or at least show why) the catenoid is diffeomorphic to the cylinder C_2 obtained by rotating the line $y = 1$ around the x-axis.

   (d) Determine (by any method), the total Gaussian curvature of the cylinder, i.e.,

   $$\iint_{C_2} K\,dA.$$

   (e) Show that the Euler characteristic of the cylinder C_2 is zero. You can use the fact that a finite interval I is diffeomorphic to the whole line $\mathbb{R}$, so the cylinder is diffeomorphic to $I \times S^1$.

   After all this, you've shown by example that the Gauss–Bonnet theorem does not hold for non-compact surfaces in the same way it holds for compact surfaces.

OVERVIEW OF FINAL PROJECTS FOR MAT 362

During the rest of the semester, you will produce an expository paper on some topic in differential geometry not covered in lectures, or at least one that is not covered in depth. The paper should be 6–10 pages long, typeset in LaTeX; if it includes lots of figures, then a couple of extra pages may be allowed. It should be written in a professional tone, with careful definitions and at least a couple of proofs given, although a little levity or the occasional gloss over details will not be considered improper. Part of the assignment's purpose is for you to practice expressing the essence of a concept in a clear and accessible manner.

Along with this paper, you will develop some multimedia component, which will be presented to the class on May 19, during the final exam period. This can take any number of forms: a poster, a collection of computer images (or animations), sculpture, a demonsration, etc. It should illustrate some key idea or example from your topic, and be presentable in 10 minutes (this is the amount of time you will have to show the class). Thus, you will not be expected to prepare a full-length talk, but only to convey expediently the interest of your subject.

Here are some possible topics (many others are available):

- Ruled surfaces
- Minimal surfaces
- Weierstrass representation
- Isothermal coordinates/uniformization
- Vector fields on surfaces (e.g., the "hairy ball" theorem)
- Line bundles on surfaces
- Riemannian geometry
- Tensor algebra
- Lie groups
- Connections on tangent bundles
- Immersed surfaces/non-orientability
- Differential forms
- Curvature flows on surfaces
- Geodesic flow on surfaces
- Differential topology
- Morse theory
- Möbius transformations
- Groups of isometries
- Knot theory
- Laplace–Beltrami operator on surfaces
- Extremal length/conformal invariants
- Shape of a drum
- Discrete differential geometry
- Applications to computer modeling
- Applications to physics: relativity, electromagnetism, mechanics

The items on this list vary from the broad to the specific, and from the easily-accessible to the esoteric. Your first job is to consult with me to find an appropriate topic for you, which may be one of the above, or a small piece of one, or something else altogether. Some of what I have listed are areas very familiar to me, and I will be able to direct you to one or more useful sources. Others are relatively outside my own experience, and will require more initiative on your part to find references. With the advent of the Information Age already behind us, it should not be hard to find at least a few relevant sources through search engines, online libraries (e.g., Google books), and Wikipedia.

As a general rule, if the topic you choose is one that appears in many differential geometry textbooks (such as minimal surfaces) but which we simply didn't get to in lecture, then you will be expected to survey lots of sources and create your own presentation from these. If you choose a topic that is less broadly covered or is a field of study unto itself (such as Morse theory), then you may focus on one or two primary sources and extract most of your material from these, although of course you must present the ideas in your own words.

Here is a rough timetable for the projects:

- *Next three weeks*: Decide on a topic and meet with me to discuss it. Find sources and begin sorting through definitions and major results.
- *April 5* (due date for first homework after the 2nd exam): Submit a summary of what you intend to cover in your project. This should be about a page long, written in LaTeX, with a list of sources.
- *April 28* (first Thursday after spring break): Present a first draft of your paper to me and to at least one other student in the class for feedback.
- *May 10* (Tuesday of last week of class): Submit final draft of paper to me electronically. (I will unfortunately be out of town.)
- *May 19* (final exam period): Present projects with the other students in the class as an audience. I will try to find a way to make this accessible to other members of the math department, as well, provided there are no objections.

DIFFERENTIALS OF C^1 MAPS $\mathbb{R}^n \to \mathbb{R}^m$

In one-variable calculus, we define the derivative of a function $f : I \to \mathbb{R}$ at a point $a \in I$ by the limit

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

It is not immediately clear how to convert this to the multivariable case. One possibility is the directional derivative. Suppose $U \subset \mathbb{R}^n$ is open and F is a function from U to $\mathbb{R}^m$. Here $n \geq 1$ and $m \geq 1$ can be arbitrary positive integers. If $\vec{a} \in U$ and $\vec{v} \in \mathbb{R}^n$, then the *directional derivative of F at $\vec{a}$ in the direction $\vec{v}$* is

$$D_{\vec{v}} F(\vec{a}) = \lim_{t \to 0} \frac{F(\vec{a} + t\vec{v}) - F(\vec{a})}{t}. \tag{1}$$

Note that the subtraction in the numerator makes sense because we can add and subtract vectors in $\mathbb{R}^m$. (Sometimes one restricts to the case where $\|\vec{v}\| = 1$, so that $t = \|t\vec{v}\|$ for $t \geq 0$. Then we can think of $t\vec{v}$ as replacing h in the numerator of the one-variable defintion and t as replacing h in the denominator. The fact that the two copies of h in the original definition turn out to play different roles in the higher-dimensional version will be important in a moment.) Assuming this limit exists for all $\vec{a} \in U$ and $\vec{v} \in \mathbb{R}^n$, we'd like to define the "differential" of F by

$$dF_{\vec{a}}(\vec{v}) = D_{\vec{v}} F(\vec{a}). \tag{2}$$

(The notation is deceptive; on the left, $dF_{\vec{a}}$ is intended to be a function we're trying to define at $\vec{v}$. On the right, $D_{\vec{v}}$ is "acting on" the function F, and the result is meant to be evaluated at $\vec{a}$. This is why $\vec{a}$ and $\vec{v}$ have moved around.) But even if the expression on the right always exists, the function defined on the left may not be linear (as a function of $\vec{v}$); more care is needed if we still want the derivative to be the "best linear approximation" of a function. To generalize this notion from one variable to many, it's useful to think of the first-order Taylor polynomial, or at least to reconsider the meaning of $f'(a)$.

The correct way to think of the derivative is suggested in the interpretation of $f'(a)$ as a slope, not just as a number. This gives $f'(a)$ geometric content: the derivative describes how a line should be placed in order to best line up with the graph of f at a, namely according to the equation $y - f(a) = f'(a)(x - a)$. When x increases a small amount, say h, then y increases by approximately $f'(a)h$. Thus $f'(a)$ is really the *coefficient* (think of it as a 1×1 coefficient matrix to see where this is going) of a linear map $\mathbb{R} \to \mathbb{R}$, defined by $h \mapsto f'(a)h$. And as h approaches zero, the discrepancy between $f(a+h)$ and $f(a) + f'(a)h$ shrinks faster than h does. This is the way to properly define the derivative in higher dimensions. First, let's rearrange the one-variable definition to say that $f'(a)$ is the number such that

$$\lim_{h \to 0} \frac{f(a+h) - \big(f(a) + f'(a)h\big)}{h} = 0.$$

Convince yourself, using the fact that $f'(a)$ is a constant, that this limit is equivalent to (1). This implies, for instance, that if such an $f'(a)$ exists, then it is unique.

Again suppose $U \subset \mathbb{R}^n$ is open, F is a function from U to $\mathbb{R}^m$, and $\vec{a} \in U$. We define the *differential of F at $\vec{a}$* to be the linear function $L = dF_{\vec{a}}$ such that

$$\lim_{\vec{h}\to\vec{0}} \frac{F(\vec{a}+\vec{h}) - \big(F(\vec{a}) + L(\vec{h})\big)}{\|\vec{h}\|} = \vec{0}, \tag{3}$$

provided such an L exists. This limit says exactly that, as $\vec{h}$ approaches zero, the discrepancy between $F(\vec{a}+\vec{h})$ and $F(\vec{a}) + dF_{\vec{a}}(\vec{h})$ shrinks faster than $\vec{h}$ does. (Note that $\vec{a}+\vec{h}$ may not always be in U, but because U is open, for small enough $\vec{h}$ it is. By the linearity property $L(\alpha\vec{v}) = \alpha L(\vec{v})$, we only need to know how L is defined on a small open ball around $\vec{0}$ in order to define it on all of $\mathbb{R}^n$.) One can prove directly from the definition that if a linear function L exists satisfying (3), then there is a unique such function. Using this definition, the multivariable chain rule

$$d(G \circ F)_{\vec{a}} = dG_{F(\vec{a})} \circ dF_{\vec{a}}$$

is very nearly self-evident. (Think about why this becomes $(g \circ f)'(a) = g'(f(a))\, f'(a)$ in the one-variable case.) It also follows that if $dF_{\vec{a}}$ exists, then it satisfies (2). Thus if the differential of F exists, all directional derivatives exist; the reverse is not true.

This does not mean that the directional derivative is not important; indeed, it is how one usually calculates the differential of a function (as the limit in (3) gives little indication of how to go about finding L). In particular, if we want to find the matrix of $dF_{\vec{a}}$ with respect to the standard basis of $\mathbb{R}^n$ and $\mathbb{R}^m$, then we compute the directional derivatives in the directions of the standard basis vectors for $\mathbb{R}^n$, which amounts to taking the partial derivatives of the component functions of F. This observation leads to the following formula for the matrix $[dF_{\vec{a}}]$ of $dF_{\vec{a}}$ (sometimes called the "Jacobian matrix"), applicable when F is C^1:

$$\big[dF_{\vec{a}}\big] = \begin{bmatrix} \dfrac{\partial F_1}{\partial x_1}(\vec{a}) & \cdots & \dfrac{\partial F_1}{\partial x_n}(\vec{a}) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F_m}{\partial x_1}(\vec{a}) & \cdots & \dfrac{\partial F_m}{\partial x_n}(\vec{a}) \end{bmatrix}.$$

More commonly, when $m = n$, this matrix is square, and its determinant is simply called the *Jacobian* (or *Jacobian determinant*) of F.

Here is an example I always find helpful: if $F : \mathbb{R}^n \to \mathbb{R}^m$ is itself linear, then it is its own "best linear approximation" at every point—i.e., $dF_{\vec{a}} = F$ for all $\vec{a} \in \mathbb{R}^n$. This is the correct interpretation of the statement "The derivative of a linear function is constant."

The next step is to define the derivative of a map between surfaces (or between manifolds, which generalize surfaces to higher dimensions). This requires rethinking the nature not only of the derivative itself, but of its *domain*. Put succinctly, if a function F sends points of a surface S_1 to points of another surface S_2, then its linear approximation at p (the derivative dF_p) sends points of the linear approximation to S_1 at p (the tangent space T_pS_1) to points of the linear approximation to S_2 at $F(p)$ (the tangent space $T_{F(p)}S_2$). Thus the condition of differentiability again lets us pass from "curvy" objects (surfaces and maps between them) to "straight" objects (vector spaces and linear maps, which are what we study in linear algebra). Defining this all properly takes a bit more care, but we gain powerful tools as well as perspective on the old derivative we learned about years ago.

DUAL SPACES, LINEAR MAPS, AND THE SPECTRAL THEOREM

Let V be a *real vector space*. This means the following:

- V is an abelian group: vectors can be added; addition $+$ is commutative and associative; there is an identity $\vec{0}$; and every element $\vec{v} \in V$ has an additive inverse $-\vec{v}$ such that $\vec{v} + (-\vec{v}) = \vec{0}$.
- elements of V can be multiplied by real numbers, subject to the following rules: for any $\vec{v}, \vec{w} \in V$ and $a, b \in \mathbb{R}$,

$$a(b\vec{v}) = (ab)\vec{v}, \qquad a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w}, \qquad (a+b)\vec{v} = a\vec{v} + b\vec{v}.$$

- For any $\vec{v} \in V$, $1\vec{v} = \vec{v}$ (that is, multiplication by 1 acts as the identity).

V is an *n-dimensional vector space* if there exists a set of n linearly independent vectors in V whose span is V; such a set is then called a *basis*.

Any vector space V has associated to it another vector space V^*, called its *dual space*, which consists of linear maps from V to $\mathbb{R}$. (Such a space of functions invariably constitutes a vector space, because the functions themselves can be added together and multiplied by real numbers.) If V is n-dimensional, then so is V^*, and any choice of basis in V induces a *dual basis* in V^*: if $\{\vec{e}_i\}$ is a basis for V, then the dual basis consists of the functions $\vec{e}_i{}^*$, which may be thought of as "projection" to the ith coordinate.

For example, if we think of vectors in $\mathbb{R}^n$ as column vectors, then elements of $(\mathbb{R}^n)^*$ are represented as row vectors, because multiplication of a column vector by a row vector (on the left) produces a scalar. We move between an element $\vec{v} \in \mathbb{R}^n$ and its dual by applying the transpose: $\vec{v}^* = \vec{v}^\top$.

Given two vector spaces V and W, their *tensor product* $V \otimes W$ is a new vector space that consists of *tensors*, which are formal linear combinations of expressions that look like $\vec{v} \otimes \vec{w}$, with $\vec{v} \in V$, $\vec{w} \in W$. Tensors are assumed to be linear in each component, meaning

$$\vec{v} \otimes (\vec{w}_1 + \vec{w}_2) = \vec{v} \otimes \vec{w}_1 + \vec{v} \otimes \vec{w}_2, \qquad (\vec{v}_1 + \vec{v}_2) \otimes \vec{w} = \vec{v}_1 \otimes \vec{w} + \vec{v}_2 \otimes \vec{w},$$
$$\text{and} \qquad (a\vec{v}) \otimes \vec{w} = \vec{v} \otimes (a\vec{w}) = a(\vec{v} \otimes \vec{w}).$$

(Basically, you assume you can "multiply" elements of V and W, then follow the rules of distributivity as you would expect.) If $\{\vec{e}_i\}$ is a basis of V and $\{\vec{f}_j\}$ is a basis of W, then the tensors of the form $\vec{e}_i \otimes \vec{f}_j$ form a basis of $V \otimes W$. This means that we can write any element of $V \otimes W$ as a unique linear combination of the elements $\vec{e}_i \otimes \vec{f}_j$; moreover, if $\dim(V) = k$ and $\dim(W) = n$, then $\dim(V \otimes W) = kn$:

$$V \otimes W = \left\{ \sum_{i=1}^{k} \sum_{j=1}^{n} a_{ij} (\vec{e}_i \otimes \vec{f}_j) \mid a_{ij} \in \mathbb{R} \right\}.$$

The space of linear maps from V to W can be expressed as the tensor product $W \otimes V^*$: for each tensor of the form $\vec{w} \otimes \alpha$, evaluate on $\vec{v} \in V$ by $(\vec{w} \otimes \alpha)(\vec{v}) = \alpha(\vec{v})\vec{w}$. Now suppose $\{\vec{e}_j\}_{j=1}^k$ is a basis for V, $\{\vec{e}_j{}^*\}_{j=1}^k$ is its dual basis, and $\{\vec{f}_i\}_{i=1}^n$ is a basis for W. Then the linear map A that sends $\vec{e}_j$ to $\sum_i a^i_j \vec{f}_i$ for all j is just the tensor $\sum_{i,j} a^i_j(\vec{f}_i \otimes \vec{e}_j{}^*)$:

$$[A] = \begin{bmatrix} a^1_1 & \cdots & a^1_k \\ \vdots & \ddots & \vdots \\ a^n_1 & \cdots & a^n_k \end{bmatrix} \quad \longleftrightarrow \quad A = \sum_{i=1}^{n}\sum_{j=1}^{k} a^i_j(\vec{f}_i \otimes \vec{e}_j{}^*).$$

Here we have written the coefficients as a^i_j instead of a_{ij} to remind ourselves of the different roles of the indices: the "down" index is matched with the input variables, and the "up" index is matched with the output variables. We will learn more about this index convention later in the course.

An element of $V^* \otimes V^*$ (i.e., a map $V \times V \to \mathbb{R}$ that is linear in each variable) is called a *bilinear form* on V. Each element of the form $\alpha \otimes \beta$ is evaluated on a pair of vectors $(\vec{v}_1, \vec{v}_2)$ according to the rule $(\alpha \otimes \beta)(\vec{v}_1, \vec{v}_2) = \alpha(\vec{v}_1)\beta(\vec{v}_2)$; this definition extends to all of $V^* \otimes V^*$ by linearity.

A bilinear form $B \in V^* \otimes V^*$ is called *symmetric* if $B(\vec{v}_1, \vec{v}_2) = B(\vec{v}_2, \vec{v}_1)$. This means that there is a basis $\{\vec{e}_i\}$ of V such that $B = \sum a_{ij}(\vec{e}_i{}^* \otimes \vec{e}_j{}^*)$, with the a_{ij} symmetric, $a_{ij} = a_{ji}$. (It does not necessarily mean that this symmetry of coefficients holds in every basis.) A bilinear form B is called *positive definite* if $B(\vec{v}, \vec{v}) > 0$ for every $\vec{v} \neq \vec{0}$. A symmetric, positive definite bilinear form on V is often called an *inner product* on V; the standard inner product on $\mathbb{R}^n$ is the prototypical example.

From our previous discussion, we see that an element of $V^* \otimes V^*$ can also be thought of as a linear map $V \to V^*$. This correspondence between bilinear forms on V and maps from V to its dual space is entirely natural (that is, independent of coordinates). Indeed, the transpose operation on matrices, which converts column vectors to row vectors and vice versa, is an instance of using the standard inner product on $\mathbb{R}^n$ to induce a map from $\mathbb{R}^n$ to its dual $(\mathbb{R}^n)^*$. (Notice that this description via transposes *does* depend on the coordinates, but it can be phrased just in terms of the standard inner product.)

Suppose V has an inner product $\langle \cdot\,, \cdot \rangle \in V^* \otimes V^*$. A linear map $A : V \to V$ (that is, an element A of $V \otimes V^*$) is called *self-adjoint* with respect to $\langle \cdot\,, \cdot \rangle$ if $\langle A\vec{v}_1, \vec{v}_2 \rangle = \langle \vec{v}_1, A\vec{v}_2 \rangle$ for all $\vec{v}_1, \vec{v}_2 \in V$. We can think of the two sides of this equality as "contracting" A with $\langle \cdot\,, \cdot \rangle$ in two different ways: if we write A in some basis $\{\vec{e}_i\}$ of V as $\sum a^i_j(\vec{e}_i \otimes \vec{e}_j{}^*)$, then we can insert each $\vec{e}_i$ in this expansion into either "slot" of $\langle \cdot\,, \cdot \rangle$ to obtain a new bilinear form in $V^* \otimes V^*$. Self-adjointness means that it does not matter which choice we make. (More generally, given a linear map $A : V \to V$, there is a unique map A^*, called the *adjoint* of A, that satisfies $\langle A\vec{v}_1, \vec{v}_2 \rangle = \langle \vec{v}_1, A^*\vec{v}_2 \rangle$. In an orthonormal basis, the matrix of the adjoint A^* is the transpose of the matrix of A, but this, too, does not hold in all bases.)

Spectral Theorem. *Let V be a real vector space with inner product $\langle \cdot\,, \cdot \rangle$. If $A : V \to V$ is self-adjoint, then all of its eigenvalues are real, and V has an orthonormal basis consisting of eigenvectors of A.*

We will give a proof for the case that V is two-dimensional. The general case is proved by induction on the dimension of V. The idea is to find eigenvalues and eigenvectors of A by looking at the restriction of the quadratic form $Q(\vec{v}) = \langle A\vec{v}, \vec{v} \rangle = \langle \vec{v}, A\vec{v} \rangle$ to the set of unit vectors in V. First, we make an observation about eigenspaces of a self-adjoint map. Recall that $\vec{v}^\perp$ is the subspace of V consisting of vectors orthogonal to $\vec{v}$.

Lemma. *If $A : V \to V$ is self-adjoint and $\vec{v}$ is an eigenvector of A, then $\vec{v}^\perp$ is A-invariant.*

Proof of lemma. Suppose $\vec{w} \in \vec{v}^{\perp}$. Then, because $A\vec{v}$ is a scalar multiple of $\vec{v}$, we have $0 = \langle \vec{v}, \vec{w} \rangle = \langle A\vec{v}, \vec{w} \rangle = \langle \vec{v}, A\vec{w} \rangle$. Thus $A\vec{w} \in \vec{v}^{\perp}$ for all $\vec{w} \in \vec{v}^{\perp}$. □

Proof of the Spectral Theorem. The differential of Q at a point $\vec{v} \in V$ is given by

$$\begin{aligned} dQ_{\vec{v}}(\vec{w}) &= \lim_{t\to 0} \frac{Q(\vec{v}+t\vec{w}) - Q(\vec{v})}{t} = \lim_{t\to 0} \frac{\big(Q(\vec{v}) + \langle A\vec{v}, t\vec{w}\rangle + \langle A(t\vec{w}), \vec{v}\rangle + t^2 Q(\vec{w})\big) - Q(\vec{v})}{t} \\ &= \lim_{t\to 0} \frac{t\big(\langle A\vec{v}, \vec{w}\rangle + \langle A\vec{w}, \vec{v}\rangle\big) + t^2 Q(\vec{w})}{t} = \lim_{t\to 0} \big(\langle A\vec{v}, \vec{w}\rangle + \langle \vec{v}, A\vec{w}\rangle + tQ(\vec{w})\big) \\ &= \langle A\vec{v}, \vec{w}\rangle + \langle A\vec{v}, \vec{w}\rangle = 2\langle A\vec{v}, \vec{w}\rangle. \end{aligned}$$

Consider the set $S^1 = \{\vec{v} \in V \mid \langle \vec{v}, \vec{v}\rangle = 1\}$. This set is closed and bounded, and therefore, by the Extreme Value Theorem, Q attains both a maximum λ_1 and a minimum λ_2 on S^1, and these are both real. For $i = 1, 2$, let $\vec{v}_i \in S^1$ be a point such that $Q(\vec{v}_i) = \lambda_i$. Then, because λ_i is an extreme value of Q, $\vec{v}_i$ must be a critical point of Q, and so $dQ_{\vec{v}_i}$ must vanish on $T_{\vec{v}_i}S^1$. (You may know this from the "method of Lagrange multipliers".) We know that $T_{\vec{v}_i}S^1 = \vec{v}_i^{\perp}$, and so for any $\vec{w} \in \vec{v}_i^{\perp}$, we have $0 = dQ_{\vec{v}_i}(\vec{w}) = 2\langle A\vec{v}_i, \vec{w}\rangle$. But this implies that $A\vec{v}_i$ is orthogonal to every element of $\vec{v}_i^{\perp}$, which means it must be a multiple of $\vec{v}_i$. Since $\lambda_i = Q(\vec{v}_i) = \langle A\vec{v}_i, \vec{v}_i\rangle$ and $\vec{v}_i$ is a unit vector, we conclude that $A\vec{v}_i = \lambda_i \vec{v}_i$. Therefore λ_1 and λ_2 are the eigenvalues of A (because it can have at most two).

By the lemma, any vector orthogonal to an eigenvector of A is itself an eigenvector of A. Thus if $\lambda_1 \neq \lambda_2$, their corresponding eigenspaces are orthogonal. □

EXISTENCE AND UNIQUENESS RESULTS FOR INITIAL VALUE PROBLEMS

We have invoked the theory of ordinary differential equations several times during the semester, and so it seems worthwhile to at least mention what results we are using and why they should be true. These notes will not provide complete proofs, but they will suggest what tools are necessary. We will also use very geometric language, as is appropriate for our immediate interests.

We consider curves in $\mathbb{R}^n$: recall that a C^k curve is a map $\gamma : I \to \mathbb{R}^n$, with $I \subseteq \mathbb{R}$ an interval, whose component functions are k times continuously differentiable. For most of our applications, and to convey the main notions, it is enough to consider curves defined on $[0, 1]$.

The first new idea we need is that of "distance" between curves. There are several ways to define such a distance; we will use the following. If γ_1 and γ_2 are two curves from $[0, 1]$ to $\mathbb{R}^n$, then their **distance** is

$$d(\gamma_1, \gamma_2) = \max_{t\in[0,1]} \|\gamma_1(t) - \gamma_2(t)\|,$$

where $\|\cdot\|$ is the usual distance in $\mathbb{R}^n$. This distance measures the *farthest* two curves are apart at any time, and so, in particular, it is zero if and only if the two curves are the same map (not just that they have the same locus).

Now we recall what it means to "solve" a differential equation. A C^k **vector field** on $\mathbb{R}^n$ is a C^k function $\vec{\xi} : \mathbb{R}^n \to \mathbb{R}^n$. Given such a vector field and a point $p \in \mathbb{R}^n$, we look for a curve $\gamma : [0, 1] \to \mathbb{R}^n$ that satisfies

$$\gamma'(t) = \vec{\xi}(\gamma(t)) \quad \text{for all } t, \qquad \gamma(0) = p.$$

That is, the field $\vec{\xi}$ specifies the tangent vectors to γ, and p provides an **initial condition**. A slightly more general situation considers **time-dependent vector fields**, which are maps $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, where the first coordinate (in $\mathbb{R}$) is thought of as "time". This actually fits better into the first context in which we encounter differential equations—namely, techniques of integration in one-variable calculus—because now we are considering initial value problems of the form

$$\gamma'(t) = \vec{\xi}(t, \gamma(t)), \qquad \gamma(0) = p;$$

when attempting to solve an equation of the form $y' = f(x)$, we are essentially solving a time-dependent differential equation where the change (i.e., the vector field $\mathbb{R} \to \mathbb{R}$) *only* depends on the "time" x, not on the current "position" y. In this case, the question of existence and uniqueness reduces to the Fundamental Theorem of Calculus and the Mean Value Theorem. Although dealing with time-dependent vector fields does not introduce any new technical difficulties, for conceptual simplicity in the rest of these notes we will restrict ourselves to vector fields that only depend on position.

Existence and Uniqueness Theorem. *Let $\vec{\xi}$ be a C^1 vector field on $\mathbb{R}^n$ such that $\|\vec{\xi}\|$ is bounded. Then for every $p \in \mathbb{R}^n$, there exists a unique curve $\gamma : [0, 1] \to \mathbb{R}^n$ such that $\gamma(0) = p$ and $\gamma'(t) = \vec{\xi}(\gamma(t))$.*

This is not the most general form of the theorem, nor is it quite sufficient for every application we need (in particular, we do not always consider vector fields defined on all of $\mathbb{R}^n$), but it does capture the essentials, and it is easy to to make quantitative adjustments that fit our needs. The proof of the theorem is constructive, in the sense that for any $\varepsilon > 0$ we can find a curve within ε of a true solution. The idea is to start with a guess, then adjust the guess so that it more closely hews to the given vector field.

First we observe that, if γ is a true solution to the initial value problem, then

$$\gamma(t) = p + \int_0^t \vec{\xi}(\gamma(s))\, ds.$$

(Here s is not meant to denote arc length, but just to be an intermediate variable.) The two appearances of γ suggest an iterative procedure. Given a curve $\eta : [0,1] \to \mathbb{R}^n$, we construct a new curve $P(\eta) : [0,1] \to \mathbb{R}^n$ as follows:

$$P(\eta) = p + \int_0^t \vec{\xi}(\eta(s))\, ds.$$

We take as a first guess the line through p with direction vector $\vec{\xi}(p)$:

$$\gamma_1(t) = p + t\,\vec{\xi}(p).$$

Now, given a guess γ_i, we apply P to get γ_{i+1}:

$$\gamma_{i+1} = P(\gamma_i) = p + \int_0^t \vec{\xi}(\gamma_i(s))\, ds.$$

That is, we use the vectors along γ_i to dictate the direction of γ_{i+1}. This algorithm is known as **Picard iteration**. The key point is that P *contracts the distance between curves*. If $\gamma, \eta : [0,1] \to \mathbb{R}^n$ are two curves, then

$$d(P(\gamma), P(\eta)) < C \cdot d(\gamma, \eta),$$

where $C < 1$ is a constant depending on how large $\|\vec{\xi}\|$ and $\|d\vec{\xi}\|$ get. This reduces the question of existence and uniqueness of solutions to a problem about contraction mappings on abstract metric spaces. This is not difficult, but it is outside of our purview.

For second-order differential equations of the form

$$\gamma''(t) = \mathbf{F}(\gamma(t), \gamma'(t))$$

(where $\mathbf{F}$ is a function $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$), there is a standard trick to reduce to the earlier case: define another curve $\eta = \gamma'$ in $\mathbb{R}^n$, set $\tilde{\gamma}(t) = (\gamma(t), \eta(t))$, and use the following first-order system for a curve in $\mathbb{R}^{2n}$:

$$\tilde{\gamma}'(t) = \begin{bmatrix} \gamma'(t) \\ \eta'(t) \end{bmatrix} = \begin{bmatrix} \eta(t) \\ \mathbf{F}(\gamma(t), \eta(t)) \end{bmatrix}.$$

Now the second "coordinate" of $\tilde{\gamma}$ is simply keeping track of the tangent vectors γ'. This system requires *two* pieces of initial data in $\mathbb{R}^n$: a starting point p and a starting vector $\vec{v}$:

$$\gamma(0) = p, \qquad \eta(0) = \gamma'(0) = \vec{v}.$$

Given these initial data, the result for first-order differential equations now also shows the existence of a unique solution to our second-order differential equation, under some appropriate smoothness and boundedness assumptions on $\mathbf{F}$.

Old homework assignments

February 1-3: curves in R^n, arclength, curvature. Read B&L 1.1-1.3 and 3.1; S1.1.
Homework due Tuesday February 8:

- B&L 1.1.3, 1.1.5, 1.2.1(a)(b), 1.2.4, 1.2.9, 1.2.10, 1.2.13 (note that b < 0), 1.3.1, 1.3.10, 1.3.13 pdf
- S 1.1: 2, 4, 8, 11, **Bonus:** 9, 12

February 8-10: torsion, Frenet frame, fundamental theorem of curves. Read B&L 1.4-1.5 and 3.2-3.4; S1.2.
Homework due Tuesday February 15:

- B&L 1.4.2, 1.4.3, 3.2.1(b)(c)(d), 3.2.5, 3.2.6, 3.2.9, 3.2.10 (cf. #19 in Shifrin 1.2), 3.3.2 pdf
- S 1.2: 3(a)(b)(c)(g), 4, 9, 14; A.1 (in the Appendix): 4
- LaTeX assignment: make sure you have LaTeX installed on your computer. (Installation for Macs, for Windows, general help)
  Download the "Short Course" from this website or this link.
  You will need it for later assignments, but for now you can just browse through it. (Appendix A has installation advice, if you need it.)

  Download the following file: 362latex1.tex Read the comments and make sure you can compile this source file as it is.
  The result should look something like this (with a different date): 362latex1.pdf
  Change the filename to include your initials at the end. Then change my name to yours, change the functions, compile the new source, and email the pdf to me.

February 15-17: applications of Green's and Stokes' theorems, global properties. Read B&L 2.1-2.2 and 4.1-4.3; S1.3 through Theorem 3.5.
Homework due Tuesday February 22:

- B&L: 2.1.1, 2.2.4, 3.2.2, 4.1.1, 4.2.1, 4.2.2, 4.3.2 pdf
- S 1.3: 2, 3, 9, **Bonus:** 6, 13
- LaTeX assignment: Download the following file: 362latex2.tex
  Again, change the filename to include your initials at the end, and change my name to yours.
  Follow the directions in the comments, and when you have finished, compile the source and send the pdf version to me.

February 22: isoperimetric inequality, four-vertex theorem. Read B&L 2.3-2.4; remainder of S1.3
No homework this week.

First exam: February 24 (in class)

March 1-3: introduction to surfaces. Read B&L 5.1-5.3; S2.1 through Example 3.
Homework due Tuesday March 8:

- B&L 5.1.2 (cf. 5.2.13), 5.1.5 (don't work harder than you have to), 5.1.6, 5.2.7, 5.2.12, 5.2.15, 5.3.2, 5.3.3
- Additional exercise (in connection with 5.2.15 above): pdf tex
- LaTeX assignment: Read pages 7-22 in the Short course from Grätzer's *More Math Into LaTeX* (chapter 2 and sections 3.1 and 3.2).
  Create the files as instructed in the text, and follow Grätzer's suggestions to learn what kinds of errors can appear.
  After you have performed all of the "experiments", email me the files note1b.tex, note1b.pdf, mathb.tex, and mathb.pdf, with your initials appended to the filenames.

March 8-10: tangent space, normal vector, first fundamental form. Read B&L 5.2, 5.4, and 6.1; remainder of S2.1.
Homework due Tuesday March 15:

- B&L 5.2.2, 5.2.8(a)(b), 5.4.1, 5.4.6, 5.4.7, 6.1.6, 6.1.15
- S 2.1 3(b)(c), 4, 8 (in 4(c), you may use the parametrization we used in class, which switches the roles of *u* and *v*)
- LaTeX assignment: Read sections 3.3 and 3.4 of Grätzer's "Short course".
  You do not have to work through all of the examples, but you should practice a few to get the hang of the various math commands.

  Download the following file, which contains some of my standard tips and tricks: 362latex3.tex
  Compile it *once*, then check the resulting pdf file for double question marks (**??**). Compile it again to see the effect of labeling and referencing.
  Comment out or delete everything between \maketitle and \end{document} in the file. Change my name to yours and append your initials to the filename.
  Typeset the paragraph following Example 5 on page 39 of Shifrin's notes, beginning with "We saw in Chapter 1..." and ending before "Suppose *M* and *M** are surfaces", as best as you can.
  Email me the .tex and .pdf files.

March 15-17: orientability, Gauss map, second fundamental form. Read B&L 5.5 and 6.2-6.3; S2.2 through Proposition 2.2.
Homework due Tuesday March 22:

- B&L 6.1.14, 6.2.1(b)(c)(d), 6.3.2(b) ff., 6.3.6, 6.3.7, 6.3.8 (To answer 6.3.7 and 6.3.8, you will need to read Definition 6.3.4; in 6.3.8 you should assume that the space curve has nonzero torsion.)
  For 6.1.14(b), include a computer-produced image of the normal variation of the sphere, using Grapher (for Mac), the WWW Interactive Multipurpose Server (website), or some other graphing calculator.
- S 2.2: 3(b)(c)(d)
- Additional exercise: find two parametrizations $\vec{X}$ and $\vec{Y}$ from the same domain U into R^3 that yield the same first fundamental form but different second fundamental forms.
- LaTeX assignment: Choose one homework exercise assigned this week to typeset in LaTeX, along with its solution.
  You may use the template provided last week, or a new document you create, or any other template you find.
  Call the file 362hw0322-XX.tex, where "XX" is replaced with your initials.
  When you are done with the exercise, email the .pdf file to Raquel, and email the .tex and .pdf files to me.

March 22: normal curvature, principal curvatures.
No homework this week.

Second exam: March 24 (in class)

March 29-31: Dupin indicatrix, asymptotic directions, Gaussian and mean curvatures. Read B&L 6.4-6.6; remainder of S2.2.
Homework due Tuesday April 5:

- B&L 6.4.2, 6.4.9, 6.5.7, 6.5.9
- S 2.2: 5, 6, 7, 8, 9(a)(b), 13, 16, 17 **Bonus:** 9(c)
- Project assignment: Provide a one-page summary of the topic you intend to cover in your paper.
  This summary should be typeset in LaTeX and include a list of at least two sources (published books

or articles, not from Wikipedia, MathWorld, PlanetMath, etc.).

April 5-7: Christoffel symbols, Gauss and Codazzi equations, Theorema egregium. Read B&L 7.2-7.3; S2.3 through page 61.
Homework due Tuesday April 12:

- B&L 7.2.2, 7.2.3, 7.2.7, 7.2.8, 7.3.3, 7.3.8
- S 2.3: 4, 5(a)(b)(c) (only check the Gauss and Codazzi equations for (b)), 16(a)-(d) **Bonus:** 16(e) (needs complex analysis)

April 12-14: Fundamental theorem of surfaces, covariant differentiation. Read B&L 7.4, 7.1; remainder of S2.3; S2.4 through Example 2.
Homework due Tuesday April 26:

- B&L 7.1.1, 7.1.5, 7.1.13(a)(b)(c) **Bonus:** 7.4.1 (needs differential equations)
- S 2.3: 7, 8, 14
- Project assignment: Prepare a first draft of your paper to be given to me and at least one other student by April 28.

Spring break: April 18-24

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back to MAT 362 main page

MAT 362 Differential Geometry
Exam 1 Solutions

1. (a) Let $k \geq 1, n \geq 1$ be integers. Show that the Lissajous curve

$$\gamma(t) = (\cos kt, \sin nt), \qquad 0 \leq t \leq 2\pi,$$

is regular if and only if k is odd.

For γ to be regular, we must have $\gamma'(t) \neq (0,0)$ for all t.

$$\gamma'(t) = (-k \sin kt, n \cos nt)$$

The first coordinate is zero if and only if kt is a multiple of π, say $kt = N\pi$, or $t = N\pi/k$. For the second coordinate to be zero, we must then have $nt = nN\pi/k$ be an odd multiple of $\pi/2$, or $2nN\pi/k$ be an odd multiple of π. Assume that k and n have no factors other than 1 in common (this should have been included in the problem's statement). If k is odd, then $2nN/k$ is never odd, and γ is therefore regular. If k is even, then n must be odd, and so $nN/(k/2)$ is odd whenever N is an odd multiple of $k/2$; therefore γ is not regular.

(b) Show that the locus of $\gamma(t) = (\cos 2t, \sin t)$ lies on a parabola. Find the equation of the parabola. Sketch the locus.

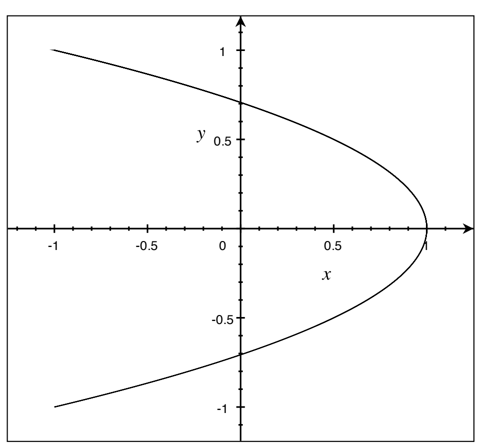
Using the double-angle formula for cosine, we have

$$\cos 2t = \cos^2 t - \sin^2 t = 1 - 2\sin^2 t.$$

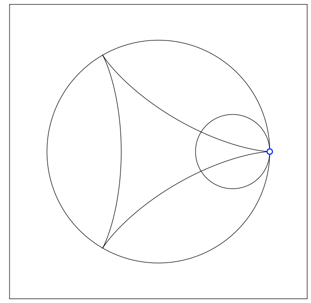
Thus the locus lies on the parabola with equation

$$x = 1 - 2y^2.$$

Its endpoints are at $(-1, 1)$ and $(-1, -1)$, as shown here:



2. (a) A *deltoid*, shown at right, is traced out by a point on the edge of a circle rolling along the inside of a circle with radius 3 times as large. Assume the inner (rolling) circle has radius 1, so that the outer (fixed) circle has radius 3, and find a parametrization for the deltoid. (*Hint:* What path does the center of the inner circle follow?)



The center of the inner circle traverses a circle of radius 2 centered at the origin. A point on its circumference turns clockwise twice as the inner circle rolls around three times its circumference, and with the normalization given by the above picture, the vector from the inner circle's center to the marked point starts at $(1, 0)$. A parametrization is given by

$$\gamma(t) = (2\cos t + \cos 2t,\ 2\sin t - \sin 2t)\,.$$

(b) Notice that the deltoid is regular except at its cusps. Assume that the deltoid has its standard (counterclockwise) orientation, and let κ_g denote the signed curvature of the deltoid, where it is defined. Explain why the integral of κ_g with respect to arc length over one-third of the deltoid (from cusp to cusp) must be $-\pi/3$. (You should not need to compute the integral.)

The integral of κ_g along a portion of the curve is the total angle that the unit vector $\vec{T}$ turns along that portion. Because

$$\gamma'(t) = (-2\sin t - 2\sin 2t,\ 2\cos t - 2\cos 2t)$$

and

$$\lim_{t\to 0} \frac{2\cos t - 2\cos 2t}{-2\sin t - 2\sin 2t} = 2\lim_{t\to 0} \frac{-\sin t + 2\sin 2t}{-\cos t - 2\cos 2t} = 0$$

(by L'Hôpital's rule), $\vec{T}(0)$ is horizontal, pointing to the left (i.e., $\vec{T}(0) = (-1, 0)$). By symmetry of the deltoid, along one third of the deltoid $\vec{T}$ turns from pointing left (in the negative x-direction) to pointing up and to the left at an angle of $2\pi/3$ from the positive x-direction. It has therefore turned through an angle of $-\pi/3$.

3. Show that the curve

$$\gamma(t) = \left(\frac{1}{2}(\sin t + \cos t),\ \frac{1}{2}(\sin t - \cos t),\ \frac{\sqrt{2}}{2}t\right), \qquad -\infty < t < \infty,$$

is parametrized by arc length. Compute its Frenet frame at $t = \dfrac{\pi}{2}$.

We have

$$\gamma'(t) = \left(\frac{1}{2}(\cos t - \sin t),\ \frac{1}{2}(\cos t + \sin t),\ \frac{\sqrt{2}}{2}\right),$$

which implies

$$\begin{aligned}
\|\gamma'(t)\|^2 &= \frac{1}{4}\big((\cos t - \sin t)^2 + (\cos t + \sin t)^2\big) + \left(\frac{\sqrt{2}}{2}\right)^2 \\
&= \frac{1}{4}(\cos^2 t - 2\cos t \sin t + \sin^2 t + \cos^2 t + 2\cos t \sin t + \sin^2 t) + \frac{1}{2} \\
&= \frac{1}{4}(2\cos^2 t + 2\sin^2 t) + \frac{1}{2} = 1.
\end{aligned}$$

This shows γ is parametrized by arc length. Hence $\vec{T}(t) = \gamma'(t)$ for all t. Thus $\vec{P}(t) = \gamma''(t)/\|\gamma''(t)\|$ for all t. We find

$$\gamma''(t) = \left(\frac{1}{2}(-\sin t - \cos t),\ \frac{1}{2}(-\sin t + \cos t),\ 0\right),$$

and so

$$\gamma'(\pi/2) = \left(\frac{1}{2}(\cos \pi/2 - \sin \pi/2), \frac{1}{2}(\cos \pi/2 + \sin \pi/2), \frac{\sqrt{2}}{2}\right) = \left(-\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right),$$

$$\gamma''(\pi/2) = \left(\frac{1}{2}(-\sin \pi/2 - \cos \pi/2),\ \frac{1}{2}(-\sin \pi/2 + \cos \pi/2),\ 0\right) = \left(-\frac{1}{2}, -\frac{1}{2}, 0\right).$$

This implies

$$\vec{T}(\pi/2) = \left(-\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right), \qquad \vec{P}(\pi/2) = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\right).$$

It only remains to find $\vec{B}(\pi/2)$, which is the cross-product of these two:

$$\vec{B}(\pi/2) = \vec{T}(\pi/2) \times \vec{P}(\pi/2) = \left(\frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{2}}{2}\right).$$

4. Let $\gamma : I \to \mathbb{R}^3$ be a C^3 curve parametrized by arc length, with curvature function $\kappa > 0$ and torsion function $\tau > 0$. Set $R(s) = 1/\kappa(s)$. Suppose the locus of γ is contained in a sphere centered at the origin. Prove that

$$\big(R(s)\big)^2 + \left(\frac{R'(s)}{\tau(s)}\right)^2 \qquad \text{is constant.}$$

(*Hint:* Differentiate the equation " $\|\gamma(s)\|^2 = \text{constant}$ " three times, and use the results to express γ as $f\,\vec{T} + g\,\vec{P} + h\,\vec{B}$ for some continuous functions f, g, h. Keep in mind that at all times the Frenet frame is an orthonormal basis of $\mathbb{R}^3$.)

From the equation $\langle \gamma(s), \gamma(s) \rangle = \text{constant}$ and the assumption that γ is parametrized by arc length, we have (by differentiating twice)

$$\langle \gamma(s), \vec{T}(s) \rangle = 0,$$
$$\langle \gamma(s), \vec{T}'(s) \rangle + \langle \vec{T}(s), \vec{T}(s) \rangle = 0.$$

By the first of the Frenet–Serret equations, this second equation becomes

$$\langle \gamma(s), \kappa(s)\vec{P}(s) \rangle + 1 = 0,$$

which implies

$$\langle \gamma(s), \vec{P}(s) \rangle = -\frac{1}{\kappa(s)} = -R(s).$$

Differentiating the next-to-last equation once more, we have

$$\langle \vec{T}(s), \kappa(s)\vec{P}(s) \rangle + \langle \gamma(s), \kappa'(s)\vec{P}(s) + \kappa(s)\vec{P}'(s) \rangle = 0.$$

The first term is zero because $\vec{T}$ and $\vec{P}$ are orthogonal, and by the second Frenet–Serret equation, the second term expands out to

$$\langle \gamma(s), \kappa'(s)\vec{P}(s) \rangle + \langle \gamma(s), -(\kappa(s))^2\vec{T}(s) \rangle + \langle \gamma(s), \kappa(s)\tau(s)\vec{B}(s) \rangle.$$

We have seen previously that the middle term is zero and the first term is $-\kappa'(s)/\kappa(s)$. This implies that

$$\langle \gamma(s), \vec{B}(s) \rangle = \frac{\kappa'(s)}{(\kappa(s))^2\tau(s)} = -\frac{R'(s)}{\tau(s)}.$$

Because the Frenet frame is orthonormal, the inner product of γ with each of the vectors $\vec{T}, \vec{P}, \vec{B}$ yields its coordinate in the basis they provide, so

$$\gamma(s) = -R(s)\,\vec{P}(s) - \frac{R'(s)}{\tau(s)}\,\vec{B}(s).$$

Moreover, the length of γ, which is constant, is obtained directly from these coefficients:

$$\text{constant} = \|\gamma(s)\|^2 = \big(R(s)\big)^2 + \left(\frac{R'(s)}{\tau(s)}\right)^2.$$

5. (a) Below, draw a closed curve whose winding number around P is 2, around Q is 1, and around R is -1. (Draw at least one arrow or tangent vector to indicate the orientation of your curve.)

$\bullet\ P$ $\qquad\qquad$ $\bullet\ Q$ $\qquad\qquad$ $\bullet\ R$

(b) Let γ be a closed, regular C^2 curve in $\mathbb{R}^2$. Assume that its (unsigned) curvature κ satisfies $0 \leq \kappa(t) \leq 1/R$ for some constant $R > 0$. Prove that, if γ has rotation index N, then

$$\text{length}(\gamma) \geq 2\pi NR.$$

Let κ_g be the signed curvature of γ. Then, using the rotation index formula,

$$N = \frac{1}{2\pi}\int_\gamma \kappa_g(s)\,ds.$$

Taking absolute values, we obtain

$$2\pi N = \left|\int_\gamma \kappa_g(s)\,ds\right| \leq \int_\gamma |\kappa_g(s)|\,ds = \int_\gamma \kappa(s)\,ds.$$

By assumption, $\kappa(s) \leq 1/R$ for some $R > 0$, which gives the inequality

$$2\pi N \leq \int_\gamma \frac{ds}{R} = \frac{1}{R}\int_\gamma ds = \frac{1}{R}\,\text{length}(\gamma),$$

which becomes the desired inequality after we multiple both sides by R.

MAT 362 Differential Geometry
Exam 2 Solutions

1. For each of the sets described below, write a "Y" in the blank if it is a regular surface; write an "N" if it is not. You are not required to give any reasons for your answer. Illegible letters will be graded as incorrect.

   $\underline{\text{Y}}$ $\{(x, y, z) \in \mathbb{R}^3 \mid x^3 + y^3 - z^3 = 1\}$

   $\underline{\text{N}}$ $\{(x, y, z) \in \mathbb{R}^3 \mid xyz = 0\}$

   $\underline{\text{N}}$ The image of the map $\vec{X}(u, v) = \big((1 + \cos u) \cos v, (1 + \cos u) \sin v, \sin u\big)$, $0 \le u \le 2\pi$, $0 \le v \le 2\pi$.

   $\underline{\text{Y}}$ The image of the map $\vec{X}(u, v) = (\cos u \sin v, 2 \sin u \sin v, 3 \cos v)$, $0 \le u \le 2\pi$, $0 \le v \le \pi$.

   $\underline{\text{Y}}$ The union of the lines $\ell(t)$, $-\infty < t < \infty$, where $\ell(t)$ joins $(0, 0, t)$ and $(1, t, 0)$.

2. For each of the statements below, write a "T" in the blank if the statement is true; write an "F" if the statement is false. You are not required to give any reasons for your answer. Illegible letters will be graded as incorrect.

   $\underline{\text{F}}$ If U is a bounded subset of $\mathbb{R}^2$, then no map from $\mathbb{R}^2$ to U can be a diffeomorphism.

   For the remaining statements, assume that $S \subset \mathbb{R}^3$ is a C^2 regular surface and $p \in S$.

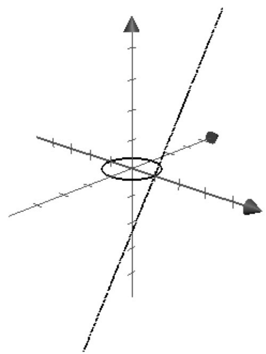
   $\underline{\text{T}}$ The first and second fundamental forms are both symmetric bilinear forms on T_pS.

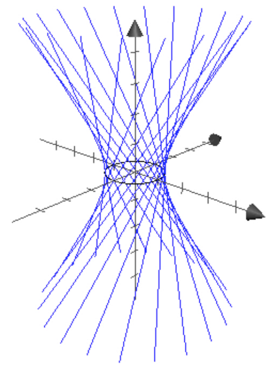
   $\underline{\text{F}}$ The first and second fundamental forms are both positive definite on T_pS.

   $\underline{\text{F}}$ If the shape operator S_p is a multiple of the identity, then there is an open set $U \subset \mathbb{R}^3$ containing p such that $S \cap U$ lies entirely on one side of T_pS.

   $\underline{\text{F}}$ If $\gamma : (-\varepsilon, \varepsilon) \to S$ is a C^2 regular curve such that $\gamma(0) = p$, then the curvature $\kappa(0)$ of γ at p is bounded by the principal curvatures of S at p.

3. (a) For each point $(\cos\theta, \sin\theta, 0)$ of the unit circle in the (x, y)-plane, consider the line through that point in the direction of the vector $(-\sin\theta, \cos\theta, 1)$.





Find a parameterization of the surface S covered by these lines for which the lines are coordinate curves. Show that each point $(x, y, z) \in S$ satisfies $x^2+y^2-z^2 = 1$.

To each point $(\cos u, \sin u, 0)$, we add all scalar multiples of $(-\sin u, \cos u, 1)$, so a parametrization is

$$\vec{X}(u, v) = (\cos u - v\sin u, \sin u + v\cos u, v).$$

The domain can be taken as $0 \le u \le 2\pi$, $-\infty < v < \infty$.
For each point in the image of $\vec{X}$, we have

$$\begin{aligned}
&(\cos u - v\sin u)^2 + (\sin u + v\cos u)^2 - v^2 \\
&= \cos^2 u - 2v\cos u\sin u + v^2\sin^2 u + \sin^2 u + 2v\cos u\sin u + v^2\cos^2 u - v^2 \\
&= \cos^2 u + \sin^2 u + v^2(\sin^2 u + \cos^2 u) - v^2 = 1,
\end{aligned}$$

which proves the final statement.

(b) Find the tangent space $T_{p_\theta}S$ at each point $p_\theta = (\cos\theta, \sin\theta, 0)$ in the (x, y)-plane, and show that it is parallel to the z-axis.

We present two ways of finding a normal vector to $T_{p_\theta}S$.
First, using the result of (a) we have

$$\vec{X}_u = (-\sin u - v\cos u, \cos u - v\sin u, 0), \qquad \vec{X}_v = (-\sin u, \cos u, 1).$$

At $(u, v) = (\theta, 0)$, this leads to

$$\vec{X}_u(\theta, 0) \times \vec{X}_v(\theta, 0) = (\cos\theta, \sin\theta, 0).$$

Second, because S is defined by $f(x, y, z) = 1$, where $f(x, y, z) = x^2 + y^2 - z^2$, the tangent space at p_θ has as a normal vector

$$\nabla f(\cos\theta, \sin\theta, 0) = (2\cos\theta, 2\sin\theta, 0).$$

Either normal vector is orthogonal to $(0, 0, 1)$, so the tangent plane is parallel to the z-axis.

(c) Find the image of the Gauss map $S \to S^2$. (It does not matter which unit normal vector you choose in this case.)

Let us take the "outward pointing" normal. Start with the hyperbola $x^2 - z^2 = 1$. This has $(1,0,0)$ as a normal vector at $(1,0,0)$, and as a point moves along the hyperbola, the normal vector approaches the direction orthogonal to the asymptote, at angle $\pm\pi/4$. Because S is obtained by revolution of this hyperbola, the Gauss map covers the area of S^2 (strictly) between the circles at $\pm\pi/4$ latitude.

(d) Show that every point $p_\theta = (\cos\theta, \sin\theta, 0) \in S$ is hyperbolic. (*Hint*: There are several ways to do this, most of which do not involve computing the full shape operator.)

We may compute from the parametrization obtained in (a):

$$\begin{aligned}
\vec{X}_{uu} &= (-\cos u + v\sin u, -\sin u - v\cos u, 0),\\
\vec{X}_{uv} &= (-\cos u, -\sin u, 0),\\
\vec{X}_{vv} &= (0,0,0)
\end{aligned}$$

When $v = 0$, the normal vector (computed in (b)) is $\vec{N} = (\cos u, \sin u, 0)$, and so

$$\begin{aligned}
L_{11}(u,0) &= \langle \vec{N}, \vec{X}_{uu}\rangle = -\cos^2 u - \sin^2 u = -1\\
L_{12}(u,0) = L_{21}(u,0) &= \langle \vec{N}, \vec{X}_{uv}\rangle = -\cos^2 u - \sin^2 u = -1\\
L_{22}(u,0) &= \langle \vec{N}, \vec{X}_{vv}\rangle = 0
\end{aligned}$$

Therefore the determinant of $\left[L_{ij}\right]$ is $(-1)(0) - (-1)^2 = -1$. This shows that every point of S in the (x,y)-plane is hyperbolic.

Alternatively, we could consider the curves given by intersecting S with the (x,y)-plane and with a vertical normal plane, giving a unit circle (having curvature 1) and a hyperbola. The hyperbola is parametrized by $(\cosh t, 0, \sinh t)$ in the (x,z)-plane, for instance, and its curvature is therefore

$$\frac{\|(\sinh t, 0, \cosh t) \times (\cosh t, 0, \sinh t)\|}{(\sinh^2 t + \cosh^2 t)^{3/2}} = \frac{1}{(\sinh^2 t + \cosh^2 t)^{3/2}},$$

which is 1 at $t = 0$. Because both of these curvatures are non-zero and their principal normal vectors point in opposite directions, the points p_θ must be hyperbolic.

As we shall see next week, the fact that every point of S has two straight lines and a curve of non-zero normal curvature passing through it implies that every point of S (on or off the (x,y)-plane) must be hyperbolic.

4. Let $\vec{X} : \mathbb{R}^2 \to S^2$ be the inverse of stereographic projection, i.e.,

$$\vec{X}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1} \right),$$

and let $\vec{Y} : [0, 2\pi] \times [0, \pi] \to S^2$ be the usual parametrization of S^2:

$$\vec{Y}(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi).$$

We have seen previously that the coefficients of the metric tensor arising from $\vec{X}$ are

$$g_{11} = g_{22} = \frac{4}{(u^2+v^2+1)^2}, \qquad g_{12} = g_{21} = 0,$$

and those arising from $\vec{Y}$ are

$$g_{11} = \sin^2\phi, \qquad g_{22} = 1, \qquad g_{12} = g_{21} = 0.$$

Let C be the cylinder $\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 = 1\}$, and let $F : S^2 \to C$ be the "axial projection"

$$F(x,y,z) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, z \right).$$

(a) What is the natural domain of $F \circ \vec{X}$? What is its image in C?

The natural domain of F is $\mathbb{R}^2 \setminus \{(0,0)\}$. Its image is the part of C contained (strictly) between the planes $z = \pm 1$.

(b) Compute the coefficients of the metric tensor arising from $F \circ \vec{Y}$.

We have

$$\begin{aligned}(F \circ \vec{Y})(\theta,\phi) &= \left(\frac{\cos\theta\sin\phi}{\sqrt{\cos^2\theta\sin^2\phi + \sin^2\phi\cos^2\phi}}, \frac{\sin\theta\sin\phi}{\sqrt{\cos^2\theta\sin^2\phi + \sin^2\phi\cos^2\phi}}, \cos\phi \right) \\ &= (\cos\theta, \sin\theta, \cos\phi).\end{aligned}$$

(Technically, we have used the fact that $\sin\phi \geq 0$ for $0 \leq \phi \leq \pi$.) Thus

$$(F \circ \vec{Y})_\theta = (-\sin\theta, \cos\theta, 0), \qquad (F \circ \vec{Y})_\phi = (0, 0, -\sin\phi).$$

This gives the coefficients

$$g_{11} = 1, \qquad g_{22} = \sin^2\phi, \qquad g_{12} = g_{21} = 0.$$

(c) Show that F is area-preserving (meaning that it does not change the determinant of the metric tensor).

The determinant of the metric tensor for both $\vec{Y}$ and $F \circ \vec{Y}$ is $\sin^2 \phi$. Therefore F is area-preserving.

(d) Use the fact that C is locally isometric to the plane $\mathbb{R}^2$ to compute the area of the unit sphere S^2.

Because the image of F covers C between the planes ± 1, the area of this image is $(2\pi)(2) = \boxed{4\pi}$. Because F is area-preserving and its domain only omits two points of the sphere, this is also the area of the unit sphere.

(e) Interpret the following integral geometrically, and evaluate it:

$$\iint_D \frac{dx\,dy}{(x^2+y^2+1)^2}, \qquad \text{where} \qquad D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}.$$

(*Hint*: You should not have to resort to formal techniques of integration.)

The integral represents one-fourth of the area of S^2 that maps to the unit disk via stereographic projection. This area is the lower hemisphere of S^2, which therefore equals $4\pi/2 = 2\pi$. One-fourth of this area is $\pi/2$.

MAT 362 Differential Geometry
Exam 3 Spring 2011 SOLUTIONS

1. Definitions. Let S be a regular surface in $\mathbb{R}^3$.

   (a) What does it mean for S to be *minimal?*

   The mean curvature of S is zero at every point.

   (b) What does it mean for a point of S to be *umbilical?*

   The principal curvatures at that point are equal.

   (c) What is a *vector field* on S?

   A function $\vec{\xi}: S \to \mathbb{R}^3$ such that $\vec{\xi}(p) \in T_pS$ for all $p \in S$.

   (d) If $\vec{X}$ is a parametrization of (a portion of) S, how are the *Christoffel symbols* Γ_{ij}^k defined?

   For $i, j, k \in \{1, 2\}$, the Christoffel symbols are the unique functions Γ_{ij}^k such that

   $$\vec{X}_{ij} = L_{ij}\vec{N} + \Gamma_{ij}^1\vec{X}_u + \Gamma_{ij}^2\vec{X}_v.$$

   This can also be written as

   $$\nabla_{\vec{X}_i}\vec{X}_j = \Gamma_{ij}^1\vec{X}_u + \Gamma_{ij}^2\vec{X}_v$$

   (e) Give at least one definition of a *geodesic* on S.

   A geodesic is a curve on S with zero geodesic curvature.
   A geodesic is a curve whose field of tangent vectors is parallel.
   A geodesic is a curve that locally minimizes distances between its points.

2. For each of the statements below, write a "T" in the blank if the statement is true; write an "F" if the statement is false. You are not required to give any reasons for your answer. Illegible letters will be graded as incorrect.

   Assume throughout this problem that S is a regular C^∞ surface in $\mathbb{R}^3$.

   __T__ If S is compact, then some point of S must have positive Gaussian curvature.

   __F__ If S is compact, then some point of S must have negative Gaussian curvature.

   __F__ If S is compact, then S cannot have any points with negative Gaussian curvature.

   __F__ If every point of S has positive Gaussian curvature, then S must be compact.

   __F__ If S has a parametrization $\vec{X}$ such that all its Christoffel symbols are zero, then the image of $\vec{X}$ is contained in a plane.

   __F__ No geodesic on S can intersect itself.

   __T__ If two simple closed geodesics on S form the boundary of a cylinder C in S, then the total Gaussian curvature of C must be zero.

   __T__ If S is compact and orientable, then its Euler characteristic is even.

3. Let $\vec{X}(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$ parametrize a surface of revolution generated by a curve $(f(t), 0, g(t))$ in the (x,z)-plane; assume this curve is parametrized by arc length, so that $(f')^2 + (g')^2 = 1$. Show that the Gaussian curvature at $\vec{X}(u,v)$ is

$$K(u,v) = \frac{g'(u)}{f(u)}\big(f'(u)g''(u) - f''(u)g'(u)\big).$$

The first derivatives of $\vec{X}$ are

$$\vec{X}_u = (f'(u)\cos v, f'(u)\sin v, g'(u)) \qquad \text{and}$$
$$\vec{X}_v = (-f(u)\sin v, f(u)\cos v, 0).$$

Observe that the unit normal at $\vec{X}(u,v)$ is a rotation of $\vec{X}_u$, namely

$$\vec{N}(u,v) = (g'(u)\cos v, g'(u)\sin v, -f'(u)),$$

and so

$$\vec{N}_u = (g''(u)\cos v, g''(u)\sin v, -f''(u)),$$
$$\vec{N}_v = (-g'(u)\sin v, g'(u)\cos v, -f'(u)).$$

Now we can compute

$$g_{11} = 1, \qquad g_{12} = 0, \qquad g_{22} = f^2,$$
$$L_{11} = f'g'' - g'f'', \qquad L_{12} = 0, \qquad L_{22} = fg'.$$

By the determinant formula for K, we have

$$K = \frac{\det[L_{ij}]}{\det[g_{ij}]} = \frac{fg'(f'g'' - g'f'')}{f^2} = \frac{g'}{f}(f'g'' - g'f''),$$

which is what we wanted.

Another method is to use the fact that $\vec{X}$ is orthogonal. Then, from the above computations, we have

$$g_{11}g_{22} = f^2, \qquad (g_{11})_v = 0, \qquad (g_{22})_u = 2ff',$$
$$\frac{\partial}{\partial v}\left(\frac{(g_{11})_v}{\sqrt{g_{11}g_{22}}}\right) = 0, \qquad \frac{\partial}{\partial u}\left(\frac{(g_{22})_u}{\sqrt{g_{11}g_{22}}}\right) = \frac{\partial}{\partial u}2f' = 2f'',$$
$$K = -\frac{1}{2\sqrt{g_{11}g_{22}}}\left[\frac{\partial}{\partial v}\left(\frac{(g_{11})_v}{\sqrt{g_{11}g_{22}}}\right) + \frac{\partial}{\partial u}\left(\frac{(g_{22})_u}{\sqrt{g_{11}g_{22}}}\right)\right] = -\frac{f''}{f}.$$

Now we just need to show that $-f'' = g'(f'g'' - f''g')$. By differentiating the condition $(f')^2 + (g')^2 = 1$, we get $f'f'' + g'g'' = 0$, or $g'g'' = -f'f''$. Thus

$$g'(f'g'' - f''g') = f'g'g'' - f''(g')^2 = -(f')^2f'' - (g')^2f'' = -f''.$$

4. Let H be the hyperboloid of one sheet, with equation $x^2 + y^2 - z^2 = 1$. Show that the unit circle in the (x, y)-plane is a geodesic on H. Show that every other geodesic on H is unbounded, i.e., has points arbitrarily far from the origin.

The unit circle is a geodesic because it is fixed by the isometry $(x, y, z) \mapsto (x, y, -z)$.

Recall that Clairaut's relation says that a geodesic on a surface of revolution satisfies the condition $r \cos\theta =$ constant, where r is the distance from the axis of revolution and θ is the angle formed with a parallel.

Now let γ be geodesic on H besides the unit circle in the (x, y)-plane. No other parallel can be a geodesic, so we conclude $\cos\theta \neq 1$ at some point of γ. Thus γ continues to extend in one direction farther from the (x, y)-plane, thus farther from the origin. Let C be the constant in Clairaut's relation. Because $r\cos\theta = C$ is constant, and r is increasing as γ moves away from the (x, y)-plane, $|\cos\theta|$ must be decreasing, i.e., the angle between γ and the parallels must be increasing. (If $\cos\theta = 0$, then γ is a meridian, i.e., a rotated copy of the hyperbola $x^2 - z^2 = 1$ in the (x, z)-plane, and we are done.) Thus γ continues to move away from the (x, y)-plane. Because it moves at constant speed, with increasing vertical component, it is unbounded.

5. Let γ be a smooth, simple closed curve on the unit sphere S^2. Assume that the total geodesic curvature of γ is zero. [Note: this is *not* the same as saying that γ is a geodesic.] Show that γ divides S^2 into two pieces of equal area.

Let R be one of the two topological disks with γ as boundary. Then, by the Gauss–Bonnet theorem,

$$2\pi = \int_\gamma \kappa_g \, ds + \iint_R K \, dA = 0 + \iint_R dA = \text{Area}(R),$$

using the fact that the curvature of the unit sphere is $K = 1$. Because this works for both "sides" of γ (also because the total area of S^2 is 4π), the result follows.

Visualizing Friendships

By Paul Butler on Monday, December 13, 2010 at 5:16pm

Visualizing data is like photography. Instead of starting with a blank canvas, you manipulate the lens used to present the data from a certain angle.

When the data is the social graph of 500 million people, there are a lot of lenses through which you can view it. One that piqued my curiosity was the locality of friendship. I was interested in seeing how geography and political borders affected where people lived relative to their friends. I wanted a visualization that would show which cities had a lot of friendships between them.

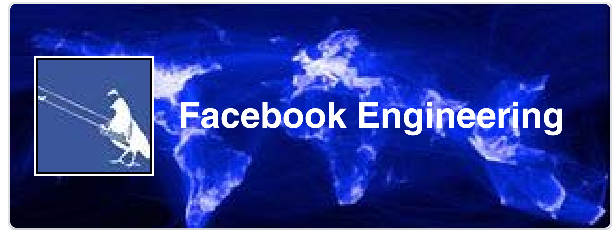
I began by taking a sample of about ten million pairs of friends from Apache Hive, our data warehouse. I combined that data with each user's current city and summed the number of friends between each pair of cities. Then I merged the data with the longitude and latitude of each city.

At that point, I began exploring it in R, an open-source statistics environment. As a sanity check, I plotted points at some of the latitude and longitude coordinates. To my relief, what I saw was roughly an outline of the world. Next I erased the dots and plotted lines between the points. After a few minutes of rendering, a big white blob appeared in the center of the map. Some of the outer edges of the blob vaguely resembled the continents, but it was clear that I had too much data to get interesting results just by drawing lines. I thought that making the lines semi-transparent would do the trick, but I quickly realized that my graphing environment couldn't handle enough shades of color for it to work the way I wanted.

Instead I found a way to simulate the effect I wanted. I defined weights for each pair of cities as a function of the Euclidean distance between them and the number of friends between them. Then I plotted lines between the pairs by weight, so that pairs of cities with the most friendships between them were drawn on top of the others. I used a color ramp from black to blue to white, with each line's color depending on its weight. I also transformed some of the lines to wrap around the image, rather than spanning more than halfway around the world.

View high-res (3.8MB)

After a few minutes of rendering, the new plot appeared, and I was a bit taken aback by what I saw. The blob had turned into a surprisingly detailed map of the world. Not only were continents visible, certain international borders were apparent as well. What really struck me, though, was knowing that the lines didn't represent coasts or rivers or political borders, but real human relationships. Each line might represent a friendship made while travelling, a family member abroad, or an old college friend pulled away by the various forces of life.



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Later I replaced the lines with great circle arcs, which are the shortest routes between two points on the Earth. Because the Earth is a sphere, these are often not straight lines on the projection.

When I shared the image with others within Facebook, it resonated with many people. It's not just a pretty picture, it's a reaffirmation of the impact we have in connecting people, even across oceans and borders.

Paul is an intern on Facebook's data infrastructure engineering team.