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MAT 351: Differential Equations: Dynamics & Chaos
Spring 2016
Schedule & Homework

Schedule

The PDF version of the schedule is available for print **here**.

Note: Z=Zhang, S=Strogatz, DHS=Devaney,Hirsch,Smale

Date	Topic	Notes	Assignments
Jan 26	Introduction: differential equations & dynamical systems		
Jan 28	First order autonomous equations Differential equations in dimension one: equilibrium & stability	**Z**3.1-3.2 **S**2.1-2.4	
Feb 2	Stability, Lyapunov function & examples	**S**2.4-2.7 Notes Bb	**HW1** (due Feb 11)
Feb 4	Existence & uniqueness of solutions Bifurcations, normal forms	**Z**3.2, **S**2.5	
Feb 9	Bifurcations: saddle-node, transcritical & examples	**Z**3.3, **S**3.1-3.2 Notes Bb	**HW2** (due Feb 18)
Feb 11	Bifurcations: transcritical, pitchfork, hysteresis	**S**3.3-3.4	
Feb 16	Dimension two: Linear systems	**Z**5, **S**5.1-5.2	
Feb 18	Classification of linear systems	**S**5.2, 6.1-6.2	**HW3** (due Feb 25)
Feb 23	Nonlinear systems: sinks, saddles, sources, stability, hyperbolicity Hartman-Grobman theorem; Examples	**S**6.3-6.5	
Feb 25	Stable/unstable manifolds, closed orbits, limit cycles An example of Hopf bifurcation	**S**7.1, 8.2	**HW4** (due Mar 8)
Mar 1	Conservative systems, energy and nonlinear centers	**S**6.5	
Mar 3	Gradient systems, Lyapunov functions and examples	**S**7.2, **Z**6.2	
Mar 8	Dulac's criterion, Bendixon's negative criterion	**S**7.1-7.3 **Z**6.3-6.4	**HW5** (due Mar 24)
Mar 10	Poincaré-Bendixon theorem	**Z**6.4-6.5	

Mar 15	*Spring break (no class)*		
Mar 17	*Spring break (no class)*		
Mar 22	Applications of Poincaré-Bendixon theorem	**S**7.3	
Mar 24	Bifurcations in two-dimensional systems	**S**8.1-8.2	**Practice problems**
Mar 29	Hopf bifurcations Review	**S**8.2-8.3	**Project Topics**
Mar 31	**Midterm (1:00-2:20pm, in class)** -- **Midterm**		
Apr 5	Hopf bifurcations; Examples	Notes Bb **DHS** Ch. 8	
Apr 7	Homoclinic bifurcations; Lorenz system	**S**8.4, **S**9.2	
Apr 12	Lorenz system & properties	**S**9.2, Notes Bb	
Apr 14	Dissipative systems, attractors, examples	**S**9.3, Notes Bb	**HW6** (due Apr 21)
Apr 19	Lorenz attractor Stable manifold of the origin: (**Video** & **Lorenz System Example** by **Alex Vladimirsky**)	**S**9.3	
Apr 21	A model for the Lorenz attractor Poincaré map	**DHS** Ch. 14 **Pictures**	
Apr 26	Chaotic attractor Reading (see Figures 6, 7): **A new twist in knot theory** Animation several trajectories (**Video**)	**DHS** Ch. 14 **Pictures**	**HW7** (due May 5)
Apr 28	Discrete dynamical systems Chaos	**S**10 **DHS** Ch. 15	
May 3	Discrete dynamical systems; Examples	**S**10 **DHS** Ch. 15	
May 5	Fractals and dimension Three-dimensional ODEs - Open Problems	**S**11	
May 16	**Projects -- due at 5:30pm in Math Tower 4-103**		

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About me

From 2013 to 2017 I was a Milnor Lecturer at the **Institute for Mathematical Sciences** at Stony Brook University. I got my Ph.D. in Mathematics from **Cornell University** in 2013, under the supervision of **John H. Hubbard**.

I started my undergraduate studies at the **University of Bucharest** and after one year I transfered to **Jacobs University Bremen**, where I earned my B.S. degree in Mathematics in 2007. I got a M.S. in Computer Science from Cornell University in 2012.

Research Interests

My interests are in the areas of Dynamical Systems (in one or several complex variables), Analysis, Topology and the interplay between these fields.

My research is focused on the study of complex Hénon maps, which are a special class of polynomial automorphisms of $\mathbb{C}^2$ with chaotic behavior. I am interested in understanding the global topology of the Julia sets J, J^- and J^+ of a complex Hénon map and the dynamics of maps with partially hyperbolic behavior such as holomorphic germs of diffeomorphisms of $(\mathbb{C}^n, 0)$ with semi-neutral fixed points. Some specific topics that I work on include: relative stability of semi-parabolic Hénon maps and connectivity of the Julia set J, regularity properties of the boundary of a Siegel disk of a semi-Siegel Hénon map, local structure of non-linearizable germs of diffeomorphisms of $(\mathbb{C}^n, 0)$.

Other activities

I was organizer for the **Dynamics Seminar** at Stony Brook University.
I have also developed projects for MEC (Math Explorer's Club): **Mathematics of Web Search** and **Billiards & Puzzles**.

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MAT 351: Differential Equations: Dynamics & Chaos
Spring 2016
Course Information

Synopsis

Dynamical systems occur in all branches of science, from the differential equations of classical mechanics in physics to the difference equations of mathematical economics and biology. This course is an introduction to the field of dynamical systems. It concerns the study of the long-term behavior of solutions to ordinary differential equations or of iterated mappings, emphasizing the distinction between stability on the one hand and sensitive dependence and chaotic behavior on the other. The course describes examples of chaotic behavior and of fractal attractors, and develops some mathematical tools for understanding them. In particular we will study the following key concepts: hyperbolicity, topological conjugacy, equilibrium, limit cycle, stability, chaos, etc.

Click here to download a copy of the course syllabus. Please visit the **course website on Blackboard** to see your grades.

Lectures

Tuesday & Thursday 1:00-2:20pm in Physics 116

Instructor

Remus Radu
Office: Math Tower 4-103
Office hours: TuTh 2:30-4:00pm in Math Tower 4-103, or by appointment

Teaching Assistant

Aleksandar Milivojevic
Office: MLC (Math Tower S-240A)
Office hours: Monday 10:00-11:00am & 1:00-2:00pm; Wednesday 10:30-11:30am in MLC

Textbook & recommended reading

- Steven Strogatz, *Nonlinear dynamics and Chaos: with applications to physics, biology, chemistry, and engineering*, **Westview press**.
- Wei-bin Zhang, *Differential equations, bifurcations, and Chaos in economics*, World Scientific 2005.
- Robert L. Devaney, Morris W. Hirsch, and Stephen Smale, *Differential Equations, Dynamical Systems, and an Introduction to Chaos*, 3rd ed., Elsevier Academic Press, 2012.
- Robert L. Devaney, *A First Course in Chaotic Dynamical Systems: Theory And Experiment*, Westview Press 1992.
- Clark Robinson, *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos*, 2nd ed., CRC Press. (more advanced)
- John Guckenheimer, Philip Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer 1983. (more advanced)

  *** We will follow the first three textbooks, but there will also be lecture notes posted on Blackboard. ***

Grading Policy

Grades will be computed using the following scheme:

- Homework – 30%
- Midterm – 35% (on **Thursday, March 31, 1:00-2:20pm**)
- Projects & presentation – 35%

Students are expected to attend class regularly and to keep up with the material presented in the lecture and the assigned reading.

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MAT 351 Differential Equations: Dynamics & Chaos
Spring 2016

Assignment 1

Due Thursday, **February 11**, in class.

Problem 1: Determine the equilibrium points of the following differential equations and discuss their stability.

a) $\dot{x} = 3x(1-x)$

b) $\dot{x} = \cos^2(x)$

c) $\dot{x} = r + x - x^3$, for various values of r. It may be useful to look at the Lyapunov function.

Problem 2: The growth of cancerous tumors can be modeled by the Gompertz law $\dot{N} = -aN\log(bN)$, where $N(t)$ is proportional to the number of cells in the tumor and $a, b > 0$ are parameters. Find the fixed points of this model and discuss their stability. Sketch the graph of the solution $N(t)$ based at $1/(2b)$.

Problem 3: Consider the equation $\dot{x} = rx + x^3$, where $r > 0$ is fixed. Show that $|x(t)| \to \infty$ in finite time, starting from any initial condition $x_0 \neq 0$.

Problem 4: Let p and q be positive integers with no common factors. Consider the initial value problem $\dot{x} = |x|^{p/q}$, $x(0) = 0$.

a) Show that there are an infinite number of solutions if $p < q$.

b) Show that there is a unique solution if $p > q$.

Problem 5: A solution $x(t)$ is a *periodic solution* of the differential equation $\dot{x} = f(x)$ if there exists $T > 0$ such that $x(t) = x(t+T)$ for all time t, but $x(t) \neq x(t+s)$ for all $0 < s < T$. Show that there are no periodic solutions to $\dot{x} = f(x)$ on the real line.

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Assignment 2

Due Thursday, **February 18**, in class.

Problem 1: Show that the following system

$$\dot{x} = \lambda + \frac{1}{2}x - \frac{x}{x+1}$$

undergoes a saddle-node bifurcation at a critical value of λ, to be determined. Sketch all the qualitatively different vector fields that occur as λ is varied. Sketch the bifurcation diagram of fixed points x^* versus λ.

Problem 2: Show that the system $\dot{x} = x(1-x^2) - 3(1-e^{-\lambda x})$ undergoes a transcritical bifurcation at $x = 0$. Find the critical value of λ for which this occurs. Find an approximate formula for the fixed point that bifurcates from $x = 0$.

Problem 3: For the following equations, find the value of λ at which bifurcations occur, and classify those as saddle-node, transcritical, pitchfork (supercritical or subcritical). Sketch the bifurcation diagram of x^* vs. λ.

a) $\dot{x} = \dfrac{\lambda - x^2}{1+x^2}x$

b) $\dot{x} = x + \tanh(\lambda x)$

Problem 4: Consider the system $\dot{x} = \lambda x - \sin(x)$, for $-4\pi \leq x \leq 4\pi$.

a) Show that for $\lambda > 1$ there is only one fixed point. Describe its stability.

b) Draw a phase portrait and a bifurcation diagram for $\frac{1}{2} \leq \lambda < \infty$. Indicate the stability of the various branches of fixed points.

c) What happens in the interval $0 < \lambda < \frac{1}{2}$? Classify all the bifurcations that occur. (You are not asked to find the exact value of λ at which bifurcations occur.)

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Assignment 3

Due Thursday, **February 25**, in class.

Problem 1: Sketch the vector field and some typical trajectories for the following linear systems. Determine whether the equilibrium is stable, asymptotically stable, or unstable.

a) $\dot{x} = x$, $\dot{y} = x + y$.

b) $\dot{x} = -x + y$, $\dot{y} = -5x + y$.

Problem 2: Sketch the phase portrait and classify the fixed point $x^* = 0$ of the following linear systems. Specify if the system is hyperbolic or not.

a) $\dot{x} = -3x + 2y$, $\dot{y} = x - 2y$.

b) $\dot{x} = y$, $\dot{y} = -x - ay$, where $-2 < a < 2$.

Problem 3: The motion of a damped harmonic oscillator is described by $m\ddot{x} + b\dot{x} + kx = 0$, where $b > 0$ is the damping coefficient. The constants $m, k > 0$.

a) Rewrite the equation as a two-dimensional linear system.

b) Classify the fixed point at the origin and sketch the phase portrait in the case when the system is: underdamped ($b^2 < 4mk$), critically damped ($b^2 = 4mk$), or overdamped ($b^2 > 4mk$).

Problem 4: Consider the system

$$\dot{x} = g(x),\ x \in \mathbb{R}^2, \quad \text{where} \quad g\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1^2 + 7x_1x_2 + x_1 + 2x_2^2 - x_2 \\ -12x_1^2 - 16x_1x_2 - 3x_1 - x_2 \end{pmatrix}.$$

a) Compute the Jacobian matrix at $x^* = (0, 0)$ and find its eigenvalues. Use this information to classify this fixed point and determine its stability.

b) Sketch the phase portrait in a neighborhood of x^*.

c) (Extra Credit - 3p) Sketch a plausible phase portrait for the whole system.

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Assignment 4

Due Tuesday, **March 8**, in class.

Problem 1: For each of the following systems, find the equilibrium points, classify them and sketch the neighboring trajectories.

a) $\dot{x} = x - y$, $\dot{y} = x^2 - 4$

b) $\dot{x} = y$, $\dot{y} = xy + x - 1$

c) $\dot{x} = x(x^2 + y^2)$, $\dot{y} = y(x^2 + y^2)$

Does the linearized system accurately describe the local behavior near the equilibrium points?

Problem 2: Consider the system $\dot{x} = \sin(y)$, $\dot{y} = \cos(x)$ in the rectangle $-3 < x < 5$, $-4 < y < 5$

a) Find all fixed points, determine their stability and sketch the neighboring trajectories.

b) Does the system have homoclinic trajectories, heteroclinic trajectories, closed orbits, or limit cycles? Sketch some of them, if they exist.

Problem 3: Consider the pendulum equation, $\ddot{x} + \sin(x) = 0$, $-8 < x < 8$.

a) Find the equilibrium points and discuss the linearization of the pendulum equation in the neighborhood of each equilibrium point.

b) Multiply both sides of the pendulum equation by $\dot{x}$ and show that the *energy* function $E(\dot{x}, x) = \frac{1}{2}(\dot{x})^2 - \cos(x)$ is constant along trajectories.

c) (Extra Credit - 3p) Sketch the phase portrait of the full nonlinear equation.

Problem 4: Discuss the local and global behavior of solutions of

$$\begin{aligned} \dot{r} &= ar - r^5 \\ \dot{\theta} &= 1 \end{aligned}$$

as the parameter a passes through 0. Does the system have a limit cycle for some value of a?

Problem 5: The relativistic equation for the orbit of a planet around the sun is

$$\frac{d^2u}{d\theta^2} + u = \alpha + \epsilon u^2$$

where $u = \dfrac{1}{r}$ and r, θ are the polar coordinates of the planet in its plane of motion. The parameter α is positive and can be found explicitly from classical Newtonian mechanics; the term ϵu^2 is Einsteins correction. Here ϵ is a very small positive parameter (so that $\epsilon\alpha \approx 0$).

a) Rewrite the equation as a system in the (u, v) phase plane, where $v = \dfrac{du}{d\theta}$.

b) Find all the equilibrium points of the system.

c) Show that one of the equilibria is a center in the (u, v) phase plane, according to the linearization. Is it a *nonlinear* center?

d) (Extra Credit - 2p) Show that the equilibrium point found in c) corresponds to a circular planetary orbit.

Problem 6: Show that the nonlinearly damped oscillator $\ddot{x} + (\dot{x})^3 + x = 0$ has no periodic solutions. *Hint:* Analyze the energy function $E(\dot{x}, x) = \frac{1}{2}\left((\dot{x})^2 + x^2\right)$.

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SPRING 2016

ASSIGNMENT 5

Due Thursday, **March 24**, in class.

Problem 1: Show that $(0, \pi)$ is a nonlinear center for the system $\dot{x} = \sin(y)$, $\dot{y} = \sin(x)$. *Hint:* Find an energy function $E(x, y)$ of the form $\alpha \sin(x) + \beta \cos(y)$, $\alpha \cos(x) + \beta \sin(y)$, $\alpha \sin(x) + \beta \sin(y)$, or $\alpha \cos(x) + \beta \cos(y)$ and show that this has a local min or max at $(0, \pi)$.

Problem 2: For each of the following systems, decide whether it is a gradient system. If so, find $V(x, y)$ and sketch the phase portrait. On a separate graph, sketch the equipotentials $V(x, y) =$ constant. If the system is not a gradient system, explain why not and go on to the next question.

a) $\dot{x} = y + x^2 y, \quad \dot{y} = -x + 2xy$

b) $\dot{x} = 2x, \quad \dot{y} = 8y$

c) $\dot{x} = -2xe^{x^2+y^2}, \quad \dot{y} = -2ye^{x^2+y^2}$

Problem 3: Consider the nonlinear system

$$\begin{aligned} \dot{x} &= -y + x(x^2 + y^2) \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right) \\ \dot{y} &= x + y(x^2 + y^2) \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right) \end{aligned}$$

a) Use polar coordinates $x = r\cos(\theta)$ and $y = r\sin(\theta)$ and show that the system becomes

$$\begin{aligned} \dot{r} &= r^3 \sin\left(\frac{1}{r}\right) \\ \dot{\theta} &= 1 \end{aligned}$$

b) Observe that if $\dot{r} = 0$ then $r = 0$ or $r = \dfrac{1}{n\pi}$ for $n = 1, 2, 3, \ldots$. The latter corresponds to closed orbits of radius $r = \dfrac{1}{n\pi}$, which are limit cycles. In this exercise you have to show that these cycles are stable for even n and unstable for odd n.

Hint: Consider $h(t) = \dfrac{1}{r(t)} - n\pi$, where $|h(t)|$ is much smaller than $n\pi$. Substitute this into the equation for r and show that $\dot{h} = -\dfrac{1}{n\pi + h}(-1)^n \sin(h)$. Then show that $\dot{h} > 0$ or $\dot{h} < 0$ when n is even. What does this tell us about $\dot{r}$? Use this information to show stability. Treat the case when n is odd similarly.

c) (Extra Credit - 2p) We demonstrated the existence of infinitely many nested limit cycles in part b). Make a sketch of the phase portrait and include at least three cycles.

Problem 4: Apply Bendixon's negative criterion or Dulac's criterion to show that there are no periodic solutions to:

a) $\dot{x} = -x + y^2, \quad \dot{y} = -y^3 + x^2$

b) $\dot{x} = -2xe^{x^2+y^2}, \quad \dot{y} = -2ye^{x^2+y^2}$

c) $\dot{x} = y, \quad \dot{y} = a_1x + a_2y + a_3x^2 + a_4y^2$, where a_1, a_2, a_3, a_4 are nonzero constants.

Problem 5: Consider the nonlinear system

$$\begin{aligned} \dot{x} &= x - y - x(x^2 + 5y^2) \\ \dot{y} &= x + y - y(x^2 + y^2) \end{aligned}$$

a) Classify the fixed point at the origin.

b) Rewrite the system in polar coordinates ($x = r\cos(\theta)$ and $y = r\sin(\theta)$).

c) Prove that the system has a limit cycle in the annular region $\dfrac{1}{\sqrt{2}} - \epsilon < r < 1 + \epsilon$. Here ϵ is a small enough positive number (e.g. $\epsilon = 0.05$).

Hint: Apply the Poincaré-Bendixson Theorem.

MAT 351 Differential Equations: Dynamics & Chaos

Spring 2016

Practice problems

Problem 1: Determine the equilibrium points of the following one-dimensional differential equations and discuss their stability:

1. $\dot{x} = e^{-x}\sin(x)$
2. $\dot{x} = x^2(6 - x)$
3. $\dot{x} = -\sinh(x)$

Problem 2: Find the critical value of λ in which bifurcations occur in the following systems. Sketch the phase portrait for various values of λ and the bifurcation diagram. Classify the bifurcation.

1. $\dot{x} = x^3 - 5x^2 - (\lambda - 8)x + \lambda - 4.$
2. $\dot{x} = 1 + \lambda x + x^2$
3. $\dot{x} = 5 - \lambda e^{-x^2}$

Problem 3: Describe the local stability behavior near equilibrium points of the following nonlinear systems. Also, draw the phase portrait near the equilibrium point.

1. $\dot{x} = y^2 - x + 2$, $\dot{y} = x^2 - y^2$
2. $\dot{x} = y + x^3$, $\dot{y} = x - y^3$ (also find a Lyapunov function)
3. $\ddot{x} + x + 4x^3 = 0$

Problem 4: Consider the system

$$\begin{aligned}\dot{x} &= -y + x(1 - 2x^2 - 3y^2)\\ \dot{y} &= x + y(1 - 2x^2 - 3y^2)\end{aligned}$$

1. Find all fixed points of the system and define their stability.
2. Transfer the system into polar coordinates (r, θ).
3. Find a trapping region of the form $R = \{(r, \theta) : a \leq r \leq b\}$ and then use Poincaré-Bendixson theorem to prove that the system has a limit cycle in the region R.

Problem 5: Discuss the local and global behavior of solutions of

$$\begin{aligned}\dot{r} &= r(a - r^2)\\ \dot{\theta} &= -1\end{aligned}$$

as the parameter a passes through 0. Does the system have a limit cycle for some value of a? What type of bifurcation does this system undergo?

MAT 351 Differential Equations: Dynamics & Chaos

Spring 2016

Project ideas

Project title due: Thursday, April 14.
The final project is due: Monday, May 16 at 5:30pm.
Note: Project 2 and Project 7 can be done in groups of 1 or 2 students. Project 3+4 (together) can be done in groups of 2 students.

Topic 1: This year we had a particularly warm winter, which was characterized by El Niño phenomenon. El Niño or, more precisely **El Niño-Southern Oscillation (ENSO)**, is a quasi-periodic climate pattern that occurs across the equatorial Pacific Ocean roughly every three to seven years. It is characterized by a change in sea surface temperatures (SSTs) in the eastern Pacific off the coast of Peru and accompanying changes in the air pressure difference between the central and western Pacific Ocean (Tahiti and Darwin, Australia). The following system of equations (with dimensionless variables and parameters) is a good model for studying ENSO:

$$\begin{aligned}\dot{x} &= -x + \frac{\lambda}{b}(bx+y) - \epsilon(bx+y)^3 \\ \dot{y} &= -ry - \alpha bx\end{aligned}$$

where $r, \alpha, b, \epsilon, \lambda$ are all positive numbers. We are looking for the oscillatory behavior that characterizes the El Niño phenomenon

1. What is a quasi-periodic solution (or function)?

1. Compute the Jacobian A at the fixed point $(0,0)$ and find its eigenvalues. Find a condition on λ such that A has a pair of complex conjugate eigenvalues $\rho_1 = \beta - i\omega$ and $\rho_2 = \beta + i\omega$. Classify the fixed point at $(0,0)$.

2. Prove that a Hopf bifurcation occurs at a critical value $\lambda = \lambda_c$ and find λ_c. Decide whether the bifurcation is subcritical or supercritical. Find the value ω_c at the critical value λ_c (note that β and ω depend on λ).

3. Find the nontrivial solutions to the linearized system $(\dot{x}, \dot{y}) = A(x,y)$ at the parameter $\lambda = \lambda_c$. Show that $y(t)$ can be written as:

$$y(t) = -\frac{\alpha b}{\sqrt{\alpha(1+r)}}x(t-\eta), \quad \text{where} \quad \eta = \frac{1}{\omega_c}\tan^{-1}\left(\frac{\omega_c}{r}\right),$$

which shows that the trajectories of x and y coincide, but y lags behind x with a lag given by η. Thus, this ENSO model predicts that the negative thermocline depth anomaly follows the same oscillatory pattern as the SST anomaly but with a time lag η.

4. Consider $r = \frac{1}{4}$, $\alpha = \frac{1}{8}$, and $\lambda = \frac{3}{4}b$. Find λ_c, ω_c, and the time lag η. Suppose the time unit is two months, what is the predicted period? (this is the period of the function $y(t)$ from above). What does the factor η predict in this case? Are there better models for studying El Niño?

A comprehensive description of El Niño and deduction of the dimensionless model can be found in Chapter 16 from:

Hans Kaper, Hans Engler, *Mathematics and Climate*, Society for Industrial and Applied Mathematics (SIAM), 2013.

It is useful to read this chapter beforehand for a better understanding of the project (especially the last question).

Topic 2: The Fitzhugh-Nagumo system is a simplified model that describes the electrochemical transmission of neuronal signals along the cell membrane. Although the model is not entirely accurate, it capture the essential behavior of nerve impulses.

The **Fitzhugh-Nagumo system** of equations is given by

$$\begin{aligned} \dot{x} &= y + x - \frac{x^3}{3} + I \\ \dot{y} &= -x + a - by \end{aligned}$$

where a and b are constants satisfying $0 < \frac{3}{2}(1-a) < b < 1$ and I is a parameter. In these equations x is similar to the voltage and represents the excitability of the system; the variable y represents a combination of other forces that tend to return the system to rest. The parameter I is a stimulus parameter that leads to excitation of the system (I is like an applied current).

1. First assume that $I = 0$. Prove that this system has a unique equilibrium point (x^*, y^*). *Hint:* Use the geometry of the nullclines for this rather than explicitly solving the equations. Also remember the restrictions placed on a and b.

2. Prove that this equilibrium point is always a sink.

3. Now suppose that $I \neq 0$. Prove that there is still a unique equilibrium point $(x^*(I), y^*(I))$ and that $x^*(I)$ varies monotonically with I.

4. Determine values of $x^*(I)$ for which the equilibrium point is a source and show that there must be a stable limit cycle in this case.

5. When $I \neq 0$, the point (x^*, y^*) is no longer an equilibrium point. Nonetheless we can still consider the solution through this point. Describe the qualitative nature of this solution as I moves away from 0. Explain in mathematical terms why biologists consider this phenomenon the "excitement" of the neuron.

6. Consider the special case where $a = I = 0$. Describe the phase plane for each $b > 0$ (no longer restrict to $b < 1$) as completely as possible. Describe any bifurcations that occur.

7. Now let I vary as well and again describe any bifurcations that occur. Describe in as much detail as possible the phase portraits that occur in the I, b-plane, with $b > 0$.

8. Extend the analysis of the previous problem to the case $b \leq 0$.

9. Now fix $b = 0$ and let a and I vary. Sketch the bifurcation plane (the I, a-plane) in this case.

This project and a brief description on neurodynamics can be found in Chapter 12.5 from:
R. Devaney, M. Hirsch, S. Smale, *Differential Equations, Dynamical Systems, and an Introduction to Chaos*, 3rd ed. Elsevier Academic Press 2013.

Topic 3: Hamiltonian systems are fundamental to classical mechanics; they provide an equivalent but more geometric version of Newtons laws. They are also central to celestial mechanics and plasma physics, where dissipation can sometimes be neglected on the time scales of interest. We restrict our attention to Hamiltonian systems in $\mathbb{R}^2$, which is a system of the form:

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial y}(x, y) \\ \dot{y} &= -\frac{\partial H}{\partial x}(x, y), \end{aligned} \tag{1}$$

where $H : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function called the Hamiltonian function.

1. Show that H is constant along every solution curve. Check that any system of the form $\ddot{x} + f(x) = 0$ is a Hamiltonian system.

2. Let (x^*, y^*) be a non-degenerate equilibrium point of a Hamiltonian system (that is, the determinant of the Jacobian at (x^*, y^*) is nonzero). Show that (x^*, y^*) is either a saddle or a center. Recall that (x^*, y^*) is a saddle for the system (1) iff it is a saddle of the Hamiltonian function $H(x, y)$ and a strict local maximum or minimum of the function $H(x, y)$ is a center for (1).

3. There is an interesting relationship between the gradient system and the Hamiltonian system. Show that the system given by $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$ is a Hamiltonian system if and only if the system *orthogonal* to it, given by $\dot{x} = g(x, y)$, $\dot{y} = -f(x, y)$ is a gradient system. To illustrate the orthogonality, consider the Hamiltonian function $H(x, y) = y \sin(x)$ and sketch the phase portraits of the Hamiltonian system and its gradient system (on the same graph).

4. Consider the equations for a nonlinear pendulum

$$\begin{aligned} \dot{\theta} &= v \\ \dot{v} &= -bv - \sin(\theta) + k. \end{aligned} \tag{2}$$

Here θ gives the angular position of the pendulum (assumed to be measured in the counterclockwise direction) and v is its angular velocity. The parameter $b > 0$ measures the damping. The parameter $k \geq 0$ is a constant torque applied to the pendulum in the counterclockwise direction.

a) Find all equilibrium points for this system and determine their stability.

b) Suppose $k > 1$. Prove that there exists a periodic solution for this system in a region R of the form $R = \{(\theta, v) : 0 < v_1 < (k - \sin(\theta))/b < v_2\}$.

c) Find a Hamiltonian function and use it to prove that when $k > 1$ there is a unique periodic solution for this system.

d) Are there any parameter values for which a stable equilibrium and a periodic solution coexist?

Useful references for this project are:

Steven Strogatz, *Nonlinear dynamics and Chaos: with applications to physics, biology, chemistry, and engineering*, 2nd ed., Addison-Wesley Pub. 2014.

R. Devaney, M. Hirsch, S. Smale, *Differential Equations, Dynamical Systems, and an Introduction to Chaos*, 3rd ed. Elsevier Academic Press 2013

Topic 4: Consider the Hamiltonian systems from Topic 3.

1. Do the first two parts of Topic 3.

2. State the Andronov-Hopf Bifurcation Theorem for a two-dimensional system.

3. Prove the **Lyapunov Center Theorem** as a consequence of the Hopf Bifurcation Theorem.

   Theorem 1 (Lyapunov Center Theorem)**.** *Assume that* $(0, 0)$ *is a center equilibrium of the Hamiltonian system* (1) *and that* $\pm\lambda i$ *are simple eigenvalues of the Jacobian* A *of the vector field at* $(0, 0)$ *(assume* $\lambda > 0$*). Then each neighborhood of the center contains periodic orbits, whose periods approaches* $2\pi/\lambda$ *as they approach the center*

The Lyapunov Center Theorem (together with a proof) and the Hopf Bifurcation Theorem can be found in:

K. Alligood, T. Sauer, J. Yorke, *Chaos: an introduction to dynamical systems*, Springer, New York, 1996.

The Hopf Bifurcation Theorem can also be found in Chapter 6 of:

Wei-bin Zhang, *Differential equations, bifurcations, and Chaos in economics*, World Scientific 2005.

Topic 5: For much of the 20th century, chemists believed that all chemical reactions tended monotonically to equilibrium. This belief was shattered in the 1950s when the Russian biochemist Belousov discovered that a certain reaction involving citric acid, bromate ions, and sulfuric acid, when combined with a cerium catalyst, could oscillate for long periods of time before settling to equilibrium. The concoction would turn yellow for a while, then fade, then turn yellow again, then fade, and on and on like this for over an hour. This reaction, now called the Belousov-Zhabotinsky reaction (the **BZ reaction**, for short), was a major turning point in the history of chemical reactions. Now, many systems are known to oscillate. Some have even been shown to behave chaotically.

One particularly simple chemical reaction is given by a chlorine dioxide-iodine-malonic acid interaction. The exact differential equations modeling this reaction are extremely complicated. However, there is a planar nonlinear system that closely approximates the concentrations of two of the reactants. The system is

$$\begin{aligned}\dot{x} &= a - x - \frac{4xy}{1+x^2}\\ \dot{y} &= bx\left(1 - \frac{y}{1+x^2}\right)\end{aligned}$$

where x and y represent the concentrations of I^- and ClO_2^-, respectively, and a and b are positive parameters.

1. Find all equilibrium points for this system. Linearize the system at your equilibria and determine the type of each equilibrium.

2. In the ab-plane, sketch the regions where you find asymptotically stable or unstable equilibria.

3. Identify the a, b-values where the system undergoes bifurcations. What kind of bifurcations are these?

4. Using the nullclines for the system together with the Poincaré-Bendixson theorem, find the a, b-values for which a stable limit cycle exists. Why do these values correspond to oscillating chemical reactions?

The project was taken from Chapter 10 of:
R. Devaney, M. Hirsch, S. Smale, *Differential Equations, Dynamical Systems, and an Introduction to Chaos*, 3rd ed. Elsevier Academic Press 2013

For more details on this reaction, see the following article: Lengyel, I., Rabai, G., and Epstein, I. *Experimental and modeling study of oscillations in the chlorine dioxide-iodine-malonic acid reaction.* J. Amer. Chem. Soc. 112 (1990), 9104.

The very interesting history of the BZ-reaction is described in: Winfree, A. T. *The prehistory of the Belousov-Zhabotinsky reaction.* J. Chem. Educ. 61 (1984), 661.

Topic 6: This project deals with the existence of periodic points of functions defined on an interval or on the real line. A point x is a periodic point of period p for the function f if $f^p(x) = x$. It is of prime period if there is no smaller number $0 < q < p$ such that $f^q(x) = x$. Here $f^p(x)$ means $f \circ f \circ f \ldots \circ f(x)$. For example $f^2(x) = f(f(x))$.

1. Explain what Sharkovskii's ordering is.

2. Give a proof of **Sharkovskii's Theorem**.

Theorem 2 (Sharkovskii)**.** *Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous map and has an orbit of prime period p. If $p \triangleright q$ in the Sharkovskii's ordering, then f has an orbit of period q.*

3. Explain the meaning of "period 3 implies chaos".

4. Give some applications of Sharkovskii's Theorem. For example, can a continuous function on $\mathbb{R}$ have a periodic point of period 176 but not one of period 96? Why? Or prove that if a continuous function $f : [0,1] \to [0,1]$ has a periodic point of period 2014, then f has a periodic point of period 100. Does Sharkovskii's Theorem hold for continuous functions $f : \mathbb{R}^2 \to \mathbb{R}^2$?

Aside from Strogatz, these are also useful references (they include proofs):

Robert Devaney, *An Introduction to Chaotic Dynamical Systems*, 2nd ed., Westview Press, 2003.

K. Alligood, T. Sauer, J. Yorke, *Chaos: an introduction to dynamical systems*, Springer, New York, 1996.

Topic 7: Another interesting project related to **Quantum Mechanical Systems** and **anisotropic Kepler problem** can be found in Chapter 13 of:

R. Devaney, M. Hirsch, S. Smale, *Differential Equations, Dynamical Systems, and an Introduction to Chaos*, 3rd ed. Elsevier Academic Press 2013

Topic 8: The subject of Differential Equations, Dynamical Systems and Chaos is a vast subject and many other topics are possible:

a) A project in **Complex Dynamics** (which requires some knowledge of Complex Analysis). This would include a description of the Julia set, the Mandelbrot set, local behavior around fixed points, a classification of the possible Fatou components, hyperbolicity (and the role of the critical points), Chaos, etc.

b) A study of the **van der Pol equation** and Liénard's Theorem.

c) The analysis of a **Lotka-Volterra** equation model of population dynamics and ecology.

d) A new topic!

Please discuss these additional topics with me to ensure that the level of difficulty is within the framework of the course.

MAT 351 Differential Equations: Dynamics and Chaos

Spring 2016

Midterm – March 31, 2016

NAME: ______________________________

Please turn off your cell phone and put it away. You are **NOT** allowed to use a calculator.

Please show your work! To receive full credit, you must explain your reasoning and neatly write the steps which led you to your final answer. If you need extra space, you can use the other side of each page.

Academic integrity is expected of all students of Stony Brook University at all times, whether in the presence or absence of members of the faculty.

PROBLEM	**SCORE**
1	
2	
3	
4	
TOTAL	

Problem 1: (22 points) Consider a two-dimensional system $\dot{x} = f(x)$, $x \in \mathbb{R}^2$ and f is a $\mathcal{C}^1$ function.

a) Give a short definition for the following notions:

hyperbolic fixed point

closed orbit

limit cycle

Hopf bifurcation

b) Give an example of a system that undergoes a Hopf bifurcation. No proof is required.

c) Sketch a phase portrait of a system that has a *stable limit cycle*, a *heteroclinic orbit*, and a *nonlinear center*. Sketch some typical trajectories for your system.

Problem 2: (22 points) Consider the differential equation

$$\dot{x} = \lambda - \frac{x^2}{1+x^2}, \qquad x \in \mathbb{R},\ \lambda \in \mathbb{R}.$$

Find the equilibrium points and discuss their stability. Find the values of λ at which a bifurcation occurs, and classify them as saddle-node, transcritical, supercritical pitchfork, or subcritical pitchfork. Sketch the bifurcation diagram of fixed points x^* vs. λ.

Problem 3: (28 points) Consider the two-dimensional system:

$$\begin{aligned} \dot{x} &= y - y^2 \\ \dot{y} &= \sin(x) \end{aligned}$$

a) Find an energy function $E(x, y)$ of the form $E(x, y) = \alpha \cos(x) + f(y)$, for some constant α and some function f. Verify that $E(x, y)$ is constant along trajectories.

b) Show that the fixed points $((2n + 1)\pi, 0)$ and $(2n\pi, 1)$ are nonlinear centers.

c) Show that the fixed points $(2n\pi, 0)$ and $((2n+1)\pi, 1)$ are saddles.

d) Sketch the phase portrait for this system.

Problem 4: (28 points) Consider the two-dimensional system

$$\begin{aligned} \dot{x} &= y + x(1 - a - x^2 - y^2) \\ \dot{y} &= -x + y(1 - x^2 - y^2) \end{aligned}$$

where a is a constant such that $0 < a < 1$.

a) Determine the equilibrium points and classify them as sinks, sources, or saddles. Draw the phase portrait near the equilibrium points.

b) Using $x = r\cos(\theta)$ and $y = r\sin(\theta)$, rewrite the system in polar coordinates.

c) Let $r_1 = \sqrt{1-a} - \epsilon$ and $r_2 = 1 + \epsilon$, for some $\epsilon > 0$ small enough. Show that there is at least one limit cycle in the region $R = \{(r, \theta) : r_1 \le r \le r_2\}$.

d) Suppose there are several limit cycles. Explain why they all must have the same period (the period will depend on the parameter a though).

MAT 351 Differential Equations: Dynamics & Chaos
Spring 2016

Assignment 6

Due Thursday, **April 21**, in class.

Problem 1: Consider the oscillator $\ddot{x} - (\mu - x^2)\dot{x} + x = 0$. Show that the system undergoes a Hopf bifurcation at $\mu = 0$. Is this subcritical or supercritical?

Problem 2: Consider the Lorenz system of equations

$$\begin{aligned} \dot{x} &= 10(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - \frac{8}{3}z. \end{aligned}$$

for $r > 0$.

a) Find the linearized system at the origin. This system has the form $\dot{X} = AX$, where A is the Jacobian matrix at $(0, 0, 0)$.

b) Compute the eigenvalues of the matrix A.

c) By studying the eigenvalues from part b), show that the origin is asymptotically stable for $r < 1$ and unstable for $r > 1$.

Problem 3: Consider the system

$$\begin{aligned} \dot{x} &= -\nu x + zy \\ \dot{y} &= -\nu y + (z - a)x \\ \dot{z} &= 1 - xy \end{aligned}$$

where $a, \nu > 0$ are parameters.

a) Show that the system is dissipative.

b) Show that the fixed points may be written in parametric form $x^* = \pm k$, $y^* = \pm \dfrac{1}{k}$, and $z^* = \nu k^2$, where k verifies the equation $\nu(k^4 - 1) = ak^2$.

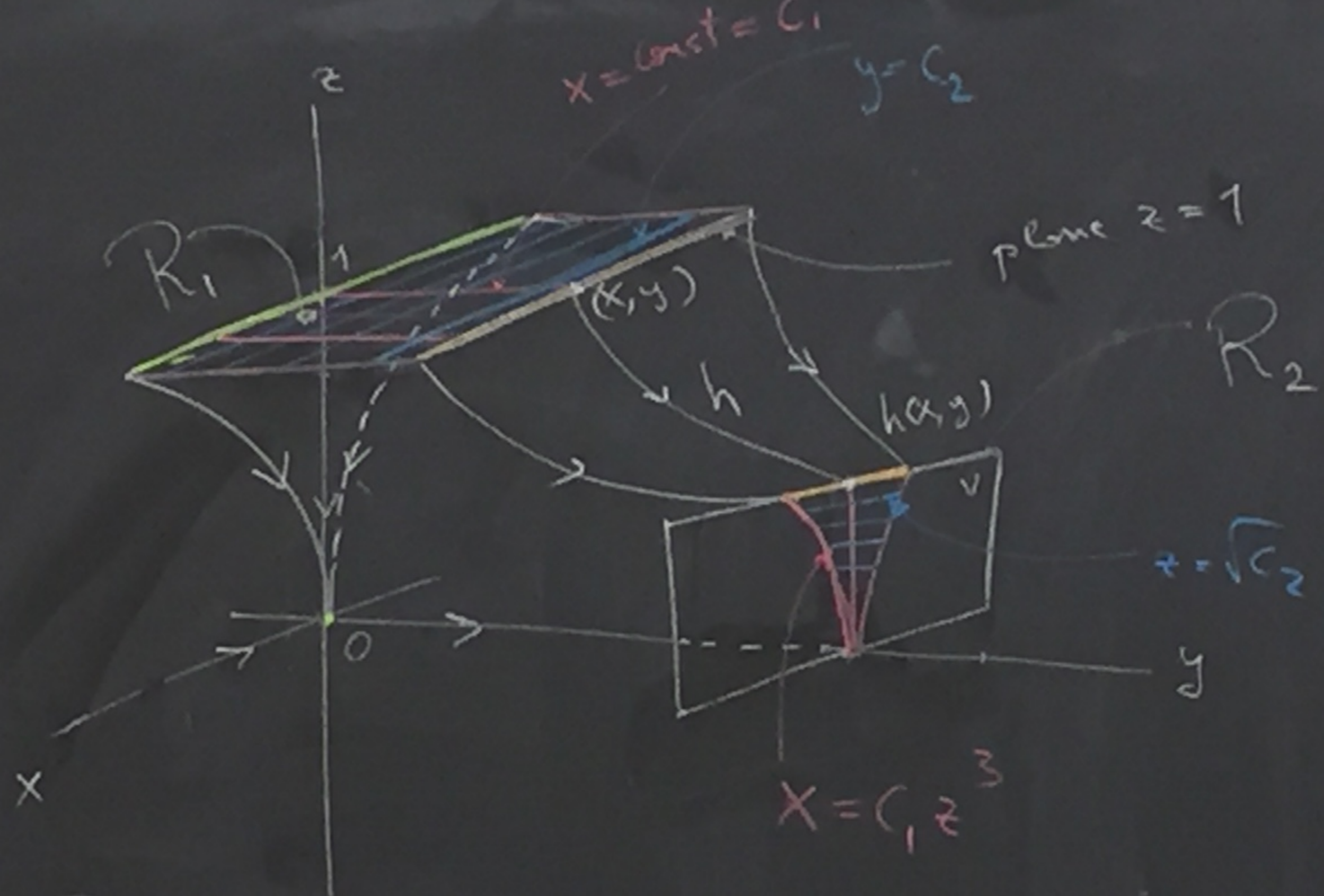
c) (Extra Credit - 3p) Classify the fixed points.

Note from Strogatz: These equations were proposed by Rikitake (1958) as a model for the self-generation of the Earths magnetic field by large current-carrying eddies in the core. Computer experiments show that the model exhibits chaotic solutions for some parameter values. These solutions are loosely analogous to the irregular reversals of the Earths magnetic field inferred from geological data.

Problem 4: Consider the following familiar system in polar coordinates: $\dot{r} = r(1 - r^2)$, $\dot{\theta} = 1$. Let A be the unit circle $x^2 + y^2 = 1$.

a) Is A an invariant set? Does A attract an open set of initial conditions?

b) Is A an attractor? If not, explain why not?



$R_1 = \{-1 \le x \le 1,\ 0 < y < \varepsilon < 1\}$ rectangles

$R_2 = \{-1 \le x \le 1,\ 0 < z \le 1\}$

Some $h: R_1 \longrightarrow R_2$, $h(x,y) = (x_1, z_1) = \begin{pmatrix} x y^{3/2} \\ y^{1/2} \end{pmatrix}$

one has to compute this using the particular form of the ODE

① blue lines $y = c_2$ in R_1 are taken by h to $h(x, c_2)$

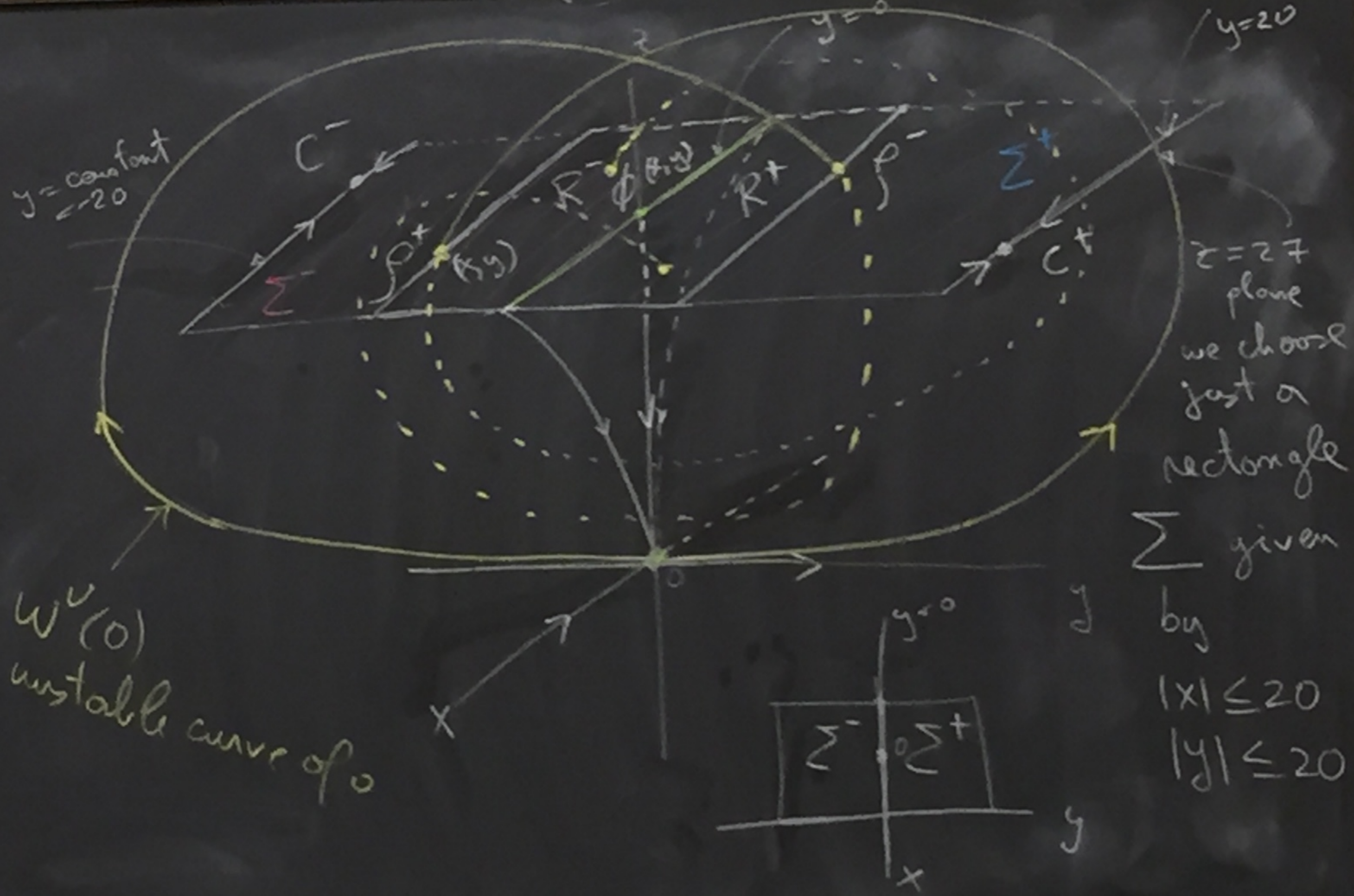
$$h(x, c_2) = \left(x c_2^{3/2},\ c_2^{\frac{1}{2}} \right) = (x_1, z_1),$$ the z-coordinate of this line is $\sqrt{c_2}$ constant

② red lines $x = c_1$ in R_1 are taken by h to $h(c_1, y) = \left(c_1 y^{3/2},\ y^{\frac{1}{2}} \right) = (x_1, z_1)$

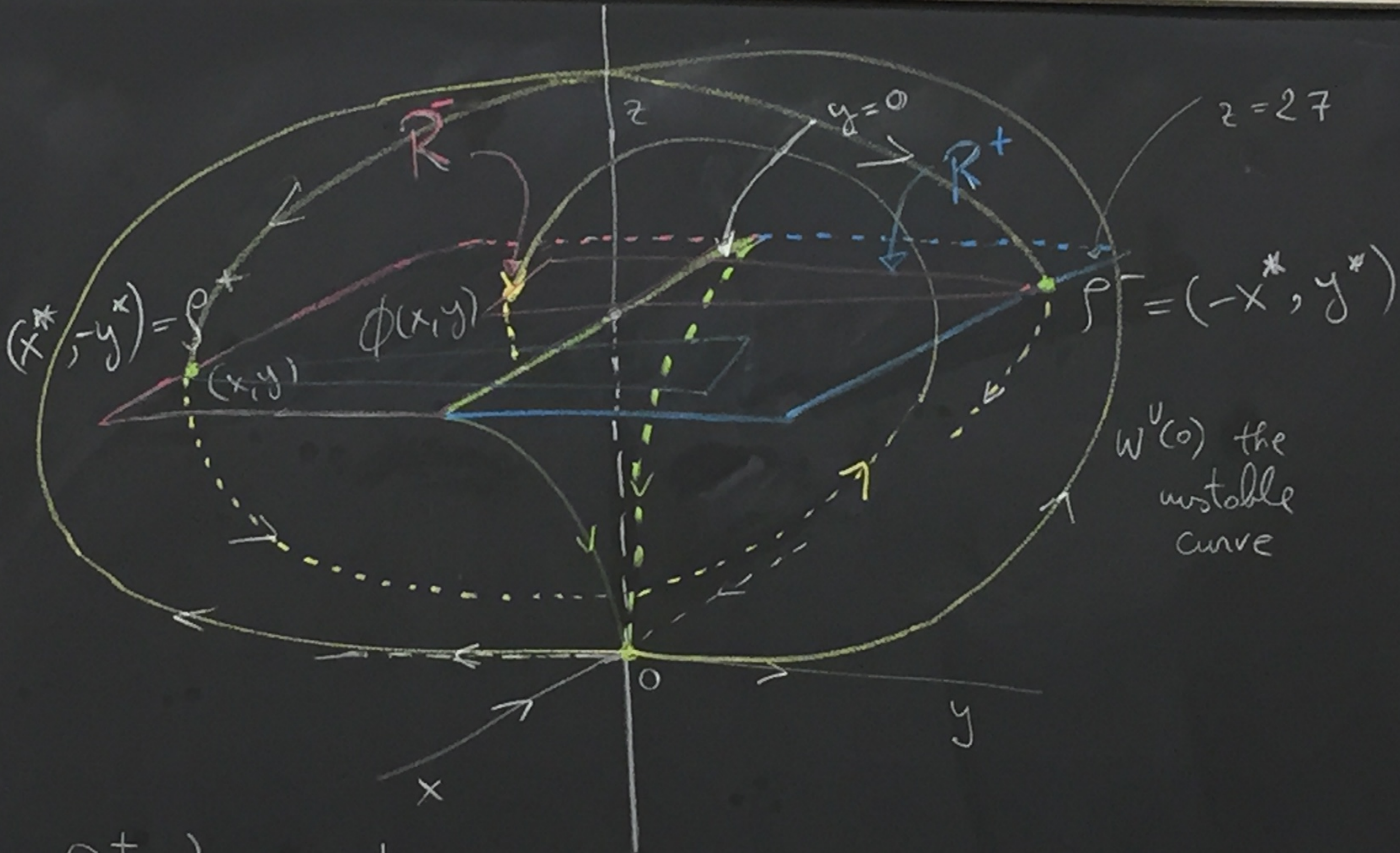
these are curves of equation $\boxed{X = c_1 z^3}$

$$x_1 = c_1 \left(y^{\frac{1}{2}}\right)^3 = c_1 z_1^3$$

$$z_1 = y^{\frac{1}{2}}$$



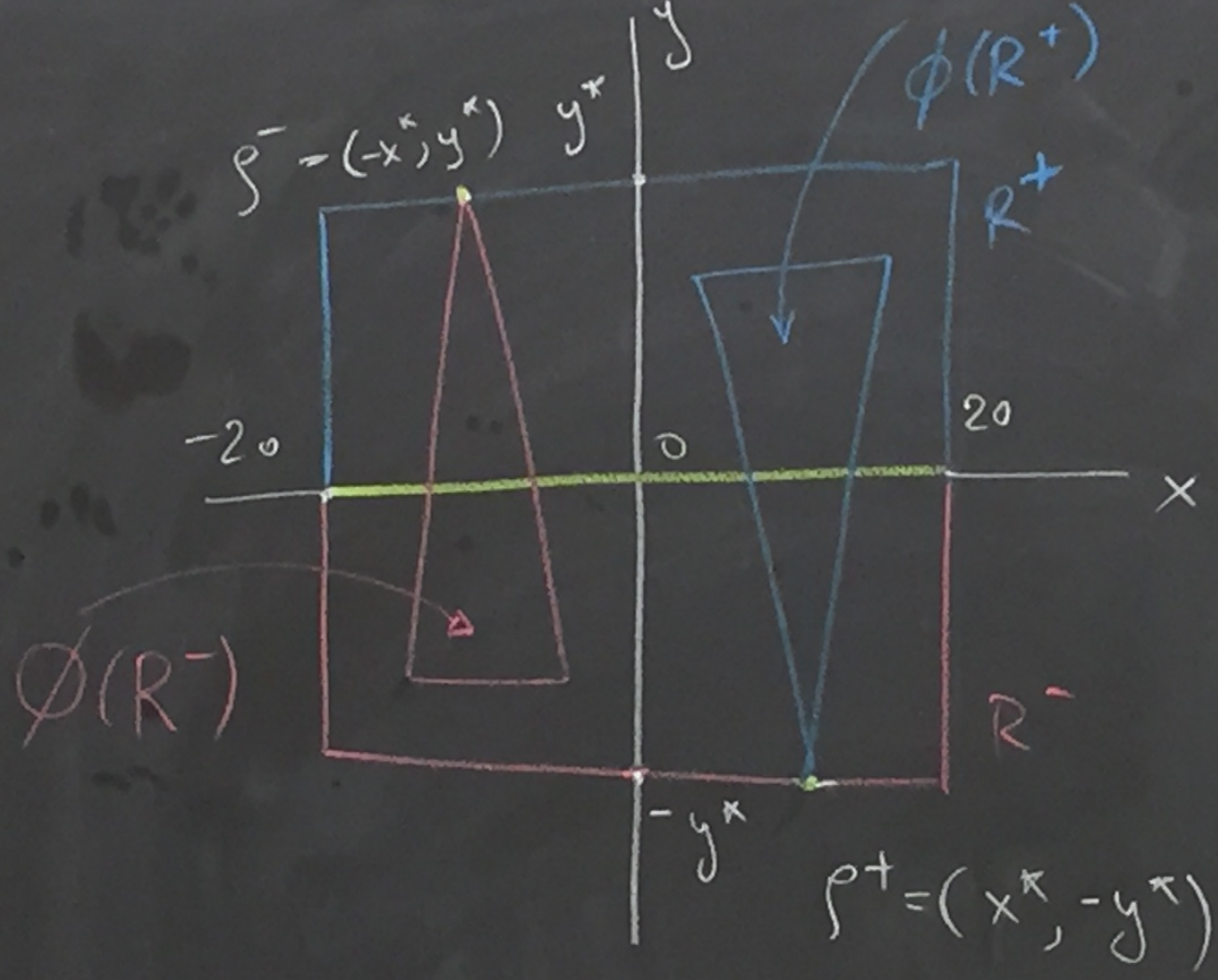
y = 0
y = 20
z
y = constant < -20
C^-
C^+
R^-
R^+
$\phi(x,y)$
ρ^+
ρ^-
(x,y)
Σ^-
Σ^+
z = 27 plane
we choose just a rectangle
Σ given by
$|x| \leq 20$
$|y| \leq 20$
0
y
x
$W^u(0)$
unstable curve of 0



$R^+ = \{(x,y) \mid |x| \le 20,\ 0 < y \le 20\}$

$R^- = \{(x,y) \mid |x| \le 20,\ -20 \le y < 0\}$

rectangles in the $z=27$ plane



y
$\phi(R^+)$
$\rho^- = (-x^*, y^*)$
y^*
R^+
-20
0
20
x
$\phi(R^-)$
R^-
$-y^*$
$\rho^+ = (x^*, -y^*)$

MAT 351 Differential Equations: Dynamics & Chaos

Spring 2016

Assignment 7

Due Thursday, **May 5**, in class.

Problem 1: Consider the system

$$\begin{aligned}\dot{x} &= 10(y-x)\\ \dot{y} &= 28x - y + xz\\ \dot{z} &= xy - \frac{8}{3}z.\end{aligned}$$

a) Consider $E = 28x^2 + 10y^2 + 10(z-56)^2$. Show that $E > 0$ and $\dot{E} > 0$ in the region

$$R = \left\{ 28x^2 + y^2 + \frac{8}{3}(z-28)^2 < \frac{8}{3}28^2 \ \text{ and } \ x > 0, y > 0, z > 0 \right\}.$$

What does this tell us about the points in the region R? The region R is the part of the solid ellipsoid $28x^2 + y^2 + \frac{8}{3}(z-28)^2 < \frac{8}{3}28^2$ where x, y, and z are all positive.

b) (Extra Credit - 5p) Show that this system is **not chaotic** in the region where x, y, and z are all positive.

Hint: It is enough to show that most solutions tend to ∞ in forward time. Note that this is not the Lorenz system: in the equation for $\dot{y}$ we have $+xz$ instead of $-xz$.

Problem 2: Determine whether $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ has sensitive dependence on initial conditions. Is the map f transitive?

Problem 3: Consider the *tent map* $T : [0,1] \to [0,1]$ defined by

$$T(x) = \begin{cases} 2x & \text{if } \ 0 \le x < \frac{1}{2} \\ 2 - 2x & \text{if } \ \frac{1}{2} \le x \le 1 \end{cases}$$

(a) Sketch the graphs of T, T^2 and T^3. What does the graph of T^n look like?

(b) Use the graph of T^n to conclude that T has exactly 2^n periodic points of period n. These points do not necessarily have least period n, but are fixed by T^n.

(c) (Extra Credit - 3p) The tent map is chaotic. In this exercise, you are asked to prove that the set of all periodic points of T is dense in $[0,1]$.

Problem 4: Consider the logistic map $G : [0,1] \to [0,1]$ defined by $G(x) = 4x(1-x)$.

(a) Prove that G is topologically conjugate to the tent map $T : [0,1] \to [0,1]$,

$$T(x) = \begin{cases} 2x & \text{if} \quad 0 \leq x < \frac{1}{2} \\ 2 - 2x & \text{if} \quad \frac{1}{2} \leq x \leq 1 \end{cases}$$

You need to verify that there exists a homeomorphism $h : [0,1] \to [0,1]$ such that $h(G(x)) = T(h(x))$. *Hint:* Consider $h(x) = (1 - \cos(\pi x))/2$.

b) Use the previous problem to conclude that the logistic map G is chaotic.

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MAT 341: Applied Real Analysis
Spring 2017
Schedule & Homework

Schedule

The PDF version of the schedule is available for print **here**.

Date	Topic	Section	Assignments	Due date
Jan 24	An introduction to Fourier series	1.1	**1.1:** 1abc, 2ad, 4, 7b, 8	**HW1** Due Jan 31
Jan 26	Determining Fourier coefficients; Examples	1.2	**1.2:** 1, 7c	
Jan 31	Even & odd extensions Convergence of Fourier series	1.2, 1.3	**1.2:**10b, 11b	**HW2** Due Feb 7
Feb 2	Uniform convergence of Fourier series	1.3, 1.4	**1.3:**1abd, 2ad, 6	
Feb 7	Fourier sine & cosine series Basic operations on Fourier series	1.4, 1.5	**1.4:** 1ae, 2, 3ab, 5bc **page 120:** 19, 20 [use a=3]	**HW3** Due Feb 14
Feb 9	*no class (snow storm)*			
Feb 14	Differentiation of Fourier series The heat equation	1.5, 2.1	**1.5:** 2, 5, 9 **2.1:** 2, 9	**HW4** Due Feb 23
Feb 16	The heat equation Steady-state & transient solutions	2.1, 2.2	**2.2:** 2, 6	
Feb 21	Fixed-end temperatures	2.3	**2.3:** 8 [use a=pi]	
Feb 23	Insulated bar; Examples Review	2.4	**2.3:** 6 **2.4:** 4 [use a=pi], 5, 8	**HW5** Due Mar 9
Feb 28	**Midterm 1** (2:30-3:50pm) Covers 1.1-1.5, 2.1-2.3 -- **Solutions** Practice exams: **Fall 2015** (**Solutions**) and **Spring 2015** (**Solutions**)			
Mar 2	Different boundary conditions	2.5	**2.5:** 4, 5 [use a=pi], 6	
Mar 7	Eigenvalues and eigenfunctions Convection	2.6, 2.7 **Notes**	**2.6:** 7, 9, 10	**HW6** Due Mar 23 **Problem 3c**
Mar 9	Sturm-Liouville problems	2.7	**2.7:** 1, 3abc, 7	

Mar 14	*no class (Spring break)*			
Mar 16	*no class (Spring break)*			
Mar 21	Series of eigenfunctions & examples Fourier integral & applications to PDEs	2.8, 1.9	**2.8:** 1 [use b=2] **1.9:** 1ab, 3a	**HW7** Due Mar 30
Mar 23	Semi-infinite rod The wave equation	2.10, 3.1	**2.10:** 3, 4	
Mar 28	The wave equation	3.2	**3.2:** 3, 4, 5, 7	**HW8** Due Apr 6 **Comments**
Mar 30	D'Alembert's solution; Examples	3.3, 3.4	**3.3:** 1, 2, 5	
Apr 4	The wave equation: generalizations Laplace's equation	3.4, 4.1	**page 255:** 18 **page 257:** 31	**HW9** Due Apr 20 **Comments**
Apr 6	Dirichlet's problem in a rectangle Examples & Review	4.2, 4.3	**4.1:** 2 **4.2:** 5 [use a=1, f(x)=sin(3pix)] **4.2:** 6	
Apr 11	**Midterm 2** (2:30-3:50pm) Covers 2.4-2.8, 2.10, 1.9, 3.1-3.4 -- **Solutions** Practice exams: **Fall 2015** (**Solutions**) and **Spring 2015** (**Solutions**) **Extra practice problems**			
Apr 13	Potential in a rectangle; Examples Potential in unbounded regions	4.3, 4.4	**4.3:** 2b **4.4:** 4a, 5ab	**HW10** Due Apr 27
Apr 18	Polar coordinates Potential in a disk	4.1, 4.5 **Notes**	**4.1:** 6 **4.5:** 1	
Apr 20	Dirichlet problem in a disk; Examples	4.5	**4.5:** 4	
Apr 25	Two-dimensional heat equation	5.3, 5.4 **Notes**	**5.3:** 1, 7c [use a=b=pi]	**HW11** Due May 4
Apr 27	Problems in polar coordinates Bessel's equation	5.5, 5.6	**5.4:** 5	
May 2	Temperature in a cylinder Applications: symmetric vibrations	5.6, 5.7	**5.6:** 3 [use a=1] **page 371:** 1	
May 4	Examples & Review	5.7		
May 15	**Final Exam** (11:15am-1:45pm) -- in class, **Melville Library E4315** The final is cumulative and covers: 1.1-1.5, 1.9, 2.1-2.8, 2.10, 3.1-3.4, 4.1-4.5, 5.3-5.6 Practice exams: **Fall 2015** and **Spring 2015**.			

MAT 351 DIFFERENTIAL EQUATIONS: DYNAMICS & CHAOS
SPRING 2016
GENERAL INFORMATION

Instructor. Remus Radu
Email: `rradu@math.stonybrook.edu`
Office: Math Tower 4-103, t: (631) 63**2-8266**
Office Hours: TuTh 2:30-4:00pm, or by appointment

Teaching Assistant. Aleksandar Milivojevic
Email: `aleksandar.milivojevic@stonybrook.edu`
Office Hours: Monday 10:00-11:00am & 1:00-2:00pm and Wednesday 10:30-11:30am in MLC

Lectures. TuTh 1:00-2:20pm in Physics P116

Blackboard. Grades and some course administration will take place on Blackboard. Please login using your NetID at `http://blackboard.stonybrook.edu`.

Course Description. Dynamical systems occur in all branches of science, from the differential equations of classical mechanics in physics to the difference equations of mathematical economics and biology.

This course is an introduction to the field of dynamical systems. It concerns the study of the long-term behavior of solutions to ordinary differential equations or of iterated mappings, emphasizing the distinction between stability on the one hand and sensitive dependence and chaotic behavior on the other. The course describes examples of chaotic behavior and of fractal attractors, and develops some mathematical tools for understanding them. In particular we will study the following key concepts: hyperbolicity, topological conjugacy, equilibrium, limit cycle, stability, chaos, etc.

Prerequisites. C or higher in the following: MAT 203 or 205 or 307 or AMS 261; MAT 303 or 305 or 308 or AMS 361; MAT 200 or permission of instructor

Recommended reading.

- Wei-bin Zhang, *Differential equations, bifurcations, and Chaos in economics*, World Scientific 2005.
- Steven Strogatz, *Nonlinear dynamics and Chaos: with applications to physics, biology, chemistry, and engineering*, 2nd ed., Addison-Wesley Pub. 2014.
- Robert Devaney, *An Introduction to Chaotic Dynamical Systems*, 2nd ed., Westview Press, 2003.
- Robert Devaney, Morris Hirsch, and Stephen Smale, *Differential Equations, Dynamical Systems, and an Introduction to Chaos*, 3rd ed., Elsevier Academic Press 2012.

Other useful materials, reading suggestions and lecture notes will be posted on Blackboard.

Exams. There will be a midterm exam on **Thursday, March 31, 1:00pm-2:20pm** in class (Physics P116). There will be no make-up exams.

Grading policy. Grades will be computed using the following scheme:

Homework 30%
Midterm 35%
Project & presentation 35%

Students are expected to attend class regularly and to keep up with the material presented in the lecture and the assigned reading. There will be (roughly) weekly homework assignments. You may work together on your problem sets, and you are encouraged to do so. However, all solutions must be written up independently. The project presentations are currently scheduled on **Monday, May 16, 5:30pm-8:00pm** in class. Project information and a list of suggested topics will be posted on Blackboard as we advance in the semester.

Extra Help. You are welcome to attend the office hours and ask questions about the lectures and about the homework assignments. In addition, math tutors are available at the MLC: http://www.math.sunysb.edu/MLC.

Special Needs. If you have a physical, psychological, medical or learning disability that may impact your course work, please contact Disability Support Services, ECC (Educational Communications Center) Building, Room 128, (631) 632-6748, or at the following website http://studentaffairs.stonybrook.edu/dss/index.shtml. They will determine with you what accommodations, if any, are necessary and appropriate. All information and documentation is confidential.

Academic integrity. Each student must pursue his or her academic goals honestly and be personally accountable for all submitted work. Representing another person's work as your own is always wrong. Faculty is required to report any suspected instances of academic dishonesty to the Academic Judiciary. Faculty in the Health Sciences Center (School of Health Technology & Management, Nursing, Social Welfare, Dental Medicine) and School of Medicine are required to follow their school-specific procedures. For more comprehensive information on academic integrity, including categories of academic dishonesty please refer to the academic judiciary website at http://www.stonybrook.edu/uaa/academicjudiciary.

Critical Incident Management. Stony Brook University expects students to respect the rights, privileges, and property of other people. Faculty are required to report to the Office of University Community Standards any disruptive behavior that interrupts their ability to teach, compromises the safety of the learning environment, or inhibits students' ability to learn. Faculty in the HSC Schools and the School of Medicine are required to follow their school-specific procedures. Further information about most academic matters can be found in the Undergraduate Bulletin, the Undergraduate Class Schedule, and the Faculty-Employee Handbook.