MAT 342 - Applied Complex Analysis

MWF 10:00am-10:53am, Frey Hall 309, Fall 2013

Organizational Information

- **Textbook:** Complex Variables and Applications by James Ward Brown and Ruel V. Churchill, Ninth Edition, McGraw-Hill, 2013.
- **Instructor:** Chi Li , Office: Math Tower: 3-120, Office Hour: T/Th 1:30-3:30pm
- **Grader:** Raquel Perales Aguilar , Office: Math Tower 3-105, Office Hour: Th 2:30-3:30pm, MLC: T/Th 1-2pm

Homework, Syllabus, Grades, Exams

- **Read the textbook:** It's very important to read (and really understand) the text book both before and after the lecture since we don't have time to cover all the details from the book.
- **Homework and Syllabus:** Doing homework is very important for understanding the materials. Note that homework takes 20% of your total scores. Try to do the rest of exercises in the book for more practices. Homework will be collected every Wednesday in the lecture. 6 homework problems will be graded. Solutions to some problems will be provided.
- **Midterm Exams :** 2 midterms in class. Tentative Schedule: Mid 1: Oct. 3 ; Mid 2: Nov. 7.
- **Final Exam** : Dec. 16, 2:15pm-5:00pm.
- **Grading Policy:** The overall numerical grade will by computed by the formula: Homework 20% + Midterm Exam 1 15% + Midterm Exam 2 15% + Final Exam 50%.

Miscellaneous

- Wikipedia articles you may find useful (from the previous course page by Professor Leon Takhtajan)
- A very useful resource is the Math Learning Center (MLC) located in room S240-A of the mathematics building basement. The Math Learning Center is open every day and most evenings. Check the schedule on the door. Another useful resource are your teachers, whose office hours are listed above.
- **Disability Support Services (DSS) Statement:** If you have a physical, psychological, medical or learning disability that may impact your course work, please contact Disability Support Services, ECC (Educational Communications Center) Building, room 128, (631) 632-6748. They will determine with you what accommodations, if any, are necessary and appropriate. All information and documentation is confidential. Students

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Syllabus

Week	Ch.	Topics From	Topics To	Homework	Notes
8/25	1	**1:** Sums/Products	**9:** Arguments	P4: 2,11; P8: 5; P13: 4,5,6,8; P16: 2,13,14; P24: 7,9,10 P4: 1,4; P7: 1; P13: 1,2,7; P16: 3,7,9,10; P23: 1,2,4,6	HW1 Solution
9/1	1	**10:** Roots	**12:** Regions	Part 1: P31: 4,7,8; P34: 1,4,5,7,8 ; Part 2 P30: 1,2,3,5,6; P34: 2,3,6,9,10	Example of types of points HW2 Solution
9/8	2	**13:** Functions/Mappings	**17:** Limits at Infinity	Part1: P43: 1,2,3,4,5,8,9; P54: 1,3,5,7,10 ; Part 2 The rest exercises	HW3 Solution
9/15	2	**18:** Continuity	**22:** Examples of Derivatives	Part1: P55: 13; P61: 2,3,4,6,8,9; P70: 1,2; Part 2	HW4 Solution
9/22	2	**23:** Differentiability	**28:** Uniquely Determined	Part 1: P71: 3,4,5,6,8; P76: 1,2,4,6,7; Part 2	HW5 Solution
9/29	3	**31:** Exponential Function	**32:** Logarithmic Function	Part 1: P79: 1,2,3; P89: 1,4,5,8,10,11,12; Part 2	HW6 Solution
		Review	**Midterm 1(solutions/statistics)**	Practice Midterm 1	Practice Solution
10/6	3	**33:** Branches of Logarithms	**38:** Zeros/Singularities	P95: 1,4,5,10,11; P99: 1; P103: 1,2,3,9; P107: 2,5,8 ;	HW7 Solution
	4	**41:** Derivatives	**42:** Definite Integrals		
10/13	4	**43:** Contours	**49:** Proof (Antiderivatives)	P119: 2,3; P124: 2,6; P132:1,3,4,5,6,10,13; P147: 2 ; Part 2	HW8 Solution The example in class
10/20	4	**50:** Cauchy-Goursat Theorem	**57:** Consequences of Extension	P147: 5; P159: 1,2,5,6,7; P170: 2,3,4,7; Part 2	HW9 Solution

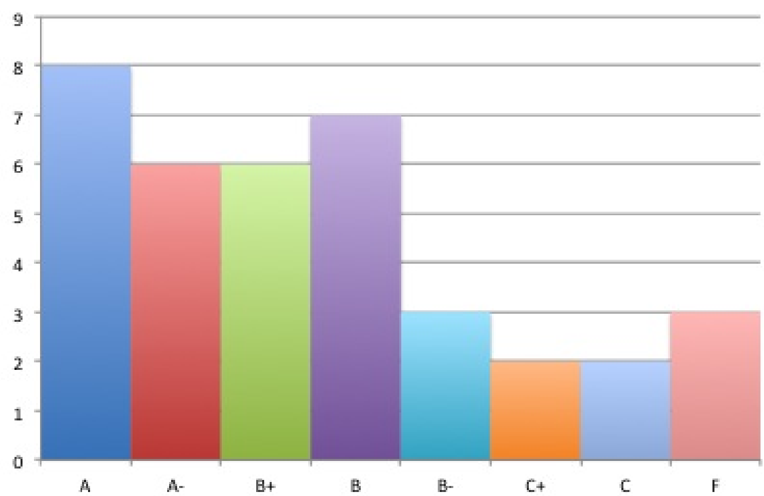
10/27	4	**58:** Louville Theorem	**59:** Maximum Modulus Principle	P138: 1,2,5; P171: 5,10; P177: 1,2,5,6,8; P185: 1,2; Part 2	HW10 Solution
	5	**60:** Sequences	**61:** Convergence of sequences		
11/3	6	**62:** Taylor series	**64** Examples	P196: 2,3,4,6,7,9,11; P205: 1,2,4,5,6,7	HW11 Solution
		Review	**Midterm 2(solutions/statistics)**	Practice Midterm 2	Practice 2 solution
11/10	6	**65:** Negative powers	**71:** Integration/Differentiation	P218: 1,3,4,6,8; P237: 1,2,4; P242: 1	HW12 Solution
11/17	7	**74:** Isolated singular points	**84:** Behavior near singularities	P246: 1,5,7; P264: 3-8; P273: 2,4,6,11	HW13 Solution
11/24	7	**85** Improper Integrals	**86** Examples		
12/1	7	**91** Integration along branch cut	**94** Roche's Theorem Review	Practice Final **Final Solutions and Overall Statistics**	Practice Final solution

Exam Information for MAT 342

Fall 2014

Final Solutions and Overall Statistics: **Final solutions**

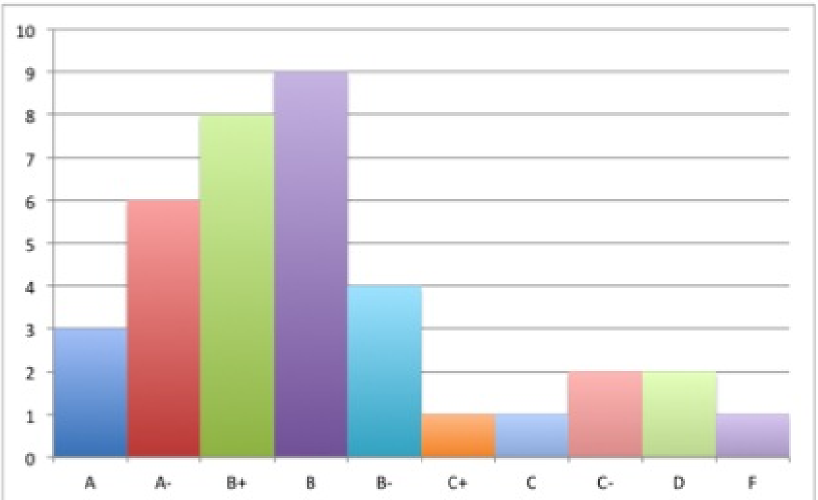
Overall Statistics



Range	Grade
>92	A
88-92	A-
81-87	B+
74-79	B
67-70	B-
62-64	C+
59-60	C
<53	F

MEAN: 78; MEDIAN: 83; High Score: 97; Low Score: 30

Midterm 1: **solution** , **SOLUTION**

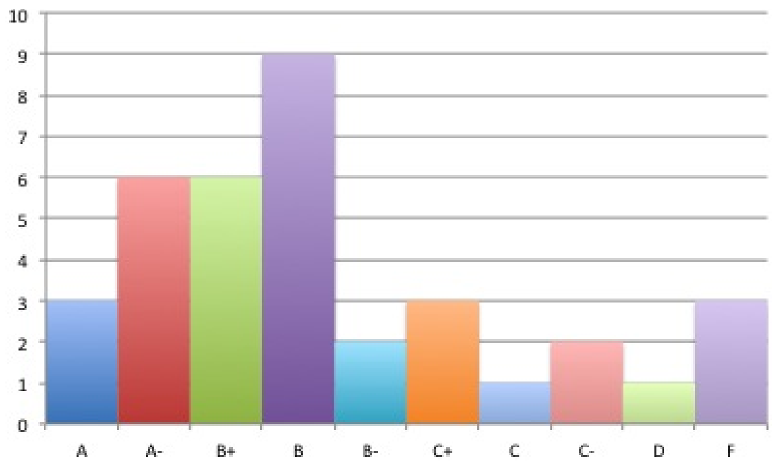


MEAN: 197.19; MEDIAN: 203; High Score: 250; Low Score: 80

Range	Grade
238-250	A
227-232	A-
214-224	B+
190-204	B
178-187	B-
164	C+
152	C
143-147	C-
115-127	D
80	F

The tentative curve for this midterm is shown in the picture. This is just to give you some idea of the distribution of the grades. The real curve will be made only after the final exam.

Midterm 2: **SOLUTION**



MEAN: 172.2; MEDIAN: 179.5; High Score: 250; Low Score: 41

Range	Grade
226-250	A
210-223	A-
198-206	B+
170-192	B
165-167	B-
140-152	C+
132	C
118-122	C-
100	D
<80	F

Final Exam

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sol1.pdf	2014-09-11 23:13	411K	

P24.

9. $S = 1 + z + z^2 + \cdots + z^n$

$z \cdot S = \quad z + z^3 + \cdots + z^n + z^{n+1} \Rightarrow (1-z)S = 1 - z^{n+1} \Rightarrow S = \dfrac{1-z^{n+1}}{1-z}$.

Let $z = e^{i\theta}$, then $1 + e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta} = \dfrac{1 - e^{(n+1)i\theta}}{1 - e^{i\theta}}$

Left hand side $= \left(1 + \cos\theta + \cos(2\theta) + \cdots + \cos(n\theta)\right) + i\left(\sin\theta + \sin(2\theta) + \cdots + \sin(n\theta)\right)$

right hand side $= \dfrac{(1 - e^{(n+1)i\theta})(1 - e^{-i\theta})}{(1 - e^{i\theta})\cdot(1 - e^{-i\theta})} = \dfrac{1 - e^{-i\theta} - e^{(n+1)i\theta} + e^{ni\theta}}{|1 - e^{i\theta}|^2}$

$$= \frac{1 - (\cos\theta - i\sin\theta) - (\cos(n+1)\theta + i\sin(n+1)\theta) + (\cos(n\theta) + i\sin(n\theta))}{(1-\cos\theta)^2 + \sin^2\theta}$$

$$= \frac{(1 - \cos\theta - \cos(n+1)\theta + \cos(n\theta)) + i(\sin\theta - \sin(n+1)\theta + \sin n\theta)}{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$= \frac{(1-\cos\theta) + 2\sin\frac{\theta}{2}\sin\frac{2n+1}{2}\theta + i(\overset{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\overset{\parallel}{\sin\theta}} - 2\cos\frac{2n+1}{2}\theta \cdot \sin\frac{\theta}{2})}{2(1-\cos\theta)}$$

(use $\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$, $\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$
$1 - \cos\theta = 2\sin^2\frac{\theta}{2}$)

$$= \left(\frac{1}{2} + \frac{\sin\frac{2n+1}{2}\theta}{2\sin\frac{\theta}{2}}\right) + i\cdot\left(\frac{\cos\frac{\theta}{2} - \cos\frac{2n+1}{2}\theta}{2\sin\frac{\theta}{2}}\right)$$

comparing the real/imaginary part of both sides, we get:

$$1 + \cos\theta + \cos(2\theta) + \cdots + \cos(n\theta) = \frac{1}{2} + \frac{\sin\frac{2n+1}{2}\theta}{2\sin\frac{\theta}{2}}$$

$$\sin\theta + \sin(2\theta) + \cdots + \sin(n\theta) = \frac{\cos\frac{\theta}{2} - \cos\frac{2n+1}{2}\theta}{2\sin\frac{\theta}{2}} = \frac{\sin(\frac{n+1}{2}\theta)\sin(\frac{n}{2}\theta)}{\sin\frac{\theta}{2}}$$

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hw2-sol.pdf	2014-09-17 13:36	2.4M	

Apache/2.4.7 (Ubuntu) Server at www.math.stonybrook.edu Port 80

Homework 2: Part 2

1. Prove the following identity and explain its geometric meaning:

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2).$$

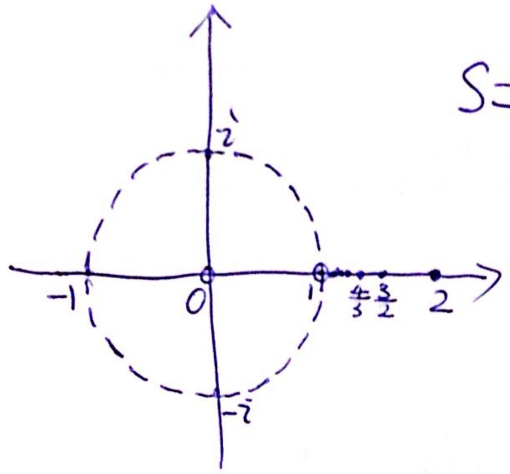
2*. Assume z_1 and z_2 are two different fixed complex numbers. Find the sets described by the following identities. (Hint: use geometric meanings)

(a)

$$\mathrm{Re}\left(\frac{z - z_1}{z - z_2}\right) = 0.$$

(b)

$$\mathrm{Im}\left(\frac{z - z_1}{z - z_2}\right) = 0.$$



$S = \{z ; 0<|z|<1\} \cup \{1+\frac{1}{n}, n=1,2,3,\cdots\}$.

- interior point z_0: all points sufficiently near z_0 (including z_0) are inside S:

$$\boxed{\{z \in \mathbb{C} ; \ 0<|z|<1\}}$$

- exterior points: all points sufficiently near z_0 (including z_0) are NOT contained in S:

$$\boxed{\{z \in \mathbb{C} ; |z|>1, \ z \neq 1+\frac{1}{n} \text{ for any } n=1,2,3,\cdots\}}$$

- boundary points: every ϵ-nbhd. contains at least one point in S and contains at least one point not in S.

$$\boxed{\{0\} \cup \{1+\frac{1}{n}, n=1,2,3,\cdots\} \cup \{z \in \mathbb{C} ; |z|=1\}}.$$

- accumulation points: there is a infinite sequence of DIFFERENT points inside S that approaches z_0.

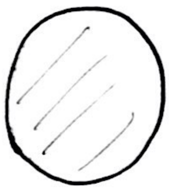
$$\boxed{\{z \in \mathbb{C} ; \ |z| \le 1\}}$$

- closure $\overline{S} = S \cup \partial S = \{z \in \mathbb{C} ; |z| \le 1\} \cup \{1+\frac{1}{n}, n=1,2,3,\cdots$

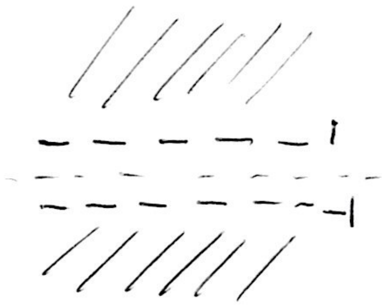
Domain: nonempty + open + connected.

Not Domain:

not open
(and not closed)
contains part of the boundary



$|z| \le 1$
(closed).



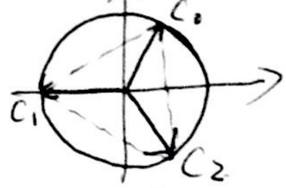
~~$|z|>1$~~ $|\mathrm{Im}\, z|>1$
(not connected)

P31

4. (a) $(-1)^{\frac{1}{3}}$ $\quad -1 = 1 \cdot e^{i\cdot\pi}$

principal root $C_0 = 1^{\frac{1}{3}} e^{\frac{i\pi}{3}} = e^{i\cdot\frac{\pi}{3}} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

3rd root of unity: $\omega_3 = e^{i\cdot\frac{2\pi}{3}}$. so we can get the other roots by rotation:

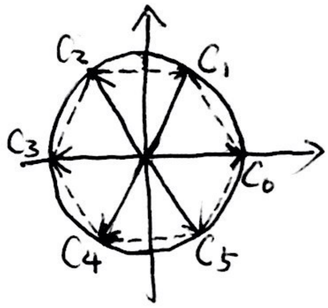


$C_1 = C_0 \cdot \omega_3 = e^{i\cdot\frac{\pi}{3}} \cdot e^{i\cdot\frac{2\pi}{3}} = e^{i\pi} = -1$

$C_2 = C_0 \cdot \omega_3^2 = e^{i\cdot\frac{\pi}{3}} \cdot e^{i\cdot\frac{4\pi}{3}} = e^{i\cdot\frac{5\pi}{3}} = e^{-i\cdot\frac{\pi}{3}} = \frac{1}{2} - i\cdot\frac{\sqrt{3}}{2}$

(b) $8^{\frac{1}{6}}$ $\quad$ principal root $C_0 = 8^{\frac{1}{6}} e^{\frac{i\cdot 0}{6}} = (2^3)^{\frac{1}{6}} = \sqrt{2}$.

6-th root of unity $\omega_6 = e^{i\cdot\frac{2\pi}{6}}$. so the other roots are:



$C_1 = \sqrt{2} e^{i\frac{\pi}{3}} = \sqrt{2}\cdot\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{1+i\sqrt{3}}{\sqrt{2}}$.

$C_2 = \sqrt{2}\, e^{i\frac{2\pi}{3}} = \frac{-1+i\sqrt{3}}{\sqrt{2}}$ $\qquad C_3 = -\sqrt{2}$

$C_4 = \sqrt{2}\cdot e^{i\frac{4\pi}{3}} = \frac{-1-i\sqrt{3}}{\sqrt{2}}$, $\qquad C_5 = \sqrt{2}\cdot e^{-i\frac{\pi}{3}} = \frac{1-i\sqrt{3}}{\sqrt{2}}$

7. $1 + C + C^2 + \cdots + C^{n-1} = \frac{C^n - 1}{C - 1} = 0$.

P31

8. (a). $az^2+bz+c=0$

$$\|$$

$$a\left(z^2+\frac{b}{a}z+\frac{c}{a}\right)=a\cdot\left(z^2+\frac{b}{a}z+\left(\frac{b}{2a}\right)^2\right)-\frac{b^2}{4a}+c.$$

$$=a\cdot\left(z+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a}$$

$$\Rightarrow \left(z+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2} \Rightarrow z+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

(b). $z^2+2z+(1-i)=0$. $a=1,\ b=2,\ c=1-i$.

$$b^2-4ac=4-4\cdot 1(1-i)=4i=4\cdot e^{i\frac{\pi}{2}}.$$

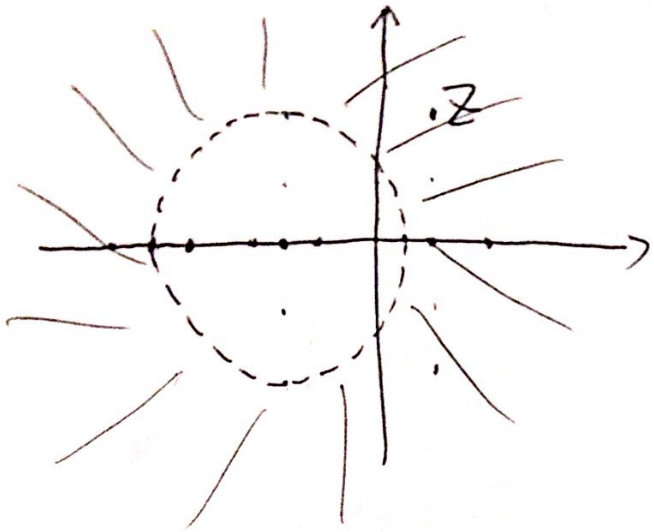
$$\sqrt{b^2-4ac}=2\cdot e^{i\left(\frac{\pi}{4}+\frac{2\pi k}{2}\right)} \underline{\underline{k=0,1}} \begin{cases} 2\cdot e^{i\frac{\pi}{4}}=\sqrt{2}+i\sqrt{2} \\ 2\cdot e^{i\frac{5\pi}{4}}=-\sqrt{2}-i\sqrt{2}. \end{cases}$$

$$\Rightarrow z=\frac{-2\pm(\sqrt{2}+i\sqrt{2})}{2}=-1\pm\frac{1+i}{\sqrt{2}}$$

So 2 roots are : $\left(-1+\frac{1}{\sqrt{2}}\right)+\frac{i}{\sqrt{2}},\ \left(-1-\frac{1}{\sqrt{2}}\right)-\frac{i}{\sqrt{2}}$

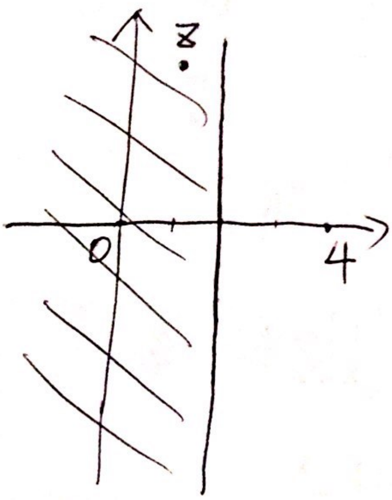
P34

1.(b) $|2z+3|>4 \iff \left|z-\left(-\frac{3}{2}\right)\right|>2$



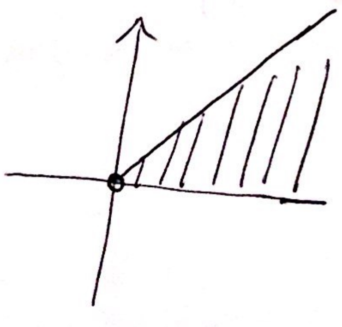
open and connected
$\Downarrow$
domain.

(f). $|z-4|\geq|z|$



closed, connected.

(e). $0\leq \arg z\leq\frac{\pi}{4}$ $(z\neq 0)$

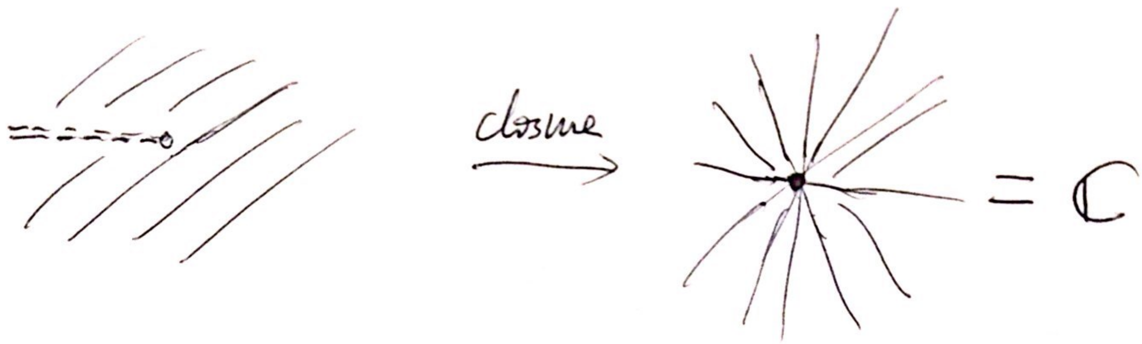


not closed (0 is bdry pt. not included).

connected.

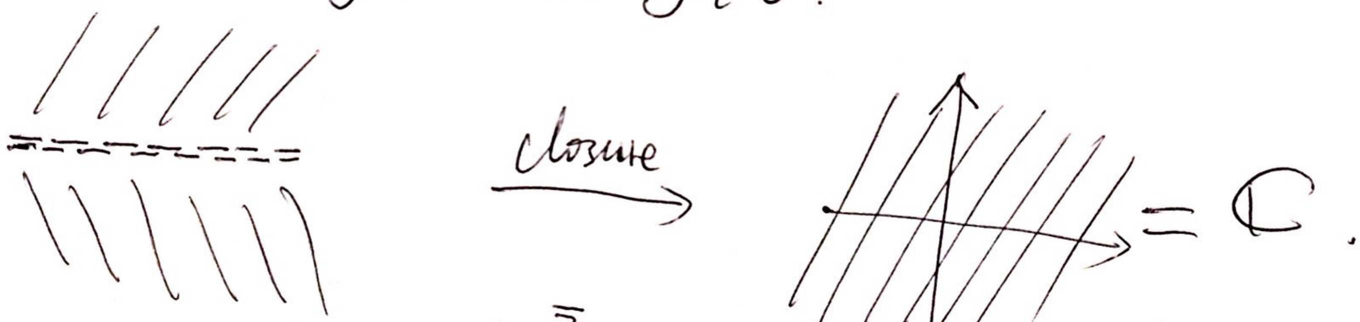
P35

4. (a) $-\pi < \operatorname{Arg} z < \pi \quad (z \neq 0)$.



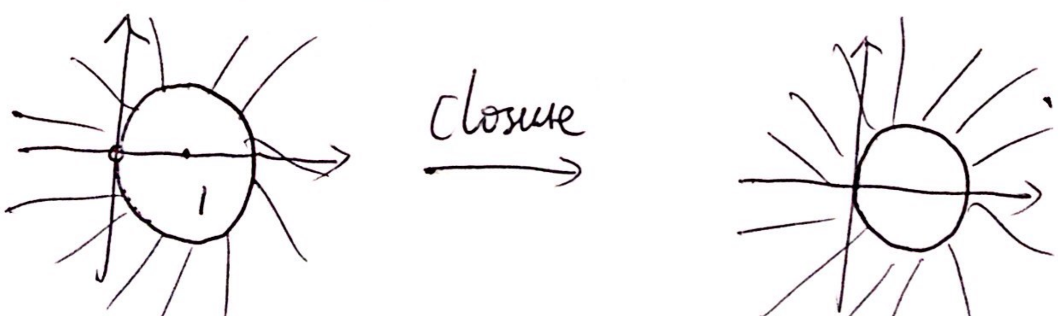
(b). $|\operatorname{Re} z| < |z| \iff |x| < \sqrt{x^2+y^2} \iff x^2 < x^2+y^2$

$\iff y^2 > 0 \iff y \neq 0$.



(c) $\operatorname{Re}\left(\frac{1}{z}\right) \le \frac{1}{2} \overset{\frac{1}{z} = \frac{\bar{z}}{|z|^2}}{\iff} \frac{x}{x^2+y^2} \le \frac{1}{2} \iff \begin{cases} x^2+y^2-2x \ge 0 \\ (x,y) \neq (0,0). \end{cases}$

$\iff \begin{cases} (x-1)^2+y^2 \ge 1 \\ (x,y) \neq (0,0). \end{cases}$

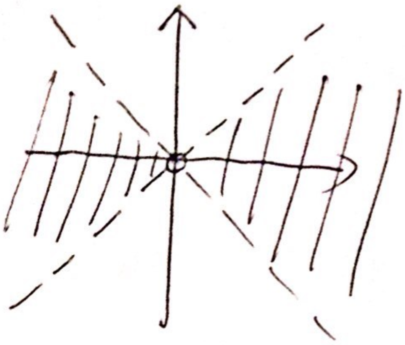


P35

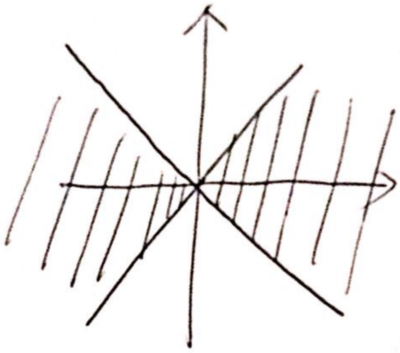
(d) $\mathrm{Re}(z^2)>0 \iff x^2-y^2>0 \iff x^2>y^2$

$\quad = \mathrm{Re}\big((x^2-y^2)+2ixy\big)$ $\qquad\qquad \Updownarrow$

$\qquad\qquad\qquad\qquad\qquad\qquad |x|>|y|.$

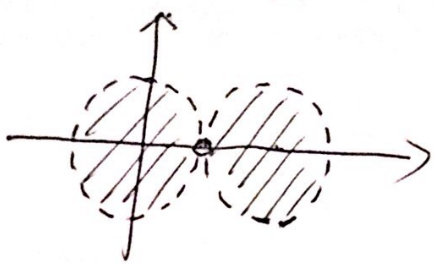


$\xrightarrow{\text{closure}}$

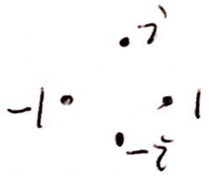


5. $|z|<1$ or $|z-2|<1$

not connected.



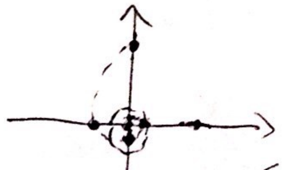
7. (a) $z_n = i^n$



no accumulation pt.

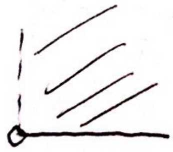
For a point z_0 to be an accumulation pt, there must be a sequence of $\boxed{\text{different}}$ pts (in the set) approaching z_0.

(b) $z_n = \frac{i^n}{n}$.



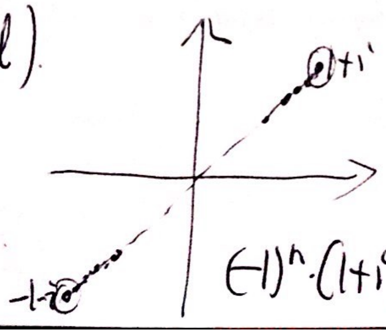
0 is the only accumulation pt.

(c). $0 \le \mathrm{Arg}\, z < \frac{\pi}{2}$



accumulation pts: all pts in the 1st quadrant.

(d)



2 accumulation pts: $1+i$ and $-1-i$.

$(-1)^n \cdot (1+i) \frac{n+1}{n}$ $(n=1,2,\ldots)$.

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P35

8. A set contains each of its accumulation pts
$\Updownarrow$
closed set.

Pf: "$\Rightarrow$" Suppose S contains each of its accumulation pts, we want to show S is closed, i.e. S contains all of its boundary points.

Let z_0 be any boundary pt. We prove z_0 is contained in S by contradiction. Suppose $\underline{z_0 \notin S}$. For any ε-nbhd. of z_0 (U_ε), there exists some point $z \in S$ in this neighborhood U_ε, because z_0 is a boundary point. Note that z is not the center since $z_0 \notin S$. So we get that any deleted ε-nbhd. U_ε^* contains some point $z \in S$.

So we know that z_0 is an accumulation point.

By our assumption, $\underline{z_0 \in S}$. Contradiction.

So. $z_0 \in S$. $\Rightarrow$ S contains each of its boundary pts.

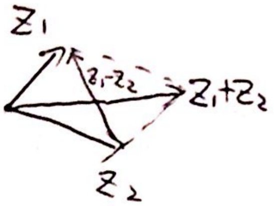
$\Rightarrow$ S is closed.

HW2 Part 2:

1. $|z_1+z_2|^2+|z_1-z_2|^2 = (z_1+z_2)(\overline{z_1}+\overline{z_2})+(z_1-z_2)(\overline{z_1}-\overline{z_2})$

$= |z_1|^2+z_1\overline{z_2}+z_2\overline{z_1}+|z_2|^2+|z_1|^2-z_1\overline{z_2}-z_2\overline{z_1}+|z_2|^2$

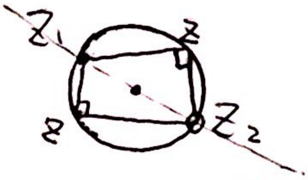
$= 2\left(|z_1|^2+|z_2|^2\right).$



For any parallelogram, the sum of squares of two diagonals is equal to the sum of squares of four sides.

2*. (a) $\mathrm{Re}\,\frac{z-z_1}{z-z_2}=0 \iff \frac{z-z_1}{z-z_2}=iy$ for some $y\in\mathbb{R}$.

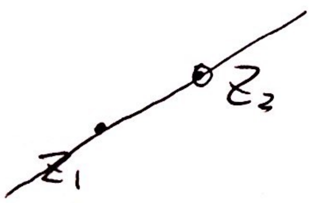
$\iff z_1-z=(iy)(z_2-z)$ for some $y\in\mathbb{R}$, and $z\neq z_2$



$\iff \overrightarrow{zz_1}\perp\overrightarrow{zz_2}$ and $z\neq z_2$.

$\iff z$ lies on the circle passing throug z_1 & z_2 that has $\overline{z_1z_2}$ as a diameter.
and $z\neq z_2$

(b) $\mathrm{Im}\,\frac{z-z_1}{z-z_2}=0 \iff \frac{z-z_1}{z-z_2}=x$ for some $x\in\mathbb{R}$



$\iff z-z_1=x(z-z_2)$ for some $x\in\mathbb{R}$ and $z\neq z_2$.

$\iff \overrightarrow{zz_1}\,//\,\overrightarrow{zz_2}$ and $z\neq z_2$

$\iff z$ lies on the straight line passing throug z_1 & z_2
and $z\neq z_2$

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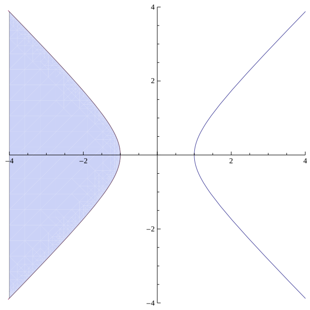
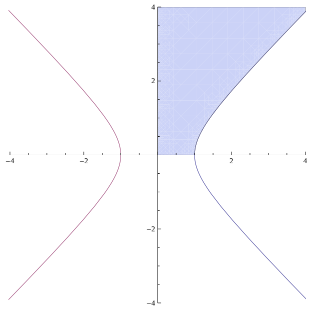
55a.JPG	2014-09-15 19:34	854K	
Week3.pdf	2014-09-08 13:38	585K	
hw3-sol.pdf	2014-09-21 23:34	753K	

Homework 3: Part 2

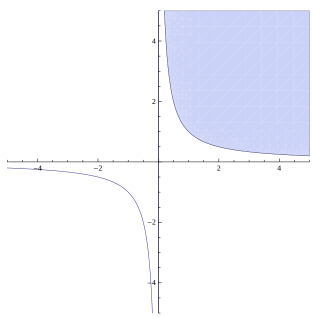
1*: Find the image of the following shaded domains on the z-plane under the map $w = z^2$.



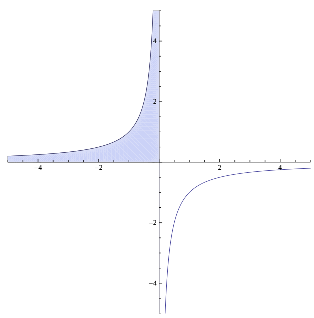
(a) $x^2 - y^2 > 1, x < 0$



(b) $x^2 - y^2 < 1, x > 0, y > 0$

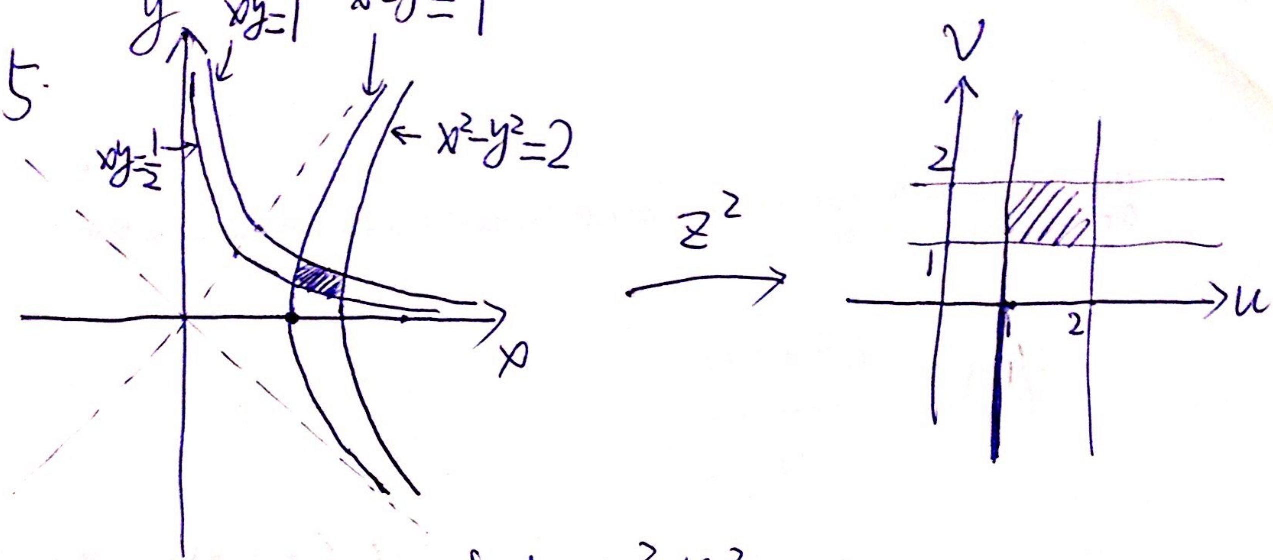


(c) $xy > 1, y > 0$



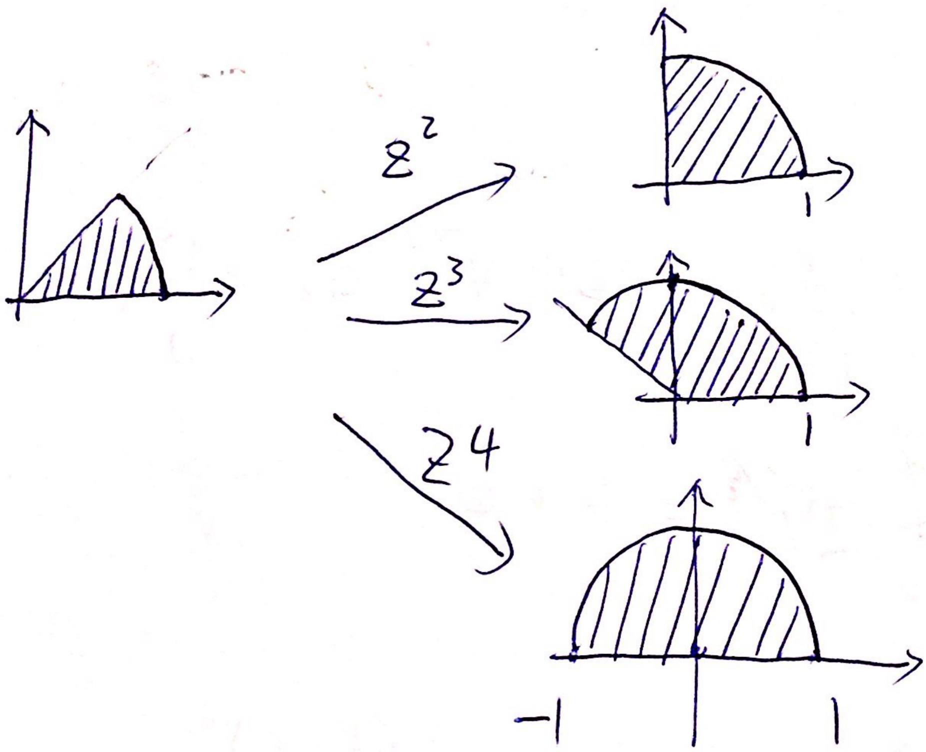
(d) $xy > -1, x < 0, y > 0$

P44.5

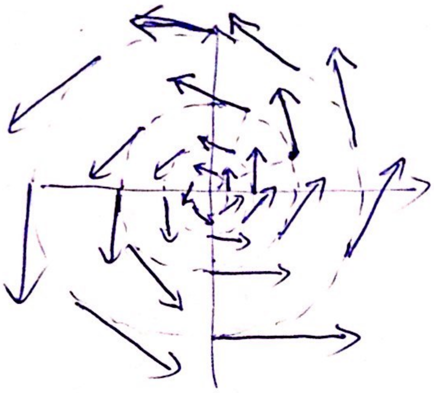


$$w = z^2 \iff \begin{cases} u = x^2 - y^2 \\ v = 2xy. \end{cases}$$

8.

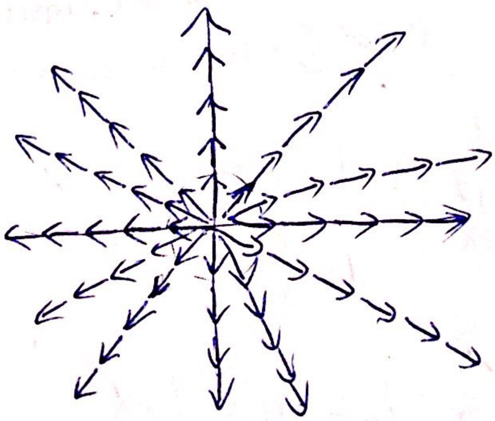


9. $w = iz = i(x+iy) = -y+ix$. vector field: $(x,y) \mapsto \langle -y, x \rangle$.

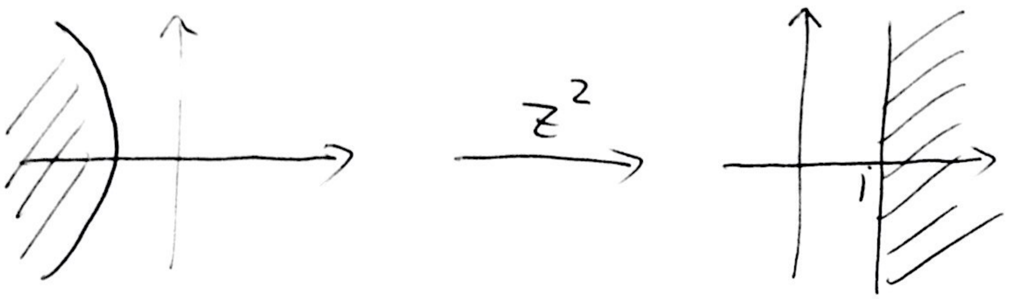


44 9(b).

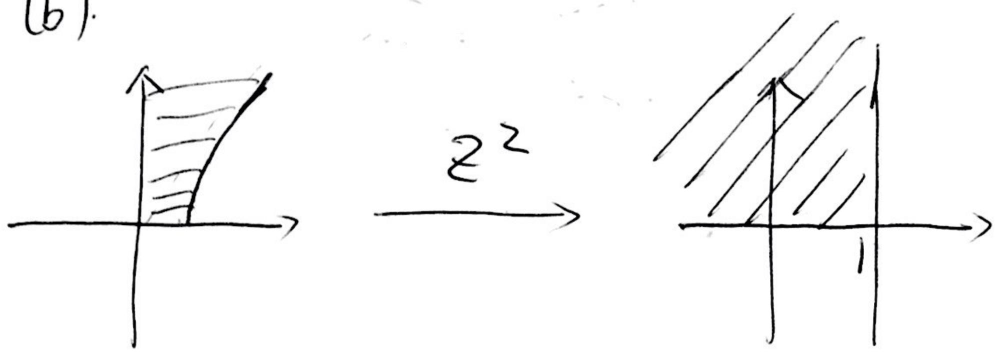
$w = \frac{z}{|z|} = \frac{x+iy}{\sqrt{x^2+y^2}}$ vector field: $(x,y) \mapsto \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$.



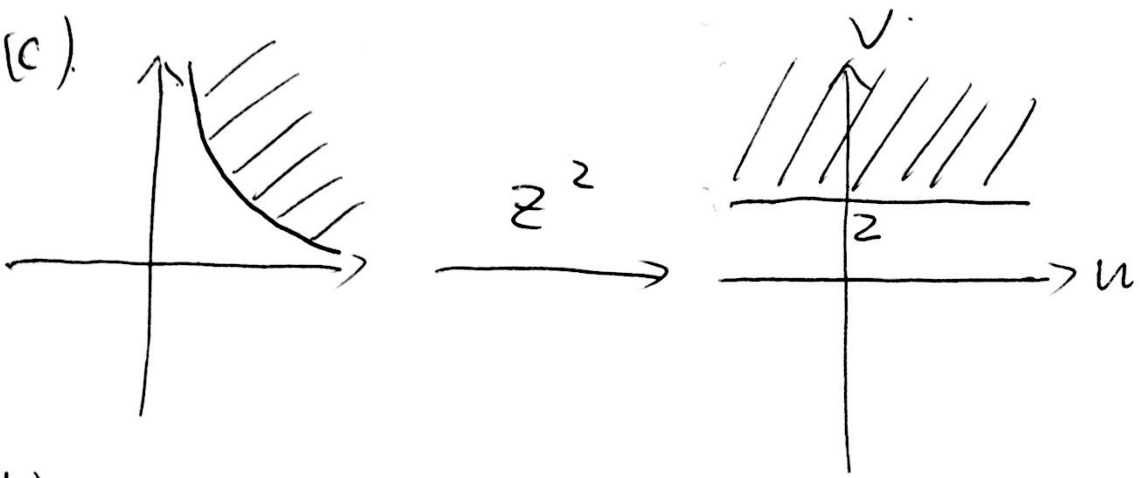
Part 2. 1*(a).



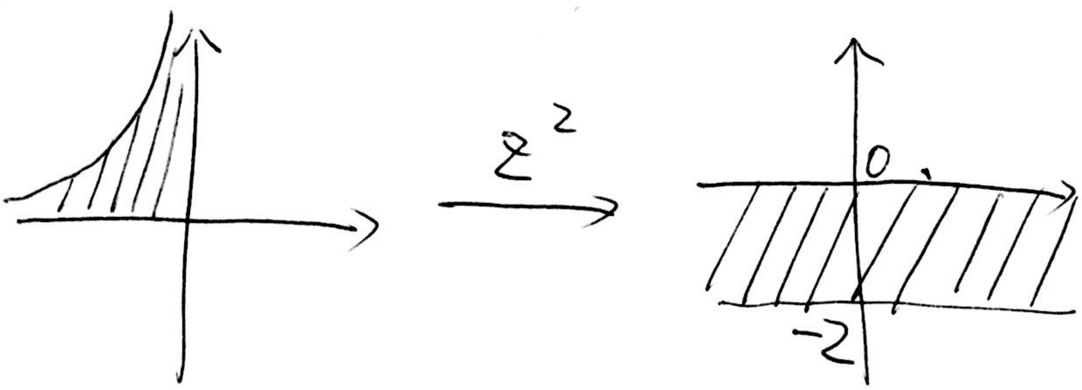
(b).



(c).



(d)



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hw4-sol.pdf	2014-09-24 13:57	2.0M	

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Homework 4: Part 2

1*: Assume that the unit sphere embedded in $\mathbb{R}^3$ is given by the equation:

$$a^2 + b^2 + c^2 = 1.$$

Assume that the complex plane sits in $\mathbb{R}^3$ as the plane given by $\{c = 0\}$.

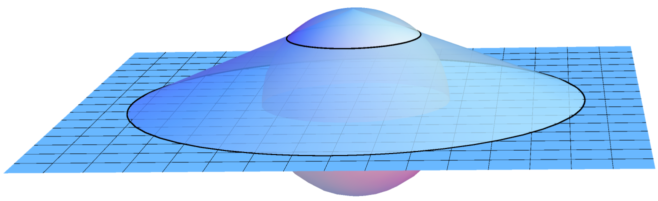

Figure 1: Stereographic Projection

(a) Show that the stereographic projection from the north pole $N = (0, 0, 1)$ is given by the following formulas:

- (from the complex plane to the sphere) $z \mapsto (a, b, c)$ is given by:

$$a = \frac{2\mathrm{Re}(z)}{1 + |z|^2}, b = \frac{2\mathrm{Im(z)}}{1 + |z|^2}, c = \frac{|z|^2 - 1}{|z|^2 + 1}.$$

- (from the sphere to the complex plane) $(a, b, c) \mapsto z$ is given by:

$$z = \frac{a + bi}{1 - c}.$$

(b) Show that, under the stenographic projection, the neighborhood at infinity $\{z; |z| > \frac{1}{\epsilon}\}$ corresponds to the following neighborhood of the north pole:

$$\left\{(a, b, c) \in \mathbb{R}^3; c > \frac{1 - \epsilon^2}{1 + \epsilon^2},\ a^2 + b^2 = 1 - c^2\right\}.$$

P55. 13. a set S is unbounded $\Leftrightarrow$ every nbhd. of the pt. at infinity contains at least one point in S.

Pf:

S unbounded $\Leftrightarrow \forall R>0, \exists z\in S$, and $|z|>R$.

$\Leftrightarrow \forall \varepsilon>0, \exists z\in S$, and $|z|>\frac{1}{\varepsilon}$.

$\Leftrightarrow$ every ε-nbhd. of ∞ contains at least 1 pt in S

$\Leftrightarrow$ every nbhd of ∞ - - - - - - - .

P61. 2 (a) $f(z)=3z^2-2z+4 \Rightarrow f'(z)=6z-2$.

(b) $f(z)=(2z^2+i)^5 \Rightarrow f'(z)=5\cdot(2z^2+i)\cdot 4z$.

(c). $f(z)=\frac{z-1}{2z+1}\ (z\neq-\frac{1}{2}) \Rightarrow f'(z)=\frac{1\cdot(2z+1)-(z-1)\cdot 2}{(2z+1)^2}=\frac{3}{(2z+1)^2}$

(d) $f(z)=\frac{(1+z^2)^4}{z^2}\ (z\neq 0) \Rightarrow f'(z)=\frac{4\cdot(1+z^2)^3\cdot 2z\cdot z^2-(1+z^2)^4\cdot 2z}{z^{\cancel{4}3}}$

$$=\frac{(1+z^2)^3}{z^3}\cdot\left(8z^2-2(1+z^2)\right)=\frac{(1+z^2)^3(6z^2-2)}{z^3}$$

4. $$\lim_{z\to z_0}\frac{f(z)}{g(z)}\overset{f(z_0)=0=g(z_0).}{=}\lim_{z\to z_0}\frac{f(z)-f(z_0)}{g(z)-g(z_0)}=\lim_{z\to z_0}\frac{\frac{f(z)-f(z_0)}{z-z_0}}{\frac{g(z)-g(z_0)}{z-z_0}}=\frac{\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}}{\lim_{z\to z_0}\frac{g(z)-g(z_0)}{z-z_0}}$$

$$=\frac{f'(z_0)}{g'(z_0)}$$

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P61.

8 (a) $f(z) = \operatorname{Re} z$.

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{z\to z_0} \frac{\operatorname{Re} z - \operatorname{Re} z_0}{z-z_0} = \lim_{z\to z_0} \frac{\operatorname{Re}(z-z_0)}{z-z_0} = \lim_{\Delta z\to 0} \frac{\operatorname{Re}\Delta z}{\Delta z}$$

horizontally 1

vertically 0.

(b) $f(z) = \operatorname{Im} z$.

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{z\to z_0} \frac{\operatorname{Im}(z)-\operatorname{Im}(z_0)}{z-z_0} = \lim_{z\to z_0} \frac{\operatorname{Im}(z-z_0)}{z-z_0} = \lim_{\Delta z\to 0} \frac{\operatorname{Im}(\Delta z)}{\Delta z}$$

0 horizontally

1 vertically.

9. $f(z) = \begin{cases} \dfrac{\bar{z}^2}{z} & z \neq 0 \\ 0 & z = 0. \end{cases}$

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z-\cancel{0}} = \lim_{z\to 0} \frac{\frac{\bar{z}^2}{z} - 0}{z} = \lim_{z\to 0} \frac{\bar{z}^2}{z^2} = \lim_{\Delta z\to 0} \frac{(\overline{\Delta z})^2}{(\Delta z)^2}$$

horizontally $\dfrac{(\overline{\Delta z})^2}{(\Delta z)^2} = \dfrac{(\Delta x)^2}{(\Delta x)^2} = 1$.

vertically $\dfrac{(\overline{\Delta z})^2}{(\Delta z)^2} = \dfrac{(-i\Delta y)^2}{(i\Delta y)^2} = \dfrac{-\Delta y^2}{-\Delta y^2} = 1$. $\Rightarrow$ limit does NOT exist.

$\Delta x = \Delta y$: $\dfrac{(\overline{\Delta z})^2}{(\Delta z)^2} = \dfrac{(\Delta x - i\Delta x)^2}{(\Delta x + i\Delta x)^2} = \dfrac{(\Delta x)^2 \cdot (-2i)}{(\Delta x)^2 \cdot 2i} = -1$.

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P70. 1. (a) $f(z)=\bar{z} \iff \begin{cases} u=x \\ v=-y \end{cases} \Rightarrow \begin{cases} u_x=1 \neq v_y=-1 \end{cases}$

$\Downarrow$

(b). $f(z)=z-\bar{z} \iff \begin{cases} u=0 \\ v=2y \end{cases} \Rightarrow \begin{cases} u_x=0 \neq v_y=2 \end{cases} \Rightarrow f'(z)$ DNE at any pt

$= (x+iy)-(x-iy) = 2iy$

(c). $f(z)=2x+ixy^2 \iff \begin{cases} u=2x \\ v=xy^2 \end{cases} \Rightarrow \begin{cases} u_x=2 & v_y=2xy \\ u_y=0 & -v_x=-y^2 \end{cases}$

CR eqs: $\begin{cases} 2=2xy \\ 0=-y^2 \end{cases} \Leftarrow$ no solution $\Rightarrow f'(z)$ DNE.

(d) $f(z)=e^x \cdot e^{-iy} = e^x(\cos y - i\sin y) \iff \begin{cases} u=e^x \cdot \cos y \\ v=-e^x \sin y. \end{cases}$

CR eq. $\begin{cases} u_x = e^x \cos y = -e^x \cdot \cos y = v_y \\ u_y = -e^x \cdot \sin y = +e^x \sin y = -v_x \end{cases} \Rightarrow \begin{cases} \cos y = -\cos y \\ -\sin y = \sin y. \end{cases}$

$\Rightarrow \cos y = \sin y = 0$ no solution $\Rightarrow f'(z)$ DNE at any pt.

2. (d) $f(z) = \cos x \cdot \cosh y - i \sin x \sinh y. \Leftrightarrow \begin{cases} u = \cos x \cdot \cosh y \\ v = -\sin x \cdot \sinh y \end{cases}$

$u_x = -\sin x \cdot \cosh y, \quad u_y = \cos x \cdot \sinh y$

$v_x = -\cos x \cdot \sinh y, \quad v_y = -\sin x \cdot \cosh y \Rightarrow \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

$\Rightarrow f'(z)$ exists and is equal to $u_x + i v_x$

$= -\sin x \cdot \cosh y - i \cdot \cos x \cdot \sinh y$

$= \tilde{u} + i \tilde{v}$

$\tilde{u}_x = -\cos x \cosh y, \quad \tilde{u}_y = -\sin x \cdot \sinh y$

$\tilde{v}_x = \sin x \cdot \sinh y, \quad \tilde{v}_y = -\cos x \cdot \cosh y \Rightarrow \begin{cases} \tilde{u}_x = \tilde{v}_y \\ \tilde{u}_y = -\tilde{v}_x \end{cases}$

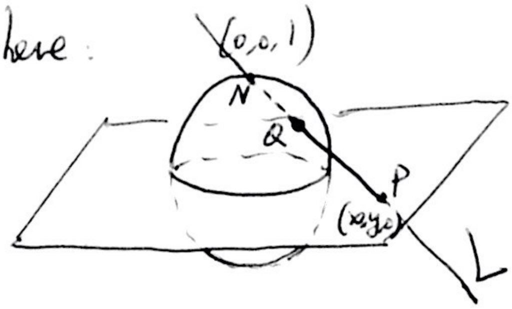
$\Rightarrow f''(z)$ exists and is equal to $\tilde{u}_x + i \tilde{v}_x$

$= -\cos x \cdot \cosh y + i \cdot \sin x \cdot \sinh y.$

$= -f(z).$

Part 2

1. (a). From the complex plane to the sphere:



$z = (x, y) \to (x, y, 0) \in \mathbb{R}^3$.

parametric equation of L:

$$a = tx,\ b = ty,\ c = 1-t \qquad \left((a,b,c) = t(x,y,0) + (1-t)(0,0,1) \right)$$

Q lies on the sphere $\Rightarrow (tx)^2 + (ty)^2 + (1-t)^2 = 1 \Rightarrow t^2(x^2+y^2+1) = 2t$

$$(tx)^2 + (ty)^2 + (1-t)^2 = t^2(x^2+y^2) + t^2 - 2t + 1$$

$\Downarrow$ $t \neq 0$ since $Q \neq N$

$$t = \frac{2}{1+x^2+y^2}$$

Coordinates of Q:

$$\Rightarrow a = \frac{2x}{1+x^2+y^2} = \frac{2\,\mathrm{Re}\,z}{1+|z|^2},\quad b = \frac{2y}{1+x^2+y^2} = \frac{2\,\mathrm{Im}\,z}{1+|z|^2},\quad c = 1 - \frac{2}{1+x^2+y^2} = \frac{x^2+y^2-1}{x^2+y^2+1} = \frac{|z|^2-1}{|z|^2+1}.$$

(b) From the sphere to the complex plane:

Fix $Q \in$ sphere, $Q = (a_0, b_0, c_0)$, Line L: $(a(t), b(t), c(t)) = t(a,b,c) + (1-t)(0,0,1) = (ta, tb, tc+1-t)$.

P satisfies: $tc + (1-t) = 0 \Rightarrow t = \dfrac{1}{1-c}$

$$tc + (1-t) = 1-(1-c)t$$

$$\Rightarrow P = \left(\frac{a}{1-c}, \frac{b}{1-c}, 0\right) = \frac{a+bi}{1-c}.$$

by (a)

(b). $|z| > \frac{1}{\varepsilon} \Leftrightarrow \left|\frac{a+bi}{1-c}\right| > \frac{1}{\varepsilon}$

$$\left|\frac{a+bi}{1-c}\right| = \frac{\sqrt{a^2+b^2}}{1-c} = \frac{\sqrt{1-c^2}}{1-c} \qquad (a^2+b^2=1-c^2).$$

on the sphere.

$$\Leftrightarrow \frac{1-c^2}{(1-c)^2} > \frac{1}{\varepsilon^2} \Leftrightarrow \frac{1+c}{1-c} > \frac{1}{\varepsilon^2}$$

$$\frac{1-c^2}{(1-c)^2} = \frac{(1+c)\cdot(1-c)}{(1-c)^2} = \frac{1+c}{1-c}$$

$$\Leftrightarrow \varepsilon^2 + c\cdot\varepsilon^2 > 1-c \Leftrightarrow c\cdot(1+\varepsilon^2) > 1-\varepsilon^2$$

$$\Leftrightarrow c > \frac{1-\varepsilon^2}{1+\varepsilon^2}. \quad \& \quad a^2+b^2=1-c^2.$$

3. From results obtained in Secs. 21 and 23, determine where $f'(z)$ exists and find its value when

(a) $f(z) = 1/z$; (b) $f(z) = x^2 + iy^2$; (c) $f(z) = z \operatorname{Im} z$.

Ans. (a) $f'(z) = -1/z^2$ $(z \neq 0)$; (b) $f'(x + ix) = 2x$; (c) $f'(0) = 0$.

4. Use the theorem in Sec. 24 to show that each of these functions is differentiable in the indicated domain of definition, and also to find $f'(z)$:

(a) $f(z) = 1/z^4$ $(z \neq 0)$;
(b) $f(z) = e^{-\theta}\cos(\ln r) + i e^{-\theta}\sin(\ln r)$ $(r > 0, 0 < \theta < 2\pi)$.

Ans. (b) $f'(z) = i\,\dfrac{f(z)}{z}$.

7 - **5.** Solve equations (2), Sec. 24 for u_x and u_y to show that

$$u_x = u_r\cos\theta - u_\theta\frac{\sin\theta}{r}, \qquad u_y = u_r\sin\theta + u_\theta\frac{\cos\theta}{r}.$$

Then use these equations and similar ones for v_x and v_y to show that in Sec. 24 equations (4) are satisfied at a point z_0 if equations (6) are satisfied there. Thus complete the verification that equations (6), Sec. 24, are the Cauchy–Riemann equations in polar form.

6. Let a function $f(z) = u + iv$ be differentiable at a nonzero point $z_0 = r_0\exp(i\theta_0)$. Use the expressions for u_x and v_x found in Exercise 5, together with the polar form (6), Sec. 24, of the Cauchy–Riemann equations, to rewrite the expression

$$f'(z_0) = u_x + iv_x$$

in Sec. 23 as

$$f'(z_0) = e^{-i\theta}(u_r + iv_r),$$

where u_r and v_r are to be evaluated at (r_0, θ_0).

7. (a) With the aid of the polar form (6), Sec. 24, of the Cauchy–Riemann equations, derive the alternative form

$$f'(z_0) = \frac{-i}{z_0}(u_\theta + iv_\theta)$$

of the expression for $f'(z_0)$ found in Exercise 6.

(b) Use the expression for $f'(z_0)$ in part *(a)* to show that the derivative of the function $f(z) = 1/z$ $(z \neq 0)$ in Exercise 3(*a*) is $f'(z) = -1/z^2$.

8. (a) Recall (Sec. 6) that if $z = x + iy$, then

$$x = \frac{z + \bar{z}}{2} \quad \text{and} \quad y = \frac{z - \bar{z}}{2i}.$$

By *formally* applying the chain rule in calculus to a function $F(x, y)$ of two real variables, derive the expression

$$\frac{\partial F}{\partial \bar{z}} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right).$$

(*b*) Define the operator

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

suggested by part (*a*), to show that if the first-order partial derivatives of the real and imaginary components of a function $f(z) = u(x, y) + iv(x, y)$ satisfy the Cauchy–Riemann equations, then

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}[(u_x - v_y) + i(v_x + u_y)] = 0.$$

Thus derive the ***complex form*** $\partial f/\partial \bar{z} = 0$ of the Cauchy–Riemann equations.

25. ANALYTIC FUNCTIONS

We are now ready to introduce the concept of an analytic function. A function f of the complex variable z is ***analytic in an open set*** S if it has a derivative everywhere in that set. It is ***analytic at a point*** z_0 if it is analytic in some neighborhood of z_0.*

Note how it follows that if f is analytic at a point z_0, it must be analytic at *each* point in some neighborhood of z_0. If we should speak of a function that is analytic in a set S that is not open, it is to be understood that f is analytic in an open set containing S.

An ***entire*** function is a function that is analytic at each point in the entire plane.

EXAMPLES. The function $f(z) = 1/z$ is analytic at each nonzero point in the finite plane since its derivative $f'(z) = -1/z^2$ exists at such a point. But the function $f(z) = |z|^2$ is not analytic anywhere since its derivative exists only at $z = 0$ and not throughout any neighborhood. (See Example 3, Sec. 19.) Finally, since the derivative of a polynomial exists everywhere, it follows that *every polynomial is an entire function.*

A necessary, but by no means sufficient, condition for a function to be analytic in a domain D is clearly the continuity of f throughout D. (See the statement in italics near the end of Sec. 19.) Satisfaction of the Cauchy–Riemann equations is also necessary, but not sufficient. Sufficient conditions for analyticity in D are provided by the theorems in Secs. 23 and 24.

Other useful sufficient conditions are obtained from the rules for differentiation in Sec. 20. The derivatives of the sum and product of two functions exist wherever the functions themselves have derivatives. Thus, *if two functions are analytic in a domain D, their sum and their product are both analytic in D.* Similarly, *their quotient is analytic in D provided the function in the denominator does not vanish at any point in D.* In particular, the quotient $P(z)/Q(z)$ of two polynomials is analytic in any domain throughout which $Q(z) \neq 0$.

*The terms *regular* and *holomorphic* are also used in the literature to denote analyticity.

where c is a real constant. If $c = 0$, it follows that $f(z) = 0$ everywhere in D. If $c \neq 0$, the property $z\bar{z} = |z|^2$ of complex numbers tells us that

$$f(z)\overline{f(z)} = c^2 \neq 0$$

and hence that $f(z)$ is never zero in D. So

$$\overline{f(z)} = \frac{c^2}{f(z)} \quad \text{for all } z \text{ in } D,$$

and it follows from this that $\overline{f(z)}$ is analytic everywhere in D. The main result in Example 3 just above thus ensures that $f(z)$ is constant throughout D.

EXERCISES

1. Apply the theorem in Sec. 23 to verify that each of these functions is entire:

(*a*) $f(z) = 3x + y + i(3y - x)$; (*b*) $f(z) = \cosh x \cos y + i \sinh x \sin y$;

(*c*) $f(z) = e^{-y} \sin x - ie^{-y} \cos x$; (*d*) $f(z) = (z^2 - 2)e^{-x}e^{-iy}$.

2. With the aid of the theorem in Sec. 21, show that each of these functions is nowhere analytic:

(*a*) $f(z) = xy + iy$; (*b*) $f(z) = 2xy + i(x^2 - y^2)$;

(*c*) $f(z) = e^y e^{ix}$.

3. State why a composition of two entire functions is entire. Also, state why any *linear combination* $c_1 f_1(z) + c_2 f_2(z)$ of two entire functions, where c_1 and c_2 are complex constants, is entire.

4. In each case, determine the singular points of the function and state why the function is analytic everywhere else:

(*a*) $f(z) = \dfrac{2z + 1}{z(z^2 + 1)}$; (*b*) $f(z) = \dfrac{z^3 + i}{z^2 - 3z + 2}$;

(*c*) $f(z) = \dfrac{z^2 + 1}{(z + 2)(z^2 + 2z + 2)}$.

Ans. (*a*) $z = 0, \pm i$; (*b*) $z = 1, 2$; (*c*) $z = -2, -1 \pm i$.

5. According to Example 2, Sec. 24, the function

$$g(z) = \sqrt{r}e^{i\theta/2} \qquad (r > 0, -\pi < \theta < \pi)$$

is analytic in its domain of definition, with derivative

$$g'(z) = \frac{1}{2\,g(z)}.$$

Show that the composite function $G(z) = g(2z - 2 + i)$ is analytic in the half plane $x > 1$, with derivative

$$G'(z) = \frac{1}{g(2z - 2 + i)}.$$

Suggestion: Observe that $\text{Re}(2z - 2 + i) > 0$ when $x > 1$.

6. Use results in Sec. 24 to verify that the function

$$g(z) = \ln r + i\theta \qquad (r > 0, 0 < \theta < 2\pi)$$

is analytic in the indicated domain of definition, with derivative $g'(z) = 1/z$. Then show that the composite function $G(z) = g(z^2 + 1)$ is analytic in the quadrant $x > 0$, $y > 0$, with derivative

$$G'(z) = \frac{2z}{z^2 + 1}.$$

Suggestion: Observe that $\text{Im}(z^2 + 1) > 0$ when $x > 0$, $y > 0$.

7. Let a function f be analytic everywhere in a domain D. Prove that if $f(z)$ is real-valued for all z in D, then $f(z)$ must be constant throughout D.

27. HARMONIC FUNCTIONS

A real-valued function H of two real variables x and y is said to be ***harmonic*** in a given domain of the xy plane if, throughout that domain, it has continuous partial derivatives of the first and second order and satisfies the partial differential equation

$$H_{xx}(x, y) + H_{yy}(x, y) = 0, \tag{1}$$

known as ***Laplace's equation***.

Harmonic functions play an important role in applied mathematics. For example, the temperatures $T(x, y)$ in thin plates lying in the xy plane are often harmonic. A function $V(x, y)$ is harmonic when it denotes an electrostatic potential that varies only with x and y in the interior of a region of three-dimensional space that is free of charges.

EXAMPLE 1. It is easy to verify that the function $T(x, y) = e^{-y} \sin x$ is harmonic in any domain of the xy plane and, in particular, in the semi-infinite vertical strip $0 < x < \pi$, $y > 0$. It also assumes the values on the edges of the strip that are indicated in Fig. 31. More precisely, it satisfies all of the conditions

$$T_{xx}(x, y) + T_{yy}(x, y) = 0,$$

$$T(0, y) = 0, \quad T(\pi, y) = 0,$$

$$T(x, 0) = \sin x, \quad \lim_{y \to \infty} T(x, y) = 0,$$

which describe steady temperatures $T(x, y)$ in a thin homogeneous plate in the xy plane that has no heat sources or sinks and is insulated except for the stated conditions along the edges.

Homework 5: Part 2

1. Let $w = f(z)$ be differentiable at any point in a domain D. Suppose $f(z)$ is one-to-one, that is $f(z_1) \neq f(z_2)$ if $z_1 \neq z_2$. We can define a inverse function $z = f^{-1}(w)$ such that it satisfies:

   $$f(f^{-1}(w)) = w \text{ and } f^{-1}(f(z)) = z.$$

   Suppose $f'(z) \neq 0$. Use the definition to prove that $z = f^{-1}(w)$ is differentiable and its derivative is given by:

   $$\frac{d}{dw}(f^{-1}(w)) = \frac{1}{f'(f^{-1}(w))}.$$

2. Any branch of the multivalued function $z^{1/n}$ can be seen as an inverse function of $f(z) = z^n$. Use Part 1 to prove that for any branch of the multivalued function $z^{1/n}$, we have:

   $$\frac{d}{dz} z^{1/n} = \frac{1}{nz^{\frac{n-1}{n}}} = \frac{1}{n} z^{\frac{1}{n}-1}.$$

   (Note that z and w are just names of variables (dummy variables) and we can interchange them)

71. 3 (a) $f(z)=\frac{1}{z}=\frac{1}{x+iy}=\frac{x-iy}{x^2+y^2} \Leftrightarrow \begin{cases} u=\frac{x}{x^2+y^2} \\ v=-\frac{y}{x^2+y^2}. \end{cases}$

$u_x=\frac{1}{x^2+y^2}-\frac{2x^2}{(x^2+y^2)^2}$, $u_y=-\frac{2xy}{(x^2+y^2)^2}$

$v_x=\frac{2y\cdot x}{(x^2+y^2)^2}$, $v_y=-\frac{1}{x^2+y^2}+\frac{2y^2}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}$

$\Rightarrow \begin{cases} u_x=v_y=\frac{y^2-x^2}{(x^2+y^2)^2} \\ u_y=-v_x=-\frac{2xy}{(x^2+y^2)^2} \end{cases} \Rightarrow f'(z)$ exists when $z\neq 0$

and $f'(z)=u_x+iv_x=\frac{y^2-x^2}{(x^2+y^2)^2}+i\cdot\frac{2xy}{(x^2+y^2)^2}=\frac{-\bar{z}^2}{|z|^2}=-\frac{1}{z^2}$ $(z\neq 0)$.

(b) $f(z)=x^2+iy^2=u+iv$

$u_x=2x$, $u_y=0$
$v_x=0$, $v_y=2y$

CR eqs. $\begin{cases} 2x=2y \\ 0=-0 \end{cases} \Rightarrow x=y.$

so $f'(z)$ exists when $x=y$ and at these points,

$f'(\underset{x+ix}{\underset{\shortparallel}{z}})=u_x+iv_x=2x+i\cdot 0=2x$

(c) $f(z)=z\cdot \mathrm{Im}\,z=(x+iy)\cdot y=xy+iy^2 \Leftrightarrow \begin{cases} u=xy \\ v=y^2 \end{cases}$ $u_x=y$, $u_y=x$, $v_x=0$, $v_y=2y$.

CR eqs $\begin{cases} y=2y \\ x=-0 \end{cases} \Rightarrow x=0=y. \Rightarrow f'(z)$ exists only when $z=0$ and

$f'(0)=(u_x+iv_x)(0)=0.$

$u = r^{-4}\cos(4\theta)$, $v = -r^{-4}\sin(4\theta)$

4. (a) $f(z) = \frac{1}{z^4} = \frac{1}{r^4 e^{i4\theta}} = r^{-4} \cdot e^{-4i\theta} = \underbrace{r^{-4}\cos(4\theta)}_{u} - \underbrace{i r^{-4}\sin(4\theta)}_{v}$

$r u_r = r \cdot (-4) \cdot r^{-5} \cdot \cos(4\theta) = -4r^{-4}\cos(4\theta)$. $u_\theta = -r^{-4} \cdot 4\sin(4\theta)$.

$r v_r = +4 \cdot r^{-4} \cdot \sin(4\theta)$, $v_\theta = -r^{-4} \cdot 4 \cdot \cos(4\theta)$

$\Rightarrow \begin{cases} r u_r = v_\theta = -4r^{-4}\cos(4\theta) \\ r v_r = -u_\theta = 4r^{-4}\sin(4\theta) \end{cases} \Rightarrow f(z)$ is differentiable when $z \neq 0$.

$f'(z) = e^{-i\theta} f_r = e^{-i\theta}(u_r + i v_r) = e^{-i\theta} \cdot (-4 \cdot r^{-5}\cos(4\theta) + i \cdot 4 \cdot r^{-5}\sin(4\theta))$

$= -4 \cdot r^{-5} \cdot e^{-i\theta} \cdot e^{-4i\theta} = -4 \cdot r^{-5} \cdot e^{-5i\theta} = -4z^{-5}$.

(b). $f(\theta) = e^{-\theta} \cdot \cos(\ln r) + i \cdot e^{-\theta} \cdot \sin(\ln r)$ $(r > 0,\ 0 < \theta < 2\pi)$.

$u_r = -e^{-\theta} \cdot \sin(\ln r) \cdot \frac{1}{r}$, $u_\theta = -e^{-\theta}\cos(\ln r)$.

$v_r = e^{-\theta} \cdot \cos(\ln r) \cdot \frac{1}{r}$, $v_\theta = -e^{-\theta} \cdot \sin(\ln r)$.

$\Rightarrow \begin{cases} r u_r = v_\theta = -e^{-\theta} \cdot \sin(\ln r) \\ r v_r = -u_\theta = e^{-\theta} \cdot \cos(\ln r) \end{cases} \Rightarrow f'(z)$ is differentiable for $\substack{r>0 \\ 0<\theta<2\pi}$

$f'(z) = e^{-i\theta}(u_r + i v_r) = e^{-i\theta}\left(-\frac{1}{r} \cdot e^{-\theta}\sin(\ln r) + \frac{i}{r} e^{-\theta} \cdot \cos(\ln r)\right)$

$= \frac{i}{re^{i\theta}} \cdot (\cos(\ln r) + i\sin(\ln r)) e^{-\theta} = \frac{i}{z} f(z)$.

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5. $\begin{cases} r=\sqrt{x^2+y^2} \\ \theta=\tan^{-1}\frac{y}{x} \end{cases} \Rightarrow r_x=\frac{x}{\sqrt{x^2+y^2}}=\frac{r\cos\theta}{r}=\cos\theta,\ r_y=\frac{y}{\sqrt{x^2+y^2}}=\sin\theta.$

$$\theta_x=\frac{-\frac{y}{x^2}}{1+\frac{y^2}{x^2}}=-\frac{y}{x^2+y^2}=-\frac{r\cdot\sin\theta}{r^2}=-\frac{\sin\theta}{r}$$

$$\theta_y=\frac{\frac{1}{x}}{1+\frac{y^2}{x^2}}=\frac{x}{x^2+y^2}=\frac{r\cdot\cos\theta}{r^2}=\frac{\cos\theta}{r}$$

so

$$u_x=u_r\cdot r_x+u_\theta\cdot\theta_x=u_r\cdot\cos\theta-u_\theta\cdot\frac{\sin\theta}{r},\quad v_x=v_r\cos\theta-v_\theta\cdot\frac{\sin\theta}{r}$$

$$u_y=u_r\cdot r_y+u_\theta\cdot\theta_y=u_r\cdot\sin\theta+u_\theta\cdot\frac{\cos\theta}{r},\quad v_y=v_r\cdot\sin\theta+v_\theta\cdot\frac{\cos\theta}{r}.$$

$$u_x-v_y=u_r\cdot\cos\theta-u_\theta\frac{\sin\theta}{r}-\left(v_r\sin\theta+v_\theta\frac{\cos\theta}{r}\right)=\frac{\cos\theta}{r}\cdot(ru_r-v_\theta)+\frac{\sin\theta}{r}(-u_\theta-rv_r)$$

$$u_y+v_x=u_r\cdot\sin\theta+u_\theta\cdot\frac{\cos\theta}{r}+v_r\cos\theta-v_\theta\frac{\sin\theta}{r}=\frac{\cos\theta}{r}\cdot(u_\theta+rv_r)+\frac{\sin\theta}{r}(ru_r-v_\theta)$$

so $\begin{cases} ru_r=v_\theta \\ u_\theta=-rv_r \end{cases} \Rightarrow \begin{cases} u_x-v_y=0 \\ u_y+v_x=0 \end{cases}$

6. $$f'(z)=u_x+iv_x=\left(u_r\cdot\cos\theta-u_\theta\cdot\frac{\sin\theta}{r}\right)+i\left(v_r\cdot\cos\theta-v_\theta\cdot\frac{\sin\theta}{r}\right).$$

$$=(\cos\theta)\cdot(u_r+iv_r)+\frac{\sin\theta}{r}\cdot(-u_\theta-iv_\theta)$$

$$=(\cos\theta)\cdot(u_r+iv_r)+\frac{\sin\theta}{r}\cdot(r\cdot v_r-i\cdot ru_r)$$

$$=(\cos\theta)\cdot(u_r+iv_r)-i\sin\theta\cdot(u_r+iv_r)$$

$$=(\cos\theta-i\sin\theta)(u_r+iv_r)=e^{-i\theta}(u_r+iv_r)$$

8. $$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}} = \frac{\partial f}{\partial x}\cdot\frac{1}{2} + \frac{\partial f}{\partial y}\cdot\left(-\frac{1}{2i}\right)$$

$$= \frac{1}{2}\cdot\frac{\partial f}{\partial x} + \frac{i}{2}\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

$$= \frac{1}{2}\cdot\left(\frac{\partial}{\partial x}(u+iv) + i\frac{\partial}{\partial y}\cdot(u+iv)\right)$$

$$= \frac{1}{2}\cdot\left(u_x + iv_x + i\cdot(u_y + iv_y)\right) = \frac{1}{2}\cdot\left(u_x + iv_x + iu_y - v_y\right)$$

$$= \frac{1}{2}\cdot(u_x - v_y) + \frac{i}{2}(u_y + v_x).$$

So $\frac{\partial f}{\partial \bar{z}} = 0 \iff \begin{cases} u_x = v_y \\ u_y = -v_x. \end{cases} \iff f$ is analytic

P76. 1. (a). $f(z)=3x+y+i(3y-x)$.

$u_x=3 \quad u_y=1$

$v_x=-1 \quad v_y=3$

$\Rightarrow$ CR eqs are satisfied for all (x,y)

$\Rightarrow f$ is entire i.e. analytic at all pts.

(b). $f(z)=\cosh x\cdot\cos y+i\sinh x\sin y$

$u_x=\sinh x\cdot\cos y,\ u_y=-\cosh x\cdot\sin y$.

$v_x=\cosh x\cdot\sin y,\ v_y=\sinh x\cdot\cos y$

$\Rightarrow$ CR eqs. satisfied for all (x,y)

$\Rightarrow f$ entire.

2. (c). $f(z)=e^y\cdot e^{ix}=e^y\cdot(\cos x+i\sin x)$.

$u_x=-e^y\cdot\sin x,\ u_y=e^y\cdot\cos x$

$v_x=e^y\cdot\cos x,\ v_y=e^y\cdot\sin x$.

CR eqs $\Rightarrow \begin{cases}-\sin x=+\sin x\\ \cos x=-\cos x\end{cases}$ no solution

$\Rightarrow$ not differentiable at any pt

$\Rightarrow$ nowhere analytic.

(a) $f(z)=xy+iy$.

$u_x=y,\ u_y=x$

$v_x=0,\ v_y=1$.

CR eqs $\begin{cases}y=1\\ x=0\end{cases}$ $\Rightarrow f$ is differentiable only at $(0,1)$

$\Rightarrow f$ is nowhere analytic.

(b). f is only differentiable at $(0,0)$ (not differentiable in a nbhd)

$f = 2xy+i(x^2-y^2)$.

$\Rightarrow$ nowhere analytic

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P76:

4. (c). $f(z)=\dfrac{z^2+1}{(z+2)(z^2+2z+2)}$

Singular points satisfy $(z+2)(z^2+2z+2)=0 \Rightarrow z=2.\ \dfrac{-2\pm\sqrt{2^2-4\times 2}}{2}$
$$\parallel$$
$$-1\pm\frac{\sqrt{-4}}{2}=-1\pm i.$$

$f(z)$ is analytic away from singular points by the differentiation rules: If two functions are analytic in a domain D, then their quotient is analytic in D provided the function in the denominator does not vanish at any point in D.

6. $g(z)=\ln r+i\theta.$ $\quad u_r=\frac{1}{r},\ u_\theta=0$, $\quad v_r=0,\ v_\theta=1$ $\Rightarrow \begin{cases} r u_r = v_\theta = 1 \\ u_\theta = -r v_r = 0 \end{cases}$

$\Rightarrow g(z)$ is analytic. and $g'(z)=e^{-i\theta}(u_r+iv_r)=e^{-i\theta}\frac{1}{r}=\frac{1}{re^{i\theta}}=\frac{1}{z}.$

For the composite $G(z)=g(z^2+1)$, $x>0, y>0$.

$1+z^2=(x^2-y^2)+2ixy+1=(1+x^2-y^2)+2ixy \xRightarrow[y>0]{x>0} \operatorname{Im}(1+z^2)>0.$

so the image of $\{x>0, y>0\}$ under (z^2+1) is contained in the upper half plane which lies in the domain $\{r>0, 0<\theta<2\pi\}$ of $g(z)$. So by chain rule, we know that $G(z)$ is analytic in $\{x>0, y>0\}$ and

$G'(z)=g'(z^2+1)\cdot 2z=\dfrac{2z}{1+z^2}.$

P77. 7. $f(z)$ is real valued $\Leftrightarrow \overline{f(z)} = f(z)$.

$f(z)$ analytic and real valued in $D \Rightarrow \overline{f(z)} = f(z)$ is analytic in D

$\Rightarrow$ $f(z)$ and $\overline{f(z)}$ are both analytic in D $\Rightarrow f(z)$ is constant.

$$\left(\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \& \begin{cases} u_x = (-v)_y \\ u_y = -(-v)_x \end{cases} \Rightarrow u_x = u_y = v_x = v_y = 0. \right).$$

Part 2: 1. For any fixed w_0.

$z = f^{-1}(w)$

$z_0 = f^{-1}(w_0)$

$$\frac{d}{dw} f^{-1}(w_0) = \lim_{w \to w_0} \frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \lim_{z \to z_0} \frac{z - z_0}{f(z) - f(z_0)}$$

$$= \frac{1}{\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}} = \frac{1}{f'(z_0)} = \frac{1}{f'(f^{-1}(w_0))}$$

2. Let $f(z) = z^n = w \Rightarrow z = f^{-1}(w) = w^{\frac{1}{n}}$

by 1, $\frac{d}{dw} w^{\frac{1}{n}} = \frac{d}{dw} f^{-1}(w) = \frac{1}{f'(f^{-1}(w))}$

$$= \frac{1}{n \cdot z^{n-1} \Big|_{z = f^{-1}(w) = w^{\frac{1}{n}}}} = \frac{1}{n \left(w^{\frac{1}{n}}\right)^{n-1}}$$

$$= \frac{1}{n \cdot w^{\frac{n-1}{n}}} = \frac{1}{n} \cdot w^{\frac{1}{n} - 1}$$

$$\Rightarrow \frac{d}{dz} z^{\frac{1}{n}} = \frac{1}{n} z^{\frac{1}{n} - 1}.$$

EXAMPLE 2. The function $f(z) = e^{-y}\sin x - ie^{-y}\cos x$ is entire, as is shown in Exercise 1(*c*), Sec. 26. Hence its real component, which is the temperature function $T(x, y) = e^{-y}\sin x$ in Example 1, must be harmonic in every domain of the xy plane.

EXAMPLE 3. Since the function $f(z) = 1/z^2$ is analytic at every nonzero point z and since

$$\frac{1}{z^2} = \frac{1}{z^2}\cdot\frac{\bar{z}^2}{\bar{z}^2} = \frac{\bar{z}^2}{(z\bar{z})^2} = \frac{\bar{z}^2}{|z^2|^2} = \frac{(x^2-y^2) - i2xy}{(x^2+y^2)^2},$$

the two functions

$$u(x, y) = \frac{x^2 - y^2}{(x^2+y^2)^2} \quad \text{and} \quad v(x, y) = -\frac{2xy}{(x^2+y^2)^2}$$

are harmonic throughout any domain in the xy plane that does not contain the origin.

Further discussion of harmonic functions related to the theory of functions of a complex variable appears in Chaps. 9 and 10, where they are needed in solving physical problems, such as in Example 1 here.

EXERCISES

1. Let the function $f(z) = u(r, \theta) + iv(r, \theta)$ be analytic in a domain D that does not include the origin. Using the Cauchy–Riemann equations in polar coordinates (Sec. 24) and assuming continuity of partial derivatives, show that throughout D the function $u(r, \theta)$ satisfies the partial differential equation

$$r^2 u_{rr}(r, \theta) + r u_r(r, \theta) + u_{\theta\theta}(r, \theta) = 0,$$

which is the ***polar form of Laplace's equation***. Show that the same is true of the function $v(r, \theta)$.

2. Let the function $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain D, and consider the families of ***level curves*** $u(x, y) = c_1$ and $v(x, y) = c_2$, where c_1 and c_2 are arbitrary real constants. Prove that these families are orthogonal. More precisely, show that if $z_0 = (x_0, y_0)$ is a point in D which is common to two particular curves $u(x, y) = c_1$ and $v(x, y) = c_2$ and if $f'(z_0) \neq 0$, then the lines tangent to those curves at (x_0, y_0) are perpendicular.

Suggestion: Note how it follows from the pair of equations $u(x, y) = c_1$ and $v(x, y) = c_2$ that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0 \quad \text{and} \quad \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dy}{dx} = 0.$$

3. Show that when $f(z) = z^2$, the level curves $u(x, y) = c_1$ and $v(x, y) = c_2$ of the component functions are the hyperbolas indicated in Fig. 32. Note the orthogonality of the two families, described in Exercise 2. Observe that the curves $u(x, y) = 0$ and $v(x, y) = 0$ intersect at the origin but are not, however, orthogonal to each other. Why is this fact in agreement with the result in Exercise 2?

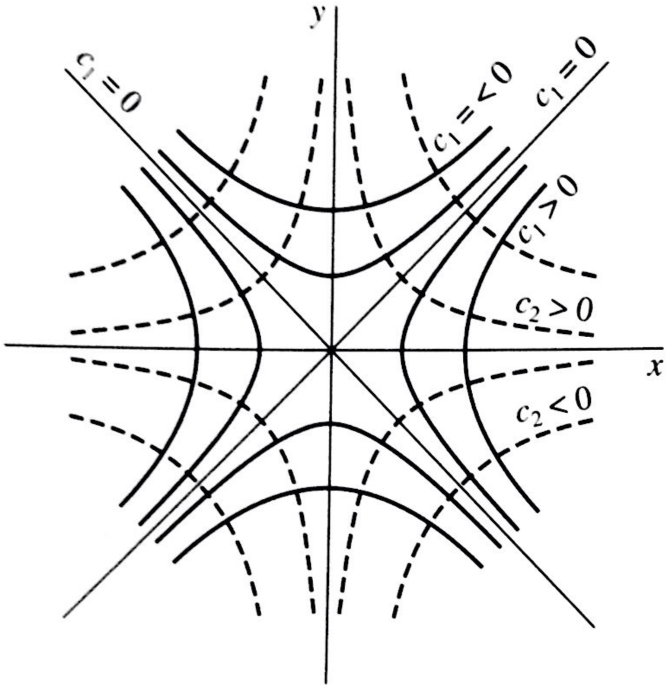


FIGURE 32

4. Sketch the families of level curves of the component functions u and v when $f(z) = 1/z$, and note the orthogonality described in Exercise 2.
5. Do Exercise 4 using polar coordinates.
6. Sketch the families of level curves of the component functions u and v when

$$f(z) = \frac{z-1}{z+1},$$

and note how the result in Exercise 2 is illustrated here.

28. UNIQUELY DETERMINED ANALYTIC FUNCTIONS

We conclude this chapter with two sections dealing with how the values of an analytic function in a domain D are affected by its values in a subdomain of D or on a line segment lying in D. While these sections are of considerable theoretical interest, they are not central to our development of analytic functions in later chapters. The reader may pass directly to Chap. 3 at this time and refer back when necessary.

Lemma. *Suppose that*

(a) a function f is analytic throughout a domain D;

(b) $f(z) = 0$ at each point z of a domain or line segment contained in D.

Then $f(z) \equiv 0$ in D; that is, $f(z)$ is identically equal to zero throughout D.

To prove this lemma, we let f be as stated in its hypothesis and let z_0 be any point of the subdomain or line segment where $f(z) = 0$. Since D is a *connected* open set (Sec. 12), there is a polygonal line L, consisting of a finite number of line segments joined end to end and lying entirely in D, that extends from z_0 to any other point P in D. We let d be the shortest distance from points on L to the boundary of D, unless D is the entire plane; in that

Some properties of e^z are, on the other hand, *not* expected. For example, since

$$e^{z+2\pi i} = e^z e^{2\pi i} \quad \text{and} \quad e^{2\pi i} = 1,$$

we find that e^z *is periodic*, with a pure imaginary period of $2\pi i$:

$$e^{z+2\pi i} = e^z. \tag{8}$$

For another property of e^z that e^x does not have, we note that while e^x is always positive, e^z can be negative. We recall (Sec. 6), for instance, that $e^{i\pi} = -1$. In fact,

$$e^{i(2n+1)\pi} = e^{i2n\pi + i\pi} = e^{i2n\pi} e^{i\pi} = (1)(-1) = -1 \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

There are, moreover, values of z such that e^z is *any* given nonzero complex number. This is shown in the next section, where the logarithmic function is developed, and is illustrated in the following example.

EXAMPLE. In order to find numbers $z = x + iy$ such that

$$e^z = 1 + \sqrt{3}i, \tag{9}$$

we write equation (9) as

$$e^x e^{iy} = 2\, e^{i\pi/3}.$$

Then, in view of the statement in italics at the beginning of Sec. 10, regarding the equality of two nonzero complex numbers in exponential form,

$$e^x = 2 \quad \text{and} \quad y = \frac{\pi}{3} + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Because $\ln(e^x) = x$, it follows that

$$x = \ln 2 \quad \text{and} \quad y = \frac{\pi}{3} + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots);$$

and so

$$z = \ln 2 + \left(2n + \frac{1}{3}\right)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{10}$$

EXERCISES

1. Show that

(*a*) $\exp(2 \pm 3\pi i) = -e^2$; (*b*) $\exp\left(\dfrac{2+\pi i}{4}\right) = \sqrt{\dfrac{e}{2}}\,(1+i)$;

(*c*) $\exp(z + \pi i) = -\exp z$.

2. State why the function $f(z) = 2z^2 - 3 - ze^z + e^{-z}$ is entire.

3. Use the Cauchy–Riemann equations and the theorem in Sec. 21 to show that the function $f(z) = \exp \bar{z}$ is not analytic anywhere.

4. Show in two ways that the function $f(z) = \exp(z^2)$ is entire. What is its derivative?

Ans. $f'(z) = 2z\exp(z^2)$.

5. Write $|\exp(2z+i)|$ and $|\exp(iz^2)|$ in terms of x and y. Then show that

$$|\exp(2z+i)+\exp(iz^2)| \le e^{2x}+e^{-2xy}.$$

6. Show that $|\exp(z^2)| \le \exp(|z|^2)$.

7. Prove that $|\exp(-2z)| < 1$ if and only if $\operatorname{Re} z > 0$.

8. Find all values of z such that

(*a*) $e^z = -2$; (*b*) $e^z = 1+i$; (*c*) $\exp(2z-1) = 1$.

Ans. (*a*) $z = \ln 2 + (2n+1)\pi i \ (n = 0, \pm 1, \pm 2, \ldots)$;

(*b*) $z = \dfrac{1}{2}\ln 2 + \left(2n + \dfrac{1}{4}\right)\pi i \ (n = 0, \pm 1, \pm 2, \ldots)$;

(*c*) $z = \dfrac{1}{2} + n\pi i \ (n = 0, \pm 1, \pm 2, \ldots)$.

9. Show that $\overline{\exp(iz)} = \exp(i\bar{z})$ if and only if $z = n\pi \ (n = 0, \pm 1, \pm 2, \ldots)$. (Compare with Exercise 4, Sec. 29.)

10. (*a*) Show that if e^z is real, then $\operatorname{Im} z = n\pi \ (n = 0, \pm 1, \pm 2, \ldots)$.

(*b*) If e^z is pure imaginary, what restriction is placed on z?

11. Describe the behavior of $e^z = e^x e^{iy}$ as (*a*) x tends to $-\infty$; (*b*) y tends to ∞.

12. Write $\operatorname{Re}(e^{1/z})$ in terms of x and y. Why is this function harmonic in every domain that does not contain the origin?

13. Let the function $f(z) = u(x,y) + iv(x,y)$ be analytic in some domain D. State why the functions

$$U(x,y) = e^{u(x,y)}\cos v(x,y), \quad V(x,y) = e^{u(x,y)}\sin v(x,y)$$

are harmonic in D.

14. Establish the identity

$$(e^z)^n = e^{nz} \quad (n = 0, \pm 1, \pm 2, \ldots)$$

in the following way.

(*a*) Use mathematical induction to show that it is valid when $n = 0, 1, 2, \ldots$.

(*b*) Verify it for negative integers n by first recalling from Sec. 8 that

$$z^n = (z^{-1})^m \quad (m = -n = 1, 2, \ldots)$$

when $z \neq 0$ and writing $(e^z)^n = (1/e^z)^m$. Then use the result in part (*a*), together with the property $1/e^z = e^{-z}$ (Sec. 30) of the exponential function.

31. THE LOGARITHMIC FUNCTION

Our motivation for the definition of the logarithmic function is based on solving the equation

$$e^w = z \tag{1}$$

Homework 6: Part 2

1: Determine and sketch the images of the following regions under the map $w = e^z$.

(a)
$$\mathrm{Re}(z) \geq 1.$$

(b)
$$\mathrm{Re}(z) < 0.$$

(c)
$$0 < \mathrm{Im}(z) < \pi$$

(d)
$$0 \leq \mathrm{Im}(z) < 2\pi$$

P79. 1. $f = u + iv$ analytic in D

$\Rightarrow$ CR equations are satisfied $\begin{cases} r u_r = v_\theta & ① \\ u_\theta = -r v_r & ② \end{cases}$

$$① \Rightarrow \underset{\substack{\| \\ u_r + r u_{rr}}}{(r u_r)_r} = v_{\theta r} \quad \Rightarrow r^2 u_{rr} + r u_r = r \cdot v_{\theta r} = r v_{r\theta} = -u_{\theta\theta}$$

$$② \Rightarrow u_{\theta\theta} = -r v_{r\theta}$$

so $\quad r^2 u_{rr} + r u_r + u_{\theta\theta} = 0.$

For V: $\quad ① \Rightarrow r u_{r\theta} = v_{\theta\theta}$

$$② \Rightarrow u_{\theta r} = -(r v_r)_r = -v_r - r v_{rr}$$

$$\Rightarrow v_{\theta\theta} = r u_{r\theta} = r u_{\theta r} = r \cdot (-v_r - r v_{rr}) \Rightarrow r^2 v_{rr} + r v_r + v_{\theta\theta} = 0$$

2. A normal vector to the level curve $u(x,y) = C_1$ is given by the gradient vector. $\nabla u = \langle u_x, u_y \rangle$, under the assumption that $\nabla u \neq 0$. Similarly, if $\nabla v \neq 0$, then $\underset{\substack{\| \\ \langle v_x, v_y \rangle}}{\nabla v}$ gives a normal vector to the level curve $v(x,y) = C_2$.

f analytic & $f'(z_0) \neq 0 \Rightarrow \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ and $f'(z_0) = u_x + i v_x \Big]_{z=z_0} \neq 0.$

$$\Rightarrow \nabla u(z_0) \cdot \nabla v(z_0) = \langle u_x, u_y \rangle \cdot \langle v_x, v_y \rangle = u_x \cdot v_x + u_y \cdot v_y$$

$$= u_x \cdot (-u_y) + u_y \cdot u_x = 0.$$

$$\left.\begin{array}{l} \nabla u(z_0) \neq 0 \\ \nabla v(z_0) \neq 0 \end{array}\right] \Rightarrow \{u = C_1\} \perp \{v = C_2\}.$$

3. $f(z) = z^2 = (x^2-y^2)+i\,2xy$ $\quad u(x,y)=x^2-y^2$

$v(x,y)=2xy.$

$$\Rightarrow \begin{cases} \nabla u = \langle 2x, -2y\rangle \\ \nabla v = \langle 2y, 2x\rangle \end{cases} \Rightarrow \nabla u \cdot \nabla v = (2x)(2y)+(-2y)(2x)=0.$$

at the origin $\nabla u(0) = \nabla v(0) = 0$. So Exercise 2 does not apply.

$\Rightarrow \{u=c_1\} \perp \{v=c_2\}$ away from the origin.

P89. 4. $f(z) = e^{z^2}$

- composition of analytic functions is analytic

$\Rightarrow$ $z \mapsto z^2 \mapsto e^{z^2}$ is analytic

- $f(z) = e^{x^2-y^2+2ixy} = \underline{e^{x^2-y^2}\cdot(\cos(2xy)+i\sin(2xy))}$

$\qquad\qquad\qquad\qquad\qquad\qquad \parallel$

$\qquad\qquad\qquad\qquad\qquad\qquad u+iv.$

$\Rightarrow$ $u = e^{x^2-y^2}\cos(2xy)$, $v(x,y) = e^{x^2-y^2}\sin(2xy)$

$$\Rightarrow \quad u_x = 2x\cdot e^{x^2-y^2}\cos(2xy) - e^{x^2-y^2}\sin(2xy)\cdot 2y$$

$$u_y = -2y\cdot e^{x^2-y^2}\cos(2xy) - e^{x^2-y^2}\sin(2xy)\cdot 2x.$$

$$v_x = 2x\cdot e^{x^2-y^2}\sin(2xy) + e^{x^2-y^2}\cos(2xy)\cdot 2y$$

$$v_y = -2y\cdot e^{x^2-y^2}\sin(2xy) + e^{x^2-y^2}\cos(2xy)\cdot 2x.$$

$$\Rightarrow \begin{cases} u_x = v_y = e^{x^2-y^2}\cdot(2x\cdot\cos(2xy) - 2y\cdot\sin(2xy)) \\ u_y = -v_x = e^{x^2-y^2}\cdot(-2y\cdot\cos(2xy) - 2x\sin(2xy)) \end{cases}$$

$\Rightarrow f$ is analytic.

8. (b) $e^{z} = 1+i = \sqrt{2}\cdot e^{\frac{\pi i}{4}}$

$\quad\ \ \| $

$e^{x}\cdot e^{iy}$

$\Longleftrightarrow \begin{cases} e^{x} = \sqrt{2} \\ y = \frac{\pi}{4} + 2n\pi. \end{cases} \Longleftrightarrow \begin{cases} x = \ln(\sqrt{2}) = \frac{1}{2}\ln 2 \\ y = \frac{\pi}{4} + 2n\pi, \quad n = 0, \pm 1, \pm 2, \cdots \end{cases}$

(c). $e^{2z-1} = 1 = 1\cdot e^{i\cdot 0}$

$\quad\ \ \|$

$e^{2x+2iy-1} = e^{2x-1}\cdot e^{i\cdot 2y}$

$\Longleftrightarrow \begin{cases} e^{2x-1} = 1 \\ 2y = 0 + n\cdot 2\pi, \quad n = 0, \pm 1, \pm 2, \cdots \end{cases}$

$\Longleftrightarrow \begin{cases} 2x-1 = \ln 1 = 0 \\ y = n\cdot\pi \end{cases} \Longleftrightarrow \begin{cases} x = \frac{1}{2} \\ y = n\pi, \quad n = 0, \pm 1, \pm 2, \cdots \end{cases}$

Or: $2z-1 = \log 1 = 0 + i(n\cdot 2\pi)$

$\Rightarrow z = \frac{1}{2} + i\cdot n\pi, \quad n = 0, \pm 1, \pm 2, \cdots$

10 (a). e^{z} is real $\Rightarrow$ $e^{z} = r > 0$

or $e^{z} = -r < 0$

$\qquad\qquad \| \; r\cdot e^{i\cdot\pi}$

$\Rightarrow z = \ln r + i\cdot n\cdot 2\pi$ or $z = \ln r + i\cdot(\pi + 2n\pi)$

$\Rightarrow \operatorname{Im}(z) = 2n\cdot\pi$ or $(2n+1)\pi. \quad n = 0, \pm 1, \pm 2, \cdots$

$\Rightarrow \operatorname{Im}(z) = n\pi. \quad (n = 0, \pm 1, \pm 2, \cdots).$

P90. 10 (b) e^z is purely imaginary

$\Rightarrow e^z = i \cdot r = r \cdot e^{\frac{\pi}{2}i}$ or $e^z = -r \cdot i = r \cdot e^{-\frac{\pi}{2}i}$

$\Rightarrow z = \ln r + i \cdot \left(\frac{\pi}{2} + n \cdot 2\pi\right)$ or $z = \ln r + i \cdot \left(-\frac{\pi}{2} + n \cdot 2\pi\right)$.

$\Rightarrow Im(z) = \frac{\pi}{2} + 2n \cdot \pi$ or $-\frac{\pi}{2} + 2n \cdot \pi = \frac{\pi}{2} + (2n-1)\pi$.

$\Rightarrow Im(z) = \frac{\pi}{2} + k \cdot \pi$. $k = 0, \pm 1, \pm 2, \ldots$ $n = 0, \pm 1, \pm 2, \ldots$

OR: e^z is purely imaginary $\Rightarrow i \cdot e^z$ is real

$$i \cdot e^z = e^{\frac{\pi}{2}i} \cdot e^z = e^{\frac{\pi}{2}i + z}$$

$\xrightarrow{\text{By (a)}} Im\left(\frac{\pi}{2}i + z\right) = n\pi \Rightarrow Im(z) = -\frac{\pi}{2} + n\pi \; (n = 0, \pm 1, \pm 2, \ldots)$

$Im\left(\frac{\pi}{2}i + z\right) = \frac{\pi}{2} + Im(z)$

$\Leftrightarrow Im(z) = \frac{\pi}{2} + m\pi \; (m = 0, \pm 1, \pm 2, \ldots)$.

11. (a) $x \to -\infty \Rightarrow |e^z| = e^x \to 0$. so $\lim_{x \to -\infty} e^z = 0$.

(b) For fixed x, as y changes, e^z rotates around the circle $|w| = e^x$

Period of 2π for y variable

12. $e^{\frac{1}{z}} = e^{\frac{1}{x+iy}} = e^{\frac{x-iy}{x^2+y^2}} = e^{\frac{x}{x^2+y^2}} \cdot e^{i \cdot \frac{-y}{x^2+y^2}}$

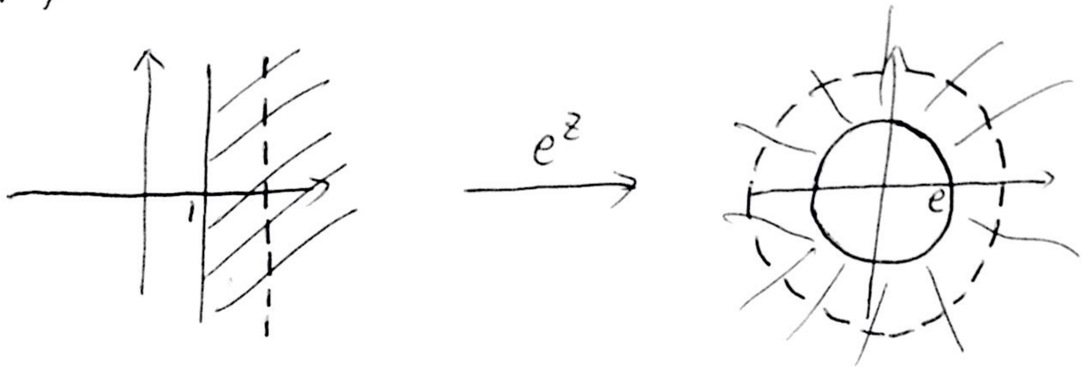
$$= e^{\frac{x}{x^2+y^2}}\left(\cos\left(\frac{y}{x^2+y^2}\right) - i \cdot \sin\left(\frac{y}{x^2+y^2}\right)\right).$$

So $\mathrm{Re}\left(e^{\frac{1}{z}}\right) = e^{\frac{x}{x^2+y^2}} \cos\left(\frac{y}{x^2+y^2}\right)$.

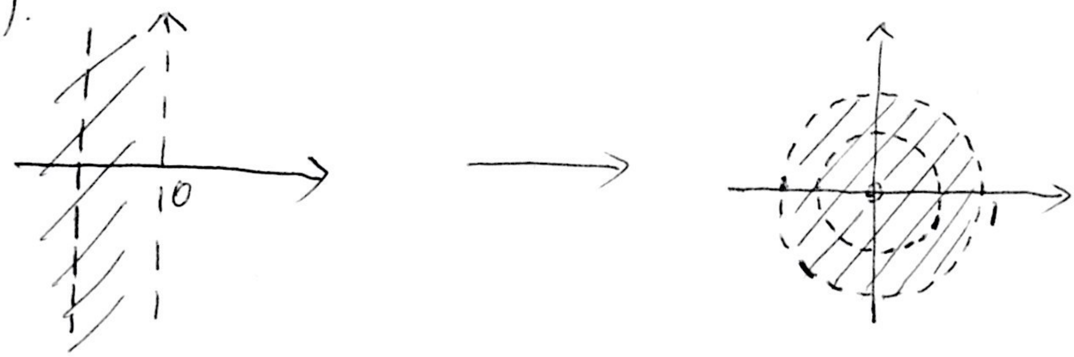
$e^{\frac{1}{z}}$ is analytic when $z \neq 0 \Rightarrow \mathrm{Re}\left(e^{\frac{1}{z}}\right)$ is harmonic in every domain that does not contain the origin.

Part 2

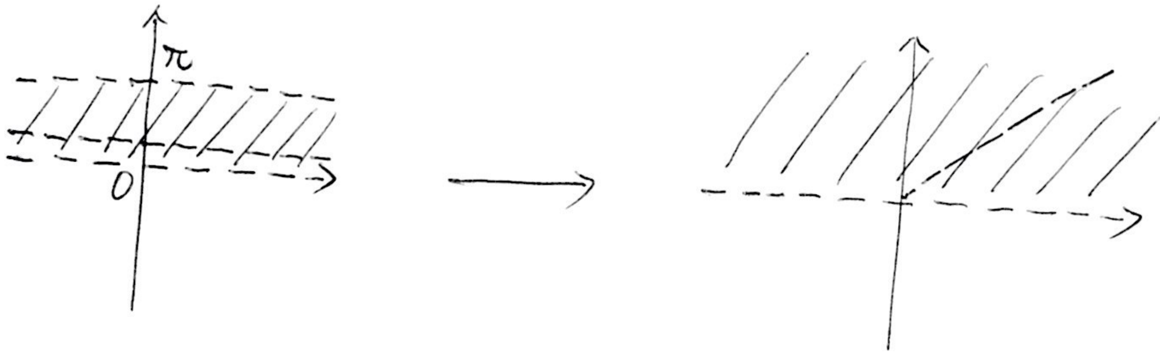
1. (a)



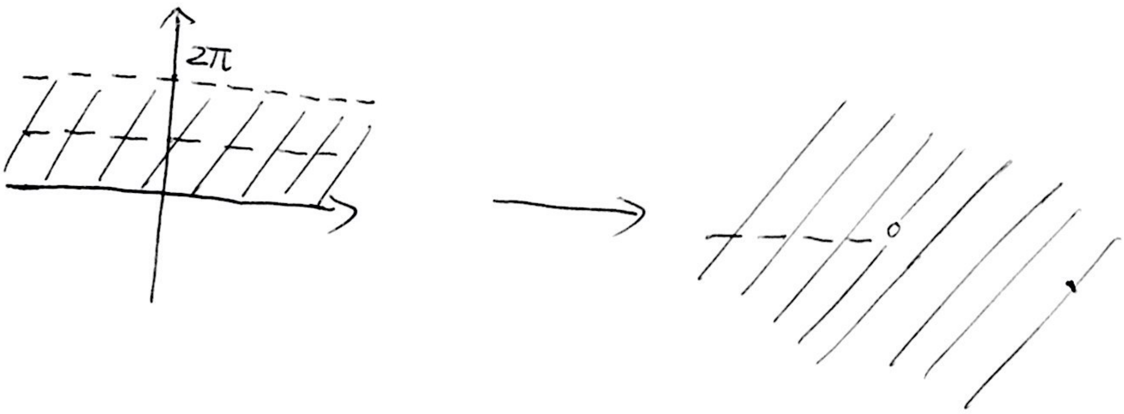
(b).



(c).



(d)



MAT 342 FALL 2014 Practice MIDTERM I

NAME : ID :

THERE ARE FIVE (5) PROBLEMS. THEY HAVE THE INDICATED VALUE.

SHOW YOUR WORK

DO NOT TEAR-OFF ANY PAGE

NO CALCULATORS NO CELLS ETC.

ON YOUR DESK: ONLY test, pen, pencil, eraser.

1		50pts
2		50pts
3		50pts
4		50pts
5		50pts
Total		250pts

!!! WRITE YOUR NAME, STUDENT ID AND LECTURE N. BELOW !!!

NAME : ID :

LECTURE N.

1. (50pts) Let $z_1 = 1 - i, z_2 = 3 - i$.

(a): Calculate $\bar{z}_1 \cdot z_2$ and z_1/z_2.

(b): Calculate $z_1^{1/3}$ and sketch the roots on a regular polygon.

2. (50pts) Calculate the limit if it exists:

(a)

$$\lim_{z \to i} \frac{z - i}{z(z^2 + 1)}.$$

(b)

$$\lim_{z \to 0} \frac{\bar{z}^4}{z^3}.$$

3. (50pts)

(1) Sketch the region given by:

$$0 \leq \mathrm{Arg}z < \frac{3\pi}{4}, \quad 1 < |z| \leq 2.$$

(2) Find the image of the above region under the mapping $w = z^2$.

4. (50pts)

(a) Explain why the following function is analytic in its domain and calculate $f'(z)$:

$$f(z) = \frac{(iz-1)^4}{(iz+1)^4}.$$

(b) If $g(z)$ is an analytic function and $f(z) = g(z) + \overline{g(z)}$ is also an analytic function, what can you say about $g(z)$? Explain your reason.

5. (50pts)

Find the points where the function is differentiable and then calculate the first order derivative of the function at those points. Is the function analytic at those points?

(a)

$$f(z) = (x^2 + (y+i)^2) + 2\,i\,x(y+i)$$

(b)

$$f(re^{i\theta}) = (\log r)^2 - \theta^2 + 2\,i\,\theta \log r, \quad r > 0, 0 < \theta < 2\pi.$$

!!! WRITE YOUR NAME, STUDENT ID AND LECTURE N. BELOW !!!

NAME : ID :

LECTURE N.

1. (50pts) Let $z_1 = 1 - i$, $z_2 = 3 - i$.

(a): Calculate $\bar{z}_1 \cdot z_2$ and z_1/z_2.

$$\bar{z}_1 \cdot z_2 = (1+i)(3-i) = 3 + i\cdot(-i) + 3i - i = 4+2i.$$

$$\frac{z_1}{z_2} = \frac{1-i}{3-i} = \frac{(1-i)\cdot(3+i)}{(3-i)\cdot(3+i)} = \frac{4-2i}{10}$$

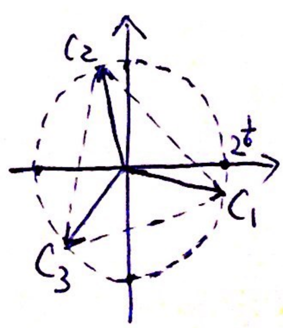
(b): Calculate $z_1^{1/3}$ and sketch the roots on a regular polygon.

$$z_1 = 1-i = \sqrt{2}\cdot e^{-\frac{i\pi}{4}}. \qquad z_1^{\frac{1}{3}} = (\sqrt{2})^{\frac{1}{3}} e^{i\left(-\frac{\pi}{12} + \frac{2k\pi}{3}\right)} \quad k=0,1,2.$$

$$c_1 = 2^{\frac{1}{6}} e^{-\frac{i\pi}{12}}, \qquad c_2 = 2^{\frac{1}{6}}\cdot e^{-\frac{i\pi}{12}}\cdot e^{i\frac{2\pi}{3}} = 2^{\frac{1}{6}} e^{\frac{7i\pi}{12}}$$

$$c_3 = c_2\cdot e^{i\frac{2\pi}{3}} = 2^{\frac{1}{6}} e^{i\left(\frac{7\pi}{12}+\frac{2\pi}{3}\right)} = 2^{\frac{1}{6}} e^{i\frac{15\pi}{12}} = 2^{\frac{1}{6}} e^{\frac{5i\pi}{4}}$$

$$= 2^{\frac{1}{6}}\cdot\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = -2^{\frac{1}{6}-\frac{1}{2}}(1+i) = -2^{-\frac{1}{3}}(1+i).$$



$$\Rightarrow c_2 = c_3\cdot e^{-i\frac{2\pi}{3}} = -2^{-\frac{1}{3}}(1+i)\cdot\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

$$= 2^{-\frac{4}{3}}\left(1-\sqrt{3} + (1+\sqrt{3})i\right)$$

$$c_1 = c_3\cdot e^{-i\frac{4\pi}{3}} = -2^{-\frac{1}{3}}(1+i)\cdot\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$= 2^{-\frac{4}{3}}\left((1+\sqrt{3}) + (1-\sqrt{3})i\right).$$

2. (50pts) Calculate the limit if it exists:

(a)

$$\lim_{z \to i} \frac{z-i}{z(z^2+1)}.$$

(b)

$$\lim_{z \to 0} \frac{\bar{z}^4}{z^3}.$$

(a) $$\lim_{z \to i} \frac{z-i}{z(z^2+1)} = \lim_{z \to i} \frac{z-i}{z(z+i)(z-i)}$$

$$= \lim_{z \to i} \frac{1}{z(z+i)} = \frac{1}{i \cdot (i+i)}$$

$$= -\frac{1}{2}$$

(b) $$\left|\frac{\bar{z}^4}{z^3}\right| = \frac{|\bar{z}|^4}{|z|^3} = |z|.$$

$$\lim_{z \to 0} \left|\frac{\bar{z}^4}{z^3}\right| = \lim_{z \to 0} |z| = 0$$

$$\Downarrow$$

$$\lim_{z \to 0} \frac{\bar{z}^4}{z^3} = 0.$$

---

or $$\lim_{z \to 0} \frac{\bar{z}^4}{z^3} = \lim_{r \to 0} \frac{r^4 e^{-4i\theta}}{r^3 e^{3i\theta}} = \lim_{r \to 0} \left(r \cdot e^{-7i\theta}\right) = 0$$

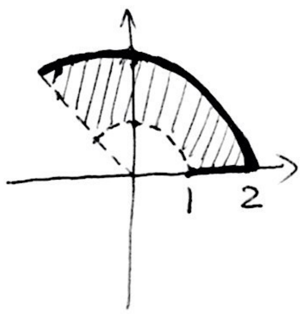
3. (50pts)

(1) Sketch the region given by:

$$0 \leq \text{Arg}z < \frac{3\pi}{4}, \quad 1 < |z| \leq 2.$$

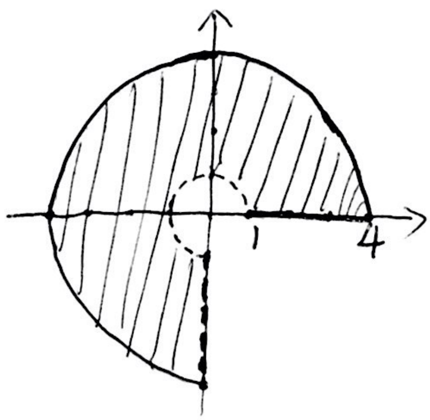
(2) Find the image of the above region under the mapping $w = z^2$.

(1)



$\downarrow z^2$

(2)



$0 \leq \text{Arg}\, w < \frac{3\pi}{2}$

$1 < |w| \leq 4.$

4. (50pts)

(a) Explain why the following function is analytic in its domain and calculate $f'(z)$:

$$f(z) = \frac{(iz-1)^4}{(iz+1)^4}.$$

Domain of $f = \{z \in \mathbb{C};\ z \neq i\}$

(f is analytic in the domain of f because the quotient of two analytic functions is analytic at points where the denominator is not zero)

$$f'(z) = \frac{4(iz-1)^3 \cdot i \cdot (iz+1)^4 - (iz-1)^4 \cdot 4 \cdot (iz+1)^3 \cdot i}{(iz+1)^8} = \frac{4i(iz-1)^3}{(iz+1)^5} \cdot \underbrace{(iz+1-(iz-1))}_{= 2}.$$

$$\boxed{\frac{8i \cdot (iz-1)^3}{(iz+1)^5}}.$$

(b) If $g(z)$ is an analytic function and $f(z) = g(z) + \overline{g(z)}$ is also an analytic function, what can you say about $g(z)$? Explain your reason.

$$f(z) = g(z) + \overline{g(z)} = 2\,\mathrm{Re}(g(z))$$

is analytic and real valued $\Rightarrow f(z) \equiv C_1$ with C_1 being a real number.

($f(z), \overline{f(z)} = f(z)$ are both analytic)

Let $g(z) = u(z) + i v(z)$, then $u(z) = \frac{f(z)}{2} \equiv \frac{C_1}{2}$.

$$\Rightarrow \begin{cases} u_x = v_y = 0 \\ u_y = -v_x = 0 \end{cases} \Rightarrow v_x = v_y = 0 \Rightarrow v \equiv C_2 \text{ is a constant}$$

$$\Rightarrow g(z) \equiv \frac{C_1}{2} + iC_2 \text{ is a constant function.}$$

5. (50pts)

Find the points where the function is differentiable and then calculate the first order derivative of the function at those points. Is the function analytic at those points?

(a)

$$f(z) = (x^2 + (y+i)^2) + 2\,i\,x(y+i)$$

(b)

$$f(re^{i\theta}) = (\log r)^2 - \theta^2 + 2\,i\,\theta \log r, \quad r > 0, 0 < \theta < 2\pi.$$

(a) $f(z) = (x^2 + (y+i)^2) + 2i x(y+i) = (x^2+y^2+2yi-1) + 2ixy - 2x$

$\underline{z = x+iy}$

$= (x^2+y^2-1-2x) + 2i\cdot(y+xy). = u+iv$

see the next page for correction

~~$\Rightarrow$ $u_x = 2x-2$, $u_y = 2y$ CR eqs $\begin{cases} 2x-2 = 1+x \\ 2y = -y \end{cases} \Rightarrow \begin{cases} x = 3 \\ y = 0 \end{cases}$~~

~~$v_x = y$, $v_y = 1+x$.~~

~~so f is differentiable at $z_0 = (3,0) = 3$, and $f'(3) = (u_x + iv_x)\big|_{z=3} = (2x-2+iy)\big|_3 = 4 + i\cdot 0 = 4.$~~

~~f is not differentiable at any other point $\Rightarrow f$ is $\boxed{\text{not analytic}}$ at any point on the complex plane.~~

(b) $u(r,\theta) = (\log r)^2 - \theta^2$, $v = 2\theta\cdot\log r$

$u_r = 2(\log r)\cdot\frac{1}{r}$, $u_\theta = -2\theta$,

$v_r = \frac{2\theta}{r}$, $v_\theta = 2\cdot\log r$

$\Rightarrow \begin{cases} r\cdot u_r = v_\theta = 2\cdot\log r \\ u_\theta = -r v_r = -2\theta \end{cases}$ i.e. CR eqs are satisfied (in polar coordinate)

$\Rightarrow$ $f(z)$ is differentiable at any point in the domain $\{r>0, 0<\theta<2\pi\}$

and $f'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\cdot\left(\frac{2\log r}{r} + i\cdot\frac{2\theta}{r}\right) = \frac{2(\log r + i\theta)}{re^{i\theta}}$

f is analytic at any point in the domain $\{r>0, 0<\theta<2\pi\}$.

As we will see later, $f(z) =$ a branch of $(\log z)^2$

$\Rightarrow f'(z) =$ a branch of $\frac{2\cdot\log z}{z}$.

5.

(a). $f(z)=(x^2+(y+i)^2)+2ix(y+i)=(x^2+y^2+2yi-1)+2ixy-2x$

$$= (x^2+y^2-1-2x)+i\cdot(2y+2xy)=u+iv.$$

$\Rightarrow$ $u_x = 2x-2,\ u_y=2y$ CR eqs. $\begin{cases} 2x-2=2+2x \\ 2y=-2y \end{cases}$ $\Rightarrow$ No solution.

$v_x = 2y,\ v_y=2+2x$

$\Rightarrow f$ is not differentiable anywhere
not analytic. at any point.

We saw in Example 5, Sec. 32, that the set of values of $\log(i^2)$ is not the set of values of $2\log i$. The following example does show, however, that equality can occur when a specific branch of the logarithm is used. In that case, of course, there is only one value of $\log(i^2)$ that is to be taken, and the same is true of $2\log i$.

EXAMPLE. In order to show that

$$\log(i^2) = 2\log i \tag{7}$$

when the branch

$$\log z = \ln r + i\theta \qquad \left(r > 0, \frac{\pi}{4} < \theta < \frac{9\pi}{4}\right)$$

is used, write

$$\log(i^2) = \log(-1) = \ln 1 + i\pi = \pi i$$

and then observe that

$$2\log i = 2\left(\ln 1 + i\frac{\pi}{2}\right) = \pi i.$$

It is interesting to contrast equality (7) with the result $\log(i^2) \neq 2\log i$ in Exercise 4, where a different branch of $\log z$ is used.

In Sec. 34, we shall consider other identities involving logarithms, sometimes with qualifications as to how they are to be interpreted. A reader who wishes to pass to Sec. 35 can simply refer to results in Sec. 34 when needed.

EXERCISES

1. Show that

(*a*) $\text{Log}(-ei) = 1 - \frac{\pi}{2}i$; (*b*) $\text{Log}(1 - i) = \frac{1}{2}\ln 2 - \frac{\pi}{4}i$.

2. Show that

(*a*) $\log e = 1 + 2n\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$;

(*b*) $\log i = \left(2n + \frac{1}{2}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$;

(*c*) $\log(-1 + \sqrt{3}i) = \ln 2 + 2\left(n + \frac{1}{3}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$.

3. Show that $\text{Log}(i^3) \neq 3\,\text{Log}\, i$.

4. Show that $\log(i^2) \neq 2\log i$ when the branch

$$\log z = \ln r + i\theta \qquad \left(r > 0, \frac{3\pi}{4} < \theta < \frac{11\pi}{4}\right)$$

is used. (Compare this with the example in Sec. 33.)

5. (*a*) Show that the two square roots of i are

$$e^{i\pi/4} \quad \text{and} \quad e^{i5\pi/4}$$

Then show that

$$\log(e^{i\pi/4}) = \left(2n + \frac{1}{4}\right)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

and

$$\log(e^{i5\pi/4}) = \left[(2n+1) + \frac{1}{4}\right]\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Conclude that

$$\log(i^{1/2}) = \left(n + \frac{1}{4}\right)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

(*b*) Show that

$$\log(i^{1/2}) = \frac{1}{2}\log i,$$

as stated in Example 5, Sec. 32, by finding the values on the right-hand side of this equation and then comparing them with the final result in part (*a*).

6. Given that the branch $\log z = \ln r + i\theta$ $(r > 0, \alpha < \theta < \alpha + 2\pi)$ of the logarithmic function is analytic at each point z in the stated domain, obtain its derivative by differentiating each side of the identity (Sec. 31)

$$e^{\log z} = z \qquad (|z| > 0, \alpha < \arg z < \alpha + 2\pi)$$

and using the chain rule.

7. Show that a branch (Sec. 33)

$$\log z = \ln r + i\theta \qquad (r > 0, \alpha < \theta < \alpha + 2\pi)$$

of the logarithmic function can be written

$$\log z = \frac{1}{2}\ln(x^2 + y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$$

in rectangular coordinates. Then, using the theorem in Sec. 23, show that the given branch is analytic in its domain of definition and that

$$\frac{d}{dz}\log z = \frac{1}{z}$$

there.

8. Find all roots of the equation $\log z = i\pi/2$.

Ans. $z = i$.

9. Suppose that the point $z = x + iy$ lies in the horizontal strip $\alpha < y < \alpha + 2\pi$. Show that when the branch $\log z = \ln r + i\theta$ $(r > 0, \alpha < \theta < \alpha + 2\pi)$ of the logarithmic function is used, $\log(e^z) = z$. [Compare with equation (5), Sec. 31.]

10. Show that

(*a*) the function $f(z) = \text{Log}(z - i)$ is analytic everywhere except on the portion $x \le 0$ of the line $y = 1$;

(*b*) the function

$$f(z) = \frac{\text{Log}(z+4)}{z^2 + i}$$

is analytic everywhere except at the points $\pm(1 - i)/\sqrt{2}$ and on the portion $x \le -4$ of the real axis.

11. Show in two ways that the function $\ln(x^2 + y^2)$ is harmonic in every domain that does not contain the origin.

12. Show that

$$\text{Re}\,[\log(z - 1)] = \frac{1}{2}\ln[(x-1)^2 + y^2] \qquad (z \ne 1).$$

Why must this function satisfy Laplace's equation when $z \ne 1$?

34. SOME IDENTITIES INVOLVING LOGARITHMS

If z_1 and z_2 denote any two nonzero complex numbers, it is straightforward to show that

$$\log(z_1 z_2) = \log z_1 + \log z_2. \tag{1}$$

This statement, involving a multiple-valued function, is to be interpreted in the same way that the statement

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \tag{2}$$

was in Sec. 9. That is, if values of two of the three logarithms are specified, then there is a value of the third such that equation (1) holds.

The verification of statement (1) can be based on statement (2) in the following way. Since $|z_1 z_2| = |z_1||z_2|$ and since these moduli are all positive real numbers, we know from experience with logarithms of such numbers in calculus that

$$\ln|z_1 z_2| = \ln|z_1| + \ln|z_2|.$$

So it follows from this and equation (2) that

$$\ln|z_1 z_2| + i\arg(z_1 z_2) = (\ln|z_1| + i\arg z_1) + (\ln|z_2| + i\arg z_2). \tag{3}$$

Finally, because of the way in which equations (1) and (2) are to be interpreted, equation (3) is the same as equation (1).

EXAMPLE 1. To illustrate statement (1), write $z_1 = z_2 = -1$ and recall from Examples 2 and 3 in Sec. 32 that

$$\log 1 = 2n\pi i \quad \text{and} \quad \log(-1) = (2n+1)\pi i,$$

for any value of $\log z$ that is taken. When $n = 1$, this reduces, of course, to relation (3), Sec. 31. Equation (5) is readily verified by writing $z = re^{i\theta}$ and noting that each side becomes $r^n e^{in\theta}$.

It is also true that when $z \neq 0$,

$$z^{1/n} = \exp\left(\frac{1}{n}\log z\right) \qquad (n = 1, 2, \ldots). \tag{6}$$

That is, the term on the right here has n distinct values, and those values are the nth roots of z. To prove this, we write $z = r\exp(i\Theta)$, where Θ is the principal value of $\arg z$. Then, in view of definition (2), Sec. 31, of $\log z$,

$$\exp\left(\frac{1}{n}\log z\right) = \exp\left[\frac{1}{n}\ln r + \frac{i(\Theta + 2k\pi)}{n}\right]$$

where $k = 0, \pm 1, \pm 2, \ldots$. Thus

$$\exp\left(\frac{1}{n}\log z\right) = \sqrt[n]{r}\exp\left[i\left(\frac{\Theta}{n} + \frac{2k\pi}{n}\right)\right] \qquad (k = 0, \pm 1, \pm 2, \ldots). \tag{7}$$

Because $\exp(i2k\pi/n)$ has distinct values only when $k = 0, 1, \ldots, n-1$, the right-hand side of equation (7) has only n values. That right-hand side is, in fact, an expression for the nth roots of z (Sec. 10), and so it can be written $z^{1/n}$. This establishes property (6), which is actually valid when n is a negative integer too (see Exercise 4).

EXERCISES

1. Show that for any two nonzero complex numbers z_1 and z_2,

$$\text{Log}(z_1 z_2) = \text{Log}\, z_1 + \text{Log}\, z_2 + 2N\pi i$$

where N has one of the values $0, \pm 1$. (Compare with Example 2 in Sec. 34.)

2. Verify expression (4), Sec. 34, for $\log(z_1/z_2)$ by

(*a*) using the fact that $\arg(z_1/z_2) = \arg z_1 - \arg z_2$ (Sec. 9);

(*b*) showing that $\log(1/z) = -\log z$ $(z \neq 0)$, in the sense that $\log(1/z)$ and $-\log z$ have the same set of values, and then referring to expression (1), Sec. 34, for $\log(z_1 z_2)$.

3. By choosing specific nonzero values of z_1 and z_2, show that expression (4), Sec. 34, for $\log(z_1/z_2)$ is not always valid when *log* is replaced by *Log*.

4. Show that property (6), Sec. 34, also holds when n is a negative integer. Do this by writing $z^{1/n} = (z^{1/m})^{-1}$ $(m = -n)$, where n has any one of the negative values $n = -1, -2, \ldots$ (see Exercise 9, Sec. 11), and using the fact that the property is already known to be valid for positive integers.

5. Let z denote any nonzero complex number, written $z = re^{i\Theta}$ $(-\pi < \Theta \leq \pi)$, and let n denote any fixed positive integer $(n = 1, 2, \ldots)$. Show that all of the values of $\log(z^{1/n})$ are given by the equation

$$\log(z^{1/n}) = \frac{1}{n}\ln r + i\frac{\Theta + 2(pn + k)\pi}{n},$$

Hence

$$(z_2z_3)^i = \left[e^{\pi/4}e^{i(\ln 2)/2}\right]\left[e^{3\pi/4}e^{i(\ln 2)/2}\right]e^{-2\pi},$$

or

$$(z_2z_3)^i = z_2^i z_3^i \, e^{-2\pi}. \tag{2}$$

EXERCISES

1. Show that

(*a*) $(1+i)^i = \exp\left(-\dfrac{\pi}{4}+2n\pi\right)\exp\left(i\dfrac{\ln 2}{2}\right) \quad (n=0,\pm1,\pm2,\ldots);$

(*b*) $\dfrac{1}{i^{2i}} = \exp[(4n+1)\pi] \quad (n=0,\pm1,\pm2,\ldots).$

2. Find the principal value of

(*a*) $(-i)^i$; (*b*) $\left[\dfrac{e}{2}(-1-\sqrt{3}i)\right]^{3\pi i}$; (*c*) $(1-i)^{4i}$.

Ans. (*a*) $\exp(\pi/2)$; (*b*) $-\exp(2\pi^2)$; (*c*) $e^{\pi}[\cos(2\ln 2)+i\sin(2\ln 2)]$.

3. Use definition (1), Sec. 35, of z^c to show that $(-1+\sqrt{3}i)^{3/2} = \pm 2\sqrt{2}$.

4. Show that the result in Exercise 3 could have been obtained by writing

(*a*) $(-1+\sqrt{3}i)^{3/2} = [(-1+\sqrt{3}i)^{1/2}]^3$ and first finding the square roots of $-1+\sqrt{3}i$;

(*b*) $(-1+\sqrt{3}i)^{3/2} = [(-1+\sqrt{3}i)^3]^{1/2}$ and first cubing $-1+\sqrt{3}i$.

5. Show that the *principal* nth root of a nonzero complex number z_0 that was defined in Sec. 10 is the same as the principal value of $z_0^{1/n}$ defined by equation (3), Sec. 35.

6. Show that if $z \neq 0$ and a is a real number, then $|z^a| = \exp(a\ln|z|) = |z|^a$, where the principal value of $|z|^a$ is to be taken.

7. Let $c = a+bi$ be a fixed complex number, where $c \neq 0, \pm1, \pm2, \ldots$, and note that i^c is multiple-valued. What additional restriction must be placed on the constant c so that the values of $|i^c|$ are all the same?

Ans. c is real.

8. Let c, c_1, c_2, and z denote complex numbers, where $z \neq 0$. Prove that if all of the powers involved are principal values, then

(*a*) $z^{c_1}z^{c_2} = z^{c_1+c_2}$; (*b*) $\dfrac{z^{c_1}}{z^{c_2}} = z^{c_1-c_2}$;

(*c*) $(z^c)^n = z^{cn} \quad (n=1,2,\ldots).$

9. Assuming that $f'(z)$ exists, state the formula for the derivative of $c^{f(z)}$.

37. THE TRIGONOMETRIC FUNCTIONS $\sin z$ AND $\cos z$

Euler's formula (Sec. 7) tells us that

$$e^{ix} = \cos x + i\sin x \quad \text{and} \quad e^{-ix} = \cos x - i\sin x$$

Observe that the quotients $\tan z$ and $\sec z$ are analytic everywhere except at the singularities (Sec. 25)

$$z = \frac{\pi}{2} + n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots),$$

which are the zeros of $\cos z$. Likewise, $\cot z$ and $\csc z$ have singularities at the zeros of $\sin z$, namely

$$z = n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

By differentiating the right-hand sides of equations (1) and (2), we obtain the anticipated differentiation formulas

$$\text{(3)} \qquad \frac{d}{dz}\tan z = \sec^2 z, \qquad \frac{d}{dz}\cot z = -\csc^2 z,$$

$$\text{(4)} \qquad \frac{d}{dz}\sec z = \sec z \tan z, \qquad \frac{d}{dz}\csc z = -\csc z \cot z.$$

The periodicity of each of the trigonometric functions defined by equations (1) and (2) follows readily from equations (10) and (11) in Sec. 37. For example,

$$\text{(5)} \qquad \tan(z + \pi) = \tan z.$$

Mapping properties of the transformation $w = \sin z$ are especially important in the applications later on. A reader who wishes at this time to learn some of those properties is sufficiently prepared to read Secs. 104 and 105 (Chap. 8), where they are discussed.

EXERCISES

1. Give details in the derivation of expressions (2), Sec. 37, for the derivatives of $\sin z$ and $\cos z$.

2. (*a*) With the aid of expression (4), Sec. 37, show that

$$e^{iz_1}e^{iz_2} = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 + i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2).$$

Then use relations (3), Sec. 37, to show how it follows that

$$e^{-iz_1}e^{-iz_2} = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 - i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2).$$

(*b*) Use the results in part (*a*) and the fact that

$$\sin(z_1 + z_2) = \frac{1}{2i}\left[e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}\right] = \frac{1}{2i}\left(e^{iz_1}e^{iz_2} - e^{-iz_1}e^{-iz_2}\right)$$

to obtain the identity

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$

in Sec. 37.

3. According to the final result in Exercise 2(*b*),

$$\sin(z + z_2) = \sin z \cos z_2 + \cos z \sin z_2.$$

By differentiating each side here with respect to z and then setting $z = z_1$, derive the expression

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

that was stated in Sec. 37.

4. Verify identity (9) in Sec. 37 using

(*a*) identity (6) and relations (3) in that section;

(*b*) the lemma in Sec. 28 and the fact that the entire function

$$f(z) = \sin^2 z + \cos^2 z - 1$$

has zero values along the x axis.

5. Use identity (9) in Sec. 37 to show that

(*a*) $1 + \tan^2 z = \sec^2 z$; (*b*) $1 + \cot^2 z = \csc^2 z$.

6. Establish differentiation formulas (3) and (4) in Sec. 38.

7. In Sec. 37, use expressions (13) and (14) to derive expressions (15) and (16) for $|\sin z|^2$ and $|\cos z|^2$.

Suggestion: Recall the identities $\sin^2 x + \cos^2 x = 1$ and $\cosh^2 y - \sinh^2 y = 1$.

8. Point out how it follows from expressions (15) and (16) in Sec. 37 for $|\sin z|^2$ and $|\cos z|^2$ that

(*a*) $|\sin z| \geq |\sin x|$; (*b*) $|\cos z| \geq |\cos x|$.

9. With the aid of expressions (15) and (16) in Sec. 37 for $|\sin z|^2$ and $|\cos z|^2$, show that

(*a*) $|\sinh y| \leq |\sin z| \leq \cosh y$; (*b*) $|\sinh y| \leq |\cos z| \leq \cosh y$.

10. (*a*) Use definitions (1), Sec. 37, of $\sin z$ and $\cos z$ to show that

$$2\sin(z_1 + z_2)\sin(z_1 - z_2) = \cos 2z_2 - \cos 2z_1.$$

(*b*) With the aid of the identity obtained in part (*a*), show that if $\cos z_1 = \cos z_2$, then at least one of the numbers $z_1 + z_2$ and $z_1 - z_2$ is an integral multiple of 2π.

11. Use the Cauchy–Riemann equations and the theorem in Sec. 21 to show that neither $\sin \bar{z}$ nor $\cos \bar{z}$ is an analytic function of z anywhere.

12. Use the reflection principle (Sec. 29) to show that for all z,

(*a*) $\overline{\sin z} = \sin \bar{z}$; (*b*) $\overline{\cos z} = \cos \bar{z}$.

13. With the aid of expressions (13) and (14) in Sec. 37, give direct verifications of the relations obtained in Exercise 12.

14. Show that

(*a*) $\overline{\cos(iz)} = \cos(i\bar{z})$ for all z;

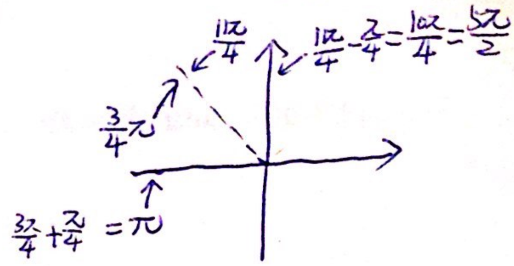
(*b*) $\overline{\sin(iz)} = \sin(i\bar{z})$ if and only if $z = n\pi i \ (n = 0, \pm 1, \pm 2, \ldots)$.

15. Find all roots of the equation $\sin z = \cosh 4$ by equating the real parts and then the imaginary parts of $\sin z$ and $\cosh 4$.

Ans. $\left(\frac{\pi}{2} + 2n\pi\right) \pm 4i \quad (n = 0, \pm 1, \pm 2, \ldots)$.

P95. 4. branch $\log z = \ln r + i\theta \quad \left(r>0, \frac{3\pi}{4}<\theta<\frac{11\pi}{4}\right)$

$\log(i^2) = \log(-1) = \ln 1 + i\cdot\pi = i\pi.$



$\log i = \ln|i| + i\cdot\frac{5\pi}{2} = i\cdot\frac{5\pi}{2}.$

$2\log i = 5\pi i \neq i\pi.$

5. (a) $i = e^{i\cdot\frac{\pi}{2}} \Rightarrow i^{\frac{1}{2}} = \left\{e^{i\frac{\pi}{4}},\ e^{i\left(\frac{\pi}{4}+\frac{2\pi}{2}\right)} = e^{i\frac{5\pi}{4}}\right\}$

$\log\left(e^{i\frac{\pi}{4}}\right) = i\cdot\left(\frac{\pi}{4}+2m\cdot\pi\right) \quad m=0,\pm1,\pm2,\cdots.$

$\log\left(e^{i\frac{5\pi}{4}}\right) = i\left(\frac{5\pi}{4}+2m\pi\right) \quad m=0,\pm1,\pm2,\cdots$

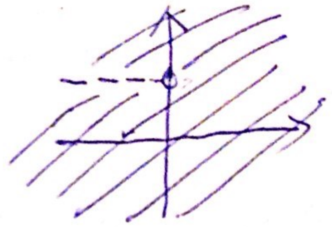
$= i\cdot\left(\frac{\pi}{4}+(2m+1)\pi\right)$

$\left(n=\overset{\text{even}}{2m} \text{ or } \overset{\text{odd}}{2m+1}\right)$

$\Rightarrow \log\left(i^{\frac{1}{2}}\right) = i\cdot\left(\frac{\pi}{4}+n\cdot\pi\right) \quad n=0,\pm1,\pm2,\cdots$

(b) $\frac{1}{2}\log i = \frac{1}{2}\cdot\left(i\left(\frac{\pi}{2}+2n\pi\right)\right) = \underset{\log\left(i^{\frac{1}{2}}\right)}{\underset{\parallel}{i\cdot\left(\frac{\pi}{4}+n\pi\right)}} \quad n=0,\pm1,\pm2,\cdots$

P97. 10. (a) $\mathrm{Log}(z-i) \overset{w=z-i}{=\!=} \mathrm{Log}(w)$.



$f(w) = \mathrm{Log}(w)$ is analytic everywhere except on the portion

$$\mathrm{Re}(w) \le 0 \ \& \ \mathrm{Im}(w) = 0$$

$\overset{w=z-i}{\Longrightarrow}$ $\mathrm{Log}(z-i)$ is analytic everywhere except on the portion

$$\mathrm{Re}(z) \le 0 \ \& \ \mathrm{Im}(w) = 1$$

$$\boxed{\begin{array}{l} \mathrm{Re}(w) = \mathrm{Re}(z) \\ \mathrm{Im}(w) = \mathrm{Im}(w) - 1 \end{array}}$$

(b) $f(z) = \dfrac{\mathrm{Log}(z+4)}{z^2+i}$

$f(z)$ is analytic every-where except at the points where

$z^2 + i = 0$ or $\{\mathrm{Re}(z+4) \le 0 \ \& \ \mathrm{Im}(z+4) = 0\}$

$\Updownarrow$

$\mathrm{Re}(z) \le -4 \ \& \ \mathrm{Im}(z) = 0$.

$\swarrow$

$$z = (-i)^{\frac{1}{2}} = \left(e^{i\cdot(-\frac{\pi}{2})}\right)^{\frac{1}{2}}$$

$$= e^{-i\cdot\frac{\pi}{4}} \text{ and } e^{i\cdot(-\frac{\pi}{4}+\frac{2\pi}{2})}$$

$$e^{-i\cdot\frac{\pi}{4}} = \frac{1}{\sqrt{2}}(1-i) \qquad e^{i\cdot(-\frac{\pi}{4}+\frac{2\pi}{2})} = e^{i\frac{3\pi}{4}} = \frac{1}{\sqrt{2}}(-1+i)$$

$\Rightarrow$ $f(z)$ is analytic everywhere except at the points $\pm\frac{1}{\sqrt{2}}(1-i)$ and on the portion $\mathrm{Re}(z) \le 4$ of the real axis.

P 97. 11. $u(x,y) = \ln(x^2+y^2) \Rightarrow u_x = \frac{2x}{x^2+y^2}$, $u_y = \frac{2y}{x^2+y^2}$

$u_{xx} = \frac{2}{x^2+y^2} - \frac{2x \cdot 2x}{(x^2+y^2)^2}$, $u_{yy} = \frac{2}{x^2+y^2} - 2y \cdot \frac{2y}{(x^2+y^2)^2}$

$\Rightarrow u_{xx} + u_{yy} = \frac{4}{x^2+y^2} - \frac{4(x^2+y^2)}{(x^2+y^2)^2} = 0$

2nd method: Consider $f(z) = \log z = \ln r + i(\mathrm{Arg}\, z + n \cdot 2\pi)$.

$= \ln(\sqrt{x^2+y^2}) + i(\mathrm{Arg}\, z + 2n\pi)$

For any point z, $= \frac{1}{2}\ln(x^2+y^2) + i(\mathrm{Arg}\, z + 2n\pi)$.

$f(z) = \log z$ is analytic around point z by an appropriate choice of branch (i.e. z does not lie on the branch locus)

$\Rightarrow 2 \cdot \mathrm{Re}(f(z)) = \ln(x^2+y^2)$ is harmonic.

P99 1. $\text{Log}(z_1z_2) = \ln|z_1z_2| + i(\text{Arg}(z_1z_2)) = \ln(|z_1|\cdot|z_2|) + i\,\text{Arg}(z_1z_2)$

$\text{Log}\,z_1 + \text{Log}\,z_2 = (\ln|z_1| + i\,\text{Arg}\,z_1) + (\ln|z_2| + i\,\text{Arg}\,z_2) = \ln(|z_1||z_2|) + i(\text{Arg}\,z_1 + \text{Arg}\,z_2)$

$\Rightarrow \text{Log}(z_1z_2) - (\text{Log}\,z_1 + \text{Log}\,z_2) = i(\text{Arg}(z_1z_2) - (\text{Arg}\,z_1 + \text{Arg}\,z_2))$.

If we take exponential on both sides, we get:

$$e^{\text{Log}(z_1z_2) - (\text{Log}\,z_1 + \text{Log}\,z_2)} = e^{i(\text{Arg}(z_1z_2) - (\text{Arg}\,z_1 + \text{Arg}\,z_2))}$$

$$e^{\text{Log}(z_1z_2)} \cdot (e^{\text{Log}\,z_1})^{-1} \cdot (e^{\text{Log}\,z_2})^{-1} = z_1 z_2 \cdot z_1^{-1} \cdot z_2^{-1} = 1$$

So $\text{Arg}(z_1z_2) - (\text{Arg}\,z_1 + \text{Arg}\,z_2) = N 2\pi$. N is an integer

Now $-\pi < \text{Arg}(z_1z_2) \le \pi$.

$-2\pi < \text{Arg}(z_1) + \text{Arg}(z_2) \le 2\pi$

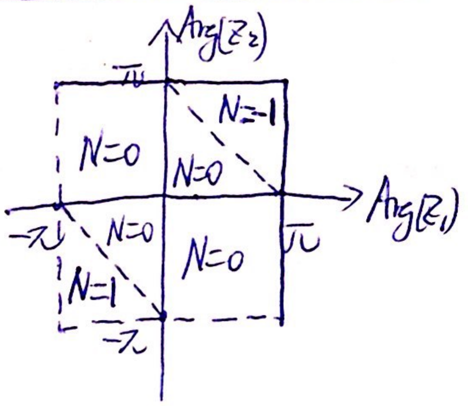
$\Rightarrow -3\pi < \text{Arg}(z_1z_2) - (\text{Arg}\,z_1 + \text{Arg}\,z_2) = 2N \cdot \pi < 3\pi$

$\Downarrow$

$-3 < 2N < 3$

$\Downarrow$ N is an integer

$N = -1, 0, 1$



Case 1: $-2\pi < \text{Arg}(z_1) + \text{Arg}\,z_2 \le -\pi \Rightarrow N = 1$

Case 2: $-\pi < \text{Arg}(z_1) + \text{Arg}(z_2) \le \pi \Rightarrow N = 0$

Case 3: $\pi < \text{Arg}(z_1) + \text{Arg}(z_2) \le 2\pi \Rightarrow N = -1$.

P103. 1. (a) $(1+i)^i = e^{(\log(1+i))i} = e^{(\ln\sqrt{2}+i(\frac{\pi}{4}+n\cdot 2\pi))\cdot i} = e^{-(\frac{\pi}{4}+2n\pi)+i\ln\sqrt{2}}$

$= e^{-\frac{\pi}{4}-2n\pi} \cdot e^{i\cdot\frac{\ln 2}{2}}$ $\quad n=0,\pm1,\pm2,\cdots$

(b). $\frac{1}{i^{2i}} = i^{-2i} = e^{-2i\cdot(\log i)} = e^{-2i\cdot(\ln 1+i\frac{\pi}{2}+2n\pi)}$

$= e^{\pi+4n\pi} = e^{(4n+1)\pi}$ $\quad n=0,\pm1,\pm2,\cdots$

3. $(-1+\sqrt{3}i)^{\frac{3}{2}} = e^{\frac{3}{2}\cdot\log(-1+\sqrt{3}i)} = e^{\frac{3}{2}(\ln 2+i\cdot(\frac{2\pi}{3}+2n\pi))}$

$= (e^{\ln 2})^{\frac{3}{2}} \cdot e^{i\cdot(\pi+3n\pi)}$ $\quad n=0,\pm1,\pm2,\cdots$

$= (2^3)^{\frac{1}{2}}\cdot(\pm1) = \pm 2\sqrt{2}.$

9. $\frac{d}{dz}(c^{f(z)}) = \frac{d}{dz}(e^{(\log c)\cdot f(z)}) = e^{(\log c)\cdot f(z)}\cdot f'(z)$

$= c^{f(z)}\cdot f'(z).$

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P107. 2. (a) $e^{iz_1}\cdot e^{iz_2} = (\cos z_1 + i\sin z_1)(\cos z_2 + i\sin z_2)$

$= (\cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2)) + i\cdot(\sin z_1 \cos z_2 + \cos z_1 \cdot \sin z_2)$

$e^{-iz_1}\cdot e^{-iz_2} = (\cos z_1 - i\sin z_1)(\cos z_2 - i\sin z_2)$

$= (\cos z_1 \cos z_2 - \sin z_1 \sin z_2) - i\cdot(\sin z_1 \cos z_2 + \cos z_1 \sin z_2)$.

(b). $\sin(z_1+z_2) = \frac{1}{2i}\cdot\left[e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}\right] = \frac{1}{2i}\left[e^{iz_1}\cdot e^{iz_2} - e^{-iz_1}\cdot e^{-iz_2}\right]$

$= \frac{2i}{2i}(\sin(z_1)\cos z_2 + \cos z_1 \sin z_2) = \sin z_1 \cdot \cos z_2 + \cos z_1 \cdot \sin z_2$.

8. $\sin z = \sin x \cdot \cosh y + i \cos x \sinh y$

$\Rightarrow |\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y = \sin^2 x(\cosh^2 y - \sinh^2 y) + \sin^2 x \sinh^2 y + \cos^2 x \cdot \sinh^2 y$

$= \sin^2 x + \sinh^2 y \geq \sin^2 x$

$\Rightarrow |\sin z| \geq |\sin x|$.

$\cos z = \cos x \cdot \cosh y - i \sin x \sinh y$

$\Rightarrow |\cos z|^2 = \cos^2 x \cosh^2 y + \sin^2 x \cdot \sinh^2 y = \cos^2 x(\cosh^2 y - \sinh^2 y) + \cos^2 x \sinh^2 y + \sin^2 x \sinh^2 y$

$= \cos^2 x + \sinh^2 y \geq \cos^2 x$

$\Rightarrow |\cos z| \geq |\cos x|$

$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{1}{2i}(e^{i(x+iy)} - e^{-i(x+iy)}) = \frac{1}{2i}(e^{-y+ix} - e^{y}\cdot e^{-ix}) = \frac{1}{2i}(e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x))$

$= \frac{1}{2i}((\cos x)(e^{-y} - e^{y}) + i\sin x(e^{-y} + e^{y})) = \sin x \cdot \frac{e^{y}+e^{-y}}{2} + i\cdot\cos x\cdot\frac{e^{y}-e^{-y}}{2}$

$= \sin x \cdot \cosh y + i \cos x \sinh y$.

we write $a = 0$, $b = 2\pi$ and use the same function $w(t) = e^{it}$ $(0 \le t \le 2\pi)$ as in Example 3, Sec. 41. It is easy to see that

$$\int_a^b w(t)\,dt = \int_0^{2\pi} e^{it}\,dt = \frac{e^{it}}{i}\Big]_0^{2\pi} = 0.$$

But, for any number c such that $0 < c < 2\pi$,

$$|w(c)(b-a)| = |e^{ic}|\,2\pi = 2\pi;$$

and we find that the left-hand side of equation (5) is zero but that the right-hand side is not.

EXERCISES

1. Use rules in calculus to establish the following rules when

$$w(t) = u(t) + iv(t)$$

is a complex-valued function of a real variable t and $w'(t)$ exists:

(*a*) $\dfrac{d}{dt}[z_0 w(t)] = z_0 w'(t)$, where $z_0 = x_0 + iy_0$ is a complex constant;

(*b*) $\dfrac{d}{dt} w(-t) = -w'(-t)$ where $w'(-t)$ denotes the derivative of $w(t)$ with respect to t, evaluated at $-t$;

Suggestion: In part (*a*). show that each side of the identity to be verified can be written

$$(x_0u' - y_0v') + i(y_0u' + x_0v').$$

2. Evaluate the following integrals:

(*a*) $\displaystyle\int_0^1 (1+it)^2\,dt$; (*b*) $\displaystyle\int_1^2 \left(\frac{1}{t} - i\right)^2 dt$;

(*c*) $\displaystyle\int_0^{\pi/6} e^{i2t}\,dt$; (*d*) $\displaystyle\int_0^{\infty} e^{-zt}\,dt$ $(\operatorname{Re} z > 0)$.

Ans. (*a*) $\dfrac{2}{3} + i$; (*b*) $-\dfrac{1}{2} - i\ln 4$; (*c*) $\dfrac{\sqrt{3}}{4} + \dfrac{i}{4}$; (*d*) $\dfrac{1}{z}$.

3. Show that if m and n are integers,

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta}\,d\theta = \begin{cases} 0 & \text{when } m \ne n, \\ 2\pi & \text{when } m = n. \end{cases}$$

4. According to definition (2), Sec. 42, of definite integrals of complex-valued functions of a real variable,

$$\int_0^{\pi} e^{(1+i)x}\,dx = \int_0^{\pi} e^x \cos x\,dx + i\int_0^{\pi} e^x \sin x\,dx.$$

Evaluate the two integrals on the right here by evaluating the single integral on the left and then using the real and imaginary parts of the value found.

Ans. $-(1+e^{\pi})/2$, $(1+e^{\pi})/2$.

5. Let $w(t) = u(t) + iv(t)$ denote a continuous complex-valued function defined on an interval $-a \le t \le a$.

(*a*) Suppose that $w(t)$ is *even*; that is, $w(-t) = w(t)$ for each point t in the given interval. Show that

$$\int_{-a}^{a} w(t)\,dt = 2\int_{0}^{a} w(t)\,dt.$$

(*b*) Show that if $w(t)$ is an *odd* function, one where $w(-t) = -w(t)$ for each point t in the given interval, then

$$\int_{-a}^{a} w(t)\,dt = 0.$$

Suggestion: In each part of this exercise, use the corresponding property of integrals of *real-valued* functions of t, which is graphically evident.

43. CONTOURS

Integrals of complex-valued functions of a *complex* variable are defined on curves in the complex plane, rather than on just intervals of the real line. Classes of curves that are adequate for the study of such integrals are introduced in this section.

A set of points $z = (x, y)$ in the complex plane is said to be an ***arc*** if

$$(1) \qquad x = x(t), \quad y = y(t) \qquad (a \le t \le b),$$

where $x(t)$ and $y(t)$ are continuous functions of the real parameter t. This definition establishes a continuous mapping of the interval $a \le t \le b$ into the xy, or z, plane; and the image points are ordered according to increasing values of t. It is convenient to describe the points of C by means of the equation

$$(2) \qquad z = z(t) \qquad (a \le t \le b),$$

where

$$(3) \qquad z(t) = x(t) + iy(t).$$

The arc C is a ***simple arc***, or a Jordan arc,* if it does not cross itself; that is, C is simple if $z(t_1) \ne z(t_2)$ when $t_1 \ne t_2$. When the arc C is simple except for the fact that $z(b) = z(a)$, we say that C is a ***simple closed curve***, or a Jordan curve. Such a curve is ***positively oriented*** when it is in the counterclockwise direction.

The geometric nature of a particular arc often suggests different notation for the parameter t in equation (2). This is, in fact, the case in the following examples.

*Named for C. Jordan (1838–1922), pronounced *jor-don'*.

enables us to write expression (13) as

$$L = \int_\alpha^\beta |Z'(\tau)|\, d\tau.$$

Thus the same length of C would be obtained if representation (10) were to be used.

If equation (2) represents a differentiable arc and if $z'(t) \neq 0$ anywhere in the interval $a < t < b$, then the unit tangent vector

$$\mathbf{T} = \frac{z'(t)}{|z'(t)|}$$

is well defined for all t in that open interval, with angle of inclination $\arg z'(t)$. Also, when $\mathbf{T}$ turns, it does so continuously as the parameter t varies over the entire interval $a < t < b$. This expression for $\mathbf{T}$ is the one learned in calculus when $z(t)$ is interpreted as a radius vector. Such an arc is said to be ***smooth***. In referring to a smooth arc $z = z(t)$ $(a \leq t \leq b)$, then, we agree that the derivative $z'(t)$ is continuous on the closed interval $a \leq t \leq b$ and nonzero throughout the open interval $a < t < b$.

A ***contour***, or piecewise smooth arc, is an arc consisting of a finite number of smooth arcs joined end to end. Hence if equation (2) represents a contour, $z(t)$ is continuous, whereas its derivative $z'(t)$ is piecewise continuous. The polygonal line (4) is, for example, a contour. When only the initial and final values of $z(t)$ are the same, a contour C is called a ***simple closed contour***. Examples are the circles (5) and (6), as well as the boundary of a triangle or a rectangle taken in a specific direction. The length of a contour or a simple closed contour is the sum of the lengths of the smooth arcs that make up the contour.

The points on any simple closed curve or simple closed contour C are boundary points of two distinct domains, one of which is the interior of C and is bounded. The other, which is the exterior of C, is unbounded. It will be convenient to accept this statement, known as the ***Jordan curve theorem***, as geometrically evident; the proof is not easy.*

EXERCISES

1. Show that if $w(t) = u(t) + iv(t)$ is continuous on an interval $a \leq t \leq b$, then

(*a*) $\displaystyle\int_{-b}^{-a} w(-t)\, dt = \int_a^b w(\tau)\, d\tau$;

(*b*) $\displaystyle\int_a^b w(t)\, dt = \int_\alpha^\beta w[\phi(\tau)]\phi'(\tau)\, d\tau$, where $\phi(\tau)$ is the function in equation (9), Sec. 43.

Suggestion: These identities can be obtained by noting that they are valid for *real-valued* functions of t.

*See pp. 115–116 of the book by Newman or Sec. 13 of the one by Thron, both of which are cited in Appendix 1. The special case in which C is a simple closed polygon is proved on pp. 281–285 of Vol. 1 of the work by Hille, also cited in Appendix 1.

2. Let C denote the right-hand half of the circle $|z| = 2$, in the counterclockwise direction, and note that two parametric representations for C are

$$z = z(\theta) = 2e^{i\theta} \qquad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$$

and

$$z = Z(y) = \sqrt{4 - y^2} + iy \qquad (-2 \le y \le 2).$$

Verify that $Z(y) = z[\phi(y)]$, where

$$\phi(y) = \arctan \frac{y}{\sqrt{4 - y^2}} \qquad \left(-\frac{\pi}{2} < \arctan t < \frac{\pi}{2}\right).$$

Also, show that this function ϕ has a positive derivative, as required in the conditions following equation (9), Sec. 43.

3. Derive the equation of the line through the points (α, a) and (β, b) in the τt plane that are shown in Fig. 37. Then use it to find the linear function $\phi(\tau)$ which can be used in equation (9), Sec. 43, to transform representation (2) in that section into representation (10) there.

$$Ans.\ \phi(\tau) = \frac{b - a}{\beta - \alpha}\tau + \frac{a\beta - b\alpha}{\beta - \alpha}.$$

4. Verify expression (14), Sec. 43, for the derivative of $Z(\tau) = z[\phi(\tau)]$.

Suggestion: Write $Z(\tau) = x[\phi(\tau)] + iy[\phi(\tau)]$ and apply the chain rule for real-valued functions of a real variable.

5. Suppose that a function $f(z)$ is analytic at a point $z_0 = z(t_0)$ lying on a smooth arc $z = z(t)$ $(a \le t \le b)$. Show that if $w(t) = f[z(t)]$, then

$$w'(t) = f'[z(t)]z'(t)$$

when $t = t_0$.

Suggestion: Write $f(z) = u(x, y) + iv(x, y)$ and $z(t) = x(t) + iy(t)$, so that

$$w(t) = u[x(t), y(t)] + iv[x(t), y(t)].$$

Then apply the chain rule in calculus for functions of two real variables to write

$$w' = (u_x x' + u_y y') + i(v_x x' + v_y y'),$$

and use the Cauchy–Riemann equations.

6. Let $y(x)$ be a real-valued function defined on the interval $0 \le x \le 1$ by means of the equations

$$y(x) = \begin{cases} x^3 \sin(\pi/x) & \text{when } 0 < x \le 1, \\ 0 & \text{when } x = 0. \end{cases}$$

(*a*) Show that the equation

$$z = x + iy(x) \qquad (0 \le x \le 1)$$

represents an arc C that intersects the real axis at the points $z = 1/n$ $(n = 1, 2, \ldots)$ and $z = 0$, as shown in Fig. 38.

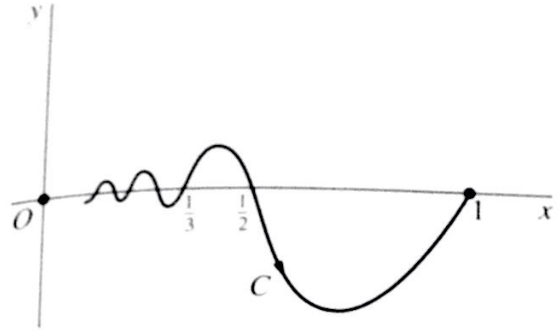


FIGURE 38

(*b*) Verify that the arc C in part (*a*) is, in fact, a *smooth* arc.

Suggestion: To establish the continuity of $y(x)$ at $x = 0$, observe that

$$0 \leq \left| x^3 \sin\left(\frac{\pi}{x}\right) \right| \leq x^3$$

when $x > 0$. A similar remark applies in finding $y'(0)$ and showing that $y'(x)$ is continuous at $x = 0$.

44. CONTOUR INTEGRALS

We turn now to integrals of complex-valued functions f of the complex variable z. Such an integral is defined in terms of the values $f(z)$ along a given contour C, extending from a point $z = z_1$ to a point $z = z_2$ in the complex plane. It is, therefore, a line integral; and its value depends, in general, on the contour C as well as on the function f. It is written

$$\int_C f(z)\, dz \quad \text{or} \quad \int_{z_1}^{z_2} f(z)\, dz,$$

the latter notation often being used when the value of the integral is independent of the choice of the contour taken between two fixed end points. While the integral can be defined directly as the limit of a sum,* we choose to define it in terms of a definite integral of the type introduced in Sec. 42.

Definite integrals in calculus can be interpreted as areas, and they have other interpretations as well. Except in special cases, no corresponding helpful interpretation, geometric or physical, is available for integrals in the complex plane.

Suppose that the equation

$$z = z(t) \quad (a \leq t \leq b) \tag{1}$$

represents a contour C, extending from a point $z_1 = z(a)$ to a point $z_2 = z(b)$. We assume that $f[z(t)]$ is *piecewise continuous* (Sec. 42) on the interval $a \leq t \leq b$ and

---

*See, for instance, pp. 245ff in Vol. I of the book by Markushevich that is listed in Appendix 1.

EXAMPLE 2. Using the principal branch

$$f(z) = z^{-1+i} = \exp[(-1+i)\text{Log}\, z] \qquad (|z| > 0, -\pi < \text{Arg}\, z < \pi)$$

of the power function z^{-1+i}, let us evaluate the integral

$$I = \int_C z^{-1+i}\, dz \tag{3}$$

where C is the positively oriented unit circle (Fig. 45)

$$z = e^{i\theta} \qquad (-\pi \le \theta \le \pi)$$

about the origin.

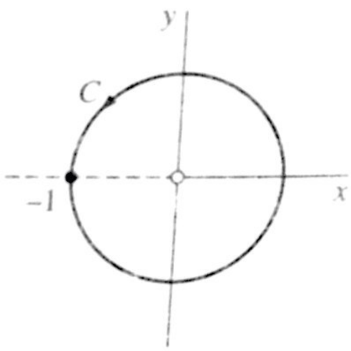


FIGURE 45

When $z(\theta) = e^{i\theta}$, it is easy to see that

$$f[z(\theta)]z'(\theta) = e^{(-1+i)(\ln 1 + i\theta)} ie^{i\theta} = ie^{-\theta}. \tag{4}$$

Inasmuch as the function (4) is piecewise continuous on $-\pi < \theta < \pi$, integral (3) exists. In fact,

$$I = i\int_{-\pi}^{\pi} e^{-\theta} d\theta = i\,[-e^{-\theta}]_{-\pi}^{\pi} = i(-e^{-\pi} + e^{\pi}),$$

or

$$I = i\,2\frac{e^{\pi} - e^{-\pi}}{2} = i\,2\sinh \pi.$$

EXERCISES

For the functions f and contours C in Exercises 1 through 8, use parametric representations for C, or legs of C, to evaluate

$$\int_C f(z)\, dz.$$

1. $f(z) = (z+2)/z$ and C is

(*a*) the semicircle $z = 2e^{i\theta}$ $(0 \le \theta \le \pi)$;

(*b*) the semicircle $z = 2e^{i\theta}$ $(\pi \le \theta \le 2\pi)$;

(*c*) the circle $z = 2e^{i\theta}$ $(0 \le \theta \le 2\pi)$.

Ans. (*a*) $-4 + 2\pi i$; (*b*) $4 + 2\pi i$; (*c*) $4\pi i$.

2. $f(z) = z - 1$ and C is the arc from $z = 0$ to $z = 2$ consisting of

(*a*) the semicircle $z = 1 + e^{i\theta}$ $(\pi \le \theta \le 2\pi)$;

(*b*) the segment $z = x$ $(0 \le x \le 2)$ of the real axis.

Ans. (*a*) 0; (*b*) 0.

3. $f(z) = \pi \exp(\pi \bar{z})$ and C is the boundary of the square with vertices at the points 0, 1, $1 + i$, and i, the orientation of C being in the counterclockwise direction.

Ans. $4(e^{\pi} - 1)$.

4. $f(z)$ is defined by means of the equations

$$f(z) = \begin{cases} 1 & \text{when } y < 0, \\ 4y & \text{when } y > 0, \end{cases}$$

and C is the arc from $z = -1 - i$ to $z = 1 + i$ along the curve $y = x^3$.

Ans. $2 + 3i$.

5. $f(z) = 1$ and C is an arbitrary contour from any fixed point z_1 to any fixed point z_2 in the z plane.

Ans. $z_2 - z_1$.

6. $f(z)$ is the principal branch

$$z^i = \exp(i \operatorname{Log} z) \quad (|z| > 0, -\pi < \operatorname{Arg} z < \pi)$$

of the power function z^i, and C is the semicircle $z = e^{i\theta}$ $(0 \le \theta \le \pi)$.

Ans. $-\dfrac{1 + e^{-\pi}}{2}(1 - i)$.

7. $f(z)$ is the principal branch

$$z^{-1-2i} = \exp[(-1 - 2i)\operatorname{Log} z] \qquad (|z| > 0, -\pi < \operatorname{Arg} z < \pi)$$

of the indicated power function, and C is the contour

$$z = e^{i\theta} \qquad \left(0 \le \theta \le \frac{\pi}{2}\right).$$

Ans. $i\dfrac{e^{\pi} - 1}{2}$.

8. $f(z)$ is the principal branch

$$z^{a-1} = \exp[(a - 1)\operatorname{Log} z] \quad (|z| > 0, -\pi < \operatorname{Arg} z < \pi)$$

of the power function z^{a-1}, where a is a nonzero real number, and C is the positively oriented circle of radius R about the origin.

Ans. $i\dfrac{2R^a}{a} \sin a\pi$, where the positive value of R^a is to be taken.

9. Let C denote the positively oriented unit circle $|z| = 1$ about the origin.

(*a*) Show that if $f(z)$ is the principal branch

$$z^{-3/4} = \exp\left[-\frac{3}{4}\operatorname{Log} z\right] \quad (|z| > 0, -\pi < \operatorname{Arg} z < \pi)$$

of $z^{-3/4}$, then

$$\int_C f(z)dz = 4\sqrt{2}\,i.$$

(*b*) Show that if $g(z)$ is the branch

$$z^{-3/4} = \exp\left[-\frac{3}{4}\log z\right] \quad (|z| > 0,\ 0 < \arg z < 2\pi)$$

of the same power function as in part (*a*), then

$$\int_C g(z)dz = -4 + 4i.$$

This exercise demonstrates how the value of an integral of a power function depends in general on the branch that is used.

10. With the aid of the result in Exercise 3, Sec. 42, evaluate the integral

$$\int_C z^m \bar{z}^n dz,$$

where m and n are integers and C is the unit circle $|z| = 1$, taken counterclockwise.

11. Let C denote the semicircular path shown in Fig. 46. Evaluate the integral of the function $f(z) = \bar{z}$ along C using the parametric representation (see Exercise 2, Sec. 43)

(*a*) $z = 2e^{i\theta}\ \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$; (*b*) $z = \sqrt{4 - y^2} + iy \quad (-2 \le y \le 2)$.

Ans. $4\pi i$.

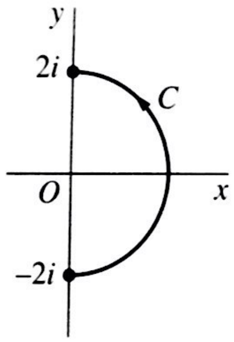


FIGURE 46

12. (*a*) Suppose that a function $f(z)$ is continuous on a smooth arc C, which has a parametric representation $z = z(t)$ $(a \le t \le b)$; that is, $f[z(t)]$ is continuous on the interval $a \le t \le b$. Show that if $\phi(\tau)$ $(\alpha \le \tau \le \beta)$ is the function described in Sec. 43, then

$$\int_a^b f[z(t)]z'(t)\,dt = \int_\alpha^\beta f[Z(\tau)]Z'(\tau)\,d\tau$$

where $Z(\tau) = z[\phi(\tau)]$.

(*b*) Point out how it follows that the identity obtained in part (*a*) remains valid when C is any contour, not necessarily a smooth one, and $f(z)$ is piecewise continuous on C. Thus show that the value of the integral of $f(z)$ along C is the same when the representation $z = Z(\tau)$ $(\alpha \le \tau \le \beta)$ is used, instead of the original one.

Suggestion: In part (*a*), use the result in Exercise 1(*b*), Sec. 43, and then refer to expression (14) in that section.

13. Let C_0 denote the circle centered at z_0 with radius R, and use the parametrization

$$z = z_0 + Re^{i\theta} \qquad (-\pi \le \theta \le \pi)$$

to show that

$$\int_{C_0} (z - z_0)^{n-1} dz = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2, \ldots, \\ 2\pi i & \text{when } n = 0. \end{cases}$$

(Put $z_0 = 0$ and then compare the result with the one in Exercise 8 when the constant a there is a nonzero integer.)

47. UPPER BOUNDS FOR MODULI OF CONTOUR INTEGRALS

We turn now to an inequality involving contour integrals that is extremely important in various applications. We present the result as a theorem but preface it with a needed lemma involving functions $w(t)$ of the type encountered in Secs. 41 and 42.

Lemma. *If $w(t)$ is a piecewise continuous complex-valued function defined on an interval $a \le t \le b$, then*

$$\left| \int_a^b w(t)\, dt \right| \le \int_a^b |w(t)|\, dt. \tag{1}$$

This inequality clearly holds when the value of the integral on the left is zero. Thus, in the verification, we may assume that its value is a *nonzero* complex number and write

$$\int_a^b w(t)\, dt = r_0 e^{i\theta_0}. \tag{2}$$

Solving for r_0, we have

$$r_0 = \int_a^b e^{-i\theta_0} w(t)\, dt. \tag{3}$$

Now the left-hand side of this equation is a real number, and so the right-hand side is too. Thus, using the fact that the real part of a real number is the number itself, we find that

$$r_0 = \text{Re} \int_a^b e^{-i\theta_0} w(t)\, dt.$$

Hence, in view of the first of properties (3) in Sec. 42,

$$r_0 = \int_a^b \text{Re}[e^{-i\theta_0} w(t)]\, dt. \tag{4}$$

But

$$\text{Re}[e^{-i\theta_0} w(t)] \le |e^{-i\theta_0} w(t)| = |e^{-i\theta_0}|\,|w(t)| = |w(t)|,$$

But f is continuous at the point z. Hence, for each positive number ε, a positive number δ exists such that

$$|f(s) - f(z)| < \varepsilon \quad \text{whenever} \quad |s - z| < \delta.$$

Consequently, if the point $z + \Delta z$ is close enough to z so that $|\Delta z| < \delta$, then

$$\left| \frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) \right| < \frac{1}{|\Delta z|} \varepsilon |\Delta z| = \varepsilon;$$

that is,

$$\lim_{\Delta z \to 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} = f(z),$$

or $F'(z) = f(z)$.

EXERCISES

1. Use an antiderivative to show that for every contour C extending from a point z_1 to a point z_2,

$$\int_C z^n \, dz = \frac{1}{n+1}(z_2^{n+1} - z_1^{n+1}) \qquad (n = 0, 1, 2, \ldots).$$

2. By finding an antiderivative, evaluate each of these integrals, where the path is any contour between the indicated limits of integration:

$$(a) \int_0^{1+i} z^2 dz; \qquad (b) \int_0^{\pi + 2i} \cos\left(\frac{z}{2}\right) dz; \qquad (c) \int_1^3 (z-2)^3 \, dz.$$

Ans. (a) $\frac{2}{3}(-1 + i)$; (b) $e + \frac{1}{e}$; (c) 0.

3. Use the theorem in Sec. 48 to show that

$$\int_{C_0} (z - z_0)^{n-1} \, dz = 0 \qquad (n = \pm 1, \pm 2, \ldots)$$

when C_0 is any closed contour which does not pass through the point z_0. (Compare with Exercise 13, Sec. 46.)

4. Find an antiderivative $F_2(z)$ of the branch $f_2(z)$ of $z^{1/2}$ in Example 4, Sec. 48, to show that integral (3) there has value $2\sqrt{3}(-1 + i)$. Note that the value of the integral of the function (2) around the closed contour $C_2 - C_1$ in that example is, therefore, $-4\sqrt{3}$.

5. Show that

$$\int_{-1}^{1} z^i \, dz = \frac{1 + e^{-\pi}}{2}(1 - i),$$

where the integrand denotes the principal branch

$$z^i = \exp(i \operatorname{Log} z) \qquad (|z| > 0, -\pi < \operatorname{Arg} z < \pi)$$

of z^i and where the path of integration is any contour from $z = -1$ to $z = 1$ that, except for its end points, lies above the real axis. (Compare with Exercise 6, Sec. 46.)

Suggestion: Use an antiderivative of the branch

$$z^i = \exp(i \log z) \qquad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

of the same power function.

Homework 8: Part 2

1: Let $f(z)$ be a complex valued function, not necessarily analytic. Let $z = z(t)$ be a smooth arc. Denote $w(t) = f(z(t))$.

(a) Prove the following chain rule:

$$w'(t) = \frac{\partial f}{\partial z}\, z'(t) + \frac{\partial f}{\partial \bar{z}}\, \overline{z'(t)}. \tag{1}$$

Note that in the above formula, we have used the following notation (compare Exercise 8 on Page 71)

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f, \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f.$$

(b) If $f(z)$ is analytic, then by (Homework Exercise 8 on Page 71) we have $\frac{\partial f}{\partial \bar{z}} = 0$. Show that in this case, equation (1) is reduced to the following formula (compare Exercise 5 on Page 124):

$$w'(t) = \frac{\partial f}{\partial z} z'(t) = f'(z(t))z'(t).$$

2*: Show that $w = \sin(z)$ maps the vertical strip $\{z \in \mathbb{C}; -\frac{\pi}{2} < \mathrm{Re}(z) < \frac{\pi}{2}\}$ to the region $\mathbb{C}\backslash((-\infty, -1] \cup [1, +\infty))$ by showing the following steps.

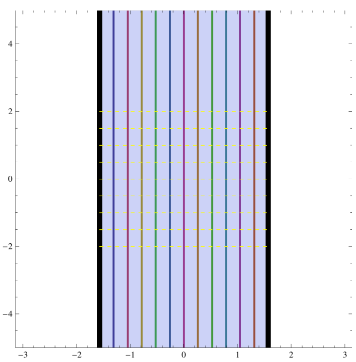
(a) Show that

$$\sin(z) = \sin x \cosh y + i \cos x \sinh y.$$

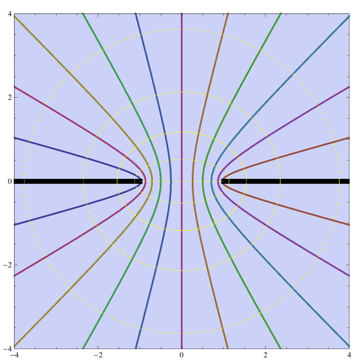
(b) Use the identity $\cosh^2 y - \sinh^2 y = 1$ to show that $w = \sin(z)$ maps the vertical line $L_c = \{\mathrm{Re}(z) = c\}$ to one branch of the following hyperbola:

$$\frac{x^2}{\sin^2(c)} - \frac{y^2}{\cos^2(c)} = 1.$$

If $c <$ (resp. $>$)0, then L_c is mapped to the left (resp. right) branch. If $c = 0$, then L_0 is the y-axis, which is mapped to the v-axis in the w-plane ($w = u + iv$).



(a) z-plane



(b) w-plane

(c) As c increases from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, visualize how the corresponding hyperbola moves continuously from the left to the right, and how the opening changes as follows:

ray $(-\infty, -1]$ $\to$ small toward to the left $\to$ large toward to the left $\to$ vertical flat $\to$ large toward the right $\to$ small toward the right $\to$ ray $[1, +\infty)$.

P119. 2. (b). $\int_1^2 \left(\frac{1}{t}-i\right)^2 dt = \int_1^2 \left(\frac{1}{t^2}-\frac{2i}{t}-1\right)dt = \left[-\frac{1}{t}-2i\ln t - t\right]_1^2$

$= \left(-\frac{1}{2}-2i\ln 2-2\right)-\left(-1-2i\ln 1-1\right)$

$= -\frac{1}{2}-i\ln 4$

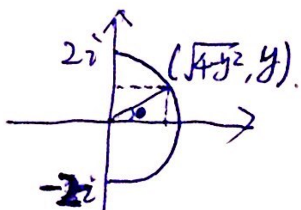
(d) $\int_0^\infty e^{-zt}\,dt \quad (\mathrm{Re}\,z>0)$

$= \int_0^\infty e^{-zt}\frac{d(zt)}{z} = -\frac{1}{z}e^{-zt}\Big]_0^\infty = -\frac{1}{z}\cdot\left[\left(\lim_{t\to\infty}e^{-zt}\right)-1\right]$

$= \frac{1}{z}$. because $\lim_{t\to\infty}|e^{-zt}| = \lim_{t\to+\infty} e^{-t\cdot \mathrm{Re}\,z} = 0 \quad (\mathrm{Re}\,z>0)$

P124. 2. $z = z(\theta) = 2\cdot e^{i\theta}$

$= Z(y) = \sqrt{4-y^2}+iy$.



$\theta = \phi(y) = \arctan\frac{y}{\sqrt{4-y^2}} \Rightarrow \tan\theta = \frac{y}{\sqrt{4-y^2}} \Rightarrow \cos\theta = \frac{\sqrt{4-y^2}}{2}$, $\sin\theta = \frac{y}{2}$.

as y increases, θ also increases $\Rightarrow \phi'(y) > 0$

$$\phi'(y) = \frac{1}{1+\frac{y^2}{4-y^2}}\left(+\frac{1}{\sqrt{4-y^2}}-\frac{1}{2}\cdot y\cdot(4-y^2)^{-\frac{3}{2}}(-2y)\right)$$

$$= \frac{4-y^2}{4}\cdot(4-y^2)^{-\frac{3}{2}}\left(+(4-y^2)+y^2\right)$$

$$= (4-y^2)^{-\frac{1}{2}} > 0.$$

P124.6 (a) $z = x + iy(x)$ represents the graph $y = y(x)$ $0 < x \leq 1$.

C intersects points where $\underset{\underset{x^3 \cdot \sin\left(\frac{\pi}{x}\right)}{\parallel}}{y(x)} = 0 \iff x = 0$ or $\frac{\pi}{x} = n\pi. \iff x = 0$ or $x = \frac{1}{n}$, $n = 1, 2, 3, \cdots$

(b). $y(x) = x^3 \sin\left(\frac{\pi}{x}\right)$ is smooth when $x \neq 0$.

$$y'(x) = 3x^2 \sin\left(\frac{\pi}{x}\right) + x^3 \cos\left(\frac{\pi}{x}\right) \cdot \left(-\frac{\pi}{x^2}\right) = 3x^2 \sin\left(\frac{\pi}{x}\right) - \pi x \cos\left(\frac{\pi}{x}\right). \quad x \neq 0.$$

at $x = 0$. $y(x)$ is continuous at 0:

$$0 \leq \left|x^3 \sin\left(\frac{\pi}{x}\right)\right| \leq x^3 \xRightarrow{\text{squeeze}} \lim_{x \to 0} x^3 \sin\left(\frac{\pi}{x}\right) = 0.$$

$$y'(0) = \lim_{x \to 0} \frac{y(x) - y(0)}{x - 0} = \lim_{x \to 0} \frac{x^3 \sin\left(\frac{\pi}{x}\right)}{x} = \lim_{x \to 0} x^2 \sin\left(\frac{\pi}{x}\right) \overset{\text{Squeeze}}{=\!=} 0$$

$y'(x)$ is also continuous at 0 because:

$$\lim_{x \to 0} y'(x) = \lim_{x \to 0} \left(3x^2 \sin\left(\frac{\pi}{x}\right) - \pi \cdot x \cos\left(\frac{\pi}{x}\right)\right) = 0 = y'(0).$$

P132. 1. $f(z)=\frac{z+2}{z}$ (b) C: $z=2e^{i\theta}$ $(\pi\le\theta\le 2\pi)$.

$z'(\theta)=2\cdot i\cdot e^{i\theta}$

$$\int_C f(z)\,dz = \int_\pi^{2\pi}\left(1+\frac{2}{2e^{i\theta}}\right)\cdot 2i\cdot e^{i\theta}\,d\theta = 2i\int_\pi^{2\pi}(e^{i\theta}+1)\,d\theta$$

$$=2i\cdot\left[\frac{1}{i}e^{i\theta}+\theta\right]_\pi^{2\pi} = 2i\left[\left(\frac{1}{i}e^{2\pi i}+2\pi\right)-\left(\frac{1}{i}e^{i\pi}+\pi\right)\right]$$

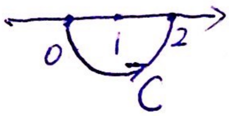
$$=2i\left[\frac{1}{i}+\frac{1}{i}+\pi\right]=4+2\pi i$$

2. $f(z)=z-1$. C: arc from $z=0$ to $z=2$ (a) $z=1+e^{i\theta}$ $(\pi\le\theta\le 2\pi)$.

$z'(\theta)=i\cdot e^{i\theta}$

$$\Rightarrow \int_C f(z)\,dz = \int_\pi^{2\pi}(1+e^{i\theta}-1)\cdot i\cdot e^{i\theta}\,d\theta = i\int_\pi^{2\pi} e^{2i\theta}\,d\theta = \frac{e^{2i\theta}}{2}\Big]_\pi^{2\pi} = \frac{1}{2}[1-1]=0.$$

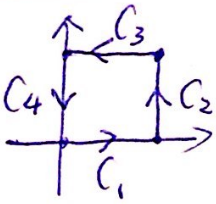
Actually $f(z)=z-1=\left(\frac{1}{2}z^2-z\right)'$ so $f(z)$ has an antiderivative.



$F(z)=\frac{1}{2}z^2-z$. so. $\int_C f(z)\,dz=\int_C F'(z)\,dz = F(z)\Big]_0^2$

for any curve from 0 to 2 $=\left(\frac{1}{2}\times 2^2-2\right)-0=0$

P133. $f(z) = \pi \cdot e^{\pi \bar{z}}$



C_1: $z(t) = t$. $0 \le t \le 1$. $z'(t) = 1$.

$$\int_{C_1} f(z)dz = \int_0^1 \pi \cdot e^{\pi \cdot t} \, 1 \cdot dt = e^{\pi t}\Big]_0^1 = e^{\pi} - 1.$$

C_2: $z(t) = 1 + ti$. $0 \le t \le 1$, $z'(t) = i$.

$$\int_{C_2} f(z)dz = \int_0^1 \pi \cdot e^{\pi \cdot (1-ti)} \, i \, dt = \pi \cdot e^{\pi} \cdot \int_0^1 e^{-\pi t i} \frac{d(\pi t i)}{\pi}$$

$$= e^{\pi} \cdot \left[- e^{\pi t i}\right]_0^1 = e^{\pi} \cdot (-e^{\pi i} + 1) = 2 \cdot e^{\pi}$$

C_3: $z(t) = (1-t) + i$. $0 \le t \le 1$. $z'(t) = -1$.

$$\underbrace{\int_{C_3} f(z)dz}_{\pi e^{\pi \bar{z}}} = \int_0^1 \pi \cdot e^{\pi \cdot ((1-t)-i)} \cdot (-1) \, dt = -\pi \cdot e^{\pi(1-i)} \cdot \int_0^1 e^{-\pi t} dt$$

$$= e^{\pi(1-i)} \, e^{-\pi t}\Big]_0^1 = \underbrace{e^{\pi(1-i)}}_{-e^{\pi}} \cdot \left[e^{-\pi} - 1\right] = -1 + e^{\pi}$$

C_4: $z(t) = (1-t)i$. $0 \le t \le 1$. $z'(t) = -i$.

$$\int_{C_4} \pi e^{\pi \bar{z}} dz = \int_0^1 \pi e^{\pi \overline{(1-t)i}} \underset{(-i)}{\cancel{dt}} \, dt = -i\pi \cdot \int_0^1 e^{-\pi(1-t)i} dt$$

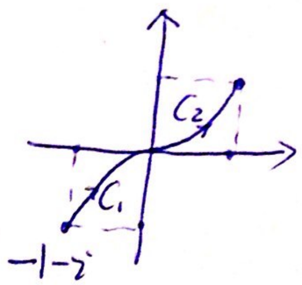
$$= -i\pi \cdot \underbrace{e^{-\pi i}}_{-1} \cdot \int_0^1 e^{\pi t i} dt = \int_0^1 e^{\pi t i} \, d(\pi i t) = \underbrace{e^{\pi i t}\Big]_0^1}_{e^{\pi i} - 1} = -2$$

So. $\int_C \pi \cdot e^{\pi \bar{z}} dz = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$

$$= (e^{\pi} - 1) + (2e^{\pi}) + (e^{\pi} - 1) + (-2) = 4e^{\pi} - 4.$$

P133

4. $f(z) = \begin{cases} 1 & \text{when } y<0 \\ 4y & \text{when } y>0. \end{cases}$



$C: z(x) = x + x^3 i. \quad -1 \le x \le 1.$

$z'(x) = 1 + 3x^2 i.$

$\int_C f(z)\,dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$

$4x^3 + 12x^5 i$

$\parallel$

$= \int_{-1}^{0} 1\cdot(1+3x^2 i)\,dx + \int_0^1 4\cdot x^3\cdot(1+3x^2 i)\,dx$

$= [x + x^3 i]\Big|_{-1}^{0} + [x^4 + 2x^6 i]_0^1$

$= [0-(-1-i)] + [1+2i-0] = 2+3i.$

5. $\int_{z_1}^{z_2} 1\,dz = z\Big]_{z_1}^{z_2} = z_2 - z_1.$

$\parallel \qquad \parallel$

$\int_{z_1}^{z_2} dx + i\,dy = [x+iy]_{z_1}^{z_2}$

P133.6. $f(z) = z^i = e^{i \operatorname{Log} z}$ $|z|>0$. $-\pi < \operatorname{Arg} z < \pi$.

C: $z = e^{i\theta}$, $0 \le \theta \le \pi$. $z'(\theta) = i \cdot e^{i\theta}$. $\operatorname{Log}(e^{i\theta}) = \ln|1| + i\theta$
$= i\theta$.

$f(e^{i\theta}) = e^{i \operatorname{Log}(e^{i\theta})} = e^{i(i\theta)} = e^{-\theta}$

$$\int_C f(z)\,dz = \int_0^{\pi} e^{-\theta} \cdot i \cdot e^{i\theta}\,d\theta = i\int_0^{\pi} e^{(-1+i)\theta}\,d\theta$$

$$= \frac{i}{-1+i} \cdot e^{(-1+i)\theta}\Big]_0^{\pi} = \frac{i(-1-i)}{2} \cdot \left[e^{(-1+i)\pi} - 1\right]$$

$$= \frac{1-i}{2} \cdot \left[-e^{-\pi} - 1\right] = -\frac{1+e^{-\pi}}{2} \cdot (1-i).$$

10.

C: $|z|=1$ counterclockwise $\rightsquigarrow$ $z = e^{i\theta}$, $0 \le \theta < 2\pi$ $z'(\theta) = ie^{i\theta}$.

$$\int_{|z|=1} z^m \bar{z}^n\,dz = \int_0^{2\pi} (e^{i\theta})^m \cdot (e^{-i\theta})^n \cdot i \cdot e^{i\theta}\,d\theta$$

$$= i \int_0^{2\pi} e^{i(m-n+1)\theta}\,d\theta$$

$$= \begin{cases} \frac{1}{m-n+1} \cdot e^{i(m-n+1)\theta}\Big]_0^{2\pi} = 0 & \text{if } m-n+1 \neq 0. \\ i \cdot \int_0^{2\pi} e^{i0}\,d\theta = 2\pi i & \text{if } m-n+1 = 0. \end{cases}$$

P147. 2. (a) $\int_0^{1+i} z^2 dz = \int_0^{1+i} \left(\frac{1}{3}z^3\right)' dz = \frac{1}{3}z^3 \Big]_0^{1+i}$

$= \frac{1}{3}\left[(1+i)^3 - 0\right] = \frac{1}{3}\cdot[1+3i-3-i] = \frac{1}{3}(-2+2i).$

(b). $\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right)dz = 2\cdot\sin\left(\frac{z}{2}\right)\Big]_0^{\pi+2i} = 2\cdot\left[\sin\left(\frac{\pi+2i}{2}\right) - \sin(0)\right]$

$= 2\cdot\sin\left(\frac{\pi}{2}+i\right) = +2\cdot\cos(i) = +2\cdot\frac{e^{i\cdot i}+e^{-i\cdot i}}{2} = e+\frac{1}{e}.$

(c) $\int_1^3 (z-2)^3 dz = \frac{1}{4}\cdot(z-2)^4\Big]_1^3 = \frac{1}{4}\cdot[1^4-(-1)^4] = 0.$

Part 2. First note that the following relations hold:

$$\begin{cases} z = x+iy \\ \bar{z} = x-iy \end{cases} \iff \begin{cases} x = \frac{1}{2}(z+\bar{z}) \\ y = \frac{1}{2i}(z-\bar{z}) \end{cases}$$ so (formally) by chain rule.

$$\begin{cases} \frac{\partial}{\partial x} = \frac{\partial z}{\partial x}\frac{\partial}{\partial z} + \frac{\partial \bar z}{\partial x}\frac{\partial}{\partial \bar z} = 1\cdot\frac{\partial}{\partial z} + 1\cdot\frac{\partial}{\partial \bar z} = \frac{\partial}{\partial z}+\frac{\partial}{\partial \bar z} \\ \frac{\partial}{\partial y} = \frac{\partial z}{\partial y}\frac{\partial}{\partial z} + \frac{\partial \bar z}{\partial y}\frac{\partial}{\partial \bar z} = i\frac{\partial}{\partial z} - i\frac{\partial}{\partial \bar z} = i\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial \bar z}\right). \end{cases}$$

reversely, we have

$$\begin{cases} \frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2i}\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial \bar z} = \frac{\partial x}{\partial \bar z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \bar z}\frac{\partial}{\partial y} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2i}\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right). \end{cases}$$

Now let (a) $w(t) = f(z(t)) = u(z(t)) + iv(z(t)) = u(x(t),y(t)) + iv(x(t),y(t))$.

Then

$$w'(t) = (u_x x' + u_y y') + i(v_x x' + v_y y') = (u_x+iv_x)x' + (u_y+iv_y)y'$$

$$= \left(\frac{\partial}{\partial x}f\right)\cdot x' + \left(\frac{\partial}{\partial y}f\right)\cdot y'$$

$$= \left(\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial \bar z}\right)f\right)\cdot \frac{d}{dt}\left(\frac{z(t)+\overline{z(t)}}{2}\right) + \left(i\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial \bar z}\right)f\right)\frac{d}{dt}\left(\frac{z(t)-\overline{z(t)}}{2i}\right)$$

$$= \left(\frac{\partial f}{\partial z}+\frac{\partial f}{\partial \bar z}\right)\cdot\frac{1}{2}\left(z'(t)+\overline{z'(t)}\right) + \left(\frac{\partial f}{\partial z}-\frac{\partial f}{\partial \bar z}\right)\cdot\frac{1}{2}\left(z'(t)-\overline{z'(t)}\right)$$

$$= \frac{1}{2}\left(\frac{\partial f}{\partial z}z' + \cancel{\frac{\partial f}{\partial z}\bar z'} + \cancel{\frac{\partial f}{\partial \bar z}z'} + \frac{\partial f}{\partial \bar z}\bar z'\right) + \frac{1}{2}\left(\frac{\partial f}{\partial z}z' - \cancel{\frac{\partial f}{\partial z}\bar z'} - \cancel{\frac{\partial f}{\partial \bar z}z'} + \frac{\partial f}{\partial \bar z}\bar z'\right)$$

$$= \frac{\partial f}{\partial z}z' + \frac{\partial f}{\partial \bar z}\bar z'.$$

a general complex valued fct

Actually, $f(z)$ should be thought of a fct. depending on z and $\bar z$, that is $f = f(z,\bar z)$. So if $w(t) = f(z(t),\overline{z(t)})$, then by chain rule, it's imediate that $w' = \frac{\partial f}{\partial z}z' + \frac{\partial f}{\partial \bar z}\bar z'$.

(b). As has been done in previous homework, we have

$$\frac{\partial f}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\cdot(u+iv) = \frac{1}{2}\cdot\left((u_x - v_y) + i(u_y + v_x)\right).$$

So f is analytic $\iff \dfrac{\partial f}{\partial \bar z} = 0 \iff \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ (CR eqs.).

In the case f is analytic, we have furthermore,

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u+iv) = \frac{1}{2}\cdot\left((u_x+v_y) + i(v_x - u_y)\right)$$

$$= \frac{1}{2}\cdot\left((u_x+u_x) + i\cdot(v_x+v_x)\right) \quad \text{(by CR eqs).}$$

$$= u_x + i v_x = f'(z).$$

So. if $w(t) = f(z(t))$, then

$$w'(t) = \frac{\partial f}{\partial z} z'(t) + \frac{\partial f}{\partial \bar z}\overline{z'(t)} = f'(z)\cdot z'(t) + 0 = f'(z)\cdot z'(t).$$

2*. (a) $\sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i} = \dfrac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}$

$$= \frac{e^{-y}\cdot e^{ix} - e^{y}\cdot e^{-ix}}{2i} = \frac{e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)}{2i}$$

$$= \frac{(\cos x)\cdot(e^{-y} - e^{y}) + i\cdot\sin x\cdot(e^{-y} + e^{y})}{2i}$$

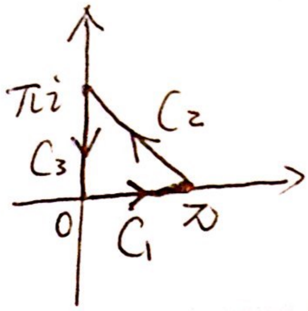
$$= \sin x \cdot \frac{1}{2}(e^{y} + e^{-y}) + i\cos x \cdot \frac{1}{2}(e^{y} - e^{-y})$$

$$= \sin x \cdot \cosh y + i\cos x \cdot \sinh y.$$

(b). $\mathrm{Re}(z) = C \overset{w=z^2}{\rightsquigarrow} (\sin C\cdot\cosh y,\ \cos C\cdot\sinh y) = (u, v).$

so $\dfrac{u^2}{\sin^2 C} - \dfrac{v^2}{\cos^2 C} = \cosh^2 y - \sinh^2 y = 1.$

Example: $\int_C \cos(\bar{z})\,dz$



$$\int_C \cos(\bar{z})\,dz = \int_{C_1}\cos(\bar{z})\,dz + \int_{C_2}\cos(\bar{z})\,dz + \int_{C_3}\cos(\bar{z})\,dz.$$

C_1: parametrization $z(t) = t + 0i \Rightarrow z'(t) = 1$. $\overline{z(t)} = t$.
$0 \le t \le \pi$

$$\int_{C_1}\cos(\bar{z})\,dz = \int_0^{\pi} \cos(t)\cdot 1\cdot dt = \sin t \Big]_0^{\pi} = 0 - 0 = 0.$$

C_2: parametrization $z(t) = (\pi - t) + ti$. $0 \le t \le \pi \Rightarrow z'(t) = -1 + i$.

$$\int_{C_2}\cos(\bar{z})\,dz = \int_0^{\pi}\cos((\pi-t)-ti)\cdot(-1+i)\,dt \qquad \begin{pmatrix}\cos(\pi+z) = -\cos(z)\\ \cos(-z) = \cos(z)\end{pmatrix} \qquad \overline{z(t)} = (\pi-t)-ti$$

$$= (1-i)\cdot\int_0^{\pi}\cos((1+i)t)\,dt = \frac{1-i}{1+i}\sin((1+i)t)\Big]_0^{\pi} = (-i)\left[\sin(\pi+\pi i) - \sin(0)\right]$$

$$= i\cdot\sin(\pi i) = i\cdot\frac{e^{i\pi i} - e^{-i(\pi i)}}{2i} = \frac{e^{-\pi} - e^{\pi}}{2} = -\sinh(\pi).$$

C_3: parametrization: $z(t) = ti$. $t: \pi \to 0$. $\overline{z(t)} = -ti$, $z'(t) = i$.

$$\int_{C_3}\cos(\bar{z})\,dz = \int_{\pi}^{0}\cos(-ti)\cdot i\,dt = -\int_0^{\pi}\cos(it)\,d(it) = -\sin(it)\Big]_0^{\pi}$$

$$= -(\sin(i\pi) - \sin(0)) = -\sin(i\pi) = -\frac{e^{i\cdot i\pi} - e^{-i\cdot i\pi}}{2i} = i\,\frac{e^{-\pi} - e^{\pi}}{2}$$

$$= -i\sinh(\pi).$$

So. $$\int_C \cos(\bar{z})\,dz = \int_{C_1} + \int_{C_2} + \int_{C_3} = 0 - \sinh(\pi) - i\sinh(\pi)$$

$$= -(1+i)\sinh(\pi).$$

EXAMPLE 1. Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant (Fig. 47). Inequality (6) can be used to show that

$$\left| \int_C \frac{z-2}{z^4+1}\, dz \right| \le \frac{4\pi}{15}. \tag{7}$$

This is done by noting first that if z is a point on C, then

$$|z-2| = |z+(-2)| \le |z| + |-2| = 2+2 = 4$$

and

$$|z^4+1| \ge ||z|^4 - 1| = 15.$$

Thus, when z lies on C,

$$\left| \frac{z-2}{z^4+1} \right| = \frac{|z-2|}{|z^4+1|} \le \frac{4}{15}.$$

By writing $M = 4/15$ and observing that $L = \pi$ is the length of C, we may now use inequality (6) to obtain inequality (7).

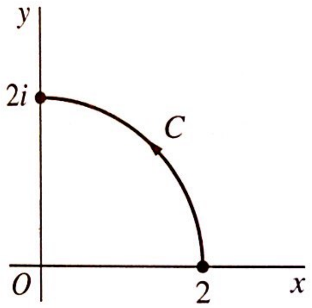


FIGURE 47

EXAMPLE 2. Let C_R denote the semicircle

$$z = Re^{i\theta} \qquad (0 \le \theta \le \pi)$$

from $z = R$ to $z = -R$, where $R > 3$ (Fig. 48). It is easy to show that

$$\lim_{R\to\infty} \int_{C_R} \frac{(z+1)\, dz}{(z^2+4)(z^2+9)} = 0 \tag{8}$$

without actually evaluating the integral. To do this, we observe that if z is a point on C_R,

$$|z+1| \le |z| + 1 = R+1,$$
$$|z^2+4| \ge ||z|^2 - 4| = R^2 - 4,$$

and

$$|z^2+9| \ge ||z|^2 - 9| = R^2 - 9.$$

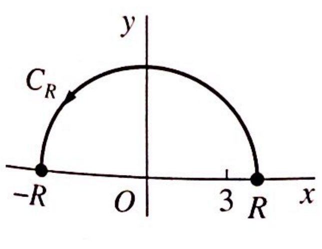


FIGURE 48

This means that if z is on C_R and $f(z)$ is the integrand in integral (8), then

$$|f(z)| = \left|\frac{z+1}{(z^2+4)\,(z^2+9)}\right| = \frac{|z+1|}{|z^2+4|\,|z^2+9|} \le \frac{R+1}{(R^2-4)\,(R^2-9)} = M_R,$$

where M_R serves as an upper bound for $|f(z)|$ on C_R. Since the length of the semicircle is πR, we may refer to the theorem in this section, using

$$M_R = \frac{R+1}{(R^2-4)\,(R^2-9)} \quad \text{and} \quad L = \pi R,$$

to write

$$\left|\int_{C_R} \frac{(z+1)\,dz}{(z^2+4)(z^2+9)}\right| \le M_R L \tag{9}$$

where

$$M_R L = \frac{\pi(R^2+R)}{(R^2-4)\,(R^2-9)} \cdot \frac{\dfrac{1}{R^4}}{\dfrac{1}{R^4}} = \frac{\pi\left(\dfrac{1}{R^2}+\dfrac{1}{R^3}\right)}{\left(1-\dfrac{4}{R^2}\right)\left(1-\dfrac{9}{R^2}\right)}.$$

This shows that $M_R L \to 0$ as $R \to \infty$, and limit (8) follows from inequality (9).

EXERCISES

1. Without evaluating the integral, show that

(a) $\displaystyle\left|\int_C \frac{z+4}{z^3-1}dz\right| \le \frac{6\pi}{7}$; (b) $\displaystyle\left|\int_C \frac{dz}{z^2-1}\right| \le \frac{\pi}{3}$

when C is the arc that was used in Example 1, Sec. 47.

2. Let C denote the line segment from $z = i$ to $z = 1$ (Fig. 49), and show that

$$\left|\int_C \frac{dz}{z^4}\right| \le 4\sqrt{2}$$

without evaluating the integral.

Suggestion: Observe that of all the points on the line segment, the midpoint is closest to the origin, that distance being $d = \sqrt{2}/2$.

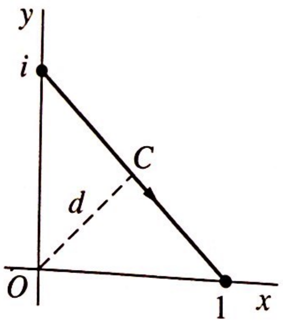


FIGURE 49

3. Show that if C is the boundary of the triangle with vertices at the points 0, $3i$, and -4, oriented in the counterclockwise direction (see Fig. 50), then

$$\left| \int_C (e^z - \bar{z})\, dz \right| \le 60.$$

Suggestion: Note that $|e^z - \bar{z}| \le e^x + \sqrt{x^2 + y^2}$ when $z = x + iy$.

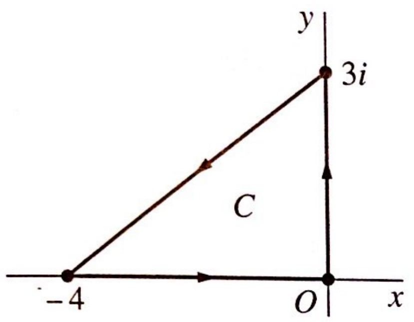


FIGURE 50

4. Let C_R denote the upper half of the circle $|z| = R$ $(R > 2)$, taken in the counterclockwise direction. Show that

$$\left| \int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4}\, dz \right| \le \frac{\pi R(2R^2 + 1)}{(R^2 - 1)(R^2 - 4)}.$$

Then, by dividing the numerator and denominator on the right here by R^4, show that the value of the integral tends to zero as R tends to infinity. (Compare with Example 2 in Sec. 47.)

5. Let C_R be the circle $|z| = R$ $(R > 1)$, described in the counterclockwise direction. Show that

$$\left| \int_{C_R} \frac{\operatorname{Log} z}{z^2}\, dz \right| < 2\pi \left(\frac{\pi + \ln R}{R} \right),$$

and then use l'Hospital's rule to show that the value of this integral tends to zero as R tends to infinity.

6. Let C_ρ denote a circle $|z| = \rho$ $(0 < \rho < 1)$, oriented in the counterclockwise direction, and suppose that $f(z)$ is analytic in the disk $|z| \le 1$. Show that if $z^{-1/2}$ represents any particular branch of that power of z, then there is a nonnegative constant M, *independent of* ρ, such that

$$\left| \int_{C_\rho} z^{-1/2} f(z)\, dz \right| \le 2\pi M \sqrt{\rho}.$$

Thus show that the value of the integral here approaches 0 as ρ tends to 0.

Suggestion: Note that since $f(z)$ is analytic, and therefore continuous, throughout the disk $|z| \le 1$, it is bounded there (Sec. 18).

But f is continuous at the point z. Hence, for each positive number ε, a positive number δ exists such that

$$|f(s) - f(z)| < \varepsilon \quad \text{whenever} \quad |s - z| < \delta.$$

Consequently, if the point $z + \Delta z$ is close enough to z so that $|\Delta z| < \delta$, then

$$\left| \frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) \right| < \frac{1}{|\Delta z|} \varepsilon |\Delta z| = \varepsilon;$$

that is,

$$\lim_{\Delta z \to 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} = f(z),$$

or $F'(z) = f(z)$.

EXERCISES

1. Use an antiderivative to show that for every contour C extending from a point z_1 to a point z_2,
$$\int_C z^n \, dz = \frac{1}{n+1}(z_2^{n+1} - z_1^{n+1}) \qquad (n = 0, 1, 2, \ldots).$$

2. By finding an antiderivative, evaluate each of these integrals, where the path is any contour between the indicated limits of integration:
$$(a) \int_0^{1+i} z^2 dz; \qquad (b) \int_0^{\pi+2i} \cos\left(\frac{z}{2}\right) dz; \qquad (c) \int_1^3 (z-2)^3 \, dz.$$
Ans. (a) $\frac{2}{3}(-1+i)$; (b) $e + \frac{1}{e}$; (c) 0.

3. Use the theorem in Sec. 48 to show that
$$\int_{C_0} (z - z_0)^{n-1} \, dz = 0 \qquad (n = \pm 1, \pm 2, \ldots)$$
when C_0 is any closed contour which does not pass through the point z_0. (Compare with Exercise 13, Sec. 46.)

4. Find an antiderivative $F_2(z)$ of the branch $f_2(z)$ of $z^{1/2}$ in Example 4, Sec. 48, to show that integral (3) there has value $2\sqrt{3}(-1+i)$. Note that the value of the integral of the function (2) around the closed contour $C_2 - C_1$ in that example is, therefore, $-4\sqrt{3}$.

5. Show that
$$\int_{-1}^{1} z^i \, dz = \frac{1 + e^{-\pi}}{2}(1 - i),$$
where the integrand denotes the principal branch
$$z^i = \exp(i \operatorname{Log} z) \qquad (|z| > 0, -\pi < \operatorname{Arg} z < \pi)$$
of z^i and where the path of integration is any contour from $z = -1$ to $z = 1$ that, except for its end points, lies above the real axis. (Compare with Exercise 6, Sec. 46.)
Suggestion: Use an antiderivative of the branch
$$z^i = \exp(i \log z) \qquad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$
of the same power function.

EXERCISES

1. Apply the Cauchy–Goursat theorem to show that

$$\int_C f(z)\,dz = 0$$

when the contour C is the unit circle $|z| = 1$, in either direction, and when

(a) $f(z) = \dfrac{z^2}{z+3}$; (b) $f(z) = z\,e^{-z}$; (c) $f(z) = \dfrac{1}{z^2+2z+2}$;

(d) $f(z) = \operatorname{sech} z$; (e) $f(z) = \tan z$; (f) $f(z) = \operatorname{Log}(z+2)$.

2. Let C_1 denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 1$, $y = \pm 1$ and let C_2 be the positively oriented circle $|z| = 4$ (Fig. 65). With the aid of the corollary in Sec. 53, point out why

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

when

(a) $f(z) = \dfrac{1}{3z^2+1}$; (b) $f(z) = \dfrac{z+2}{\sin(z/2)}$; (c) $f(z) = \dfrac{z}{1-e^z}$.

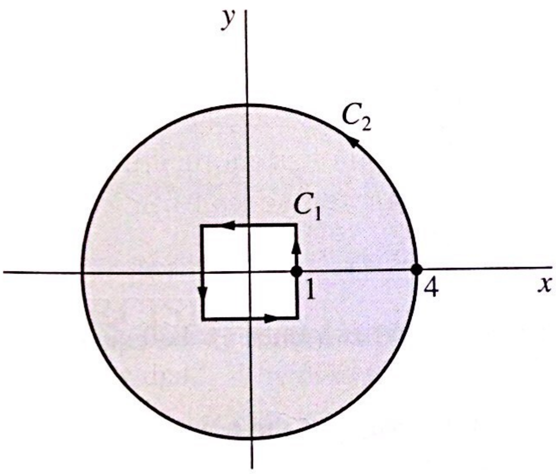


FIGURE 65

3. If C_0 denotes a positively oriented circle $|z - z_0| = R$, then

$$\int_{C_0} (z-z_0)^{n-1}\,dz = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2, \ldots, \\ 2\pi i & \text{when } n = 0, \end{cases}$$

according to Exercise 13, Sec. 46. Use that result and the corollary in Sec. 53 to show that if C is the boundary of the rectangle $0 \le x \le 3$, $0 \le y \le 2$, described in the positive sense, then

$$\int_C (z-2-i)^{n-1}\,dz = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2, \ldots, \\ 2\pi i & \text{when } n = 0. \end{cases}$$

4. Use the following method to derive the integration formula

$$\int_0^\infty e^{-x^2} \cos 2bx\,dx = \frac{\sqrt{\pi}}{2} e^{-b^2} \qquad (b > 0).$$

(a) Show that the sum of the integrals of e^{-z^2} along the lower and upper horizontal legs of the rectangular path in Fig. 66 can be written

$$2\int_0^a e^{-x^2}dx - 2e^{b^2}\int_0^a e^{-x^2}\cos 2bx\,dx$$

and that the sum of the integrals along the vertical legs on the right and left can be written

$$ie^{-a^2}\int_0^b e^{y^2}e^{-i2ay}dy - ie^{-a^2}\int_0^b e^{y^2}e^{i2ay}dy.$$

Thus, with the aid of the Cauchy–Goursat theorem, show that

$$\int_0^a e^{-x^2}\cos 2bx\,dx = e^{-b^2}\int_0^a e^{-x^2}dx + e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin 2ay\,dy.$$

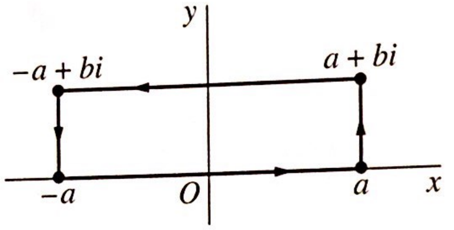


FIGURE 66

(b) By accepting the fact that*

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

and observing that

$$\left|\int_0^b e^{y^2}\sin 2ay\,dy\right| \le \int_0^b e^{y^2}dy,$$

obtain the desired integration formula by letting a tend to infinity in the equation at the end of part (a).

5. According to Exercise 6, Sec. 43, the path C_1 from the origin to the point $z = 1$ along the graph of the function defined by means of the equations

$$y(x) = \begin{cases} x^3\sin(\pi/x) & \text{when } 0 < x \le 1, \\ 0 & \text{when } x = 0 \end{cases}$$

is a smooth arc that intersects the real axis an infinite number of times. Let C_2 denote the line segment along the real axis from $z = 1$ back to the origin, and let C_3 denote any smooth arc from the origin to $z = 1$ that does not intersect itself and has only its end points in common with the arcs C_1 and C_2 (Fig. 67). Apply the Cauchy–Goursat

*The usual way to evaluate this integral is by writing its square as

$$\int_0^\infty e^{-x^2}dx\int_0^\infty e^{-y^2}dy = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dx\,dy$$

and then evaluating this iterated integral by changing to polar coordinates. Details are given in, for example, A. E. Taylor and W. R. Mann, "Advanced Calculus," 3d ed., pp. 680–681, 1983.

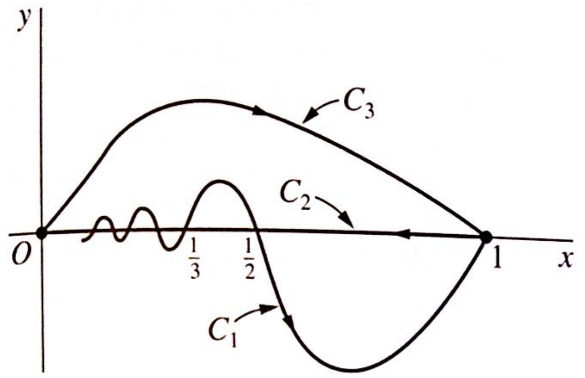


FIGURE 67

theorem to show that if a function f is entire, then

$$\int_{C_1} f(z)\,dz = \int_{C_3} f(z)\,dz \quad \text{and} \quad \int_{C_2} f(z)\,dz = -\int_{C_3} f(z)\,dz.$$

Conclude that even though the closed contour $C = C_1 + C_2$ intersects itself an infinite number of times,

$$\int_C f(z)\,dz = 0.$$

6. Let C denote the positively oriented boundary of the half disk $0 \le r \le 1, 0 \le \theta \le \pi$, and let $f(z)$ be a continuous function defined on that half disk by writing $f(0) = 0$ and using the branch

$$f(z) = \sqrt{r}e^{i\theta/2} \qquad \left(r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\right)$$

of the multiple-valued function $z^{1/2}$. Show that

$$\int_C f(z)\,dz = 0$$

by evaluating separately the integrals of $f(z)$ over the semicircle and the two radii which make up C. Why does the Cauchy–Goursat theorem not apply here?

7. Show that if C is a positively oriented simple closed contour, then the area of the region enclosed by C can be written

$$\frac{1}{2i}\int_C \bar{z}\,dz.$$

Suggestion: Note that expression (4), Sec. 50, can be used here even though the function $f(z) = \bar{z}$ is not analytic anywhere [see Example 2, Sec. 19].

8. *Nested Intervals.* An infinite sequence of closed intervals $a_n \le x \le b_n$ $(n = 0, 1, 2, \ldots)$ is formed in the following way. The interval $a_1 \le x \le b_1$ is either the left-hand or right-hand half of the first interval $a_0 \le x \le b_0$, and the interval $a_2 \le x \le b_2$ is then one of the two halves of $a_1 \le x \le b_1$, etc. Prove that there is a point x_0 which belongs to every one of the closed intervals $a_n \le x \le b_n$.

Suggestion: Note that the left-hand end points a_n represent a bounded nondecreasing sequence of numbers, since $a_0 \le a_n \le a_{n+1} < b_0$; hence they have a limit A as n tends to infinity. Show that the end points b_n also have a limit B. Then show that $A = B$, and write $x_0 = A = B$.

maximum value of $|f(z)|$ *on* C_R, *then*

$$|f^{(n)}(z_0)| \leq \frac{n!M_R}{R^n} \quad (n = 1, 2, \ldots). \tag{2}$$

Inequality (2) is called ***Cauchy's inequality*** and is an immediate consequence of the expression

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_R} \frac{f(z)\,dz}{(z - z_0)^{n+1}} \quad (n = 1, 2, \ldots),$$

in the theorem in Sec. 55 when n is a positive integer. We need only apply the theorem in Sec. 47, which gives upper bounds for the moduli of the values of contour integrals, to see that

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \cdot \frac{M_R}{R^{n+1}} 2\pi R \quad (n = 1, 2, \ldots),$$

where M_R is as in the statement of Theorem 3. This inequality is, of course, the same as inequality (2).

EXERCISES

1. Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of these integrals:

(a) $\displaystyle\int_C \frac{e^{-z}\,dz}{z - (\pi i/2)}$; $\quad (b)$ $\displaystyle\int_C \frac{\cos z}{z(z^2 + 8)}\,dz$; $\quad (c)$ $\displaystyle\int_C \frac{z\,dz}{2z + 1}$;

(d) $\displaystyle\int_C \frac{\cosh z}{z^4}\,dz$; $\quad (e)$ $\displaystyle\int_C \frac{\tan(z/2)}{(z - x_0)^2}\,dz \quad (-2 < x_0 < 2)$.

Ans. $(a)\ 2\pi$; $(b)\ \pi i/4$; $(c)\ -\pi i/2$; $(d)\ 0$; $(e)\ i\pi \sec^2(x_0/2)$.

2. Find the value of the integral of $g(z)$ around the circle $|z - i| = 2$ in the positive sense when

(a) $g(z) = \dfrac{1}{z^2 + 4}$; $\quad (b)$ $g(z) = \dfrac{1}{(z^2 + 4)^2}$.

Ans. $(a)\ \pi/2$; $(b)\ \pi/16$.

3. Let C be the circle $|z| = 3$, described in the positive sense. Show that if

$$g(z) = \int_C \frac{2s^2 - s - 2}{s - z}\,ds \qquad (|z| \neq 3),$$

then $g(2) = 8\pi i$. What is the value of $g(z)$ when $|z| > 3$?

4. Let C be any simple closed contour, described in the positive sense in the z plane, and write

$$g(z) = \int_C \frac{s^3 + 2s}{(s - z)^3}\,ds.$$

Show that $g(z) = 6\pi i z$ when z is inside C and that $g(z) = 0$ when z is outside.

5. Show that if f is analytic within and on a simple closed contour C and z_0 is not on C, then

$$\int_C \frac{f'(z)\,dz}{z-z_0} = \int_C \frac{f(z)\,dz}{(z-z_0)^2}.$$

6. Let f denote a function that is *continuous* on a simple closed contour C. Following the procedure used in Sec. 56, prove that the function

$$g(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{s-z}$$

is *analytic* at each point z interior to C and that

$$g'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^2}$$

at such a point.

7. Let C be the unit circle $z = e^{i\theta}$ $(-\pi \le \theta \le \pi)$. First show that for any real constant a,

$$\int_C \frac{e^{az}}{z}\,dz = 2\pi i.$$

Then write this integral in terms of θ to derive the integration formula

$$\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = \pi.$$

8. Show that $P_n(-1) = (-1)^n$ $(n = 0, 1, 2, \ldots)$, where $P_n(z)$ are the Legendre polynomials in Example 3, Sec. 55.

Suggestion: Note that

$$\frac{(s^2-1)^n}{(s+1)^{n+1}} = \frac{(s-1)^n}{s+1}.$$

9. Follow the steps below to verify the expression

$$f''(z) = \frac{1}{\pi i}\int_C \frac{f(s)\,ds}{(s-z)^3}$$

in Sec. 56.

(*a*) Use expression (2) in Sec. 56 for $f'(z)$ to show that

$$\frac{f'(z+\Delta z) - f'(z)}{\Delta z} - \frac{1}{\pi i}\int_C \frac{f(s)\,ds}{(s-z)^3} = \frac{1}{2\pi i}\int_C \frac{3(s-z)\Delta z - 2(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3} f(s)\,ds.$$

(*b*) Let D and d denote the largest and smallest distances, respectively, from z to points on C. Also, let M be the maximum value of $|f(s)|$ on C and L the length of C. With the aid of the triangle inequality and by referring to the derivation of expression (2) in Sec. 56 for $f'(z)$, show that when $0 < |\Delta z| < d$, the value of the integral on the right-hand side in part (*a*) is bounded from above by

$$\frac{(3D|\Delta z| + 2|\Delta z|^2)M}{(d-|\Delta z|)^2 d^3} L.$$

(*c*) Use the results in parts (*a*) and (*b*) to obtain the desired expression for $f''(z)$.

Homework 9: Part 2

1: Calculate the following integrals ($\{|z - z_0| = R\}$ denotes the circle with the anti-clockwise orientation)

(a)

$$\int_{|z|=\frac{1}{2}} \frac{z-3}{z^2-1} dz.$$

(b)

$$\int_{|z-1|=1} \frac{z-3}{z^2-1} dz.$$

(c)

$$\int_{|z|=2} \frac{(z-3)dz}{(z-1)^2}.$$

(d)

$$\int_{|z|=2} \frac{dz}{(z-1)^2(z-3)}.$$

2*: C is a simple closed curve and D is the interior region of C. f is a complex valued function defined on D. Assume that the real part and imaginary part of f have continuous partial derivatives. Show that the calculation of Section 50 (using Green's formula) is equivalent to the following Stokes' theorem,

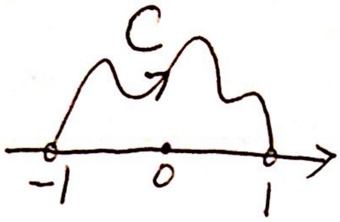
$$\int_C f(z)dz = \int_D \frac{\partial f}{\partial \bar{z}} d\bar{z} \wedge dz.$$

Note that, by definition,

$$d(f(z)dz) = df \wedge dz = \left(\frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z} \right) \wedge dz = \frac{\partial f}{\partial \bar{z}} d\bar{z} \wedge dz.$$

In particular, if furthermore f is analytic, then $\frac{\partial f}{\partial \bar{z}} = 0$ and we have the Cauchy-Goursat Theorem.

P147 5. $z^i = e^{i \cdot \text{Log}\, z}$



The branch z^i has an antiderivative:

$$\frac{1}{i+1} z^{(i+1)} = \frac{1}{i+1} e^{(i+1)\text{Log}\, z}$$

because: $$\frac{d}{dz}\left(\frac{1}{i+1} e^{(i+1)\text{Log}\, z}\right) = \frac{1}{i+1} e^{(i+1)\text{Log}\, z} \cdot (i+1) \frac{d}{dz}\text{Log}\, z = e^{i \cdot \text{Log}\, z} e^{\text{Log}\, z} \frac{1}{z}$$

$$= z^i.$$

So $$\int_C z^i \, dz = \int_C \left(\frac{1}{i+1} z^{(i+1)}\right)' dz = \frac{1}{i+1} z^{i+1} \Big]_{-1}^{1} = \frac{1}{i+1} e^{(i+1)\text{Log}\, z} \Big]_{-1}^{1}$$

$$= \frac{1}{i+1}\left(e^{(i+1)\text{Log}\, 1} - e^{(i+1)\text{Log}(-1)}\right) = \frac{1-i}{2}\left(e^{(i+1)\cdot 0} - e^{(i+1)\cdot(0+\pi i)}\right)$$

$$= \frac{1-i}{2}\left(1 - e^{-\pi} \cdot e^{\pi i}\right) = \frac{1-i}{2}\left(1 + e^{-\pi}\right).$$

P159. 1. $\int_{|z|=1} f(z) = 0$ if f is analytic inside $|z|=1$ (no singularity inside $|z|=1$)

2. $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$ if f is analytic between C_1 and C_2.

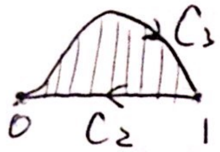
that is, there are no singularities between C_1 and C_2.

In other words, C_1 can be deformed to C_2 without touching singularities of f (deformation principal).

P160. 5.

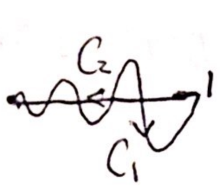
f entire $\Rightarrow \int_{C_1} f(z)dz = \int_{C_3} f(z)dz.$

$+$

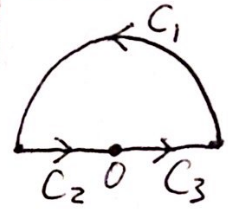


f entire $\Rightarrow \int_{C_2} f(z)dz = -\int_{C_3} f(z)dz.$

$\Downarrow$



$\int_{C_1} f(z)dz = -\int_{C_2} f(z)dz \iff \underline{\int_{C_1+C_2} f(z)dz = \int_C f(z)dz = 0}.$



6. Along C_1: $z = e^{i\theta}$, $0 \le \theta \le \pi$. $z'(\theta) = i\cdot e^{i\theta}$. $f(e^{i\theta}) = e^{\frac{i\theta}{2}}$

so $\int_{C_1} f(z)dz = \int_0^{\pi} e^{\frac{i\theta}{2}}\cdot i e^{i\theta} d\theta = i\int_0^{\pi} e^{\frac{3i\theta}{2}} d\theta = \frac{2}{3} e^{\frac{3i\theta}{2}}\Big]_0^{\pi}$

$= \frac{2}{3}\left(e^{\frac{3\pi i}{2}} - e^0\right) = \frac{2}{3}(-i-1).$

Along C_2: $z = x$, $-1 \le x \le 0 \Rightarrow \underset{\frac{dz}{dx}}{\overset{\shortparallel}{z'}} = 1$. $f(x) = f((-x)\cdot e^{i\pi}) = (-x)^{\frac{1}{2}} e^{\frac{i\pi}{2}}$

so $\int_{C_2} f(z)dz = \int_{-1}^{0} (-x)^{\frac{1}{2}}\cdot e^{\frac{i\pi}{2}}\cdot 1\, dx = i\cdot\left[-\frac{2}{3}(-x)^{\frac{3}{2}}\right]_{-1}^{0} = i\cdot\frac{2}{3}\cdot 1^{\frac{3}{2}} = \frac{2}{3}i$

Along C_3: $z = x$, $0 \le x \le 1 \Rightarrow z' = 1$. $f(x) = f(x\cdot e^{i0}) = x^{\frac{1}{2}}$

so $\int_{C_3} f(z)dz = \int_0^1 x^{\frac{1}{2}} dx = \frac{2}{3}x^{\frac{3}{2}}\Big]_0^1 = \frac{2}{3}.$

so $\int_C f(z)dz = \int_{C_1} + \int_{C_2} + \int_{C_3} = \frac{2}{3}(-i-1) + \frac{2}{3}i + \frac{2}{3} = 0.$

The Cauchy-Goursat theorem does not apply because $f(z) = \sqrt{r}\cdot e^{\frac{i\theta}{2}}$ is not analytic at the origin 0 which is on the boundary C.

P161. 7. By Stoke's formula:

$$\frac{1}{2i}\int_C \bar{z}\,dz = \frac{1}{2i}\iint_D d(\bar{z}\,dz) = \frac{1}{2i}\iint_D d\bar{z}\wedge dz$$

$$= \frac{1}{2i}\iint_D (dx-i\,dy)\wedge(dx+i\,dy) = \frac{1}{2i}\iint_D 2i\,dx\wedge dy$$

$$= \iint_D dx\wedge dy = \iint_D dA = \text{Area}(D).$$

Or write it out:

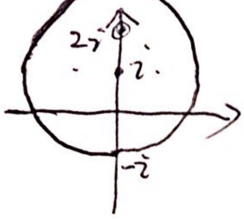
$$\frac{1}{2i}\int_C \bar{z}\,dz = \frac{1}{2i}\int_C (x-iy)(dx+i\,dy) = \frac{1}{2i}\int_C (x-iy)\,dx + i(x-iy)\,dy$$

$$\overset{\text{Green's formula}}{=\!=\!=} \frac{1}{2i}\iint_D\left(\frac{\partial}{\partial x}\big(i(x-iy)\big) - \frac{\partial}{\partial y}(x-iy)\right)dx\wedge dy$$

$$= \frac{1}{2i}\iint_D \big(i-(-i)\big)\,dx\wedge dy = \iint_D dx\wedge dy = \iint_D dA = \text{Area}(D).$$

P170 2. (a)

$$\int_{|z-i|=2}\frac{1}{z^2+4}\,dz = \int_{|z-i|=2}\frac{dz}{(z+2i)(z-2i)}$$



$$= 2\pi i\cdot\frac{1}{z+2i}\Big|_{z=2i} = 2\pi i\cdot\frac{1}{4i} = \frac{\pi}{2}$$

(b).

$$\int_{|z-i|=2}\frac{dz}{(z^2+4)^2} = \int_{|z-i|=2}\frac{dz}{(z+2i)^2(z-2i)^2} = \frac{2\pi i}{1!}\,\frac{d}{dz}(z+2i)^{-2}\Big|_{z=2i}$$

$$= 2\pi i\cdot(-2)\cdot(z+2i)^{-3}\Big|_{z=2i} = -\frac{4\pi i}{(4i)^3} = -\frac{\pi i}{-16i} = \frac{\pi}{16}.$$

P170. 3.

$$\int_{|z|=3} \frac{2s^2-s-2}{s-2}\,ds = 2\pi i\cdot(2s^2-s-2)\Big|_{s=2} = 2\pi i(8-2-2) = 8\pi i$$

$$\int_{|z|=3} \frac{2s^2-s-2}{s-z}\,ds = 0$$ when $|z|>3$ because $\frac{2s^2-s-2}{s-z}$ is analytic inside $|s|\le 3$.

4. when z is inside C, then

$$g(z) = \int_C \frac{s^3+2s}{(s-z)^3}\,ds = \frac{2\pi i}{2!}\cdot\frac{d^2}{ds^2}(s^3+2s)\Big|_{s=z} = \pi i\cdot 6s\Big|_{s=z} = 6\pi i z.$$

when z is outside C, then $\frac{s^3+2s}{(s-z)^3}$ is analytic on the region bounded by C.

so. $g(z)=0$ by Cauchy-Goursat theorem.

7.

$$\int_{|z|=1} \frac{e^{az}}{z}\,dz = 2\pi i\cdot e^{az}\Big|_{z=0} = 2\pi i\cdot 1 = 2\pi i.$$

$|z|=1 \Leftrightarrow z=e^{i\theta}, -\pi\le\theta\le\pi.$ $z'(\theta)=i\cdot e^{i\theta}$. so

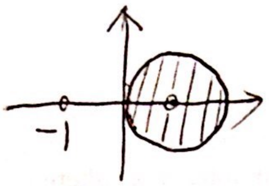
$$2\pi i = \int_{|z|=1}\frac{e^{az}}{z}\,dz = \int_{-\pi}^{\pi}\frac{e^{ae^{i\theta}}}{e^{i\theta}}\cdot ie^{i\theta}\,d\theta = i\cdot\int_{-\pi}^{\pi} \underbrace{e^{a(\cos\theta+i\sin\theta)}}_{e^{a\cos\theta}\cdot e^{i\cdot a\sin\theta}}\,d\theta$$

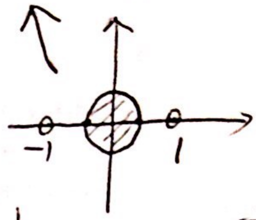
$$= i\int_{-\pi}^{\pi} e^{a\cos\theta}\cdot(\cos(a\sin\theta)+i\sin(a\sin\theta))\,d\theta$$

$$\Rightarrow 2\pi = \int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta \underset{\text{even function}}{=} 2\cdot\int_0^{\pi} e^{a\cos\theta}\cdot\cos(a\sin\theta)\,d\theta$$

$$\Rightarrow \int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = \pi.$$ $\left(\int_{-\pi}^{\pi} e^{a\cos\theta}\cdot\sin(a\sin\theta)\,d\theta = 0\right.$ because of odd fct.$\left.\right)$.

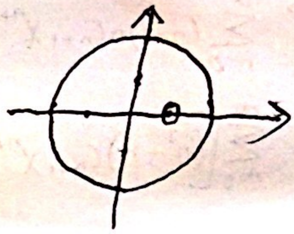
Part 2 (a) $\displaystyle\int_{|z|=\frac{1}{2}} \frac{z-3}{z^2-1}\,dz = 0$ because $\dfrac{z-3}{z^2-1}$ is analytic for $|z|\le\frac{1}{2}$





(b) $\displaystyle\int_{|z-1|=1} \frac{z-3}{z^2-1}\,dz = \int_{|z-1|=1} \frac{z-3}{(z+1)(z-1)}\,dz = 2\pi i\cdot \frac{z-3}{z+1}\Big|_{z=1} = 2\pi i\cdot\frac{-2}{2} = -2\pi i$

(c)



$$\int_{|z|=2} \frac{(z-3)}{(z-1)^2}\,dz = \frac{2\pi i}{1!}\,\frac{d}{dz}(z-3)\Big|_{z=1} = 2\pi i\cdot 1 = 2\pi i$$

(d)

$$\int_{|z|=2} \frac{1}{(z-1)^2(z-3)}\,dz = \frac{2\pi i}{1!}\,\frac{d}{dz}\left(\frac{1}{z-3}\right)\Big|_{z=1} = 2\pi i\cdot \frac{-1}{(z-3)^2}\Big|_{z=1}$$

$$= 2\pi i\cdot\frac{1}{-4} = -\frac{\pi i}{2}.$$

2^*.

$$\int_C f(z)\,dz = \int_C (u+iv)\cdot(dx+i\,dy) = \int_C (u+iv)\,dx + i(u+iv)\,dy$$

$$\overset{\text{Green's formula}}{=} \iint_D \left(\frac{\partial}{\partial x}\big(i(u+iv)\big) - \frac{\partial}{\partial y}(u+iv)\right) dx\wedge dy$$

$$= \iint_D \left[\frac{\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)}{2}(u+iv)\right]\cdot(2i\,dx\wedge dy)$$

$$= \iint_D \left(\frac{\partial}{\partial \bar z} f(z)\right) d\bar z\wedge dz$$

$d\bar z\wedge dz = (dx-i\,dy)\wedge(dx+i\,dy) = 2i\,dx\wedge dy$

EXAMPLE 1. Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant (Fig. 47). Inequality (6) can be used to show that

$$\left| \int_C \frac{z-2}{z^4+1}\, dz \right| \le \frac{4\pi}{15}. \tag{7}$$

This is done by noting first that if z is a point on C, then

$$|z-2| = |z+(-2)| \le |z| + |-2| = 2+2 = 4$$

and

$$|z^4+1| \ge ||z|^4 - 1| = 15.$$

Thus, when z lies on C,

$$\left| \frac{z-2}{z^4+1} \right| = \frac{|z-2|}{|z^4+1|} \le \frac{4}{15}.$$

By writing $M = 4/15$ and observing that $L = \pi$ is the length of C, we may now use inequality (6) to obtain inequality (7).

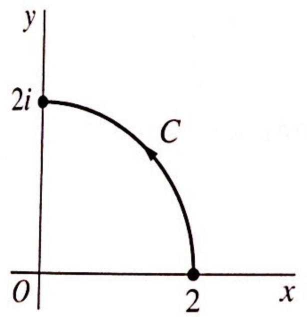


FIGURE 47

EXAMPLE 2. Let C_R denote the semicircle

$$z = Re^{i\theta} \qquad (0 \le \theta \le \pi)$$

from $z = R$ to $z = -R$, where $R > 3$ (Fig. 48). It is easy to show that

$$\lim_{R\to\infty} \int_{C_R} \frac{(z+1)\, dz}{(z^2+4)(z^2+9)} = 0 \tag{8}$$

without actually evaluating the integral. To do this, we observe that if z is a point on C_R,

$$|z+1| \le |z| + 1 = R+1,$$
$$|z^2+4| \ge ||z|^2 - 4| = R^2 - 4,$$

and

$$|z^2+9| \ge ||z|^2 - 9| = R^2 - 9.$$

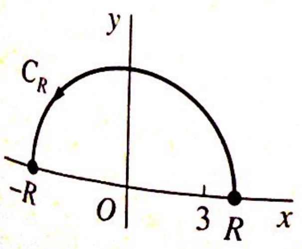


FIGURE 48

This means that if z is on C_R and $f(z)$ is the integrand in integral (8), then

$$|f(z)| = \left|\frac{z+1}{(z^2+4)(z^2+9)}\right| = \frac{|z+1|}{|z^2+4|\,|z^2+9|} \le \frac{R+1}{(R^2-4)(R^2-9)} = M_R,$$

where M_R serves as an upper bound for $|f(z)|$ on C_R. Since the length of the semicircle is πR, we may refer to the theorem in this section, using

$$M_R = \frac{R+1}{(R^2-4)(R^2-9)} \quad \text{and} \quad L = \pi R,$$

to write

$$\left|\int_{C_R} \frac{(z+1)\,dz}{(z^2+4)(z^2+9)}\right| \le M_R L \tag{9}$$

where

$$M_R L = \frac{\pi(R^2+R)}{(R^2-4)(R^2-9)} \cdot \frac{\frac{1}{R^4}}{\frac{1}{R^4}} = \frac{\pi\left(\frac{1}{R^2}+\frac{1}{R^3}\right)}{\left(1-\frac{4}{R^2}\right)\left(1-\frac{9}{R^2}\right)}.$$

This shows that $M_R L \to 0$ as $R \to \infty$, and limit (8) follows from inequality (9).

EXERCISES

1. Without evaluating the integral, show that

$$(a)\ \left|\int_C \frac{z+4}{z^3-1}\,dz\right| \le \frac{6\pi}{7}; \qquad (b)\ \left|\int_C \frac{dz}{z^2-1}\right| \le \frac{\pi}{3}$$

when C is the arc that was used in Example 1, Sec. 47.

2. Let C denote the line segment from $z = i$ to $z = 1$ (Fig. 49), and show that

$$\left|\int_C \frac{dz}{z^4}\right| \le 4\sqrt{2}$$

without evaluating the integral.

Suggestion: Observe that of all the points on the line segment, the midpoint is closest to the origin, that distance being $d = \sqrt{2}/2$.

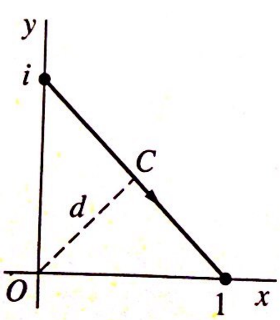


FIGURE 49

3. Show that if C is the boundary of the triangle with vertices at the points 0, $3i$, and -4, oriented in the counterclockwise direction (see Fig. 50), then

$$\left| \int_C (e^z - \bar{z})\, dz \right| \leq 60.$$

Suggestion: Note that $|e^z - \bar{z}| \leq e^x + \sqrt{x^2 + y^2}$ when $z = x + iy$.

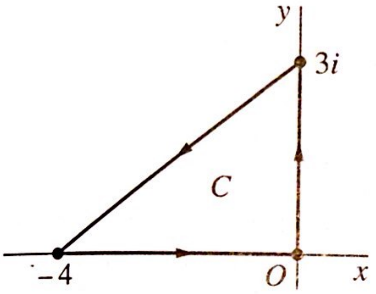


FIGURE 50

4. Let C_R denote the upper half of the circle $|z| = R$ $(R > 2)$, taken in the counterclockwise direction. Show that

$$\left| \int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4}\, dz \right| \leq \frac{\pi R(2R^2 + 1)}{(R^2 - 1)(R^2 - 4)}.$$

Then, by dividing the numerator and denominator on the right here by R^4, show that the value of the integral tends to zero as R tends to infinity. (Compare with Example 2 in Sec. 47.)

5. Let C_R be the circle $|z| = R$ $(R > 1)$, described in the counterclockwise direction. Show that

$$\left| \int_{C_R} \frac{\operatorname{Log} z}{z^2}\, dz \right| < 2\pi \left(\frac{\pi + \ln R}{R} \right),$$

and then use l'Hospital's rule to show that the value of this integral tends to zero as R tends to infinity.

6. Let C_ρ denote a circle $|z| = \rho$ $(0 < \rho < 1)$, oriented in the counterclockwise direction, and suppose that $f(z)$ is analytic in the disk $|z| \leq 1$. Show that if $z^{-1/2}$ represents any particular branch of that power of z, then there is a nonnegative constant M, *independent of* ρ, such that

$$\left| \int_{C_\rho} z^{-1/2} f(z)\, dz \right| \leq 2\pi M \sqrt{\rho}.$$

Thus show that the value of the integral here approaches 0 as ρ tends to 0.

Suggestion: Note that since $f(z)$ is analytic, and therefore continuous, throughout the disk $|z| \leq 1$, it is bounded there (Sec. 18).

maximum value of $|f(z)|$ *on* C_R, *then*

$$|f^{(n)}(z_0)| \leq \frac{n!M_R}{R^n} \quad (n = 1, 2, \ldots). \tag{2}$$

Inequality (2) is called ***Cauchy's inequality*** and is an immediate consequence of the expression

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{C_R} \frac{f(z)\,dz}{(z - z_0)^{n+1}} \quad (n = 1, 2, \ldots),$$

in the theorem in Sec. 55 when n is a positive integer. We need only apply the theorem in Sec. 47, which gives upper bounds for the moduli of the values of contour integrals, to see that

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi}\cdot\frac{M_R}{R^{n+1}}2\pi R \quad (n = 1, 2, \ldots).$$

where M_R is as in the statement of Theorem 3. This inequality is, of course, the same as inequality (2).

EXERCISES

1. Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of these integrals:

(*a*) $\displaystyle\int_C \frac{e^{-z}\,dz}{z - (\pi i/2)}$; (*b*) $\displaystyle\int_C \frac{\cos z}{z(z^2 + 8)}\,dz$; (*c*) $\displaystyle\int_C \frac{z\,dz}{2z + 1}$;

(*d*) $\displaystyle\int_C \frac{\cosh z}{z^4}\,dz$; (*e*) $\displaystyle\int_C \frac{\tan(z/2)}{(z - x_0)^2}\,dz \quad (-2 < x_0 < 2)$.

Ans. (*a*) 2π; (*b*) $\pi i/4$; (*c*) $-\pi i/2$; (*d*) 0; (*e*) $i\pi \sec^2(x_0/2)$.

2. Find the value of the integral of $g(z)$ around the circle $|z - i| = 2$ in the positive sense when

(*a*) $g(z) = \dfrac{1}{z^2 + 4}$; (*b*) $g(z) = \dfrac{1}{(z^2 + 4)^2}$.

Ans. (*a*) $\pi/2$; (*b*) $\pi/16$.

3. Let C be the circle $|z| = 3$, described in the positive sense. Show that if

$$g(z) = \int_C \frac{2s^2 - s - 2}{s - z}\,ds \qquad (|z| \neq 3),$$

then $g(2) = 8\pi i$. What is the value of $g(z)$ when $|z| > 3$?

4. Let C be any simple closed contour, described in the positive sense in the z plane, and write

$$g(z) = \int_C \frac{s^3 + 2s}{(s - z)^3}\,ds.$$

Show that $g(z) = 6\pi i z$ when z is inside C and that $g(z) = 0$ when z is outside.

5. Show that if f is analytic within and on a simple closed contour C and z_0 is not on C, then

$$\int_C \frac{f'(z)\,dz}{z - z_0} = \int_C \frac{f(z)\,dz}{(z - z_0)^2}.$$

6. Let f denote a function that is *continuous* on a simple closed contour C. Following the procedure used in Sec. 56, prove that the function

$$g(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{s - z}$$

is *analytic* at each point z interior to C and that

$$g'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s - z)^2}$$

at such a point.

7. Let C be the unit circle $z = e^{i\theta}$ $(-\pi \le \theta \le \pi)$. First show that for any real constant a,

$$\int_C \frac{e^{az}}{z}\,dz = 2\pi i.$$

Then write this integral in terms of θ to derive the integration formula

$$\int_0^\pi e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = \pi.$$

8. Show that $P_n(-1) = (-1)^n$ $(n = 0, 1, 2, \ldots)$, where $P_n(z)$ are the Legendre polynomials in Example 3, Sec. 55.

Suggestion: Note that

$$\frac{(s^2 - 1)^n}{(s + 1)^{n+1}} = \frac{(s - 1)^n}{s + 1}.$$

9. Follow the steps below to verify the expression

$$f''(z) = \frac{1}{\pi i}\int_C \frac{f(s)\,ds}{(s - z)^3}$$

in Sec. 56.

(*a*) Use expression (2) in Sec. 56 for $f'(z)$ to show that

$$\frac{f'(z + \Delta z) - f'(z)}{\Delta z} - \frac{1}{\pi i}\int_C \frac{f(s)\,ds}{(s - z)^3} = \frac{1}{2\pi i}\int_C \frac{3(s - z)\Delta z - 2(\Delta z)^2}{(s - z - \Delta z)^2(s - z)^3} f(s)\,ds.$$

(*b*) Let D and d denote the largest and smallest distances, respectively, from z to points on C. Also, let M be the maximum value of $|f(s)|$ on C and L the length of C. With the aid of the triangle inequality and by referring to the derivation of expression (2) in Sec. 56 for $f'(z)$, show that when $0 < |\Delta z| < d$, the value of the integral on the right-hand side in part *(a)* is bounded from above by

$$\frac{(3D|\Delta z| + 2|\Delta z|^2)M}{(d - |\Delta z|)^2 d^3} L.$$

(*c*) Use the results in parts (*a*) and (*b*) to obtain the desired expression for $f''(z)$.

10. Let f be an entire function such that $|f(z)| \le A|z|$ for all z, where A is a fixed positive number. Show that $f(z) = a_1 z$, where a_1 is a complex constant.

Suggestion: Use Cauchy's inequality (Sec. 57) to show that the second derivative $f''(z)$ is zero everywhere in the plane. Note that the constant M_R in Cauchy's inequality is less than or equal to $A(|z_0| + R)$.

58. LIOUVILLE'S THEOREM AND THE FUNDAMENTAL THEOREM OF ALGEBRA

Cauchy's inequality in Theorem 3 of Sec. 57 can be used to show that no entire function except a constant is bounded in the complex plane. Our first theorem here, which is known as ***Liouville's theorem***, states this result in a slightly different way.

Theorem 1. *If a function f is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.*

To start the proof, we assume that f is as stated and note that since f is entire, Theorem 3 in Sec. 57 can be applied with any choice of z_0 and R. In particular, Cauchy's inequality (2) in that theorem tells us that when $n = 1$,

$$|f'(z_0)| \le \frac{M_R}{R}. \tag{1}$$

Moreover, the boundedness condition on f tells us that a nonnegative constant M exists such that $|f(z)| \le M$ for all z; and, because the constant M_R in inequality (1) is always less than or equal to M, it follows that

$$|f'(z_0)| \le \frac{M}{R}, \tag{2}$$

where R can be arbitrarily large. Now the number M in inequality (2) is independent of the value of R that is taken. Hence that inequality holds for arbitrarily large values of R only if $f'(z_0) = 0$. Since the choice of z_0 was arbitrary, this means that $f'(z) = 0$ everywhere in the complex plane. Consequently, f is a constant function, according to the theorem in Sec. 25.

The following theorem is called the ***fundamental theorem of algebra*** and follows readily from Liouville's theorem.

Theorem 2. *Any polynomial*

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n \ne 0)$$

of degree n ($n \ge 1$) has at least one zero. That is, there exists at least one point z_0 such that $P(z_0) = 0$.

The proof here is by contradiction. Suppose that $P(z)$ is *not* zero for any value of z. Then the quotient $1/P(z)$ is clearly entire. It is also bounded in the complex plane.

EXAMPLE. Consider the function $f(z) = (z+1)^2$ defined on the closed triangular region R with vertices at the points

$$z = 0, \quad z = 2, \quad \text{and} \quad z = i.$$

A simple geometric argument can be used to locate points in R at which the modulus $|f(z)|$ has its maximum and minimum values. The argument is based on the interpretation of $|f(z)|$ as the square of the distance d between -1 and any point z in R:

$$d^2 = |f(z)| = |z - (-1)|^2.$$

As one can see in Fig. 74, the maximum and minimum values of d, and therefore $|f(z)|$, occur at boundary points, namely $z = 2$ and $z = 0$, respectively.

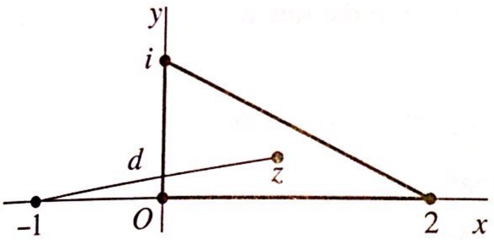


FIGURE 74

EXERCISES

1. Suppose that $f(z)$ is entire and that the harmonic function $u(x, y) = \text{Re}[f(z)]$ has an upper bound u_0; that is, $u(x, y) \le u_0$ for all points (x, y) in the xy plane. Show that $u(x, y)$ must be constant throughout the plane.
   Suggestion: Apply Liouville's theorem (Sec. 58) to the function $g(z) = \exp[f(z)]$.

2. Let a function f be continuous on a closed bounded region R, and let it be analytic and not constant throughout the interior of R. Assuming that $f(z) \ne 0$ anywhere in R, prove that $|f(z)|$ has a *minimum value* m in R which occurs on the boundary of R and never in the interior. Do this by applying the corresponding result for maximum values (Sec. 59) to the function $g(z) = 1/f(z)$.

3. Use the function $f(z) = z$ to show that in Exercise 2 the condition $f(z) \ne 0$ anywhere in R is necessary in order to obtain the result of that exercise. That is, show that $|f(z)|$ *can* reach its minimum value at an interior point when the minimum value is zero.

4. Let R region $0 \le x \le \pi, 0 \le y \le 1$ (Fig. 75). Show that the modulus of the entire function $f(z) = \sin z$ has a maximum value in R at the boundary point $z = (\pi/2) + i$.
   Suggestion: Write $|f(z)|^2 = \sin^2 x + \sinh^2 y$ (see Sec. 37) and locate points in R at which $\sin^2 x$ and $\sinh^2 y$ are the largest.

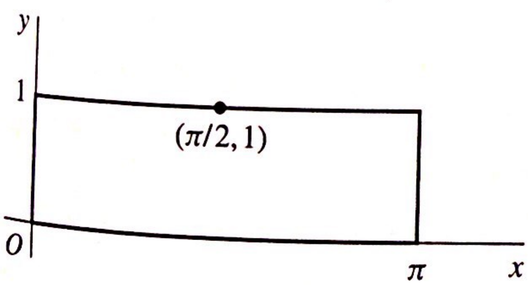


FIGURE 75

5. Let $f(z) = u(x, y)+iv(x, y)$ be a function that is continuous on a closed bounded region R and analytic and not constant throughout the interior of R. Prove that the component function $u(x, y)$ has a minimum value in R which occurs on the boundary of R and never in the interior. (See Exercise 2.)

6. Let f be the function $f(z) = e^z$ and R the rectangular region $0 \le x \le 1, 0 \le y \le \pi$. Illustrate results in Sec. 59 and Exercise 5 by finding points in R where the component function $u(x, y) = \text{Re}[f(z)]$ reaches its maximum and minimum values.

Ans. $z = 1, z = 1 + \pi i$.

7. Let the function $f(z) = u(x, y) + iv(x, y)$ be continuous on a closed bounded region R, and suppose that it is analytic and not constant in the interior of R. Show that the component function $v(x, y)$ has maximum and minimum values in R which are reached on the boundary of R and never in the interior, where it is harmonic.

Suggestion: Apply results in Sec. 59 and Exercise 5 to the function $g(z) = -if(z)$.

8. Let z_0 be a zero of the polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n \neq 0)$$

of degree n ($n \ge 1$). Show in the following way that

$$P(z) = (z - z_0)Q(z)$$

where $Q(z)$ is a polynomial of degree $n - 1$.

(*a*) Verify that

$$z^k - z_0^k = (z - z_0)(z^{k-1} + z^{k-2}z_0 + \cdots + z\,z_0^{k-2} + z_0^{k-1}) \qquad (k = 2, 3, \ldots).$$

(*b*) Use the factorization in part (*a*) to show that

$$P(z) - P(z_0) = (z - z_0)Q(z)$$

where $Q(z)$ is a polynomial of degree $n - 1$, and deduce the desired result from this.

then,

$$\rho_N(z) = S(z) - S_N(z) = \frac{z^N}{1-z} \qquad (z \neq 1).$$

Thus

$$|\rho_N(z)| = \frac{|z|^N}{|1-z|},$$

and it is clear from this that the remainders $\rho_N(z)$ tend to zero when $|z| < 1$ but not when $|z| \geq 1$. Summation formula (10) is, therefore, established.

EXERCISES

1. Use definition (1), Sec. 60, of limits of sequences to show that

$$\lim_{n\to\infty}\left(\frac{1}{n^2} + i\right) = i.$$

2. Let Θ_n $(n = 1, 2, \ldots)$ denote the principal arguments of the numbers

$$z_n = 1 + i\frac{(-1)^n}{n^2} \qquad (n = 1, 2, \ldots),$$

and point out why

$$\lim_{n\to\infty} \Theta_n = 0.$$

(Compare with Example 2, Sec. 60.)

3. Use the inequality (see Sec. 5) $||z_n| - |z|| \leq |z_n - z|$ to show that

$$\text{if} \quad \lim_{n\to\infty} z_n = z, \quad \text{then} \quad \lim_{n\to\infty} |z_n| = |z|.$$

4. Write $z = re^{i\theta}$, where $0 < r < 1$, in the summation formula (10), Sec. 61. Then, with the aid of the theorem in Sec. 61, show that

$$\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{r\cos\theta - r^2}{1 - 2r\cos\theta + r^2} \quad \text{and} \quad \sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1 - 2r\cos\theta + r^2}$$

when $0 < r < 1$. (Note that these formulas are also valid when $r = 0$.)

5. Show that a limit of a convergent sequence of complex numbers is unique by appealing to the corresponding result for a sequence of real numbers.

6. Show that

$$\text{if} \quad \sum_{n=1}^{\infty} z_n = S, \quad \text{then} \quad \sum_{n=1}^{\infty} \overline{z_n} = \overline{S}.$$

7. Let c denote any complex number and show that

$$\text{if} \quad \sum_{n=1}^{\infty} z_n = S, \quad \text{then} \quad \sum_{n=1}^{\infty} c z_n = cS.$$

Homework 10: Part 2

1. Let $w(t)$ be a complex valued function of a real variable t. Prove the following inequality using the definition of integrals via Riemann sums:

$$\left|\int_a^b w(t)dt\right| \leq \int_a^b |w(t)|dt.$$

   Note that a different proof using a rotation trick was given in Section 47 of the textbook.

2. Let $f(z)$ be a complex valued function of a complex variable z. Prove the following inequality by either using the definition of contour integrals via Riemann sum or use a parametrization to reduce to the above case:

$$\left|\int_C f(z)dz\right| \leq \int_C |f(z)||dz|.$$

   Note that the right hand side is calculated by using a parametrization as:

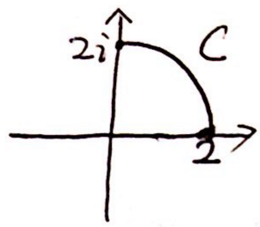
$$\int_C |f(z)||dz| = \int_a^b |f(z(t))||z'(t)|dt$$

   with

$$|dz| = |z'(t)|dt = \sqrt{x'(t)^2 + y'(t)^2}\ dt = ds$$

   being the arc-length differential element.

P138. 1. (a).



$$\left|\int_C \frac{z+4}{z^3-1}dz\right| \leq \int_C \left|\frac{z+4}{z^3-1}\right| |dz|.$$

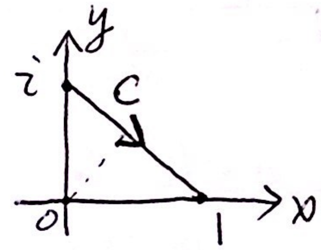
$$\left(\text{along } C: \quad \left|\frac{z+4}{z^3-1}\right| \leq \frac{|z|+4}{|z|^3-1} = \frac{2+4}{2^3-1} = \frac{6}{7}\right).$$

$$\leq \int_C \frac{6}{7}|dz| = \frac{6}{7}\cdot\frac{2\pi\cdot 2}{4} = \frac{6}{7}\pi.$$

(b). along C, $\left|\frac{1}{z^2-1}\right| \leq \frac{1}{|z|^2-1} = \frac{1}{2^2-1} = \frac{1}{3}$

$$\Rightarrow \left|\int_C \frac{dz}{z^2-1}\right| \leq \int_C \left|\frac{1}{z^2-1}\right||dz| \leq \frac{1}{3}\cdot\int_C |dz| = \frac{1}{3}\pi.$$

2. Along C:

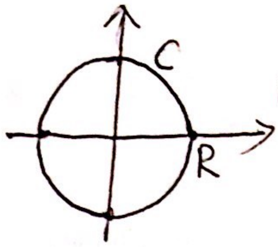


$$\left|\frac{1}{z^4}\right| = \frac{1}{|z|^4} \leq \frac{1}{(dist(0,C))^4} = \frac{1}{(\frac{1}{\sqrt{2}})^4} = (\sqrt{2})^4 = 4.$$

length of $C = \sqrt{2}$

$$\Rightarrow \left|\int_C \frac{dz}{z^4}\right| \leq \int_C \left|\frac{1}{z^4}\right||dz| \leq 4\cdot\sqrt{2}.$$

P139. 5. Along C.



$R>1$.

$$\left|\frac{\mathrm{Log}\, z}{z^2}\right| = \frac{|\mathrm{Log}\, z|}{|z|^2} = \frac{|\ln R + i\,\mathrm{Arg}(z)|}{R^2}$$

$$\leq \frac{\ln R + |\mathrm{Arg}(z)|}{R^2} \leq \frac{\ln R + \pi}{R^2}$$

$$< \frac{\ln R + \pi}{R^2} \quad \text{计 } z \neq -1.$$

$$\Rightarrow \left|\int_{C_R} \frac{\mathrm{Log}\, z}{z^2}\, dz\right| \leq \int_{C_R} \left|\frac{\mathrm{Log}\, z}{z^2}\right| |dz| < \frac{\ln R + \pi}{R^2} \cdot 2\pi R = 2\pi\left(\frac{\pi + \ln R}{R}\right)$$

P171.5 $\displaystyle\int_C \frac{f'(z)}{z-z_0}\, dz = f'(z) \cdot 2\pi i = \int_C \frac{f(z)}{(z-z_0)^2}\, dz.$

10. Cauchy's formula for 2nd order derivative: $\displaystyle f''(z) = \frac{2!}{2\pi i} \int_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^3}\, dz$

$$\Rightarrow |f''(z_0)| \leq \left|\frac{2}{2\pi i}\right| \cdot \int_{|z-z_0|=R} \left|\frac{f(z)}{(z-z_0)^3}\right| |dz| \leq \frac{1}{\pi} \int_{|z-z_0|=R} \frac{A|z|}{R^3}\, |dz|.$$

$$\leq \frac{A}{\pi R^3} \int_{|z-z_0|} (|z-z_0| + |z_0|) \cdot |dz| = \frac{A}{\pi R^3} (R + |z_0|) \cdot 2\pi R.$$

$$= \frac{2A}{R^2} (R + |z_0|) \xrightarrow{R \to +\infty} 0 \quad \text{for any } z_0 \in \mathbb{C}.$$

$$\Rightarrow f''(z) \equiv 0 \Rightarrow f'(z) = a_1 \text{ constant.} \Rightarrow f(z) = a_1 z + a_0.$$

Now $\underset{a_0}{\underset{\parallel}{|f(0)|}} \leq A \cdot 0 = 0 \Rightarrow a_0 = 0$ so $f(z) \doteq a_1 z$.

P177. 1. $u = Re(f(z)) \le u_0$

$$\Rightarrow |e^f| = |e^{u+iv}| = e^u \le e^{u_0}$$

Because e^f is entire and bounded, by Liouville theorem

$$e^f = \text{constant} \Rightarrow f = \text{constant} \Rightarrow u = \text{constant.}$$
$$= C_1 \qquad\qquad \overset{\shortparallel}{\log C_1}$$

2. $f(z) \ne 0$ in R $\Rightarrow$ $\frac{1}{f(z)}$ is analytic in R.
and continuous in R

$\xrightarrow[f \text{ not constant}]{\text{maximum principle}}$ $\left|\frac{1}{f(z)}\right|$ has a maximum value M in R which occurs on the boundary of R and never in the interior.

$\Rightarrow$ $|f(z)|$ has a minimum value m in R which occurs on the boundary of R and never in the interior.

5. $f = u + iv$ continuous, analytic, not constant

$\Rightarrow$ $e^{-f} = e^{-(u+iv)} = e^{-u} \cdot e^{-iv}$ is continuous, analytic, not constant

$\xrightarrow{\text{maximum principle}}$ $|e^{-f}| = e^{-u}$ has a maximum in R which occurs on the boundary of R and never in the interior

$\Rightarrow$ u has a minimum in R which occurs on the boundary of R and never in the interior.

6. $f(z) = e^z = e^{x+iy} = e^x(\cos y + i\sin y)$

$u(x,y) = e^x \cos y$. $\begin{cases} u_x = e^x\cos y = 0 \\ u_y = -e^x \sin y = 0 \end{cases} \Rightarrow \begin{cases} \cos y = 0 \\ \sin y = 0 \end{cases}$ No solution

$\Rightarrow$ no critical points in the interior $\Rightarrow$ max. and min occur on the boundary.

P185. 2. $z_n = 1 + i\dfrac{(-1)^n}{n!}$ $(n=1,2,\cdots)$.

$$\lim_{n\to\infty} \operatorname{Arg}(z_n) = \operatorname{Arg}(1) = 0.$$

Part 2: 1. $$\left|\int_a^b w(t)\,dt\right| = \left|\lim_{N\to\infty}\sum_{i=1}^{N} w(t_i)\cdot\Delta t_i\right| = \lim_{N\to\infty}\left|\sum_{i=1}^{N} w(t_i)\Delta t_i\right|$$

$$\le \lim_{N\to\infty}\sum_{i=1}^{N}|w(t_i)|\Delta t_i = \int_a^b |w(t)|\,dt.$$

2. $$\left|\int_C f(z)\,dz\right| = \left|\lim_{N\to\infty}\sum_{i=1}^{N} f(z_i)\,\Delta z_i\right| = \lim_{N\to\infty}\left|\sum_{i=1}^{N} f(z_i)\,\Delta z_i\right|$$

$$\le \lim_{N\to\infty}\sum_{i=1}^{N}|f(z_i)|\cdot|\Delta z_i| = \int_C |f(z)|\cdot|dz|$$

or $$\left|\int_C f(z)\,dz\right| = \left|\int_C f(z(t))\,z'(t)\,dt\right| \le \int_C |f(z(t))|\,|z'(t)|\,dt$$

$$= \int_C |f(z)|\,|dz|.$$

7

2. Obtain the Taylor series

$$e^z = e\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \qquad (|z-1| < \infty)$$

for the function $f(z) = e^z$ by

(a) using $f^{(n)}(1)$ $(n = 0, 1, 2, \ldots)$; (b) writing $e^z = e^{z-1}e$.

3. Find the Maclaurin series expansion of the function

$$f(z) = \frac{z}{z^4+4} = \frac{z}{4} \cdot \frac{1}{1+(z^4/4)}.$$

Ans. $f(z) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+2}} z^{4n+1} \quad (|z| < \sqrt{2}).$

4. With the aid of the identity (see Sec. 37)

$$\cos z = -\sin\left(z - \frac{\pi}{2}\right),$$

expand $\cos z$ into a Taylor series about the point $z_0 = \pi/2$.

5. Use the identity $\sinh(z + \pi i) = -\sinh z$, verified in Exercise 7(a), Sec. 39, and the fact that $\sinh z$ is periodic with period $2\pi i$ to find the Taylor series for $\sinh z$ about the point $z_0 = \pi i$.

Ans. $-\displaystyle\sum_{n=0}^{\infty} \frac{(z-\pi i)^{2n+1}}{(2n+1)!} \quad (|z-\pi i| < \infty).$

6. What is the largest circle within which the Maclaurin series for the function $\tanh z$ converges to $\tanh z$? Write the first two nonzero terms of that series.

7. Show that if $f(z) = \sin z$, then

$$f^{(2n)}(0) = 0 \quad \text{and} \quad f^{(2n+1)}(0) = (-1)^n \qquad (n = 0, 1, 2, \ldots).$$

Thus give an alternative derivation of the Maclaurin series (3) for $\sin z$ in Sec. 64.

8. Rederive the Maclaurin series (4) in Sec. 64 for the function $f(z) = \cos z$ by

(a) using the definition

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

in Sec. 37 and appealing to the Maclaurin series (2) for e^z in Sec. 64;

(b) showing that

$$f^{(2n)}(0) = (-1)^n \quad \text{and} \quad f^{(2n+1)}(0) = 0 \qquad (n = 0, 1, 2, \ldots).$$

9. Use representation (3), Sec. 64, for $\sin z$ to write the Maclaurin series for the function

$$f(z) = \sin(z^2),$$

and point out how it follows that

$$f^{(4n)}(0) = 0 \quad \text{and} \quad f^{(2n+1)}(0) = 0 \qquad (n = 0, 1, 2, \ldots).$$

10. Derive the expansions

(a) $$\frac{\sinh z}{z^2} = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+3)!} \qquad (0 < |z| < \infty);$$

(b) $$\frac{\sin(z^2)}{z^4} = \frac{1}{z^2} - \frac{z^2}{3!} + \frac{z^6}{5!} - \frac{z^{10}}{7!} + \cdots \qquad (0 < |z| < \infty).$$

11. Show that when $0 < |z| < 4$,

$$\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}.$$

66. LAURENT SERIES

We turn now to a statement of ***Laurent's theorem***, which enables us to expand a function $f(z)$ into a series involving positive and negative powers of $(z - z_0)$ when the function fails to be analytic at z_0.

Theorem. *Suppose that a function f is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$, centered at z_0, and let C denote any positively oriented simple closed contour around z_0 and lying in that domain* (Fig. 80). *Then, at each point in the domain, $f(z)$ has the series representation*

(1) $$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \qquad (R_1 < |z - z_0| < R_2),$$

where

(2) $$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{n+1}} \qquad (n = 0, 1, 2, \ldots)$$

and

(3) $$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{-n+1}} \qquad (n = 1, 2, \ldots).$$

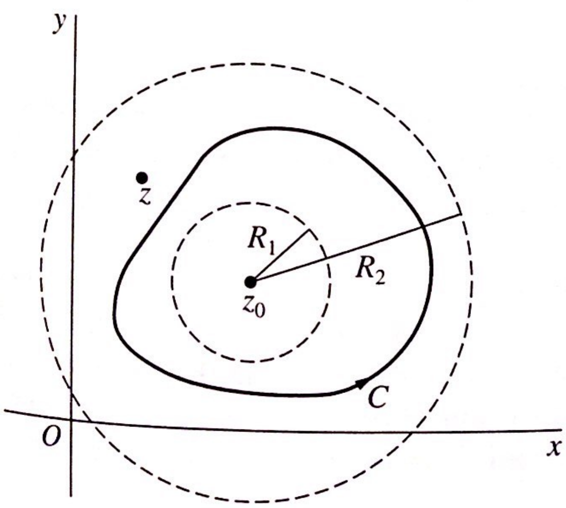


FIGURE 80

where C is any positively oriented simple closed contour around the origin. Since $b_1 = 1$, then,

$$\int_C e^{1/z}\,dz = 2\pi i.$$

This method of evaluating certain integrals around simple closed contours will be developed in considerable detail in Chap. 6 and then used extensively in Chap. 7.

EXAMPLE 4. The function $f(z) = 1/(z-i)^2$ is already in the form of a Laurent series, where $z_0 = i$. That is,

$$\frac{1}{(z-i)^2} = \sum_{n=-\infty}^{\infty} c_n(z-i)^n \qquad (0 < |z-i| < \infty)$$

where $c_{-2} = 1$ and all of the other coefficients are zero. From expression (5), Sec. 66, for the coefficients in a Laurent series, we know that

$$c_n = \frac{1}{2\pi i}\int_C \frac{dz}{(z-i)^{n+3}} \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

where C is, for instance, any positively oriented circle $|z-i| = R$ about the point $z_0 = i$. Thus [compare with Exercise 13, Sec. 46]

$$\int_C \frac{dz}{(z-i)^{n+3}} = \begin{cases} 0 & \text{when } n \neq -2, \\ 2\pi i & \text{when } n = -2. \end{cases}$$

EXERCISES

1. Find the Laurent series that represents the function

$$f(z) = z^2 \sin\left(\frac{1}{z^2}\right)$$

in the domain $0 < |z| < \infty$.

Ans. $1 + \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{1}{z^{4n}}$.

2. Find a representation for the function

$$f(z) = \frac{1}{1+z} = \frac{1}{z} \cdot \frac{1}{1+(1/z)}$$

in negative powers of z that is valid when $1 < |z| < \infty$.

Ans. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^n}$.

3. Find the Laurent series that represents the function $f(z)$ in Example 1, Sec. 68, when $1 < |z| < \infty$.

Ans. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^{2n+1}}$.

2, 5, 7

4. Give two Laurent series expansions in powers of z for the function

$$f(z) = \frac{1}{z^2(1-z)},$$

and specify the regions in which those expansions are valid.

Ans. $\displaystyle\sum_{n=0}^{\infty} z^n + \frac{1}{z} + \frac{1}{z^2} \quad (0 < |z| < 1); \quad -\sum_{n=3}^{\infty} \frac{1}{z^n} \quad (1 < |z| < \infty).$

5. The function

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2},$$

which has the two singular points $z = 1$ and $z = 2$, is analytic in the domains (Fig. 84)

$$D_1 : |z| < 1, \quad D_2 : 1 < |z| < 2, \quad D_3 : 2 < |z| < \infty.$$

Find the series representation in powers of z for $f(z)$ in each of those domains.

Ans. $\displaystyle\sum_{n=0}^{\infty}(2^{-n-1} - 1)z^n$ in D_1; $\displaystyle\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} + \sum_{n=1}^{\infty}\frac{1}{z^n}$ in D_2; $\displaystyle\sum_{n=1}^{\infty}\frac{1-2^{n-1}}{z^n}$ in D_3.

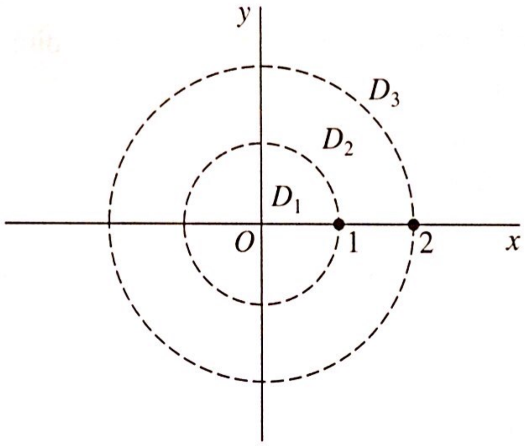


FIGURE 84

6. Show that when $0 < |z-1| < 2$,

$$\frac{z}{(z-1)(z-3)} = -3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}.$$

7. (*a*) Let a denote a real number, where $-1 < a < 1$, and derive the Laurent series representation

$$\frac{a}{z-a} = \sum_{n=1}^{\infty}\frac{a^n}{z^n} \qquad (|a| < |z| < \infty).$$

(*b*) After writing $z = e^{i\theta}$ in the equation obtained in part (*a*), equate real parts and then imaginary parts on each side of the result to derive the summation formulas

$$\sum_{n=1}^{\infty} a^n \cos n\theta = \frac{a\cos\theta - a^2}{1 - 2a\cos\theta + a^2} \quad \text{and} \quad \sum_{n=1}^{\infty} a^n \sin n\theta = \frac{a\sin\theta}{1 - 2a\cos\theta + a^2},$$

where $-1 < a < 1$. (Compare with Exercise 4, Sec. 61.)

P196. 3.

$$\frac{z}{z^4+4} = \frac{z}{4}\cdot\frac{1}{1+\frac{z^4}{4}} = \frac{z}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z^4}{4}\right)^n = \sum_{n=0}^{\infty}(-1)^n\cdot\frac{z^{4n+1}}{4\cdot 4^n}$$

$$= \sum_{n=0}^{\infty}(-1)^n\cdot\frac{z^{4n+1}}{2^{2n+2}} \qquad \left|\frac{z^4}{4}\right|<1 \Leftrightarrow |z|<4^{\frac14}=\sqrt{2}.$$

4. $$\cos z = -\sin\left(z-\frac{\pi}{2}\right) = -\sum_{n=0}^{\infty}(-1)^n\frac{\left(z-\frac{\pi}{2}\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{\left(z-\frac{\pi}{2}\right)^{2n+1}}{(2n+1)!}.\ \forall z\in\mathbb{C}.$$

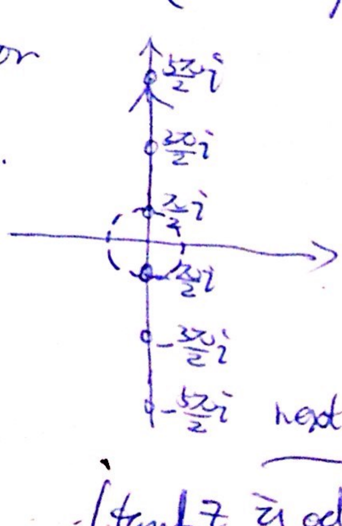
6. $$\tanh z = \frac{\sinh z}{\cosh z} \qquad \cosh z = \frac{e^z+e^{-z}}{2}$$

$$\cosh z = 0 \Leftrightarrow \frac{e^z+e^{-z}}{2}=0 \Leftrightarrow e^{-z}\left((e^z)^2+1\right)=0 \Leftrightarrow (e^z)^2=-1$$

$$\Leftrightarrow e^z=\pm i \Leftrightarrow z=\log(\pm i)=\ln 1 + i\cdot\arg(\pm i) = 0+i\left(\pm\frac{\pi}{2}+2\pi\cdot n\right)$$

$$\Leftrightarrow z = i\left(\frac{4n+1}{2}\pi\right) \text{ or } i(4n-1)\frac{\pi}{2} \Leftrightarrow z=i(2m+1)\cdot\frac{\pi}{2} = i\left(m\cdot\pi+\frac{\pi}{2}\right).$$

$\Rightarrow$ Largest circle within which the Maclaurin series for $\tanh(z)$ converges to ~~tanz~~ $\tanh z$ is $\left\{|z|=\frac{\pi}{2}\right\}$.



$$\tanh z = C_0 + C_1 z + C_2 z^2 + \cdots \qquad |z|<\frac{\pi}{2}$$

$$C_0 = \tanh(0)=0. \quad C_1 = (\tanh(z))'(0) = \frac{1}{\cosh^2 z}\Big]_{z=0} = 1.$$

next page

$$C_2 = \frac{1}{2!}\frac{d^2}{dz^2}\tanh(z)\Big]_{z=0} = \frac{1}{2}\cdot(-2)(\cosh z)^{-3}\cdot\sinh z\Big]_{z=0} = 0$$ (tanh z is odd fct)

$$\frac{d}{dz}\tanh z = \frac{d}{dz}\left(\frac{\sinh z}{\cosh z}\right) = \frac{\cosh z\cosh z - \sinh z\cdot\sinh z}{\cosh^2 z} = \frac{1}{\cosh^2 z}$$

$$\frac{d^2}{dz^2}\tanh z = \frac{d}{dz}(\cosh z)^{-2} = -2(\cosh z)^{-3}\cdot\sinh z$$

$$\frac{d^3}{dz^3}\tanh z = \frac{d}{dz}\left(-2(\cosh z)^{-3}\sinh z\right) = 6(\cosh z)^{-4}\sinh^2 z - 2(\cosh z)^{-3}\cosh z.$$

$\tanh z$ is odd fct, so $C_{2n} = 0$.

$$C_1 = f'(0) = \frac{1}{\cosh^2 z}\Big]_{z=0} = 1. \quad C_3 = \frac{1}{3!}f'''(0) = \frac{1}{6}(6\cdot 0 - 2\cdot 1) = -\frac{1}{3}$$

so. $\tanh z = z - \frac{1}{3}z^3 + \cdots\cdots, \quad |z| < \frac{\pi}{2}.$

9. $$f(z) = \sin(z^2) = \sum_{n=0}^{\infty}(-1)^n\frac{(z^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\cdot\frac{z^{4n+2}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n$$

$$\Rightarrow \quad f^{(4n)}(0) = 0, \text{ and } f^{(2n+1)}(0) = 0$$

because only terms with exponents $4n+2$ are presented.

11. $$\frac{1}{4z - z^2} = \frac{1}{z}\cdot\frac{1}{4-z} = \frac{1}{z}\cdot\frac{1}{4(1-\frac{z}{4})} = \frac{1}{z}\cdot\frac{1}{4}\cdot\sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n \quad \left(\left|\frac{z}{4}\right| < 1\right).$$

$$= \frac{1}{4z}\left(1 + \sum_{n=1}^{\infty}\frac{z^n}{4^n}\right) = \frac{1}{4z} + \sum_{n=1}^{\infty}\frac{z^{n-1}}{4^{n+1}}$$

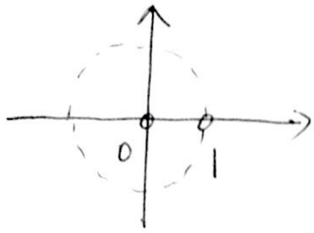
$$\overset{m=n-1}{=} \frac{1}{4z} + \sum_{m=0}^{\infty}\frac{z^m}{4^{m+2}} = \frac{1}{4z} + \sum_{n=0}^{\infty}\frac{z^n}{4^{n+2}} \quad \text{for } 0 < |z| < 4.$$

P205. 1. $z^2\sin\left(\frac{1}{z^2}\right) = z^2\cdot\sum_{n=0}^{\infty}(-1)^n\cdot\frac{1}{(2n+1)!}\left(\frac{1}{z^2}\right)^{2n+1}$

$$= z^2\cdot\left(\frac{1}{z^2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}\cdot\frac{1}{z^{4n+2}}\right)$$

$$= 1+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}\cdot\frac{1}{z^{4n}}.$$

4. $f(z)=\frac{1}{z^2(1-z)}$.



In the annulus $0<|z|<1$:

$$f(z)=\frac{1}{z^2}\cdot\frac{1}{1-z}=\frac{1}{z^2}\cdot\sum_{n=0}^{\infty}z^n=\frac{1}{z^2}\left(1+z+\sum_{n=2}^{\infty}z^n\right)=\frac{1}{z^2}+\frac{1}{z}+\sum_{n=2}^{\infty}z^{n-2}$$

$$=\frac{1}{z^2}+\frac{1}{z}+\sum_{n=0}^{\infty}z^n. \qquad 0<|z|<1.$$

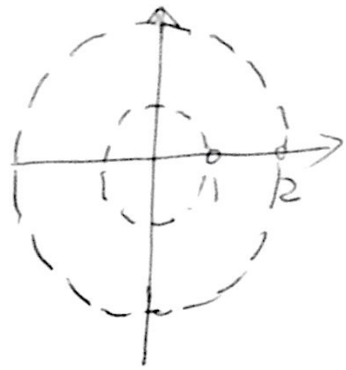
In the annulus $|z|>1$:

$$f(z)=\frac{1}{z^2}\cdot\frac{1}{z\left(\frac{1}{z}-1\right)}=-\frac{1}{z^3}\cdot\frac{1}{1-\frac{1}{z}}=-\frac{1}{z^3}\cdot\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n=-\sum_{n=0}^{\infty}\frac{1}{z^{n+3}}$$

$$=-\sum_{n=3}^{\infty}\frac{1}{z^n}. \qquad |z|>1$$

5. $f(z) = \frac{-1}{(z-1)(z-2)}$

$= \frac{1}{z-1} - \frac{1}{z-2}.$



• annulus $\overset{\shortparallel}{\text{disk}}$ $|z|<1$: $\frac{1}{z-1} = -\frac{1}{1-z} = -\sum_{n=0}^{\infty} z^n$

$\frac{1}{z-2} = -\frac{1}{2(1-\frac{z}{2})} = -\frac{1}{2}\cdot\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$

so. $f(z) = -\sum_{n=0}^{\infty} z^n - \left(-\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}\right) = \sum_{n=0}^{\infty}\left(-1+\frac{1}{2^{n+1}}\right) z^n.$

• $1<|z|<2$. $\frac{1}{z-1} = +\frac{1}{z(1-\frac{1}{z})} = \frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}} = \frac{1}{z}\cdot\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = \sum_{n=1}^{\infty}\frac{1}{z^n}$

$\frac{1}{z-2} = -\frac{1}{2(1-\frac{z}{2})} = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$

$\Rightarrow f(z) = \sum_{n=1}^{\infty}\frac{1}{z^n} - \left(-\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}\right) = \sum_{n=1}^{\infty}\frac{1}{z^n} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}.$

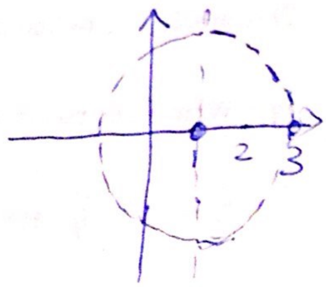
• $|z|>2$: $\frac{1}{z-1} = \frac{1}{z\cdot(1-\frac{1}{z})} \overset{|\frac{1}{z}|<1}{=} \frac{1}{z}\cdot\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = \sum_{n=1}^{\infty}\left(\frac{1}{z}\right)^n$

$\frac{1}{z-2} = \frac{1}{z(1-\frac{2}{z})} \overset{|\frac{2}{z}|<1}{=} \frac{1}{z}\cdot\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n = \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}} = \sum_{n=1}^{\infty}\frac{2^{n-1}}{z^n}$

$\Rightarrow f(z) = \sum_{n=1}^{\infty}\left(\frac{1}{z}\right)^n - \sum_{n=1}^{\infty}\frac{2^{n-1}}{z^n} = \sum_{n=1}^{\infty}\frac{1-2^{n-1}}{z^n}$

6. $f(z) = \frac{z}{(z-1)\cdot(z-3)} = \frac{A}{z-1} + \frac{B}{z-3} \Rightarrow \begin{cases} A+B=1 \\ -3A-B=0 \end{cases} \Rightarrow \begin{cases} A=-\frac{1}{2} \\ B=\frac{3}{2} \end{cases}$

$\Rightarrow f(z) = -\frac{1}{2}\cdot\frac{1}{z-1} + \frac{3}{2}\cdot\frac{1}{z-3}.$



when $0<|z-1|<2$: $\frac{1}{z-1} = (z-1)^{-1}.$

$\frac{1}{z-3} = \frac{1}{(z-1)-2} = \frac{1}{2\left(\frac{z-1}{2}-1\right)} = -\frac{1}{2}\cdot\frac{1}{1-\frac{z-1}{2}}$ $\quad \left|\frac{z-1}{2}\right|<1.$

$= -\frac{1}{2}\cdot\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n = -\sum_{n=0}^{\infty}\cdot\frac{(z-1)^n}{2^{n+1}}$

$\Rightarrow f(z) = -\frac{1}{2}(z-1)^{-1} + \frac{3}{2}\cdot\left(-\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+1}}\right) = -\frac{1}{2(z-1)} - 3\cdot\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}}.$

$0<|z-1|<2$

7. (a). $\frac{a}{z-a} = \frac{a}{z\left(1-\frac{a}{z}\right)} \overset{\left|\frac{a}{z}\right|<1}{=\!=\!=} \frac{a}{z}\cdot\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n = \sum_{n=1}^{\infty}\frac{a^n}{z^n}.$

$|a|<|z|<\infty$

(b). $z=e^{i\theta} \Rightarrow \frac{a}{z-a} = \frac{a}{e^{i\theta}-a} = \frac{a}{(\cos\theta+i\sin\theta)-a}$

$= \frac{a\cdot((\cos\theta-a)-i\sin\theta)}{((\cos\theta-a)+i\sin\theta)((\cos\theta-a)-i\sin\theta)} = \frac{a(\cos\theta-a)-ia\sin\theta}{(\cos\theta-a)^2+\sin^2\theta}$

$= \frac{(a\cos\theta-a^2)-i\cdot a\sin\theta}{1-2a\cdot\cos\theta+a^2}.$

righthandside: $\sum_{n=1}^{\infty}\frac{a^n}{z^n} = \sum_{n=1}^{\infty}\frac{a^n}{e^{in\theta}} = \sum_{n=1}^{\infty}a^n\cdot(\cos(n\theta)-i\sin(n\theta)) = \left(\sum_{n=1}^{\infty}a^n\cos(n\theta)\right).$
$-i\cdot\left(\sum_{n=0}^{\infty}a^n\sin(n\theta)\right)$

compare both sides to get:

$\sum_{n=1}^{\infty}a^n\cos(n\theta) = \frac{a\cos\theta-a^2}{1-2a\cos\theta+a^2},$ $\quad \sum_{n=1}^{\infty}a^n\sin(n\theta) = \frac{a\cdot\sin\theta}{1-2a\cos\theta+a^2}$

real part $\qquad$ imaginary part.

MAT 342 FALL 2014 Practice MIDTERM II

NAME : ID :

THERE ARE SIX (6) PROBLEMS. THEY HAVE THE INDICATED VALUE.

SHOW YOUR WORK

DO NOT TEAR-OFF ANY PAGE

NO CALCULATORS NO CELLS ETC.

ON YOUR DESK: ONLY test, pen, pencil, eraser.

1		50pts
2		50pts
3		50pts
4		50pts
5		50pts
6		20pts
Total		270pts

!!! WRITE YOUR NAME, STUDENT ID AND LECTURE N. BELOW !!!

NAME : ID :

1. (50pts)

(1) Calculate i^i, $|i^i|$ and $|i|^i$. What's the principal value of i^i?

(2) Does $f(z) = 1/z$ have an antiderivative on the region $\mathbb{C}\backslash[1, +\infty)$? How about the region $\mathbb{C}\backslash[-1, +\infty)$?

2. (50pts) Denote by $D = \{|z| \leq 1, \mathrm{Re}(z) \geq 0\}$ and by $C = \partial D$ the boundary of D with positive orientation with respect to D. Calculate the following integrals:

(1)

$$\int_C \bar{z}^2 dz.$$

(2)

$$\int_C |\bar{z}^2||dz|.$$

(3)

$$\int_C \sin(z) dz.$$

3. (50pts)

Calculate the following integrals:

(1)

$$\int_{|z+i|=1} \frac{e^{\pi z}}{z^2+1} dz.$$

(2)

$$\int_{|z|=2} \frac{dz}{(z-1)^3(z-3)^3}.$$

4. (50pts)

(1) Assume that $f(z)$ is an entire function satisfying $\mathrm{Im}(f(z)) > 1$. What can you say about the function $f(z)$? Explain the reason. How about with different assumption $\mathrm{Im}(f(z)) < 1$ or $\mathrm{Re}(f(z)) > 1$?

(2) Assume $f(z) = u(z) + iv(z)$ is analytic and continuous on the closed disk $\{|z| \leq 1\}$. Assume that $u(z)$ obtains a local minimum or local maximum at $z = 0$. What can you say about the function $f(z)$? Explain the reason.

5. (50pts)

(1) Calculate the series:

$$\sum_{n=1}^{+\infty} \frac{1}{(2i)^n}$$

(2) Calculate the limit:

$$\lim_{n\to+\infty} \operatorname{Arg}\left(i + (-1)^n \frac{100}{n}\right).$$

6. (20pts)(Extra credit)

Estimate the following quantity from above without calculating it:

$$\left| \int_{|z|=10} \frac{z-i}{z^2+z+1} dz \right|.$$

!!! WRITE YOUR NAME, STUDENT ID ...

NAME : ID :

1. (50pts)

(1) Calculate i^i, $|i^i|$ and $|i|^i$. What's the principal value of i^i?

(2) Does $f(z) = 1/z$ have an antiderivative on the region ~~$\mathbb{C}\backslash[1,+\infty)$~~ $\mathbb{C}\backslash(\{0\}\cup[1,+\infty))$? How about the region $\mathbb{C}\backslash[-1,+\infty)$?

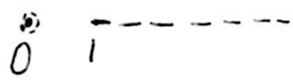
(1) $i^i = e^{i\log i} = e^{i(\ln 1 + i(\frac{\pi}{2}+2\pi\cdot n))} = e^{-(\frac{\pi}{2}+2\pi n)}$ $n=0,\pm1,\pm2,\dots$

$|i^i| = e^{-(\frac{\pi}{2}+2\pi n)}$ $n=0,\pm1,\pm2,\dots$

$|i|^i = 1^i = e^{i\log 1} = e^{i(\ln 1 + i\cdot 2\pi n)} = e^{-2\pi n}$. $n=0,\pm1,\pm2,\dots$

Principal value of i^i: $P.V.\, i^i = e^{i\cdot \mathrm{Log}\, i} = e^{i(\ln 1 + i\frac{\pi}{2})} = e^{-\frac{\pi}{2}}$.

(2) $\frac{1}{z}$ does NOT have an



antiderivative on the region $\mathbb{C}\backslash([1,+\infty)\cup\{0\}) = D_1$

Otherwise, the integral of $\frac{1}{z}$ along any closed curve in D_1 is zero. However $\int_{|z|=\frac{1}{2}} \frac{1}{z}dz \underset{z=\frac{1}{2}e^{i\theta}}{=} \int_0^{2\pi} i\,d\theta = 2\pi i \neq 0$.

On the region $D_2 = \mathbb{C}\backslash[-1,\infty)$

$\frac{1}{z}$ has an antiderivative given by the following branch of $\log z$:

$$\log z = \ln|z| + i\cdot\arg z. \quad 0<\arg z<2\pi.$$

or any branch s.t. $2n\pi < \arg < (2n+2)\pi$

2. (50pts) Denote by $D = \{|z| \le 1, \mathrm{Re}(z) \ge 0\}$ and by $C = \partial D$ the boundary of D with positive orientation with respect to D. Calculate the following integrals:

(1)

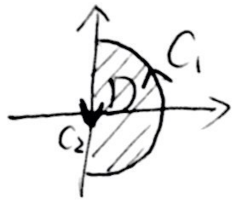
$$\int_C \bar{z}^2 dz.$$

(2)

$$\int_C |\bar{z}^2||dz|.$$

(3)

$$\int_C \sin(z) dz.$$



$C = C_1 + C_2$. C_1: $z = e^{i\theta}$ $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. $z'(\theta) = i \cdot e^{i\theta}$ $|z'(\theta)| = 1$.

C_2: $z = \cancel{(1-t)i. \ 0 \le t \le 2}$ $-ti$ $-1 \le t \le 1$. $z'(t) = -i$. $|z'(t)| = 1$.

(1). $$\int_{C_1} \bar{z}^2 dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (e^{-i\theta})^2 \cdot i e^{i\theta} d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-i\theta} d(i\theta) = -e^{-i\theta}\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= -e^{-i \cdot \frac{\pi}{2}} + e^{i \frac{\pi}{2}} = -(-i) + i = 2i.$$

$$\int_{C_2} \bar{z}^2 dz = \int_{-1}^{1} (ti)^2 \cdot (-i) \cdot dt = +i \cdot \int_{-1}^{1} t^2 dt = i \cdot \frac{t^3}{3}\Big]_{-1}^{1} = i \cdot \frac{2}{3}.$$

So $$\int_C \bar{z}^2 dz = \int_{C_1} + \int_{C_2} = 2i + \frac{2i}{3} = \frac{8i}{3}$$

(2). $$\int_{C_1} |\bar{z}^2||dz| = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |(e^{-i\theta})^2| \cdot |i \cdot e^{i\theta}| d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = \theta\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi.$$

$$\int_{C_2} |\bar{z}^2||dz| = \int_{-1}^{1} |(ti)^2| \, 1 \cdot dt = \int_{-1}^{1} |t^2| dt = 2 \int_0^1 t^2 dt = \frac{2}{3} t^3\Big]_0^1 = \frac{2}{3}$$

So $$\int_C |\bar{z}^2||dz| = \pi + \frac{2}{3}.$$

(3) Because $\sin(z)$ is analytic on $\bar{D}$, by Cauchy-Goursat theorem

$$\int_C \sin(z) dz = 0.$$

3. (50pts)

Calculate the following integrals:

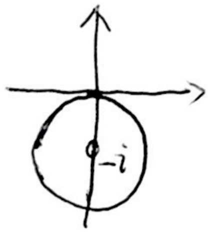
(1)

$$\int_{|z+i|=1} \frac{e^{\pi z}}{z^2+1} dz.$$

(2)

$$\int_{|z|=2} \frac{dz}{(z-1)^3(z-3)^3}.$$

(1).

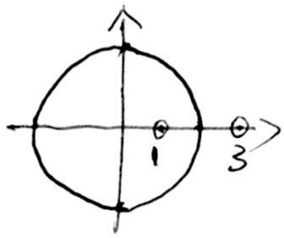


$$\int_{|z+i|=1} \frac{e^{\pi z}}{z^2+1} dz = \int_{|z+i|=1} \frac{e^{\pi z}}{(z-i)(z+i)} dz$$

$$= 2\pi i \cdot \frac{e^{\pi z}}{z-i}\Big]_{z=-i} = 2\pi i \cdot \frac{e^{-\pi i}}{-i-i} = -\pi e^{-\pi i}$$

$$= \pi.$$

(2)



$$\int_{|z|=2} \frac{dz}{(z-1)^3(z-3)^3} \cancel{dz} = \int_{|z|=2} \frac{\frac{1}{(z-3)^3}}{(z-1)^3} dz$$

$$= \frac{2\pi i}{2!} \frac{d^2}{dz^2} \frac{1}{(z-3)^3}\Big]_{z=1} = \pi i \cdot (-3) \cdot (-4) \cdot (z-3)^{-5}\Big]_{z=1}$$

$$= +12\pi i \cdot \frac{1}{(-2)^5} = -\frac{12\pi i}{32} = \frac{-3\pi i}{8}$$

4. **(50pts)**

(1) Assume that $f(z)$ is an entire function satisfying $\mathrm{Im}(f(z)) > 1$. What can you say about the function $f(z)$? Explain the reason. How about with different assumption $\mathrm{Im}(f(z)) < 1$ or $\mathrm{Re}(f(z)) > 1$?

(2) Assume $f(z) = u(z) + iv(z)$ is analytic and continuous on the closed disk $\{|z| \leq 1\}$. Assume that $u(z)$ obtains a local minimum or local maximum at $z = 0$. What can you say about the function $f(z)$? Explain the reason.

(1) $\mathrm{Im}(f(z)) = v(z) > 1 \Rightarrow -v < -1$

$f(z) = u + iv \Rightarrow i \cdot f(z) = -v + iu \Rightarrow e^{i \cdot f(z)} = e^{-v} \cdot e^{iu}$

$\Rightarrow |e^{i \cdot f(z)}| = e^{-v} < e^{-1}$. Note that $e^{i f(z)}$ is also entire

By Liouville Theorem for entire functions, we get $e^{i f(z)} = \text{const} = C_1$

$\Rightarrow f(z) = \frac{1}{i} \log C_1$ is constant.

Similarly, • when $\mathrm{Im}(f(z)) < 1$, consider $e^{-if} = e^{v - iu}$

then $|e^{-if}| = e^{v} < e$ & e^{-if} entire $\Rightarrow e^{-if} = \text{constant}$ $\Downarrow$

• when $\mathrm{Re}(f(z)) \cancel{<} > 1$, consider $e^{-f} = e^{-u - iv}$ instead. f is constant.

(2) Consider $e^{f} = e^{u} \cdot e^{iv}$. $|e^{f}| = e^{u}$

If u obtains a (local) maximum at $z=0$, then e^{f} obtains a (local) maximum at $z=0$. e^{f} is also analytic on D. So by the Maximum Modulus principle, $e^{f} = \text{constant} \Rightarrow f = \text{constant}$.

If u obtains a (local) minimum at $z=0$, then consider $e^{-f} = e^{-u-iv}$ and apply the same argument. We get again $e^{-f} = \text{constant}. \Rightarrow f = \text{const}.$

5. (50pts)

(1) Calculate the series:

$$\sum_{n=1}^{+\infty} \frac{1}{(2i)^n}$$

(2) Calculate the limit:

$$\lim_{n\to+\infty} \operatorname{Arg}\left(i + (-1)^n \frac{100}{n}\right).$$

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2i)^n} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{(2i)^n} = \lim_{N\to\infty} \frac{\frac{1}{2i} - \left(\frac{1}{2i}\right)^{N+1}}{1-\frac{1}{2i}}$

$\displaystyle = \frac{\frac{1}{2i}}{1-\cancel{\frac{1}{2i}}} = \frac{1}{2i-1} = \frac{-1-2i}{(-1+2i)\cdot(-1-2i)}$

$\displaystyle = \frac{1}{5}(-1-2i).$

(2). $\displaystyle\lim_{n\to+\infty} \operatorname{Arg}\left(i + (-1)^n \frac{100}{n}\right) = \operatorname{Arg}(i) = \frac{\pi}{2}.$

6. (20pts)(Extra credit)

Estimate the following quantity from above without calculating it:

$$\left|\int_{|z|=10} \frac{z-i}{z^2+z+1} dz\right|.$$

$$\left|\int_{|z|=10} \frac{z-i}{z^2+z+1} dz\right| \le \int_{|z|=10} \left|\frac{z-i}{z^2+z+1}\right| |dz|$$

$$\le \int_{|z|=10} \frac{|z|+|i|}{|z|^2-|z|-1} |dz|$$

$$= \int_{|z|=10} \frac{10+1}{10^2-10-1} |dz|$$

$$= \frac{11}{89} \cdot 2\pi \cdot 10 = \frac{220\pi}{89}.$$

converges to $f(z)$ at all points in some annular domain about z_0, then it is the Laurent series expansion for f in powers of $z - z_0$ for that domain.

The method of proof here is similar to the one used in proving Theorem 1. The hypothesis of this theorem tells us that there is an annular domain about z_0 such that

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

for each point z in it. Let $g(z)$ be as defined by equation (4), but now allow n to be a negative integer too. Also, let C be any circle around the annulus, centered at z_0 and taken in the positive sense. Then, using the index of summation m and adapting Theorem 1 in Sec. 71 to series involving both nonnegative *and* negative powers of $z - z_0$ (Exercise 10), write

$$\int_C g(z) f(z)\, dz = \sum_{m=-\infty}^{\infty} c_m \int_C g(z)(z - z_0)^m\, dz,$$

or

$$\frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z - z_0)^{n+1}} = \sum_{m=-\infty}^{\infty} c_m \int_C g(z)(z - z_0)^m\, dz. \tag{9}$$

Since equations (6) are also valid when the integers m and n are allowed to be negative, equation (9) reduces to

$$\frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z - z_0)^{n+1}} = c_n \qquad (n = 0, \pm 1, \pm 2, \ldots),$$

which is expression (5), Sec. 66, for the coefficients c_n in the Laurent series for f in the annulus.

EXERCISES

1. By differentiating the Maclaurin series representation

$$\frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1),$$

obtain the expansions

$$\frac{1}{(1 - z)^2} = \sum_{n=0}^{\infty} (n + 1)\, z^n \qquad (|z| < 1)$$

and

$$\frac{2}{(1 - z)^3} = \sum_{n=0}^{\infty} (n + 1)(n + 2)\, z^n \qquad (|z| < 1).$$

2. By substituting $1/(1-z)$ for z in the expansion

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1)\, z^n \qquad (|z| < 1),$$

found in Exercise 1, derive the Laurent series representation

$$\frac{1}{z^2} = \sum_{n=2}^{\infty} \frac{(-1)^n (n-1)}{(z-1)^n} \qquad (1 < |z-1| < \infty).$$

(Compare with Example 2, Sec. 71.)

3. Find the Taylor series for the function

$$\frac{1}{z} = \frac{1}{2+(z-2)} = \frac{1}{2} \cdot \frac{1}{1+(z-2)/2}$$

about the point $z_0 = 2$. Then, by differentiating that series term by term, show that

$$\frac{1}{z^2} = \frac{1}{4} \sum_{n=0}^{\infty} (-1)^n (n+1) \left(\frac{z-2}{2}\right)^n \qquad (|z-2| < 2).$$

4. Show that the function defined by means of the equations

$$f(z) = \begin{cases} (1-\cos z)/z^2 & \text{when } z \neq 0, \\ 1/2 & \text{when } z = 0 \end{cases}$$

is entire. (See Example 1, Sec. 71.)

5. Prove that if

$$f(z) = \begin{cases} \dfrac{\cos z}{z^2 - (\pi/2)^2} & \text{when } z \neq \pm\pi/2, \\[2ex] -\dfrac{1}{\pi} & \text{when } z = \pm\pi/2, \end{cases}$$

then f is an entire function.

6. In the w plane, integrate the Taylor series expansion (see Example 1, Sec. 64)

$$\frac{1}{w} = \sum_{n=0}^{\infty} (-1)^n (w-1)^n \qquad (|w-1| < 1)$$

along a contour interior to its circle of convergence from $w = 1$ to $w = z$ to obtain the representation

$$\text{Log}\, z = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (z-1)^n \qquad (|z-1| < 1).$$

7. Use the result in Exercise 6 to show that if

$$f(z) = \frac{\text{Log}\, z}{z-1} \qquad \text{when } z \neq 1$$

and $f(1) = 1$, then f is analytic throughout the domain

$$0 < |z| < \infty, \quad -\pi < \text{Arg}\, z < \pi.$$

8. Prove that if f is analytic at z_0 and $f(z_0) = f'(z_0) = \cdots = f^{(m)}(z_0) = 0$, then the function g defined by means of the equations

$$g(z) = \begin{cases} \dfrac{f(z)}{(z-z_0)^{m+1}} & \text{when } z \neq z_0, \\[2ex] \dfrac{f^{(m+1)}(z_0)}{(m+1)!} & \text{when } z = z_0 \end{cases}$$

is analytic at z_0.

9. Suppose that a function $f(z)$ has a power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

inside some circle $|z - z_0| = R$. Use Theorem 2 in Sec. 71, regarding term by term differentiation of such a series, and mathematical induction to show that

$$f^{(n)}(z) = \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} a_{n+k} (z-z_0)^k \qquad (n = 0, 1, 2, \ldots)$$

when $|z - z_0| < R$. Then, by setting $z = z_0$, show that the coefficients a_n $(n = 0, 1, 2, \ldots)$ are the coefficients in the Taylor series for f about z_0. Thus give an alternative proof of Theorem 1 in Sec. 72.

10. Consider two series

$$S_1(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{and} \quad S_2(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n},$$

which converge in some annular domain centered at z_0. Let C denote any contour lying in that annulus, and let $g(z)$ be a function which is continuous on C. Modify the proof of Theorem 1, Sec. 71, which tells us that

$$\int_C g(z) S_1(z)\, dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z-z_0)^n\, dz,$$

to prove that

$$\int_C g(z) S_2(z)\, dz = \sum_{n=1}^{\infty} b_n \int_C \frac{g(z)}{(z-z_0)^n}\, dz.$$

Conclude from these results that if

$$S(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n},$$

EXAMPLE. It is easy to see that the singularities of the function

$$f(z) = \frac{z^3(1-3z)}{(1+z)(1+2z^4)}$$

all lie inside the positively oriented circle C centered at the origin with radius 3. In order to use the theorem in this section, we write

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z} \cdot \frac{z-3}{(z+1)(z^4+2)}. \tag{8}$$

Inasmuch as the quotient

$$\frac{z-3}{(z+1)(z^4+2)}$$

is analytic at the origin, it has a Maclaurin series representation whose first term is the nonzero number $-3/2$. Hence, in view of expression (8),

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z}\left(-\frac{3}{2} + a_1 z + a_2 z^2 + a_3 z^3 + \cdots\right) = -\frac{3}{2} \cdot \frac{1}{z} + a_1 + a_2 z + a_3 z^2 + \cdots$$

for all z in some punctured disk $0 < |z| < R_0$. It is now clear that

$$\operatorname*{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = -\frac{3}{2},$$

and so

$$\int_C \frac{z^3(1-3z)}{(1+z)(1+2z^4)} dz = 2\pi i \left(-\frac{3}{2}\right) = -3\pi i. \tag{9}$$

EXERCISES

1. Find the residue at $z = 0$ of the function

(a) $\dfrac{1}{z+z^2}$; (b) $z\cos\left(\dfrac{1}{z}\right)$; (c) $\dfrac{z-\sin z}{z}$; (d) $\dfrac{\cot z}{z^4}$; (e) $\dfrac{\sinh z}{z^4(1-z^2)}$.

Ans. (a) 1; (b) $-1/2$; (c) 0; (d) $-1/45$; (e) $7/6$.

2. Use Cauchy's residue theorem (Sec. 76) to evaluate the integral of each of these functions around the circle $|z| = 3$ in the positive sense:

(a) $\dfrac{\exp(-z)}{z^2}$; (b) $\dfrac{\exp(-z)}{(z-1)^2}$; (c) $z^2 \exp\left(\dfrac{1}{z}\right)$; (d) $\dfrac{z+1}{z^2-2z}$.

Ans. (a) $-2\pi i$; (b) $-2\pi i/e$; (c) $\pi i/3$; (d) $2\pi i$.

3. In the example in Sec. 76, two residues were used to evaluate the integral

$$\int_C \frac{4z-5}{z(z-1)} dz$$

where C is the positively oriented circle $|z| = 2$. Evaluate this integral once again by using the theorem in Sec. 77 and finding only one residue.

4. Use the theorem in Sec. 77, involving a single residue, to evaluate the integral of each of these functions around the circle $|z| = 2$ in the positive sense:

$$(a)\ \frac{z^5}{1-z^3}; \qquad (b)\ \frac{1}{1+z^2}; \qquad (c)\ \frac{1}{z}.$$

Ans. $(a)\ -2\pi i$; $(b)\ 0$; $(c)\ 2\pi i$.

5. Let C denote the circle $|z| = 1$, taken counterclockwise, and use the following steps to show that

$$\int_C \exp\left(z + \frac{1}{z}\right) dz = 2\pi i \sum_{n=0}^{\infty} \frac{1}{n!\,(n+1)!}.$$

(*a*) By using the Maclaurin series for e^z and referring to Theorem 1 in Sec. 71, which justifies the term by term integration that is to be used, write the above integral as

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int_C z^n \exp\left(\frac{1}{z}\right) dz.$$

(*b*) Apply the theorem in Sec. 76 to evaluate the integrals appearing in part (*a*) to arrive at the desired result.

6. Suppose that a function f is analytic throughout the finite plane except for a finite number of singular points $z_1, z_2, \ldots, z_n$. Show that

$$\operatorname*{Res}_{z=z_1} f(z) + \operatorname*{Res}_{z=z_2} f(z) + \cdots + \operatorname*{Res}_{z=z_n} f(z) + \operatorname*{Res}_{z=\infty} f(z) = 0.$$

7. Let the degrees of the polynomials

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n \neq 0)$$

and

$$Q(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_m z^m \qquad (b_m \neq 0)$$

be such that $m \geq n + 2$. Use the theorem in Sec. 77 to show that if all of the zeros of $Q(z)$ are interior to a simple closed contour C, then

$$\int_C \frac{P(z)}{Q(z)}\, dz = 0.$$

[Compare with Exercise 4(*b*).]

78. THE THREE TYPES OF ISOLATED SINGULAR POINTS

We saw in Sec. 75 that the theory of residues is based on the fact that if f has an isolated singular point at z_0, then $f(z)$ has a Laurent series representation

$$(1)\qquad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_n}{(z - z_0)^n} + \cdots$$

In the remaining sections of this chapter, we shall develop in greater depth the theory of the three types of isolated singular points just illustrated. The emphasis will be on useful and efficient methods for identifying poles and finding the corresponding residues.

EXERCISES

1. In each case, write the principal part of the function at its isolated singular point and determine whether that point is a removable singular point, an essential singular point, or a pole:

(a) $z \exp\left(\dfrac{1}{z}\right)$; (b) $\dfrac{z^2}{1+z}$; (c) $\dfrac{\sin z}{z}$; (d) $\dfrac{\cos z}{z}$; (e) $\dfrac{1}{(2-z)^3}$.

2. Show that the singular point of each of the following functions is a pole. Determine the order m of that pole and the corresponding residue B.

(a) $\dfrac{1-\cosh z}{z^3}$; (b) $\dfrac{1-\exp(2z)}{z^4}$; (c) $\dfrac{\exp(2z)}{(z-1)^2}$.

Ans. (a) $m = 1$, $B = -1/2$; (b) $m = 3$, $B = -4/3$; (c) $m = 2$, $B = 2e^2$.

3. Suppose that a function f is analytic at z_0, and write $g(z) = f(z)/(z - z_0)$. Show that

(a) if $f(z_0) \neq 0$, then z_0 is a simple pole of g, with residue $f(z_0)$;

(b) if $f(z_0) = 0$, then z_0 is a removable singular point of g.

Suggestion: As pointed out in Sec. 62, there is a Taylor series for $f(z)$ about z_0 since f is analytic there. Start each part of this exercise by writing out a few terms of that series.

4. Write the function

$$f(z) = \frac{8a^3z^2}{(z^2+a^2)^3} \qquad (a > 0)$$

as

$$f(z) = \frac{\phi(z)}{(z-ai)^3} \quad \text{where} \quad \phi(z) = \frac{8a^3z^2}{(z+ai)^3}.$$

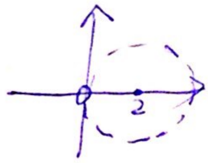
Point out why $\phi(z)$ has a Taylor series representation about $z = ai$, and then use it to show that the principal part of f at that point is

$$\frac{\phi''(ai)/2}{z-ai} + \frac{\phi'(ai)}{(z-ai)^2} + \frac{\phi(ai)}{(z-ai)^3} = -\frac{i/2}{z-ai} - \frac{a/2}{(z-ai)^2} - \frac{a^2 i}{(z-ai)^3}.$$

80. RESIDUES AT POLES

When a function f has an isolated singularity at a point z_0, the basic method for identifying z_0 as a pole and finding the residue there is to write the appropriate Laurent series and to note the coefficient of $1/(z - z_0)$. The following theorem provides an alternative characterization of poles and a way of finding residues at poles that is often more convenient.

P219. 3 Taylor series of $\frac{1}{z}$ centered at $z_0=2$:



$$\frac{1}{z}=\frac{1}{(z-2)+2}=\frac{1}{2\left(1+\frac{z-2}{2}\right)}=\frac{1}{2}\cdot\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-2}{2}\right)^n=\sum_{n=0}^{\infty}(-1)^n\cdot\frac{(z-2)^n}{2^{n+1}}\qquad \left|\frac{z-2}{2}\right|<1$$

$$\Updownarrow$$

$$|z-2|<2$$

Differentiate term by term:

$$-\frac{1}{z^2}=\sum_{n=1}^{\infty}(-1)^n\cdot\frac{n\cdot(z-2)^{n-1}}{2^{n+1}}=\sum_{n=0}^{\infty}(-1)^{n+1}\cdot\frac{(n+1)\cdot(z-2)^n}{2^{n+2}}$$

$$\Rightarrow \frac{1}{z^2}=\sum_{n=0}^{\infty}(-1)^n\cdot\frac{(n+1)\cdot(z-2)^n}{2^{n+2}}=\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n(n+1)\cdot\left(\frac{z-2}{2}\right)^n.\qquad |z-2|<2.$$

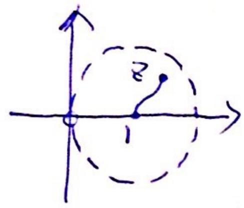
4.

$$f(z)=\begin{cases}\dfrac{1-\cos z}{z^2} & z\neq 0\\[2mm] \dfrac{1}{2} & z=0\end{cases}\qquad \cos z=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}=1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots$$

$$\frac{1-\cos z}{z^2}=\frac{1-\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots\right)}{z^2}=\frac{1}{2!}-\frac{z^2}{4!}+\frac{z^4}{6!}-\cdots$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n\cdot z^n}{(n+2)!}$$ analytic at any $z\in\mathbb{C}$ so entire

6. $$\frac{1}{w}=\frac{1}{1+w-1}=\sum_{n=0}(-1)^n\cdot(w-1)^n\qquad |w-1|<1$$



Integrate term by term:

$$\int_1^z\frac{1}{w}\,dw=\sum_{n=0}^{\infty}(-1)^n\cdot\int_1^z(w-1)^n\,dw\qquad |z-1|<1$$

$$\text{Log}\,z-\text{Log}\,1 = \text{Log}\,z$$

$$\sum_{n=0}^{\infty}(-1)^n\cdot\frac{(w-1)^{n+1}}{n+1}\Bigg]_1^z=\sum_{n=0}^{\infty}(-1)^n\cdot\frac{(z-1)^{n+1}}{n+1}=\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{(z-1)^n}{n}$$

$$\Rightarrow \text{Log}\,z=\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{(z-1)^n}{n}\qquad |z-1|<1.$$

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P220 8. $f(z)$ analytic at $z_0 \Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$

$f(z_0) = f'(z_0) = \cdots = f^{(m)}(z_0) = 0 \Rightarrow f(z) = \frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0)^{m+1} + \frac{f^{(m+2)}}{(m+2)!}(z-z_0)^{m+2} + \cdots$

$$\Rightarrow \frac{f(z)}{(z-z_0)^{m+1}} = \frac{f^{(m+1)}(z_0)}{(m+1)!} + \frac{f^{(m+2)}(z_0)}{(m+2)!}(z-z_0) + \frac{f^{(m+3)}(z_0)}{(m+3)!}(z-z_0)^2 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(m+1+n)}(z_0)}{(m+1+n)!}(z-z_0)^n$$ is analytic at z_0.

(differentiable in a nbhd of z_0)

P237. 1. (a). $\frac{1}{z+z^2} = \frac{1}{z} \cdot \frac{1}{1+z} = \frac{1}{z}(1 - z + z^2 - \cdots) = \frac{1}{z} - 1 + z - \cdots$

$\Rightarrow \underset{z=0}{\text{Res}} \frac{1}{z+z^2} = 1$.

(b) $z \cdot \cos\left(\frac{1}{z}\right) = z \cdot \left(1 - \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{4!}\left(\frac{1}{z}\right)^4 - \cdots\right) = z - \frac{1}{2z} + \frac{1}{24z^3} - \cdots$

$\Rightarrow \underset{z=0}{\text{Res}}\, z \cdot \cos\left(\frac{1}{z}\right) = -\frac{1}{2}$.

(c). $\frac{z - \sin z}{z} = \frac{z - \left(z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \cdots\right)}{z} = \frac{1}{3!} z^2 - \frac{1}{5!} z^4 - \cdots$

$\Rightarrow \underset{z=0}{\text{Res}} \frac{z - \sin z}{z} = 0$.

P237. 1 (d) $\frac{\cot z}{z^4} = \frac{\cos z}{z^4 \cdot \sin z} = \frac{1-\frac{1}{2!}z^2+\frac{1}{4!}z^4-\cdots}{z^4\cdot\left(z-\frac{1}{3!}z^3+\frac{1}{5!}z^5-\cdots\right)}$.

$= \frac{1}{z^5}\cdot\frac{1-\frac{1}{2}z^2+\frac{1}{24}z^4-\cdots}{1-\frac{1}{6}z^2+\frac{1}{120}z^4\cdots} = \frac{1}{z^5}\cdot\frac{1-\frac{1}{2}z^2+\frac{1}{24}z^4-\cdots}{1-\left(\frac{1}{6}z^2-\frac{1}{120}z^4+\cdots\right)}$

$= \frac{1}{z^5}\cdot\left(1-\frac{1}{2}z^2+\frac{1}{24}z^4-\cdots\right)\cdot\left(1+\left(\frac{1}{6}z^2-\frac{1}{120}z^4+\cdots\right)+\left(\frac{1}{36}z^4+\cdots\right)\right)$

$= \frac{1}{z^5}\cdot\left(1+\left(-\frac{1}{2}+\frac{1}{6}\right)z^2+\left(-\frac{1}{120}+\frac{1}{36}-\frac{1}{12}+\frac{1}{24}\right)z^4+O(z^6)\right)$

$= \frac{1}{z^5}\cdot\left(1-\frac{1}{3}z^2+\frac{-3+10-30+15}{360}z^4+O(z^6)\right) = \frac{1}{z^5}\cdot\left(1-\frac{1}{3}z^2+\frac{-8}{360}z^4+O(z^6)\right)$

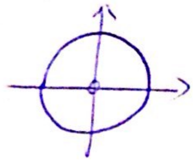
$= \frac{1}{z^5}-\frac{1}{3}\cdot\frac{1}{z^3}-\frac{1}{45}\cdot\frac{1}{z}+O(z) \Rightarrow \operatorname{Res}_{z=0}\frac{\cot z}{z^4} = -\frac{1}{45}$.

(e). $\frac{\sinh z}{z^4\cdot(1-z^2)} = \frac{1}{z^4}\cdot\frac{z+\frac{1}{3!}z^3+\frac{1}{5!}z^5+\cdots}{1-z^2} = \frac{1}{z^3}\cdot\left(1+\frac{1}{6}z^2+\frac{1}{120}z^4+\cdots\right)\cdot\left(1+z^2+z^4+\cdots\right)$

$= \frac{1}{z^3}\cdot\left(1+\left(\frac{1}{6}+1\right)z^2+O(z^4)\right) = \frac{1}{z^3}+\frac{7}{6}\cdot\frac{1}{z}+O(z)$

$\Rightarrow \operatorname{Res}_{z=0}\frac{\sinh z}{z^4\cdot(1-z^2)} = \frac{7}{6}$.

P237. 2 (a). $f(z)=\dfrac{e^{-z}}{z^2}$. $C: |z|=3$



There is one singularity inside the circle: $z=0$.

$$\frac{e^{-z}}{z^2}=\frac{1-z+\frac{1}{2!}(z)^2-\cdots}{z^2}=\frac{1}{z^2}-\frac{1}{z}+\frac{1}{2}-\cdots \Rightarrow \operatorname*{Res}_{z=0} f(z)=-1$$

$$\Rightarrow \int_{|z|=3}\frac{e^{-z}}{z^2}dz=2\pi i\cdot \operatorname*{Res}_{z=0}\frac{e^{-z}}{z^2}=2\pi i\cdot(-1)=-2\pi i.$$

(b). $f(z)=\dfrac{e^{-z}}{(z-1)^2}$. There is one singularity $z=1$:

$$\frac{e^{-z}}{(z-1)^2}=\frac{e^{-(z-1)-1}}{(z-1)^2}=e^{-1}\cdot\frac{1-(z-1)+\frac{1}{2!}\cdot(-(z-1))^2-\cdots}{(z-1)^2}=\frac{1}{e}\left(\frac{1}{(z-1)^2}-\frac{1}{z-1}+\frac{1}{2}-\cdots\right)$$

$$\Rightarrow \operatorname*{Res}_{z=0}\frac{e^{-z}}{(z-1)^2}=-\frac{1}{e}\Rightarrow \int_{|z|=3}\frac{e^{-z}}{(z-1)^2}dz=2\pi i\cdot\operatorname*{Res}_{z=1}\frac{e^{-z}}{(z-1)^2}=-\frac{2\pi i}{e}.$$

(c). $z^2\cdot e^{\frac{1}{z}}$. Singularity: $z=0$.

$$z^2\cdot e^{\frac{1}{z}}=z^2\left(1+\frac{1}{z}+\frac{1}{2!}\left(\frac{1}{z}\right)^2+\frac{1}{3!}\left(\frac{1}{z}\right)^3+\cdots\right)=z^2+z+\frac{1}{2}+\frac{1}{6}\cdot\frac{1}{z}+\cdots$$

$$\Rightarrow \operatorname*{Res}_{z=0}\left(z^2e^{\frac{1}{z}}\right)=\frac{1}{6}\Rightarrow\int_{|z|=3}z^2e^{\frac{1}{z}}dz=2\pi i\operatorname*{Res}_{z=0}\left(z^2e^{\frac{1}{z}}\right)=\frac{2\pi i}{6}=\frac{\pi i}{3}$$

(d). $\dfrac{z+1}{z^2-2z}$. There are 2 singularities inside the circle: $z_1=0$, $z_2=2$.

around $z=0$:
$$\frac{z+1}{z^2-2z}=\frac{1+z}{z(z-2)}=\frac{1}{z}(1+z)\cdot\frac{-1}{2(1-\frac{z}{2})}=\frac{-1}{z}\cdot(1+z)\cdot\frac{1}{2}\cdot\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right)$$

$$=-\frac{1}{2z}\cdot\left(1+\frac{3z}{2}+\cdots\right)\Longrightarrow \operatorname*{Res}_{z=0}\frac{z+1}{z^2-2z}=-\frac{1}{2}$$

4. (a) $f(z) = \frac{z^5}{1-z^3}$

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2} \frac{\frac{1}{z^5}}{1-\frac{1}{z^3}} = \frac{1}{z^2} \cdot \frac{1}{z^5 - z^2}$$

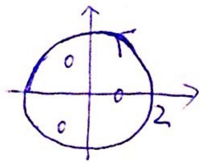
$$= \frac{1}{z^4} \frac{-1}{1-z^3} = -\frac{1}{z^4} \cdot (1 + z^3 + z^6 + \cdots) = -\frac{1}{z^4} - \frac{1}{z} - z^2 \cdots$$

So

$$\operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = 1$$

There are no singularities outside $|z|=2$

$$\Rightarrow \int_{|z|=2} \frac{z^5}{1-z^3}\, dz = -2\pi i \cdot \operatorname*{Res}_{z=\infty} f(z) = -2\pi i.$$



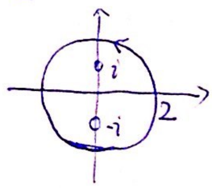
(b) $f(z) = \frac{1}{1+z^2}$.

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2} \cdot \frac{1}{1+\frac{1}{z^2}} = \frac{1}{z^2+1} = 1 - z^2 + z^4 - \cdots$$

$$\Rightarrow \operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = 0$$

There are no singularities outside $|z|=2$

$$\Rightarrow \int_{|z|=2} \frac{1}{1+z^2}\, dz = -2\pi i \operatorname*{Res}_{z=\infty} f(z) = 0$$



(c) $f(z) = \frac{1}{z}$.

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2} \cdot \frac{1}{\frac{1}{z}} = \frac{1}{z}$$

$$\Rightarrow \operatorname*{Res}_{z=\infty} f(z) = -1 \Rightarrow \int_{|z|=2} \frac{1}{z}\, dz = -2\pi i(-1) = 2\pi i.$$

P242. (a). $z\exp\left(\frac{1}{z}\right) = z\cdot\left(1+\frac{1}{z}+\frac{1}{2!}\left(\frac{1}{z}\right)^2+\frac{1}{3!}\left(\frac{1}{z}\right)^3+\cdots\right)$

$= z+1+\frac{1}{2!}\cdot\frac{1}{z}+\frac{1}{3!}\frac{1}{z^2}+\cdots$ isolated singularity $z=0$.

$= z+1+\sum_{n=1}^{\infty}\frac{1}{(n+1)!}\cdot\frac{1}{z^n}$

principal part = singular part = $\sum_{n=1}^{\infty}\frac{1}{(n+1)!}\cdot\frac{1}{z^n}$ has infinitely many terms $\Rightarrow$ $z=0$ is an essential singularity.

(b) $\frac{z^2}{1+z}$ singularity: $z=-1$.

$\frac{(z+1-1)^2}{1+z} = \frac{(z+1)^2-2(z+1)+1}{z+1} = (z+1)-2+\frac{1}{z+1}$.

principal part (singular part) $= \frac{1}{z+1}$ $\Rightarrow$ $z=-1$ is a pole of order 1.

(c) $\frac{\sin z}{z} = \frac{z-\frac{1}{3!}z^3+\frac{1}{5!}z^5-\cdots}{z} = 1-\frac{1}{6}z^2+\frac{1}{120}z^4-\cdots$

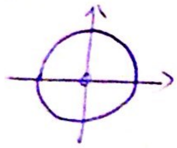
principal part $=0$ $\Rightarrow$ $z=0$ is a removable singularity.

(d). $\frac{\cos z}{z} = \frac{1-\frac{1}{2!}z^2+\frac{1}{4!}z^4-\cdots}{z} = \frac{1}{z}-\frac{1}{2}z+\frac{1}{24}z^3-\cdots$

principal part $=\frac{1}{z}$ $\Rightarrow$ $z=0$ is a pole of order 1.

(e) $\frac{1}{(2-z)^3} = -(z-2)^{-3}$ principal part $=\frac{-1}{(z-2)^3}$ $\Rightarrow$ $z=2$ is a pole of order 3.

P237.2 (a). $f(z)=\dfrac{e^{-z}}{z^2}$. $C: |z|=3$



There is one singularity inside the circle: $z=0$.

$$\frac{e^{-z}}{z^2} = \frac{1-z+\frac{1}{2!}(z)^2-\cdots}{z^2} = \frac{1}{z^2}-\frac{1}{z}+\frac{1}{2}-\cdots \Rightarrow \operatorname*{Res}_{z=0} f(z) = -1$$

$$\Rightarrow \int_{|z|=3}\frac{e^{-z}}{z^2}\,dz = 2\pi i\cdot \operatorname*{Res}_{z=0}\frac{e^{-z}}{z^2} = 2\pi i\cdot(-1) = -2\pi i.$$

(b). $f(z)=\dfrac{e^{-z}}{(z-1)^2}$. There is one singularity $z=1$:

$$\frac{e^{-z}}{(z-1)^2} = \frac{e^{-(z-1)-1}}{(z-1)^2} = e^{-1}\cdot\frac{1-(z-1)+\frac{1}{2!}(-(z-1))^2-\cdots}{(z-1)^2} = \frac{1}{e}\left(\frac{1}{(z-1)^2}-\frac{1}{z-1}+\frac{1}{2}-\cdots\right)$$

$$\Rightarrow \operatorname*{Res}_{z=0}\frac{e^{-z}}{(z-1)^2} = -\frac{1}{e} \Rightarrow \int_{|z|=3}\frac{e^{-z}}{(z-1)^2}\,dz = 2\pi i\cdot\operatorname*{Res}_{z=1}\frac{e^{-z}}{(z-1)^2} = -\frac{2\pi i}{e}.$$

(c). $z^2\cdot e^{\frac{1}{z}}$. Singularity: $z=0$.

$$z^2\cdot e^{\frac{1}{z}} = z^2\left(1+\frac{1}{z}+\frac{1}{2!}\left(\frac{1}{z}\right)^2+\frac{1}{3!}\left(\frac{1}{z}\right)^3+\cdots\right) = z^2+z+\frac{1}{2}+\frac{1}{6}\cdot\frac{1}{z}+\cdots$$

$$\Rightarrow \operatorname*{Res}_{z=0}\left(z^2e^{\frac{1}{z}}\right) = \frac{1}{6} \Rightarrow \int_{|z|=3} z^2e^{\frac{1}{z}}\,dz = 2\pi i\operatorname*{Res}_{z=0}\left(z^2e^{\frac{1}{z}}\right) = \frac{2\pi i}{6} = \frac{\pi i}{3}$$

(d). $\dfrac{z+1}{z^2-2z}$. There are 2 singularities inside the circle: $z_1=0$, $z_2=2$.

around $z=0$: $$\frac{z+1}{z^2-2z} = \frac{1+z}{z(z-2)} = \frac{1}{z}(1+z)\cdot\frac{-1}{2(1-\frac{z}{2})} = \frac{-1}{z}\cdot(1+z)\cdot\frac{1}{2}\cdot\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right)$$

$$= -\frac{1}{2z}\cdot\left(1+\frac{3z}{2}+\cdots\right) \Longrightarrow \operatorname*{Res}_{z=0}\frac{z+1}{z^2-2z} = -\frac{1}{2}$$

Around $z=2$: $\dfrac{z+1}{z^2-2z} = \dfrac{z+1}{z\cdot(z-2)}$. $\dfrac{z+1}{z}$ is analytic ~~around~~ at $z=2$

$$\Rightarrow \operatorname*{Res}_{z=2}\frac{z+1}{z^2-2z} = \left.\frac{z+1}{z}\right]_{z=2} = \frac{3}{2}$$

So. $$\int_{|z|=3}\frac{z+1}{z^2-2z}\,dz = 2\pi i\cdot\left(\operatorname*{Res}_{z=0}\frac{z+1}{z^2-2z}+\operatorname*{Res}_{z=2}\frac{z+1}{z^2-2z}\right) = 2\pi i\left(-\frac{1}{2}+\frac{3}{2}\right) = 2\pi i.$$

the desired residue is to write out a few terms in the Laurent series

$$f(z) = \frac{1}{z^3}\left[1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right)\right] = \frac{1}{z^3}\left(\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \cdots\right)$$

$$= \frac{1}{2!}\cdot\frac{1}{z} - \frac{z}{4!} + \frac{z^3}{6!} - \cdots \qquad (0 < |z| < \infty).$$

This shows that $f(z)$ has a *simple pole* at $z = 0$, not a pole of order 3, the residue at $z = 0$ being $B = 1/2$.

EXAMPLE 5. Since $z^2 \sinh z$ is entire and its zeros are (Sec. 39)

$$z = n\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots),$$

the point $z = 0$ is clearly an isolated singularity of the function

$$f(z) = \frac{1}{z^2 \sinh z}.$$

Here it would be a mistake to write

$$f(z) = \frac{\phi(z)}{z^2} \quad \text{where} \quad \phi(z) = \frac{1}{\sinh z}$$

and try to use the theorem in Sec. 80 with $m = 2$. This is because the function $\phi(z)$ is not even defined at $z = 0$. The needed residue, namely $B = -1/6$, follows at once from the Laurent series

$$\frac{1}{z^2 \sinh z} = \frac{1}{z^3} - \frac{1}{6}\cdot\frac{1}{z} + \frac{7}{360}z + \cdots \qquad (0 < |z| < \pi)$$

that was obtained in Exercise 5, Sec.73. The singularity at $z = 0$ is, of course, a pole of the *third* order, not the second order.

EXERCISES

1. In each case, show that any singular point of the function is a pole. Determine the order m of each pole, and find the corresponding residue B.

(*a*) $\dfrac{z+1}{z^2+9}$; (*b*) $\dfrac{z^2+2}{z-1}$; (*c*) $\left(\dfrac{z}{2z+1}\right)^3$; (*d*) $\dfrac{e^z}{z^2+\pi^2}$.

Ans. (*a*) $m = 1, B = \dfrac{3 \pm i}{6}$; (*b*) $m = 1, B = 3$; (*c*) $m = 3, B = -\dfrac{3}{16}$;

(*d*) $m = 1, B = \pm\dfrac{i}{2\pi}$.

2. Show that

(*a*) $\displaystyle\operatorname*{Res}_{z=-1} \frac{z^{1/4}}{z+1} = \frac{1+i}{\sqrt{2}} \quad (|z| > 0, 0 < \arg z < 2\pi)$;

(*b*) $\displaystyle\operatorname*{Res}_{z=i} \frac{\operatorname{Log} z}{(z^2+1)^2} = \frac{\pi + 2i}{8}$;

(*c*) $\displaystyle\operatorname*{Res}_{z=i} \frac{z^{1/2}}{(z^2+1)^2} = \frac{1-i}{8\sqrt{2}} \quad (|z| > 0, 0 < \arg z < 2\pi)$.

3. In each case, find the order m of the pole and the corresponding residue B at the singularity $z = 0$:

$$(a)\ \frac{\sinh z}{z^4}; \qquad (b)\ \frac{1}{z(e^z - 1)}.$$

Ans. $(a)\ m = 3,\ B = \dfrac{1}{6}$; $(b)\ m = 2,\ B = -\dfrac{1}{2}$.

4. Find the value of the integral

$$\int_C \frac{3z^3 + 2}{(z-1)(z^2+9)}\, dz,$$

taken counterclockwise around the circle $(a)\ |z - 2| = 2$; $(b)\ |z| = 4$.
Ans. $(a)\ \pi i$; $(b)\ 6\pi i$.

5. Find the value of the integral

$$\int_C \frac{dz}{z^3(z+4)},$$

taken counterclockwise around the circle $(a)\ |z| = 2$; $(b)\ |z + 2| = 3$.
Ans. $(a)\ \pi i/32$; $(b)\ 0$.

6. Evaluate the integral

$$\int_C \frac{\cosh \pi z}{z(z^2+1)}\, dz$$

when C is the circle $|z| = 2$, described in the positive sense.
Ans. $4\pi i$.

7. Use the theorem in Sec. 77, involving a single residue, to evaluate the integral of $f(z)$ around the positively oriented circle $|z| = 3$ when

$$(a)\ f(z) = \frac{(3z+2)^2}{z(z-1)(2z+5)}; \qquad (b)\ f(z) = \frac{z^3 e^{1/z}}{1 + z^3}.$$

Ans. $(a)\ 9\pi i$; $(b)\ 2\pi i$.

8. Let z_0 be an isolated singular point of a function f and suppose that

$$f(z) = \frac{\phi(z)}{(z - z_0)^m},$$

where m is a positive integer and $\phi(z)$ is analytic and nonzero at z_0. By applying the extended form (3), Sec. 55, of the Cauchy integral formula to the function $\phi(z)$, show that

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!},$$

as stated in the theorem of Sec. 80.

Suggestion: Since there is a neighborhood $|z - z_0| < \varepsilon$ throughout which $\phi(z)$ is analytic (see Sec. 25), the contour used in the extended Cauchy integral formula can be the positively oriented circle $|z - z_0| = \varepsilon/2$

Next, we show that the integral on the right in equation (3) tends to 0 as R tends to ∞. To do this, we observe that when $R > 1$,

$$|z^6 + 1| \geq ||z|^6 - 1| = R^6 - 1.$$

So, if z is any point on C_R,

$$|f(z)| = \frac{1}{|z^6 + 1|} \leq M_R \quad \text{where} \quad M_R = \frac{1}{R^6 - 1};$$

and this means that

$$\left| \int_{C_R} f(z)\, dz \right| \leq M_R \pi R, \tag{4}$$

πR being the length of the semicircle C_R. (See Sec. 47.) Since the number

$$M_R \pi R = \frac{\pi R}{R^6 - 1}$$

is a quotient of polynomials in R and since the degree of the numerator is less than the degree of the denominator, that quotient must tend to zero as R tends to ∞. More precisely, if we divide both numerator and denominator by R^6 and write

$$M_R \pi R = \frac{\dfrac{\pi}{R^5}}{1 - \dfrac{1}{R^6}},$$

it is evident that $M_R \pi R$ tends to zero. Consequently, in view of inequality (4),

$$\lim_{R \to \infty} \int_{C_R} f(z)\, dz = 0.$$

It now follows from equation (3) that

$$\lim_{R \to \infty} \int_{-R}^{R} \frac{dx}{x^6 + 1} = \frac{2\pi}{3},$$

or

$$\text{P.V.} \int_{-R}^{R} \frac{dx}{x^6 + 1} = \frac{2\pi}{3}.$$

Since the integrand here is even, we know from equation (7) in Sec. 85 that

$$\int_{0}^{\infty} \frac{dx}{x^6 + 1} = \frac{\pi}{3}. \tag{5}$$

EXERCISES

Use residues to derive the integration formulas in Exercises 1 through 6.

1. $\displaystyle\int_0^\infty \frac{dx}{x^2 + 1} = \frac{\pi}{2}.$

2. $\displaystyle\int_0^\infty \frac{dx}{(x^2 + 1)^2} = \frac{\pi}{4}.$

3. $\displaystyle\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}.$

4. $\displaystyle\int_0^\infty \frac{x^2\,dx}{x^6+1} = \frac{\pi}{6}.$

5. $\displaystyle\int_0^\infty \frac{x^2\,dx}{(x^2+1)(x^2+4)} = \frac{\pi}{6}.$

6. $\displaystyle\int_0^\infty \frac{x^2\,dx}{(x^2+9)(x^2+4)^2} = \frac{\pi}{200}.$

Use residues to find the Cauchy principal values of the integrals in Exercises 7 and 8.

7. $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^2+2x+2}.$

8. $\displaystyle\int_{-\infty}^\infty \frac{x\,dx}{(x^2+1)(x^2+2x+2)}.$

   Ans. $-\pi/5$.

9. Use a residue and the contour shown in Fig. 101, where $R > 1$, to establish the integration formula

$$\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi}{3\sqrt{3}}.$$

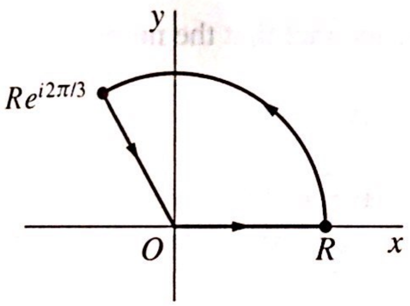


FIGURE 101

10. Let m and n be integers, where $0 \le m < n$. Follow the steps below to derive the integration formula

$$\int_0^\infty \frac{x^{2m}}{x^{2n}+1}\,dx = \frac{\pi}{2n}\csc\left(\frac{2m+1}{2n}\pi\right).$$

(*a*) Show that the zeros of the polynomial $z^{2n}+1$ lying above the real axis are

$$c_k = \exp\left[i\frac{(2k+1)\pi}{2n}\right] \qquad (k = 0, 1, 2, \ldots, n-1)$$

and that there are none on that axis.

(*b*) With the aid of Theorem 2 in Sec. 83, show that

$$\operatorname*{Res}_{z=c_k} \frac{z^{2m}}{z^{2n}+1} = -\frac{1}{2n}e^{i(2k+1)\alpha} \qquad (k = 0, 1, 2, \ldots, n-1)$$

EXERCISES

Use residues to derive the integration formulas in Exercises 1 through 5.

1. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right) \quad (a > b > 0).$

2. $\displaystyle\int_{0}^{\infty} \frac{\cos ax}{x^2+1}\,dx = \frac{\pi}{2}e^{-a} \quad (a > 0).$

3. $\displaystyle\int_{0}^{\infty} \frac{\cos ax}{(x^2+b^2)^2}\,dx = \frac{\pi}{4b^3}(1+ab)e^{-ab} \quad (a > 0, b > 0).$

4. $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin ax}{x^4+4}\,dx = \frac{\pi}{2}e^{-a}\sin a \quad (a > 0).$

5. $\displaystyle\int_{-\infty}^{\infty} \frac{x^3\sin ax}{x^4+4}\,dx = \pi e^{-a}\cos a \quad (a > 0).$

Use residues to evaluate the integrals in Exercises 6 and 7.

6. $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin x\,dx}{(x^2+1)(x^2+4)}.$

7. $\displaystyle\int_{0}^{\infty} \frac{x^3\sin x\,dx}{(x^2+1)(x^2+9)}.$

Use residues to find the Cauchy principal values of the improper integrals in Exercises 8 through 11.

8. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x\,dx}{x^2+4x+5}.$

   Ans. $-\dfrac{\pi}{e}\sin 2.$

9. $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin x\,dx}{x^2+2x+2}$

   Ans. $\dfrac{\pi}{e}(\sin 1 + \cos 1).$

10. $\displaystyle\int_{-\infty}^{\infty} \frac{(x+1)\cos x}{x^2+4x+5}\,dx.$

    Ans. $\dfrac{\pi}{e}(\sin 2 - \cos 2).$

11. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x+a)^2+b^2} \quad (b > 0).$

12. Follow the steps below to evaluate the ***Fresnel integrals***, which are important in diffraction theory:

$$\int_0^{\infty} \cos(x^2)\,dx = \int_0^{\infty} \sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

P246. 1 (a) $\frac{z+1}{z^2+9} = \frac{z+1}{(z+3i)(z-3i)}$ poles: $\pm 3i$ order $=1$.

$$\operatorname*{Res}_{z=3i} f(z) = \frac{z+1}{z+3i}\Big|_{3i} = \frac{3i+1}{6i} = \frac{3-i}{6}$$

$$\operatorname*{Res}_{z=-3i} f(z) = \frac{z+1}{z-3i}\Big|_{-3i} = \frac{-3i+1}{-6i} = \frac{3+i}{6}.$$

(b) $\frac{z^2+2}{z-1}$ pole: $z=1$. order $m=1$

$$\operatorname*{Res}_{z=1} \frac{z^2+2}{z-1} = z^2+2\Big]_{z=1} = 3.$$

(c) $\left(\frac{z}{2z+1}\right)^3$ pole: $z=-\frac{1}{2}$. order $m=3$. $f(z) = \frac{\left(\frac{z}{2}\right)^3}{\left(z+\frac{1}{2}\right)^3}$

$$\Rightarrow \operatorname*{Res}_{z=-\frac{1}{2}} f = \frac{1}{2!}\frac{d^2}{dz^2}\left(\frac{z}{2}\right)^3\Big]_{z=-\frac{1}{2}} = \frac{1}{16}\cdot 6z\Big|_{z=-\frac{1}{2}} = -\frac{3}{16}.$$

(d) $\frac{e^z}{z^2+\pi^2} = \frac{e^z}{(z+\pi i)(z-\pi i)}$ poles: $z=\pm\pi i$.

$$\operatorname*{Res}_{z=\pi i} f(z) = \frac{e^z}{z+\pi i}\Big|_{z=\pi i} = \frac{e^{\pi i}}{2\pi i} = \frac{-1}{2\pi i} = \frac{i}{2\pi}$$

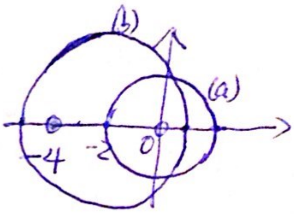
$$\operatorname*{Res}_{z=-\pi i} f(z) = \frac{e^z}{z-\pi i}\Big|_{z=-\pi i} = \frac{e^{-\pi i}}{-2\pi i} = \frac{-1}{-2\pi i} = \frac{-i}{2\pi}.$$

P247. 5. $\int_C \frac{dz}{z^3(z+4)}$ $f(z)=\frac{1}{z^3(z+4)}$ poles 0 -4

order 3 1

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{2!}\frac{d^2}{dz^2}\left(\frac{1}{z+4}\right)\Big|_{z=0} = \frac{1}{2}\cdot(-1)\cdot(-2)\cdot\frac{1}{(z+4)^3}\Big|_{z=0} = \frac{1}{64}$$

$$\operatorname*{Res}_{z=-4} f(z) = \frac{1}{z^3}\Big|_{z=-4} = -\frac{1}{64}.$$



(a) $\int_C f(z)\,dz = 2\pi i\cdot\operatorname*{Res}_{z=0} f(z) = \frac{2\pi i}{64} = \frac{\pi i}{32}$

(b) $\int_C f(z)\,dz = 2\pi i\left(\operatorname*{Res}_{z=0} + \operatorname*{Res}_{z=-4}\right) = 2\pi i\cdot\left(\frac{1}{64}-\frac{1}{64}\right) = 0.$

7. (a). $f(z) = \frac{(3z+2)^2}{z(z-1)(2z+5)}$. singularities: $0, 1, -\frac{5}{2}$ all inside $|z|=3$.

$$\frac{1}{z^2}f\left(\frac{1}{z}\right) = \frac{1}{z^2}\cdot\frac{\left(\frac{3}{z}+2\right)^2}{\frac{1}{z}\left(\frac{1}{z}-1\right)\left(\frac{2}{z}+5\right)} = \frac{1}{z^2}\frac{\frac{(3+2z)^2}{z^2}}{\frac{(1-z)(2+5z)}{z^3}} = \frac{(3+2z)^2}{(1-z)(2+5z)z}.$$

$z=0$: pole of order 1.

$$\Rightarrow \operatorname*{Res}_{z=0}\frac{1}{z^2}f\left(\frac{1}{z}\right) = \frac{(3+2z)^2}{(1-z)(2+5z)}\Big|_{z=0} = \frac{9}{2}. = -\operatorname*{Res}_{z=\infty} f(z)$$

$$\Rightarrow \int_{|z|=3} f(z)\,dz = -2\pi i\cdot\operatorname*{Res}_{z=\infty} f(z) = 2\pi i\cdot\frac{9}{2} = 9\pi i.$$

(b). $f(z) = \frac{z^3 e^{\frac{1}{z}}}{1+z^3}$. $\frac{1}{z^2}f\left(\frac{1}{z}\right) = \frac{1}{z^2}\frac{\frac{1}{z^3}e^z}{1+\left(\frac{1}{z}\right)^3} = \frac{e^z}{(z^3+1)z^2}$ $z=0$: pole of order 2.

$$\Rightarrow \operatorname*{Res}_{z=0}\frac{1}{z^2}f\left(\frac{1}{z}\right) = \frac{d}{dz}\left(\frac{e^z}{z^3+1}\right)\Big|_{z=0} = \frac{e^z(z^3+1)-e^z\cdot 3z^2}{(z^3+1)^2}\Big|_{z=0} = 1 \Rightarrow \operatorname*{Res}_{z=\infty} f(z) = -1.$$

$$\Rightarrow \int_{|z|=3} f(z)\,dz \underset{\uparrow}{=} 2\pi i\cdot\left(-\operatorname*{Res}_{z=\infty} f(z)\right) = 2\pi i.$$

singularities $z=(-1)^{\frac{1}{3}}$ are contained inside $|z|=3$.

P264. 3. $\int_0^{\infty}\frac{dx}{x^4+1} = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{dx}{x^4+1} = \frac{1}{2}\lim_{R\to+\infty}\int_{-R}^{R}\frac{dx}{x^4+1}.$

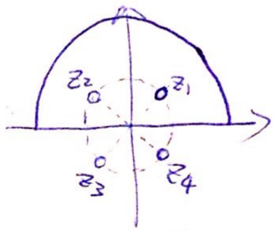
$$\underset{\downarrow R\to\infty \atop \int_{-\infty}^{+\infty}\frac{dx}{x^4+1}}{\int_{-R}^{R}\frac{dx}{x^4+1}} + \underset{\downarrow \atop 0}{\int_{C_R}\frac{dz}{z^4+1}} = 2\pi i\cdot\left(\operatorname*{Res}_{z=z_1} + \operatorname*{Res}_{z=z_2}\right)$$

$z_1 = e^{\frac{i\pi}{4}} = \frac{1}{\sqrt{2}}(1+i)$

$z_2 = e^{\frac{3\pi i}{4}} = \frac{1}{\sqrt{2}}(1-i).$

$$\operatorname*{Res}_{z=e^{\frac{i\pi}{4}}} f(z) = \frac{1}{(z^2+i)(z-e^{\frac{5\pi}{4}i})}\Bigg|_{z=e^{\frac{i\pi}{4}}} = \frac{1}{2i\cdot\sqrt{2}(1+i)} = \frac{1}{2\sqrt{2}\cdot(-1+i)}$$

$$= \frac{-1-i}{2\sqrt{2}\cdot(-1+i)(-1-i)} = \frac{-1-i}{4\sqrt{2}}$$



$z_3 = e^{\frac{i5\pi}{4}} = \frac{1}{\sqrt{2}}(-1-i)$

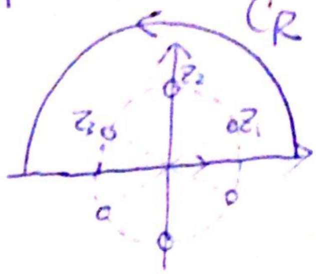
$z_4 = e^{\frac{-i\pi}{4}} = \frac{1}{\sqrt{2}}(1-i).$

$$\operatorname*{Res}_{z=e^{\frac{3\pi}{4}}} f(z) = \frac{1}{(z^2-i)\cdot(z-e^{-\frac{i\pi}{4}})}\Bigg|_{z=e^{\frac{3\pi}{4}}} = \frac{1}{(-2i)\cdot(+\sqrt{2})(-1+i)} = \frac{1}{2\sqrt{2}\cdot(1+i)} = \frac{1-i}{4\sqrt{2}}$$

$$\Rightarrow \int_{-\infty}^{+\infty}\frac{dx}{x^4+1} = 2\pi i\cdot\left(\frac{(-1-i)+(1-i)}{4\sqrt{2}}\right) = 2\pi i\cdot\frac{-2i}{4\sqrt{2}} = \frac{\pi}{\sqrt{2}}$$

$$\Rightarrow \int_0^{\infty}\frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}.$$

4. $\displaystyle\int_0^{\infty}\frac{x^2dx}{x^6+1}=\frac{1}{2}\int_{-\infty}^{+\infty}\frac{x^2dx}{x^6+1}$ let $f(z)=\dfrac{z^2}{z^6+1}$



$$\operatorname*{Res}_{z=z_k} f(z) \overset{\text{Thm 2 on pg. 251}}{=\!=\!=} \frac{z^2}{6z^5}\Big|_{z=z_k}=\frac{z_k^2\cdot z_k}{6z_k^5 z_k}=\frac{z_k^3}{-6}$$

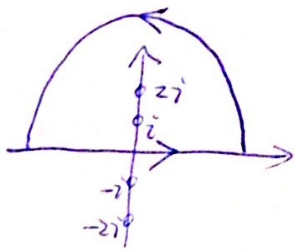
$$\sum_{k=1}^{3}\operatorname*{Res}_{z=z_k} f(z)=-\frac{1}{6}\left((e^{\frac{i\pi}{6}})^3+(e^{\frac{i\pi}{2}})^3+(e^{\frac{5\pi i}{6}})^3\right)=-\frac{1}{6}\left(e^{\frac{i\pi}{2}}+e^{\frac{3i\pi}{2}}+e^{\frac{5\pi i}{2}}\right)$$

$$=-\frac{1}{6}(i+(-i)+i)=-\frac{i}{6}.$$

$$\Rightarrow \int_{-\infty}^{+\infty}\frac{x^2dx}{x^6+1}=2\pi i\cdot\sum_{k=1}^{3}\operatorname*{Res}_{z=z_k} f(z)=2\pi i\cdot\left(-\frac{i}{6}\right)=\frac{\pi}{3}$$

$$\Rightarrow \int_0^{\infty}\frac{x^2dx}{x^6+1}=\frac{\pi}{6}.$$

5. $\displaystyle\int_0^{\infty}\frac{x^2dx}{(x^2+1)(x^2+4)}=\frac{1}{2}\int_{-\infty}^{+\infty}\frac{x^2dx}{(x^2+1)(x^2+4)}$ $\quad f(z)=\dfrac{z^2}{(z^2+1)(z^2+4)}$



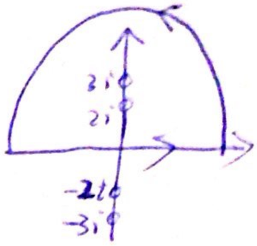
$$\operatorname*{Res}_{z=i} f(z)=\frac{z^2}{(z^2+4)(z+i)}\Big|_{z=i}=\frac{-1}{3\cdot 2i}=\frac{i}{6}$$

$$\operatorname*{Res}_{z=2i} f(z)=\frac{z^2}{(z^2+1)(z+2i)}\Big|_{z=2i}=\frac{-4}{(-3)\cdot 4i}=\frac{1}{3i}=-\frac{i}{3}$$

$$\Rightarrow \int_0^{\infty}\frac{x^2dx}{(x^2+1)(x^2+4)}=\frac{1}{2}2\pi i\left(\operatorname*{Res}_{z=i}+\operatorname*{Res}_{z=2i}\right)=\pi i\cdot\left(\frac{i}{6}-\frac{i}{3}\right)=\pi i\cdot\left(-\frac{i}{6}\right)=\frac{\pi}{6}.$$

6. $\int_0^{\infty} \frac{x^2\,dx}{(x^2+9)(x^2+4)^2} = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x^2\,dx}{(x^2+9)(x^2+4)^2}$. $\quad f(z)=\frac{z^2}{(z^2+9)(z^2+4)^2}$

$$\operatorname*{Res}_{z=2i} f(z) = \frac{d}{dz}\left(\frac{z^2}{(z^2+9)\cdot(z+2i)^2}\right)\Bigg|_{z=2i}$$



$$= \left[\frac{2z}{(z^2+9)(z+2i)^2} - \frac{z^2\cdot 2z}{(z^2+9)^2(z+2i)^2} - 2\cdot\frac{z^2\cdot}{(z^2+9)(z+2i)^3}\right]\Bigg|_{z=2i}$$

$$= \frac{4i}{5\times(-16)} - \frac{-8i\times 2}{5^2\cdot(-16)} - 2\cdot\frac{-4}{5\times(-64i)}$$

$$= -\frac{i}{20} - \frac{i}{25} + \frac{i}{40} = \frac{i}{200}(-10-8+5) = -\frac{13i}{200}$$

$$\operatorname*{Res}_{z=3i} f(z) = \frac{z^2}{(z^2+4)^2(z+3i)}\Bigg|_{z=3i} = \frac{-9}{(-5)^2\times 6i} = \frac{3i}{50}$$

$$\Rightarrow \int_0^{\infty} \frac{x^2\,dx}{(x^2+9)(x^2+4)^2} = \frac{1}{2}2\pi i\left(\operatorname*{Res}_{z=2i} f(z) + \operatorname*{Res}_{z=3i} f(z)\right) = \pi i\cdot\left(-\frac{13}{200}+\frac{12}{200}\right)i$$

$$= \frac{\pi}{200}.$$

7. $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$ $\qquad f(z)=\frac{1}{z^2+2z+2}.$ $\qquad$ Singularity: $z^2+2z+2=0$

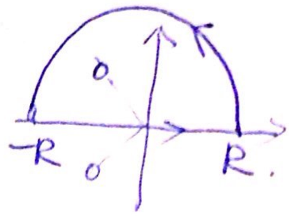
$$\Downarrow$$

$$z=\frac{-2\pm\sqrt{2^2-8}}{2}$$

$$=-1\pm i$$

$$\operatorname*{Res}_{z=-1+i} f(z)= \left.\frac{1}{z-(-1-i)}\right|_{z=-1+i} = \frac{1}{2i} = -\frac{i}{2}.$$

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{dx}{x^2+2x+2} = 2\pi i \cdot \operatorname*{Res}_{z=-1+i} f(z) = 2\pi i \cdot \frac{1}{2i} = \pi.$$



8. $\int_{-\infty}^{+\infty} \frac{x\,dx}{(x^2+1)(x^2+2x+2)}$ $\qquad f(z)=\frac{z}{(z^2+1)(z^2+2z+2)}$

Singularities

pole	$\pm i$	$-1\pm 2i$
order	1	1

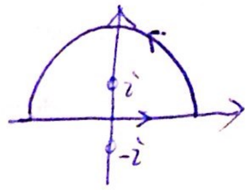
$$\operatorname*{Res}_{z=i} f(z)= \left.\frac{z}{(z+i)(z^2+2z+2)}\right|_{z=i} = \frac{i}{(2i)\cdot(-1+2i+2)} = \frac{1-2i}{2\cdot(1+2i)(1-2i)} = \frac{1-2i}{10}$$

$$\operatorname*{Res}_{z=-1+i} f(z)= \left.\frac{z}{(z^2+1)(z-(-1-i))}\right|_{z=-1+i} = \frac{-1+i}{(1-2i)\cdot(2i)} = \frac{1}{2}\cdot\frac{(-1+i)\cdot(2-i)}{(2+i)(2-i)} = \frac{-1+3i}{10}$$

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{x\,dx}{(x^2+1)(x^2+2x+2)} = 2\pi i\left(\frac{1-2i}{10}+\frac{-1+3i}{10}\right) = 2\pi i\cdot\frac{i}{10} = -\frac{\pi}{5}$$

P273. 2. $\int_0^{\infty} \frac{\cos(ax)}{x^2+1}dx$ $f(z) = \frac{e^{iaz}}{z^2+1} = \frac{e^{ia(x+iy)}}{z^2+1} = \frac{e^{-ay} \cdot e^{iax}}{z^2+1}$

$\int_{-\infty}^{+\infty} \frac{\cos(ax)+i\sin(ax)}{x^2+1}dx = 2\pi i \cdot \operatorname{Res}_{z=i} f(z).$



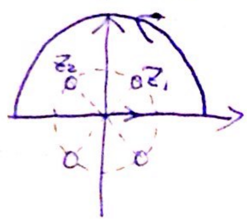
$= 2\pi i \cdot \left.\frac{e^{iaz}}{z+i}\right|_{z=i} = 2\pi i \cdot \frac{e^{-a}}{2i} = \pi \cdot e^{-a}$

$\Rightarrow \int_0^{\infty} \frac{\cos(ax)}{x^2+1}dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\cos(ax)}{x^2+1}dx = \frac{\pi}{2}e^{-a}.$

4. $\int_{-\infty}^{+\infty} \frac{x \cdot \sin(ax)}{x^4+4}dx.$ $f(z) = \frac{z \cdot e^{iaz}}{z^4+4} = \frac{z}{z^4+4} \cdot e^{-ay}(\cos(ax)+i\sin(ax))$

singularities are simple poles.

$\int_{-\infty}^{+\infty} \frac{xe^{+iax}}{x^4+4}dx = 2\pi i\left(\operatorname{Res}_{z=z_1} f(z) + \operatorname{Res}_{z=z_2} f(z)\right)$



$\operatorname{Res}_{z=z_i} \frac{z \cdot e^{iaz}}{z^4+4} = \left.\frac{z \cdot e^{iaz}}{4z^3}\right|_{z=z_i} = \frac{e^{iaz_i}}{4z_i^2}$

$z_1 = (1+i)$
$z_2 = (-1+i)$

$\operatorname{Res}_{z=z_1} f = \frac{e^{ia(1+i)}}{4 \cdot 2i} = \frac{e^{-a} \cdot e^{ia}}{8i}$

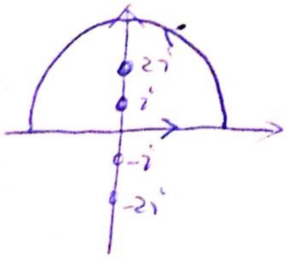
$\operatorname{Res}_{z=z_2} f = \frac{e^{ia(-1+i)}}{4 \cdot (-2i)} = -\frac{e^{-a} \cdot e^{-ia}}{8i}$

$\Rightarrow \operatorname{Res}_{z=z_1} + \operatorname{Res}_{z=z_2} = \frac{1}{8i}e^{-a} \cdot (e^{ia} - e^{-ia}) = \frac{e^{-a}}{4}\sin(a).$

$\Rightarrow \int_{-\infty}^{+\infty} \frac{x(\cos(ax)+i\sin(ax))dx}{x^4+4} = 2\pi i \cdot \frac{e^{-a}}{4}\sin(a)$

$\Rightarrow \int_{-\infty}^{+\infty} \frac{x \cdot \sin(ax)}{x^4+4}dx = \frac{\pi}{2}e^{-a}\sin(a).$

6. $\int_{-\infty}^{+\infty} \frac{x\cdot\sin x\, dx}{(x^2+1)(x^2+4)}$ $\qquad f(z) = \frac{z\cdot e^{iz}}{(z^2+1)(z^2+4)}$.



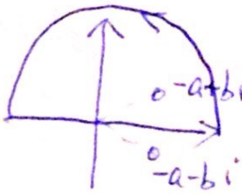
$$\int_{-\infty}^{+\infty} \frac{x\cdot e^{ix} dx}{(x^2+1)(x^2+4)} = 2\pi i\cdot\left(\operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=2i} f(z)\right) = 2\pi i\left(\frac{1}{6e} - \frac{1}{6e^2}\right)$$

$$\operatorname*{Res}_{z=i} f(z) = \left.\frac{z e^{iz}}{(z^2+4)(z+i)}\right|_{z=i} = \frac{i\cdot e^{-1}}{3\cdot 2i} = \frac{1}{6e}.$$

$$\operatorname*{Res}_{z=2i} f(z) = \frac{z\cdot e^{iz}}{(z^2+1)(z+2i)} = \frac{2i\cdot e^{-2}}{(-3)\cdot 4i} = -\frac{1}{6e^2}$$

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{x\cdot\sin x\, dx}{(x^2+1)(x^2+4)} = \frac{2\pi}{6e^2}(e-1) = \frac{\pi}{3e^2}(e-1).$$

11. $\int_{-\infty}^{+\infty} \frac{\cos x\cdot dx}{(x+a)^2+b^2}$ $\qquad f(z) = \frac{e^{iz}}{(z+a)^2+b^2}$



$(b>0)$

$$\operatorname*{Res}_{z=-a+bi} f(z) = \left.\frac{e^{iz}}{z-(-a-bi)}\right|_{-a+bi} = \frac{e^{i(-a+bi)}}{2bi} = \frac{e^{-b}e^{-ia}}{2bi}$$

~~Res z=b~~

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{e^{ix} dx}{(x+a)^2+b^2} = 2\pi i\cdot\operatorname*{Res}_{z=-a+bi} f(z) = 2\pi i\cdot\frac{e^{-b}e^{-ia}}{2bi} = \frac{\pi e^{-b}}{b}\cdot e^{-ia}$$

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{\cos x\, dx}{(x+a)^2+b^2} = \frac{\pi e^{-b}}{b}\cos(a) = \frac{\pi\cdot\cos(a)}{b\cdot e^b}$$

$$\int_{-\infty}^{+\infty} \frac{\sin x\, dx}{(x+a)^2+b^2} = \frac{\pi\cdot e^{-b}}{b}\cdot(-\sin(a)) = -\frac{\pi\cdot\sin(a)}{b\cdot e^b}$$

MAT 342 FALL 2014 PRACTICE FINAL EXAM

NAME : ID :

THERE ARE NINE (9) PROBLEMS. THEY HAVE THE INDICATED VALUE.

SHOW YOUR WORK

DO NOT TEAR-OFF ANY PAGE

NO CALCULATORS NO CELLS ETC.

ON YOUR DESK: ONLY test, pen, pencil, eraser.

1		50pts
2		50pts
3		50pts
4		50pts
5		50pts
6		50pts
7		50pts
8		50pts
9		50pts
Total		450pts

!!! WRITE YOUR NAME, STUDENT ID. BELOW !!!

NAME : ID :

1. (50pts) (a): Find complex numbers z such that $e^{1/z} = 2(1-i)$.
(b): Solve for z such that $\cos(z) = 2$.

2. (50pts)

(a): What's the image D_2 of the region $D_1 = \{z \in \mathbb{C}; 0 < \mathrm{Re}(z) < \pi\}$ under the map $w = iz$?

(b): What's the image of the region D_2 (from above) under the map $w = e^z$?

3. (50pts)

(a): Suppose $f(z) = u + iv$ is analytic. If we know that $u(z) = x^3 - 3xy^2$, what equations does v satisfy? Solve them to get $v = v(z)$.

(b): Assume that f is an entire function. If there is an analytic function $g(z)$ satisfying $f(z) = e^{g(z)}$, show that $f(z)$ has no zero point. Calculate $g'(z)$ in terms of the function $f(z)$. Reversely if $f(z)$ has no zero point on $\mathbb{C}$, does there exist such a $g(z)$?

4. (50pts) Calculate the following contour integrals.
(a):

$$\int_{|z|=3} \frac{\cos(z)}{z^5} dz.$$

(b):

$$\int_0^{2\pi i} \frac{1}{\cos^2(z)} dz$$

along any path from 0 to $2\pi i$.
(c):

$$\int_{|z|=3} \bar{z} dz.$$

5. (50pts) (a): Find the Taylor series of the following function centered at 0.

$$\frac{z}{(z-2)^2}.$$

What's the radius of convergence?
(b): Find the Taylor series of the above function centered at 1. What's the convergence of radius?

6. (50pts) Find the Laurent series centered at 0 of the following function in the given region.

$$\frac{2}{(z-1)^2(z-2)}$$

$$(a)|z| < 1 \quad (b)1 < |z| < 2 \quad (c)|z| > 2$$

7. (50pts) Calculate the contour integrals using residues:
(a):

$$\int_{|z|=3} \frac{2}{(z-1)^2(z-2)} dz.$$

(b):

$$\int_{|z|=10} \frac{z^9}{z^5+1} dz.$$

8. (50pts) Classify the isolated singularities and calculate their residues:

(a) $\dfrac{\mathrm{Log} z}{z-1}$ at $z = 1$.

(b) $\cos(1/z)$ at $z = 0$.

(c) $\dfrac{\sin(z)}{(z-\pi)^4}$ at $z = \pi$.

9. (50pts) Calculate the following integrals (a):

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx.$$

(b):

$$\int_{0}^{\infty} \frac{\cos(2x)}{(x^2+1)(x^2+4)} dx.$$

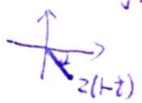
(c):

$$\int_{0}^{2\pi} \frac{d\theta}{3+2\cos\theta}.$$

Scratch paper

1. (a) $e^{\frac{1}{z}} = 2(1-i)$.

$2(1-i) = 2\sqrt{2} \cdot e^{-i\frac{\pi}{4}} \Rightarrow \frac{1}{z} = \log(2(1-i)) = \ln 2\sqrt{2} + i\left(-\frac{\pi}{4} + 2\pi \cdot n\right)$



$= \frac{3}{2}\ln 2 + i\left(-\frac{\pi}{4} + 2\pi \cdot n\right)$ $n = 0, \pm 1, \pm 2, \dots$

$\Rightarrow z = \dfrac{1}{\frac{3}{2}\ln 2 + i\left(-\frac{\pi}{4} + 2\pi \cdot n\right)}$ $n = 0, \pm 1, \pm 2, \dots$

(b) $\cos(z) = 2$. $\cos(z) = \dfrac{e^{iz} + e^{-iz}}{2}$

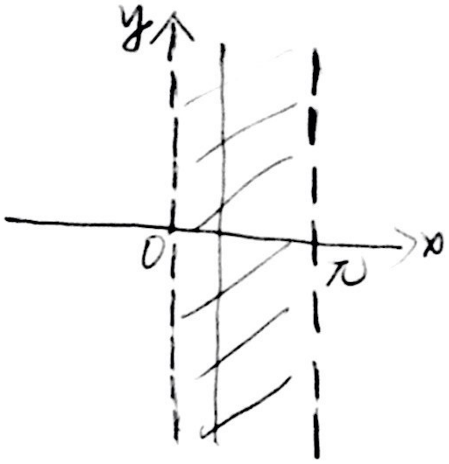
$\Rightarrow \dfrac{e^{iz} + e^{-iz}}{2} = 2 \Rightarrow e^{iz} - 4 + e^{-iz} = 0$. let $e^{iz} = w$, then $e^{-iz} = w^{-1}$

$\Rightarrow w^2 - 4w + 1 = 0 \Rightarrow w = \dfrac{4 \pm \sqrt{4^2 - 4}}{2} = \dfrac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3} > 0$.

$\Rightarrow iz = \log w = \ln(2 \pm \sqrt{3}) + i(0 + 2\pi \cdot n)$ $n = 0, \pm 1, \pm 2, \dots$

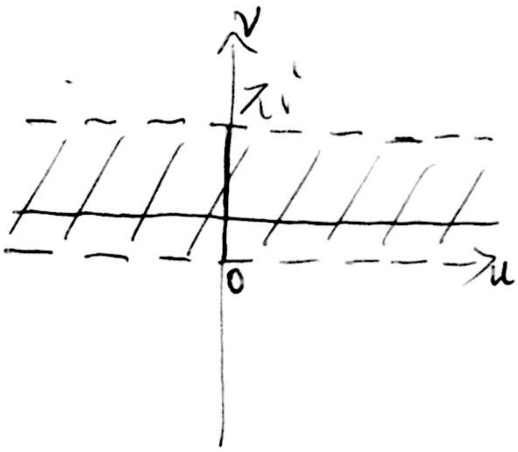
$\Rightarrow z = -i \cdot \ln(2 \pm \sqrt{3}) + 2\pi \cdot n = \pm i \ln(2 + \sqrt{3}) + 2\pi n$. $n = 0, \pm 1, \pm 2, \dots$.

$\left(\ln(2 - \sqrt{3}) = -\ln(2 + \sqrt{3})\right)$.



$\downarrow iz$

rotation by 90° degrees

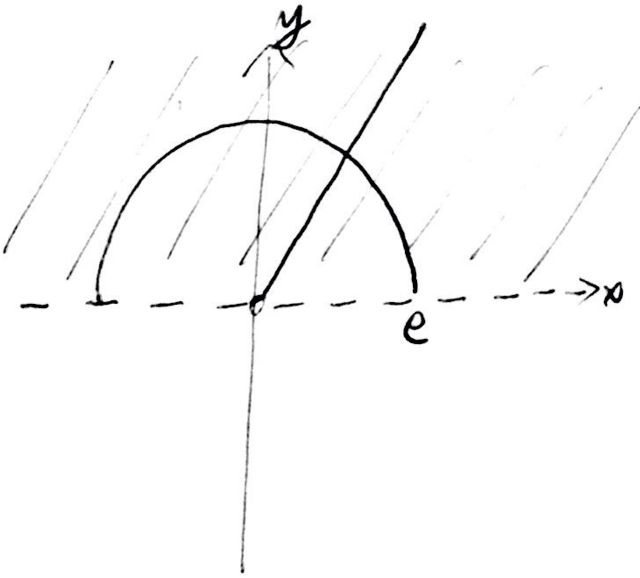


$\downarrow$ $e^{w} = e^{u} \cdot e^{iv}$

horizontal line → ray

vertical line → semicircle

$0 < \underbrace{\mathrm{Arg}(e^{w})}_{= v} < \pi$



open upper half plane

3. (a) $f(z) = u + iv$ analytic $\Rightarrow$ Cauchy-Riemann equations $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

$u = x^3 - 3xy^2 \Longrightarrow \begin{cases} v_y = 3x^2 - 3y^2 \quad ① \Rightarrow v(x,y) = 3x^2y - y^3 + C(x) \\ v_x = -(u_y) = -(-6xy) = 6xy \quad ② \end{cases}$

$\Downarrow$

$v_x = 6xy + C'(x) \overset{②}{=} 6xy$

$\Downarrow$

$C'(x) = 0 \Rightarrow C(x) = \text{constant}$

$\Rightarrow v(x,y) = 3x^2y - y^3 + \text{constant}$. $\left(\Rightarrow f(z) = (x^3 - 3xy^2) + i(3x^2y - y^3) + \text{const}) = z^3 + \text{constant} \right)$

(b). If $g(x,y) = u(z) + iv(z)$, then $e^{g(z)} = e^{u(z)} \cdot e^{iv(z)}$. $\Rightarrow f(z) \neq 0$.

$g(x,y) \overset{\shortparallel}{=} g(z)$, $|e^{g(z)}| = e^{u(z)} \neq 0$

$f(z) = e^{g(z)} \Rightarrow f'(z) = e^{g(z)} \cdot g'(z) = f(z) \cdot g'(z) \Rightarrow \boxed{g'(z) = \frac{f'(z)}{f(z)}}$

If $f(z)$ has no zero point in $\mathbb{C}$, then $\frac{f'(z)}{f(z)}$ is an entire (analytic) function

Define $\tilde{g}(z) = \int_0^z \frac{f'(w)}{f(w)} dw$ (along any path from 0 to z)

then $\tilde{g}'(z) = \frac{f'(z)}{f(z)} = (\log f(z))'$ locally. So the analytic function $\tilde{g}(z)$ differs from $\log f(z)$. (any branch) by a constant that is equal to $\log f(0) - \tilde{g}(0)$ ($\overset{\shortparallel}{=} \log f(0)$).

so $\boxed{g(z) = \tilde{g}(z) + \log f(0) = \int_0^z \frac{f'(w)}{f(w)} dw + \text{Log} f(0) \ (+(2\pi n)i) \quad \text{any } n \in \mathbb{Z}}$

4.(a). $\int_{|z|=3} \frac{\cos(z)}{z^5} = \frac{2\pi i}{4!} \frac{d^4}{dz^4} \cos z \Big|_{z=0} = \frac{2\pi i}{24} \frac{d^2}{dz^2}(-\cos z)\Big|_{z=0}$

$= \frac{\pi i}{12} \cdot \cos z \Big|_{z=0} = \frac{\pi i}{12}$ by Cauchy's formula for derivatives (or Residue Theorem).

(b). $\frac{1}{\cos^2 z} = \sec^2 z$ has an antiderivative $\tan(z)$:

$(\tan(z))' = \left(\frac{\sin z}{\cos z}\right)' = \frac{\cos^2 z + \sin^2 z}{\cos^2 z} = \frac{1}{\cos^2 z}$

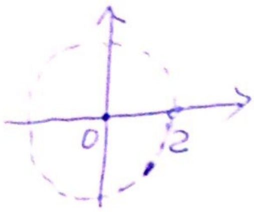
$\Rightarrow \int_0^{2\pi i} \frac{1}{\cos^2 z} dz = \tan(z)\Big|_0^{2\pi i} = \frac{\sin(2\pi i)}{\cos(2\pi i)} - \frac{\sin(0)}{\cos(0)} = \frac{\frac{e^{i(2\pi i)} - e^{-i(2\pi i)}}{2i}}{\frac{e^{i(2\pi i)} + e^{-i(2\pi i)}}{2}} - 0$

$= \frac{\frac{e^{-2\pi} - e^{2\pi}}{2i}}{e^{-2\pi} + e^{2\pi}} = i \cdot \frac{e^{2\pi} - e^{-2\pi}}{e^{2\pi} - e^{-2\pi}} \left(= i \tanh(2\pi)\right).$

(c). $|z|=3$: $z = 3e^{i\theta}$ $0 \le \theta \le 2\pi$. $z'(\theta) = 3e^{i\theta} \cdot i d\theta$.

$\Rightarrow \int_{|z|=3} \bar{z}\, dz = \int_0^{2\pi} 3e^{-i\theta} 3e^{i\theta}\, i d\theta = 9\int_0^{2\pi} i d\theta = 18\pi i.$

5. (a). $\frac{z}{(z-2)^2} = z \cdot \frac{1}{(z-2)^2}$



$$\frac{1}{z-2} = -\frac{1}{2(1-\frac{z}{2})} = -\frac{1}{2} \cdot \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$\Rightarrow -\frac{1}{(z-2)^2} = \frac{d}{dz}\left(-\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}\right) = -\sum_{n=1}^{\infty} \frac{n \cdot z^{n-1}}{2^{n+1}} = -\sum_{n=0}^{\infty} \frac{(n+1) z^n}{2^{n+2}}$$

$$\frac{d}{dz}\left(\frac{1}{z-2}\right)$$

$$\Rightarrow \frac{z}{(z-2)^2} = z \cdot \sum_{n=0}^{\infty} \frac{(n+1) z^n}{2^{n+2}} = \sum_{n=0}^{\infty} \frac{(n+1) \cdot z^{n+1}}{2^{n+2}} = \sum_{n=1}^{\infty} \frac{n \cdot z^n}{2^{n+1}}.$$

$$\lim_{n\to\infty} \left|\frac{C_{n+1}}{C_n}\right| = \lim_{n\to\infty} \frac{\frac{(n+1)}{2^{n+2}}}{\frac{n}{2^{n+1}}} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2} \Rightarrow \text{Radius of convergence} = \left(\frac{1}{2}\right)^{-1} = 2.$$

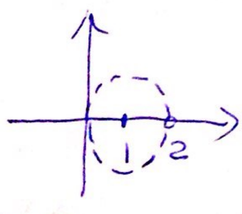
(b). $\frac{z}{(z-2)^2} = \frac{(z-1)+1}{(z-2)^2} = (z-1) \cdot \frac{1}{(z-2)^2} + \frac{1}{(z-2)^2}.$

$$\frac{1}{z-2} = \frac{1}{(z-1)-1} = -\frac{1}{1-(z-1)} = -\sum_{n=0}^{\infty} (z-1)^n \xRightarrow{\text{differentiate}} -\frac{1}{(z-2)^2} = -\sum_{n=1}^{\infty} n \cdot (z-1)^{n-1} = -\sum_{n=0}^{\infty} (n+1)(z-1)^n.$$

$$\Rightarrow \frac{z}{(z-2)^2} = (z-1) \cdot \sum_{n=1}^{\infty} n \cdot (z-1)^{n-1} + \sum_{n=0}^{\infty} (n+1)(z-1)^n = \sum_{n=1}^{\infty} n \cdot (z-1)^n + \sum_{n=0}^{\infty} (n+1) \cdot (z-1)^n$$

$$= \sum_{n=0}^{\infty} (2n+1) \cdot (z-1)^n.$$

$$\lim_{n\to\infty} \left|\frac{C_{n+1}}{C_n}\right| = \lim_{n\to\infty} \frac{2(n+1)+1}{2n+1} = 1 \Rightarrow \text{Radius of convergence} = 1^{-1} = 1$$



6. $$\frac{2}{(z-1)^2(z-2)} = \frac{A}{(z-1)^2} + \frac{B}{z-1} + \frac{C}{z-2} \quad \left(\text{or } \frac{Az+B}{(z-1)^2} + \frac{C}{z-2} \right)$$

$$= \frac{A(z-2)+B(z-1)(z-2)+C\cdot(z-1)^2}{(z-1)^2(z-2)} = \frac{(B+C)z^2+(A-3B-2C)z+(-2A+2B+C)}{(z-1)^2(z-2)}$$

$$\Rightarrow \begin{cases} B+C=0 & ① \\ A-3B-2C=0 & ② \\ -2A+2B+C=2 & ③ \end{cases} \Rightarrow C=-B \xrightarrow{②} A-B=0,\ \xrightarrow{③} -2A+B=2 \Rightarrow A=-2 \Rightarrow \begin{cases} A=-2 \\ B=-2 \\ C=2. \end{cases}$$

$$\Rightarrow \boxed{\frac{2}{(z-1)^2(z-2)} = -\frac{2}{(z-1)^2} - \frac{2}{z-1} + \frac{2}{z-2}}.$$

(a). $|z|<1$

$$\frac{1}{1-z} = 1+z+z^2+\cdots = \sum_{n=0}^{\infty} z^n \Rightarrow \frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n\cdot z^{n-1} = \sum_{n=0}^{\infty}(n+1)z^n.$$

$\frac{d}{dz}\left(\frac{1}{1-z}\right)$

$$\frac{2}{z-2} = -\frac{2}{2}\cdot\frac{1}{1-\frac{z}{2}} = -\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = -\sum_{n=0}^{\infty}\frac{z^n}{2^n} \ (\leftarrow |z|<2)$$

So $$\frac{2}{(z-1)^2(z-2)} = -2\cdot\sum_{n=0}^{\infty}(n+1)z^n + 2\cdot\sum_{n=0}^{\infty} z^n - \sum_{n=0}^{\infty}\frac{z^n}{2^n} = \boxed{\sum_{n=0}^{\infty}\left(-2n-\frac{1}{2^n}\right)\cdot z^n}$$

$|z|<1.$

(b) $1<|z|<2.$ $$\frac{1}{z-1} = \frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}} = \frac{1}{z}\cdot\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} = \sum_{n=1}^{\infty}\frac{1}{z^n}$$

$$\Rightarrow \frac{1}{(z-1)^2} = -\frac{d}{dz}\frac{1}{(z-1)} = -\frac{d}{dz}\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} = \sum_{n=0}^{\infty}(n+1)\cdot z^{-(n+2)} = \sum_{n=2}^{\infty}\frac{(n-1)}{z^n}.$$

$$\Rightarrow \frac{2}{(z-1)^2(z-2)} = -2\cdot\sum_{n=2}^{\infty}\frac{n-1}{z^n} - 2\cdot\sum_{n=1}^{\infty}\frac{1}{z^n} + 1\cdot\left(-\sum_{n=0}^{\infty}\frac{z^n}{2^n}\right)$$

$$= \sum_{n=1}^{\infty}\frac{1}{z^n}\cdot\left(-2(n-1)-2\right) - \sum_{n=0}^{\infty}\frac{z^n}{2^n} = \boxed{-\sum_{n=1}^{\infty}\frac{2n}{z^n} - \sum_{n=0}^{\infty}\frac{z^n}{2^n}.}$$

$1<|z|<2.$

6.(c). $\dfrac{2}{(z-1)^2(z-2)} = -\dfrac{2}{(z-1)^2} - \dfrac{2}{z-1} + \dfrac{2}{z-2}$

$|z|>2$. As before. $-\dfrac{2}{(z-1)^2} - \dfrac{2}{z-1} = -\sum_{n=1}^{\infty} \dfrac{2n}{z^n}$.

$$\frac{2}{z-2} = \frac{2}{z}\cdot\frac{1}{1-\frac{2}{z}} = \frac{2}{z}\cdot\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n = \sum_{n=0}^{\infty}\frac{2^{n+1}}{z^{n+1}} = \sum_{n=1}^{\infty}\frac{2^n}{z^n}.$$

$$\Rightarrow \frac{2}{(z-1)^2(z-2)} = -\sum_{n=1}^{\infty}\frac{2n}{z^n} + \sum_{n=1}^{\infty}\frac{2^n}{z^n} = \boxed{\sum_{n=1}^{\infty}\frac{(-2n+2^n)}{z^n}. \quad (|z|>2)}$$

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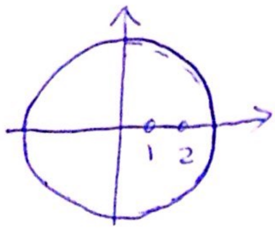
7. (a) $\displaystyle\int_{|z|=3} \frac{2}{(z-1)^2(z-2)}\,dz$. $\quad f(z)=\dfrac{2}{(z-1)^2(z-2)}$.



$$= 2\pi i\cdot\left(\operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=2} f(z)\right)$$

$$\operatorname*{Res}_{z=1} f(z) = \frac{1}{1!}\cdot\frac{d}{dz}\left(\frac{2}{z-2}\right)\Big|_{z=1} = -\frac{2}{(z-2)^2}\Big|_{z=1} = -2.$$

$$\operatorname*{Res}_{z=2} f(z) = \frac{2}{(z-1)^2}\Big|_{z=2} = 2.$$

$$\Rightarrow \int_{|z|=3} f(z)\,dz = 2\pi i\cdot(-2+2) = 0.$$

(b). $\displaystyle\int_{|z|=10} \frac{z^9}{z^5+1}\,dz$. because all the singularities of $f(z)=\dfrac{z^9}{z^5+1}$ are contained inside $|z|=10$.

$$= -2\pi i\cdot \operatorname*{Res}_{z=\infty} \frac{z^9}{z^5+1}$$

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2}\cdot\frac{\frac{1}{z^9}}{\frac{1}{z^5}+1} = \frac{1}{z^{11}}\cdot\frac{z^5}{z^5+1} = \frac{1}{z^6}\cdot\frac{1}{1+z^5}$$

$$= \frac{1}{z^6}\cdot(1-z^5+z^{10}-z^{15}+\cdots)$$

$$= \frac{1}{z^6} - \frac{1}{z} + z^4 - z^9 + \cdots$$

$$\operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right) = -(-1) = 1.$$

$$\Downarrow$$

$$\int_{|z|=10} \frac{z^9}{z^5+1}\,dz = -2\pi i\cdot 1 = -2\pi i.$$

8. (a). $\frac{\mathrm{Log}\, z}{z-1}$.

$$\lim_{z\to1}\frac{\mathrm{Log}\, z}{z-1} \underset{\text{L'Hopital}}{=} \lim_{z\to1}\frac{\frac{1}{z}}{1}=1$$

$\Rightarrow z=1$ is a removable singularity $\Rightarrow \operatorname*{Res}_{z=1}\frac{\mathrm{Log}\, z}{z-1}=0$.

Actually: $\mathrm{Log}\, z=\mathrm{Log}\,(1+(z-1))=(z-1)-\frac{(z-1)^2}{2}+\frac{(z-1)^3}{3}-\cdots \quad |z-1|<1$.

$\Rightarrow \frac{\mathrm{Log}\, z}{z-1}=1-\frac{z-1}{2}+\frac{(z-1)^2}{3}-\cdots\cdots$ no singular part.

(b). $\cos\left(\frac{1}{z}\right)=\sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{1}{z}\right)^{2n}}{(2n)!}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\frac{1}{z^{2n}}=1-\frac{1}{2!}\frac{1}{z^2}+\frac{1}{4!}\frac{1}{z^4}-\cdots$

infinitely many singular terms $\Rightarrow$ essential singularity.

$\operatorname*{Res}_{z=0}\cos\left(\frac{1}{z}\right)=0$ because the coefficient $C_{-1}=0$ (no $\frac{1}{z}$ term).

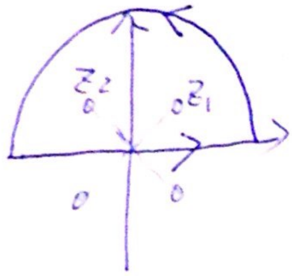
(c). $\frac{\sin z}{(z-\pi)^4}=\frac{\sin(\pi-z)}{(z-\pi)^4}=-\frac{\sin(z-\pi)}{(z-\pi)^4}=-\frac{1}{(z-\pi)^4}\sum_{n=0}^{\infty}(-1)^n\cdot\frac{(z-\pi)^{2n+1}}{(2n+1)!}$

$=-\frac{1}{(z-\pi)^4}\left((z-\pi)-\frac{1}{3!}(z-\pi)^3+\frac{1}{5!}(z-\pi)^5-\cdots\right)=-\frac{1}{(z-\pi)^3}+\frac{1}{6}\frac{1}{(z-\pi)}-\frac{1}{120}(z-\pi)^2+\cdots$

finitely many singular terms $\Rightarrow$ poles (of order 2).
(3)

$\operatorname*{Res}_{z=\pi}\frac{\sin z}{(z-\pi)^4}=C_{-1}=+\frac{1}{6}$. or $\operatorname*{Res}_{z=\pi}\frac{\sin z}{(z-\pi)^4}=\frac{1}{3!}\frac{d^3}{dz^3}\sin(z)\Big|_{z=\pi}=\frac{1}{6}(-\cos z)\Big|_{z=\pi}$
$=\frac{1}{6}$.

9. (a). $\int_{-\infty}^{+\infty}\frac{x^2}{x^4+1}dx$ $\quad f(z)=\frac{z^2}{z^4+1}$.



$$=\lim_{R\to\infty}\int_{-R}^{R}\frac{z^2}{z^4+1}dz$$

$$=\lim_{R\to\infty}\left(\int_{-R}^{R}\frac{z^2}{z^4+1}dz+\int_{C_R}\frac{z^2}{z^4+1}dz\right)=2\pi i\cdot\left(\operatorname*{Res}_{z=z_1}f(z)+\operatorname*{Res}_{z=z_2}f(z)\right)$$

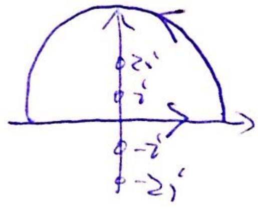
$$\operatorname*{Res}_{z=z_1}\frac{z^2}{z^4+1}=\left.\frac{z^2}{4z^3}\right|_{z=z_1}=\frac{1}{4z_1}=\frac{1}{4}\cdot e^{-i\frac{\pi}{4}}=\frac{1}{4}\frac{1}{\sqrt{2}}(1-i)$$

$$\operatorname*{Res}_{z=z_2}\frac{z^2}{z^4+1}=\left.\frac{z^2}{4z^3}\right|_{z=z_2}=\frac{1}{4z_2}=\frac{1}{4}\cdot e^{-i\frac{3\pi}{4}}=\frac{1}{4}\frac{1}{\sqrt{2}}(-1-i).$$

$$\Rightarrow\int_{-\infty}^{+\infty}\frac{x^2}{x^4+1}dx=2\pi i\cdot\left(\frac{1}{4\sqrt{2}}\cdot(1-i)+\frac{1}{4\sqrt{2}}(-1-i)\right)=\frac{\pi i}{2\sqrt{2}}(-2i)=\frac{\pi}{\sqrt{2}}.$$

(b). $\int_0^{\infty}\frac{\cos(2x)}{(x^2+1)(x^2+4)}=\frac{1}{2}\int_{-\infty}^{+\infty}\frac{\cos(2x)}{(x^2+1)(x^2+4)}dx$ $\quad f(z)=\frac{e^{2iz}}{(z^2+1)(z^2+4)}=\frac{e^{-2y}\cdot e^{i\cdot(2x)}}{(z^2+1)(z^2+4)}.$

$$\operatorname*{Res}_{z=i}f(z)=\left.\frac{e^{2iz}}{(z^2+4)(z+i)}\right|_{z=i}=\frac{e^{-2}}{3\cdot 2i}=\frac{e^{-2}}{6i}.$$

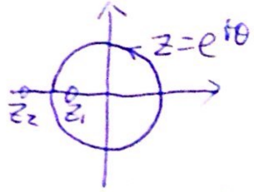


$$\operatorname*{Res}_{z=2i}f(z)=\left.\frac{e^{2iz}}{(z^2+1)(z+2i)}\right|_{z=2i}=\frac{e^{-4}}{(-3)\cdot 4i}=-\frac{e^{-4}}{12i}$$

$$\Rightarrow\int_{-\infty}^{\infty}\frac{\cos(2x)\,dx}{(x^2+1)(x^2+4)}=2\pi i\left(\operatorname*{Res}_{z=i}+\operatorname*{Res}_{z=2i}\right)=2\pi i\cdot\frac{1}{i}\left(\frac{e^{-2}}{6}-\frac{e^{-4}}{12}\right)=\frac{2\pi\cdot e^{-4}}{12}(2e^2-1)\Rightarrow\frac{\pi}{6}\cdot\frac{2e^2-1}{e^4}$$

$$\Rightarrow\int_0^{\infty}\frac{\cos(2x)\,dx}{(x^2+1)(x^2+4)}=\frac{\pi}{12}\cdot\frac{2e^2-1}{e^4}.$$

9. (c) $\displaystyle\int_0^{2\pi} \frac{d\theta}{3+2\cos\theta}$. $\quad z=e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta = iz\,d\theta$



$\|$ $\quad \Rightarrow d\theta = \dfrac{dz}{iz}$, and $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2} = \dfrac{z+\frac{1}{z}}{2}$

$$\int_{|z|=1} \frac{\frac{dz}{iz}}{3+2\cdot\frac{z+\frac{1}{z}}{2}} = \frac{1}{i}\cdot\int_{|z|=1}\frac{dz}{z^2+3z+1} \qquad f(z)=\frac{1}{z^2+3z+1}.$$

Singularity of $f(z)$: $z^2+3z+1=0 \Rightarrow z = \dfrac{-3\pm\sqrt{3^2-4}}{2} = \dfrac{-3\pm\sqrt{5}}{2}$

$z_1 = \dfrac{-3+\sqrt{5}}{2}$ is contained inside $|z|=1$. $\quad z_2 = \dfrac{-3-\sqrt{5}}{2}$ is outside

$$\operatorname*{Res}_{z=z_1}\frac{1}{z^2+3z+1} = \left.\frac{1}{2z+3}\right|_{z=\frac{-3+\sqrt{5}}{2}} = \frac{1}{(-3+\sqrt{5})+3} = \frac{1}{\sqrt{5}}.$$

$$\Rightarrow \int_{|z|=1}\frac{dz}{z^2+3z+1} = 2\pi i\cdot\frac{1}{\sqrt{5}}$$

$$\Rightarrow \int_0^{2\pi}\frac{d\theta}{3+2\cos\theta} = \frac{1}{i}\cdot 2\pi i\cdot\frac{1}{\sqrt{5}} = \frac{2\pi}{\sqrt{5}}.$$

1.(a). $z = i^{\frac{1}{2}} = 1^{\frac{1}{2}} e^{i\cdot\frac{(\frac{\pi}{2}+2\pi\cdot n)}{2}}$ $n=0,1$ 5+5

$= e^{i((\frac{\pi}{4})+0\pi)}, \ e^{i(\frac{\pi}{4}+\pi)}$

$= e^{i\frac{\pi}{4}}, e^{i\frac{5\pi}{4}} = \pm\frac{1}{\sqrt{2}}(1+i).$ 10

(b). 5

$\sin(z) = \frac{e^{iz}-e^{-iz}}{2i} = 2i \Rightarrow e^{iz}-e^{-iz} = -4.$

$w = e^{iz} \Rightarrow w+4-\frac{1}{w} = 0 \Leftrightarrow w^2+4w-1=0.$ 5

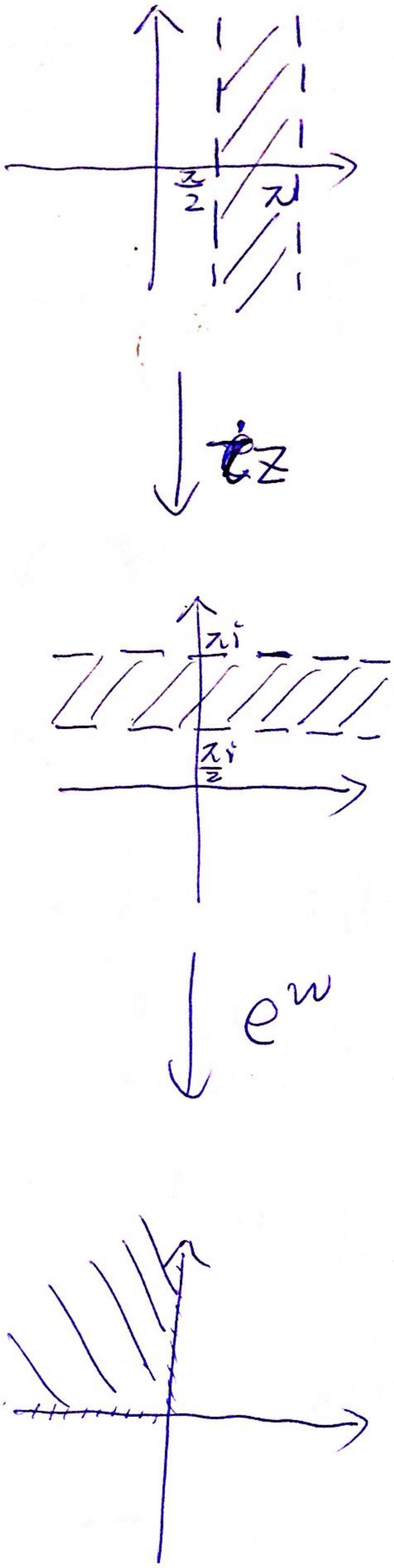
$\Rightarrow w = \frac{-4\pm\sqrt{4^2+4}}{2} = \frac{-4\pm 2\sqrt{5}}{2} = -2\pm\sqrt{5}. = e^{iz}$ 5 +5

$\Rightarrow iz = \log(-2\pm\sqrt{5}) = \begin{cases} \ln(-2+\sqrt{5}) + i\cdot 2\pi n. \\ \ln(2+\sqrt{5}) + i(\pi+2\pi n) \end{cases}$ $n=0,\pm1,\pm2,\cdots$

10

$\Rightarrow z = \begin{cases} -i\ln(\sqrt{5}-2) + 2\pi n. \\ -i\ln(\sqrt{5}+2) + (2n+1)\pi = i\ln(\sqrt{5}-2) + (2n+1)\pi. \end{cases}$

2.



25

25

(a). $\begin{cases} u_x = v_y = -2y \quad ① \\ u_y = -v_x = -2x \quad ② \end{cases} \Rightarrow u(x,y) = -2xy + c(y).$ 10 5

$$\Downarrow ①$$

$$u_y = -2x + c'(y) \overset{②}{=\!=} -2x$$ 5

$$\Downarrow$$

$$c'(y) = 0 \Rightarrow c(y) = \text{constant}$$

So $u(x,y) = -2xy + \text{real constant}$ 5 $\Rightarrow f(z) = (-2xy) + i\cdot(x^2-y^2)$

$$= i\cdot\left[(x^2-y^2) + i(2xy)\right] = i\cdot z^2.$$

(b). $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 4 \neq 0 \Rightarrow v$ is not harmonic $\Rightarrow \nexists f(z)$ s.t. $v = \mathrm{Im} f(z)$.

25.

4.(a). $\displaystyle\int_{|z-\pi|=3} \frac{\sin(z)}{(z-\pi)^4}\,dz = \frac{2\pi i}{3!}\frac{d^3}{dz^3}\sin(z)\Big|_{z=\pi}$ 15

$= \frac{2\pi i}{6}\cdot(-\cos z)\Big|_{z=\pi} = \frac{2\pi i}{6}(-(-1)) = \frac{\pi i}{3}.$

5 5

(b). $\frac{1}{z+1} = \frac{d}{dz}\mathrm{Log}(z+1).$

15 5 5

$\displaystyle\int_0^i \frac{1}{z+1}\,dz = \mathrm{Log}(z+1)\Big|_0^i = \mathrm{Log}(i+1) - \mathrm{Log}\,1 = \boxed{\ln\sqrt{2} + i\cdot\frac{\pi}{4}}$

or $z(t) = ti.\quad 0\le t\le 1.\quad z'(t) = i.$ 5

$\displaystyle\int_0^i \frac{1}{z+1}\,dz =$ 10 $\displaystyle\int_0^1 \frac{i\cdot dt}{ti+1} = i\cdot\int_0^1 \frac{1-it}{1+t^2}\,dt.$

5 $\displaystyle= i\cdot\int_0^1 \frac{dt}{1+t^2} + \int_0^1 \frac{t\,dt}{1+t^2}$

$= i\cdot\tan^{-1}t\Big]_0^1 + \frac{1}{2}\ln(1+t^2)\Big|_0^1$

5

$= i\cdot\frac{\pi}{4} + \frac{1}{2}\ln 2.$

5. (a). $\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n \cdot z^n$ 10.

5 5

$\Rightarrow \text{Log}(1+z) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{n+1}}{n+1} = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{z^n}{n} = \left(z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots\right)$

$\lim_{n\to\infty} \left|\frac{C_{n+1}}{C_n}\right| = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = 1 \Rightarrow$ radius of convergence $= 1$. 5.

(b). $\frac{1}{1+z} = \frac{1}{2+(z-1)} = \frac{1}{2} \cdot \frac{1}{1+\frac{z-1}{2}} = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-1}{2}\right)^n$

$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(z-1)^n}{2^{n+1}}$ 10.

$\Rightarrow \text{Log}(1+z) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(z-1)^{n+1}}{(n+1)2^{n+1}} + \text{Log}\, 2.$

10

$= \ln 2 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(z-1)^n}{n \cdot 2^n}$

$\lim_{n\to\infty} \left|\frac{C_{n+1}}{C_n}\right| = \lim_{n\to\infty} \frac{\frac{1}{(n+1)\cdot 2^{n+1}}}{\frac{1}{n \cdot 2^n}} = \lim_{n\to\infty} \frac{n}{2(n+1)} = \frac{1}{2}$ 5.

$\Rightarrow$ radius of convergence $= \left(\frac{1}{2}\right)^{-1} = 2.$

OR: $\text{Log}(1+z) = \text{Log}\,(2+(z-1)) = \text{Log}\, 2 + \text{Log}\left(1 + \frac{z-1}{2}\right)$. 5 5

$= \text{Log}\, 2 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(z-1)^n}{n \cdot 2^n}$ 10

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6. $$\frac{1}{(z-1)(z-2)} = -\frac{1}{z-1} + \frac{1}{z-2} = f(z)$$ 10

(a) $|z|<1$, $$-\frac{1}{z-1} = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n.$$ 10

$$\frac{1}{z-2} = -\frac{1}{2(1-\frac{z}{2})} = -\frac{1}{2}\cdot\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$$ 10

$$\Rightarrow f(z) = \sum_{n=0}^{\infty}\left(1-\frac{1}{2^{n+1}}\right)z^n.$$ 5

(b). $1<|z|<2$. $$-\frac{1}{z-1} = -\frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}} = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = -\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} = -\sum_{n=1}^{\infty}\frac{1}{z^n}.$$

$$\frac{1}{z-2} = \frac{-1}{2}\cdot\frac{1}{1-\frac{z}{2}} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$$ 10

$$\Rightarrow f(z) = -\sum_{n=1}^{\infty}\frac{1}{z^n} - \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}.$$ 5

7. (a). $\int_{|z|=4} \frac{dz}{(z-3)^2(z-1)}$ $\quad f(z) = \frac{1}{(z-3)^2(z-1)}$

$\operatorname{Res}_{z=1} f(z) = \frac{1}{(z-3)^2}\Big|_{z=1} = \frac{1}{(-2)^2} = \frac{1}{4}$. 10

$\operatorname{Res}_{z=3} f(z) = \frac{d}{dz}\left(\frac{1}{z-1}\right)\Big|_{z=3} = -\frac{1}{(z-1)^2}\Big|_{z=3} = -\frac{1}{2^2} = -\frac{1}{4}$. 10

$\Rightarrow \int_{|z|=3} \frac{dz}{(z-3)^2(z-1)} = 2\pi i \cdot \left(\operatorname{Res}_{z=1} f(z) + \operatorname{Res}_{z=3} f(z)\right) = 0$. 5

(b). 10

$f(z) = \frac{z^9}{z^5-2}$ $\quad \frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{\frac{1}{z^9}}{\frac{1}{z^5}-2} \cdot \frac{1}{z^2} = \frac{\frac{1}{z^4}}{1-2z^5} \cdot \frac{1}{z^2} = \frac{1}{z^6} \cdot \frac{1}{1-2z^5}$

$= \frac{1}{z^6} \cdot \left(1 + 2z^5 + 4z^{10} + \cdots\right)$

$\Rightarrow \operatorname{Res}_{z=\infty} f(z) = -\operatorname{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right)$ $\quad = \frac{1}{z^6} + \frac{2}{z} + 4z^4 + \cdots$ 10

$= 2$.

$\Rightarrow \int_{|z|=10} f(z)\,dz = 2\pi i \cdot \operatorname{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right) = 2\pi i \cdot 2 = 4\pi i$. 5

$z=0$ 5 5 5

8. (a). $\dfrac{1-\cos z}{z^3} = \dfrac{1-\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots\right)}{z^3} = \dfrac{1}{2z} - \dfrac{z}{4!} + \dfrac{z^3}{6!} - \cdots$

+0 5 5

$\Rightarrow$ $z=0$ is a pole of order 1. $\operatorname{Res}_{z=0} f(z) = \frac{1}{2}$.

5 5

(b). $z^6 e^{\frac{1}{z}} = z^6 \cdot \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot z^{6-n}$.

5 5

infinitely many singular terms $\Rightarrow$ $z=0$ is an essential singularity.

$\operatorname{Res}_{z=0} f(z) = \frac{1}{7!}$ 5

(c) $f(z) = \dfrac{1}{\sin z}$. singular points $z = \pi \cdot n$. $n = 0, \pm 1, \pm 2, \cdots$ 2

$z=0$. $f(z) = \left(\dfrac{z}{\sin z}\right) \cdot \dfrac{1}{z} = \dfrac{g(z)}{z}$. 2 $g(z) = \dfrac{z}{\sin z}$ has removable singularity at $z=0$. 2

$\Rightarrow$ $z=0$ is a pole of order 1. $\operatorname{Res}_{z=0} f(z) = 1$. $\lim_{z\to 0} g(z) = \lim_{z\to 0} \frac{z}{\sin z} = 1$.

Similarly

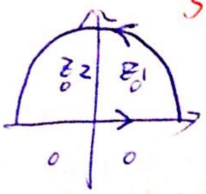
~~Because~~ $\sin z = \sin(z - 2m\pi)$. So. $z = 2m\pi$ is pole of order 1.

$\operatorname{Res}_{z=2m\pi} f(z) = 1$

$\sin(z) = -\sin(z-(2m+1)\pi)$. $\Rightarrow$ $z = (2m+1)\pi$ is pole of order 1.

$\operatorname{Res}_{z=(2m+1)\pi} f(z) = -1$.

(a). $\int_{-\infty}^{+\infty}\frac{1}{x^4+4}dx$ $\quad f(z)=\frac{1}{z^4+4}$ 5



5 $\quad z_1=\sqrt{2}\cdot e^{\frac{i\pi}{4}}$

$z_2=\sqrt{2}\cdot e^{\frac{i3\pi}{4}}$

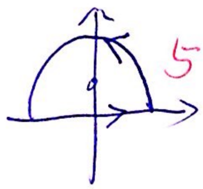
$\operatorname{Res}_{z=z_1} f(z)=\left.\frac{1}{4\cdot z^3}\right|_{z=z_1}=\frac{1}{4\cdot z_1^3}=\frac{z_1}{4\cdot z_1^4}=-\frac{z_1}{16}=-\frac{1}{16}(1+i)$ 5

$\operatorname{Res}_{z=z_2} f(z)=\left.\frac{1}{4\cdot z^3}\right|_{z=z_2}=\frac{1}{4\cdot z_2^3}=\frac{z_2}{4z_2^4}=-\frac{z_2}{16}=-\frac{1}{16}(-1+i)$. 5

$\Rightarrow \int_{-\infty}^{+\infty}\frac{1}{x^4+4}dx=2\pi i\cdot\left(\operatorname{Res}_{z=z_1}f(z)+\operatorname{Res}_{z=z_2}f(z)\right)=2\pi i\cdot -\frac{i}{8}=\frac{\pi}{4}$. 5

(b) $\int_0^{\infty}\frac{\cos(2x)}{x^2+1}dx=\frac{1}{2}\int_{-\infty}^{+\infty}\frac{\cos(2x)}{x^2+1}dx$. 3 $\quad f(z)=\frac{e^{2iz}}{z^2+1}=\frac{e^{-2y}e^{i2x}}{z^2+1}$. 5

$\operatorname{Res}_{z=i} f(z)=\left.\frac{e^{2iz}}{z+i}\right|_{z=i}=\frac{e^{-2}}{2i}$. 5



5

$\Rightarrow \underbrace{\int_{-R}^{+R}\frac{e^{2iz}\ \cancel{\cos(2x)}}{\cancel{x}\,z^2+1}dx}_{\|} + \underbrace{\int_{C_R} f(z)dz}_{\downarrow\ 0}=2\pi i\cdot\operatorname{Res}_{z=i}f(z)=2\pi i\cdot\frac{e^{-2}}{2i}=\frac{\pi}{e^2}$. 5

$\int_{-R}^{+R}\frac{\cos(2x)+i\sin(2x)}{x^2+1}dx$

$\Rightarrow \int_0^{\infty}\frac{\cos(2x)}{x^2+1}dx=\frac{\pi}{2e^2}$. 2

9(c)

$\cancel{Z}$. $\int_0^{2\pi} \frac{d\theta}{2+\sin\theta}$. $z=e^{i\theta}$ $dz = ie^{i\theta}d\theta = i\cdot z\, d\theta$

$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-\frac{1}{z}}{2i}$.

2

$$= \int_{|z|=1} \frac{\frac{dz}{iz}}{2+\frac{z-\frac{1}{z}}{2i}} = 2\int_{|z|=1} \frac{dz}{z^2+4iz-1}$$ 2

Singularity: $z^2+4iz-1=0 \Rightarrow z = \frac{-4i \pm \sqrt{(4i)^2+4}}{2}$ 2

$f(z) = \frac{1}{z^2+4iz-1}$

$$= \frac{-4i \pm \sqrt{-12}}{2} = \frac{-4i \pm 2\sqrt{3}i}{2}$$

$$= (-2 \pm \sqrt{3})i.$$

$$\operatorname*{Res}_{z=(-2+\sqrt{3})i} f(z) = \frac{1}{2z+4i}\Big|_{z=(-2+\sqrt{3})i} = \frac{1}{2\sqrt{3}i}$$ 2

$$\Rightarrow 2\int_{|z|=1} \frac{dz}{z^2+4iz-1} = 2\cdot 2\pi i \cdot \operatorname*{Res}_{z=(-2+\sqrt{3})i} f(z) = 4\pi i \cdot \frac{1}{2\sqrt{3}i} = \frac{2\pi}{\sqrt{3}}.$$

$\parallel$

$\int_0^{2\pi} \frac{d\theta}{2+\sin\theta}$.

2

!!! WRITE YOUR NAME, STUDENT ID BELOW !!!

NAME :

ID :

1. (50pts) Let $z_1 = -1 + i$, $z_2 = \sqrt{2}\, e^{i\frac{\pi}{4}}$.
(a): Calculate $\bar{z}_1 \cdot z_2$ (write the result in the form of $a + bi$).

$\bar{z}_1 = -1-i$, $z_2 = \sqrt{2}\cdot(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = \sqrt{2}(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) = 1+i$. 10

so $\bar{z}_1 \cdot z_2 = (-1-i)\cdot(1+i) = -1-i-i+1 = -2i$. 10

or $z_1 = \sqrt{2}\cdot e^{\frac{3\pi}{4}i}$, $\bar{z}_1 \cdot z_2 = \sqrt{2}\cdot e^{-\frac{3\pi}{4}i}\cdot\sqrt{2}\cdot e^{i\frac{\pi}{4}} = 2\cdot e^{-\frac{\pi}{2}i} = -2i$

(b): Calculate $z_1^{1/3}$ and sketch the roots on a regular polygon.

$z_1 = -1+i = \sqrt{2}\cdot e^{i\cdot\frac{3\pi}{4}}$ 5

$\Rightarrow z_1^{\frac{1}{3}} = 2^{\frac{1}{6}} e^{i\cdot\frac{1}{3}(\frac{3\pi}{4}+2\pi k)} = 2^{\frac{1}{6}}\cdot e^{i\frac{\pi}{4}}\cdot(e^{\frac{2i\pi}{3}})^k$ $k=0,1,2$.

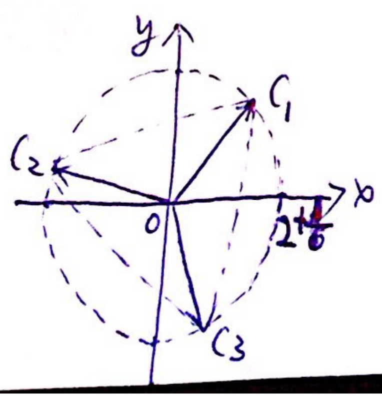
$k=0$: $c_1 = 2^{\frac{1}{6}}\cdot\frac{1}{\sqrt{2}}(1+i) = 2^{\frac{1}{6}-\frac{3}{6}}(1+i) = 2^{-\frac{1}{3}}(1+i)$ 7

$k=1$: $c_2 = c_1\cdot e^{\frac{2i\pi}{3}} = 2^{-\frac{1}{3}}\cdot(1+i)\cdot(-\frac{1}{2}+\frac{\sqrt{3}}{2}i)$ 5

$= 2^{-\frac{4}{3}}\cdot(-1+\sqrt{3}i-i-\sqrt{3}) = 2^{-\frac{4}{3}}\cdot((-1-\sqrt{3})+i(\sqrt{3}-1))$.

$k=2$: $c_3 = c_1(e^{\frac{2i\pi}{3}})^2 = c_1 e^{-\frac{2i\pi}{3}} = 2^{-\frac{1}{3}}(1+i)\cdot(-\frac{1}{2}-\frac{\sqrt{3}}{2}i)$ 5

$= 2^{-\frac{4}{3}}\cdot(-1-\sqrt{3}i-i+\sqrt{3}) = 2^{-\frac{4}{3}}\cdot((-1+\sqrt{3})+i\cdot(-\sqrt{3}-1))$.



8

$\frac{\pi}{4}$, $\frac{\pi}{4}+\frac{2\pi}{3} = \frac{11\pi}{12}$, $\frac{\pi}{4}+\frac{4\pi}{3} = \frac{19\pi}{12} = (\frac{\pi}{4}-\frac{2\pi}{3})+2\pi$

$= -\frac{5\pi}{12}+2\pi$

2. (50pts) Determine whether the limit exists or not. If it exist, then calculate it.

(a)

$$\lim_{z\to i} \frac{\overline{(z-i)^2}}{(z-i)^2}.$$

$$\lim_{z\to i} \frac{\overline{(z-i)^2}}{(z-i)^2} = \lim_{(z-i)\to 0} \frac{\overline{(z-i)^2}}{(z-i)^2} = \lim_{w\to 0} \frac{\overline{w}^2}{w^2} \overset{w=re^{i\theta}}{=\!=} \lim_{\substack{|w|\to 0 \\ \| \\ r}} \frac{\overline{r^2 e^{2i\theta}}}{r^2 e^{2i\theta}}$$

$$= \lim_{r\to 0} \frac{r^2 \cdot e^{-2i\theta}}{r^2 e^{2i\theta}} = \lim_{r\to 0} e^{-4i\theta} = e^{-4i\theta}$$

The limits along different directions are not the same $\Longrightarrow$ the limit does not exist.

(b)

$$\lim_{z\to e^{i\frac{\pi}{4}}} \frac{z^2-i}{z^4+1}.$$

$$\lim_{z\to e^{i\frac{\pi}{4}}} \frac{z^2-i}{z^4+1} = \lim_{z\to e^{i\frac{\pi}{4}}} \frac{z^2-i}{(z^2+i)(z^2-i)} = \lim_{z\to e^{i\frac{\pi}{4}}} \frac{1}{z^2+i}$$

$$= \frac{1}{e^{i\frac{\pi}{2}}+i} = \frac{1}{2i} = -\frac{i}{2}$$

or

$$\lim_{z\to e^{i\frac{\pi}{4}}} \frac{z^2-i}{z^4+1} \overset{\text{L'Hopistal}}{=\!=\!=} \lim_{z\to e^{i\frac{\pi}{4}}} \frac{2z}{4z^3} = \lim_{z\to e^{i\frac{\pi}{4}}} \frac{1}{2z^2}$$

$$= \frac{1}{2\cdot e^{\frac{i\pi}{2}}} = \frac{1}{2i} = -\frac{i}{2}$$

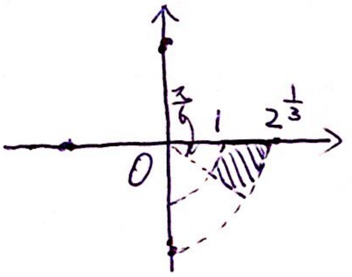
3. (50pts)

(1) Sketch the region given by:

$$-\frac{\pi}{6} < \mathrm{Arg} z \leq 0, \quad 1 < |z| < 2^{1/3}.$$

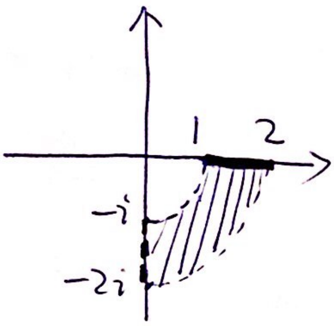
(2) Find the image of the above region under the mapping $w = z^3$.

1.



$\downarrow$ $w=z^3$

2.



$-\frac{\pi}{2} < \mathrm{Arg}\, w \leq 0.$

$1 < |w| < 2.$

4. (50pts)

(a) Find the domain of the following function. Explain why the following function is analytic in its domain and calculate $f'(z)$:

$$f(z) = e^{\frac{z+1}{z-1}}$$

- Domain $= \{z \in \mathbb{C} \,;\, z \neq 1\} = \mathbb{C} \setminus \{1\}$. 10

- $z+1$, $z-1$ analytic $\Rightarrow \frac{z+1}{z-1}$ analytic $\xRightarrow[\text{composition}]{e^z \text{ analytic}} e^{\frac{z+1}{z-1}}$ analytic 5

Chain rule: $f'(z) = e^{\frac{z+1}{z-1}} \cdot \left(\frac{z+1}{z-1}\right)' = e^{\frac{z+1}{z-1}} \cdot \frac{1\cdot(z-1)-(z+1)\cdot 1}{(z-1)^2}$

$= e^{\frac{z+1}{z-1}} \cdot \left(-\frac{2}{(z-1)^2}\right) = -\frac{2}{(z-1)^2} e^{\frac{z+1}{z-1}}$. 10

(b) If $g(z)$ is an analytic function analytic function in a domain D and $\mathrm{Im}(g(z))$ is constant on D , what can you say about $g(z)$? Explain your reason.

Let $g(z) = u(z) + i v(z)$.

By assumption, $\mathrm{Im}(g(z)) = v(z) = \text{const} = C_2$ 5

$\Downarrow$

$v_x = v_y = 0$

By Cauchy-Riemann equations:

10 $u_x = v_y = 0$, $u_y = -v_x = 0$ $\Rightarrow du = 0$ 5 $\Rightarrow u = \text{constant} = C_1$ 5

So $g(z) = C_1 + iC_2$ is a constant function.

5. (50pts) Find the points where the function $f(z)$ is differentiable and then calculate $f'(z)$ at those points. Is the function analytic at those points? $(z = x+yi = re^{i\theta})$

(a) $f(z) = (x^2+y^2) + (x-y)i$

$u(x,y) = x^2+y^2$, $v = x-y$. $u_x = 2x$, $u_y = 2y$ 8
$v_x = 1$, $v_y = -1$.

Cauchy-Riemann equations:

$$\begin{cases} 2x = -1 \\ 2y = -1 \end{cases} \Rightarrow \begin{cases} x = -\frac{1}{2} \\ y = -\frac{1}{2} \end{cases}$$ 5

so $f(z)$ is differentiable only at $z = -\frac{1}{2} - \frac{1}{2}i$. 2

$f'(-\frac{1}{2}-\frac{1}{2}i) = u_x + iv_x \big|_{-\frac{1}{2}-\frac{1}{2}i} = (2x+i)\big|_{-\frac{1}{2}-\frac{1}{2}i} = -1+i$. 5

$f(z)$ is not differential in any neighborhood $\Rightarrow f(z)$ is not analytic at any point. 5

(b) $f(re^{i\theta}) = e^\theta \cos(\ln r) - i \cdot e^\theta \sin(\ln r)$, $r > 0, 0 < \theta < 2\pi$.

$u(r,\theta) = e^\theta \cdot \cos(\ln r)$, $v(r,\theta) = -e^\theta \sin(\ln r)$.

$u_r = -e^\theta \cdot \sin(\ln r) \cdot \frac{1}{r}$. $u_\theta = e^\theta \cdot \cos(\ln r)$.

$v_r = -e^\theta \cos(\ln r) \cdot \frac{1}{r}$, $v_\theta = -e^\theta \sin(\ln r)$ 8

$$\Rightarrow \begin{cases} r u_r = v_\theta = -e^\theta \sin(\ln r). \\ u_\theta = -r v_r = e^\theta \cdot \cos(\ln r) \end{cases}$$

i.e. CR eqs are satisfied for $r>0$, $0<\theta<2\pi$ 5

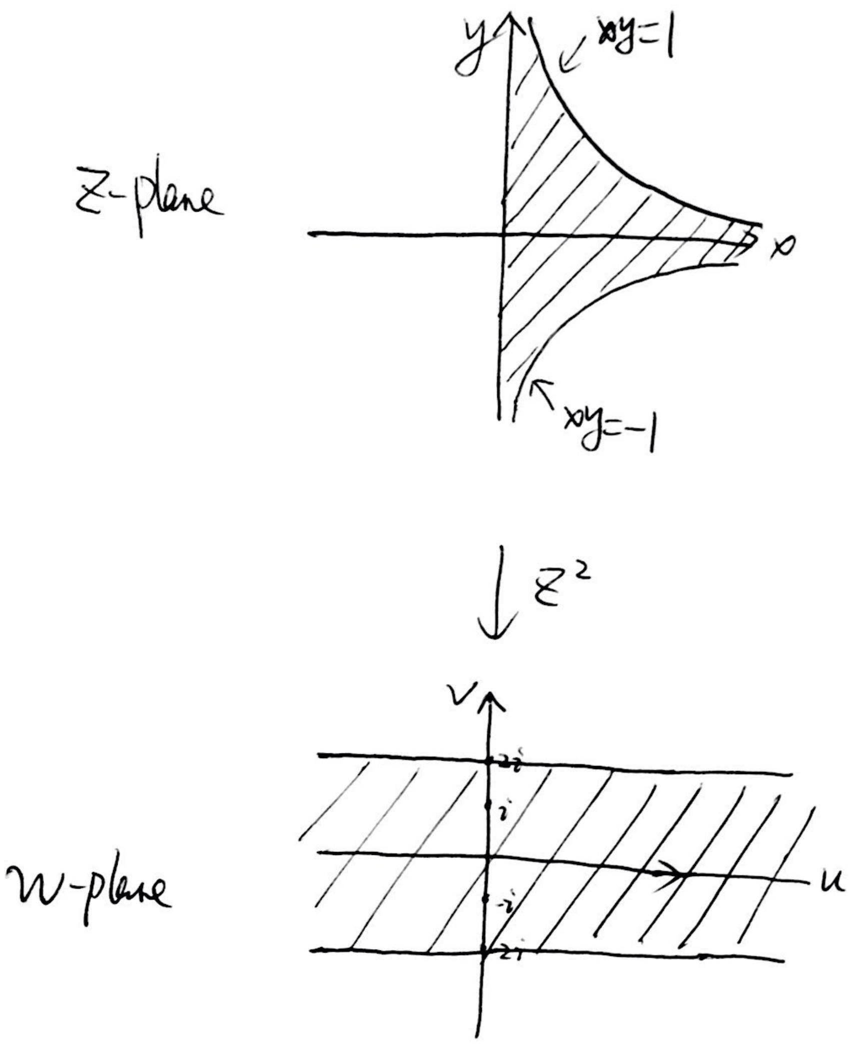
$\Rightarrow f(z)$ is analytic (differentiable) at any point in the domain $\{r>0, 0<\theta<2\pi\}$ 5

$$f'(re^{i\theta}) = e^{-i\theta} \cdot (u_r + iv_r) = e^{-i\theta} \cdot \left(\frac{-e^\theta \cdot \sin(\ln r)}{r} + i \cdot \frac{-e^\theta \cdot \cos(\ln r)}{r}\right).$$

$$= -\frac{e^\theta(\sin(\ln r) + i \cdot \cos(\ln r))}{r \cdot e^{i\theta}} = -i \cdot \frac{f(z)}{z}.$$ 7

$$\left(f(z) = e^\theta \cdot e^{-i\ln r} = e^{-i\log z} \Rightarrow f'(z) = e^{-i\log z} \cdot \frac{-i}{z} = -i\frac{f(z)}{z}\right)$$

Extra Credit Problem



!!! WRITE YOUR NAME, STUDENT ID BELOW !!!

NAME :   ID :

1. (50pts) Let $z_1 = -1 - i$, $z_2 = \sqrt{2}\, e^{i\frac{\pi}{4}}$.

(a): Calculate $\bar{z}_1 \cdot z_2$ (write the result in the form of $a + bi$).

$$\bar z_1 = -1+i, \quad z_2=\sqrt2\left(\cos\tfrac{\pi}{4}+i\sin\tfrac{\pi}{4}\right)=\sqrt2\cdot\left(\tfrac{1}{\sqrt2}+i\tfrac{1}{\sqrt2}\right)=1+i.$$

So $\bar z_1\cdot z_2=(-1+i)\cdot(1+i)=-1-i+i-1=-2$.

or $z_1=\sqrt2\cdot e^{-\frac{3\pi}{4}i}$, $\bar z_1\cdot z_2=\sqrt2\cdot e^{\frac{3\pi}{4}i}\cdot\sqrt2\cdot e^{i\frac{\pi}{4}}=2\cdot e^{\pi i}=-2$.

(b): Calculate $z_1^{1/3}$ and sketch the roots on a regular polygon.

$$z_1=-1-i=\sqrt2\cdot e^{-i\frac{3\pi}{4}}.$$

$$\Rightarrow z_1^{\frac13}=2^{\frac16}\cdot e^{i\frac13\left(-\frac{3\pi}{4}+2\pi k\right)}=2^{\frac16}\cdot e^{-\frac{i\pi}{4}}\left(e^{\frac{2\pi i}{3}}\right)^k \quad k=0,1,2.$$

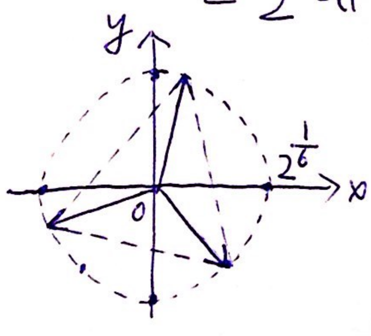
$k=0$: $c_1=2^{\frac16}\cdot e^{-\frac{i\pi}{4}}=2^{\frac16}\cdot\frac{1}{\sqrt2}(1-i)=2^{-\frac13}\cdot(1-i)$.

$k=1$: $c_2=c_1\cdot e^{\frac{2i\pi}{3}}=2^{\frac16}\cdot e^{i\left(-\frac{\pi}{4}+\frac{2\pi}{3}\right)}=2^{\frac16}e^{i\frac{5\pi}{12}}=2^{\frac16}e^{i\frac{5\pi}{12}}$

$=2^{-\frac13}(1-i)\cdot\left(-\frac12+\frac{\sqrt3}{2}i\right)=2^{-\frac43}\cdot\left((-1+\sqrt3)+i\cdot(\sqrt3+1)\right)$

$k=2$: $c_3=c_1\cdot e^{-\frac{2\pi i}{3}}=2^{\frac16}\cdot e^{i\left(-\frac{\pi}{4}-\frac{2\pi}{3}\right)}=2^{\frac16}e^{-i\cdot\frac{11\pi}{12}}=2^{\frac16}e^{i\frac{13\pi}{12}}$

$=2^{-\frac13}(1-i)\cdot\left(-\frac12-\frac{\sqrt3}{2}i\right)=2^{-\frac43}\cdot\left((-1-\sqrt3)+i\cdot(-\sqrt3+1)\right)$



2. (50pts) Determine whether the limit exists or not. If it exist, then calculate it.

(a)

$$\lim_{z\to 1} \frac{\overline{(z-1)^2}}{(z-1)^2}.$$

$$\lim_{z\to 1} \frac{\overline{(z-1)^2}}{(z-1)^2} = \lim_{(z-1)\to 0} \frac{\overline{(z-1)^2}}{(z-1)^2} = \lim_{w\to 0} \frac{\overline{w^2}}{w^2} \overset{w=re^{i\theta}}{=\!=} \lim_{r\to 0} \frac{r^2e^{-2i\theta}}{r^2e^{2i\theta}}$$

$$= \lim_{r\to 0} e^{-4i\theta} = e^{-4i\theta}.$$

The limits along different directions are not the same $\Longrightarrow$ the limit does not exist.

(b)

$$\lim_{z\to e^{i\frac{\pi}{4}}} \frac{z^2-i}{z^4+1}.$$

$$\lim_{z\to e^{i\frac{\pi}{4}}} \frac{z^2-i}{z^4+1} = \lim_{z\to e^{i\frac{\pi}{4}}} \frac{z^2-i}{(z^2+i)(z^2-i)} = \lim_{z\to e^{i\frac{\pi}{4}}} \frac{1}{z^2+i}$$

$$= \frac{1}{e^{i\frac{\pi}{2}}+i} = \frac{1}{2i} = -\frac{i}{2}$$

or $$\lim_{z\to e^{i\frac{\pi}{4}}} \frac{z^2-i}{z^4+1} \overset{\text{L'Hospital}}{=\!=\!=} \lim_{z\to e^{i\frac{\pi}{4}}} \frac{2z}{4z^3} = \lim_{z\to e^{i\frac{\pi}{4}}} \frac{1}{2z^2}$$

$$= \frac{1}{2\cdot e^{i\frac{\pi}{2}}} = \frac{1}{2i} = -\frac{i}{2}$$

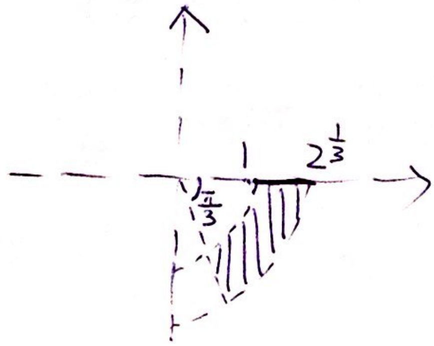
3. (50pts)

(1) Sketch the region given by:

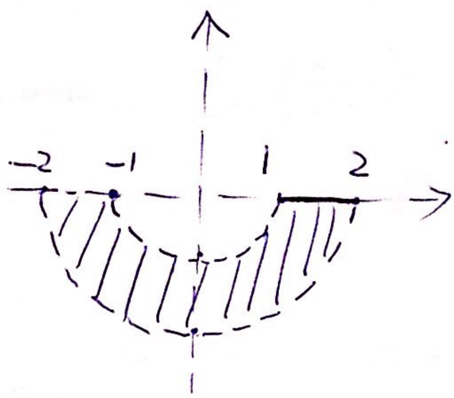
$$-\frac{\pi}{3} < \text{Arg}z \leq 0, \quad 1 < |z| < 2^{1/3}.$$

(2) Find the image of the above region under the mapping $w = z^3$.

1.



$\downarrow\ w = z^3$



$-\pi < \text{Arg}\, w \leq 0$

$1 < |w| < 2.$

4. (50pts)

(a) Find the domain of the following function. Explain why the following function is analytic in its domain and calculate $f'(z)$:

$$f(z) = e^{\frac{z-1}{z+1}}$$

- Domain $= \{z \in \mathbb{C};\ z \neq -1\} = \mathbb{C} \setminus \{-1\}$.
- $z+1$, $z-1$ analytic $\Rightarrow \frac{z-1}{z+1}$ analytic $\xrightarrow{e^z \text{ analytic}}$ composition $e^{\frac{z+1}{z-1}}$ analytic

Chain rule: $f'(z) = e^{\frac{z-1}{z+1}} \cdot \left(\frac{z-1}{z+1}\right)' = e^{\frac{z-1}{z+1}} \frac{1\cdot(z+1)-(z-1)\cdot 1}{(z+1)^2}$

$$= \frac{2}{(z+1)^2} e^{\frac{z-1}{z+1}}$$

(b) If $g(z)$ is an analytic function analytic function in a domain D and $\text{Im}(g(z))$ is constant on D , what can you say about $g(z)$? Explain your reason.

Let $g(z) = u(z) + i v(z)$.

By assumption, $\text{Im}(g(z)) = v(z) = \text{const} = C_2$.

$\Downarrow$

By Cauchy-Riemann equations. $v_x = v_y = 0$

$u_x = v_y = 0$
$u_y = -v_x = 0$ $\Rightarrow du = 0 \Rightarrow u = \text{constant} = C_1$

so $g(z) = C_1 + i C_2$ is a constant function.

5. (50pts) Find the points where the function $f(z)$ is differentiable and then calculate $f'(z)$ at those points. Is the function analytic at those points? $(z = x+yi = re^{i\theta})$

(a) $f(z) = (x^2 - y^2) + (x + y)i$

$u(x,y) = x^2 - y^2$, $v(x,y) = x+y$. $u_x = 2x$, $u_y = -2y$
$v_x = 1$. $v_y = 1$.

Cauchy-Riemann equations:

$$\begin{cases} 2x = 1 \\ -2y = -1 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2} \\ y = \frac{1}{2} \end{cases}$$

so $f(z)$ is differentiable only at $z = \frac{1}{2} + \frac{1}{2}i$.

$$f'\left(\tfrac{1}{2} + \tfrac{1}{2}i\right) = u_x + iv_x \Big|_{z=\frac{1}{2}+\frac{1}{2}i} = (2x+i)\Big|_{z=\frac{1}{2}+\frac{1}{2}i} = 1+i.$$

$f(z)$ is not differentiable in any neighborhood $\Rightarrow$ $f(z)$ is not analytic at any point.

(b) $f(re^{i\theta}) = e^{\theta}\sin(\ln r) + ie^{\theta}\cos(\ln r)$, $r > 0, 0 < \theta < 2\pi$.

$u(r,\theta) = e^{\theta}\cdot\sin(\ln r)$, $v(r,\theta) = e^{\theta}\cdot\cos(\ln r)$

$u_r = e^{\theta}\cdot\cos(\ln r)\cdot\frac{1}{r}$, $u_\theta = e^{\theta}\cdot\sin(\ln r)$.

$v_r = -e^{\theta}\cdot\sin(\ln r)\cdot\frac{1}{r}$, $v_\theta = e^{\theta}\cdot\cos(\ln r)$.

$$\Rightarrow \begin{cases} ru_r = v_\theta = e^{\theta}\cdot\cos(\ln r). \\ u_\theta = -rv_r = e^{\theta}\sin(\ln r) \end{cases}$$ i.e. CR eqs. are satisfied.

$\Rightarrow$ $f(z)$ is analytic (differentiable) at any point in the domain $\{r>0, 0<\theta<2\pi\}$

$$f'(re^{i\theta}) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\cdot\left(\frac{e^{\theta}\cos(\ln r)}{r} + i\cdot\frac{-e^{\theta}\sin(\ln r)}{r}\right)$$

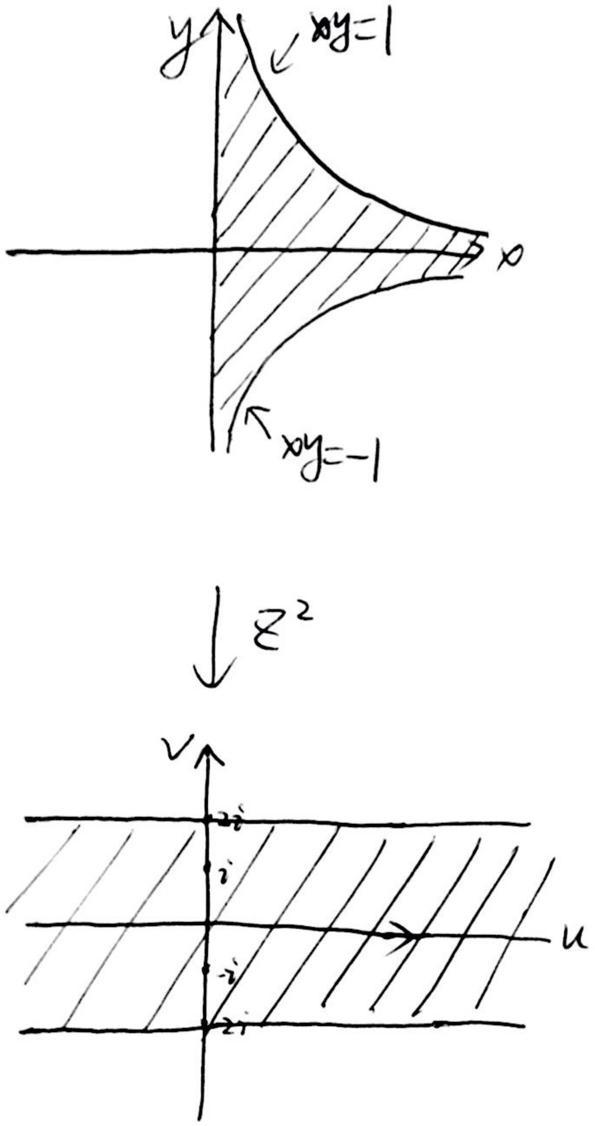
$$= \frac{e^{\theta}\cdot\cos(\ln r) - i\,e^{\theta}\sin(\ln r)}{re^{i\theta}} = -i\cdot\frac{f(z)}{z}$$

$$\left(f(z) = i\cdot e^{\theta}\cdot(\cos(\ln r) - i\cdot\sin(\ln r)) = ie^{\theta}\cdot e^{-i\ln r} = i\cdot e^{-i\log z}\right)$$

$\Rightarrow f'(z) = ie^{-i\log z}\cdot\frac{-i}{z} = \frac{-i\,f(z)}{z}$

Extra Credit Problem

z-plane



w-plane

!!! WRITE YOUR NAME, STUDENT ID AND LECTURE N. BELOW !!!

NAME :

ID :

1. (50pts) (a): Calculate the principal value of $(1+i)^{(1+i)}$.

$$1+i=\sqrt{2}\cdot e^{i\frac{\pi}{4}} \Rightarrow \mathrm{Log}(1+i)=\ln\sqrt{2}+i\,\mathrm{Arg}(1+i)=\frac{1}{2}\ln 2+i\cdot\frac{\pi}{4}$$

$$\text{P.V. } (1+i)^{1+i}=e^{(1+i)\cdot \mathrm{Log}(1+i)}=e^{(1+i)\cdot(\frac{1}{2}\ln 2+i\cdot\frac{\pi}{4})}$$

$$=e^{(\frac{1}{2}\ln 2-\frac{\pi}{4})+i(\frac{1}{2}\ln 2+\frac{\pi}{4})}$$

$$=e^{(\frac{1}{2}\ln 2-\frac{\pi}{4})}\left(\cos\left(\frac{1}{2}\ln 2+\frac{\pi}{4}\right)+i\cdot\sin\left(\frac{1}{2}\ln 2+\frac{\pi}{4}\right)\right)$$

(b): Choose **the branch of** $\log z$ as

$$\log z=\ln|z|+i\,\theta, -\frac{5\pi}{3}\le\theta<\frac{\pi}{3}.$$

Calculate $\log(1+i)^2$ and $2\log(1+i)$. Are they equal to each other?

$$(1+i)^2=2i \Rightarrow \log(1+i)^2=\ln 2+i\cdot\left(\frac{\pi}{2}+2\pi\cdot n\right).$$

$$\parallel$$

$$(1+i)\cdot(1+i)=(1-1)+(i+i)$$

To fit in the chosen branch, we need to take $n=-1$ to get $\left(-\frac{5\pi}{3}<-\frac{3\pi}{2}<\frac{\pi}{3}\right)$

$$\log(1+i)^2=\ln 2+i\cdot\left(\frac{\pi}{2}-2\pi\right)=\ln 2-i\frac{3\pi}{2}$$

On the other hand, $\left(\text{since } -\frac{5\pi}{3}<\frac{\pi}{4}<\frac{\pi}{3}\right)$

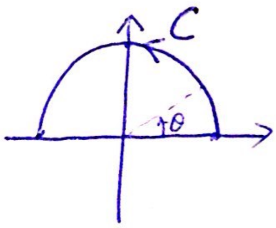
$$2\log(1+i)=2\cdot\left(\ln\sqrt{2}+i\cdot\frac{\pi}{4}\right)=\ln 2+i\cdot\frac{\pi}{2}$$

so

$$\log(1+i)^2=\ln 2-i\cdot\frac{3\pi}{2}\neq\ln 2+i\cdot\frac{\pi}{2}=2\cdot\log(1+i)$$

2.(50pts) Let C denote the **upper semicircle** of the circle $|z| = \pi$ oriented **anti-clockwisely**. **(a).** Calculate

$$\int_C (\bar{z})^{-2} dz.$$



$C:\ z = \pi \cdot e^{i\theta} \quad 0 \le \theta \le \pi.$

$z'(\theta) = \pi i \cdot e^{i\theta}, \quad \overline{z(\theta)} = \pi \cdot e^{-i\theta}$

$$\int_C (\bar{z})^{-2} dz = \int_0^{\pi} \pi^{-2} \cdot (e^{-i\theta})^{-2} \cdot \pi \cdot i e^{i\theta} d\theta = \int_0^{\pi} \pi^{-1} \cdot e^{3i\theta} d(i\theta)$$

$$= \frac{1}{\pi} \cdot \frac{e^{3i\theta}}{3} \Big]_0^{\pi} = \frac{1}{3\pi}(e^{3i\pi} - e^0) = -\frac{2}{3\pi}.$$

(b). Calculate the contour integral: $\int_C \sin(z)dz$.

$\sin(z) = -(\cos z)'$, so

$$\int_C \sin(z) dz = \int_C -(\cos z)' dz = -\cos z \Big]_{-\pi}^{\pi}$$

$$= -(\cos \pi - \cos(-\pi)) = -(-1-(-1)) = 0.$$

3. (50pts) (a). Factorize the polynomial $z^3 - 1$ into linear factors.

$$z^3-1=(z-1)\cdot(z^2+z+1)=(z-1)\cdot\left(z-\left(-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)\right)\cdot\left(z-\left(-\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i\right)\right).$$

Or find cubic roots $1^{\frac{1}{3}} = 1^{\frac{1}{3}}\cdot e^{i\cdot\left(\frac{0}{3}+\frac{2\pi}{3}\cdot k\right)}$ $\quad k=0,1,2$

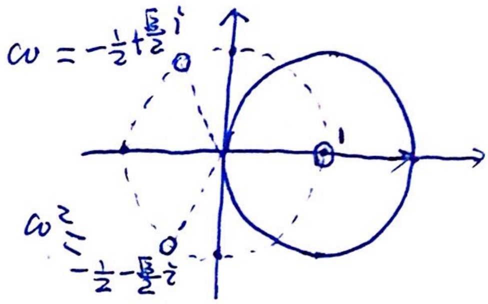
$$= 1,\ e^{i\frac{2\pi}{3}},\ e^{i\cdot\frac{4\pi}{3}}$$

$$= 1,\ -\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i,\ -\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i$$

$$\Rightarrow z^3-1=(z-1)\cdot\left(z-\left(-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)\right)\left(z-\left(-\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i\right)\right)$$

(b). Calculate the integral:

$$\int_{|z-1|=1}\frac{dz}{z^3-1}.$$



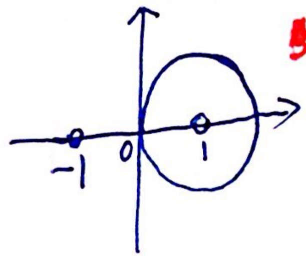
$$\int_{|z-1|=1}\frac{dz}{z^3-1}=\int_{|z-1|=1}\frac{dz}{(z-1)(z-\omega)(z-\omega^2)}$$

$$=\int_{|z-1|=1}\frac{dz}{(z-1)\cdot(z^2+z+1)}$$

$$=2\pi i\cdot\frac{1}{z^2+z+1}\Bigg]_{z=1}=\frac{2\pi i}{3}$$

4. (50pts)(a). Calculate the integral:

$$\int_{|z-1|=1} \frac{dz}{(z^2-1)^3}.$$



$$\int_{|z-1|=1} \frac{dz}{(z^2-1)^3} = \int_{|z-1|=1} \frac{dz}{(z+1)^3(z-1)^3} \quad 3$$

$$= \frac{2\pi i}{2!} \cdot \frac{d^2}{dz^2}(z+1)^{-3}\Big]_{z=1}^{10} = \pi i \cdot (-3)\cdot(-4)\cdot(z+1)^{-5}\Big]_{z=1} \quad 5$$

$$= \frac{12\pi i}{2^5} = \frac{12\pi i}{32} = \frac{3\pi i}{8} \quad 2$$

(b). Calculate the series:

$$\sum_{n=1}^{+\infty} \frac{1-i^n}{2^n}$$

$$\sum_{n=1}^{+\infty} \frac{1-i^n}{2^n} = \sum_{n=1}^{+\infty} \left(\frac{1}{2}\right)^n - \sum_{n=1}^{+\infty} \left(\frac{i}{2}\right)^n \quad \cancel{8}6$$

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}} \cancel{+} - \frac{\frac{i}{2}}{1-\frac{i}{2}} \quad 12$$

$$= 1 - \frac{i}{2-i} = 1 - \frac{i\cdot(2+i)}{(2-i)\cdot(2+i)} \quad 5$$

$$= 1 - \frac{-1+2i}{5} = \frac{6}{5} - \frac{2}{5}i. \quad 2$$

5.(50pts) (a). Assume $f(z) = u(z) + iv(z)$ is analytic on the closed disk $\{|z| \leq 2\}$. Assume that $u(z)$ obtains a minimum at $z = 1$. What can you say about the function $f(z)$? Explain your reason.

By assumptions, $u(z) \geq u(1)$, for any $z \in D = \{|z| \leq 2\}$.

Consider the function $g(z) = e^{-f(z)}$

$= e^{-u-iv}$ 5

Then $u(z) \geq u(1) \Rightarrow -u(z) \leq -u(1)$.

$|g(z)| = |e^{-u} \cdot e^{-iv}| = e^{-u(z)} \leq e^{-u(1)}$ 5

f analytic on $D \Rightarrow g(z)$ is also analytic on D.

10

So the analytic function $g(z)$ obtains its maximum at the interior point $z = 1 \in D$. By maximum modulus principle, $g(z)$ is a constant function. So

$f(z) = -\log g(z)$ must also be a constant function.

5

(b). Assume that $f(z)$ is an entire function satisfying $|f(z)| < A|z| + B$ for every $z \in \mathbb{C}$, for some uniform constants $A > 0$ and $B > 0$. What can you say about the function $f(z)$? Explain your reason.

This is problem P171.10 of homework 10.

$f(z)$ is a linear function because of the following reasons.

By Cauchy's formula for the 2nd order derivative:

$$f''(z_0) = \frac{2!}{2\pi i}\int_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^3}\,dz \quad \text{for any } z_0 \in \mathbb{C} \text{ and } R>0 \quad 5$$

$$\Rightarrow |f''(z_0)| = \left|\frac{2}{2\pi i}\right| \cdot \int_{|z-z_0|=R} \left|\frac{f(z)}{(z-z_0)^3}\right| |dz| \quad 5$$

$$\le \frac{1}{\pi}\int_{|z-z_0|=R} \frac{A\cdot|z|+B}{R^3}|dz| \le \frac{1}{\pi R^3}\int_{|z-z_0|=R} \left(A(|z-z_0|+|z_0|)+B\right)|dz| \quad 5$$

$$= \frac{1}{\pi R^3}\cdot\left(A(R+|z_0|)+B\right)2\pi R = \frac{2(AR+A|z_0|+B)}{R^2} \xrightarrow{R\to+\infty} 0 \text{ for any } z_0\in\mathbb{C} \quad 5$$

So $f''(z_0) = 0$ for any $z_0 \in \mathbb{C}$.

$$\Rightarrow f'(z) = a_1 \text{ constant} \Rightarrow \boxed{f(z) = a_1\cdot z + b.} \quad 5$$

$$\left(|f(z)| = \underset{\underset{|a||z|-|b|}{\vee}}{|az+b|} \le A|z| + B \quad \forall z\in\mathbb{C} \Rightarrow |a| \le A \right)$$

6. (10pts)(Extra credit)

Estimate the following quantity from above without calculating it:

$$\left|\int_{|z|=10} \frac{z-10}{(z-1)(z-2)} dz\right|.$$

$$\left|\int_{|z|=10} \frac{z-10}{(z-1)(z-2)} dz\right| \leq \int_{|z|=10} \left|\frac{z-10}{(z-1)\cdot(z-2)}\right| |dz|$$ 2

$$= \int_{|z|=10} \frac{|z-10|}{|z-1|\cdot|z-2|} |dz|$$ 1

$$\leq \int_{|z|=10} \frac{|z|+10}{(|z|-1)\cdot(|z|-2)} |dz| \overset{6}{=} \int_{|z|=10} \frac{10+10}{(10-1)(10-2)} \cdot |dz|$$

$$= \frac{20}{9\times 8} \times 2\pi 10 = \frac{50\pi}{9}$$ 1

or estimate as $$\left|\frac{z-10}{(z-1)(z-2)}\right| = \left|\frac{z-10}{z^2-3z+2}\right| \leq \frac{|z|+10}{|z|^2-3|z|-2}$$

$$\overset{|z|=10}{=} \frac{10+10}{100-30-2} = \frac{20}{68} = \frac{5}{17}$$

to get $$\left|\int_{|z|=10} \frac{z-10}{(z-1)(z-2)} dz\right| \leq \frac{5}{17} \cdot 2\pi 10 = \frac{100\pi}{17}.$$

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for every complex number z. Furthermore, 0 and 1 are the only complex numbers with such properties (see Exercise 8).

There is associated with each complex number $z = (x, y)$ an additive inverse

$$-z = (-x, -y), \tag{5}$$

satisfying the equation $z + (-z) = 0$. Moreover, there is only one additive inverse for any given z, since the equation

$$(x, y) + (u, v) = (0, 0)$$

implies that

$$u = -x \quad \text{and} \quad v = -y.$$

For any *nonzero* complex number $z = (x, y)$, there is a number z^{-1} such that $zz^{-1} = 1$. This multiplicative inverse is less obvious than the additive one. To find it, we seek real numbers u and v, expressed in terms of x and y, such that

$$(x, y)(u, v) = (1, 0).$$

According to equation (4), Sec. 1, which defines the product of two complex numbers, u and v must satisfy the pair

$$xu - yv = 1, \quad yu + xv = 0$$

of linear simultaneous equations; and simple computation yields the unique solution

$$u = \frac{x}{x^2 + y^2}, \quad v = \frac{-y}{x^2 + y^2}.$$

So *the* multiplicative inverse of $z = (x, y)$ is

$$z^{-1} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right) \qquad (z \neq 0). \tag{6}$$

The inverse z^{-1} is not defined when $z = 0$. In fact, $z = 0$ means that $x^2 + y^2 = 0$; and this is not permitted in expression (6).

EXERCISES

1. Verify that

   (*a*) $(\sqrt{2} - i) - i(1 - \sqrt{2}i) = -2i$;

   (*b*) $(2, -3)(-2, 1) = (-1, 8)$;

   (*c*) $(3, 1)(3, -1)\left(\frac{1}{5}, \frac{1}{10}\right) = (2, 1)$.

2. Show that

(a) $\operatorname{Re}(iz) = -\operatorname{Im} z$;

(b) $\operatorname{Im}(iz) = \operatorname{Re} z$.

3. Show that $(1+z)^2 = 1 + 2z + z^2$.

4. Verify that each of the two numbers $z = 1 \pm i$ satisfies the equation $z^2 - 2z + 2 = 0$.

5. Prove that multiplication of complex numbers is commutative, as stated at the beginning of Sec. 2.

6. Verify

(a) the associative law for addition of complex numbers, stated at the beginning of Sec. 2;

(b) the distributive law (3), Sec. 2.

7. Use the associative law for addition and the distributive law to show that

$$z(z_1 + z_2 + z_3) = zz_1 + zz_2 + zz_3.$$

8. (a) Write $(x, y) + (u, v) = (x, y)$ and point out how it follows that the complex number $0 = (0, 0)$ is unique as an additive identity.

(b) Likewise, write $(x, y)(u, v) = (x, y)$ and show that the number $1 = (1, 0)$ is a unique multiplicative identity.

9. Use $-1 = (-1, 0)$ and $z = (x, y)$ to show that $(-1)z = -z$.

10. Use $i = (0, 1)$ and $y = (y, 0)$ to verify that $-(iy) = (-i)y$. Thus show that the additive inverse of a complex number $z = x + iy$ can be written $-z = -x - iy$ without ambiguity.

11. Solve the equation $z^2 + z + 1 = 0$ for $z = (x, y)$ by writing

$$(x, y)(x, y) + (x, y) + (1, 0) = (0, 0)$$

and then solving a pair of simultaneous equations in x and y.

Suggestion: Use the fact that no real number x satisfies the given equation to show that $y \neq 0$.

Ans. $z = \left(-\dfrac{1}{2}, \pm\dfrac{\sqrt{3}}{2}\right)$.

3. FURTHER ALGEBRAIC PROPERTIES

In this section, we mention a number of other algebraic properties of addition and multiplication of complex numbers that follow from the ones already described in Sec. 2. Inasmuch as such properties continue to be anticipated because they also apply to real numbers, the reader can easily pass to Sec. 4 without serious disruption.

We begin with the observation that the existence of multiplicative inverses enables us to show that *if a product z_1z_2 is zero, then so is at least one of the factors z_1 and z_2*. For suppose that $z_1z_2 = 0$ and $z_1 \neq 0$. The inverse z_1^{-1} exists; and any complex number times zero is zero (Sec. 1). Hence

$$z_2 = z_2 \cdot 1 = z_2\left(z_1z_1^{-1}\right) = \left(z_1^{-1}z_1\right)z_2 = z_1^{-1}(z_1z_2) = z_1^{-1} \cdot 0 = 0.$$

There are some expected properties involving quotients that follow from the relation

$$\frac{1}{z_2} = z_2^{-1} \qquad (z_2 \neq 0), \tag{9}$$

which is equation (2) when $z_1 = 1$. Relation (9) enables us, for instance, to write equation (2) in the form

$$\frac{z_1}{z_2} = z_1\left(\frac{1}{z_2}\right) \qquad (z_2 \neq 0). \tag{10}$$

Also, by observing that (see Exercise 3)

$$(z_1z_2)(z_1^{-1}z_2^{-1}) = (z_1z_1^{-1})(z_2z_2^{-1}) = 1 \qquad (z_1 \neq 0, z_2 \neq 0),$$

and hence that $z_1^{-1}z_2^{-1} = (z_1z_2)^{-1}$, one can use relation (9) to show that

$$\left(\frac{1}{z_1}\right)\left(\frac{1}{z_2}\right) = z_1^{-1}z_2^{-1} = (z_1z_2)^{-1} = \frac{1}{z_1z_2} \qquad (z_1 \neq 0, z_2 \neq 0). \tag{11}$$

Another useful property, to be derived in the exercises, is

$$\left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right) = \frac{z_1z_2}{z_3z_4} \qquad (z_3 \neq 0, z_4 \neq 0). \tag{12}$$

Finally, we note that the ***binomial formula*** involving real numbers remains valid with complex numbers. That is, if z_1 and z_2 are any two nonzero complex numbers, then

$$(z_1 + z_2)^n = \sum_{k=0}^{n}\binom{n}{k} z_1^k z_2^{n-k} \qquad (n = 1, 2, \ldots) \tag{13}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad (k = 0, 1, 2, \ldots, n)$$

and where it is agreed that $0! = 1$. The proof is left as an exercise. Because addition of complex numbers is commutative, the binomial formula can, of course, be written

$$(z_1 + z_2)^n = \sum_{k=0}^{n}\binom{n}{k} z_1^{n-k} z_2^k \qquad (n = 1, 2, \ldots). \tag{14}$$

EXERCISES

1. Reduce each of these quantities to a real number:

(*a*) $\dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i}$; (*b*) $\dfrac{5i}{(1-i)(2-i)(3-i)}$; (*c*) $(1-i)^4$.

Ans. (*a*) $-\dfrac{2}{5}$; (*b*) $-\dfrac{1}{2}$; (*c*) -4.

2. Show that

$$\frac{1}{1/z} = z \qquad (z \neq 0).$$

3. Use the associative and commutative laws for multiplication to show that

$$(z_1 z_2)(z_3 z_4) = (z_1 z_3)(z_2 z_4).$$

4. Prove that if $z_1 z_2 z_3 = 0$, then at least one of the three factors is zero.

Suggestion: Write $(z_1 z_2) z_3 = 0$ and use a similar result (Sec. 3) involving two factors.

5. Derive expression (6), Sec. 3, for the quotient z_1/z_2 by the method described just after it.

6. With the aid of relations (10) and (11) in Sec. 3, derive the identity

$$\left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right) = \frac{z_1 z_2}{z_3 z_4} \qquad (z_3 \neq 0, z_4 \neq 0).$$

7. Use the identity obtained in Exercise 6 to derive the cancellation law

$$\frac{z_1 z}{z_2 z} = \frac{z_1}{z_2} \qquad (z_2 \neq 0, z \neq 0).$$

8. Use mathematical induction to verify the binomial formula (13) in Sec. 3. More precisely, note that the formula is true when $n = 1$. Then, assuming that it is valid when $n = m$ where m denotes any positive integer, show that it must hold when $n = m + 1$.

Suggestion: When $n = m + 1$, write

$$(z_1 + z_2)^{m+1} = (z_1 + z_2)(z_1 + z_2)^m = (z_2 + z_1)\sum_{k=0}^{m} \binom{m}{k} z_1^k z_2^{m-k}$$

$$= \sum_{k=0}^{m} \binom{m}{k} z_1^k z_2^{m+1-k} + \sum_{k=0}^{m} \binom{m}{k} z_1^{k+1} z_2^{m-k}$$

and replace k by $k - 1$ in the last sum here to obtain

$$(z_1 + z_2)^{m+1} = z_2^{m+1} + \sum_{k=1}^{m} \left[\binom{m}{k} + \binom{m}{k-1}\right] z_1^k z_2^{m+1-k} + z_1^{m+1}.$$

Finally, show how the right-hand side here becomes

$$z_2^{m+1} + \sum_{k=1}^{m} \binom{m+1}{k} z_1^k z_2^{m+1-k} + z_1^{m+1} = \sum_{k=0}^{m+1} \binom{m+1}{k} z_1^k z_2^{m+1-k}.$$

4. VECTORS AND MODULI

It is natural to associate any nonzero complex number $z = x + iy$ with the directed line segment, or vector, from the origin to the point (x, y) that represents z in the complex plane. In fact, we often refer to z as the point z or the vector z. In Fig. 2 the numbers $z = x + iy$ and $-2 + i$ are displayed graphically as both points and radius vectors when $z \neq 0$. Next, we multiply through equation (7) by z^n:

$$wz^n = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}.$$

This tells us that

$$|w||z|^n \le |a_0| + |a_1||z| + |a_2||z|^2 + \cdots + |a_{n-1}||z|^{n-1},$$

or

$$|w| \le \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \frac{|a_2|}{|z|^{n-2}} + \cdots + \frac{|a_{n-1}|}{|z|}. \tag{9}$$

Now that a sufficiently large positive number R can be found such that each of the quotients on the right in inequality (9) is less than the number $|a_n|/(2n)$ when $|z| > R$, and so

$$|w| < n\frac{|a_n|}{2n} = \frac{|a_n|}{2} \quad \text{whenever} \quad |z|>R.$$

Consequently,

$$|a_n + w| \ \ge \ ||a_n| - |w|| \ > \frac{|a_n|}{2} \quad \text{whenever} \quad |z| > R;$$

and, in view of equation (8),

$$|P_n(z)| = |a_n + w||z|^n > \frac{|a_n|}{2}|z|^n > \frac{|a_n|}{2}R^n \quad \text{whenever} \quad |z| > R. \tag{10}$$

Statement (6) follows immediately from this.

EXERCISES

1. Locate the numbers $z_1 + z_2$ and $z_1 - z_2$ vectorially when
   (*a*) $z_1 = 2i, \quad z_2 = \dfrac{2}{3} - i$;
   (*b*) $z_1 = (-\sqrt{3}, 1), \quad z_2 = (\sqrt{3}, 0)$;
   (*c*) $z_1 = (-3, 1), \quad z_2 = (1, 4)$;
   (*d*) $z_1 = x_1 + iy_1, \quad z_2 = x_1 - iy_1$.
2. Verify inequalities (3), Sec. 4, involving $\operatorname{Re} z$, $\operatorname{Im} z$, and $|z|$.
3. Use established properties of moduli to show that when $|z_3| \neq |z_4|$,
$$\frac{\operatorname{Re}(z_1 + z_2)}{|z_3 + z_4|} \le \frac{|z_1| + |z_2|}{||z_3| - |z_4||}.$$
4. Verify that $\sqrt{2}\,|z| \ge |\operatorname{Re} z| + |\operatorname{Im} z|$.

   Suggestion: Reduce this inequality to $(|x| - |y|)^2 \ge 0$.
5. In each case, sketch the set of points determined by the given condition:
   (*a*) $|z - 1 + i| = 1$; (*b*) $|z + i| \le 3$; (*c*) $|z - 4i| \ge 4$.

6. Using the fact that $|z_1 - z_2|$ is the distance between two points z_1 and z_2, give a geometric argument that $|z-1| = |z+i|$ represents the line through the origin whose slope is -1.

7. Show that for R sufficiently large, the polynomial $P(z)$ in Example 3, Sec. 5, satisfies the inequality

$$|P(z)| < 2|a_n||z|^n \quad \text{whenever} \quad |z| > R.$$

Suggestion: Observe that there is a positive number R such that the modulus of each quotient in inequality (9), Sec. 5, is less than $|a_n|/n$ when $|z| > R$.

8. Let z_1 and z_2 denote any complex numbers

$$z_1 = x_1 + iy_1 \quad \text{and} \quad z_2 = x_2 + iy_2.$$

Use simple algebra to show that

$$|(x_1 + iy_1)(x_2 + iy_2)| \quad \text{and} \quad \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$

are the same and then point out how the identity

$$|z_1 z_2| = |z_1||z_2|$$

follows.

9. Use the final result in Exercise 8 and mathematical induction to show that

$$|z^n| = |z|^n \qquad (n = 1, 2, \ldots),$$

where z is any complex number. That is, after noting that this identity is obviously true when $n = 1$, assume that it is true when $n = m$ where m is any positive integer and then show that it must be true when $n = m + 1$.

6. COMPLEX CONJUGATES

The ***complex conjugate***, or simply the conjugate, of a complex number $z = x + iy$ is defined as the complex number $x - iy$ and is denoted by $\bar{z}$; that is,

$$\bar{z} = x - iy. \tag{1}$$

The number $\bar{z}$ is represented by the point $(x, -y)$, which is the reflection in the real axis of the point (x, y) representing z (Fig. 5). Note that

$$\bar{\bar{z}} = z \quad \text{and} \quad |\bar{z}| = |z|$$

for all z.

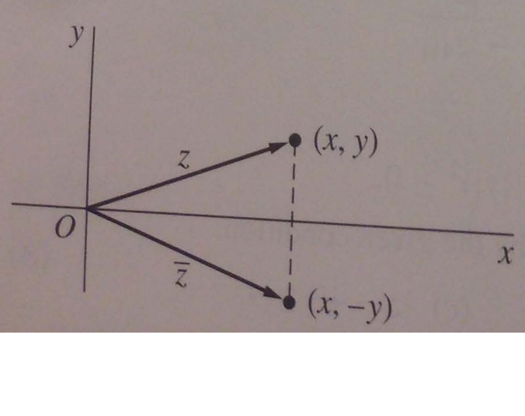

FIGURE 5

and recalling that a modulus is never negative. Property (9) can be verified in a similar way.

EXAMPLE 2. Property (8) tells us that $|z^2| = |z|^2$ and $|z^3| = |z|^3$. Hence if z is a point inside the circle centered at the origin with radius 2, so that $|z| < 2$, it follows from the generalized triangle inequality (4) in Sec. 5 that

$$|z^3 + 3z^2 - 2z + 1| \le |z|^3 + 3|z|^2 + 2|z| + 1 < 25.$$

EXERCISES

1. Use properties of conjugates and moduli established in Sec. 6 to show that

(*a*) $\overline{\bar{z} + 3i} = z - 3i$; (*b*) $\overline{iz} = -i\bar{z}$;

(*c*) $\overline{(2+i)^2} = 3 - 4i$; (*d*) $|(2\bar{z}+5)(\sqrt{2}-i)| = \sqrt{3}\,|2z+5|$.

2. Sketch the set of points determined by the condition

(*a*) $\operatorname{Re}(\bar{z} - i) = 2$; (*b*) $|2\bar{z} + i| = 4$.

3. Verify properties (3) and (4) of conjugates in Sec. 6.

4. Use property (4) of conjugates in Sec. 6 to show that

(*a*) $\overline{z_1 z_2 z_3} = \bar{z}_1\,\bar{z}_2\,\bar{z}_3$; (*b*) $\overline{z^4} = \bar{z}^4$.

5. Verify property (9) of moduli in Sec. 6.

6. Use results in Sec. 6 to show that when z_2 and z_3 are nonzero,

(*a*) $\overline{\left(\dfrac{z_1}{z_2 z_3}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2\,\bar{z}_3}$; (*b*) $\left|\dfrac{z_1}{z_2 z_3}\right| = \dfrac{|z_1|}{|z_2||z_3|}$.

7. Show that

$$|\operatorname{Re}(2 + \bar{z} + z^3)| \le 4 \qquad \text{when } |z| \le 1.$$

8. It is shown in Sec. 3 that if $z_1 z_2 = 0$, then at least one of the numbers z_1 and z_2 must be zero. Give an alternative proof based on the corresponding result for real numbers and using identity (8), Sec. 6.

9. By factoring $z^4 - 4z^2 + 3$ into two quadratic factors and using inequality (2), Sec. 5, show that if z lies on the circle $|z| = 2$, then

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right| \le \frac{1}{3}.$$

10. Prove that

(*a*) z is real if and only if $\bar{z} = z$;

(*b*) z is either real or pure imaginary if and only if $\bar{z}^2 = z^2$.

11. Use mathematical induction to show that when $n = 2, 3, \ldots$,

(*a*) $\overline{z_1 + z_2 + \cdots + z_n} = \bar{z}_1 + \bar{z}_2 + \cdots + \bar{z}_n$;

(*b*) $\overline{z_1 z_2 \cdots z_n} = \bar{z}_1\,\bar{z}_2 \cdots \bar{z}_n$.

12. Let $a_0, a_1, a_2, \ldots, a_n$ ($n \geq 1$) denote *real* numbers, and let z be any complex number. With the aid of the results in Exercise 11, show that

$$\overline{a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n} = a_0 + a_1 \overline{z} + a_2 \overline{z}^2 + \cdots + a_n \overline{z}^n.$$

13. Show that the equation $|z - z_0| = R$ of a circle, centered at z_0 with radius R, can be written

$$|z|^2 - 2\,\mathrm{Re}(z\overline{z_0}) + |z_0|^2 = R^2.$$

14. Using expressions (6), Sec. 6, for $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$, show that the hyperbola $x^2 - y^2 = 1$ can be written

$$z^2 + \overline{z}^2 = 2.$$

15. Follow the steps below to give an algebraic derivation of the triangle inequality (Sec. 5)

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

(*a*) Show that

$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) = z_1\overline{z_1} + (z_1\overline{z_2} + \overline{z_1\overline{z_2}}) + z_2\overline{z_2}.$$

(*b*) Point out why

$$z_1\overline{z_2} + \overline{z_1\overline{z_2}} = 2\,\mathrm{Re}(z_1\overline{z_2}) \leq 2|z_1||z_2|.$$

(*c*) Use the results in parts (*a*) and (*b*) to obtain the inequality

$$|z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2,$$

and note how the triangle inequality follows.

7. EXPONENTIAL FORM

Let r and θ be polar coordinates of the point (x, y) that corresponds to a *nonzero* complex number $z = x + iy$. Since $x = r\cos\theta$ and $y = r\sin\theta$, the number z can be written in ***polar form*** as

$$z = r(\cos\theta + i\sin\theta). \tag{1}$$

If $z = 0$, the coordinate θ is undefined; and so it is understood that $z \neq 0$ whenever polar coordinates are used.

In complex analysis, the real number r is not allowed to be negative and is the length of the radius vector for z; that is, $r = |z|$. The real number θ represents the angle, measured in radians, that z makes with the positive real axis when z is interpreted as a radius vector (Fig. 6). As in calculus, θ has an infinite number of possible values, including negative ones, that differ by integral multiples of 2π. Those values can be determined from the equation $\tan\theta = y/x$, where the quadrant containing the poin corresponding to z must be specified. Each value of θ is called an ***argument*** of z, an the set of all such values is denoted by $\arg z$. The ***principal value*** of $\arg z$, denoted b

Statement (2) tells us that

$$\arg\left(\frac{z_1}{z_2}\right) = \arg\left(z_1 z_2^{-1}\right) = \arg z_1 + \arg\left(z_2^{-1}\right);$$

and, since (Sec. 8)

$$z_2^{-1} = \frac{1}{r_2} e^{-i\theta_2},$$

one can see that

$$\arg\left(z_2^{-1}\right) = -\arg z_2. \tag{3}$$

Hence

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2. \tag{4}$$

Statement (3) is, of course, to be interpreted as saying that the set of all values on the left-hand side is the same as the set of all values on the right-hand side. Statement (4) is, then, to be interpreted in the same way that statement (2) is.

EXAMPLE 2. In order to illustrate statement (4), let us use it to find the principal value of $\operatorname{Arg} z$ when

$$z = \frac{i}{-1-i}.$$

We start by writing

$$\arg z = \arg i - \arg(-1-i).$$

Since

$$\operatorname{Arg} i = \frac{\pi}{2} \quad \text{and} \quad \operatorname{Arg}(-1-i) = -\frac{3\pi}{4},$$

one value of $\arg z$ is $5\pi/4$. But this is not a *principal* value Θ, which must lie in the interval $-\pi < \Theta \leq \pi$. We can, however, obtain that value by adding some integral multiple, possibly negative, of 2π:

$$\operatorname{Arg}\left(\frac{i}{-1-i}\right) = \frac{5\pi}{4} - 2\pi = -\frac{3\pi}{4}.$$

EXERCISES

1. Find the principal argument $\operatorname{Arg} z$ when

(*a*) $z = \dfrac{-2}{1+\sqrt{3}i}$; (*b*) $z = \left(\sqrt{3}-i\right)^6$.

Ans. (*a*) $2\pi/3$; (*b*) π.

2. Show that (*a*) $|e^{i\theta}| = 1$; (*b*) $\overline{e^{i\theta}} = e^{-i\theta}$.

3. Use mathematical induction to show that

$$e^{i\theta_1}e^{i\theta_2}\cdots e^{i\theta_n}=e^{i(\theta_1+\theta_2+\cdots+\theta_n)} \qquad (n=2,3,\ldots).$$

4. Using the fact that the modulus $|e^{i\theta}-1|$ is the distance between the points $e^{i\theta}$ and 1 (see Sec. 4), give a geometric argument to find a value of θ in the interval $0\le\theta<2\pi$ that satisfies the equation $|e^{i\theta}-1|=2$.

Ans. π.

5. By writing the individual factors on the left in exponential form, performing the needed operations, and finally changing back to rectangular coordinates, show that

(*a*) $i(1-\sqrt{3}i)(\sqrt{3}+i)=2(1+\sqrt{3}i)$; (*b*) $5i/(2+i)=1+2i$;

(*c*) $(\sqrt{3}+i)^6=-64$; (*d*) $(1+\sqrt{3}\,i)^{-10}=2^{-11}(-1+\sqrt{3}\,i)$.

6. Show that if $\operatorname{Re} z_1>0$ and $\operatorname{Re} z_2>0$, then

$$\operatorname{Arg}(z_1z_2)=\operatorname{Arg} z_1+\operatorname{Arg} z_2,$$

where principal arguments are used.

7. Let z be a nonzero complex number and n a negative integer $(n=-1,-2,\ldots)$. Also, write $z=re^{i\theta}$ and $m=-n=1,2,\ldots$. Using the expressions

$$z^m=r^me^{im\theta} \quad\text{and}\quad z^{-1}=\left(\frac{1}{r}\right)e^{i(-\theta)},$$

verify that $(z^m)^{-1}=(z^{-1})^m$ and hence that the definition $z^n=(z^{-1})^m$ in Sec. 7 could have been written alternatively as $z^n=(z^m)^{-1}$.

8. Prove that two nonzero complex numbers z_1 and z_2 have the same moduli if and only if there are complex numbers c_1 and c_2 such that $z_1=c_1c_2$ and $z_2=c_1\overline{c_2}$.

Suggestion: Note that

$$\exp\left(i\frac{\theta_1+\theta_2}{2}\right)\exp\left(i\frac{\theta_1-\theta_2}{2}\right)=\exp(i\theta_1)$$

and [see Exercise 2(*b*)]

$$\exp\left(i\frac{\theta_1+\theta_2}{2}\right)\overline{\exp\left(i\frac{\theta_1-\theta_2}{2}\right)}=\exp(i\theta_2).$$

9. Establish the identity

$$1+z+z^2+\cdots+z^n=\frac{1-z^{n+1}}{1-z} \qquad (z\ne 1)$$

and then use it to derive ***Lagrange's trigonometric identity:***

$$1+\cos\theta+\cos 2\theta+\cdots+\cos n\theta=\frac{1}{2}+\frac{\sin[(2n+1)\theta/2]}{2\sin(\theta/2)} \qquad (0<\theta<2\pi).$$

Suggestion: As for the first identity, write $S=1+z+z^2+\cdots+z^n$ and consider the difference $S-zS$. To derive the second identity, write $z=e^{i\theta}$ in the first one.

10. Use de Moivre's formula (Sec. 8) to derive the following trigonometric identities:

(*a*) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$;

(*b*) $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

11. (*a*) Use the binomial formula (14), Sec. 3, and de Moivre's formula (Sec. 8) to write

$$\cos n\theta + i\sin n\theta = \sum_{k=0}^{n}\binom{n}{k}\cos^{n-k}\theta\,(i\sin\theta)^k \qquad (n = 0, 1, 2, \ldots).$$

Then define the integer m by means of the equations

$$m = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (n-1)/2 & \text{if } n \text{ is odd} \end{cases}$$

and use the above summation to show that [compare with Exercise 10(*a*)]

$$\cos n\theta = \sum_{k=0}^{m}\binom{n}{2k}(-1)^k\cos^{n-2k}\theta\sin^{2k}\theta \qquad (n = 0, 1, 2, \ldots).$$

(*b*) Write $x = \cos\theta$ in the final summation in part (*a*) to show that it becomes a polynomial*

$$T_n(x) = \sum_{k=0}^{m}\binom{n}{2k}(-1)^k x^{n-2k}(1-x^2)^k$$

of degree n ($n = 0, 1, 2, \ldots$) in the variable x.

10. ROOTS OF COMPLEX NUMBERS

Consider now a point $z = re^{i\theta}$, lying on a circle centered at the origin with radius r (Fig. 10). As θ is increased, z moves around the circle in the counterclockwise direction. In particular, when θ is increased by 2π, we arrive at the original point; and the same is true when θ is decreased by 2π. It is, therefore, evident from Fig. 10 that *two nonzero complex numbers*

$$z_1 = r_1 e^{i\theta_1} \quad \textit{and} \quad z_2 = r_2 e^{i\theta_2}$$

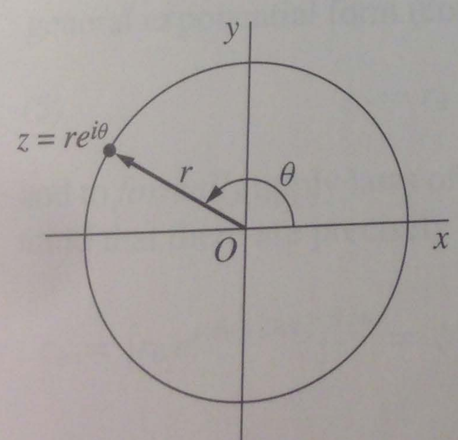

FIGURE 10

*These are called ***Chebyshev polynomials*** and are prominent in approximation theory.

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expression (5) can be put in the form

$$c_0 = \sqrt{A}\left(\sqrt{\frac{1+\cos\alpha}{2}} + i\sqrt{\frac{1-\cos\alpha}{2}}\right). \tag{6}$$

But $\cos\alpha = a/A$, and so

$$\sqrt{\frac{1\pm\cos\alpha}{2}} = \sqrt{\frac{1\pm(a/A)}{2}} = \sqrt{\frac{A\pm a}{2A}}. \tag{7}$$

Consequently, it follows from expression (6) and (7), as well as the relation $c_1 = -c_0$, that the two square roots of $a + i\,(a > 0)$ are (see Fig. 14)

$$\pm\frac{1}{\sqrt{2}}\left(\sqrt{A+a} + i\sqrt{A-a}\right). \tag{8}$$

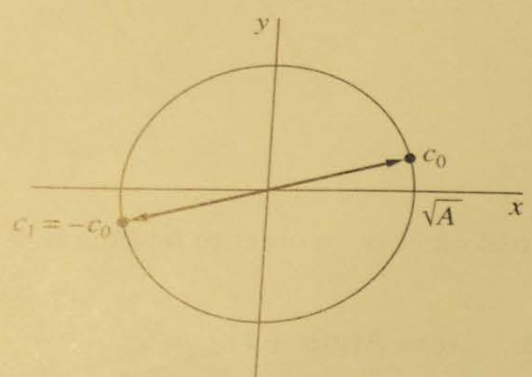

FIGURE 14

EXERCISES

1. Find the square roots of (a) $2i$; (b) $1 - \sqrt{3}i$ and express them in rectangular coordinates.

   Ans. (a) $\pm(1+i)$; (b) $\pm\dfrac{\sqrt{3}-i}{\sqrt{2}}$.

2. Find the three cube roots c_k $(k = 0, 1, 2)$ of $-8i$, express them in rectangular coordinates and point out why they are as shown in Fig. 15.

   Ans. $\pm\sqrt{3} - i,\ 2i$.

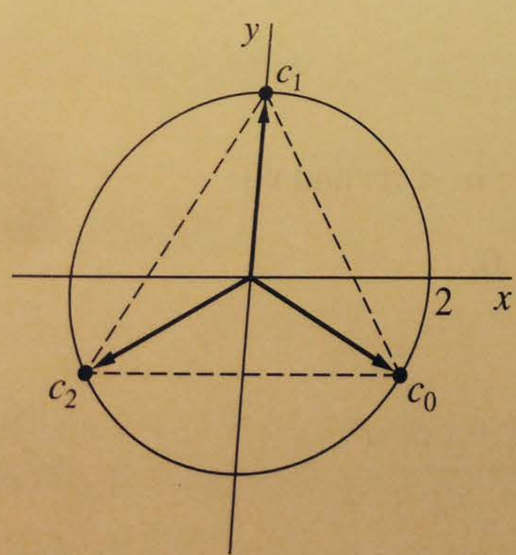

FIGURE 15

3. Find $(-8-8\sqrt{3}i)^{1/4}$, express the roots in rectangular coordinates, exhibit them as the vertices of a certain square, and point out which is the principal root.

Ans. $\pm(\sqrt{3}-i),\ \pm(1+\sqrt{3}i)$.

4. In each case, find all of the roots in rectangular coordinates, exhibit them as vertices of certain regular polygons, and identify the principal root:

(a) $(-1)^{1/3}$; *(b)* $8^{1/6}$.

Ans. (b) $\pm\sqrt{2},\ \pm\dfrac{1+\sqrt{3}i}{\sqrt{2}},\ \pm\dfrac{1-\sqrt{3}i}{\sqrt{2}}$.

5. According to Sec. 10, the three cube roots of a nonzero complex number z_0 can be written $c_0, c_0\omega_3, c_0\omega_3^2$ where c_0 is the principal cube root of z_0 and

$$\omega_3 = \exp\left(i\frac{2\pi}{3}\right) = \frac{-1+\sqrt{3}i}{2}.$$

Show that if $z_0 = -4\sqrt{2}+4\sqrt{2}i$, then $c_0 = \sqrt{2}(1+i)$ and the other two cube roots are, in rectangular form, the numbers

$$c_0\omega_3 = \frac{-(\sqrt{3}+1)+(\sqrt{3}-1)i}{\sqrt{2}}, \qquad c_0\omega_3^2 = \frac{(\sqrt{3}-1)-(\sqrt{3}+1)i}{\sqrt{2}}.$$

6. Find the four zeros of the polynomial z^4+4, one of them being

$$z_0 = \sqrt{2}\,e^{i\pi/4} = 1+i.$$

Then use those zeros to factor z^2+4 into quadratic factors with real coefficients.

Ans. $(z^2+2z+2)(z^2-2z+2)$.

7. Show that if c is any nth root of unity other than unity itself, then

$$1+c+c^2+\cdots+c^{n-1}=0.$$

Suggestion: Use the first identity in Exercise 9, Sec. 9.

8. *(a)* Prove that the usual formula solves the quadratic equation

$$az^2+bz+c=0 \qquad (a\neq 0)$$

when the coefficients a, b, and c are complex numbers. Specifically, by completing the square on the left-hand side, derive the ***quadratic formula***

$$z = \frac{-b+(b^2-4ac)^{1/2}}{2a},$$

where both square roots are to be considered when $b^2-4ac\neq 0$,

(b) Use the result in part *(a)* to find the roots of the equation $z^2+2z+(1-i)=0$.

Ans. (b) $\left(-1+\dfrac{1}{\sqrt{2}}\right)+\dfrac{i}{\sqrt{2}},\quad \left(-1-\dfrac{1}{\sqrt{2}}\right)-\dfrac{i}{\sqrt{2}}$.

So inequality (4) represents the region interior to the circle (Fig. 18)

$$(x-0)^2 + \left(y+\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2,$$

centered at $z = -i/2$ and with radius $1/2$.

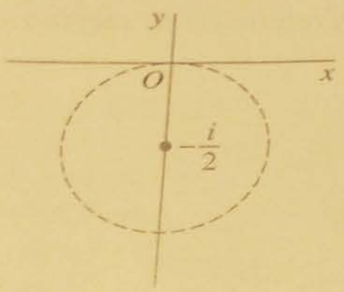

FIGURE 18

A point z_0 is said to be an ***accumulation point***, or limit point, of a set S if each deleted neighborhood of z_0 contains at least one point of S. It follows that if a set S is closed, then it contains each of its accumulation points. For if an accumulation poin z_0 were not in S, it would be a boundary point of S; but this contradicts the fact th a closed set contains all of its boundary points. It is left as an exercise to show th the converse is, in fact, true. Thus a set is closed if and only if it contains all of accumulation points.

Evidently, a point z_0 is *not* an accumulation point of a set S whenever there ex some deleted neighborhood of z_0 that does not contain at least one point in S. that the origin is the only accumulation point of the set

$$z_n = \frac{i}{n} \quad (n = 1, 2, \ldots).$$

EXERCISES

1. Sketch the following sets and determine which are domains:

(*a*) $|z - 2 + i| \le 1$; (*b*) $|2z + 3| > 4$;

(*c*) $\operatorname{Im} z > 1$; (*d*) $\operatorname{Im} z = 1$;

(*e*) $0 \le \arg z \le \pi/4 \ (z \ne 0)$; (*f*) $|z - 4| \ge |z|$.

Ans. (*b*), (*c*) are domains.

2. Which sets in Exercise 1 are neither open nor closed?

Ans. (*e*).

3. Which sets in Exercise 1 are bounded?

Ans. (*a*).

4. In each case, sketch the closure of the set:

(*a*) $-\pi < \arg z < \pi \ (z \neq 0)$; (*b*) $|\text{Re}\, z| < |z|$;

(*c*) $\text{Re}\left(\frac{1}{z}\right) \leq \frac{1}{2}$; (*d*) $\text{Re}(z^2) > 0$.

5. Let S be the open set consisting of all points z such that $|z| < 1$ or $|z - 2| < 1$. State why S is not connected.

6. Show that a set S is open if and only if each point in S is an interior point.

7. Determine the accumulation points of each of the following sets:

(*a*) $z_n = i^n \ (n = 1, 2, \ldots)$; (*b*) $z_n = i^n/n \ (n = 1, 2, \ldots)$;

(*c*) $0 \leq \arg z < \pi/2 \ (z \neq 0)$; (*d*) $z_n = (-1)^n(1+i)\dfrac{n-1}{n} \ (n = 1, 2, \ldots)$.

Ans. (*a*) None; (*b*) 0; (*d*) $\pm(1 + i)$.

8. Prove that if a set contains each of its accumulation points, then it must be a closed set.

9. Show that any point z_0 of a domain is an accumulation point of that domain.

10. Prove that a finite set of points $z_1, z_2, \ldots, z_n$ cannot have any accumulation points.

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the entire w plane. However, in this case, the transformation is not one to one since both the positive and negative real axes in the z plane are mapped onto the positive real axis in the w plane.

When n is a positive integer greater than 2, various mapping properties of the transformation $w = z^n$, or $w = r^n e^{in\theta}$, are similar to those of $w = z^2$. Such a transformation maps the entire z plane onto the entire w plane, where each nonzero point in the w plane is the image of n distinct points in the z plane. The circle $r = r_0$ is mapped onto the circle $\rho = r_0^n$; and the sector $r \le r_0, 0 \le \theta \le 2\pi/n$ is mapped onto the disk $\rho \le r_0^n$, but not in a one to one manner.

Other, but somewhat more involved, mappings by $w = z^2$ appear in Example 1, Sec. 107, and Exercises 1 through 4 Sec. 108.

EXERCISES

1. For each of the functions below, describe the domain of definition that is understood:

(a) $f(z) = \dfrac{1}{z^2+1}$; $\quad$ (b) $f(z) = \operatorname{Arg}\left(\dfrac{1}{z}\right)$;

(c) $f(z) = \dfrac{z}{z+\overline{z}}$; $\quad$ (d) $f(z) = \dfrac{1}{1-|z|^2}$.

Ans. (a) $z \ne \pm i$; $\quad$ (b) $\operatorname{Re} z \ne 0$.

2. In each case, write the function $f(z)$ in the form $f(z) = u(x, y) + iv(x, y)$:

(a) $f(z) = z^3 + z + 1$; $\quad$ (b) $f(z) = \dfrac{\overline{z}^2}{z}$ $\quad (z \ne 0)$.

Suggestion: In part (b), start by multiplying the numerator and denominator by $\overline{z}$.

Ans. (a) $f(z) = (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y)$;

(b) $f(z) = \dfrac{x^3 - 3xy^2}{x^2+y^2} + i\dfrac{y^3 - 3x^2y}{x^2+y^2}$.

3. Suppose that $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$, where $z = x + iy$. Use the expressions (see Sec. 6)

$$x = \frac{z+\overline{z}}{2} \quad \text{and} \quad y = \frac{z-\overline{z}}{2i}$$

to write $f(z)$ in terms of z, and simplify the result.

Ans. $f(z) = \overline{z}^2 + 2iz$.

4. Write the function

$$f(z) = z + \frac{1}{z} \qquad (z \ne 0)$$

in the form $f(z) = u(r, \theta) + iv(r, \theta)$.

Ans. $f(z) = \left(r + \dfrac{1}{r}\right)\cos\theta + i\left(r - \dfrac{1}{r}\right)\sin\theta$.

5. By referring to the discussion in Sec. 14 related to Fig. 19 there, find a domain in the z plane whose image under the transformation $w = z^2$ is the square domain in the w plane bounded by the lines $u = 1$, $u = 2$, $v = 1$, and $v = 2$. (See Fig. 2, Appendix 2.)

6. Find and sketch, showing corresponding orientations, the images of the hyperbolas

$$x^2 - y^2 = c_1 \ (c_1 < 0) \quad \text{and} \quad 2xy = c_2 \ (c_2 < 0)$$

under the transformation $w = z^2$.

7. Use rays indicated by dashed half lines in Fig. 21 to show that the transformation $w = z^2$ maps the first quadrant onto the upper half plane, as shown in Fig. 21.

8. Sketch the region onto which the sector $r \le 1$, $0 \le \theta \le \pi/4$ is mapped by the transformation (*a*) $w = z^2$; (*b*) $w = z^3$; (*c*) $w = z^4$.

9. One interpretation of a function $w = f(z) = u(x, y) + iv(x, y)$ is that of a ***vector field*** in the domain of definition of f. The function assigns a vector w, with components $u(x, y)$ and $v(x, y)$, to each point z at which it is defined. Indicate graphically the vector fields represented by

(*a*) $w = iz$; (*b*) $w = \dfrac{z}{|z|}$.

15. LIMITS

Let a function f be defined at all points z in some deleted neighborhood of a point z_0. The statement that $f(z)$ has a ***limit*** w_0 as z approaches z_0, or that

$$\lim_{z \to z_0} f(z) = w_0, \tag{1}$$

means that the point $w = f(z)$ can be made arbitrarily close to w_0 if we choose the point z close enough to z_0 but distinct from it. We now express the definition of limit in a precise and usable form.

Statement (1) means that for each positive number ε, there is a positive number δ such that

$$|f(z) - w_0| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta. \tag{2}$$

Geometrically, this definition says that for each ε neighborhood $|w - w_0| < \varepsilon$ of w_0, there is a deleted δ neighborhood $0 < |z - z_0| < \delta$ of z_0 such that every point z in it has an image w lying in the ε neighborhood (Fig. 22). Note that even though all points in the deleted neighborhood $0 < |z - z_0| < \delta$ are to be considered, their images need not fill up the entire neighborhood $|w - w_0| < \varepsilon$. If f has the constant value w_0, for instance, the image of z is always the center of that neighborhood. Note

Theorem 4. *If a function f is continuous throughout a region R that is both closed and bounded, there exists a nonnegative real number M such that*

$$|f(z)| \leq M \qquad \text{for all points } z \text{ in } R, \tag{6}$$

where equality holds for at least one such z.

To prove this, we assume that the function f in equation (5) is continuous and note how it follows that the function

$$\sqrt{[u(x, y)]^2 + [v(x, y)]^2}$$

is continuous throughout R and thus reaches a maximum value M somewhere in R.* Inequality (6) thus holds, and we say that f is ***bounded on R***.

EXERCISES

1. Use definition (2), Sec. 15, of limit to prove that

(*a*) $\lim_{z \to z_0} \operatorname{Re} z = \operatorname{Re} z_0$; (*b*) $\lim_{z \to z_0} \bar{z} = \overline{z_0}$; (*c*) $\lim_{z \to 0} \dfrac{\bar{z}^2}{z} = 0$.

2. Let a, b, and c denote complex constants. Then use definition (2), Sec. 15, of limit to show that

(*a*) $\lim_{z \to z_0} (az + b) = az_0 + b$; (*b*) $\lim_{z \to z_0} (z^2 + c) = z_0^2 + c$;

(*c*) $\lim_{z \to 1-i} [x + i(2x + y)] = 1 + i \quad (z = x + iy)$.

3. Let n be a positive integer and let $P(z)$ and $Q(z)$ be polynomials, where $Q(z_0) \neq 0$. Use Theorem 2 in Sec. 16, as well as limits appearing in that section, to find

(*a*) $\lim_{z \to z_0} \dfrac{1}{z^n} \; (z_0 \neq 0)$; (*b*) $\lim_{z \to i} \dfrac{iz^3 - 1}{z + i}$; (*c*) $\lim_{z \to z_0} \dfrac{P(z)}{Q(z)}$.

Ans. (*a*) $1/z_0^n$; (*b*) 0; (*c*) $P(z_0)/Q(z_0)$.

4. Use mathematical induction and property (9), Sec. 16, of limits to show that

$$\lim_{z \to z_0} z^n = z_0^n$$

when n is a positive integer $(n = 1, 2, \ldots)$.

5. Show that the function

$$f(z) = \left(\frac{z}{\bar{z}}\right)^2$$

has the value 1 at all nonzero points on the real and imaginary axes, where $z = (x, 0)$ and $z = (0, y)$, respectively, but that it has the value -1 at all nonzero points on the line $y = x$, where $z = (x, x)$. Thus show that the limit of $f(z)$ as z tends to 0 does

*See, for instance, A. E. Taylor and W. R. Mann, "*Advanced Calculus*," 3d ed., pp. 125–126 and p. 529, 1983.

not exist. [Note that it is not sufficient to simply consider nonzero points $z = (x, 0)$ and $z = (0, y)$, as it was in Example 2, Sec. 15.]

6. Prove statement (8) in Theorem 2 of Sec. 16 using

(*a*) Theorem 1 in Sec. 16 and properties of limits of real-valued functions of two real variables;

(*b*) definition (2), Sec. 15, of limit.

7. Use definition (2), Sec. 15, of limit to prove that

$$\text{if} \quad \lim_{z \to z_0} f(z) = w_0, \quad \text{then} \quad \lim_{z \to z_0} |f(z)| = |w_0|.$$

Suggestion: Observe how inequality (2), Sec. 5, enables one to write

$$||f(z)| - |w_0|| \leq |f(z) - w_0|.$$

8. Write $\Delta z = z - z_0$ and show that

$$\lim_{z \to z_0} f(z) = w_0 \quad \text{if and only if} \quad \lim_{\Delta z \to 0} f(z_0 + \Delta z) = w_0.$$

9. Show that

$$\lim_{z \to z_0} f(z)g(z) = 0 \quad \text{if} \quad \lim_{z \to z_0} f(z) = 0$$

and if there exists a positive number M such that $|g(z)| \leq M$ for all z in some neighborhood of z_0.

10. Use the theorem in Sec. 17 to show that

(*a*) $\displaystyle\lim_{z \to \infty} \frac{4z^2}{(z-1)^2} = 4$; (*b*) $\displaystyle\lim_{z \to 1} \frac{1}{(z-1)^3} = \infty$; (*c*) $\displaystyle\lim_{z \to \infty} \frac{z^2+1}{z-1} = \infty$.

11. With the aid of the theorem in Sec. 17, show that when

$$T(z) = \frac{az+b}{cz+d} \qquad (ad - bc \neq 0),$$

(*a*) $\displaystyle\lim_{z \to \infty} T(z) = \infty$ if $c = 0$;

(*b*) $\displaystyle\lim_{z \to \infty} T(z) = \frac{a}{c}$ and $\displaystyle\lim_{z \to -d/c} T(z) = \infty$ if $c \neq 0$.

12. State why limits involving the point at infinity are unique.

13. Show that a set S is unbounded (Sec. 12) if and only if every neighborhood of the point at infinity contains at least one point in S.

19. DERIVATIVES

Let f be a function whose domain of definition contains a neighborhood $|z - z_0| < \varepsilon$ of a point z_0. The ***derivative*** of f at z_0 is the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}, \tag{1}$$

and the function f is said to be ***differentiable*** at z_0 when $f'(z_0)$ exists.

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Parent Directory		-	
hw4-sol.pdf	2014-09-24 13:57	2.0M	
Week4.pdf	2014-09-15 14:28	330K	
61-62-70.pdf	2014-09-15 19:32	1.4M	
55.JPG	2014-09-15 19:34	854K	

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6. Prove statement (8) in Theorem 2 of Sec. 16 using
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and the function f is said to be ***differentiable*** at z_0 when $f'(z_0)$ exists.

EXERCISES

8-3 **1.** Use definition (3), Sec. 19, to give a direct proof that

$$\frac{dw}{dz} = 2z \quad \text{when} \quad w = z^2.$$

2. Use results in Sec. 20 to find $f'(z)$ when

(a) $f(z) = 3z^2 - 2z + 4$;   (b) $f(z) = (2z^2 + i)^5$;

(c) $f(z) = \dfrac{z-1}{2z+1} \quad \left(z \neq -\dfrac{1}{2}\right)$;   (d) $f(z) = \dfrac{(1+z^2)^4}{z^2} \quad (z \neq 0)$.

3. Using results in Sec. 20, show that

(a) a polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n \neq 0)$$

of degree n $(n \geq 1)$ is differentiable everywhere, with derivative

$$P'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1};$$

(b) the coefficients in the polynomial $P(z)$ in part (a) can be written

$$a_0 = P(0), \quad a_1 = \frac{P'(0)}{1!}, \quad a_2 = \frac{P''(0)}{2!}, \quad \ldots, \quad a_n = \frac{P^{(n)}(0)}{n!}.$$

4. Suppose that $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$ and $g'(z_0)$ exist, where $g'(z_0) \neq 0$. Use definition (1), Sec. 19, of derivative to show that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

5. Derive expression (3), Sec. 20, for the derivative of the sum of two functions.

6. Derive expression (2), Sec. 20, for the derivative of z^n when n is a positive integer by using

(a) mathematical induction and expression (4), Sec. 20, for the derivative of the product of two functions;

(b) definition (3), Sec. 19, of derivative and the binomial formula (Sec. 3).

7. Prove that expression (2), Sec. 20, for the derivative of z^n remains valid when n is a negative integer $(n = -1, -2, \ldots)$, provided that $z \neq 0$.

Suggestion: Write $m = -n$ and use the rule for the derivative of a quotient of two functions.

8. Use the method in Example 2, Sec. 19, to show that $f'(z)$ does not exist at any point z when

(a) $f(z) = \text{Re}\, z$;   (b) $f(z) = \text{Im}\, z$.

9. Let f denote the function whose values are

$$f(z) = \begin{cases} \bar{z}^2/z & \text{when} \quad z \neq 0, \\ 0 & \text{when} \quad z = 0. \end{cases}$$

Show that if $z = 0$, then $\Delta w/\Delta z = 1$ at each nonzero point on the real and imaginary axes in the Δz, or $\Delta x\, \Delta y$, plane. Then show that $\Delta w/\Delta z = -1$ at each nonzero point $(\Delta x, \Delta x)$ on the line $\Delta y = \Delta x$ in that plane (Fig. 29). Conclude from these observations that $f'(0)$ does not exist. Note that to obtain this result, it is not sufficient to consider only horizontal and vertical approaches to the origin in the Δz plane. (Compare with Exercise 5, Sec. 18, as well as Example 2, Sec. 19.)

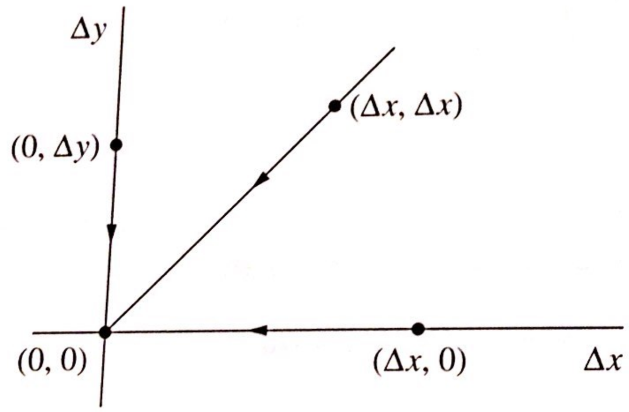

FIGURE 29

10. With the aid of the binomial formula (13) in Sec. 3, point out why each of the functions

$$P_n(z) = \frac{1}{n!2^n} \frac{d^n}{dz^n}(z^2 - 1)^n \qquad (n = 0, 1, 2, \ldots)$$

is a polynomial (Sec. 13) of degree n*. (We use the convention that the derivative of order zero of a function is the function itself.)

21. CAUCHY–RIEMANN EQUATIONS

In this section, we obtain a pair of equations that the first-order partial derivatives of the component functions u and v of a function

$$f(z) = u(x, y) + iv(x, y) \tag{1}$$

must satisfy at a point $z_0 = (x_0, y_0)$ when the derivative of f exists there. We also show how to express $f'(z_0)$ in terms of those partial derivatives.

Starting with the assumption that $f'(z_0)$ exists, we write

$$z_0 = x_0 + iy_0, \quad \Delta z = \Delta x + i\Delta y,$$

where $z \neq 0$, the component functions are

$$u = \frac{\cos 2\theta}{r^2} \quad \text{and} \quad v = -\frac{\sin 2\theta}{r^2}.$$

Since

$$ru_r = -\frac{2\cos 2\theta}{r^2} = v_\theta, \qquad u_\theta = -\frac{2\sin 2\theta}{r^2} = -r\,v_r$$

and since the other conditions in the theorem are satisfied at every nonzero point $z = re^{i\theta}$, the derivative of f exists when $z \neq 0$. Moreover, according to the theorem,

$$f'(z) = e^{-i\theta}\left(-\frac{2\cos 2\theta}{r^3} + i\frac{2\sin 2\theta}{r^3}\right) = -2e^{-i\theta}\frac{e^{-i2\theta}}{r^3} = -\frac{2}{(re^{i\theta})^3} = -\frac{2}{z^3}.$$

EXAMPLE 2. The theorem can be used to show that any branch

$$f(z) = \sqrt{r}\,e^{i\theta/2} \quad (r > 0,\ \alpha < \theta < \alpha + 2\pi)$$

of the square root function $z^{1/2}$ has a derivative everywhere in its domain of definition. Here

$$u(r,\theta) = \sqrt{r}\cos\frac{\theta}{2} \quad \text{and} \quad v(r,\theta) = \sqrt{r}\sin\frac{\theta}{2}.$$

Inasmuch as

$$ru_r = \frac{\sqrt{r}}{2}\cos\frac{\theta}{2} = v_\theta \quad \text{and} \quad u_\theta = -\frac{\sqrt{r}}{2}\sin\frac{\theta}{2} = -rv_r$$

and since the remaining conditions in the theorem are satisfied, the derivative $f'(z)$ exists at each point where $f(z)$ is defined. The theorem also tells us that

$$f'(z) = e^{-i\theta}\left(\frac{1}{2\sqrt{r}}\cos\frac{\theta}{2} + i\frac{1}{2\sqrt{r}}\sin\frac{\theta}{2}\right);$$

and this reduces to

$$f'(z) = \frac{1}{2\sqrt{r}}e^{-i\theta}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) = \frac{1}{2\sqrt{r}\,e^{i\theta/2}} = \frac{1}{2f(z)}.$$

EXERCISES

1. Use the theorem in Sec. 21 to show that $f'(z)$ does not exist at any point if
   (a) $f(z) = \bar{z}$; (b) $f(z) = z - \bar{z}$;
   (c) $f(z) = 2x + ixy^2$; (d) $f(z) = e^x e^{-iy}$.

2. Use the theorem in Sec. 23 to show that $f'(z)$ and its derivative $f''(z)$ exist everywhere, and find $f''(z)$ when
   (a) $f(z) = iz + 2$; (b) $f(z) = e^{-x}e^{-iy}$;
   (c) $f(z) = z^3$; (d) $f(z) = \cos x \cosh y - i \sin x \sinh y$.

   Ans. (b) $f''(z) = f(z)$; (d) $f''(z$