MAT 310 (Linear Algebra)

Fall 2006

Department of Mathematics, Stony Brook University

Welcome to MAT 310!

This is the final course in the linear algebra sequence. We cover all the material in MAT 211 (or AMS 210) as well as some material (for example the Cayley-Hamilton theorem and the Jordan canonical form) that will be new to most of you. The course will probably have quite a different emphasis from your previous courses on this subject: specific calculations will be of far less importance than understanding the statement of the main theorems and precisely why they are true. Because one of the aims of this course is to teach you how to write proofs, the homework and recitations are an integral part of the course.

This is the link to the CURRENT HOMEWORK. This page will also contain links to solutions.

Course Notes:

- *(posted Dec 17)* The exam will be in the usual room. I will have office hours on Tuesday 10am -1pm.
- *(posted Dec 17)* Here are solutions to the review sheet. p1 p2 p3 p4 Solutions to the HWs are also posted.
- *(posted Dec 13)* Just to repeat: I will NOT be in the dept tomorrow-- I told one student today that I would be, but I forgot that I am going to the city. I have done some of the Final Review which is now posted HERE. I will update this on Thursday morning. (now done)
- *(posted Dec 13)* I forgot to say in HW 11 Ex 2 that \$V\$ has dim \$n\$.
- *(posted Dec 10)* The homework is now posted; sorry it is late. Vincent will be at Stony Brook on Wed Dec 13 and will be in his office most of the day. I will be away that day, though.
- *(posted Dec 5)* There was a mistake in Ex 1(ii) in the HW this week. It is now corrected. Also, Vincent sent you all a rather incoherent message about recitations. The recitations this week as are usual. His message concerns next week when WED Dec 13 follows MONDAY's schedule. This means that there is no recitation on Wed 13. He invites all of you who are interested to come to his Thursday class instead.

- *(posted Nov 29)* Either of the midterms (carefully rewitten) would be suitable for the proof part of the writing requirement. If you are interested in this, please submit your rewrite to me fairly soon (eg next week) so I can check it over. Also, your projects would do for another part of the writing requirement (and would do for the proof part if they contained enough proofs...) If you are interested in this, please write on your project when you submit it that you want me to consider it for the writing requirement.

- *(posted Nov 19)* I will give back the exams in Tuesday's class. Most of you have been working really hard, and it shows in the results. Many of you did really well. There was a wide spread in the results: the distribution was: < 30: 9; 30-39: 4; 40-49: 3; 50-59 4; 60-69: 6; 70-79: 6; 80-89: 5; 90-110: 7. a total of 44 exams. Again, I will give the 16 students who got < 50 a chance to improve their grade. They must hand in a rewritten exam IN CLASS ON TUESDAY NOV 27 together with their old exam. I will add half the extra points to the exam grade up to a maximum of 50. (The exam is posted on the HW page: HW for next week is also posted there.) I will spend Tuesday's class discussing the exam and answering any questions there may be about the projects. On Tuesday Nov 27 I will start lecturing on the characteristic polynomial (Ch 8).

- *(posted Nov 12)* I am leaving on the morning of Wed Nov 15 and so will not be able to hold my usual office hours then, sorry. I will be having office hours on Tuesday as usual (12-1 in UG office and 1-2 in my office.)

- *(posted Nov 11)* Here are the projects. Project 1 concerns eigenvectors and eigenvalues; Project 2 is a minimization problem; Project 3 is about Fibonacci numbers; Project 4 is a basic project intended for students who are aiming for a C (or C+); Project 5 is about the Jordan normal form. The deadline for all projects is Thursday, December 14 at 5pm. You should give them to me in my office (or under the door, if I am not there.) The projects should be your own work and written in your own words. However, you can come to me or Vincent to discuss any questions you might have about them. Each student should do at most ONE project.

- *(posted Nov 11)* Here is the review sheet for Midterm 2 (added at 8:45pm: slightly revised). We will discuss it in Tuesday's class, together with any other questions you have. Since there will also be review time in recitation this week, it might be best if on Tuesday I tried to answer your more fundamental questions to give you a basis for doing the review sheet. So please, if there any subject that you really have no clue about, do ask me to go over it.

- *(posted Nov 10)* The HW solutions are now posted.

- *(posted Oct 31)* I am leaving tomorrow (Wed) in the morning for a meeting in Washington. So I won't be able to manage tomorrow office hours, sorry. Here is the Worksheet for Thursday. Most of the problems on HW 8 will be very much like these. I haven't quite written it yet, but I plan just to change numbers etc. in many problems. So Thursday's class should be an opportunity for you to get a head start on the HW for next week.

- *(posted Oct 31)* There was a small typo in Ex 5 on HW 7 that I have just corrected.

- *(posted Oct 26)* I just got an email saying that USG (the Undergrad student government) is starting a new program called PASS to make free one on one tutoring available to all students. You might want to check it out (though I'm not sure they would have tutors equipped to handle MAT 310... -- it would be better to come to my (or Vincent's) office hours...

- *(posted Oct 26)* I added HW problems from Ch 5 and 6 to the syllabus. I will be away on Nov 2; Vincent will give another workshop on some topics from Ch 6.

- *(posted Oct 12)* The average in the exam was 60%; with distribution: (90-100) 5; (80-89) 11; (70-79) 8; (60-69) 3; (50-59) 5; (40-49) 8; (30-39) 3; (0-29) 8. VERY rough grade equivalents are: A for 85 and above, B to A- for 70-84; C to B- for 50 to 69; D to C- for 40 to 49. For those of you who did not do so well the important thing is to keep working. I will allow the grade of your rewrite of the exam to improve your exam grade up to a maximum of 50. So, the rewrite will count as a HW, but also I will add 1/2 of the difference between the new grade and the old to your old grade, to give you a revised grade of no more than 50. (So this will NOT affect those of you who got over 50). For fairness, I will keep note of both the revised and the old midterm grades.

- *(posted Oct 11)* Vincent I and will get the exam graded by class time on Thursday at the latest. (Some of you may get it back in recitation.) From my bit of grading, I think you have mostly done very well. The homework for this week is to rewrite the exam. You need not rewrite any question for which you got 90% or more on a question. Please hand in the rewrite as well as your original exam paper in the usual way next week. Thanks.

- *(posted Oct 11)* Here are the two versions of the exam. Version 1 Version 2 Also, I corrected two types that were pointed out to me in the list of theorems. You might want to look at the new list (posted below.)

- *(posted Oct 11)* The lecture tomorrow will be on polynomials. Then we move on to Ch 5.

- *(Posted Oct 6)* Solutions to HW 4 are now posted.

- *(posted Oct 5)* Here is the final version of the review sheet, and also the final list of theorems. Please bring these to class today if you can.

- *(posted Oct 3)* I will prepare a review sheet today or tomorrow to discuss in class on Thursday. I will post it on this page as soon as it is ready.

- *(posted Oct 3)* I realised that part (ii) of Ex 3 was wrong and changed it.

- *(posted Sep 27)* Here is the worksheet for the class on Thursday Sept 28.

- *(posted Sep 20)* Here are some notes to amplify some parts of Ch 2 and Ch 3.

- *(posted Sep 20)* I have extended my Wednesday office hours to 11:15 - 12:30. I also have office hours on Tuesday. The department wants me to hold some of these in the Undergraduate office. So I will be 12-1 in UG Office (P143) and 1-2:15 in my office (3-111).

- *(posted Sep 18)* Someone asked me to post the tex file of the HW. I have now done this for HW 2. It is on the HW web page.

- *(posted Sep 12)* In today's class some students pointed out some inaccuracies in HW 1. So I have slightly revised it, correcting Ex 2 (iii) and making Ex 6 more clear. I also added a bonus problem. (Note: if you ever think one of the HW problems is wrong etc, please email me and I'll see if it needs correcting.)

- *(posted Sep 12)* I just sent an email message to all the students who gave me their address last week. Please email me at dusa at math.sunysb.edu if you are registered for the course and do NOT receive this email by the morning of Wed Sep 13. Those who got it need do nothing (unless you want me to take you off the list.) Thanks.

- *(posted Sep 12)* I wrote out a model proof for you to guide you in doing your homework.

- *(posted Sep 12)* I will be away on Sept 14. The lecture will be given by Professor Phillips. Topic: first half of Ch 2 especially the proof of 2.4. This is a crucial result.

- *(posted Sep 7)* Here is the Exercise Sheet for this week's recitations. You are not expected to hand anything in. The material in Ex 3-5 is review for next week and is not relevant to this week's work.

- *(posted Aug 30)* Please go to your first recitation, even though this will be before the first lecture. This will be a review class to remind you of some of the basics from MAT 211.

Instructor: Dusa McDuff

Instructor's Office Hours: held in Undergraduate Office (Math P143): *Tu 12-1*, and in Math Building 3-111: *Tu 1:00-2:15, Wed 11:15-12:30* You are always welcome to contact me by email (dusa at math.sunysb.edu) either to ask a short question or to set up an appointment to see me.

TA: Vincent Graziano

TA's Office: Math Building 2-116

TA's Office Hours: *TBA*

Textbook: S. Axler, *Linear Algebra done right, 2nd Ed.,* Springer Undergraduate Text

(one copy is available on reserve in the Math/Physics/Astronomy Library; later on in the semester there should be two)

Course Format:

The course meets for lecture Tuesdays and Thursdays in Harriman Hall 108 from 2:20 pm to 3:40 pm. You will get most out of these classes if you prepare beforehand by reading the relevant section in the textbook before class. My aim in these lectures will be to explain and illustrate the arguments in the book. I am always glad to answer questions during class. (If you have more questions, please talk to me after class or come to my office hours.)

There are recitations on Wednesday or Thursday. Their main aim is to help you understand the lecture and do the homework. The syllabus below suggests problems from the book for you to try on your own, but this is not the assigned homework. I will write out homework sets for you each week that you can access from this page. Homework should be handed in to the TA Vincent Graziano in recitation. He will tell you exactly when it is due. Late homework will be penalized (and if it is too late, e.g. after the solutions are posted, it will not be accepted.) He will also tell you exactly how the homework grade will be computed.

It is fine for you to work with others on your homework. But the work you hand in must be written in your own words. Do not copy other's work or let your own work be copied; both parties are penalized for copying.

Some links of interest There are many places online where you can get useful information, eg MathWorld and Wikipedia. Here are some other suggestions.

A useful online linear algebra text with many worked examples and exercises with solutions.

An online linear algebra tutorial by our own Avi Goldstein. Some of this is too computational to be very relevant, but there are many helpful worked examples.

A discussion of the many applications of linear algebra. We will not have time to discuss detailed applications, so you might want to look at this to see the range of possibilities.

A nice expository paper on the use of linear algebra in search engines.

Examinations:

There will be two in-class midterms (on **October 10** and **November 16**). The final, which will be cumulative, is on **December 19, 5:00-7:30 pm** . It is your responsibility to make sure that you can manage these times; tell me of any problems well beforehand. Incomplete grades will be granted only if **documented circumstances beyond your control** prevent you from completing 50% or more of all class assignments.

Optional projects:

These will be similar in spirit to the projects done in Fall 05, i.e. written papers of less than 5 pages. You may submit work on at most two of them; the best will be graded and will add at most an extra 10% to your grade. Further details will be given out later (by Nov 21 at the latest).

Grading:

Your grade will be based on your examination performance, homework and project (if any), weighted as follows:

Exam I	20%

Exam II	20%
Final Exam	30%
Homework	30%
Project	10% extra

DSS advisory:

If you have a physical, psychological, medical, or learning disability that may affect your course work, please contact Disability Support Services (DSS) office: ECC (Educational Communications Center) Building, room 128, telephone (631) 632-6748/TDD. DSS will determine with you what accommodations are necessary and appropriate. Arrangements should be made early in the semester (before the first exam) so that your needs can be accommodated. All information and documentation of disability is confidential. Students requiring emergency evacuation are encouraged to discuss their needs with their professors and DSS. For procedures and information, go to the following web site http://www.ehs.sunysb.edu and search Fire safety and Evacuation and Disabilities.

Schedule (tentative):

The following is the basic syllabus. Please read the relevant parts of the book **before** class.

Days	**Sections covered**	**Suggested exercises**
Sep 7, 12	Ch 1	p 19: 2-5, 7, 8, 11, 13
Sep 14, 19	Ch 2	p 35: 1, 2, 5, 6, 9, 11, 13, 14
Sep 21 - Oct 3	Ch 3	p 59: 1-3, 5, 7, 9, 10, 12, 15, 16, 19, 20, 22, 26.
Sep 28		a workshop instead of lecture
Oct 5	review	
Oct 10	**Midterm I**	on Ch 1,2,3.
Oct 12	Ch 4	p 73, 1-3
Oct 17, 19, 24	Ch 5	4, 5, 7,8,9,10,15,16,18,19,21,23
Oct 26 -Nov 7	Ch 6	1,2,3,4,7,8*,9,10,11,13,15,18, 21,24,27,29,30
Nov 9	Ch 7 (part)	
Nov 14	review	
Nov 16	**Midterm II**	on Ch 4,5,6
Nov 21	ch 7 ctd	projects given out and discussed
Nov 28, 30,	ch 8 (parts)	TBA

Dec 5		
Dec 7, 12	ch 10 (parts)	TBA
Dec 14	review	
Dec 19	**Final Exam**	(Cumulative)

MAT 310 Homework Assignments

Fall 2006

	Comments	Due Date
Homework 1		9/13/06 or 9/14/06
Homework 2		9/20/06 or 9/21/06
Homework 3	Brief solutions	9/27/06 or 9/28/06
Homework 4 (revised)	Solutions	10/4/06 or 10/5/06
Homework 5: rewrite the exam	see notes on web page for details	10/18/06 or 10/19/06
Homework 6	Solutions	10/25/06 or 10/26/06
Homework 7	Solutions	11/1/06 or 11/2/06
Homework 8	Solutions	11/8/06 or 11/9/06
For Homework 9 rewrite the questions on the exam for which you got less than 90 %	some of you should hand this in to me on Tues Nov 26 (see notes on web page). The others give to Vincent as usual (together with the original exam).	11/29/06 or 11/30/06
Homework 10	Solutions (I realised I called some of this HW11, but it is HW10.)	12/6/06 or 12/7/06
Homework 11	Brief Solutions (only one page)	12/14/06

Review.

1 (ii) $\det(A-\lambda I) = \det\begin{vmatrix} 2-\lambda & 0 & 2 \\ 1 & -\lambda & 1 \\ 0 & 3 & -\lambda \end{vmatrix} = (2-\lambda)(\lambda^2-3) + 6$

$= -\lambda^3 + 2\lambda^2 + 3\lambda - 6 + 6$

$= -\lambda(\lambda^2 - 2\lambda - 3)$

$= -\lambda(\lambda-3)(\lambda+1)$

eigenvalues $0, 3, -1$

iii) eigenvectors $\lambda = 0$ $v_1 = \begin{pmatrix}1\\0\\-1\end{pmatrix}$; $\lambda = -1$ $\begin{pmatrix}3 & 0 & 2\\ 1 & 1 & 1 \\ 0 & 3 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = 0$

$\lambda = 3$ $\left(\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 1 & -3 & 1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right)$

$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -1 & 0 \end{array}\right) \Rightarrow \begin{pmatrix}-2\\-1\\3\end{pmatrix} = v_2$

$\to \left(\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \Rightarrow \begin{pmatrix}2\\1\\1\end{pmatrix} = v_3$

If $\mathcal{B} = v_1, v_2, v_3$ $\mathcal{M}(A, \mathcal{B}_3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

2. If $\operatorname{span}(v_1, \ldots, v_k) \neq V$ $\exists\, w \in V :$ $w \notin sp(v_1, \ldots, v_k)$

Claim $(v_1, \ldots, v_k, w)$ is lin indep list.

Proof Suppose $a_1 v_1 + \cdots + a_k v_k + a_{k+1} w = 0$

If $a_{k+1} \neq 0$ $w = -\frac{1}{a_{k+1}}(a_1 v_1 + \cdots + a_k v_k) \in sp(v_1, \ldots, v_k)$

not true

$\therefore a_{k+1} = 0$ $\therefore a_1 v_1 + \cdots + a_k v_k = 0 \Rightarrow a_1 = 0 = \cdots = a_k$

since $v_1 \cdots v_k$ lin indep.

$\therefore$ all $a_i = 0$ $\therefore (v_1, \ldots, v_k, w)$ is lin indep.

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3. Suppose $av + bTv + cT^2 v = 0$.

Then $T^2(av + bTv + cT^2 v) = 0$ ie $aT^2 v = 0 \Rightarrow a = 0$ (since $T^2 v \neq 0$)

Similarly $T(bTv + cT^2 v) = bT^2 v = 0 \Rightarrow b = 0$

$\therefore$ also $c = 0$ $\therefore$ The only lin relation between $v, Tv, T^2 v$

has all zero coeffs. $\therefore$ They are lin indep

Ex4 (i) There are many examples. eg any operator T_A with a diagonal matrix with $\underline{\text{distinct}}$ eigenvalues – $A = \begin{bmatrix} 1&0&0&0\\ 0&2&0&0\\ 0&0&3&0\\ 0&0&0&4 \end{bmatrix}$

ii) $T_A = \begin{bmatrix} 1&1&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&2 \end{bmatrix}$ has min poly $(z-1)^2(z-2)$
- it has 2 eigenvectors for $\lambda=1$ namely e_1, e_3
and one extra element in $\text{null}\,(T_A - I)^2$ namely e_2.

char poly $T_A = (z-1)^3(z-2)$

Ex 5. (i) We know $\text{null}\, T^n = \text{null}\, T^{n+1} = \cdots = \text{null}\, T^{2n}$

Suppose $v \in \text{Null}\,T^n \cap \text{Range}\, T^n$,

So $v = T^n w$ and $T^n v = T^{2n} w = 0$

$\therefore w \in \text{Null}\,T^{2n} = \text{Null}\,T^n$. $\therefore T^n w = 0$ ie $v = 0$.

ii) $\text{null}\,(T - \lambda I) = \{v : Tv = \lambda v\}$ $\therefore$ each $v \in \text{null}\,(T - \lambda I)$ satisfies

$v = \frac{1}{\lambda} Tv = \frac{1}{\lambda^n} T^n v \therefore v \in \text{Range}\, T^n$.

now suppose $\text{null}\,(T - \lambda I)^{k-1} \subseteq \text{Range}\, T^n$.

Let $v \in \text{Null}\,(T - \lambda I)^k$. Then $(T - \lambda I) v = w \in \text{null}\,(T - \lambda I)^{k-1} \in \text{Range}\, T^n$

$\therefore (T - \lambda I) v = T^n u$ for some $u \in V$

$$\therefore \quad v = \frac{1}{\lambda} Tv - T^n u_1 \quad \text{for some } u_1 \in V$$

apply $\frac{1}{\lambda} T$ to both sides, then $\frac{1}{\lambda^2} T^2$ etc to get

$$\frac{1}{\lambda} T v = \frac{1}{\lambda^2} T^2 v - \frac{1}{\lambda} T^{n+1} u_1$$

$$\frac{1}{\lambda^2} T^2 v = \frac{1}{\lambda^3} T^3 v - \frac{1}{\lambda^2} T^{n+2} u_1$$

$$\cdots \quad \frac{1}{\lambda^{n-1}} T^{n-1} v = \frac{1}{\lambda^n} T^n v - \frac{1}{\lambda^{n-1}} T^{2n-1} u_1$$

add all these equations to get

$$v + \frac{1}{\lambda} Tv + \frac{1}{\lambda^2} T^2 v + \cdots + \frac{1}{\lambda^{n-1}} T^{n-1} v = \frac{1}{\lambda} Tv + \cdots + \frac{1}{\lambda^{n-1}} T^{n-1} v + w$$

where $w \in \text{Range}\, T^n$

$\therefore \quad v \in \text{Range}\, T^n$.

$\therefore \text{null}\,(T - \lambda I)^k \subseteq \text{Range}\, T^n$, which completes inductive step

Ex5 iii) By (ii) we know $V_1 \oplus \cdots \oplus V_k \subseteq \text{Range}\, T^n$.

Also $V_0 = \text{Null}\, T^n$ by def

$\therefore$ $\text{Null}\, T^n + \text{Range}\, T^n = V$ since $V = V_0 \oplus V_1 \oplus \cdots \oplus V_k$

but $(\text{Null}\, T^n) \cap (\text{Range}\, T^n) = \{0\}$

$\therefore$ $V = \text{Null}\, T^n \oplus \text{Range}\, T^n$.

iv) Take T_A with $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. $\text{Null}\, T = sp(e_1)$

$\text{Range}\, T = sp(e_1, e_3)$

Ex 6. i) U is defined by two lin indep equations $\therefore$ has dim 3.

a basis is $(2,0,0,1,0)$, $(0,6,0,0,1)$, $(0,0,1,0,0)$

ii) $(1,0,0,-2,0)$, $(0,1,0,0,-6)$

Ex7. Extend e_1, e_2 to an onb $e_1, \ldots, e_n$ of V.

Write $v = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$

Then $\langle v, e_i \rangle = a_i$ since e_i onb.

and $\|v\|^2 = \sum |a_i|^2$ by Pythagoras.

$\therefore$ if $\|v\|^2 = a_1^2 + a_2^2$ we must have $a_3 = \cdots = a_n = 0$

$\therefore$ $v = a_1 e_1 + a_2 e_2 \in sp(e_1, e_2)$.

Conversely, if $v \in sp(e_1, e_2)$ $v = b_1 e_1 + b_2 e_2$ for some b_1, b_2

and $(a_1, a_2, \ldots, a_n) = (b_1, b_2, 0, \ldots, 0)$ by uniqueness of the expression for v

$\therefore$ $\|v\|^2 = \|b_1 e_1 + b_2 e_2\|^2 = b_1^2 + b_2^2 = |\langle v, e_1 \rangle|^2 + |\langle v, e_2 \rangle|^2$.

Ex 8 Suppose $U \neq V$ and choose $y \in V$, $y \notin U$.

Choose a basis $u_1, \ldots, u_k$ for V and apply Gram-Schmidt to get

onb $e_1, \ldots, e_k$ for U. Then consider $w = y - (\langle y, e_1 \rangle e_1 + \cdots + \langle y, e_k \rangle e_k)$.

Note $\langle w, e_i \rangle = \langle y, e_i \rangle - \langle y, e_i \rangle \langle e_i, e_i \rangle = 0$ for all i

$\therefore$ $w \in U^\perp$. Also $w \neq 0$ since $y \neq \sum \langle y, e_i \rangle e_i \in sp(U)$ $\therefore$ $U^\perp \neq \{0\}$

Conversely suppose $U^\perp \neq \{0\}$ and let $w \in U^\perp$, $w \neq 0$ Then $\langle w, u \rangle = 0$ for all $u \in U$

If $U = V$ then $w \in U$ and we get $\langle w, w \rangle = \|w\|^2 = 0 \implies w = 0$, contrary

to hypothesis. $\therefore$ $U \neq V$.

Ex 9. Let $U = sp(v)$. Then $TU = U \Rightarrow Tv = \lambda v$ for some λ.

Suppose $\exists_{\wedge}^{\text{nonzero}} v \neq w$ s.t. $Tv = \lambda v$ and $Tw = \lambda' w$, $\lambda \neq \lambda'$

Then $T(v-w) = \lambda v - \lambda' w$. But also $T(v-w) = \mu(v-w)$ since $sp(v-w)$ is invariant.

$$\therefore \lambda v - \lambda' w = \mu v - \mu w$$
$$\Rightarrow (\lambda - \mu) v = (\lambda' - \mu) w.$$

Since $\lambda \neq \lambda'$ at least one of $\lambda - \mu$, $\lambda' - \mu$ is nonzero, say $\lambda - \mu \neq 0$.

$\therefore v = \dfrac{\lambda' - \mu}{\lambda - \mu} w$. Since we assumed $v, w \neq 0$ we get that v is a nonzero multiple of w which implies that $\lambda = \lambda'$. A contradiction.

$\therefore \exists$ a constant $\lambda = c$ such that $Tv = cv$ for $\underline{\underline{\text{all}}}$ v.

10. We know $\text{Range}\, T^2 \leq \text{Range}\, T$ always.

Suppose $\text{Range}\, T^2 \subsetneq \text{Range}\, T$. Then the map $T : \text{Range}\, T \longrightarrow \text{Range}\, T^2$ is not injective. (apply the statement $\dim W = \dim \text{Null}\, S + \dim \text{Range}\, S$ to the map $S : W = \text{Range}\, T \rightarrow$ ~~Range~~ V given by restricting T to $W = \text{Range}\, T$.)

ie $\exists\, v \in \text{Range}\, T$ s.t. $Tv = 0 \implies \text{Null}\, T \cap \text{Range}\, T \neq \{0\}$

$\therefore$ if $\text{Null}\, T \cap \text{Range}\, T = \{0\}$, we must have $\text{Range}\, T = \text{Range}\, T^2$.

Math 310: Review for the Final, Dec 19, 5–7:30

Final version

The exam will be cumulative. You are expected to know all the previous definitions and also:

multiplicity of an eigenvalue;

generalized eigenvalues;

the characteristic and minimal polynomials;

the definition of trace and determinant of a linear map;

the formula for the determinant of an $n \times n$ matrix (using permutations).

the formula for the characteristic polynomial in terms of the determinant (see Ex 1 below, and the book Thm (10.17).)

All this JUST WHEN $\mathbb{F} = \mathbb{C}$. I will also NOT ask anything about the adjoint; or anything from Ch 7. From Ch 8 we did not do square roots or the Jordan normal form. I will ask you to prove some easy (and short) statements. (as in Ex 2); and also to compute some examples.

The most important theorems: (8.6), (8.9), (8.18), (8.19), (8.23), (8.28), (8.36); (10.3), (10.17), (10.33).

Here are some sample problems.

Ex 1 (i) First, some unfinished business from the lecture on Tuesday. Let V be a finite dimensional vector space over $\mathbb{C}$. I did not have time in class today to point out that $T \in \mathcal{L}(V)$ has 0 as an eigenvalue iff $\det T = 0$. (This is obvious because $\det T$ is the product of the eigenvalues of T.) Using this statement show that λ is an eigenvalue for T iff $\det(\lambda I - T) = 0$. Deduce from this that the characteristic polynomial of T is $\det(zI - T) = 0$. (Show that this is a monic polynomial $q(z)$ that has the same roots as the characteristic polynomial.) This justifies one of the standard definitions of the characteristic polynomial. It also justifies a standard computational method for finding eigenvalues. (Of course I would not ask you to prove this on an exam, but it's good review...)
(ii) Use this determinantal formula to compute the eigenvalues of $T_A : \mathbb{C}^3 \to \mathbb{C}^3$, where $A = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix}$.
(iii) Find a basis $\mathcal{B} = v_1, v_2, v_3$ with respect to which T_A is represented by a diagonal matrix $\mathcal{M}(T_A, \mathcal{B})$, and calculate $\mathcal{M}(T_A, \mathcal{B})$.

Ex 2. Define a linearly independent list. Suppose that the list $v_1, \dots, v_k$ of vectors in V is linearly independent but does NOT span V. Show that there is $v_{k+1} \in V$ such that the list $v_1, \dots, v_k, v_{k+1}$ is linearly independent.
Note: Prove this just using the basic definitions, DO NOT quote any theorems from the book.

Ex 3. Suppose $T \in \mathcal{L}(V)$, m is a positive integer and $v \in V$ is such that $T^2 v \neq 0$ but $T^3 v = 0$. Show that $(v, Tv, T^2 v)$ is linearly independent. (Hint: Suppose this is false; write down a linear relation and play with it.) This generalizes; cf p 188 ex 3.)

Ex 4 (i) Give an example of an operator on $\mathbb{C}^4$ whose minimal polynomial and characteristic polynomial are equal.
(ii) Give an example of an operator on $\mathbb{C}^4$ with minimal polynomial $(z-1)^2(z-2)$ and characteristic polynomial. $(z-1)^3(z-2)$.

Ex 5 (i) Suppose V is an n-dimensional complex vector space and let $T \in \mathcal{L}(V)$. Show that $(\operatorname{null} T^n) \cap (\operatorname{range} T^n) = \{0\}$. (Use (8.6).)
(ii) Suppose that the eigenvalues of T are $\lambda_0 = 0, \lambda_1, \ldots, \lambda_k$ and write $V = V_0 \oplus \cdots \oplus V_k$ where $V_i = \operatorname{null}(T - \lambda_i I)^n$ as in Thm (8.23). Show that $V_i \subseteq \operatorname{range} T^n$ for all $i > 0$.
Hint: Show by induction on k that $\operatorname{null}(T - \lambda I)^k \subset \operatorname{range} T^n$ whenever $\lambda \neq 0$.
(iii) Deduce from (i) and (ii) that $V = \operatorname{null} T^n \oplus \operatorname{range} T^n$.
(iv) Give an example of a T with eigenvalues 0 and 3 such that $V \neq \operatorname{null} T \oplus \operatorname{range} T$.

Ex 6 (i) Let U be the subspace of $\mathbb{R}^5$ defined by

$$U = \{(x_1, x_2, \ldots, x_5) : x_1 = 2x_4, x_2 = 6x_5\}.$$

Find a basis for U.
(ii) Find a basis for a subspace W such that $\mathbb{R}^5 = U \oplus W$. Hint: first decide what dimension W should have.)

Ex 7. Suppose that e_1, e_2 is an orthonormal list in a real inner product space V. Let $v \in V$. Show that $\|v\|^2 = |\langle v, e_1 \rangle|^2 + |\langle v, e_2 \rangle|^2$ iff $v \in \operatorname{span}(e_1, e_2)$.

Ex 8. Suppose U is a subspace of a real inner product space V. Show that $U^\perp = \{0\}$ iff $U = V$.
Note: in both ex 6 and ex 7 you should argue from the definition of inner product. Do not just quote results from the book. (eg if you assume that $V = U \oplus U^\perp$ then 7. is obvious, but what's the shortest argument that gives what you want, if assume you do NOT know this?)

Ex 9 Suppose that $T \in \mathcal{L}(V)$ is such that every subspace of V is invariant under T. Show that $T = cI$ for some scalar c.

Ex 10. Suppose $T \in \mathcal{L}(V)$ is such that $\operatorname{null} T \cap \operatorname{range} T = \{0\}$. What can you say about $\operatorname{range} T^2$?

Project 1

(due by 12/14/06 – 5:00pm)

Discuss the problem below in a concise and precise essay, at most 5 typed pages long. Whenever you use a reference, quote it and do not copy. Use your own words.

Consider the linear operator $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ which has the matrix

$$\begin{bmatrix} -4 & 3 & 1 & -1 \\ -6 & 5 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & -1 & 4 \end{bmatrix}$$

with respect to the standard basis.

1. Argue that there is a basis v_1, v_2, v_3, v_4 of $\mathbb{R}^4$ for which the matrix of T is upper-triangular by explicit construction, using the methods of Chapter 5. In particular, all eigenvalues are real. Give them and compute the matrix.

2. Find all invariant subspaces of T. Why is there no basis of eigenvectors, so T is not diagonalizable?

3. Finally, apply the Gram-Schmidt process to the above basis and construct an orthonormal basis e_1, e_2, e_3, e_4 and the corresponding upper-triangular matrix for T according to Schur's Theorem (Corollary (6.27) in the text book.).

Project 2

(due by 12/14/06 – 5:00pm)

This project concerns applications of Linear Algebra to approximating continuous functions by polynomials. Discuss the problem below in a concise and precise essay, at most 5 typed pages long. Whenever you use a reference, quote it and do not copy. Use your own words.

We want to use the idea of orthogonal projection to approximate a function as in Chapter 6, pp111-116. Consider the function cosh defined by

$$\cosh x = \frac{e^x + e^{-x}}{2} \ .$$

Recall some basic properties of this important function (including its graph, which is also known as a *catenary*, since a homogeneous chain freely suspended beween two points will hang conforming to this shape. Note in particular that cosh is an *even* function. We now restrict attention to the fixed interval $[-2, 2]$. On the real vector space $C[-2, 2]$ of all continuous function $f\colon [-2, 2] \to \mathbb{R}$ we work with the inner product

$$< f, g >= \int_{-2}^{2} f(x)g(x)dx \ ,$$

as usual. Let U denote the subspace of all polynomial functions in $C[-2, 2]$ of degree at most 5. Let $v \in C[-2, 2]$ be the restriction of cosh to the interval $[-2, 2]$ and $P_U v$ the othogonal projection of v in U.

1. Discuss first that x, x^3, x^5 are all orthogonal to v on $[-2, 2]$. Why is no explicit computation necessary here? Exploit that odd-degree monomials are odd functions.

2. Why does the previous result guarantee that $P_U v$ actually lies in the subspace $W \subset U$ spanned by $1, x^2, x^4$. Now compute $P_U v = P_W v = u$ by applying the Gram-Schmidt process to $1, x^2, x^4$ and obtain an orthonormal basis e_1, e_2, e_3 of W. So this best approximation of v on $[-2, 2]$ by polynomials of degree at most 5 will be of the form $u(x) = a + bx^2 + cx^4$.

3. As in our text, plot v and u in one graph. To obtain reasonably good graphics you need to use Maple, Mathematica, or similar programs for plotting, which you should try even if you have never done it before. Also plot v against its Taylor polynomial about 0, $p(x) = 1 + \frac{x^2}{2} + \frac{x^4}{24}$. In any case, decide either from the graphs or by any other method, which of these 2 polynomials u, p is the better approximation. Of course, both of them are very good. Discuss any other observations you might have made.

Project 3

(due by 12/14/06 – 5:00pm)

This project concerns some simple applications of Linear Algebra to Fibonacci Numbers. Discuss the problem below in a concise and precise essay at most 5 typed pages long. Whenever you use a reference, quote it and do not copy. Use your own words.

Let a_n denote the basic sequence of Fibonacci numbers defined by the recursive relation

$$a_{n+2} = a_n + a_{n+1}, \quad a_0 = 1, a_1 = 1\,.$$

Work out the problems below and find an explicit formula for a_n.

1. Consider an operator T on the vector space $\mathbb{R}^2$ such that T maps the vector (x, y) to the vector $(y, x+y)$. Show that T maps (a_{n-2}, a_{n-1}) to (a_{n-1}, a_n), where a_n is the Fibonacci sequence. Write the matrix A of T in the standard basis and prove that

$$A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}.$$

2. Diagonalize the operator T by finding its eigenvalues and eigenvectors. Show that the eigenvectors v_1, v_2 form a basis in $\mathbb{R}^2$. If B denotes the diagonal matrix of T with respect to this basis, verify that $A = P^{-1}BP$, where the columns of the transition matrix P are v_1, v_2 in terms of the coordinates with respect to the standard basis.

3. Find B^n and conclude that $A^n = P^{-1}B^nP$ from $A = P^{-1}BP$. Now easily find A^n and write an explicit formula for the n-th Fibonacci number.

4. Suppose that the numbers $b_n, n \geq 0$, are defined by the same recursive relation, but with $b_0 = 1, b_2 = 3$. Thus $b_2 = 4, b_3 = 7 \ldots$. Find an explicit formula for b_n. Check your answer for b_{10} (using a calculator).

Special Basic Project 4

(due by 12/14/06 – 5:00pm)

This project concerns only very basic aspects of Linear Algebra. It is meant to give those a chance who are in need of extra credit toward passing at a C level. It can **not** *be counted as credit toward the course grades A and B. Whenever you use a reference, quote it and do not copy. Use your own words.*

1. Let U and W be linear subspaces of a finite dimensional vector space V. Prove that the following three conditions are equivalent:
(a) $U + W = V$ and $U \cap W = 0$.
(b) For each vector $v \in V$ there are *unique* vectors $u \in U$ and $w \in W$ such that $v = u + w$.
(c) There exists a basis in V such that each vector in this basis belongs either to U or to W.

2. Consider the vectors in $\mathbb{R}^4$ defined by

$$v_1 = (1,0,1,1),\;\; v_2 = (1,0,2,1),\;\; v_3 = (1,2,0,1)\;\; v_4 = (3,2,3,3)$$

(a) What is the dimension of the subspace W of $\mathbb{R}^4$ spanned by the four given vectors? Find a basis for W and extend it to a basis of $\mathbb{R}^4$.
(b) Use a basis of $\mathbb{R}^4$ as in (a) to characterize all linear transformations $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ that have the null space equal to W. What can you say about the rank of such a T? (Note: rank = dimension of the range.)
(c) Give an explicit example of an operator $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ such that the range of T is W.

3. Prove that the vectors

$$v_1 = (1,1,1,1),\;\; v_2 = (1,1,2,1),\;\; v_3 = (0,1,0,1),\;\; v_4 = (1,1,1,0)$$

form a basis for $\mathbb{R}^4$. What are the coordinates of the vector (a,b,c,d) in this basis?

4. Let V be the vector space over $\mathbb{R}$ of all real polynomial functions p of degree at most 2.
(a) What are the coordinates of the polynomial function $a+bx+cx^2$ with respect to the ordered basis $\{1-x^2,\;\; 1+x+x^2,\;\; 1\}$ in V?
(b) For any fixed $h \in \mathbb{R}$ consider the *shift operator* $T\colon V \to V$ with $(Tp)(x) = p(x+h)$. Consider also the differentiation operator $D\colon V \to V$ with $Dp = p'$. Find the range, null space, rank and nullity of the operators TD, DT, D^2 and T^2. Which of these operators are isomorphisms? Write down the matrices of the operators TD, D^2 and T^2 with respect to the ordered basis $1, x, x^2$.

5. Let T be the linear operator on $\mathbb{R}^2$ defined by $T(x_1,x_2) = (-\frac{\sqrt{2}}{2}(x_1+x_2), \frac{\sqrt{2}}{2}(x_1-x_2))$.
(a) What is the matrix of T in the standard ordered basis for $\mathbb{R}^2$?
(b) Interpret the operation of T geometrically.
(c) What is the matrix of T in the ordered basis v_1, v_2, where $v_1 = (1,1)$ and $v_2 = (2,0)$?
(d) Prove that for every real number λ the operator $(T - \lambda I)$ is invertible.
(e) Find all *complex* numbers λ such that the operator $(T - \lambda I)$ is *not* invertible.

6. Let $T\colon V \to V$ be a linear operator on the vector space V with null space W_1 and range W_2. Suppose that $S\colon V \to V$ is another linear operator on V commuting with T, i.e. $ST = TS$. Prove that W_1, W_2 are invariant subspaces of both T and S.

Project 5

(due by 12/14/06 – 5:00pm)

This project concerns some problems related to the Jordan form of a linear operator – you should look a little into Chapter 8, notably the secion on square roots. But not many details will be necessary.
Discuss the problems below in a concise and precise essay, at most 5 typed pages long. Whenever you use a reference, quote it and do not copy. Use your own words.

1. Suppose $T\colon \mathbb{C}^4 \to \mathbb{C}^4$ is defined by $T(z_1, z_2, z_3, z_4) = (z_2, z_3, z_4, 0)$. Prove that T has no square root. More precisely, prove that there does not exist a linear operator $S\colon \mathbb{C}^4 \to \mathbb{C}^4$ such that $S^2 = T$.

2. Define $N\colon \mathbb{F}^5 \to \mathbb{F}^5$ by

$$N(x_1, x_2, x_3, x_4, x_5) = (2x_2, 3x_3, -x_4, 4x_5, 0)\,.$$

Find a square root of $I + N$.

3. Prove that if V is a complex vector space, then every invertible operator on V has a cubic root.

Math 310: Review for Midterm II, Nov 16

The exam will concentrate on Ch 5 and 6, except for p 91-93, and p 113-117 (about orthogonal projections and minimization problems). Obviously you also need to know how to use most of the concepts of Ch 1-3, but there will be no specific questions on this material. You are expected to know the following definitions, know some relevant examples, and to be able to do simple calculations using them. You are also expected to know the statements of the main theorems so that you can quote them in your answers. I will NOT distribute a list since there aren't so many this time.

invariant subspace for operator T;

eigenvalue, eigenvector for T;

upper triangular matrix;

inner product; orthogonal vectors, orthonormal basis;

orthogonal complement; orthogonal projection of a vector onto a subspace

adjoint T^* of $T \in \mathcal{L}(V, W)$.

Most important results:

existence and basic properties of eigenvalues/vectors: (5.6), (5.10), (5.13), (5.16), (5.20)
the properties of inner product: (6.3), (6,6), (6.9), and Gram-Schmidt. Orthogonal projection of a vector on a subspace. Elementary properties of adjoint.

Here are some sample problems. Note also that on the syllabus I suggested some problems from the book.

Ex 1: (i) Let V be a vector space over $\mathbb{C}$ with subspace U. What does it meant to say that U is invariant under $T \in \mathcal{L}(V)$?

(ii) Let $V = \mathbb{C}^\infty$ and $T : V \to V$ be the right shift: $T(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$. Find a T-invariant subspace U that is NOT equal to $\{0\}$ or V.

(iii) What are the eigenvalues and eigenvectors of T?

Ex 2: Let

$$A := \begin{bmatrix} 2 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \qquad B := \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Let $T_A, T_B \in \mathcal{L}(\mathbb{R}^3)$ be the linear maps given by multiplication by the matrices A, B.

(i) Questions on T_A:

(1) Find all eigenvalues and eigenvectors for T_A.

(2) Find TWO different bases $\mathcal{B} := (v_1, v_2, v_3)$ of $\mathbb{R}^3$ such that the matrix $\mathcal{M}(T)$ representing T with respect to $\mathcal{B}$ is upper triangular. (Here $\mathcal{B} := (v_1, v_2, v_3)$ is DIFFERENT from $\mathcal{B}' := (v_1', v_2', v_3')$ if there do not exist scalars a_1, a_2, a_3 so that $v_i' = a_i v_i$ for all i.)

(3) How much choice do you have in choosing such a basis $\mathcal{B}$? (I wouldn't ask this in an exam ... but it is good to help you understand.)

(ii) Questions on T_B:

(1) Find a subspace U of $\mathbb{R}^3$ that is not $\{0\}$ or $\mathbb{R}^3$ and is invariant by T_B. **Hint:** look at the zero entries of B – what does B do to the elements of the standard basis??
(2) Find the eigenvalues and eigenvectors for $T_B : U \to U$.
(3) Find all eigenvalues and eigenvectors for $T_B : \mathbb{R}^3 \to \mathbb{R}^3$.
(4) Find an invertible matrix Q such that $Q^{-1}AQ$ is diagonal.

Ex 3: (slightly revised) This exercise asks you to find an eigenvector for $T : \mathbb{C}^3 \to \mathbb{C}^3$ using the method in the proof of (5.10). Let $T = T_A$ where

$$A := \begin{bmatrix} 3 & 1 & -2 \\ 2 & 0 & -6 \\ 1 & -2 & 1 \end{bmatrix}.$$

(1) Let $v = (1, 0, 0)$. Calculate Tv, T^2v, T^3v.
(2) Find $a, b, c, d \in \mathbb{C}$ so that $(aT^3 + bT^2 + cT + dI)v = 0$.
(3) Find an eigenvector and eigenvalue for T. Check your answer. **Note:** the calculations here should not be too hard. If they are, you've made an arithmetic error. Also: in fact you should be able to find at least two eigenvalues and eigenvectors from this calculation.

Ex 4: (i) Let U be a subspace of an inner product space $(V, \langle,\rangle)$. What is its orthogonal complement $U^\perp$? Given any subspaces U, W of V show that $U \subseteq W$ iff $W^\perp \subseteq U^\perp$.

(ii) Let U be the subspace of $\mathbb{C}^4$ given by the equations $x_1 + x_2 + x_3 + x_4 = 0$ and $x_1 - x_2 + 2x_3 + x_4 = 0$.

(1) Find a basis for $U^\perp$. **Note:** this is an easy question that requires NO calculation! You just need to understand what the equations for U tell you. But this subspace is the one considered in Ex1 on HW8, so all calculations are done anyway.
(2) Find an orthonormal basis for $U^\perp$.
(3) Let $w = (1, 2, 3, 0)$. Write $w = u + v$ where $u \in U$ and $v \in U^\perp$.
(4) Check that $\|w^2\| = \|u\|^2 + \|v\|^2$.

Ex 5: (i) Let $(V, \langle,\rangle)$ and $(W, \langle,\rangle)$ be finite dimensional inner product spaces. Let $T \in \mathcal{L}(V, W)$. Define the adjoint T^* of T.

(ii) Define $T : \mathbb{C}^2 \to \mathbb{C}^2$ by $T(z, w) = (iz - w, 2z + (1 + i)w)$. Use the definition in (i) to calculate T^*.

(iii) Now go back to the general case with V, W, T as in (i).

(1) Show that $(aT)^* = \overline{a}T^*$.
(2) Show that $\operatorname{range} T^* = (\operatorname{null} T)^\perp$. Note: it is easy to show that $\operatorname{range} T^* \subseteq (\operatorname{null} T)^\perp$ (DO THIS), but not so easy to give a direct argument why $(\operatorname{null} T)^\perp \subseteq \operatorname{range} T^*$. But note that $U \subseteq W$ iff $W^\perp \subseteq U^\perp$. Therefore instead of proving $(\operatorname{null} T)^\perp \subseteq \operatorname{range} T^*$ it is enough to show that $(\operatorname{range} T^*)^\perp \subseteq \operatorname{null} T$. For some (slightly mysterious) reason, this is easier! (And you note that this what the book does in Prop 6.46.)
(3) Show that $\operatorname{range} T$ and $\operatorname{range} T^*$ have the same dimension. Give an example where $\operatorname{null} T$ and $\operatorname{null} T^*$ do NOT have the same dimension.

Math 310: Worksheet 2

Nov 2 2006

(1) (*an exercise on the Gram-Schmidt process*) Find an orthonormal basis for the subspace of $\mathbb{R}^4$ generated by the following vectors:
  (a) $(1,1,0,0)$ and $(1,-1,1,1)$.
  (b) $(1,1,0,0),(1,-1,1,1)$ and $(-1,0,2,1)$

(2) *Some geometry*
  (a) Check that the subspace U in Ex 1 (b) is given by the equation $x-y+3z-5t=0$.
  (b) Check that $v:=(3,-2,0,5)\in U$. Find its coordinates with respect to the basis e_1,e_2,e_3 for U that you found in Ex 1 (b).
  Hint: Remember that if $e_1,\ldots,e_k$ is an orthonormal basis for a subspace U and $u\in U$ then the coordinates of u are given by $u=\langle u,e_1\rangle e_1+\cdots+\langle u,e_k\rangle e_k$.
  (c) Let $w\in\mathbb{R}^4$ be any vector and define
$$u:=\langle u,e_1\rangle e_1+\langle u,e_2\rangle e_2+\langle u,e_3\rangle e_3.$$
  Show that the vector $w-u$ is perpendicular to U, i.e. $\langle w-u,u'\rangle=0$ for all $u\in U$. (It is enough to check this for $u'=e_1,e_2,e_3$. Why?) Thus we may decompose w as the sum $u+(w-u)$ where $u\in U$ and $w-u\in U^\perp$.
  (d) Calculate the decomposition of $w=(1,1,1,1)$ as a sum $u+v$ where $u\in U$ and $v\in U^\perp$.

(3) Let V be the vector space of continuous real-valued functions on the interval $[0,1]$. Define the inner product of two such functions f,g by the rule
$$\langle f,g\rangle=\int_0^1 f(t)g(t)dt.$$
Note: Sometimes people (e.g. Vincent...) say *scalar product* instead of inner product. This means the same.
  (a) Using the standard properties of the integral, verify that this is an inner product; i.e. check that for all $f,g,h\in V$ and $c\in\mathbb{R}$
   - $\langle f,f\rangle\geq 0$ and $=0$ only if $f=0$;
   - $\langle cf,g\rangle=c\langle f,g\rangle$;
   - $\langle f+h,g\rangle=\langle f,g\rangle+\langle h,g\rangle$;
   - $\langle f,g\rangle=\langle g,f\rangle$.

  Note: You will find that it is much easier to check positivity by arguing indirectly from properties of the integral rather than by calculating $\langle f,f\rangle\geq 0$!
  (b) Let U be the subspace of V generated by the two functions $f(t)=t$ and $g(t)=t^2$. Find an orthonormal basis for U.
  (c) Let U' be the subspace generated by the three functions $1,t,t^2$ (where 1 is the constant function). Find an orthonormal basis for U'.

Math 310: Midterm 1

October 10, 2006

Name: **ID number:**

There are 5 questions worth a total of 100 points, plus one small bonus question worth 10 points. Please justify all your statements, and write neatly so that we can read and follow your answers. Continue your answers on the back of the pages. Also, please turn off cell phones.

Question 1. (20 points) (i) Define a subspace of a vector space.

(ii) Suppose that V is a finite dimensional vector space and that $W \subseteq V$ is a subspace such that $\dim W = \dim V$. Prove carefully that $W = V$.

1	20pt	
2	20pt	
3	20pt	
4	30pt	
5	10pt	
Total	100pt	
bonus	10pt	

Question 2. (20 points) (i) Let $(v_1, \ldots, v_n)$ be a list of vectors in V. What does it mean to say that this list is linearly independent? Give the formal definition.

(ii) Give an example of a list of three vectors in $\mathbb{R}^4$ that is linearly independent and another that is linearly dependent. (You only need to give brief explanations.)

(iii) Suppose that the list v_1, v_2, v_3 is linearly independent. Show that the list $v_1 + 2v_2 + v_3, v_2 + v_3, v_3$ is also linearly independent.

Question 3. (20 points) (i) Find a basis for the subspace

$$U := \{(y_1, y_2, y_3, y_4) \in \mathbb{F}^4 : y_1 = y_2 = y_2 + y_3 + y_4\},$$

and prove that the elements you give do form a basis.

(ii) Suppose that W is another subspace of $\mathbb{F}^4$ such that $U + W = \mathbb{F}^4$. What can you say about $\dim W$?

Question 4. (30 points) (i) Show that a linear map $T : V \to W$ is injective if and only if $\operatorname{Null} T = \{0\}$.

(ii) Let $V := \mathcal{P}(3)$ the polynomials of degree ≤ 3 and coefficients in $\mathbb{F}$. Define $T : V \to V$ by $T(f) = (z^2 + z)f''$, where f'' denotes the second derivative of f. Describe $\operatorname{Null} T$ and $\operatorname{Range} T$. What are their dimensions?

(iii) Find the matrix $\mathcal{M}(T)$ that represents T with respect to the standard basis $f_0 := 1,\ f_i = z^i, i = 1, 2, 3.$

Question 5. (10 points, plus 10 points bonus) Let $L : V \to W$ be a linear map.

(i) Suppose that $w_1, \ldots, w_n$ is a linearly independent list in V and that L is injective. Show that the list $(Lw_1, \ldots, Lw_n)$ is linearly independent.

(ii) **Bonus:** Is it possible for the list $w_1, \ldots, w_n$ to be linearly dependent while $(Lw_1, \ldots, Lw_n)$ is linearly independent?

Math 310: Midterm 1

October 10, 2006

Name: **ID number:**

There are 5 questions worth a total of 100 points, plus one small bonus question worth 10 points. Please justify all your statements, and write neatly so that we can read and follow your answers. Continue your answers on the back of the pages. Also, please turn off cell phones.

Question 1. (20 points) (i) Find a basis for the subspace

$$W := \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_1 = x_2 - 3x_3 = x_3 + x_4\},$$

and prove that the elements you give do form a basis.

(ii) Suppose that U is another subspace of $\mathbb{F}^4$ such that $U + W = \mathbb{F}^4$. What can you say about $\dim U$?

1	20pt	
2	20pt	
3	20pt	
4	30pt	
5	10pt	
Total	100pt	
bonus	10pt	

Question 2. (20 points) (i) Let $(w_1, \dots, w_n)$ be a list of vectors in V. What does it mean to say that this list is linearly independent? Give the formal definition.

(ii) Give an example of a list of three vectors in $\mathbb{R}^3$ that is linearly independent and another that is linearly dependent. (You only need to give brief explanations.)

(iii) Suppose that the list w_1, w_2, w_3 is linearly independent. Show that the list $w_1 - w_2 + 3w_3, w_2 - w_3, w_3$ is also linearly independent.

Question 3. (20 points) (i) Define a subspace of a vector space.

(ii) Suppose that V is a finite dimensional vector space and that $U \subseteq V$ is a subspace such that $\dim U = \dim V$. Prove carefully that $U = V$.

Question 4. (30 points) (i) Show that a linear map $T : W \to V$ is injective if and only if $\operatorname{Null} T = \{0\}$.

(ii) Let $W := \mathcal{P}(3)$ the polynomials of degree ≤ 3 and coefficients in $\mathbb{F}$. Define $T : W \to W$ by $T(f) = (z-1)f''$, where f'' denotes the second derivative of f. Describe $\operatorname{Null} T$ and $\operatorname{Range} T$. What are their dimensions?

(iii) Find the matrix $\mathcal{M}(T)$ that represents T with respect to the standard basis $g_0 := 1$, $g_i = z^i, i = 1, 2, 3$.

Question 5. (10 points, plus 10 points bonus) Let $T : V \to W$ be a linear map.

(i) Suppose that $v_1, \dots, v_n$ is a linearly independent list in V and that T is injective. Show that the list $(Tv_1, \dots, Tv_n)$ is linearly independent.

(ii) **Bonus:** Is it possible for the list $v_1, \dots, v_n$ to be linearly dependent while $(Tv_1, \dots, Tv_n)$ is linearly independent?

Math 310: Review for Midterm I, Oct 4

You are expected to know the following definitions, know some relevant examples, and to be able to do simple calculations using them.

subspace; sum of two subspaces $U + W$; direct sum $U \oplus W$;

linear (in)dependence of a list $(v_1, \dots, v_n)$; span of a list $(v_1, \dots, v_n)$;

finite and infinite dimensional vector spaces;

basis and dimension of a finite dimensional vector space;

linear map $T : V \to W$; null space and range of T; surjective, injective, invertible;

the matrix $\mathcal{M}(T)$ of a linear map $T : V \to W$ with respect to given bases of V and W.

In the midterm I will ask you to give some definitions. I will also ask you to prove some statements. In your answers you may quote any theorem from the list, unless you are given instructions not to.

Here are some sample problems.

Ex 1: Let V be a finite dimensional space.
(i) Define $\dim V$.
(ii) Suppose $\dim V = n$ and $v_1, \dots, v_n$ spans V. Give a careful proof that $v_1, \dots, v_n$ is a basis for V.

You may use any theorem on the list. Any other statement must be proved.

Ex 2: (i) Find a basis for the subspace

$$V := \{(x_1, \dots, x_4) \in \mathbb{F}^4 : x_1 - x_2 + x_4 = 0, x_2 = x_3\}.$$

(ii) Extend this to a basis for $\mathbb{F}^4$.
(iii) Define a linear map $T : \mathbb{F}^4 \to \mathbb{F}^3$ with null $T = V$. Justify all your claims.

Ex 3: (i) Define an infinite dimensional space.
(ii) Give an example of an infinite dimensional space V and a linear map $T : V \to V$ that is

(1) injective but not surjective
(2) surjective but not injective.

(iii) Do such linear maps exist when V is finite dimensional? Give examples or a careful proof that such examples cannot exist.

Ex 4: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map $T(x, y) = (x, 3x + 2y)$. Let $\mathcal{B} = (v_1, v_2)$ be the basis with $v_1 = (1, -3), v_2 = (0, 1)$. Find the matrix $\mathcal{M}(T)$ that represents T with respect to this basis.

Ex 5: Let $V = \mathcal{P}_3(\mathbb{R})$ the vector space of real polynomials with degree ≤ 3. Let $\mathcal{B}$ be the basis:

$$f_1 = 1, \quad f_2 = z, \quad f_3 = z^2, \quad f_4 = z^3.$$

Define $T : V \to V$ by $T(f) = zf'(z)$, where f' denotes the derivative of f.
(i) Show that T is linear.

(ii) Give an example of a map $S : V \to V$ that is NOT linear, explaining your answer.

(iii) Find the matrix $\mathcal{M}(T)$ that represents T with respect to the basis $\mathcal{B}$.

Note: I added an extra one of these for practice; there certainly will NOT be more than one question on the exam about this.

Ex 6: Let V be as in Ex 5.

(i) What is a subspace of the vector space V?.

(ii) Show that the subset $W := \{f \in V : f(1) = 0\}$ is a subspace of V.

(iii) Find a basis for W.

(iv) Let $g = 1 - z + z^2 - z^3$. Write g as a linear combination of the elements of the basis that you found in (iii).

Ex 7: (i) What does it mean to say that the list $v_1, \dots, v_n$ spans the vector space V?

(ii) Suppose that the list v_1, v_2, v_3 spans V. Show that the list $v_1, v_2 + v_1, v_3 + v_1$ also spans V.

(iii) Show that your argument in (i) fails with the list $v_1 + v_2, v_2 + v_3, v_3 - v_1$.

(iv) Is it possible to find 3 vectors v_1, v_2, v_3 that span $\mathbb{R}^2$ but are such that $v_1 + v_2, v_2 + v_3, v_3 - v_1$ do not?

Note: I wouldn't put a question like (iv) on the exam because it is too open ended. But it is a good review question.

Ex 8: (i) What does it mean to say that the list $v_1, \dots, v_n$ is linearly dependent?

(ii) Use this definition to prove (2.4):

If $(v_1, \dots, v_m)$ is linearly dependent in V and $v_1 \neq 0$ then there exists $j \in \{2, \dots, m\}$ such that the following hold:

(1) $v_j \in \operatorname{span}(v_1, \dots, v_{j-1})$;

(2) *if the jth term is removed from $(v_1, \dots, v_m)$, the span of the remaining list equals* $\operatorname{span}(v_1, \dots, v_m)$.

(iii) Give a list $v_1, v_2, v_3 \in \mathbb{R}^2$ that is linearly dependent, but is such that any pair of vectors from this list is linearly independent.

Ex 9: (i) Let U, W be subspaces of V. Define $U + W$.

(ii) Suppose that U, V are 2-dimensional subspaces of $\mathbb{R}^4$ such that $U \cap W = \{0\}$. Show that $U + W = \mathbb{R}^4$.

Ex 10: (i) Let $T : V \to W$ be a linear map. Define null T and range T.

(ii) Let v_1, v_2, v_3 be a basis for a vector space V and define a linear map $T : V \to \mathbb{R}^3$ by setting

$$T(v_1) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad T(v_2) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad T(v_3) = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}.$$

Find a basis for null (T) and a basis for range T.

(iii) Is T injective?

Math 310: List of theorems

This list will be provided with the midterm.

(1.2,3) A vector space V has a unique additive identity 0. Every element $v \in V$ has a unique additive inverse $-v$.

(1.4,5,6) $0v = 0$ and $(-1)v = -v$ for every $v \in V$. $a0 = 0$ for every $a \in \mathbb{F}$.

(1.9) Suppose that U and W are subspaces of V. Then $V = U \oplus W$ if and only if $V = U + W$ and $U \cap W = \emptyset$.

(2.4) If $(v_1, \ldots, v_m)$ is linearly dependent in V and $v_1 \neq 0$ then there exists $j \in \{2, \ldots, m\}$ such that the following hold:

(1) $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$;

(2) if the jth term is removed from $(v_1, \ldots, v_m)$, the span of the remaining list equal $\operatorname{span}(v_1, \ldots, v_m)$.

(2.6) In a finite dimensional vector space the length of every linearly independent list is less than or equal to the length of every spanning list of vectors.

(2.8) A list $(v_1, \ldots, v_n)$ of vectors in V is a basis for V if and only if every $v \in V$ can be written uniquely in the form $v = a_1 v_1 + \cdots + a_n v_n$.

(2.12) Every linearly independent list of vectors in a finite dimensional vector space can be extended to a basis of the vector space.

(2.14) Any two bases of a finite dimensional vector space have the same length.

(3.0) Let $(v_1, \ldots, v_n)$ be a basis of V and $w_1, \ldots, w_n$ be any elements in W. Then there is a unique linear map $T : V \to W$ such that

$$T(v_i) = w_i, \quad \text{for } i = 1, \ldots, n.$$

(3.1) If $T : V \to W$ is a linear map, $\operatorname{null} T$ is a subspace of V.

(3.2) A linear map T is injective if and only if $\operatorname{null} T = \{0\}$.

(3.3) If $T : V \to W$ is a linear map, $\operatorname{range} T$ is a subspace of V.

(3.4) If V is finite dimensional and $T : V \to W$ is a linear map, then $\operatorname{range} T$ is a finite dimensional subspace of W and

$$\dim V = \dim \ T + \dim \operatorname{range} T.$$

(3.17) A linear map is invertible if and only if it is injective and surjective.

(3.18) Two finite dimensional vector spaces are isomorphic if and only if they have the same dimension.

Math 310: Workshop; Sep 28

(1) What is the dimension of the space of $m \times n$ matrices? Give a basis for this space.
Hint Do this first for $m \times n = 2 \times 3$ and then generalize. If you find the general case very hard, go onto the next problems, doing them in the case $n = 2, 3, 4$.

(2) What is the dimension of the space of symmetric $n \times n$ matrices? Give a basis for this space. Recall that a matrix A is called symmetric if $A = {}^tA$. Here ${}^tA$ is the transpose of A (sometimes also written as A^T); thus it is obtained from A by reflecting A in the main diagonal.
Hint: To do this, write down some symmetric nmatrices when $n = 2, 3, \ldots$. What choices can you make?

(3) An $n \times n$ matrix A is called *skew-symmetric* if ${}^tA = -A$. Show that any matrix A can be written as a sum
$$A = B + C$$
where B is symmetric and C is skew-symmetric. [Hint: Let $B = (A + {}^tA)/2$.] Show that if $A = B_1 + C_1$, where B_1 is symmetric and C_1 is skew-symmetric, then $B = B_1$ and $C = C_1$.

(4) Let M be the space of all $n \times n$ matrices. Let
$$P \colon M \to M$$
be the map such that
$$P(A) = \frac{A + {}^tA}{2}.$$
(a) Show that P is linear.
(b) Show that the kernel of P consists of the space of skew-symmetric matrices.
(c) What is the dimension of the kernel of P?
(d) What is Range (T) and its dimension?
(e) Check that your answer here is consistent with your answers to (1), (2), (3) and with Thm (3.4).

(5) Let V, W be finite dimensional spaces of the same dimension and $L \colon V \to W$ be a linear map. Deduce from Thm (3.4) that
(a) L injective implies that L is surjective.
(b) L surjective implies that L is injective.
(c) Construct an isomorphism from the space of all $m \times n$ complex matrices to $\mathbb{C}^d$, where d is your answer to Question (1).

Note: A linear map that is both injective and surjective is called an isomorphism.

Math 310: Notes, Sept 20

Here is the Lemma that I proved in class.

Lemma 1. *Suppose that* $(v_1, \ldots, v_n)$ *is a linearly independent list of vectors in* V *and that* $v_{n+1} \in V$. *Then the list* $(v_1, \ldots, v_{n+1})$ *is linearly independent iff* $v_{n+1} \notin \mathrm{sp}(v_1, \ldots, v_n)$.

NOTE: "iff" means "if and only if"

We saw in class that it was hard to give a direct proof of the lemma. – How do you express that v_{n+1} is NOT in $\mathrm{sp}(v_1, \ldots, v_n)$? In class, instead of proving P is equivalent to Q, I proved NOT P is equivalent to NOT Q. (Logically these are the same.) In other words, I proved

Lemma 2. *Suppose that* $(v_1, \ldots, v_n)$ *is a linearly independent list of vectors in* V *and that* $v_{n+1} \in V$. *Then the list* $(v_1, \ldots, v_{n+1})$ *is linearly dependent iff* $v_{n+1} \in \mathrm{sp}(v_1, \ldots, v_n)$.

Proof ($\Rightarrow$) Suppose that $(v_1, \ldots, v_{n+1})$ is linearly dependent. Then there is a relation

$$a_1 v_1 + \cdots + a_{n+1} v_{n+1} = 0$$

in which the scalars $a_1, \ldots, a_{n+1}$ are not all 0.

If $a_{n+1} = 0$ we would have a relation $a_1 v_1 + \cdots + a_n v_n = 0$. By the linear independence of $(v_1, \ldots, v_n)$ this means that $a_i = 0$ for all $i = 1, \ldots n$. But the a_i do not all vanish. So this is impossible. Hence $a_{n+1} \neq 0$.

Therefore we can divide the above relation by a_{n+1} to get

$$v_{n+1} = -\Big(\frac{a_1}{a_{n+1}} v_1 = \cdots + \frac{a_n}{a_{n+1}} v_n\Big) \in \mathrm{sp}(v_1, \ldots, v_n).$$

($\Leftarrow$) The proof of the converse implication is left to you. (It's easier.) You must show that if $v_{n+1} \in \mathrm{sp}(v_1, \ldots, v_n)$ then $(v_1, \ldots, v_{n+1})$ is linearly dependent. $\square$.

Chapter 3 contains some very important facts about linear transformations that are not formulated into Propositions. The next result is taken from p 39-40.

Prop 3 *Let* $(v_1, \ldots, v_n)$ *be a basis of* V *and* $w_1, \ldots, w_n$ *be ANY elements in* W. *Then there is a UNIQUE linear map* $T : V \to W$ *such that*

$$T(v_i) = w_i, \quad \text{for } i = 1, \ldots, n.$$

Proof First, let us prove that T exists. To do this, we must define $T(v)$ for every $v \in V$ and show that T has the two properties: **additivity** $T(u + v) = Tu + Tv$ for all $v, w \in V$

homogeneity $T(av) = aT(v)$ for all $a \in \mathbb{F}, v \in V$.

Definition of Tv: For each $v \in V$ there are unique scalars a_i so that $v = a_1 v_1 + \cdots + a_n v_n$. Define

$$T(v) := a_1 w_1 + \cdots + a_n w_n.$$

Since $v_1 = 1v_1 + 0v_2 + \ldots 0v_n$, we find $Tv_1 = w_1$. Similarly, $Tv_i = w_i$ for all i.

Check additivity: Suppose that $v = a_1 v_1 + \cdots + a_n v_n$ and $w = b_1 v_1 + \cdots + b_n v_n$. Then

$$v + w = (a_1 + b_1)v_1 + \cdots + (a_n + b_n)v_n.$$

Therefore by the definition of T,

$$T(v + w) = (a_1 + b_1)w_1 + \cdots + (a_n + b_n)w_n.$$

But by definition

$$Tv = a_1 w_1 + \cdots + a_n w_n, \quad Tw = b_1 w_1 + \cdots + b_n w_n.$$

Therefore $T(v + w) = Tv + Tw$.

Check homogeneity This is similar. Write out $T(av)$ and compare it with aTv.

We have now defined $T : V \to W$ with the required properties. We need to check that it is unique. But this is immediate. The properties (*Additivity*) and (*Homogeneity*) imply that for any vectors $v_i \in V$ and scalars $a_i \in \mathbb{F}$ we have

$$T(a_1 v_1 + \cdots + a_n v_n) = a_1 Tv_1 + \ldots a_n Tv_n.$$

Therefore, if we are told that $Tv_i = w_i$ for all i, $T(a_1 v_1 + \cdots + a_n v_n)$ must equal $a_1 w_1 + \ldots a_n w_n$. Therefore if $v = a_1 v_1 + \cdots + a_n v_n$, Tv must be given by the formula

$$Tv = a_1 Tv_1 + \ldots a_n Tv_n.$$

Thus there is only one linear map T that satisfies the given conditions, i.e. T is unique.

NOTE: If $Tv_i = w_i$ for all i as above, then the image (or range) of T is the span of the vectors $w_1, \ldots, w_n$. Can you prove this?

Example *Find a nonzero linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ with image contained in the subspace $W = \{(x, y, z) : x + 2z = 0\}$. What is the range of T? Calculate $T(2, 3)$.*

Let $v_1 = (1, 0), v_2 = (0, 1)$ be the standard basis of $\mathbb{R}^2$. Take $w_1 = w_2 = w = (2, 0, -1) \in W$. Consider the map T such that $T(1, 0) = (2, 0, -1) \in W$ and $T(0, 1) = (2, 0, -1) \in W$. Then $T(a_1 v_1 + a_2 v_2) = a_1 w + a_2 w = (a_1 + a_2)w$. In particular $T(2, 3) = 5w = (10, 0, -5)$. Note that $Tv \in W$ for all $v \in \mathbb{R}^2$, as we would expect. But the vectors Tv do not span W. Rather they span the line $\{aw : a \in \mathbb{R}\}$. This line is the span of w_1, w_2 in this case (since $w = w_1 = w_2$). Hence this agrees with the NOTE above.

Math 310: Model Proof

Sept 12 2006

When you write a proof is it very important to explain clearly the logic of your argument; what are you assuming? what are you trying to prove? how do you justify each step?

Some of the proofs in the book are rather too short. Here is an example to explain what I mean. This is Prop 1.6 in the book. (The proof in the book is a bit longer than the first one below, but the explanation is still rather brief.)

Proposition 1.6 $(-1)v = -v$ for every $v \in V$.

Proof. For every $v \in V$
$$v + (-1)v = 1v + (-1)v = (1 + (-1))v = 0v = 0.$$
Therefore $(-1)v = -v$. □

This proof is too short. For one, there is no explanation of the conclusion: "Therefore $(-1)v = -v$". This step is justified by Prop 1.3 that says that every element v has a unique additive inverse, i.e. if $v+w = 0$ then w is the additive inverse $-v$. Secondly the steps in the first statement $v+(-1)v = 1v+(-1)v = (1+(-1))v = 0v = 0$ are not explained. These are justified by the familiar axioms of addition and multiplication together with Prop 1.4.

Later on in the semester it would be fine for you to use the axioms and results in Ch 1 without comment. In this first week it would be best to explain each step.

Here is a fuller version of the proof.

Proposition 1.6 $(-1)v = -v$ for every $v \in V$.

Proof. For every $v \in V$
$$\begin{aligned} v + (-1)v &= 1v + (-1)v && \text{by the Mult. identity axiom} \\ &= (1 + (-1))v && \text{by the distributive properties} \\ &= 0v \\ &= 0 && \text{by Prop 1.4,} \end{aligned}$$
Therefore $(-1)v = -v$ by Prop 1.3. □

Math 310: Review Exercises Sept 5, 2006

Ex 1. Review complex numbers.
(i) Calculate (with answers in the form $z = a + ib$)

$$(2+3i)(4-i), \quad (2-3i)^{-1}, \quad \frac{1-i}{2+3i}.$$

(ii) Recall that for the complex number $z = a + ib \in \mathbb{C}$, the modulus $|z|$ is defined to be $\sqrt{a^2+b^2}$. Calculate $|2+3i|, |1+i|$.
(iii) Check that $|(2+3i)(1+i)| = |2+3i||1+i|$., i.e. the modulus of a product is the product of the moduli.

Ex 2. Review matrices, especially matrix multiplication. Calculate
$A+B, A-C, AC, CB$ and BA (if they are defined) when

$$A := \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}, \quad B := \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & -2 \end{bmatrix}, \quad C := \begin{bmatrix} 0 & 1 & 5 \\ 2 & -1 & 1 \end{bmatrix}.$$

Some linear algebra in (real) 3-space $\mathbb{R}^3$: Definitions
A 3-**vector** $\mathbf{v}$ or (vector in $\mathbb{R}^3$) is an ordered triple of 3 real numbers, usually written horizontally. eg $\mathbf{v} = (1,2,3)$ but sometimes vertically: $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.
The **span** of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ consists of all linear combinations

$$\mathbf{v} := a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k, \quad a_i \in \mathbb{R}.$$

The vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ are said to be **linearly independent** if and only if

$$a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k = 0 \text{ implies } a_i = 0 \ \ \forall i,$$

i.e. the only linear combination $\sum_i a_i\mathbf{v}_i$ that equals 0 is the trivial combination with all coefficients equal to 0. If $\mathbf{v}_1, \dots, \mathbf{v}_k$ are not linearly independent then they are said to be **linearly dependent**.
Note: The book gives very slightly different (but equivalent) definitions; we will discuss this in class.

Here are some questions that use these concepts. Try to make careful arguments, not just a list of scraps of calculations. (This is good practice for what is coming...)
Ex 3. (i) Let $\mathbf{v}_1 = (1,2,1), \mathbf{v}_2 = (1,0,1), \mathbf{v}_3 = (0,1,-1)$. Do these vectors span $\mathbb{R}^3$?
(ii) Are these vectors linearly independent?

Ex 4. Same questions for the vectors $\mathbf{v}_1 = (1,-2,1), \mathbf{v}_2 = (1,0,-1), \mathbf{v}_3 = (0,1,-1)$.

Ex 5. Show that the span of the vectors $\mathbf{v}_1 = (1,-2,-1), \mathbf{v}_2 = (1,0,1), \mathbf{v}_3 = (0,1,1)$ is the plane $x + y - z = 0$.

Math 310: Homework 1 (slightly revised)

due Sept 13,14 2006 in recitation

Ex 1. (i) Calculate (with answers in the form $z = a + ib$)

$$(1+i)^2, \ \ (1+i)^4.$$

Draw a diagram of these points on the plane.

(ii) Find $z = a + ib$ with $a, b > 0$ such that $z^8 = 1$.

In the following exercises, let V be a vector space over $\mathbb{F}$ (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.) You may use any proposition from Ch 1 provided that you say where it is used.

Ex 2. (i) Let $v, w \in V$ be such that $v + w = v$. Show that $w = 0$.

(ii) Let $v \in V$ and $a \in \mathbb{F}$ be such that $av = 0$. Show that either $a = 0$ or $v = 0$.

(iii) Let $v \in V$ and $a \in \mathbb{F}$ be such that $av = v$. Show that $a = 1$ or $v = 0$.

Ex 3. For each of the following subsets U of $\mathbb{R}^3$, determine whether it is a subspace of $\mathbb{F}^3$. If U is a subspace find W such that $U \oplus W = \mathbb{R}^3$. Explain your answer carefully.
(a) $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 - 2x_2 + x_3 = 1\}$.

(b) $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1)^2 - x_2 + x_3 = 0\}$.

(c) $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 - x_2 = 3x_3\}$.

Ex 4. Give an example of a subset U of $\mathbb{R}^2$ that is closed under addition and taking additive inverses, but is not a vector space.

Ex 5. Let $\mathcal{P}_2(\mathbb{R})$ be the vector space of polynomials in z of degree at most 2 with real coefficients. Thus $\mathcal{P}_2(\mathbb{R}) = \{a + bz + cz^2 : a, b, c \in \mathbb{R}\}$.

(i) Give an example of a subset U of $\mathcal{P}_2(\mathbb{R})$ that is closed under multiplication by scalars but is not a subspace.

(ii) Give an example of a subspace of U of $\mathcal{P}_2(\mathbb{R})$ that is *proper*, i.e. not equal to $\{0\}$ or to the whole space $\mathcal{P}_2(\mathbb{R})$.

(iii) For the subspace U you found in (ii) describe another subspace W such that $\mathcal{P}_2(\mathbb{R}) = U \oplus W$.

Ex 6. Are there subspaces U_1, U_2, W of $\mathbb{R}^2$ such that $U_1 \oplus W = U_2 \oplus W$ but $U_1 \neq U_2$? Give an example or a proof that no such subspaces exist.

Bonus problem (added on Tuesday Sep 12)
(i) Suppose that U_1, U_2, U_3 are subspaces of V such that $V = U_1 + U_2 + U_3$. Formulate a condition in terms of intersections of suitable subspaces that is equivalent to the condition that $V = U_1 \oplus U_2 \oplus U_3$.
(ii) The same question for k fold sums.

Math 310: Homework 2

due in recitation on Sept 20/21

Ex 1. Prove that if the list (v_1, v_2, v_3) spans V then so does the list $(v_1 + 2v_2, v_2 - v_3, v_3)$.

Ex 2. Prove that if the list (v_1, v_2, v_3) is linearly independent in V then so is the list $(v_1 + 2v_2, v_2 - v_3, v_3)$.

Ex 3. Find a basis for the vector space

$$V = \{(x_1, \ldots, x_4) \in \mathbb{F}^4 : x_1 + 2x_2 + 3x_3 + 4x_4 = 0\}.$$

What is the dimension of V?

Ex 4. Suppose that $(v_1, \ldots, v_n)$ is linearly independent in V.

(i) Suppose that for some $w \in V$ the list $(v_1 - w, v_2 - w, \ldots, v_n - w)$ is linearly dependent. Show that $w \in \text{span}(v_1, \ldots, v_n)$.

(ii) Is the converse true? That is, if $w \neq 0$ is in $\text{span}(v_1, \ldots, v_n)$ must it be true that the list $(v_1 - w, v_2 - w, \ldots, v_n - w)$ is linearly dependent?
Hint: What does this say when $n = 1, 2$? Try these cases first.

Ex 5. Let $\mathcal{P}(\mathbb{F})$ be the space of polynomials with coefficients in $\mathbb{F}$.

(i) Find two different 2-dimensional subspaces of $\mathcal{P}(\mathbb{F})$.

(ii) Find an infinite dimensional *proper* subspace of $\mathcal{P}(\mathbb{F})$ (i.e. a subspace that does not equal the whole of $\mathcal{P}(\mathbb{F})$.)

Ex 6. (i) Let U, V be subspaces of $\mathbb{F}^7$ such that $U \oplus V = \mathbb{F}^7$. If $\dim U = 3$ show that $\dim V = 4$.

(ii) Does this statement remain true if all you know is that $U + V = \mathbb{F}^7$? Give a proof or counterexample.

Note: In this question you may use all the results numbered up to and including 2.12. Anything else should be proved. Try to find the most econimical argument that you can.

Math 310: Homework 3

due Sept 27,28 2006 in recitation

Many of these exercises ask you to construct linear maps with certain properties. For this kind of problem Prop 3 (on the sheet I distributed this week) is often useful.

Ex 1. (i) Let V be a real vector space with basis v_1, v_2, v_3. Construct a linear map $T : V \to W = \mathbb{R}^2$ that is surjective and has the property that $T(v_1 + v_2 - 3v_3) = 0$.

(ii) Is it possible to construct a linear map with these properties if $W = \mathbb{R}^3$? Give an example, or explain why not.

Ex 2. (i) Let U be a subspace of V, and suppose that $T : U \to W$ is a linear map. Show that it is always possible to extend T to a linear map $T' : V \to W$. i.e. show that there is a linear map $T' : V \to W$ such that $T'(u) = T(u)$ for all $u \in U$.

(ii) Suppose that $U = \mathrm{sp}(e_1, e_2) \subset \mathbb{R}^5 = V$, $W = \mathbb{R}^4$ and that $T : U = \mathbb{R}^2 \to \mathbb{R}^4$ is given by

$$T(e_1) = \sum_{j=1}^{4} e_j, \qquad T(e_2) = e_1.$$

(Here $e_1, \ldots, e_n$ denotes the standard basis of $\mathbb{R}^n$; see p 27.)

(a) Since T is a map $\mathbb{R}^2 \to \mathbb{R}^4$, it is given by multiplication by a matrix A. Write down this matrix A.

(b) Write down a matrix for T'. Choose this matrix so that T' is injective (if possible) and surjective (if possible). Explain your answer.

(iii) Now go back to the general problem in (i). Under what conditions on U, V, W, T can you choose T' to be injective? Under what conditions can you choose T' to be surjective? Give the most general conditions you can find.

Ex 3. Suppose that $T : V \to W$ is a linear map and that $v_1, \ldots, v_n$ is a basis for V. Suppose that the list $Tv_1, \ldots, Tv_n$ is linearly dependent in W. Show that T is not injective.
Note: For this question, you may use any result in the book up to and including Theorem 3.4.

Ex 4. (i) Prove that there does not exist a linear map $T : \mathbb{R}^6 \to \mathbb{R}^2$ with null space equal to

$$\{(x_1, \ldots, x_6) : x_1 + x_2 + x_3 = 0, x_2 = -x_4 = x_6\}.$$

(ii) Give the matrix of a linear map $T : \mathbb{R}^6 \to \mathbb{R}^2$ with null space

$$\{(x_1, \ldots, x_6) : x_1 + x_2 + x_3 = 0, x_2 = -x_4\}.$$

We saw in class that the space $\mathcal{L}(V, W)$ of linear maps from V to W is always a vector space. Take $W = \mathbb{F}$. We then get the space $V^* := \mathcal{L}(V, \mathbb{F})$ of $\mathbb{F}$-linear maps $V \to \mathbb{F}$. This is called the **dual space** of V. The next two exercises ask you to explore its structure.

Ex 5. Let $V = \mathbb{F}^2$ with basis e_1, e_2. Define elements $e_1^*, e_2^* \in V^*$ by:

$$e_1^*(e_1) = 1,\ e_1^*(e_2) = 0, \qquad e_2^*(e_1) = 0,\ e_2^*(e_2) = 1.$$

Show that e_1^*, e_2^* form a basis for V^*. Deduce that $\dim(\mathbb{F}^2)^* = 2$.

Bonus ex 6: (i) Show that if V is a vector space of dimension n then V^* also has dimension n.

(ii) If V has infinite dimension then so does V^*. However, even if we have a basis for V it is not easy to define a basis for V^*. For example suppose that V is the set of infinite sequences that are eventually 0:

$$V := \{(x_1, x_2, x_3, \dots) : x_i \neq 0 \text{ for only finitely many } i\}.$$

Then V has the basis $e_i, i \in \mathbb{N}$, where e_i has 1 in the ith place and zeros elsewhere. As before we can define $e_i^* \in V^*$ which equals 1 on e_i and 0 on all other e_j. Find an element of V^* that is NOT in $\mathrm{sp}(e_1, e_2, e_3, \dots)$.

Math 310: Homework 4 (revised)

due Oct 4,5 2006 in recitation

NOTE: Ex 3 (ii) is changed, and I added a hint to Ex 2.

Ex 1. (i) Let $L\colon V \to W$ be a linear map. Let w_0 be an element of W. Let v_0 be an element of V such that $L(v_0) = w_0$. Show that any solution of the equation $L(X) = w_0$ is of type $v_0 + u$, where u is an element of the kernel of L.
Hint: You might find it easier to do (ii) and (iii) before (i)!

(ii) Consider the system of linear equations

$$\begin{aligned} 2x_1 + 3x_2 + 2x_3 &= 1 \\ x_1 + x_2 + x_3 &= 1. \end{aligned}$$

Find a linear map $L : V \to W$ and element $w_0 \in W$ such that the solution set of this system of equations can be identified with the set of vectors v such that $Lv = w_0$.

(iii) Solve the equations in (ii) and express the solutions in the form v_0+u as explained in (i).

Ex 2. Let $A = (a_{ij})$ be an $n \times n$ matrix. Define the *trace* of A to be the sum of the diagonal elements, that is

$$tr(A) = \sum_{i=1}^{n} a_{ii}.$$

(1) What is the dimension of the space of $n\times n$ traceless matrices (i.e., $tr(A) = 0$)?
(2) Show that the trace is a linear map of the space of $n \times n$ matrices into $\mathbb{F}$.
(3) If A, B are $n \times n$ matrices, show that $tr(AB) = tr(BA)$.
(4) Prove that there are no matrices A, B such that

$$AB - BA = I_n$$

.

Hint: Part (3) is an exercise in using the double summation notation: if $A = (a_{ij})$ and $B = (b_{ij})$ then $AB = (c_{ik})$ where $c_{ik} = \sum_j a_{ij}b_{jk}$. If you think of c_{ik} as the dot product of the ith row of A with the kth col of B it is not so easy to see why the trace has this symmetry.

Ex 3. (i) Find the matrix of a nonzero linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that $L^2 = 0$.

(ii) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map such that $L \neq 0$ but $L^2 = 0$. What are the dimensions of Null$L$ and Range$L$? Is there any relation between these two spaces? (Understanding this will help with the bonus question.)

(iii) Let $L\colon V \to V$ be a linear mapping such that $L^2 = 0$. Show that $I - L$ is invertible. (I is the identity mapping on V.)

Hint: Show that Null $(I - L) = \{0\}$. (There is another proof that finds an algebraic formula for the inverse. This argument works also when V is infinite dimensional.)

Ex 4. Let $V = \mathbb{R}^3$ with basis $\mathcal{B} := (v_1, v_2, v_3)$ where

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The basis $\mathcal{B}$ determines a unique isomorphism $\mathcal{M} : V \to \mathbb{R}^3$ such that $\mathcal{M}(v_1) = e_1, \mathcal{M}(v_2) = e_2$, and $\mathcal{M}(v_3) = e_3$.

(i) Calculate $\mathcal{M}(v)$ for $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$.

(ii) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be multiplication by the matrix

$$A := \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

Calculate $\mathcal{M}(T)$ where $\mathcal{M} := \mathcal{M}(A, (v_1, v_2, v_3), (v_1, v_2, v_3))$. (cf pp 48–53: I will lecture on this next Tuesday)

Hint: You should be able to use your answer to (i) when doing this.

(iii) Calculate Tv, where v is as in (i) and T is as in (ii). Also calculate $\mathcal{M}(Tv)$.

(iv) Check that $\mathcal{M}(Tv) = \mathcal{M}(T)\mathcal{M}(v)$.

Bonus question 5. (i) Find a linear map $L : \mathbb{R}^3 \to \mathbb{R}^3$ such that $L^2 \neq 0$ but $L^3 = 0$.

(ii) Is there a linear map $L : \mathbb{R}^3 \to \mathbb{R}^3$ such that $L^3 \neq 0$ but $L^4 = 0$?

Math 310: Homework 6

due Oct 25,26 2006 in recitation

In this homework V is a finite dimensional vector space over $\mathbb{C}$.

Ex 1. (i) Let $T \in \mathcal{L}(V)$. Show that the subspaces $\text{Null}(T - \lambda I)$ and $\text{Range}(T - \lambda I)$ are invariant under T.

(ii) Let $S \in \mathcal{L}(V)$ be any operator such that $TS = ST$. Show that $\text{Null}S$ and $\text{Range}S$ are invariant under T. Give a second proof of (i) by using this statement.

Ex 2. (i) Suppose that the subspaces U, W of V are invariant under T. Show that $U + W$ and $U \cap W$ are also invariant.

(ii) Show that if every 2-dimensional subspace of V is invariant under T then every one dimensional subspace is also invariant. Deduce that T is a scalar multiple of the identity.

Ex 3. Find all eigenvectors and eigenvalues for the following operators. Also, find a basis for which the corresponding matrix is upper triangular as in Theorem (5.13).

(i) Let $V = \mathcal{P}_4(\mathbb{C})$, the polynomials of degree ≤ 4, and $T : V \to V$ is $T(f) = f'' + 3f$.
(ii) $V = \mathbb{R}^3$ and $T(x, y, z) = (2y, x, 5z)$.

Ex 4. (i) Suppose that there is a basis $v_1, \ldots, v_n$ of V such that the matrix $\mathcal{M}(T)$ representing $T \in \mathcal{L}(V)$ with respect to this basis is upper triangular. Show there are subspaces $U_1 \subset U_2 \subset \ldots U_{n-1} \subset V$ such that
(a) $\dim U_i = i$ for all i, and
(b) $T(U_i) \subset U_i$ for all i.
Hint: define these subspaces in terms of the basis $v_1, \ldots, v_n$.

(ii) Find such subspaces U_i in both the examples in Ex 3.

(iii) Suppose in the situation of (i) that $T(U_i) \subset U_{i-1}$ for some i. Show that T is not invertible.

Ex 5. (i) Let U, W be subspaces of V with bases $(u_1, \ldots, u_k)$ and $(w_1, \ldots, w_m)$ respectively. Show that $V = U \oplus W$ iff $(u_1, \ldots, u_k, w_1, \ldots, w_m)$ is a basis for V.

(ii) (BONUS) Let U_i be a subspace of V for $i = 1, \ldots, m$. Suppose that U_i has dimension k_i with basis $v_{i1}, \ldots, v_{ik_i}$. Show that $V = U_1 \oplus U_2 \oplus \cdots \oplus U_m$ iff the list

$$(v_{11}, \ldots, v_{1k_1}, \ldots, v_{m1}, \ldots, v_{mk_m})$$

is a basis for V.
Note: Remember that $V = U_1 \oplus U_2 \oplus \cdots \oplus U_m$ iff every element of V may be written UNIQUELY in the form $u_1 + u_2 + \cdots + u_m$ where $u_i \in U_i$ for all i.

(iii) Suppose in the situation of (ii) that each subspace U_i is invariant under T. Show that T has at least m linearly independent eigenvectors.

Math 310: Homework 7 (revised)

due Nov 1,2 2006 in recitation

In this homework V is a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. I corrected a typo in Ex 5.

Ex 1: *A question about invertibility.* (i) Recall that $T \in \mathcal{L}(V)$ is said to be invertible if there is $S \in \mathcal{L}(V)$ such that $ST = TS = I$. In fact, it is enough to assume that there is $R \in \mathcal{L}(V)$ satisfying just the first identity: $RT = I$. Prove this, explaining each step of the argument.

(ii) Prove that if $S, T \in \mathcal{L}(V)$ satisfy $ST = I$ then $TS = I$.

(iii) Deduce carefully from (ii) that if A, B are any $n \times n$ matrices over $\mathbb{F}$ such that $AB = I$ then $BA = I$.

Note: This fact always seems to me to be to be surprising. It is an elementary computational fact, but I don't see a way to prove it by a simple calculation.

Ex 2: Prove that if $\mathbf{x}, \mathbf{y}$ are nonzero vectors in $\mathbb{R}^2$ then $\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\|\|\mathbf{y}\| \cos\theta$, where θ is the angle between $\mathbf{x}$ and $\mathbf{y}$. (Think of $\mathbf{x}, \mathbf{y}$ as the sides OX, OY of a triangle, and use the law of cosines.)

Ex 3: (i) Let $A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Show that for any vector $\mathbf{x} \in \mathbb{R}^2$, $\|A_\theta \mathbf{x}\| = \|\mathbf{x}\|$ and the angle between $\mathbf{x}$ and $A_\theta(\mathbf{x})$ is θ. Thus this matrix A_θ represents the rotation through angle θ. It is a simple example of an *orthogonal* matrix.
(ii) Verify that $A_\theta A_\phi = A_{\theta+\phi}$.
(iii) Find the (complex) eigenvalues and eigenvectors of A_θ. (Here you must think of the linear transformation $v \mapsto A_\theta v$ as an element of $\mathcal{L}(\mathbb{C}^2)$.)

Ex 4: (i) Let $R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$. Show that for any vector $\mathbf{x} \in \mathbb{R}^2$, $\|R_\theta \mathbf{x}\| = \|\mathbf{x}\|$. Find the (real) eigenvectors and eigenvalues for R_θ and describe the linear map $\mathbf{x} \mapsto R_\theta \mathbf{x}$ geometrically.
(ii) Suppose that B is any 2×2 real matrix such that $\|B\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$. Show that B either equals A_θ or R_θ for some θ.

Ex 5: (i) Define $\|(x_1, x_2)\| := |x_1| + |x_2|$ for $(x_1, x_2) \in \mathbb{R}^2$. Check that with this definition the triangle inequality holds, i.e. $\|(x_1+y_1, x_2+y_2)\| \le \|(x_1, x_2)\| + \|(y_1, y_2)\|$.
(ii) Check that the parallelogram rule given as (6.14) in the book does NOT hold.
(iii) Deduce that there is no inner product on $\mathbb{R}^2$ for which this is the associated norm. (Note: Since $\|\cdot\|$ is positive and homogeneous, it satisfies the axioms for a *norm*. Ex 8 on p 123 – which relies on Ex 6 and Ex 7– shows that if the parallelogram rule holds for some norm then it does come from an inner product.)

Ex 6: (i) Given an example of a linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ that has only one real eigenvalue. (**Hint:** Use Ex 3.)

(Bonus question). (ii) Give the shortest proof you can that *every* linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ has at least one real eigenvalue. (You can try to adapt the proof of (5.26) to the 3-dimensional case.)

Math 310: Homework 8

due Nov 8,9 2006 in recitation

Ex 1 Let $U \subset \mathbb{R}^4$ be the subspace given by the equations $x_1 + x_2 + x_3 + x_4 = 0$, $x_1 - x_2 + 2x_3 + x_4 = 0$.

(i) Find a basis of U. (Make the calculations easier by giving the vectors lots of zeros...)

(ii) Find an orthonormal basis of U.

(iii) Extend this to an orthonormal basis for $\mathbb{R}^4$. (First find any extension and then apply Gram–Schmidt.)

(iv) Let $v = (1, 2, 3, 4)$. Find the coordinates of V with respect to the basis you found in (iii).

Ex 2. Let $e_1, \dots, e_n$ be any basis of an inner product space V. Define $U := \text{span}(e_1, \dots, e_k)$ and $W = \text{span}(e_{k+1}, \dots, e_n)$. Also define $U^\perp := \{v : \langle v, u \rangle = 0, \ \forall u \in U\}$. ($U^\perp$ is called the orthogonal complement to U.)

(i) Show that $V = U \oplus W$.
(ii) Show that $U^\perp$ is a subspace. Show also that $U^\perp = W$.
(iii) Deduce that for any subspace U of V, $V = U \oplus U^\perp$.
(iv) Now assume that $V = \mathbb{R}^4$ and that U is the subspace defined in Ex 1. Calculate the decomposition of $v = (1, 1, 1, 1)$ as a sum $u + w$ where $u \in U$ and $w \in U^\perp$.

Ex 3 Let V be the vector space of all $n \times n$ matrices over $\mathbb{R}$, and given any two matrices $A, B \in V$ define

$$\langle A, B \rangle = \text{trace}(AB) = \sum_{i,j} a_{ij} b_{ji}.$$

(i) Show that this satisfies all axioms for an inner product except possibly for positivity and nondegeneracy. (e.g. give an example (with $n = 2$) such that $A \neq 0$ but $\text{trace} A^2 = 0$.)

(ii) If A is a real symmetric matrix, show that $\text{trace}(A^2) \geqq 0$, and $\text{trace}(A^2) > 0$ if $A \neq 0$. Thus the trace defines an inner product on the space of real symmetric matrices.

(iii) Let V be the symmetric space of real $n \times n$ symmetric matrices. What is $\dim V$? What is the dimension of the subspace W consisting of those matrices A such that $\text{trace}(A) = 0$? What is the dimension of the orthogonal complement $W^\perp$ relative to the inner product defined above?

Ex 4 Let A be an $n \times n$ matrix, and define $T \in \mathfrak{L}(\mathbb{F}^n)$ by $Tv = Av$.

(i) Show that T is diagonalizable iff there exists an invertible matrix Q such that $Q^{-1}AQ$ is a diagonal matrix.
(ii) How can you interpret the columns of the matrix Q? (Hint: think of these as vectors. What relation do they have to the operator T?)

Ex 5 Two linear operators S and T on a finite-dimensional vector space V are called *simultaneously diagonalizable* if there exists a basis $\mathcal{B}$ for V such that both $\mathcal{M}(S, \mathcal{B})$ and $\mathcal{M}(T, \mathcal{B})$ are diagonal matrices. This is equivalent to saying that there is a basis for V consisting of vectors that are eigenvectors for both S and T.

(i) Prove that if S and T are simultaneously diagonalizable operators then S and T commute. (Hint: see what the operators ST and TS do to a suitable basis for V.)

(ii) (Bonus) Prove also that if S and T are diagonalizable operators that commute then they are simultaneously diagonalizable.

(iii) Let $T_A, T_B \in \mathcal{L}(\mathbb{F}^n)$ be the operators defined by multiplication by the matrices A, B. Show that T_A, T_B are simultaneously diagonalizable iff there is an invertible matrix Q such that both $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal matrices. (cf Ex 4).

(iv) (Bonus) Deduce that if the matrices A, B commute there is an invertible matrix Q such that both $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal matrices.

Math 310: Midterm 2 (slightly edited)

November 16, 2006

Name: **ID number:**

There are 4 questions worth a total of 100 points, plus one small bonus question worth 10 points. Please justify all your statements, and write neatly so that we can read and follow your answers. Any theorems that you use in your arguments should be carefully stated. Continue your answers on the back of the pages. Also, please turn off cell phones.

Question 1. (30 points) Let $(V, \langle , \rangle)$ be a finite dimensional inner product space over $\mathbb{R}$.
(i) Define the length $\|v\|$ of a vector $v \in V$.
(ii) Show from this definition that $\|v\|^2 + \|w\|^2 = \|v + w\|^2$ if and only if the vectors $v, w \in V$ are orthogonal.

(iii) Find an orthonormal basis for the subspace $x_1 + 2x_2 - x_3 = 0$ of $\mathbb{R}^3$.

(iv) Find the orthogonal projection of $y = (1, 1, 1)$ onto this subspace.

Notes: When proving (ii) work with the inner product $\langle v, w\rangle$. The subspace U in (iii) is a plane, so the basis should have two vectors in it. (Many of you gave me one vector, a multiple of $(1, 2, -1)$, i.e. a basis for $U^\perp$.)

1	30pt	
2	20pt	
3	35pt	
4	15pt	
Total	100pt	
bonus	10pt	
Grand	Total	

Question 2: (20 points) Let V be a finite dimensional vector space over $\mathbb{C}$ and suppose that $S, T \in \mathcal{L}(V)$ commute.
(i) Show that $\operatorname{null} S$ and $\operatorname{range} S$ are invariant under T.
(ii) Suppose in addition that $V = \operatorname{null} S \oplus \operatorname{range} S$ where both $\operatorname{null} S$ and $\operatorname{range} S$ have nonzero dimension. Show that T has at least two linearly independent eigenvectors.

Note: (ii) is an easy deduction from one of the theorems in the book; you should say which one an explain what is going on.

Question 3: (25 points) (i) Let $A = \begin{bmatrix} 0 & -1 \\ 4 & 4 \end{bmatrix}$. Find a basis of $\mathbb{C}^2$ such that the operator T_A defined by $T_A v = Av$ is represented by an upper triangular matrix M with respect to this basis. What is M?

(ii) (10 points). Suppose that A is a 3×3 matrix of the form $\begin{bmatrix} 1 & * & * \\ 0 & -1 & * \\ 0 & 0 & 4 \end{bmatrix}$. where the entries $*$ are all nonzero. Is there always a basis of $\mathbb{C}^3$ with respect to which $T_A \in \mathcal{L}(\mathbb{C}^3)$ can be represented by a diagonal matrix?

Note: for (i) you should begin by finding the eigenvalues and eigenvectors of A. It turns out that these do not form a basis for $\mathbb{C}^2$, so you have to figure out what to do to complete the proof. (ii) is an easy deduction from some theorems.

Question 4: (15 points) (i) Let $(V, \langle , \rangle)$ be a finite dimensional inner product space, and let $T \in \mathcal{L}(V)$. Define the adjoint T^* of T.

(ii) Show that if the subspace U of V is invariant under T then $U^\perp$ is invariant under T^*.

(iii) (Bonus) (10 points) Give an example of an operator $T \in \mathcal{L}(\mathbb{C}^2)$ that has a 1-dimensional invariant subspace U such that $U^\perp$ is NOT invariant under T.

Note: in (ii) you should work from the definition you gave in (i).

Math 310: Homework 10 (revised)

due Dec 6,7 2006 in recitation

Ex 1 (a review question) (i) Define a linearly independent list $(v_1, \dots, v_n)$.

(ii) Using this definition, show that if the lists $(v_1, \dots, v_m, u_1, \dots, u_k)$ and $(v_1, \dots, v_m, w_1 \dots, w_\ell)$ are linearly independent and if also

$$span(u_1, \dots, u_k, v_1, \dots, v_m) \cap span(w_1, \dots, w_\ell) = \{0\}$$

then $(v_1, \dots, v_m, u_1, \dots, u_k, w_1 \dots, w_\ell)$ is also linearly independent.

(iii) Deduce from (ii) that for any subspaces U, W of a finite dimensional vector space V,

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

Ex 2. (i) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation $T(x, y, z) = (2x + y, 3y + z, 2z)$. Write down the matrix that represents T with respect to the standard basis.

(ii) Find bases $\mathcal{B}_1 := v_1, v_2, v_3$ and $\mathcal{B}_2 := w_1, w_2, w_3$ such that the matrices $\mathcal{M}(T, \mathcal{B}_i)$ that represent T with respect to these bases are:

$$\mathcal{M}(T, \mathcal{B}_1) := \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \qquad \mathcal{M}(T, \mathcal{B}_2) := \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(iii) Is there a basis $\mathcal{B}_3$ such that $\mathcal{M}(T, \mathcal{B}_3) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$?

Ex 3 (i) Let $\mathcal{B} = v_1, \dots, v_4$ be a basis for V and $T \in \mathcal{L}(V)$. Suppose that $\mathcal{M}(T, \mathcal{B}) = \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$. Let $R = T - I$ and $S = T - 2I$. Write down $\mathcal{M}(R, \mathcal{B})$ and $\mathcal{M}(S, \mathcal{B})$.

(ii) If $U = span(v_1, v_2, v_3)$ then the proof of (8.10) in the book implies that $\operatorname{Null} R^4 \subseteq U$ and $\operatorname{Null} S^4 \subsetneq U$. Why? Prove these statements by calculating $\operatorname{Null} R^4$ and $\operatorname{Null} S^4$. **Hint:** What are the dimensions of these spaces? The calculation will be easier if you remember that if $\dim \operatorname{Null} R^4 = k < 4$ then $\dim \operatorname{Null} R^4 = \dim \operatorname{Null} R^k$.

(iii) Find a basis for $\operatorname{Null} R^4$ and $\operatorname{Null} S^4$. You should get 4 vectors in all that form a basis for V. Which theorem in the book does this follow from?

(iv) Let $k = \dim \operatorname{Null} R^4$. Then (8.5) and the proof of (8.9) imply that $\operatorname{Range} R^k = \operatorname{Range} R^{k+1}$. Calculate these two spaces and check this. Do you notice anything about this space (e.g. is it equal to any other space you have recently calculated?)

Ex 4 (i) Let $A := \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$. Find the eigenvalues of A (using any convenient method) and hence the characteristic polynomial $P(z)$ of A.

(ii) Calculate A^2 and check that $p(A) = 0$.

Ex 5 Let $A := \begin{bmatrix} 2 & 1 & -2 \\ -2 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix}$. Calculate A^2 and A^3. Find a linear relation between A^3, A^2, A and I. Hence find a polynomial $p(z)$ such that $p(A) = 0$.
Hint: look at $A^3 + 2A$.

(ii) If $p(z)$ is the characteristic polynomial of A, what does that tell you about the eigenvalues of A? Find these eigenvalues by some other method and compare answers.

(iii) (Bonus) Why must $p(z)$ be the characteristic polynomial of A?

Math 310: Homework 11

due Dec 14 2006 in recitation or class

Ex 1 Let $T_1, T_2, T_3 \in \mathcal{L}(\mathbb{C}^3)$ be the linear maps given by the following matrices:

$$A_1 := \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \; A_2 := \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, A_3 := \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Find their minimal and characteristic polynomials.

(ii) Let $T \in \mathcal{L}(V)$ where $\dim V = n$. Suppose that $q(t)$ is a monic polynomial of degree n such that $q(T) = 0$. What conditions guarantee that $q(t)$ is the characteristic polynomial of T? Discuss this question using the maps in (i) as examples.

Ex 2 (i) Let $\dim V = n$. Suppose that for some vector $v \in V$ the list $v_0 := v, v_1 := Tv, \ldots, v_{n-1} := T^{n-1}v$ is linearly independent. Why is there a linear relation

$$T^n v = a_0 v_0 + a_1 v_1 + \cdots + a_{n-1} v_{n-1}?$$

(ii) Let $q(t) = t^n - a_{n-1}t^{n-1} - \cdots - a_1 t - a_0$. Show that $q(T)v_i = 0$ for all i. (Use the fact that $v_i = T^i(v_0)$ for all i.) Hence deduce that $q(T) = 0$.
(iii) Show also that there is no polynomial m of degree $< n$ such that $m(T) = 0$. Hence deduce that $q(t)$ is the minimal *and* the characteristic polynomial of T. (Compare Ex 1.)
(iv) Use this method to find the characteristic polynomial of $T_A : \mathbb{C}^3 \to \mathbb{C}^3$ where

$$A := \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \text{ and } v = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

(v) What are the eigenvalues and eigenvectors of T_A? (You can use any method here.)

Ex 3 (i) Consider the permutations on 8 letters given by the arrays

$$\mathbf{m} := [m_1, \ldots, m_8] = [3,5,1,6,7,2,8,4], \quad \mathbf{n} := [n_1, \ldots, n_8] = [3,8,1,6,7,2,5,4].$$

Calculate sign$\mathbf{m}$ and sign$\mathbf{n}$. Why do you expect sign $\mathbf{m}$ = -sign $\mathbf{n}$?
(ii) A permutation $\mathbf{m}$ is a map $\{1, \ldots, 8\} \to \{1, \ldots, 8\}$ given by $\mathbf{m}(1) = m_1, \ldots, \mathbf{m}(8) = m_8$. To emphasize this one can describe $\mathbf{m}$ by two rows:

$$\mathbf{m} := \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 1 & 6 & 7 & 2 & 8 & 4 \end{bmatrix}.$$

Another way of describing a permutation is in terms of cycles $(d_1, d_2, \ldots d_r)$: this is the map that takes d_1 to d_2, d_2 to d_3 and so on, finally taking d_r back to d_1. In this notation, when one writes a product one does the right hand one first. Thus:

$$(134)(24) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{bmatrix}.$$

Note that $(134)(24) = (1342)$.
(a) Write down $(15)(245)(34)$ in the double row format.

(b) Hence write (15)(245)(34) as a product of *disjoint* cycles (ie so that no number occurs in more than one cycle.)

(c) Repeat the above two steps with (23)(245)(34)

(d) Write down the permutation $\mathbf{m}$ above as a product of disjoint cycles.

(iii) Use (a) (b) above to write down (15)(245)(34) as a product of transpositions in two different ways. Check that the number of transpositions has the same parity in both cases. (A transposition is a cycle of length 2. We saw in class that any permutation can be written as a product of transpositions. For example $(1234) = (43)(42)(41)$.)

(iv) Use (d) to write $\mathbf{m}$ as a product of k transpositions. Check that $(-1)^k = \operatorname{sign}\mathbf{m}$.

Ex 4 Let

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 2 \\ 3 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Recall that

$$\det B = \sum_{\mathbf{m} \in \mathrm{perm}_5} \operatorname{sign}\mathbf{m}\; b_{m_1,1} b_{m_2,2} b_{m_3,3} b_{m_4,4} b_{m_5,5}.$$

Since $\mathbf{m}$ is a permutation, each term in this sum contains just one element from each row and each column of B. For B as above, which permutations give you nontrivial terms in this sum? (Hint: since for example there is just one element in the first row which lies in position b_{12} we must have $m_2 = 1$.) List all these permutations, and hence calculate $\det B$.

Ex 5 (i) The trace $\mathrm{tr}A$ of a matrix A is the sum of its diagonal entries. Explain why $\mathrm{tr}A$ is the sum of the eigenvalues of the linear map T_A.

(ii) Suppose that A is a complex $n \times n$ matrix such that $A^2 = A$. Show that $\mathrm{tr}A$ is a nonnegative integer. (Hint: What can you say about the eigenvalues of A?)

Ex 6 (Bonus) Suppose that in the situation of Ex 1 the span of the $T^i v, i = 0, \ldots, n-1$, has dimension $< n$. You can still find a polynomial $f(t)$ such that $f(T)(T^i v) = 0$ for all i. What can you say about its roots? What relation does this have to the minimal or characteristic polynomial? You could experiment starting with the matrix

$$A := \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}, \text{ and } v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$