MAT 303: Calculus IV with Applications
Fall 2016

Schedule & Homework

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Schedule

This is our tentative weekly schedule and it will be updated as we advance in the semester, please check regularly. Students are expected to attend class regularly and to keep up with the material presented in the lecture and the assigned reading. As a general rule, the written homework assignment is **due Fridays** (for example, HW1 is due Friday, Sept 9, Week 2), unless otherwise stated.

Week	Lectures (Assigned Reading)	Assignments
Week 1 8/29 - 9/4	**1.1:** Differential equations and math models **1.2:** Integrals as general and particular solutions **1.4:** Separable differential equations	**1.1:** 26, 48 **1.2:** 8, 10, 44 **1.4:** 18, 28, 32, 36, 37, 39
Week 2 9/5 - 9/11	No class on Sept 5 (Labor Day) **1.5:** Linear first order differential equations **1.3:** Slope fields and solutions curves	**1.3:** 11, 14, 18, 27, 29, 30 **1.5:** 1, 6, 16, 19, 27, 38
Week 3 9/12 - 9/18	**1.3:** Local Existence and Uniqueness of Solutions **Intro:** Diff Eq and Phase Portraits in Mathematica (**.nb**) **1.6:** Substitution Methods and Exact Equations	**1.6:** 8, 11, 21, 27, 29, 44 **1.6:** 47, 57, 58, ***Mathematica***
Week 4 9/19 - 9/25	**1.6:** Substitution Methods and Exact Equations (**exact.pdf**)	**due Monday, Oct 3** **1.6:** 33, 37, 38 **2.1:** 13, 23, 33 **2.2:** 8, 9 **2.6:** 25, 27 from **exact.pdf**
Week 5 9/26 - 10/2	**2.1:** Population models **2.2:** Equilibrium solutions and stability **2.3:** Acceleration-Velocity models	
	Midterm 1 – Wednesday, October 5, 12 - 12:53pm, in class; **Midterm I Solutions** **covers Chapter 1 and Sections 2.1 and 2.2** **Practice Problems (Spring 2013)** with **Solutions**	
Week 6 10/3 - 10/9	**2.4:** Numerical approximation: Euler's Method **3.1:** Second order eq. with constant coeff. (see also **3.3**)	**2.2:** 19, 22 **2.4:** 1, 27 **3.1:** 39, 51, 52, 56 **3.3:** 9, 21, 23
Week 7 10/10 - 10/16	**3.1:** Second order linear equations **3.2:** N^{th}-order linear equations **3.3:** N^{th}-order equations with constant coefficients	**3.1:** 19, 29, 30, 32, **3.2:** 28, 35, 36, 38, 43 **3.3:** 12, 15, 20
Week 8 10/17 - 10/23	**3.4:** Mechanical vibrations **3.5:** Nonhomogeneous eq. and undetermined coefficients	**3.4:** 3, 14, 15 **3.5:** 5, 19, 21, 22, 34 **Additional Exercises**
Week 9 10/24 - 10/30	**3.6:** Forced Oscillations and Resonance **3.8:** Endpoint Problems and Eigenvalues	**3.6:** 1, 8, 11, 26, 27 **3.8:** 3, 6, 7, 13, 16

Week 10 10/31 - 11/6	**4.1:** First Order Systems & The Elimination Method in **4.2** **5.1:** Matrices and Linear Systems **5.2:** The Eigenvalue Method for Homogeneous Systems	**4.1:** 19 **5.1:** 3, 23 **4.2:** 2 **5.2:** 1, 6, 17, 19
Week 11 11/7 - 11/13	**4.1:** Fundamental Set of Solutions **5.5:** The Eigenvalue Method: Repeated Eigenvalues	
	Midterm 2 – Wednesday, November 16, 12 - 12:55pm, in class **Midterm II Solutions** **covers Sections 2.2, 2.4, Chapter 3, and Sections 4.1, 4.2, 5.1, 5.2** **Practice Problems I** with **Solutions** **Practice Problems II** with **Solutions**	
Week 12 11/14 - 11/20	Review on Monday **5.5:** Repeated Eigenvalues	TBA, due Friday, Dec 2 **5.5:** 13, 16, 20, 33 **5.6:** 23, 25, 34, 35 **Additional Exercises**
Week 13 11/21 - 11/27	**5.6:** Matrix Exponentials and Linear Systems Thanksgiving Break!	
Week 14 11/28 - 12/4	**5.6:** Matrix Exponentials and Linear Systems **5.7:** Nonhomogeneous Systems; Variation of Parameters Jordan canonical forms **Lecture Notes!** **Mathematica Tutorial!**	Read the **Lecture Notes** attached and the **Mathematica Tutorial**! **5.7:** 24, 31 ***Instructions*** **6.2:** 1, 3, 10, 13, 15, 29, 33
Week 15 12/5 - 12/11	**6.1:** Stability and the Phase Plane **6.2:** Linear and Almost Linear Systems	
	Final exam – Thursday, December 15, 5:30 - 8pm, in Harriman Hall 137 ***the final covers everything!*** **Practice Final (2013)** with **Solutions** ***Review Session on Monday, December 12, 11am - 1pm, in Javits 111***	

Last updated December 7, 2016

MAT 303: Calculus IV with Applications
Fall 2016

Course Information

Course Description

This course is an introduction to differential equations, with particular emphasis on scientific applications. Topics we will cover include homogeneous and non homogeneous linear differential equations, systems of linear differential equations, non-linear systems, Laplace transforms, series solutions to equations. We study standard techniques for solving ordinary differential equations, including numerical methods, and their applications to engineering, physics, biology, chemistry, economics, social sciences.

Click here to download a copy of the course syllabus. Please visit also the **course website on Blackboard**.

Textbook

The course textbook is *Differential Equations with Boundary Value Problems: Computing and Modelling (5th edition)*, by Edwards & Penney.

Office Hours

Instructor: Raluca Tanase
Office hours: Mondays 1-2pm in MLC, Thursdays 1:30-2:30pm in Math Tower 4-120
Wednesdays 1-2:30pm in Math Tower 4-120, or by appointment.

Recitations

LEC 01	MWF 12 - 12:53pm	Harriman Hall 137	Raluca Tanase
R 01	W 10:00am-10:53am	Library W4535	Dyi-Shing Ou
R 02	F 1:00pm - 1:53pm	Library E4330	Timothy Ryan
R 03	Tu 5:30pm - 6:23pm	Library W4525	Timothy Ryan
R 04	W 7:00pm - 7:53pm	Library W4525	Alexandra Viktorova
R 05	M 5:30pm - 6:23pm	Lgt Engr Lab 152	Jiasheng Teh

Blackboard

Almost all course administration will take place on Blackboard. Your exam and homework grades will also be reported on Blackboard. On this webpage, under **Schedule & Homework** you will find the most up-to-date version of the weekly course schedule and the written homework assigments.

Grading Policy

Grades will be computed using the following scheme:

- Recitation & Homework – 20%
- Each Midterm – 20%
- Final Exam – 40%

Students are expected to attend the lectures and recitations regularly and to keep up with the material presented in the lecture and the assigned reading.

Homework

You cannot learn differential equations without working problems. Each week, you will be given a set of problems, due the next Friday, in class. Do all of the assigned problems, as well as additional ones to study. Most of the homework problems will be analytic exercises, whose solutions will require only pen and paper. A few of them (clearly marked) will require the use of a computer program like *Mathematica* . Your solutions for the homework should be written neatly and legibly in grammatically correct mathematical English, and all steps should be clearly outlined. For the ones that require computer software, the generating code should also be turned in.

Software

We will use *Mathematica*, which is a computational software program developed by Wolfram Research and used in many scientific, engineering, mathematical and computing fields, based on symbolic mathematics. *Mathematica* has a **comprehensive documentation**.

Stony Brook students can download the Windows/Mac/Linux version of *Mathematica 10.3* from **Softweb**. You need your Stony Brook netID and netID password to log in to Softweb. To obtain an Activation Key for *Mathematica* you must visit the **Wolfram User Portal**. If it's your first time visiting the Wolfram User Portal, you must create a Wolfram ID and follow the steps in there to request an **Activation Key**.

In addition, you can use any of the campus SINC sites, or you can access the **Virtual SINC site**.

(b) If the substances P and Q are the same, then $p = q$ and Eq. (i) is replaced by

$$dx/dt = \alpha(p - x)^2. \tag{ii}$$

If $x(0) = 0$, determine the limiting value of $x(t)$ as $t \to \infty$ without solving the differential equation. Then solve the initial value problem and determine $x(t)$ for any t.

2.6 Exact Equations and Integrating Factors

For first order equations there are a number of integration methods that are applicable to various classes of problems. The most important of these are linear equations and separable equations, which we have discussed previously. Here, we consider a class of equations known as exact equations for which there is also a well-defined method of solution. Keep in mind, however, that those first order equations that can be solved by elementary integration methods are rather special; most first order equations cannot be solved in this way.

EXAMPLE 1

Solve the differential equation

$$2x + y^2 + 2xyy' = 0. \tag{1}$$

The equation is neither linear nor separable, so the methods suitable for those types of equations are not applicable here. However, observe that the function $\psi(x, y) = x^2 + xy^2$ has the property that

$$2x + y^2 = \frac{\partial \psi}{\partial x}, \qquad 2xy = \frac{\partial \psi}{\partial y}. \tag{2}$$

Therefore, the differential equation can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{3}$$

Assuming that y is a function of x, we can use the chain rule to write the left side of Eq. (3) as $d\psi(x, y)/dx$. Then Eq. (3) has the form

$$\frac{d\psi}{dx}(x, y) = \frac{d}{dx}(x^2 + xy^2) = 0. \tag{4}$$

By integrating Eq. (4) we obtain

$$\psi(x, y) = x^2 + xy^2 = c, \tag{5}$$

where c is an arbitrary constant. The level curves of $\psi(x, y)$ are the integral curves of Eq. (1). Solutions of Eq. (1) are defined implicitly by Eq. (5).

In solving Eq. (1) the key step was the recognition that there is a function ψ that satisfies Eqs. (2). More generally, let the differential equation

$$M(x, y) + N(x, y)y' = 0 \tag{6}$$

be given. Suppose that we can identify a function $\psi(x, y)$ such that

$$\frac{\partial\psi}{\partial x}(x, y) = M(x, y), \qquad \frac{\partial\psi}{\partial y}(x, y) = N(x, y), \tag{7}$$

and such that $\psi(x, y) = c$ defines $y = \phi(x)$ implicitly as a differentiable function of x. Then

$$M(x, y) + N(x, y)y' = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = \frac{d}{dx}\psi[x, \phi(x)]$$

and the differential equation (6) becomes

$$\frac{d}{dx}\psi[x, \phi(x)] = 0. \tag{8}$$

In this case Eq. (6) is said to be an **exact** differential equation. Solutions of Eq. (6), or the equivalent Eq. (8), are given implicitly by

$$\psi(x, y) = c, \tag{9}$$

where c is an arbitrary constant.

In Example 1 it was relatively easy to see that the differential equation was exact and, in fact, easy to find its solution, at least implicitly, by recognizing the required function ψ. For more complicated equations it may not be possible to do this so easily. How can we tell whether a given equation is exact, and if it is, how can we find the function $\psi(x, y)$? The following theorem answers the first question, and its proof provides a way of answering the second.

Theorem 2.6.1

Let the functions M, N, M_y, and N_x, where subscripts denote partial derivatives, be continuous in the rectangular[17] region $R: \alpha < x < \beta, \gamma < y < \delta$. Then Eq. (6)

$$M(x, y) + N(x, y)y' = 0$$

is an exact differential equation in R if and only if

$$M_y(x, y) = N_x(x, y) \tag{10}$$

at each point of R. That is, there exists a function ψ satisfying Eqs. (7),

$$\psi_x(x, y) = M(x, y), \qquad \psi_y(x, y) = N(x, y),$$

if and only if M and N satisfy Eq. (10).

[17]It is not essential that the region be rectangular, only that it be simply connected. In two dimensions this means that the region has no holes in its interior. Thus, for example, rectangular or circular regions are simply connected, but an annular region is not. More details can be found in most books on advanced calculus.

The proof of this theorem has two parts. First, we show that if there is a function ψ such that Eqs. (7) are true, then it follows that Eq. (10) is satisfied. Computing M_y and N_x from Eqs. (7), we obtain

$$M_y(x,y) = \psi_{xy}(x,y), \qquad N_x(x,y) = \psi_{yx}(x,y). \tag{11}$$

Since M_y and N_x are continuous, it follows that ψ_{xy} and ψ_{yx} are also continuous. This guarantees their equality, and Eq. (10) is valid.

We now show that if M and N satisfy Eq. (10), then Eq. (6) is exact. The proof involves the construction of a function ψ satisfying Eqs. (7)

$$\psi_x(x,y) = M(x,y), \qquad \psi_y(x,y) = N(x,y).$$

We begin by integrating the first of Eqs. (7) with respect to x, holding y constant. We obtain

$$\psi(x,y) = Q(x,y) + h(y), \tag{12}$$

where $Q(x,y)$ is any differentiable function such that $\partial Q(x,y)/\partial x = M(x,y)$. For example, we might choose

$$Q(x,y) = \int_{x_0}^{x} M(s,y)\,ds, \tag{13}$$

where x_0 is some specified constant in $\alpha < x_0 < \beta$. The function h in Eq. (12) is an arbitrary differentiable function of y, playing the role of the arbitrary constant. Now we must show that it is always possible to choose $h(y)$ so that the second of Eqs. (7) is satisfied—that is, $\psi_y = N$. By differentiating Eq. (12) with respect to y and setting the result equal to $N(x,y)$, we obtain

$$\psi_y(x,y) = \frac{\partial Q}{\partial y}(x,y) + h'(y) = N(x,y).$$

Then, solving for $h'(y)$, we have

$$h'(y) = N(x,y) - \frac{\partial Q}{\partial y}(x,y). \tag{14}$$

In order for us to determine $h(y)$ from Eq. (14), the right side of Eq. (14), despite its appearance, must be a function of y only. One way to show that this is true is to show that its derivative with respect to x is zero. Thus we differentiate the right side of Eq. (14) with respect to x, obtaining

$$\frac{\partial N}{\partial x}(x,y) - \frac{\partial}{\partial x}\frac{\partial Q}{\partial y}(x,y). \tag{15}$$

By interchanging the order of differentiation in the second term of Eq. (15), we have

$$\frac{\partial N}{\partial x}(x,y) - \frac{\partial}{\partial y}\frac{\partial Q}{\partial x}(x,y),$$

or, since $\partial Q/\partial x = M$,

$$\frac{\partial N}{\partial x}(x,y) - \frac{\partial M}{\partial y}(x,y),$$

which is zero on account of Eq. (10). Hence, despite its apparent form, the right side of Eq. (14) does not, in fact, depend on x. Then we find $h(y)$ by integrating Eq. (14), and upon substituting this function in Eq. (12), we obtain the required function $\psi(x, y)$. This completes the proof of Theorem 2.6.1.

It is possible to obtain an explicit expression for $\psi(x, y)$ in terms of integrals (see Problem 17), but in solving specific exact equations, it is usually simpler and easier just to repeat the procedure used in the preceding proof. That is, integrate $\psi_x = M$ with respect to x, including an arbitrary function of $h(y)$ instead of an arbitrary constant, and then differentiate the result with respect to y and set it equal to N. Finally, use this last equation to solve for $h(y)$. The next example illustrates this procedure.

EXAMPLE 2

Solve the differential equation

$$(y\cos x + 2xe^y) + (\sin x + x^2 e^y - 1)y' = 0. \tag{16}$$

By calculating M_y and N_x, we find that

$$M_y(x, y) = \cos x + 2xe^y = N_x(x, y),$$

so the given equation is exact. Thus there is a $\psi(x, y)$ such that

$$\psi_x(x, y) = y\cos x + 2xe^y,$$
$$\psi_y(x, y) = \sin x + x^2 e^y - 1.$$

Integrating the first of these equations, we obtain

$$\psi(x, y) = y\sin x + x^2 e^y + h(y). \tag{17}$$

Setting $\psi_y = N$ gives

$$\psi_y(x, y) = \sin x + x^2 e^y + h'(y) = \sin x + x^2 e^y - 1.$$

Thus $h'(y) = -1$ and $h(y) = -y$. The constant of integration can be omitted since any solution of the preceding differential equation is satisfactory; we do not require the most general one. Substituting for $h(y)$ in Eq. (17) gives

$$\psi(x, y) = y\sin x + x^2 e^y - y.$$

Hence solutions of Eq. (16) are given implicitly by

$$y\sin x + x^2 e^y - y = c. \tag{18}$$

EXAMPLE 3

Solve the differential equation

$$(3xy + y^2) + (x^2 + xy)y' = 0. \tag{19}$$

We have

$$M_y(x, y) = 3x + 2y, \qquad N_x(x, y) = 2x + y;$$

since $M_y \neq N_x$, the given equation is not exact. To see that it cannot be solved by the procedure described above, let us seek a function ψ such that

$$\psi_x(x, y) = 3xy + y^2, \qquad \psi_y(x, y) = x^2 + xy. \tag{20}$$

Integrating the first of Eqs. (20) gives

$$\psi(x, y) = \tfrac{3}{2}x^2 y + xy^2 + h(y), \tag{21}$$

where h is an arbitrary function of y only. To try to satisfy the second of Eqs. (20), we compute ψ_y from Eq. (21) and set it equal to N, obtaining

$$\tfrac{3}{2}x^2 + 2xy + h'(y) = x^2 + xy$$

or

$$h'(y) = -\tfrac{1}{2}x^2 - xy. \tag{22}$$

Since the right side of Eq. (22) depends on x as well as y, it is impossible to solve Eq. (22) for $h(y)$. Thus there is no $\psi(x, y)$ satisfying both of Eqs. (20).

Integrating Factors. It is sometimes possible to convert a differential equation that is not exact into an exact equation by multiplying the equation by a suitable integrating factor. Recall that this is the procedure that we used in solving linear equations in Section 2.1. To investigate the possibility of implementing this idea more generally, let us multiply the equation

$$M(x, y) + N(x, y)y' = 0 \tag{23}$$

by a function μ and then try to choose μ so that the resulting equation

$$\mu(x, y)M(x, y) + \mu(x, y)N(x, y)y' = 0 \tag{24}$$

is exact. By Theorem 2.6.1, Eq. (24) is exact if and only if

$$(\mu M)_y = (\mu N)_x. \tag{25}$$

Since M and N are given functions, Eq. (25) states that the integrating factor μ must satisfy the first order partial differential equation

$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0. \tag{26}$$

If a function μ satisfying Eq. (26) can be found, then Eq. (24) will be exact. The solution of Eq. (24) can then be obtained by the method described in the first part of this section. The solution found in this way also satisfies Eq. (23), since the integrating factor μ can be canceled out of Eq. (24).

A partial differential equation of the form (26) may have more than one solution; if this is the case, any such solution may be used as an integrating factor of Eq. (23). This possible nonuniqueness of the integrating factor is illustrated in Example 4.

Unfortunately, Eq. (26), which determines the integrating factor μ, is ordinarily at least as hard to solve as the original equation (23). Therefore, although in principle integrating factors are powerful tools for solving differential equations, in practice they can be found only in special cases. The most important situations in which simple integrating factors can be found occur when μ is a function of only one of the variables x or y, instead of both.

Let us determine conditions on M and N so that Eq. (23) has an integrating factor μ that depends on x only. If we assume that μ is a function of x only, then the partial derivative μ_x reduces to the ordinary derivative $d\mu/dx$ and $\mu_y = 0$. Making these substitutions in Eq. (26), we find that

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu. \tag{27}$$

If $(M_y - N_x)/N$ is a function of x only, then there is an integrating factor μ that also depends only on x; further, $\mu(x)$ can be found by solving Eq. (27), which is both linear and separable.

A similar procedure can be used to determine a condition under which Eq. (23) has an integrating factor depending only on y; see Problem 23.

EXAMPLE 4

Find an integrating factor for the equation

$$(3xy + y^2) + (x^2 + xy)y' = 0 \tag{19}$$

and then solve the equation.

In Example 3 we showed that this equation is not exact. Let us determine whether it has an integrating factor that depends on x only. On computing the quantity $(M_y - N_x)/N$, we find that

$$\frac{M_y(x, y) - N_x(x, y)}{N(x, y)} = \frac{3x + 2y - (2x + y)}{x^2 + xy} = \frac{1}{x}. \tag{28}$$

Thus there is an integrating factor μ that is a function of x only, and it satisfies the differential equation

$$\frac{d\mu}{dx} = \frac{\mu}{x}. \tag{29}$$

Hence

$$\mu(x) = x. \tag{30}$$

Multiplying Eq. (19) by this integrating factor, we obtain

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0. \tag{31}$$

Equation (31) is exact, since

$$\frac{\partial}{\partial y}(3x^2y + xy^2) = 3x^2 + 2xy = \frac{\partial}{\partial x}(x^3 + x^2y).$$

Thus there is a function ψ such that

$$\psi_x(x, y) = 3x^2y + xy^2, \qquad \psi_y(x, y) = x^3 + x^2y. \tag{32}$$

Integrating the first of Eqs. (32), we obtain

$$\psi(x, y) = x^3y + \tfrac{1}{2}x^2y^2 + h(y).$$

Substituting this expression for $\psi(x, y)$ in the second of Eqs. (32), we find that

$$x^3 + x^2y + h'(y) = x^3 + x^2y,$$

so $h'(y) = 0$ and $h(y)$ is a constant. Thus the solutions of Eq. (31), and hence of Eq. (19), are given implicitly by

$$x^3y + \tfrac{1}{2}x^2y^2 = c. \tag{33}$$

Solutions may also be found in explicit form since Eq. (33) is quadratic in y.

You may also verify that a second integrating factor for Eq. (19) is

$$\mu(x, y) = \frac{1}{xy(2x + y)}$$

and that the same solution is obtained, though with much greater difficulty, if this integrating factor is used (see Problem 32).

PROBLEMS

Determine whether each of the equations in Problems 1 through 12 is exact. If it is exact, find the solution.

1. $(2x+3)+(2y-2)y'=0$
2. $(2x+4y)+(2x-2y)y'=0$
3. $(3x^2-2xy+2)+(6y^2-x^2+3)y'=0$
4. $(2xy^2+2y)+(2x^2y+2x)y'=0$
5. $\dfrac{dy}{dx}=-\dfrac{ax+by}{bx+cy}$
6. $\dfrac{dy}{dx}=-\dfrac{ax-by}{bx-cy}$
7. $(e^x\sin y-2y\sin x)+(e^x\cos y+2\cos x)y'=0$
8. $(e^x\sin y+3y)-(3x-e^x\sin y)y'=0$
9. $(ye^{xy}\cos 2x-2e^{xy}\sin 2x+2x)+(xe^{xy}\cos 2x-3)y'=0$
10. $(y/x+6x)+(\ln x-2)y'=0, \quad x>0$
11. $(x\ln y+xy)+(y\ln x+xy)y'=0; \quad x>0, \quad y>0$
12. $\dfrac{x}{(x^2+y^2)^{3/2}}+\dfrac{y}{(x^2+y^2)^{3/2}}\dfrac{dy}{dx}=0$

In each of Problems 13 and 14, solve the given initial value problem and determine at least approximately where the solution is valid.

13. $(2x-y)+(2y-x)y'=0, \quad y(1)=3$
14. $(9x^2+y-1)-(4y-x)y'=0, \quad y(1)=0$

In each of Problems 15 and 16, find the value of b for which the given equation is exact, and then solve it using that value of b.

15. $(xy^2+bx^2y)+(x+y)x^2y'=0$
16. $(ye^{2xy}+x)+bxe^{2xy}y'=0$
17. Assume that Eq. (6) meets the requirements of Theorem 2.6.1 in a rectangle R and is therefore exact. Show that a possible function $\psi(x,y)$ is

$$\psi(x,y)=\int_{x_0}^{x}M(s,y_0)\,ds+\int_{y_0}^{y}N(x,t)\,dt,$$

where (x_0,y_0) is a point in R.

18. Show that any separable equation

$$M(x)+N(y)y'=0$$

is also exact.

In each of Problems 19 through 22, show that the given equation is not exact but becomes exact when multiplied by the given integrating factor. Then solve the equation.

19. $x^2y^3+x(1+y^2)y'=0, \quad \mu(x,y)=1/xy^3$
20. $\left(\dfrac{\sin y}{y}-2e^{-x}\sin x\right)+\left(\dfrac{\cos y+2e^{-x}\cos x}{y}\right)y'=0, \quad \mu(x,y)=ye^x$
21. $y+(2x-ye^y)y'=0, \quad \mu(x,y)=y$
22. $(x+2)\sin y+(x\cos y)y'=0, \quad \mu(x,y)=xe^x$
23. Show that if $(N_x-M_y)/M=Q$, where Q is a function of y only, then the differential equation

$$M+Ny'=0$$

has an integrating factor of the form

$$\mu(y)=\exp\int Q(y)\,dy.$$

24. Show that if $(N_x - M_y)/(xM - yN) = R$, where R depends on the quantity xy only, then the differential equation
$$M + Ny' = 0$$
has an integrating factor of the form $\mu(xy)$. Find a general formula for this integrating factor.

In each of Problems 25 through 31, find an integrating factor and solve the given equation.

25. $(3x^2y + 2xy + y^3) + (x^2 + y^2)y' = 0$
26. $y' = e^{2x} + y - 1$
27. $1 + (x/y - \sin y)y' = 0$
28. $y + (2xy - e^{-2y})y' = 0$
29. $e^x + (e^x \cot y + 2y \csc y)y' = 0$
30. $[4(x^3/y^2) + (3/y)] + [3(x/y^2) + 4y]y' = 0$
31. $\left(3x + \dfrac{6}{y}\right) + \left(\dfrac{x^2}{y} + 3\dfrac{y}{x}\right)\dfrac{dy}{dx} = 0$

    Hint: See Problem 24.
32. Solve the differential equation
$$(3xy + y^2) + (x^2 + xy)y' = 0$$
using the integrating factor $\mu(x, y) = [xy(2x + y)]^{-1}$. Verify that the solution is the same as that obtained in Example 4 with a different integrating factor.

2.7 Numerical Approximations: Euler's Method

Recall two important facts about the first order initial value problem
$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0. \tag{1}$$
First, if f and $\partial f/\partial y$ are continuous, then the initial value problem (1) has a unique solution $y = \phi(t)$ in some interval surrounding the initial point $t = t_0$. Second, it is usually not possible to find the solution ϕ by symbolic manipulations of the differential equation. Up to now we have considered the main exceptions to the latter statement: differential equations that are linear, separable, or exact, or that can be transformed into one of these types. Nevertheless, it remains true that solutions of the vast majority of first order initial value problems cannot be found by analytical means, such as those considered in the first part of this chapter.

Therefore, it is important to be able to approach the problem in other ways. As we have already seen, one of these ways is to draw a direction field for the differential equation (which does not involve solving the equation) and then to visualize the behavior of solutions from the direction field. This has the advantage of being a relatively simple process, even for complicated differential equations. However, it does not lend itself to quantitative computations or comparisons, and this is often a critical shortcoming.

For example, Figure 2.7.1 shows a direction field for the differential equation
$$\frac{dy}{dt} = 3 - 2t - 0.5y. \tag{2}$$

Stony Brook University

NAME: *Midterm Solutions*

RECITATION #: ____

MAT 303 – Fall 2016 – Midterm 1

October 5, 2016

INSTRUCTIONS – PLEASE READ

Please turn off your cell phone and put it away.
➪ Please write your name and your section number right now.
➪ This is a closed book exam. You are NOT allowed to use a calculator or any other electronic device or aid.
➪ Show your work. To receive full credit, your answers must be neatly written and logically organized. If you need more space, write on the back side of the preceding sheet, but be sure to label your work clearly. You do not need to simplify your answers unless explicitly instructed to do so.
➪ Academic integrity is expected of all Stony Brook University students at all times, whether in the presence or absence of members of the faculty.

PROBLEM	SCORE
1.	
2.	
3.	
4.	
Total	

LEC 01	MWF	12:00pm-12:53pm	Harriman Hall 137	Raluca Tanase
R01	W	10:00am-10:53am	Library W4535	Dyi-Shing Ou
R02	F	1:00pm- 1:53pm	Library E4330	Timothy Ryan
R03	Tu	5:30pm- 6:23pm	Library W4525	Timothy Ryan
R04	W	7:00pm- 7:53pm	Library W4525	Alexandra Viktorova
R05	M	5:30pm- 6:23pm	Lgt Engr Lab 152	Jiasheng Teh

Problem 1. (25 points) Find the solution to the given initial value problem:

a) $\frac{dy}{dt} - 2y = -2e^t, \quad y(0) = 1.$

Sol. This is a linear equation. An integrating factor is $\mu(t) = e^{\int -2dt} = e^{-2t}$

$$\underbrace{e^{-2t}\frac{dy}{dt} - 2e^{-2t}y}\ = -2e^{-t}$$

$$\frac{d}{dt}\left(y\cdot e^{-2t}\right) = -2e^{-t} \Rightarrow y\cdot e^{-2t} = \int -2e^{-t}dt = 2e^{-t} + c$$

$$\Rightarrow y(t) = e^{2t}(2e^{-t}+c) = 2e^t + ce^{2t}$$

$$y(0) = 1 \Rightarrow 2 + c = 1 \Rightarrow c = -1$$

$$\Rightarrow y(t) = 2e^t - e^{2t}$$

b) $\frac{dy}{dt} + y = e^t y^3, \quad y(0) = -1.$

Sol. This is a Bernoulli equation with exponent $n = 3$.

if $y \neq 0$ we can divide by y^3

$$\frac{1}{y^3}\frac{dy}{dt} + \frac{1}{y^2} = e^t$$

Do the substitution $v = y^{1-3} = \frac{1}{y^2}$

$$\frac{dv}{dt} = \frac{d}{dt}\left(\frac{1}{y^2}\right) = -\frac{2}{y^3}\frac{dy}{dt}$$

$\Rightarrow \frac{dv}{dt} - 2v = -2e^t$, which is exactly the equation from part (a).

$$\Rightarrow v(t) = 2e^t + ce^{2t}$$

$$v = y^{-2} \Rightarrow y^2 = \frac{1}{v} = \frac{1}{2e^t + ce^{2t}}$$

$$y(0) = -1$$

$$\Rightarrow \frac{1}{2+c} = 1 \Rightarrow 2 + c = 1 \Rightarrow c = -1$$

$y^2 = \frac{1}{2e^t - e^{2t}} \Rightarrow y(t) = -\frac{1}{\sqrt{2e^t - e^{2t}}}$, which is well defined if

$$2e^t > e^{2t}$$

$$\ln(2e^t) > \ln(e^{2t})$$

$$(\ln 2) + t > 2t$$

$$\ln 2 > t$$

Since $y(0)$ is negative, we select the negative sol.

Problem 2. (25 points) Determine whether the following equation is exact. If it is exact, find its solutions.

$$(2xy-1)\frac{dy}{dx} = 3x^2 - y^2$$

Solution: $3x^2 - y^2 - (2xy-1)\frac{dy}{dx} = 0$

$M(x,y) = 3x^2 - y^2$
$N(x,y) = -2xy + 1$ $\Rightarrow$ $\frac{\partial M}{\partial y}(x,y) = -2y = \frac{\partial N}{\partial x}(x,y)$ $\Rightarrow$ the equation is exact.

There exists a function $F(x,y)$ such that $\begin{cases} \frac{\partial F}{\partial x} = M \\ \frac{\partial F}{\partial y} = N \end{cases}$

$F(x,y) = \int \frac{\partial F}{\partial x}(x,y)\,dx = \int (3x^2 - y^2)\,dx = x^3 - xy^2 + h(y)$

$\frac{\partial F}{\partial y}(x,y) = -2xy + h'(y)$
$\frac{\partial F}{\partial y}(x,y) = N(x,y) = -2xy + 1$ $\Rightarrow h'(y) = 1 \Rightarrow h(y) = y + c_1$

$F(x,y) = x^3 - xy^2 + y + c_1$

The solution set of the differential equation is $F(x,y) = c$, that is
$x^3 - xy^2 + y = c$, where c is any real constant.

Problem 3. (25 points)

a) Find all values of a and b for which the initial value problem

$$x\frac{dy}{dx} = y, \quad y(a) = b$$

has (i) a unique local solution, (ii) no solution, or (iii) infinitely many local solutions.

Sol. $\left.\begin{array}{l} F(x,y) = \frac{y}{x} \\ \frac{\partial F}{\partial y}(x,y) = \frac{1}{x} \end{array}\right\}$ continuous when $x \neq 0$. $\Rightarrow$ by the existence & uniqueness theorem, for any initial condition $y(a)=b$ with $a \neq 0$, there exists (locally around $x=a$) a unique solution $y(x)$ of the initial value problem.

If $a=0$, then the theorem does not apply, so we need to solve the diff. eq. first. Using part b) we know that the general solution is $y(x) = C\cdot x$, so we conclude the following:

If $a=0$ and $b\neq 0$, there is no solution as $\left.\begin{array}{l} y(x) = Cx \\ y(0) = b \end{array}\right| \Rightarrow C\cdot 0 = b \Rightarrow b = 0$ (contradiction)

If $a=0$ and $b=0$, the IVP has infinitely many solutions. Every solution $y(x) = Cx$ satisfies the initial condition $y(0)=0$.

b) Solve the differential equation $x\frac{dy}{dx} = y$.

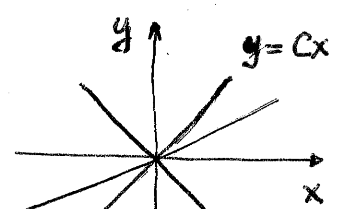
Sol. This is a separable equation with a singular solution $y=0$.

When $y\neq 0$ we write $\frac{1}{y}dy = \frac{1}{x}dx \Rightarrow \int \frac{1}{y}dy = \int \frac{1}{x}dx$

$\Rightarrow \ln|y| = \ln|x| + C_1 \quad \Rightarrow |y| = |x|e^{C_1}$

$\Rightarrow y = x\cdot C$, where $C = \pm e^{C_1}$ is any random constant.

General solution: $y(x) = C\cdot x$

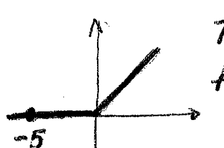


c) How many global solutions can you define which satisfy the initial condition $y(-5) = 0$?

Sol. We first solve $\begin{cases} y(x) = C\cdot x \\ y(-5) = 0 \end{cases}$ and get $y(x) = 0$. Since $-5 \neq 0$, by part a) we know that the solution $y(x)=0$ is locally unique around the point $x=-5$.

This solution is also globally unique, because we cannot glue it to another solution to obtain a differentiable function. Any glueing defines a function of the form

$y_c(x) = \begin{cases} 0, & x<0 \\ Cx, & x\geq 0 \end{cases}$. However y_c is not differentiable at 0 unless $C=0$.



Graph of $y_c(x)$

Problem 4. (25 points) Consider the following model for the growth rate of the deer population as a function of time:

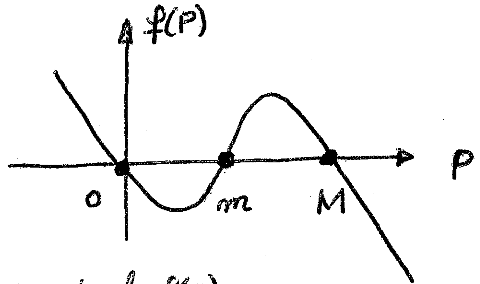
$$\frac{dP}{dt} = P(M-P)(P-m),$$

where P is the deer population and $M > m > 0$ are positive constants. It is known that if the deer population rises above the carrying capacity M, the population will decrease back to M through disease and malnutrition.

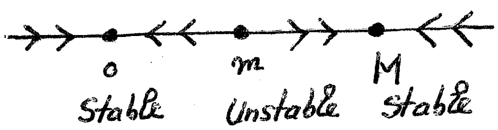
a) Determine the equilibrium points of this model and classify each one as stable or unstable. Sketch several graphs of solutions of the differential equation.

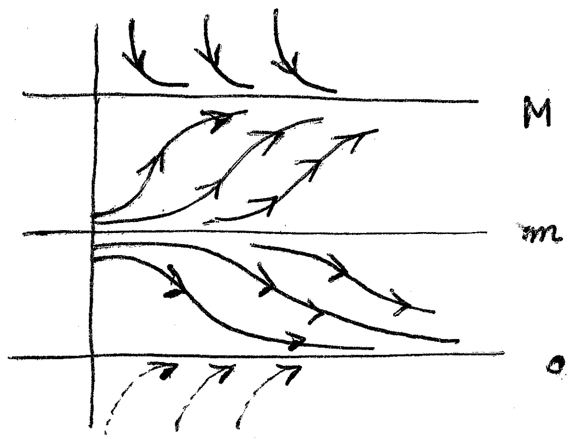
Sol. Define $f(P) = P(M-P)(P-m)$

Critical points: $f(P) = 0 \Rightarrow P = 0$, or $P = M$, or $P = m$



Graph of $f(P)$





Typical Solution Curves

b) What happens to the deer population in the long run if the initial population $P(0)$ is between 0 and M? (You are not required to solve the differential equation).

if $P(0) = 0, m, M$ then the population does not change.

if $0 < P(0) < m$ then the population will become extinct.

if $m < P(0) < M$ then the population will increase to the carrying capacity M.

Midterm #1 Practice Problems

1. Solve the following initial value problems:

(a) $xy' = y + x^2$, $y(1) = 0$

(b) $y' = 6e^{2x-y}$, $y(0) = 0$

(c) $y' = -\frac{2}{x}y + \frac{1}{x^2}$, $y(1) = 2$

(d) $(1 + x)y' = 4y$, $y(0) = 1$

(e) $v' - \frac{1}{x}v = xv^6$, $v(1) = 1$

(f) $2xyy' = x^2 + 2y^2$, $y(1) = 2$

2. Find the (complete) general solution to each of the following differential equations:

(a) $y' + 2xy + 6x = 0$

(b) $(3x^2 + 2y^2)\,dx + (4xy + 6y^2)\,dy = 0$

(c) $xy'' + y' = 12x^2$

(d) $9y' = xy^2 + 5xy - 14x$

(e) $y' + \frac{2}{3x}y + 3y^{-2} = 0$

(f) $y' + y\cot x = \cos x$

(g) $x^2y' + \frac{1}{x}y^3 = 2y^2$

(h) $(xy + y^2)\,dx + x^2\,dy = 0$

(i) $y' = (\frac{4}{x} + 1)y$

(j) $(2x - \frac{\ln y}{x^2})\,dx + \frac{1}{xy}\,dy = 0$

3. Consider the differential equation $\dfrac{dy}{dx} = \dfrac{4y}{x^2 - 4}$.

(a) Find all values a and b such that this equation with the initial condition $y(a) = b$ is guaranteed to have a unique solution.

(b) Find the general solution to the differential equation.

(c) Sketch a slope field for this equation, along with several representative solutions. Are there any points where solutions exist but are not unique?

4. A cold drink at 36°F is placed in a sweltering conference room at 90°F. After 15 minutes, its temperature is 54°F.

(a) Find the temperature $T(t)$ of the drink after t minutes, assuming it obeys Newton's law of cooling.

(b) How long does it take for the drink to reach 74°F?

5. A tank with capacity 500 liters originally contains 200 l of water with 100 kg of salt in solution. Water containing 1 kg salt per liter is entering at a rate of 3 l/min, and the mixture is allowed to flow out of the tank at a rate of 2 l/min.

(a) Find the amount of salt in the tank at any time t before it starts to overflow.

(b) Find the concentration (in kg/l) of salt in the tank at the time when the tank overflows.

(c) If the tank has infinite volume, what is the limiting concentration in the tank?

6. Suppose that at time $t = 0$, half of a "logistic" population of 100,000 persons have heard a certain rumor, and that the number of those who have heard it is then increasing at the rate of 1000 persons per day. How long will it take for this rumor to spread to 80% of the population?

7. Consider the differential equation $y' + \tan y = 0$.

(a) Find all equilibrium solutions to this equation, and characterize their stability, with justification.

(b) Find the general solution to the differential equation, and sketch a few representative solutions.

8. Consider the differential equation $y' = y^2 - 4y - 4k^2 + 8k$ with parameter k.

(a) Given $k = 2$, find the critical points of the differential equation, draw a phase diagram, and determine the stability of the critical points.

(b) Draw the bifurcation diagram for this differential equation, and label the stability of the equilibria.

9. A boat moving at 27 m/s suddenly loses power and starts to coast. The water resistance slows the boat down with force proportional to the 4/3-power of its velocity. Suppose the boat takes 20 seconds to slow down to 8 m/s. What is the total distance the boat travels as it slows down to a stop?

Solutions to Midterm #1 Practice Problems

1. Solve the following initial value problems:

(a) $xy' = y + x^2$, $y(1) = 0$

Solution: Writing the DE as $xy' - y = x^2$, we recognize it as linear. Normalizing, we have $y' - \frac{1}{x}y = x$, so the integrating factor is $\mu(x) = e^{\int -\frac{1}{x}\,dx} = e^{-\ln|x|} = \frac{1}{x}$. Multiplying the normalized DE by this factor, we obtain

$$\left(\frac{1}{x}y\right)' = \frac{1}{x}y' - \frac{1}{x^2}y = 1.$$

Integrating, $\frac{1}{x}y = x + C$, so $y = x^2 + Cx$ is the general solution. We apply the initial condition: $y(1) = 1 + C = 0$, so $C = -1$, and the particular solution is $y = x^2 - x$.

(b) $y' = 6e^{2x-y}$, $y(0) = 0$

Solution: Rewriting the exponential, the DE becomes $y' = 6e^{2x}e^{-y}$, which is separable. Separating, $e^y y' = 6e^{2x}$, which we integrate to obtain $e^y = 3e^{2x} + C$. Applying the initial condition, $e^0 = 3e^0 + C$, so $1 = 3 + C$, and $C = -2$. Hence, $e^y = 3e^{2x} - 2$, so $y = \ln(3e^{2x} - 2)$.

(c) $y' = -\frac{2}{x}y + \frac{1}{x^2}$, $y(1) = 2$

Solution: We observe that this equation is already linear and normalized: $y' + \frac{2}{x}y = \frac{1}{x^2}$. The integrating factor is then $\mu(x) = e^{\int \frac{2}{x}\,dx} = x^2$, so $(x^2y)' = x^2y' + 2xy = 1$. Integrating, $x^2y = x + C$, so $y = \frac{1}{x} + \frac{C}{x^2}$. Applying the initial condition, $y(1) = 1 + C = 2$, so $C = 1$. Then $y = \frac{1}{x} + \frac{1}{x^2}$.

(d) $(1 + x)y' = 4y$, $y(0) = 1$

Solution: This DE is separable, and we can rewrite it as $\frac{1}{y}y' = \frac{4}{1+x}$. Integrating,

$$\ln|y| = 4\ln|1 + x| + C = \ln(1 + x)^4 + C$$

and, redefining C, $y = C(1 + x)^4$. Applying the initial condition, $1 = C(1)^4$, so $C = 1$, and $y = (1 + x)^4$.

(e) $v' - \frac{1}{x}v = xv^6$, $v(1) = 1$

Solution: Since this equation is linear except for the xv^6 term, we recognize it as a Bernoulli equation, with $n = 6$. We therefore make the substitution $u = v^{1-n} = v^{-5}$. Then $v = u^{-1/5}$, so $v' = -\frac{1}{5}u^{-6/5}u'$, and the DE becomes

$$-\frac{1}{5}u^{-6/5}u' - \frac{1}{x}u^{-1/5} = xu^{-6/5}.$$

Multiplying by $-5u^{6/5}$, this equation normalizes to $u' + \frac{5}{x}u = -5x$, with is linear in $u(x)$. The integrating factor is x^5, so we have $(x^5u)' = -5x^6$, which integrates to $x^5u = -\frac{5}{7}x^7 + C$. Dividing by x^5,

$$v^{-5} = u = \frac{C}{x^5} - \frac{5x^2}{7}.$$

We apply the initial condition, so $1 = C - \frac{5}{7}$, and $C = \frac{12}{7}$. Then $v^{-5} = \frac{12}{7x^5} - \frac{5x^2}{7} = \frac{12-5x^7}{7x^5}$, so

$$v = \sqrt[5]{\frac{7x^5}{12-5x^7}}.$$

(f) $2xyy' = x^2 + 2y^2, y(1) = 2$
Solution: We try normalizing this DE, dividing by $2xy$ to obtain $y' = \frac{x}{2y} + \frac{y}{x}$. We observe that y' is expressed entirely in terms of y/x, so we let $v = y/x$. Then $y = xv$ and $y' = xv' + v$, so this DE becomes

$$xv' + v = \frac{1}{2v} + v.$$

Hence, $xv' = \frac{1}{2v}$, which is separated, so $2vv' = \frac{1}{x}$. Integrating, $v^2 = \ln|x| + C$, so $v = \pm\sqrt{\ln|x| + C}$, and $y = \pm x\sqrt{\ln|x| + C}$.

Applying the initial condition, $2 = \pm 1\sqrt{C}$, so we must take the positive branch of the square root, and $C = 2^2 = 4$. Then the solution to the IVP is

$$y = x\sqrt{\ln x + 4},$$

and is valid for $x > 0$.

2. Find the (complete) general solution to each of the following differential equations:

(a) $y' + 2xy + 6x = 0$
Solution: We observe that this DE is linear, with integrating factor $\mu(x) = e^{\int 2x\,dx} = e^{x^2}$. Then the DE becomes

$$\left(e^{x^2}y\right)' = -6xe^{x^2} \quad \xrightarrow{\int} \quad e^{x^2}y = -3e^{x^2} + C \quad \Rightarrow \quad y = Ce^{-x^2} - 3.$$

(b) $(3x^2 + 2y^2)\,dx + (4xy + 6y^2)\,dy = 0$
Solution: Given the format of the DE, we check whether the coefficient functions $M(x,y) = 3x^2 + 2y^2$ and $N(x,y) = 4xy + 6y^2$ make it exact. Since $M_y = 4y$ and

$N_x = 4y$, it is. We can then integrate M with respect to x to find a function $F(x,y)$ such that this DE is $dF = 0$, up to pure y-indeterminacy in F:

$$F(x,y) = \int 3x^2 + 2y^2 \, dx = x^3 + 2xy^2 + g(y).$$

Then $F_y = 4xy + g'(y)$, but this is also $N = 4xy + 6y^2$, so $g'(y) = 6y^2$. Then $g(y) = 2y^3 + C$, so one value for F is $x^3 + 2xy^2 + 2y^3$. Finally, solutions to the DE are the level sets

$$x^3 + 2xy^2 + 2y^3 = C,$$

which cannot easily be solved explicitly for y.

(c) $xy'' + y' = 12x^2$

Solution: Since y is not present in this DE, we recognize it as being a reducible second-order equation. We take $p(x) = y'$, so $y'' = p'$. Thus, we obtain the linear DE $xp' + p = 12x^2$. Since the left-hand side is already $(xp)'$, we integrate to get $xp = 4x^3 + C$, so $y' = p = 4x^2 + \frac{C}{x}$. Integrating again, $y = \frac{4}{3}x^3 + C \ln x + D$.

(d) $9y' = xy^2 + 5xy - 14x$

Solution: Factoring an x out of the terms on the right-hand side, $9y' = x(y^2 + 5y - 14)$, which reveals that the DE is separable. Hence, we separate this to

$$\frac{9}{y^2 + 5y - 14} y' = x,$$

which excludes the constant solutions $y = 2$ and $y = -7$. We use partial fractions to decompose the fraction on the left-hand side. Since $y^2 + 5y - 14 = (y+7)(y-2)$, we have

$$\frac{A}{y-2} + \frac{B}{y+7} = \frac{9}{y^2 + 5y - 14} \quad \Rightarrow \quad A(y+7) + B(y-2) = 9.$$

Plugging in $y = -7$ and $y = 2$, we see that $A = 1$ and $B = -1$. Then integrating,

$$\int \frac{1}{y-2} - \frac{1}{y+7} \, dy = \int x \, dx \quad \Rightarrow \quad \ln|y-2| - \ln|y+7| = \frac{1}{2}x^2 + C$$

Combining the difference of the logs and exponentiating, $\frac{y-2}{y+7} = Ce^{x^2/2}$. Solving for y,

$$y = \frac{9}{1 - Ce^{x^2/2}} - 7.$$

Setting $C = 0$ recovers the solution $y = 2$, but no value of C gives $y = -7$, so we record this as a singular solution not incorporated into the general solution.

(e) $y' + \frac{2}{3x}y + 3y^{-2} = 0$

Solution: We recognize this as a Bernoulli equation with $n = -2$, so we make the substitution $v = y^{1-n} = y^3$. Then $y = v^{1/3}$, so $y' = \frac{1}{3}v^{-2/3}v'$. Thus, the DE becomes

$$\frac{1}{3}v^{-2/3}v' + \frac{2}{3x}v^{1/3} + 3v^{-2/3} = 0 \quad \Rightarrow \quad v' + \frac{2}{x}v = -9.$$

This DE is linear in v with integrating factor $\mu(x) = x^2$, so $(x^2v)' = -9x^2$. Integrating, $x^2v = -3x^3 + C$, so $v = -3x + C/x^2$. Since $y = v^{1/3}$,

$$y = \sqrt[3]{\frac{C}{x^2} - 3x}.$$

(f) $y' + y\cot x = \cos x$

Solution: We see that this equation is already linear, with $p(x) = \cot x = \frac{\cos x}{\sin x}$, so its integrating factor is $\mu(x) = e^{\int \cot x\, dx} = e^{\ln|\sin x|} = \sin x$. Multiplying by this integrating factor, we get

$$(y \sin x)' = \sin x \cos x,$$

so integrating gives $y\sin x = \frac{1}{2}\sin^2 x + C$, and $y = \frac{1}{2}\sin x + \frac{C}{\sin x}$.

(g) $x^2y' + \frac{1}{x}y^3 = 2y^2$

Solution: We start by normalizing the DE: $y' + \frac{y^3}{x^3} = 2\frac{y^2}{x^2}$. Then the entire DE can be written in terms of y' and y/x, so it is homogeneous. Making the standard substitution $v = y/x$, $y' = xv' + v$, we have

$$xv' + v + v^3 = 2v^2,$$

so $xv' = -v^3 + 2v^2 - v = -v(v-1)^2$. Separating variables, we have $-\frac{1}{v(v-1)^2}v' = \frac{1}{x}$. By partial fraction theory, we expect to be able to decompose this fraction as

$$-\frac{1}{v(v-1)^2} = \frac{A}{v} + \frac{B}{v-1} + \frac{C}{(v-1)^2}.$$

Then $-1 = A(v-1)^2 + Bv(v-1) + Cv$, so we set $v = 0$ and $v = 1$ to determine $A = -1$ and $C = -1$. Then $-1 = -(v-1)^2 - v + Bv(v-1)$, so expanding and cancelling terms we have that $B = 1$. Thus, we integrate:

$$\int \frac{1}{v-1} - \frac{1}{v} - \frac{1}{(v-1)^2}\, dv = \int \frac{1}{x}\, dx \quad \Rightarrow \quad \ln|v-1| - \ln|v| + \frac{1}{v-1} = \ln|x| + C.$$

Exponentiating and using $v = y/x$, we obtain the implicit solution

$$\left(1 - \frac{x}{y}\right) e^{\frac{x}{y-x}} = Cx.$$

(h) $(xy + y^2)\,dx + x^2\,dy = 0$

Solution: We check whether this DE is exact: letting $M(x,y) = xy + y^2$ and $N(x,y) = x^2$, we compute $M_y = x + 2y$ and $N_x = 2x$. Since these are not equal, the DE is not exact!

We instead divide by $x^2\,dx$ to obtain $\frac{y}{x} + \frac{y^2}{x^2} + \frac{dy}{dx} = 0$, which is homogeneous. Then letting $v = y/x$, we have $v + v^2 + xv' + v = 0$, so $xv' = -2v - v^2 = -v(v+2)$. Separating variables, $-\frac{1}{v(v+2)v'} = \frac{1}{x}$, and by partial fractions, $\frac{1}{v(v+2)} = \frac{1}{2}(\frac{1}{v} - \frac{1}{v+2})$. Therefore, $(\frac{1}{v+2} - \frac{1}{v})v' = \frac{2}{x}$, so integrating gives

$$\ln|v+2| - \ln|v| = C + 2\ln|x|.$$

Exponentiating, $\frac{v+2}{v} = Cx^2$, so $\frac{2}{v} = Cx^2 - 1$, and $v = \frac{2}{Cx^2-1}$. We also potentially omitted the solutions $v = 0$ and $v = -2$ when we divided above, but the case $C = 0$ recovers $v = -2$. Finally, we recover y, as

$$y = xv = \frac{2x}{Cx^2 - 1},$$

or $y = 0$.

(i) $y' = (\frac{4}{x} + 1)y$

Solution: This equation is separable, so we separate variables to get $\frac{1}{y}y' = \frac{4}{x} + 1$. Integrating, $\ln|y| = \ln x^4 + x + C$, so $y = Cx^4e^x$. Dividing by y above excludes the solution $y = 0$, but we recover it when $C = 0$.

(j) $(2x - \frac{\ln y}{x^2})\,dx + \frac{1}{xy}\,dy = 0$

Solution: We check for exactness: $M(x,y) = 2x - \frac{\ln y}{x^2}$ and $N(x,y) = \frac{1}{xy}$, so $M_y = -\frac{1}{x^2y}$ and $N_x = -\frac{1}{x^2y}$. Hence, the DE is exact.

We integrate M with respect to x:

$$F(x,y) = \int 2x - \frac{\ln y}{x^2}\,dy = x^2 + \frac{\ln y}{x} + g(y).$$

Then $F_y = \frac{1}{xy} + g'(y) = N = \frac{1}{xy}$, so $g'(y) = 0$, and $g(y)$ is constant. Hence, one choice for $F(x,y)$ is $x^2 + \frac{\ln y}{x}$, and the solutions are level sets

$$x^2 + \frac{\ln y}{x} = C.$$

Solving for y, $\ln y = Cx - x^3$, so $y = e^{Cx-x^3}$.

3. Consider the differential equation $\dfrac{dy}{dx} = \dfrac{4y}{x^2-4}$.

(a) Find all values a and b such that this equation with the initial condition $y(a) = b$ is guaranteed to have a unique solution.

Solution: We let $f(x,y) = \frac{4y}{x^2-4}$, so that this DE is $y' = f(x,y)$. Then both f and its derivative $f_y(x,y) = \frac{4}{x^2-4}$ are continuous for $x \neq 2$ and $x \neq -2$. Hence, for all a not equal to 2 or -2 and all b, the DE with the initial condition $y(a) = b$ has a unique solution.

(b) Find the general solution to the differential equation.

Solution: We note that this equation is separable, so we separate it into

$$\frac{1}{y}y' = \frac{4}{x^2-4} = \frac{1}{x-2} - \frac{1}{x+2},$$

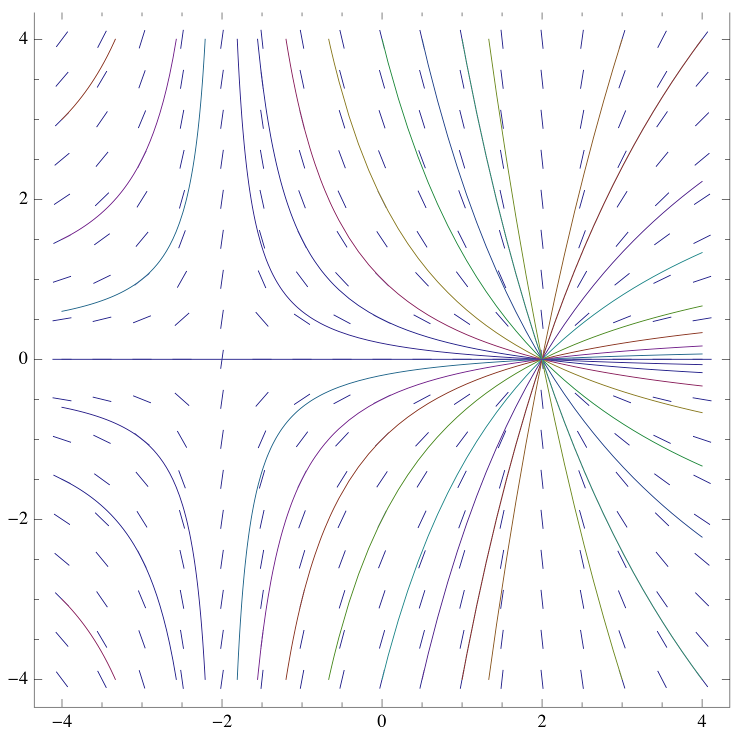
which excludes $y = 0$ as a solution. Integrating, $\ln|y| = \ln|x-2| - \ln|x+2| + C$. Exponentiating,

$$y = C\frac{x-2}{x+2} = C\left(1 - \frac{4}{x+2}\right),$$

with the case $C = 0$ corresponding to the solution $y = 0$ that we would otherwise have missed. These are hyperbolas along the axes $x = -2$ and $y = C$.

(c) Sketch a slope field for this equation, along with several representative solutions. Are there any points where solutions exist but are not unique?

Solution: Below is a slope field with several solutions:



We see that all of the parabolas to the right of $x = -2$ intersect at the point $(2, 0)$, so this is a point where uniqueness fails but existence does not.

4. A cold drink at 36°F is placed in a sweltering conference room at 90°F. After 15 minutes, its temperature is 54°F.

(a) Find the temperature $T(t)$ of the drink after t minutes, assuming it obeys Newton's law of cooling.

Solution: The DE governing the temperature $T(t)$ is $T' = -k(T - A)$, where $A = 90$. Writing this as a normalized linear DE, then, it becomes $T' + kT = 90k$, so the integrating factor is e^{kt}. Hence, $(e^{kt}T)' = 90ke^{kt}$, so integrating yields $e^{kt}T = 90e^{kt} + C$, or $T = 90 + Ce^{-kt}$. Finally, applying the initial condition $T(0) = 36$ gives that $C = 36 - 90 = -54$. Hence, the temperature is given by

$$T(t) = 90 - 54e^{-kt}.$$

We now determine k: $T(15) = 90 - 54e^{-15k} = 54$, so $e^{-15k} = \frac{-36}{-54} = \frac{2}{3}$, and $k = \frac{1}{15} \ln \frac{3}{2}$. Therefore,

$$T(t) = 90 - 54e^{-\frac{t}{15} \ln \frac{3}{2}} = 90 - 54 \left(\frac{2}{3}\right)^{\frac{t}{15}}.$$

(b) How long does it take for the drink to reach 74°F?

Solution: We set $T = 74$ and solve for t: $90 - 54 \left(\frac{2}{3}\right)^{\frac{t}{15}} = 74$, so $\left(\frac{2}{3}\right)^{\frac{t}{15}} = \frac{-16}{-54} = \frac{8}{27} = (\frac{2}{3})^3$. Then $t/15 = 3$, so $t = 45$ min.

5. A tank with capacity 500 liters originally contains 200 l of water with 100 kg of salt in solution. Water containing 1 kg salt per liter is entering at a rate of 3 l/min, and the mixture is allowed to flow out of the tank at a rate of 2 l/min.

(a) Find the amount of salt in the tank at any time t before it starts to overflow.

Solution: We set up a differential equation for the amount $x(t)$ of salt in the tank at time t. The volume of solution in the tank is $V(t) = 200 + (3 - 2)t = 200 + t$, so $\frac{dx}{dt}$ is given by

$$\frac{dx}{dt} = r_{in}c_{in} - r_{out}c_{out} = (3)(1) - 2\frac{x(t)}{200 + t} = 3 - \frac{2}{200 + t}x.$$

Then $x' + \frac{2}{200+t}x = 3$, which is linear, with integrating factor $\mu(t) = e^{\int \frac{2}{200+t}\,dt} = e^{2 \ln |200+t|} = (200 + t)^2$. Then $((200 + t)^2 x)' = 3(200 + t)^2$, so integrating gives

$$(200 + t)^2 x = (200 + t)^3 + C \quad \Rightarrow \quad x(t) = 200 + t + \frac{C}{(200 + t)^2}.$$

At time $t = 0$, $x(t) = 100$, so $100 = 200 + \frac{C}{200^2}$, and $C = -100(200)^2 = -4 \times 10^6$. Thus,

$$x(t) = 200 + t - \frac{4{,}000{,}000}{(200+t)^2}.$$

(b) Find the concentration (in kg/l) of salt in the tank at the time when the tank overflows.

Solution: The concentration in the tank is given by

$$c(t) = \frac{x(t)}{V(t)} = 1 - \frac{4{,}000{,}000}{(200+t)^3}.$$

The tank overflows when $V(t) = 500$, so at this time

$$c = 1 - \frac{4{,}000{,}000}{500^3} = 1 - \frac{4}{125} = 1 - 0.032 = 0.968 \text{ kg/l}.$$

(c) If the tank has infinite volume, what is the limiting concentration in the tank?

Solution: We have that

$$\lim_{t\to\infty} c(t) = \lim_{t\to\infty} 1 - \frac{4{,}000{,}000}{(200+t)^3} = 1 \text{ kg/l}.$$

6. Suppose that at time $t = 0$, half of a "logistic" population of 100,000 persons have heard a certain rumor, and that the number of those who have heard it is then increasing at the rate of 1000 persons per day. How long will it take for this rumor to spread to 80% of the population?

Solution: Let $P(t)$ be the number of persons in the population who have heard the rumor. Then $P(0) = P_0 = 50{,}000$, and $P' = kP(M - P)$, with $M = 100{,}000$. Furthermore, $P'(0) = 1000$, so

$$1000 = k(50{,}000)(100{,}000 - 50{,}000) = k(50{,}000)^2 = (2.5 \times 10^9)k,$$

and $k = \frac{1}{2.5\times 10^6} = 4 \times 10^{-7}$. Thus, $kM = (4 \times 10^{-7})(100{,}000) = 4 \times 10^{-2} = 1/25$.

Recalling that $P(t)$ is given by

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}},$$

we solve $P(t) = 80{,}000$ for t. Then

$$80{,}000 = \frac{(100{,}000)(50{,}000)}{50{,}000 + (50{,}000)e^{-t/25}} \quad \Rightarrow \quad \frac{4}{5} = \frac{1}{1 + e^{-t/25}} \quad \Rightarrow \quad e^{-t/25} = \frac{5}{4} - 1 = \frac{1}{4}.$$

Then $-t/25 = \ln \frac{1}{4} = -2 \ln 2$, so $t = 50 \ln 2 \approx 34.7$ days.

7. Consider the differential equation $y' + \tan y = 0$.

(a) Find all equilibrium solutions to this equation, and characterize their stability, with justification.

Solution: Since the DE is $y' = f(y)$, with $f(y) = -\tan y$, we look for solutions to $-\tan y = 0$. This occurs when $\sin y = 0$, or when $y = n\pi$ for any integer n. Hence, we expect equilibria at these y values. Furthermore, for $n\pi - \frac{\pi}{2} < y < n\pi$, $-\tan y$ is positive, and for $n\pi < y < n\pi + \frac{\pi}{2}$, $-\tan y$ is negative, so each equilibrium $y = n\pi$ is stable.

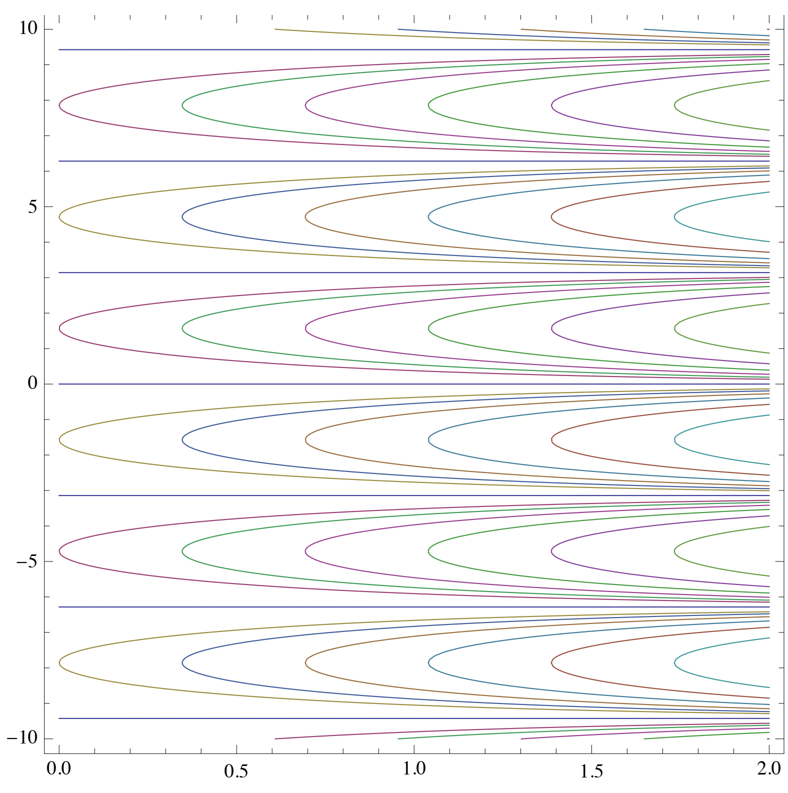
Alternately, since $f(y) = -\tan y$ is differentiable at each $n\pi$, we examine $f'(y) = -\sec^2 y = -\frac{1}{\cos^2 y}$ there: $\cos n\pi = \pm 1$, so $f(n\pi) = -\frac{1}{1} = -1$. Thus, the derivative is negative, so $f(y)$ is decreasing across this equilibrium, and it is therefore stable.

(b) Find the general solution to the differential equation, and sketch a few representative solutions.

Solution: We note that this DE is separable, since it is autonomous, and we separate it into $-\frac{1}{\tan y}y' = -\frac{\cos y}{\sin y}y' = 1$. Integrating, and using u-substitution with $u = \sin y$,

$$-\ln|\sin y| = x + C.$$

Then $\sin y = Ce^{-x}$, redefining C, and $y = \sin^{-1}(Ce^{-x})$. We note that there is some ambiguity in $\sin^{-1}$: canonically, it takes values between $-\pi/2$ and $\pi/2$, but it may be translated by $2n\pi$, or translated by $(2n+1)\pi$ with a sign reversal. In any case, we obtain the following representative graphs for y for different choices of C and the arcsine branch:



8. Consider the differential equation $y' = y^2 - 4y - 4k^2 + 8k$ with parameter k.

(a) Given $k = 2$, find the critical points of the differential equation, draw a phase diagram, and determine the stability of the critical points.

Solution: When $k = 2$, the DE is $y' = y^2 - 4y - 16 + 16 = y^2 - 4y = y(y-4)$. Thus, the equilibria are $y = 0$ and $y = 4$. Since $y^2 - 4y$ is a parabola opening upwards, its values are negative between 0 and 4 and positive otherwise. Hence, we have the following stability diagram:

$$\begin{array}{cccccc} + & 0 & - & 0 & + & y' \text{ sign} \\ \hline & \bullet & & \bullet & & \rightarrow y \\ \longrightarrow & 0 & \longleftarrow & 4 & \longrightarrow & \end{array}$$

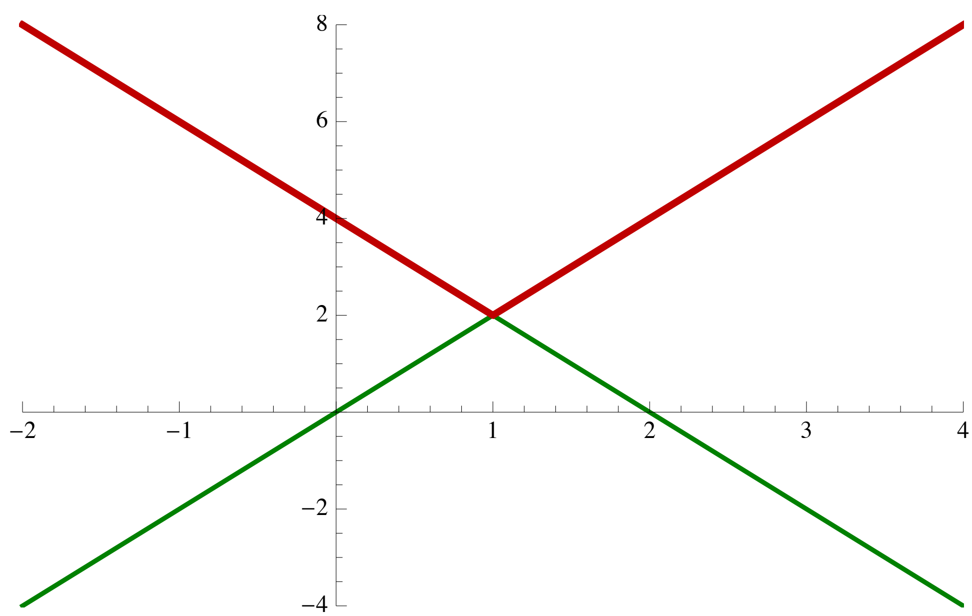
Thus, the equilibrium $y = 0$ is stable, and the equilibrium $y = 4$ is unstable.

(b) Draw the bifurcation diagram for this differential equation, and label the stability of the equilibria.

Solution: We solve $y^2 - 4y - 4k^2 + 8k = 0$ to find the equilibria for general k: using the quadratic formula,

$$y = \frac{4 \pm \sqrt{16 + 16k^2 - 32k}}{2} = \frac{4 \pm 4\sqrt{k^2 - 2k + 1}}{2} = 2 \pm 2(k+1).$$

Thus, the equilibria are $y = 2 + 2k - 2 = 2k$ and $2 - 2k + 2 = 4 - 2k$, which are straight lines in the yk-plane. By the stability analysis above, the upper branch is unstable, and the lower branch is stable. Finally, when the branches meet at $k = 1$ and $y = 2$, that equilibrium is semistable. We illustrate this below (red is unstable, green is stable):



9. A boat moving at 27 m/s suddenly loses power and starts to coast. The water resistance slows the boat down with force proportional to the 4/3-power of its velocity. Suppose the boat takes 20 seconds to slow down to 8 m/s. What is the total distance the boat travels as it slows down to a stop?

Solution: The DE governing the velocity of the boat is $v' = -kv^{4/3}$, which is separable. Then $v^{-4/3}v' = -k$, so integrating gives $-3v^{-1/3} = C - kt$. Hence, $v^{-1/3} = \frac{1}{v^{1/3}} = C + \frac{1}{3}kt$. Applying the initial condition $v(0) = v_0 = 27 = 3^3$, $\frac{1}{3} = C + 0$, so $C = \frac{1}{3}$. Therefore, $v^{1/3} = \frac{1}{C+\frac{1}{3}kt} = \frac{1}{\frac{1}{3}+\frac{1}{3}kt} = \frac{3}{1+kt}$, and

$$v(t) = \frac{27}{(1+kt)^3}.$$

We now determine k. At $t = 20$, $v(t) = 8$, so $8 = \frac{27}{(1+20k)^3}$. Then $(1+20k)^3 = \frac{27}{8} = (\frac{3}{2})^3$, so $1 + 20k = \frac{3}{2}$, and $k = \frac{1}{2} \cdot \frac{1}{20} = \frac{1}{40}$.

We determine $x(t)$, the distance that the boat travels over time. Integrating $v(t) = 27(1+kt)^{-3}$ from 0 to t,

$$x(t) - x_0 = \int_0^t 27(1+kt)^{-3} = \left[27\frac{1}{-2}\frac{1}{k}(1+kt)^{-2}\right]_0^t = \frac{27}{2k}\left(1 - \frac{1}{(1+kt)^2}\right).$$

Then $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} = \frac{27}{2k}\left(1 - \frac{1}{(1+kt)^2}\right) = \frac{27}{2k}$. With $k = 1/40$, this is $27(20) = 540$ m.

MAT 303 - HW7 - Additional Problems

Exercise 1. [Reduction of Order] Show that if y_1 is a solution of the differential equation

$$y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = 0$$

then $y_2 = y_1 v$ is a new solution of the differential equation above, provided that v satisfies the following second order equation in v':

$$y_1 v''' + (3y_1' + p_1 y_1)v'' + (3y_1'' + 2p_1 y_1' + p_2 y_1)v' = 0$$

Exercise 2. Consider the differential equation

$$t^2(t+3)y''' - 3t(t+2)y'' + 6(1+t)y' - 6y = 0, \quad t > 0$$

Assume that a solution of this equation is already known, $y_1(t) = t^2$.

a) Use the method of reduction of order to find a fundamental set of solutions for the differential equations (that is, three linearly independent solutions).

b) Check that the solutions obtained at part a) are linearly independent in two ways: by using the definition of linear independence, and by using the Wronskian test.

c) Find the general solution of the differential equation.

Exercise 3. Consider the differential equation

$$y''' - 3y'' + 3y' - y = 0$$

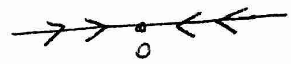
which has characteristic polynomial $p(r) = (r-1)^3$. By the general theory of equations with constant coefficients, we know that $y_1(t) = e^t$ is a solution of this differential equation. Use the method of reduction of order to find a fundamental set of solutions for this differential equation.

If you need help with 3x3 determinants, you can browse through the following lecture notes:
http://ksuweb.kennesaw.edu/~plaval//math3260/det1.pdf
http://ksuweb.kennesaw.edu/~plaval//math3260/det2.pdf

1. $x' = \dfrac{\lambda x - x^3}{1+x^2}$

Critical points: $\dfrac{\lambda x - x^3}{1+x^2} = 0 \iff \lambda x - x^3 = 0 \iff x = 0, \pm\sqrt{\lambda}$

<u>When $\lambda = 0$</u> one critical point $x=0$, which is stable

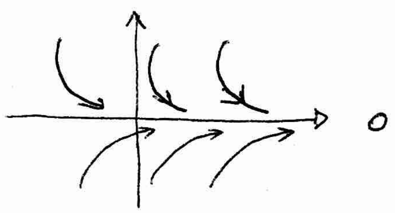


since the function $f(x) = \dfrac{\lambda x - x^3}{1+x^2} = -\dfrac{x^3}{1+x^2}$ is $\begin{cases} \text{positive when } x<0 \\ \text{negative when } x>0. \end{cases}$

<u>When $\lambda < 0$</u> one stable critical point $x=0$.

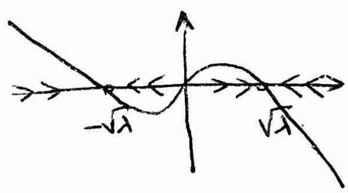
$f(x) = \dfrac{\lambda x - x^3}{1+x^2} = x \cdot \underbrace{\dfrac{\lambda - x^2}{1+x^2}}_{<0}$ is $\begin{cases} \text{positive when } x<0 \\ \text{negative when } x>0. \end{cases}$

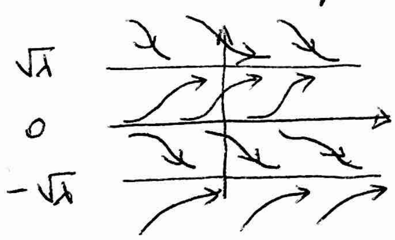
<u>Phase Portrait for $\lambda \le 0$</u>



<u>When $\lambda > 0$</u> three critical points: $x=0$ unstable, $x = \pm\sqrt{\lambda}$ stable

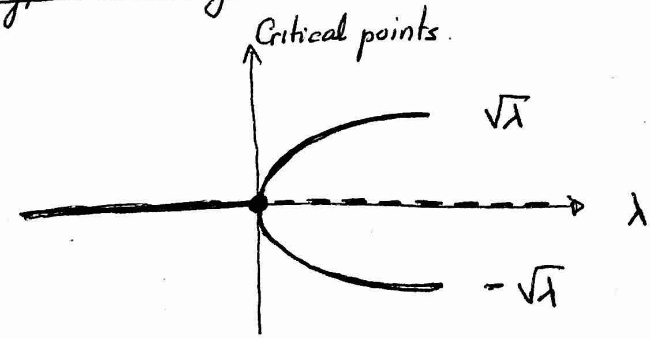
since the function $\lambda x - x^3$ looks like





Phase Portrait for $\lambda > 0$.

<u>Bifurcation Diagram:</u>



Pitchfork Bifurcation.

2. $t^2y'' + 2ty' - 12y = 0, \ t>0 \quad \Rightarrow \quad y'' + \underbrace{\left(\frac{2}{t}\right)}_{=p} y' - \frac{12}{t^2} y = 0$

a) $y_1(t) = t^3$ is a solution since $y_1'(t) = 3t^2$, $y_1''(t) = 6t$, and.

$$t^2(6t) + 2t(3t^2) - 12(t^3) = 6t^3 + 6t^3 - 12t^3 = 0$$

b) We look for a solution $y_2(t) = v(t)\, y_1(t)$, where v satisfies the equation

$$y_1 v'' + (2y_1' + p y_1) v' = 0$$

$$\Rightarrow t^3 v'' + \left(6t^2 + \frac{2}{t}\cdot t^3\right) v' = 0 \quad \Rightarrow \quad t v'' + 8v' = 0.$$

Using the substitution $u = v'$ and separation of variables, we get

$$\frac{du}{u} = -\frac{8}{t}\,dt \Rightarrow \ln|u| = -8\ln|t| + C_0$$

or $u = 0$.

$\Rightarrow |u(t)| = |t|^{-8} \cdot e^{C_0}$ or $u = 0$ $\Big| \Rightarrow u(t) = t^{-8} \cdot C_1$ where C_1 is $\pm e^{C_0}$ or 0, so C_1 is any random constant.

$$\Rightarrow v(t) = \int t^{-8} C_1 = -\frac{1}{8} t^{-7} C_1 + C_2$$

We can choose $v(t) = t^{-7} \Rightarrow y_2(t) = v y_1 = t^{-7} \cdot t^3 = t^{-4}$

c) $W(y_1, y_2)(t) = \begin{vmatrix} t^3 & t^{-4} \\ 3t^2 & -4t^{-5} \end{vmatrix} = -4t^{-2} - 3t^{-2} = -7t^{-2} < 0$ for all $t > 0$.

$\Rightarrow y_1, y_2$ are linearly independent.

Alternatively if there exist constants a, b such that

$$at^3 + bt^{-4} = 0 \text{ for all } t > 0$$

then $at^7 + b = 0 \Rightarrow 7at^6 = 0 \Rightarrow a = 0 \Rightarrow a = b = 0.$

So y_1, y_2 are linearly independent.

3.
$$y'' + w_0^2 y = F_0 \cos(wt).$$

Homogeneous Equation: $y'' + w_0^2 y = 0$. Characteristic equation $r^2 + w_0^2 = 0$ with complex conjugate roots $r_{1,2} = \pm i w_0$

General Solution $y_h(t) = C_1 \cos(w_0 t) + C_2 \sin(w_0 t)$

a) $\underline{W \neq W_0}$ We look for a particular solution $y_p(t) = A\cos(wt)$. There is no need to consider $y_p(t) = A\cos(wt) + B\sin(wt)$ since y' is not present in the differential equation.

$$y_p'' + w_0^2 y_p = F_0 \cos(wt)$$

$$\Rightarrow -Aw^2 \cos(wt) + Aw_0^2 \cos(wt) = F_0 \cos(wt) \Rightarrow A = \frac{F_0}{w_0^2 - w^2}$$

General Solution: $y(t) = y_h(t) + y_p(t) = C_1 \cos(w_0 t) + C_2 \sin(w_0 t) + \dfrac{F_0}{w_0^2 - w^2} \cos(wt)$

Initial Conditions $\begin{cases} y(0) = 0 \Rightarrow C_1 = -\dfrac{F_0}{w_0^2 - w^2} \\ y'(0) = 0 \Rightarrow C_2 = 0 \end{cases}$

$$\Rightarrow y(t) = \frac{F_0}{w_0^2 - w^2} \left(\cos(wt) - \cos(w_0 t) \right)$$

b) $\underline{W = W_0}$ We look for a particular solution $y_p(t) = t\left(A\cos(w_0 t) + B\sin(w_0 t)\right)$

We compute $y_p' = \left(-Aw_0 \sin(w_0 t) + Bw_0 \cos(w_0 t)\right)t + \left(A\cos(w_0 t) + B\sin(w_0 t)\right)$

$$y_p'' = \left(-Aw_0^2 \cos(w_0 t) - Bw_0^2 \sin(w_0 t)\right) \cdot t +$$
$$\left(-Aw_0 \sin(w_0 t) + Bw_0 \cos(w_0 t)\right) +$$
$$\left(-Aw_0 \sin(w_0 t) + Bw_0 \cos(w_0 t)\right)$$

$$\Rightarrow y_p'' + w_0^2 y_p = -2Aw_0 \sin(w_0 t) + 2Bw_0 \cos(w_0 t) = F_0 \cos(w_0 t)$$

$$\left.\begin{array}{l} \Rightarrow A = 0 \\ B = \dfrac{F_0}{2w_0} \end{array}\right| \Rightarrow y_p(t) = t \cdot \frac{F_0}{2w_0} \sin(w_0 t)$$

General Solution $y(t) = y_h(t) + y_p(t) = C_1 \cos(w_0 t) + C_2 \sin(w_0 t) + \dfrac{F_0}{2w_0} t \sin(w_0 t)$

Initial Conditions $\begin{cases} y(0) = 0 \Rightarrow C_1 = 0 \\ y'(0) = 0 \Rightarrow C_2 = 0 \end{cases} \Rightarrow y(t) = \dfrac{F_0}{2w_0} t \sin(w_0 t)$

4. $y'' - 2y' + (1+\lambda)y = 0 \qquad y(0)=0 \;,\; y(\pi)=0$

Characteristic Equation: $\quad r^2 - 2r + (1+\lambda) = 0$ with roots $r_{1,2} = \dfrac{2 \pm \sqrt{4-4(1+\lambda)}}{2}$

$\Rightarrow r_{1,2} = 1 \pm \sqrt{-\lambda}$

$\underline{\lambda=0}$ One repeated root $r=1$

General Solution $y(x) = c_1 e^x + c_2 x e^x$

BVP Conditions $\begin{cases} y(0)=0 \Rightarrow c_1 = 0 \\ y(\pi)=0 \Rightarrow c_2 \pi e^{\pi} = 0 \Rightarrow c_2 = 0 \end{cases}$ $\Rightarrow$ $y=0$ is the only solution $\Rightarrow \lambda=0$ is not an eigenvalue

$\underline{\lambda>0}$ Complex Conjugate Roots $r_{1,2} = 1 \pm i\mu$

$\lambda = \mu^2$

General Solution $y(x) = c_1 e^x \cos(\mu x) + c_2 e^x \sin(\mu x)$.

BVP Conditions $\begin{cases} y(0)=0 \Rightarrow c_1 = 0 \\ y(\pi)=0 \Rightarrow c_2 \sin(\mu\pi) = 0 \Rightarrow \mu\pi = k\pi,\ k \in \mathbb{Z} \end{cases}$ or $c_2 = 0$.

$\Rightarrow \mu = k \Rightarrow \lambda = k^2, \quad k \in \mathbb{Z}^*$

$\underline{\lambda<0}$ ~~Complex~~ Distinct Real Roots $r_{1,2} = 1 \pm \mu$

$\lambda = -\mu^2$

General Solution $y(x) = c_1 e^{(1+\mu)x} + c_2 e^{(1-\mu)x}$

BVP Conditions $y(0)=0 \Rightarrow c_1 + c_2 = 0$

$y(\pi)=0 \Rightarrow c_1 e^{(1+\mu)\pi} + c_2 e^{(1-\mu)\pi} = 0 \Rightarrow$

$\Rightarrow c_1 e^{2\mu\pi} + c_2 = 0$

$\Rightarrow c_1 = c_2 = 0 \Rightarrow$ the only solution is $y(x)=0$.

b) Non-trivial Solutions corresponding to $\lambda = k^2$, $k = \cancel{0}, 1, 2, 3, \ldots$ not an eigenvalue.

$y_\lambda(x) = c_2 e^x \sin(kx)$.

MAT 303: Calculus IV with Applications
FALL 2016

Practice problems for Midterm 2

Problem 1:

a) Find the general solution of the ODE $y'' + 4y = 4\cos(2t)$.

b) Make a sketch of y_p vs. t, where $y_p(t)$ denotes the particular solution found in part a). What is the pseudo-period of the oscillation and the time varying amplitude?

Problem 2: Consider the 4th order ODE $y^{(4)} + 4y'' = f(x)$.

a) Obtain the homogeneous solution.

b) For each case given below, give the general form of the particular solution using the method of undetermined coefficients. Do not evaluate the coefficients.

1. $f(x) = 5 + 8x^3$
2. $f(x) = x\sin(5x)$
3. $f(x) = \cos(2x)$
4. $f(x) = 2\sin^2(x)$

Problem 3: Consider the boundary value problem (BVP):

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + \lambda y = 0, \quad 1 < t < e, \quad y(1) = \frac{dy}{dt}(e) = 0.$$

a) Find all positive values of $\lambda \in (0, \infty)$ such that the BVP has a nontrivial solution.

b) Determine a nontrivial solution corresponding to each of the values of λ found in part a).

c) For what values of $\lambda \in (0, \infty)$ does the BVP admit a unique solution? What is that solution.

Problem 4: Consider the ODE

$$t^2y'' + ty' + \lambda y = 0, t > 0. \tag{1}$$

a) For $\lambda = 4$, find two solutions of (1), calculate their Wronskian and thus deduce that they form a fundamental set of solutions.

b) Verify your answer for the Wronskian using Abel's Theorem and a convenient initial condition from part a).

c) Solve the eigenvalue problem (1) on $1 < t < e$, subject to $y(1) = y'(e) = 0$, that is find all values of λ such that the boundary value problem has a nontrivial solution and in that case determine the latter.

Problem 5: Find the general solution of the system

$$\begin{aligned} x_1' &= 4x_1 + x_2 + x_3 \\ x_2' &= x_1 + 4x_2 + x_3 \\ x_3' &= x_1 + x_2 + 4x_3. \end{aligned}$$

Problem 6: Consider the differential equation

$$x^2y'' + xy' - 9y = 0, \quad x > 0.$$

We know that $y_1(x) = x^3$ is a solution to this ODE. Use the method of reduction of order to find a second solution y_2. Show that y_1 and y_2 form a fundamental set of solutions of the differential equation (that is, show that they are linearly independent).

Problem 7: Find the critical value of λ in which bifurcations occur in the system

$$\dot{x} = 1 + \lambda x + x^2.$$

Sketch the phase portrait for various values of λ and the bifurcation diagram. Classify the bifurcation.

MAT 303: Calculus IV with Applications
Fall 2016

Practice problems for Midterm 2
Solutions

Problem 1:

a) Find the general solution of the ODE $y'' + 4y = 4\cos(2t)$.

b) Make a sketch of y_p vs. t, where $y_p(t)$ denotes the particular solution found in part a). What is the pseudo-period of the oscillation and the time varying amplitude?

Solution. The characteristic equation for the homogeneous ODE is $r^2 + 4 = 0$, which has solutions $r = \pm 2i$. The homogeneous solution is $y_h(t) = C_1\cos(2t) + C_2\sin(2t)$. We look for particular solutions $y_p(t) = t(A\cos(2t) + B\sin(2t))$. We compute

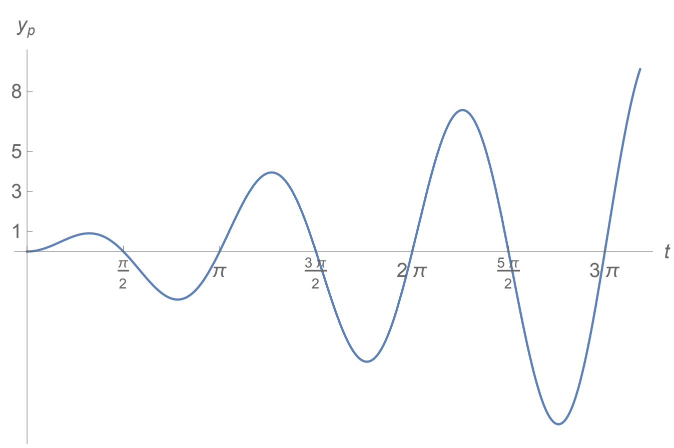
$$y_p'(t) = A\cos(2t) + B\sin(2t) + 2t(-A\sin(2t) + B\cos(2t))$$

$$y_p''(t) = -4A\sin(2t) + 4B\cos(2t) - 4t(A\cos(2t) + B\sin(2t)).$$

Plugging these in the initial ODE we find

$$y_p'' + 4y_p = -4A\sin(2t) + 4B\cos(2t) = 4\cos(2t),$$

which gives $A = 0$ and $B = 1$. Hence a particular solution is $y_p(t) = t\sin(2t)$. The amplitude is $A(t) = t$, the frequency is $\omega = 2$, so the period is $T = \dfrac{2\pi}{\omega} = \pi$. The general solution is $y = y_h + y_p = C_1\cos(2t) + (C_2 + t)\sin(2t)$.



$\square$

Problem 2: Consider the 4th order ODE $y^{(4)} + 4y'' = f(x)$.

a) Obtain the homogeneous solution.

b) For each case given below, give the general form of the particular solution using the method of undetermined coefficients. Do not evaluate the coefficients.

1. $f(x) = 5 + 8x^3$
2. $f(x) = x\sin(5x)$
3. $f(x) = \cos(2x)$
4. $f(x) = 2\sin^2(x)$

SOLUTION.

a) The characteristic equation is $r^4 + 4r^2 = 0$, which has roots $r = 0$ (repeated root of order 2) and $r = \pm 2i$. The homogeneous solution is

$$y_h(x) = C_1 + C_2 x + C_3\cos(2x) + C_4\sin(2x).$$

b)
1. $y_p = x^2(a_0 + a_1 x + a_2 x^2 + a_3 x^3)$
2. $y_p = (a_0 + a_1 x)\cos(5x) + (b_0 + b_1 x)\sin(5x)$
3. $y_p = x(A\cos(2x) + B\sin(2x))$
4. Note that $2\sin^2(x) = 1 - \cos(2x)$, hence $y_p = a_0 x^2 + x(A\cos(2x) + B\sin(2x))$.

□

Problem 3: Consider the boundary value problem (BVP):

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + \lambda y = 0, \quad 1 < t < e, \quad y(1) = \frac{dy}{dt}(e) = 0.$$

a) Find all positive values of $\lambda \in (0, \infty)$ such that the BVP has a nontrivial solution.

b) Determine a nontrivial solution corresponding to each of the values of λ found in part a).

c) For what values of $\lambda \in (0, \infty)$ does the BVP admit a unique solution? What is that solution.

SOLUTION. We make the change of variables $x = \ln(t)$. Note that $\ln(1) = 0$ and $\ln(e) = 1$. The equivalent BVP is

$$\frac{d^2y}{dx^2} + \lambda y = 0, \quad 0 < x < 1, \quad y(0) = y'(1) = 0.$$

a) Let $\lambda > 0$. The general solution is $y = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$. We have $y(0) = c_1 = 0$ and $y'(1) = -c_2\sqrt{\lambda}\cos(\sqrt{\lambda}) = 0$. This gives $\sqrt{\lambda} = \dfrac{(2n-1)\pi}{2}$, $n = 1, 2, \ldots$.

b) $y = c_2 \sin\left(\frac{(2n-1)\pi x}{2}\right) = c_2 \sin\left(\frac{(2n-1)\pi}{2}\ln(t)\right)$, $n = 1, 2, \ldots$.

c) For $\lambda \neq \frac{(2n-1)\pi}{2}$, $n = 1, 2, \ldots$, the unique solution is $y = 0$.

$\square$

Problem 4: Consider the ODE

$$t^2 y'' + t y' + \lambda y = 0, t > 0. \tag{1}$$

a) For $\lambda = 4$, find two solutions of (1), calculate their Wronskian and thus deduce that they form a fundamental set of solutions.

b) Verify your answer for the Wronskian using Abel's Theorem and a convenient initial condition from part a).

c) Solve the eigenvalue problem (1) on $1 < t < e$, subject to $y(1) = y'(e) = 0$, that is find all values of λ such that the boundary value problem has a nontrivial solution and in that case determine the latter.

Solution.

a) For $\lambda = 4$, the equation becomes $t^2 y'' + t y' + \lambda y = 0, t > 0$. We make a change of variables $x = \ln(t)$ and obtain the ODE $y'' + 4y = 0$. The fundamental solutions are $y_1 = \cos(2x)$ and $y_2 = \sin(2x)$ or $y_1(t) = \cos(2\ln(t))$ and $y_2(t) = \sin(2\ln(t))$. By differentiating with respect to t, we find $y_1'(t) = -\frac{2}{t}\sin(2\ln(t))$ and $y_2'(t) = \frac{2}{t}\cos(2\ln(t))$. For $t > 0$, the Wronskian is

$$W(y_1, y_2) = \frac{2}{t}\cos^2(2\ln(t)) + \frac{2}{t}\sin^2(2\ln(t)) = \frac{2}{t}.$$

Clearly $W \neq 0$ so y_1, and y_2 are linearly independent and form a fundamental set of solutions.

b) We put the original ODE in the form

$$y'' + \frac{1}{t}y' + \frac{4}{t^2}y = 0, \ t > 0.$$

By Abel's theorem we get $W = C\exp\left(-\int \frac{1}{t}dt\right) = C\exp(-\ln(t)) = \frac{C}{t}$. From part a), $W(1) = 2$, which gives $C = 2$.

c) By making a change of variables $x = \ln(t)$, we have to solve the eigenvalue problem

$$y'' + 4y = 0, \quad y(0) = 0, \quad y'(1) = 0.$$

We find eigenvalues $\lambda_n = \left(\frac{(2n-1)\pi}{2}\right)^2$, for $n = 1, 2, \ldots$ and corresponding eigenfunctions $y_n(x) = \sin\left(\frac{(2n-1)\pi x}{2}\right)$, $n = 1, 2, \ldots$.

$\square$

Problem 5: Find the general solution of the system

$$\begin{aligned} x_1' &= 4x_1 + x_2 + x_3 \\ x_2' &= x_1 + 4x_2 + x_3 \\ x_3' &= x_1 + x_2 + 4x_3. \end{aligned}$$

Solution. The system can be written as $X' = AX$, where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}.$$

The characteristic polynomial of A is

$$\begin{aligned} \begin{vmatrix} 4-\lambda & 1 & 1 \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} &= \begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} = (6-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} \\ &= (6-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (6-\lambda)(3-\lambda)^2. \end{aligned}$$

The eigenvalues are $\lambda_1 = 6$ (of algebraic multiplicity 1) and $\lambda_2 = 3$ (of algebraic multiplicity 2). The eigenvectors for the eigenvalue $\lambda_2 = 3$ are given by the equation $(A - 3I_3)v = 0$. We write

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and obtain $v_1 + v_2 + v_3 = 0$, hence $v_3 = -v_1 - v_2$. Thus

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ -v_1 - v_2 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + v_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

The geometric multiplicity is 2. Two linearly independent eigenvectors of $\lambda_2 = 3$ are

$$w_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \text{and} \quad w_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

The eigenvectors for $\lambda_1 = 6$ are solutions of

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We find the following system of equations:

$$\begin{aligned} -2v_1 + v_2 + v_3 &= 0 \\ v_1 - 2v_2 + v_3 &= 0 \\ v_1 + v_2 - 2v_3 &= 0 \end{aligned}$$

The third equation is redundant. Subtracting the second equation from the first we get $-3v_1 + 3v_2 = 0$, so $v_1 = v_2$. Substituting this in the first equation yields $v_3 = v_1$. It follows that

$$w_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

is an eigenvector for $\lambda_1 = 6$. The general solution for the given system of equations is

$$x(t) = c_1 e^{3t} w_1 + c_2 e^{3t} w_2 + c_3 w_3 e^{6t}.$$

□

Problem 6: Consider the differential equation

$$x^2 y'' + x y' - 9y = 0, \quad x > 0.$$

We know that $y_1(x) = x^3$ is a solution to this ODE. Use the method of reduction of order to find a second solution y_2. Show that y_1 and y_2 are linearly independent.

Solution. Substitute $y = vx^3$ in the given equation and simplify. We get the differential equation $xv'' + 7v' = 0$, which is separable. We write $\frac{v''}{v'} = -\frac{7}{x}$ and integrate. This gives $\ln v' = -7\ln x + \ln A$, which yields $v' = \frac{A}{x^7}$ and finally $v(x) = -\frac{A}{6x^6} + B$. With $A = -6$ and $B = 0$ we get $v(x) = \frac{1}{x^6}$, so $y_2(x) = \frac{1}{x^3}$.

To show linear independence, assume that $ax^3 + b\frac{1}{x^3} = 0$ for all $x > 0$. This is equivalent to $ax^6 + b = 0$. When $x = 1$ we get $a + b = 0$. When $x = 2$ we get $64a + b = 0$, so the only values of a and b for which both conditions are satisfied is $a = b = 0$. In conclusion, y_1 and y_2 are two linearly independent solutions. □

Problem 7: Find the critical value of λ in which bifurcations occur in the system

$$\dot{x} = 1 + \lambda x + x^2.$$

Sketch the phase portrait for various values of λ and the bifurcation diagram. Classify the bifurcation.

Solution. The critical points c_1 and c_2 of the system verify $1 + \lambda x + x^2 = 0$, so

$$c_{1,2} = \frac{-\lambda \pm \sqrt{\lambda^2 - 4}}{2}.$$

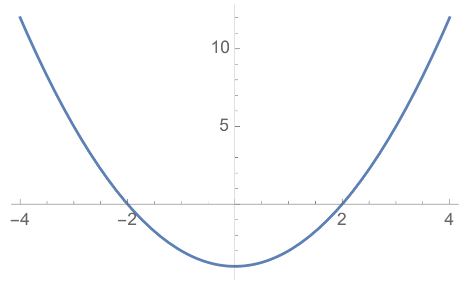


Figure 1: The graph of $\lambda^2 - 4$.

We have three cases to consider. First, suppose $\lambda^2 = 4$. Then $\lambda = \pm 2$. For $\lambda = 2$, the system has one critical point $c = -\dfrac{\lambda}{2} = -1$, which is semi-stable, since $f(x) = 1 + 2x + x^2 = (1+x)^2 \geq 0$ for all x. Similarly, for $\lambda = -2$, the system has one critical point $c = -\dfrac{\lambda}{2} = 1$, which is semi-stable, since $f(x) = 1 - 2x + x^2 = (1-x)^2 \geq 0$ for all x.

If $\lambda^2 < 4$, then $-2 < \lambda < 2$ and there are no critical points.

If $\lambda^2 > 4$, then $\lambda > 2$ or $\lambda < -2$. The system has two distinct critical points:

$$\begin{aligned} c_1 &= \frac{-\lambda - \sqrt{\lambda^2 - 4}}{2} \quad \text{(stable)} \\ c_2 &= \frac{-\lambda + \sqrt{\lambda^2 - 4}}{2} \quad \text{(unstable)} \end{aligned}$$

The function $f(x) = 1 + \lambda x + x^2$ is positive when $x < c_1$ or $x > c_2$, and negative when $c_1 < x < c_2$.

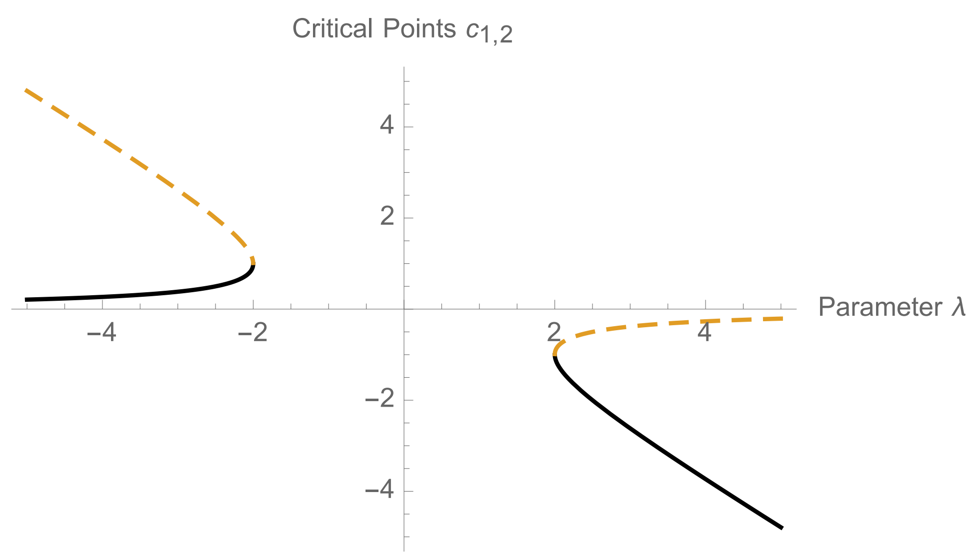


Figure 2: The bifurcation diagram.

The system undergoes a saddle-node bifurcation. $\square$

Midterm #2 — April 12, 2013, 10:00 to 10:53 AM

Name: ______________________________

Circle your recitation:

R01 (Claudio · Fri) R02 (Xuan · Wed) R03 (Claudio · Mon)

- **You have a maximum of 53 minutes.** This is a closed-book, closed-notes exam. No calculators or other electronic aids are allowed.
- Read each question carefully. Show your work and justify your answers for full credit. You do not need to simplify your answers unless instructed to do so.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

Grading

1	/30
2	/15
3	/20
4	/15
5	/20
Total	/100

1. (*30 points*) Find the general solution to each of the following differential equations:

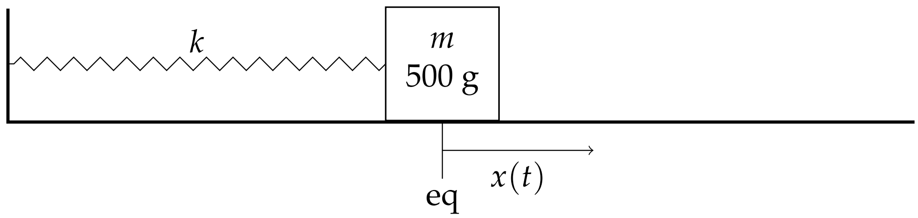
(a) (*10 points*) $y'' - 4y' + 3y = 0$

(b) (*10 points*) $y'' - 4y' + 4y = 0$

(c) (*10 points*) $y'' - 4y' + 5y = 0$

2. (*15 points*) Two solutions to the DE $y'' - 3y' - 10y = 0$ are $y_1 = e^{5x}$ and $y_2 = e^{-2x}$. Find a solution to this DE satisfying the initial conditions $y(0) = 7$ and $y'(0) = 7$.

3. (*20 points*) A 500-gram test mass m is attached to a spring of unknown spring constant k and allowed to settle into its equilibrium position, as shown:



The mass is struck sharply at time $t = 0$, and the resulting displacement measured to be

$$x(t) = 0.25e^{-3t}\sin 5t \qquad (x \text{ in meters, } t \text{ in seconds})$$

(a) (*5 points*) Is this system underdamped, critically damped, or overdamped? Explain.

(b) (*10 points*) Find the spring constant k of the spring and the damping constant c resulting from natural friction in the system. Include appropriate units.

(c) (*5 points*) Suppose we then apply a periodic force $f(t) = 10\cos\omega t$ (in newtons) to the spring-mass system, where we may vary the forcing frequency ω. Find the value of ω at which the system exhibits practical resonance, or explain why it does not occur.

4. (*15 points*) Find a particular solution to the nonhomogeneous DE

$$y^{(3)} - 6y'' + 12y' = 40e^{2x} - 24.$$

5. (*20 points*) Let $y_1(x) = x$ and $y_2(x) = x^4$.

(a) (*10 points*) Show that y_1 and y_2 are both solutions to the DE $x^2y'' - 4xy' + 4y = 0$.

(b) (*5 points*) Show that y_1 and y_2 are linearly independent functions on the entire real line.

(c) (*5 points*) Find the general solution to the nonhomogeneous DE

$$x^2y'' - 4xy' + 4y = 6x^4 - 9x.$$

Midterm #2 — April 12, 2013, 10:00 to 10:53 AM

Name: Solution Key

Circle your recitation:

R01 (Claudio · Fri) R02 (Xuan · Wed) R03 (Claudio · Mon)

- **You have a maximum of 53 minutes.** This is a closed-book, closed-notes exam. No calculators or other electronic aids are allowed.
- Read each question carefully. Show your work and justify your answers for full credit. You do not need to simplify your answers unless instructed to do so.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

Grading

1	/30
2	/15
3	/20
4	/15
5	/20
Total	/100

1. (*30 points*) Find the general solution to each of the following differential equations:

(a) (*10 points*) $y'' - 4y' + 3y = 0$

Solution: The characteristic equation of this homogeneous, constant-coefficient DE is $r^2 - 4r + 3 = 0$, which factors as $(r-1)(r-3) = 0$. Therefore, the roots are $r = 1$ and $r = 3$, so the general solution is $y = c_1 e^x + c_2 e^{3x}$.

(b) (*10 points*) $y'' - 4y' + 4y = 0$

Solution: The characteristic equation of this DE is $r^2 - 4r + 4 = 0$, which factors as $(r-2)^2 = 0$. Therefore, the only root is a double root $r = 2$, so the general solution is $y = c_1 e^{2x} + c_2 x e^{2x}$.

(c) (*10 points*) $y'' - 4y' + 5y = 0$

Solution: The characteristic equation of this DE is $r^2 - 4r + 5 = 0$, which has discriminant $4^2 - 4(1)(5) = -4$, and therefore has complex roots. By the quadratic formula, these roots are $r = \frac{1}{2}(4 \pm \sqrt{-4}) = 2 \pm i$, so the general solution is $y = c_1 e^{2x} \cos x + c_2 e^{2x} \sin x$.

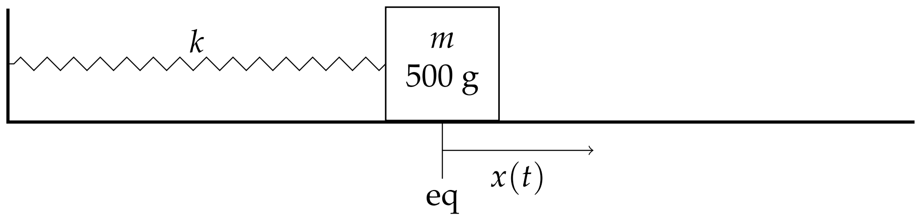
2. (*15 points*) Two solutions to the DE $y'' - 3y' - 10y = 0$ are $y_1 = e^{5x}$ and $y_2 = e^{-2x}$. Find a solution to this DE satisfying the initial conditions $y(0) = 7$ and $y'(0) = 7$.

Solution: Since this DE is second-order, homogeneous, and linear, and since y_1 and y_2 are clearly linearly independent, $y(x) = c_1 e^{5x} + c_2 e^{-2x}$ is the general solution to this DE. We then match it and its derivative $y'(x) = 5c_1 e^{5x} - 2c_2 e^{-2x}$ to the initial conditions at $x = 0$. We obtain the linear system

$$c_1 + c_2 = 7, \qquad 5c_1 - 2c_2 = 7.$$

Then $c_2 = 7 - c_1$, so $5c_1 - 14 + 2c_1 = 7$, and $c_1 = 21/7 = 3$. Hence, $c_2 = 7 - 3 = 4$, so $y(x) = 3e^{5x} + 4e^{-2x}$ solves this IVP.

3. (*20 points*) A 500-gram test mass m is attached to a spring of unknown spring constant k and allowed to settle into its equilibrium position, as shown:



The mass is struck sharply at time $t = 0$, and the resulting displacement measured to be

$$x(t) = 0.25e^{-3t} \sin 5t \qquad (x \text{ in meters, } t \text{ in seconds})$$

(a) (*5 points*) Is this system underdamped, critically damped, or overdamped? Explain.

Solution: Since the displacement of the mass in this free system takes the form of a function $e^{-at} \sin bt$, which contains a sinusoidal factor, the system is underdamped.

(b) (*10 points*) Find the spring constant k of the spring and the damping constant c resulting from natural friction in the system. Include appropriate units.

Solution: We reconstruct the differential equation $mx'' + cx' + kx = 0$ governing the motion. First, since the motion is a multiple of $e^{-3t} \sin 5t$, this DE has complex roots $r = -3 \pm 5i$. Therefore, its characteristic equation is a multiple of $(r + 3 - 5i)(r + 3 + 5i) = (r + 3)^2 + 5^2 = r^2 + 6r + 34 = 0$. The coefficient on the r^2-term must be $m = 1/2$ (in mks units), though, so we multiply through by this to obtain $\frac{1}{2}r^2 + 3r + 17$. Hence, $c = 3$ N-s/m, and $k = 17$ N/m.

Many students tried to use the relation $\omega_0 = \sqrt{k/m}$ to determine k, but since damping is present in the system, ω_0 does not come directly from the frequency of the sine factor in the displacement, and therefore is not 5 rad/s. The frequency in the displacement is instead ω_1, and $\omega_0^2 = \omega_1^2 + p^2$, where $p = 3$ is the exponent in the displacement's exponential factor e^{-pt}.

(c) (*5 points*) Suppose we then apply a periodic force $f(t) = 10\cos\omega t$ (in newtons) to the spring-mass system, where we may vary the forcing frequency ω. Find the value of ω at which the system exhibits practical resonance, or explain why it does not occur.

Solution: We recall that, as a function of ω and the parameters m, c, k, and F_0, the amplitude of the steady-state solution is

$$C(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}.$$

(We can arrive at this solution by assuming a steady periodic displacement of the form $A\cos\omega t + B\sin\omega t$; see pp. 219–220 for details.) By minimizing the denominator, we can show that, if $c^2 < 2km$, $C(\omega)$ has a maximum at $\omega_m = \sqrt{\dfrac{2km - c^2}{2m^2}}$. Since $c^2 = 3^2 = 9$ and $2km = 2(17)(\frac{1}{2}) = 17$, c is small enough for practical resonance to occur at

$$\omega_m = \sqrt{\frac{2km - c^2}{2m^2}} = \sqrt{\frac{17 - 9}{2(1/2)^2}} = \sqrt{16} = 4\frac{\text{rad}}{\text{s}}.$$

4. (*15 points*) Find a particular solution to the nonhomogeneous DE

$$y^{(3)} - 6y'' + 12y' = 40e^{2x} - 24.$$

Solution: From the form of the forcing term $40e^{2x} - 24$, we expect the particular solution to contain terms corresponding to the roots $r = 2$ (from the e^{2x}) and $r = 0$ (from the constant term). Before we construct our guess, though, we must also check whether these roots also contribute to the complementary solution. The characteristic equation for the associated homogeneous DE is

$$r^3 - 6r^2 + 12r = r(r^2 - 6r + 12) = 0,$$

which then has $r = 0$ as a single root. We check for overlap with $r = 2$: since $(2)^2 - 6(2) + 12 = 4 - 12 + 12 = 4 \neq 0$, 2 is not a root. Hence, to avoid overlaps on $r = 0$, we guess a particular solution of the form

$$y = Ae^{2x} + Bx.$$

(Notice we have shifted only the constant term by a power of x, and not the non-overlapping e^{2x} term.) Then $y' = 2Ae^{2x} + B$, $y'' = 4Ae^{2x}$, and $y''' = 8Ae^{2x}$, which we plug into the nonhomogeneous DE:

$$8Ae^{2x} - 6(4Ae^{2x}) + 12(2Ae^{2x} + B) = 40e^{2x} - 24.$$

Simplifying, $8Ae^{2x} + 12B = 40e^{2x} - 24$, so $8A = 40$ and $12B = -24$. Then $A = 5$ and $B = -2$, so a particular solution is

$$y = 5e^{2x} - 2x.$$

5. (*20 points*) Let $y_1(x) = x$ and $y_2(x) = x^4$.

(a) (*10 points*) Show that y_1 and y_2 are both solutions to the DE $x^2y'' - 4xy' + 4y = 0$.

Solution: We compute the first and second derivatives of these functions:

$$y_1' = 1, \quad y_1'' = 0, \quad y_2' = 4x^3, \quad y_2'' = 12x^2.$$

Plugging these into the homogeneous DE,

$$\begin{aligned} x^2y_1'' - 4xy_1' + 4y_1 &= x^2(0) - 4x(1) + 4(x) = -4x + 4x = 0, \\ x^2y_2'' - 4xy_2' + 4y_2 &= x^2(12x^2) - 4x(4x^2) + 4(x^4) = (12 - 16 + 4)x^4 = 0. \end{aligned}$$

(b) (*5 points*) Show that y_1 and y_2 are linearly independent functions on the entire real line.

Solution: We compute the Wronskian $W(x)$ of y_1 and y_2:

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^4 \\ 1 & 4x^3 \end{vmatrix} = 4x^4 - x^4 = 3x^4.$$

Since this function is nonzero on the real line (in fact, everywhere but $x = 0$), these functions are linearly independent.

(c) (*5 points*) Find the general solution to the nonhomogeneous DE

$$x^2y'' - 4xy' + 4y = 6x^4 - 9x.$$

Solution: We use variation of parameters to solve this nonhomogeneous DE. First, we write the *normalized* version of the DE, dividing by x^2 to obtain

$$y'' - \frac{4}{x}y' + \frac{4}{x^2}y = 6x^2 - 9x^{-1}.$$

Then $f(x) = 6x^2 - 9x^{-1}$, which we include in the formula for variation of parameters:

$$\begin{aligned}
y &= -y_1 \int \frac{y_2(x)f(x)}{W(x)}\,dx + y_2 \int \frac{y_1(x)f(x)}{W(x)}\,dx \\
&= -x \int \frac{x^4(6x^2 - 9x^{-1})}{3x^4}\,dx + x^4 \int \frac{x(6x^2 - 9x^{-1})}{3x^4}\,dx \\
&= -x \int 2x^2 - \frac{3}{x}\,dx + x^4 \int \frac{2}{x} - \frac{3}{x^4}\,dx \\
&= -x\left(\frac{2}{3}x^3 - 3\ln x\right) + x^4(2\ln x + x^{-3}) \\
&= 3x\ln x - \frac{2}{3}x^4 + 2x^4\ln x + x.
\end{aligned}$$

The general solution to the homogeneous equation is $y_c = c_1y_1 + c_2y_2 = c_1x + c_2x^4$, so adding this to the particular solution above produces the general solution to the nonhomogeneous equation:

$$y = c_1x + c_2x^4 + 3x\ln x - \frac{2}{3}x^4 + 2x^4\ln x + x = C_1x + C_2x^4 + 3x\ln x + 2x^4\ln x,$$

where $C_1 = c_1 + 1$ and $C_2 = c_2 - \frac{2}{3}$ are new arbitrary constants that absorb the y_1 and y_2 terms coming from the integration. (We also observe that if we had included the constants of integration, we would have obtained the general solution directly from the variation of parameters formula.)

MAT 303 - HW10 - Additional Problems
due Friday, December 2
All homework problems are mandatory!

Exercise 1. Consider the system of differential equations $x' = Ax$, where $A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$ and λ is an arbitrary real number.

a) Compute A^2 and A^3. Use an inductive argument to show that $A^n = \begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}$.

b) Determine the exponential e^{At} using the computations in part a).

c) Find the exponential e^{At} by first writing A as a sum of a diagonal matrix and a nilpotent matrix $A = \lambda I_3 + C$, then computing e^{At} as a product of two exponential matrices $e^{\lambda I_3 t} e^{Ct}$.

d) Find the general solution of the system $x' = Ax$ using the exponential e^{At}.

e) Find the eigenvalues and eigenvectors of the matrix A.

f) Find the general solution of the system $x' = Ax$ using part e), then compare it to the answer you got for part d).

Exercise 2. Consider the system of differential equations $x' = Bx$, where $B = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$ and λ is an arbitrary real number.

a) Compute B^2, B^3. Use an inductive argument to show that $B^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \frac{n(n-1)}{2}\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}$.

b) Determine the exponential e^{Bt} using part a).

c) Find the exponential e^{Bt}, using a different approach: write B as a sum of a scalar multiple of the identity matrix and a nilpotent matrix $B = \lambda I_3 + C$. Then use the fact that $e^{Bt} = e^{\lambda I_3 t} e^{Ct}$.

d) Find the general solution of the system $x' = Bx$ using the exponential matrix e^{Bt}.

e) Find the eigenvalues and eigenvectors of the matrix B.

f) Find the general solution of the system $x' = Bx$ using part e), then compare it to the answer you got for part d).

MAT 303: Calculus IV with Applications
Fall 2016

Lecture Notes - Friday, December 2

Let A be an $n \times n$ matrix. Let λ_1 be an eigenvalue of A. Let's recall first some definitions:

Algebraic Multiplicity: The algebraic multiplicity of λ_1 is the number of times the factor $\lambda - \lambda_1$ appears in the characteristic polynomial $p(\lambda) = \det(A - \lambda I_n)$.

Geometric Multiplicity: The geometric multiplicity of λ_1 is the maximum number of linearly independent eigenvectors corresponding to the eigenvalue λ_1.

Defect: The defect of the eigenvalue λ_1 is equal to the Algebraic Multiplicity of λ_1 minus its Geometric Multiplicity.

Invertible: An $n \times n$ square matrix S is called invertible if there exists an $n \times n$ matrix S^{-1}, called the inverse of S, such that $S^{-1}S = SS^{-1} = I_n$.

It can be shown that S is invertible if and only if the determinant of S is different from 0. It can also be shown that if S is invertible, then its inverse S^{-1} is unique.

The inverse of a 2×2 matrix: If $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\det(S) = ad - bc \neq 0$, then

$$S^{-1} = \frac{1}{\det(S)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

DiagonalizationTheorem : Let A be an $n \times n$ matrix. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A (not necessarily distinct). Assume that all eigenvalues of A have algebraic multiplicity equal to their geometric multiplicity. Then A has n linearly independent eigenvectors $v_1, \ldots, v_n$. If we denote by S the matrix whose columns are the eigenvectors $v_1, \ldots, v_n$, then S is an invertible matrix, and

$$S^{-1}AS = D,$$

where $D = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \lambda_n \end{pmatrix}$ is a diagonal matrix whose diagonal entries are precisely the eigenvalues of A. We say that A is a diagonalizable matrix.

Remark: It can be shown that if A has repeated eigenvalues with geometric multiplicity different from the algebraic multiplicity (so A has fewer than n eigenvectors!), then A cannot be diagonalized. It is still possible however to transform A into a nearly diagonal matrix called the Jordan canonical form.

Remark: Matrix multiplication is not commutative in general! Hence $S^{-1}AS$ is not the same as $AS^{-1}S = AI_n = A$!

Jordan Block: Let λ be an eigenvalue of some matrix A. An $m \times m$ matrix B is called a *Jordan block of size* m corresponding to λ if

$$B = \begin{pmatrix} \lambda & 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda & 1 & 0 \\ 0 & 0 & \dots & 0 & \lambda & 1 \\ 0 & 0 & \dots & 0 & 0 & \lambda \end{pmatrix}, \tag{1}$$

that is, the entries on the main diagonal of B are all equal to λ, and the entries right above the main diagonal are equal to 1. All the other entries are equal to 0.

The exponential matrix of a Jordan Block: If B has the form from Equation (1), then

$$e^{Bt} = e^{\lambda t} \begin{pmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{3!} & \dots & \frac{t^{m-1}}{(m-1)!} \\ 0 & 1 & t & \frac{t^2}{2} & \dots & \vdots \\ \dots & \dots & \dots & \dots & \dots & \vdots \\ 0 & 0 & \dots & 1 & t & \frac{t^2}{2} \\ 0 & 0 & \dots & 0 & 1 & t \\ 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}$$

Proof. To make the computations easier, let's give the proof for $m = 4$. The Jordan block can be written as $B = \lambda I_4 + C$, where $C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is a nilpotent matrix. The matrix C has the property that $C^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $C^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $C^4 = 0_4$.

The matrix exponential e^{Ct} can be computed as

$$e^{Ct} = I_4 + \sum_{n=1}^{\infty} \frac{C^n t^n}{n!} = I_4 + Ct + C^2\frac{t^2}{2} + C^3\frac{t^3}{3!} = \begin{pmatrix} 1 & t & t^2/2 & t^3/3! \\ 0 & 1 & t & t^2/2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrix $D = \lambda I_4$ has the property that $D^n = I_4 \lambda^n$ for every $n > 0$. Therefore

$$e^{Dt} = I_4 + \sum_{n=1}^{\infty} \frac{D^n t^n}{n!} = I_4 + \sum_{n=1}^{\infty} \frac{I_4 \lambda^n t^n}{n!} = I_4 (1 + \sum_{n=1}^{\infty} \frac{\lambda^n t^n}{n!}) = I_4 e^{\lambda t}$$

In conclusion $e^{Bt} = e^{\lambda I_4 t + Ct} = e^{Dt}e^{Ct} = e^{Dt}I_4 e^{\lambda t} = e^{Dt}e^{\lambda t} = e^{\lambda t}\begin{pmatrix} 1 & t & t^2/2 & t^3/3! \\ 0 & 1 & t & t^2/2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

□

Examples:

a) If $B = (\lambda)$ is a Jordan block of size 1, then $e^{Bt} = e^{\lambda t}(1) = (e^{\lambda t})$.

a) If $B = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, then $e^{Bt} = e^{\lambda t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$.

c) $B = \begin{pmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix}$, $e^{Bt} = e^{5t}\begin{pmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{3!} \\ 0 & 1 & t & \frac{t^2}{2} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{5t} & te^{5t} & t^2e^{5t}/2 & t^3e^{5t}/6 \\ 0 & e^{5t} & te^{5t} & t^2e^{5t}/2 \\ 0 & 0 & e^{5t} & te^{5t} \\ 0 & 0 & 0 & e^{5t} \end{pmatrix}$.

Jordan Canonical Form Theorem :

Let A be any $n \times n$ matrix. There exists an invertible matrix S such that

$$S^{-1}AS = J, \quad \text{where} \quad J = \begin{pmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \dots & \\ 0 & & & J_s \end{pmatrix} \tag{2}$$

J is a block diagonal matrix, where $J_1, J_2, \dots, J_s$ are Jordan blocks corresponding to the eigenvalues of A, as in Equation (1). The columns of the matrix S are chosen as follows: Each column of S that corresponds to the first column of each Jordan block J_k is an eigenvector v_k of A. The rest of the columns are generalized eigenvectors.

Remarks: An eigenvalue can produce several Jordan blocks (the number of Jordan blocks for one eigenvalue is equal to the geometric multiplicity of the eigenvalue). The size of one Jordan block is equal to the length of a chain of generalized eigenvectors.

Examples: Let A be a 2×2 matrix.

a) If A has a repeated eigenvalue λ_1 with algebraic and geometric multiplicity 2, then its Jordan canonical form is the diagonal matrix $J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$. In this case, J has two Jordan blocks, each of size 1.

b) If A has a repeated eigenvalue λ_1 with algebraic multiplicity 2, and geometric multiplicity 1 then its Jordan canonical form is the block diagonal matrix $J = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}$. The Jordan matrix has a single Jordan block, of size 2.

Examples: Let A be a 7×7 matrix. Assume that A has eigenvalues λ (with algebraic multiplicity 2 and geometric multiplicity 1) and μ (with algebraic multiplicity 5 and geometric multiplicity 2). Matrix A has 3 linearly independent eigenvectors v_1, w_1, u_1: one corresponding to the eigenvalue λ, and two corresponding to the eigenvalue μ. Assume that after doing some computations, we found 3 chains of generalized eigenvectors

$$\begin{aligned}
\{v_1, v_2\}, &\quad where \quad (A-\lambda I_7)v_1 = 0, (A-\lambda I_7)v_2 = v_1\\
\{w_1, w_2\}, &\quad where \quad (A-\mu I_7)w_1 = 0, (A-\mu I_7)w_2 = w_1\\
\{u_1, u_2, u_3\}, &\quad where \quad (A-\mu I_7)u_1 = 0, (A-\mu I_7)u_2 = u_1, (A-\mu I_7)u_3 = u_2
\end{aligned}$$

If we let S be the 7×7 matrix with columns $S = [v_1, v_2, w_1, w_2, u_1, u_2, u_3]$, then

$$S^{-1}AS = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & \lambda & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \mu & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & \mu & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \mu & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & \mu & 1\\ 0 & 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix} \tag{3}$$

This is the Jordan canonical form of A. It has 3 Jordan blocks, one corresponding to the eigenvalue λ, and two corresponding to the eigenvalue μ. The canonical form is unique up to permutations of the Jordan blocks. If we write the columns of S is a different order, for example $S = [v_1, v_2, u_1, u_2, u_3, w_1, w_2]$, we get

$$S^{-1}AS = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & \lambda & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \mu & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & \mu & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & \mu & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \mu & 1\\ 0 & 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix} \tag{4}$$

Applications of the Jordan canonical form: Assume that $S^{-1}AS = J$, where J is the Jordan canonical form.

1. Exponential of a Jordan matrix: If $J_1, \ldots J_s$ are the Jordan blocks of J, then

$$e^{Jt} = \begin{pmatrix} e^{J_1 t} & & & 0\\ & e^{J_2 t} & & \\ & & \cdots & \\ 0 & & & e^{J_s t} \end{pmatrix}.$$

The exponential matrix of each block can then be computed as in Equation (1).

2. Computing powers of the matrix A: It can be easily seen that $A = SJS^{-1}$ and so $A^n = S^{-1}J^nS$ for every $n > 0$.

3. Finding the exponential matrix of A: $e^{At} = Se^{Jt}S^{-1}$. Indeed,

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} = \sum_{n=0}^{\infty} \frac{(SJ^nS^{-1})t^n}{n!} = S(\sum_{n=0}^{\infty} \frac{J^n t^n}{n!})S^{-1} = Se^{Jt}S^{-1}.$$

4. Solving the system of differential equations $x' = Ax$: The general solution of the system $x' = Ax$ is given by $x(t) = e^{At}c$, where c is a vector of coefficients. If J is the Jordan canonical form of A, then the solution can also be written as:

$$x(t) = Se^{Jt}S^{-1}c = Se^{Jt}\overline{c},$$

where $\overline{c} = S^{-1}c$ is a new vector of random constants. The matrix $\Phi(t) = Se^{Jt}$ is therefore a fundamental matrix of the system $x' = Ax$. When solving the system $x' = Ax$ we prefer to work with the fundamental matrix $\Phi(t)$ because it does not require finding the inverse of the matrix S, unlike e^{At}.

5. Reducing a system of differential equations Consider the system of differential equations

$$x' = Ax. \tag{5}$$

We do the variable substitution $x = Sy$, which gives $x' = Sy'$ and obtain the equivalent system

$$Sy' = ASy$$

After multiplication by S^{-1}, we end up with

$$y' = S^{-1}ASy = Jy \tag{6}$$

A fundamental matrix of the system $y' = Jy$ is e^{Jt}. The general solution is then $y(t) = e^{Jt}c$. Since the relation between Systems (5) and (6) is given by the substitution $x = Sy$, it follows that the general solution of System (5) is $x(t) = Sy(t) = Se^{Jt}c$. Therefore $\Phi(t) = Se^{Jt}$ is a fundamental matrix for System (5).

Solving a nonhomogeneous system $x' = Ax + f(t)$: As proven in class, using integrating factors, the general solution is given by the Variation of Parameters formula:

$$x(t) = e^{At}\int e^{-At}f(t)dt = \Phi(t)\int \Phi^{-1}(t)f(t)dt,$$

where $\Phi(t)$ is any fundamental matrix of $x' = Ax$.

Example 1: Based on Exercise 5.5.26 from the textbook. We will solve the system $x' = Ax$, where $A = \begin{pmatrix} 5 & -1 & 1 \\ 1 & 3 & 0 \\ -3 & 2 & 1 \end{pmatrix}$ using two different methods.

SOLUTION. The eigenvalues of A are the roots of the characteristic polynomial

$$\begin{aligned}
p(\lambda) &= \begin{vmatrix} 5-\lambda & -1 & 1 \\ 1 & 3-\lambda & 0 \\ -3 & 2 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 5-\lambda & -1 & 1-1 \\ 1 & 3-\lambda & 0+(3-\lambda) \\ -3 & 2 & (1-\lambda)+2 \end{vmatrix} = \begin{vmatrix} 5-\lambda & -1 & 0 \\ 1 & 3-\lambda & 3-\lambda \\ -3 & 2 & 3-\lambda \end{vmatrix} \\
&= \begin{vmatrix} 5-\lambda & -1 & 0 \\ 1-(-3) & (3-\lambda)-2 & (3-\lambda)-(3-\lambda) \\ -3 & 2 & 3-\lambda \end{vmatrix} = \begin{vmatrix} 5-\lambda & -1 & 0 \\ 4 & 1-\lambda & 0 \\ -3 & 2 & 3-\lambda \end{vmatrix} \\
&= (3-\lambda)\begin{vmatrix} 5-\lambda & -1 \\ 4 & 1-\lambda \end{vmatrix} = (3-\lambda)\left((5-\lambda)(1-\lambda)+4\right) = (3-\lambda)(9-6\lambda+\lambda^2) = (3-\lambda)^3.
\end{aligned}$$

Therefore $\lambda = 3$ is the only eigenvalue of A, and it has algebraic multiplicity 3. To find its geometric multiplicity, we solve the equation

$$(A-3I_3)v = 0, \text{ where } v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

This gives

$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -3 & 2 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which gives the system of equations:

$$\begin{aligned}
2a - b + c &= 0 \\
a &= 0 \\
-3a + 2b - 2c &= 0
\end{aligned}$$

The third equation is redundant. The first equation gives $c = b$. Therefore, all eigenvectors corresponding to the eigenvalue $\lambda = 3$ are of the form

$$v = \begin{pmatrix} 0 \\ b \\ b \end{pmatrix} = b\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

All eigenvectors are therefore scalar multiples of the eigenvector $v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and they belong to the line spanned by v_1 in $\mathbb{R}^3$. Since we can find at most one linearly independent eigenvector, it follows that the geometric multiplicity of $\lambda = 3$ is 1.

We need to find two generalized eigenvectors v_2 and v_3, starting from v_1,by successively solving the equations

$$(A-3I_3)v_2 = v_1 \tag{7}$$
$$(A-3I_3)v_3 = v_2. \tag{8}$$

$\{v_1, v_2, v_3\}$ will then be a chain of length 3 for the eigenvalue λ. For Equation (7), we solve

$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -3 & 2 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

and get the system

$$\begin{aligned} 2a - b + c &= 0 \\ a &= 1 \\ -3a + 2b - 2c &= 1 \end{aligned}$$

which reduces to $a = 1$ and $c = b - 2$. Therefore, $v_2 = \begin{pmatrix} 1 \\ b \\ b-2 \end{pmatrix} = b \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$. The first term of the sum can be ignored, because it is an eigenvector of A, and we can choose $v_2 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$.

We now solve Equation (8).

$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -3 & 2 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix},$$

which gives the system

$$\begin{aligned} 2a - b + c &= 1 \\ a &= 0 \\ -3a + 2b - 2c &= -2 \end{aligned}$$

which is equivalent to $a = 0$ and $c = b+1$. Hence $v_2 = \begin{pmatrix} 0 \\ b \\ b+1 \end{pmatrix} = b \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. The first term can be ignored since it is an eigenvector of A, so we choose the second generalized eigenvector to be $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

We have obtained three linearly independent vectors, forming a chain of length 3:

$$\{v_1, v_2, v_3\} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Now we go back to solving the system $x' = Ax$.

Method 1: Three linearly independent solutions can be found by setting

$$\begin{aligned} x_1(t) &= e^{3t}v_1 = e^{3t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\ x_2(t) &= e^{3t}(v_2 + tv_1) = e^{3t}\begin{pmatrix} 1 \\ t \\ -2+t \end{pmatrix} \\ x_3(t) &= e^{3t}(v_3 + tv_2 + \frac{t^2}{2}v_1) = e^{3t}\begin{pmatrix} t \\ t^2/2 \\ 1-2t+t^2/2 \end{pmatrix} \end{aligned}$$

The general solution is $x(t) = c_1x_1(t) + c_2x_2(t) + c_3x_3(t)$, where c_1, c_2, c_3 are real numbers. In expanded form this gives:

$$x(t) = e^{3t}\begin{pmatrix} c_2 + c_3 t \\ c_1 + c_2 t + c_3\frac{t^2}{2} \\ c_1 + c_2(t-2) + c_3(\frac{t^2}{2} - 2t + 1) \end{pmatrix}.$$

Method 2: Consider the matrix $S = [v_1, v_2, v_3] = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -2 & 1 \end{pmatrix}$.

Since A has a repeated eigenvalue $\lambda = 3$, with algebraic multiplicity 3, and geometric multiplicity 1, the Jordan canonical form J of A will contain a single Jordan block:

$$J = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

Then $e^{Jt} = e^{\lambda t}\begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$, so a fundamental matrix for $x' = Ax$ is given by

$$\Phi(t) = Se^{Jt} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -2 & 1 \end{pmatrix} e^{\lambda t}\begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} = e^{\lambda t}\begin{pmatrix} 0 & 1 & t \\ 1 & t & \frac{t^2}{2} \\ 1 & t-2 & \frac{t^2}{2} - 2t + 1 \end{pmatrix}.$$

The general solution of $x' = Ax$ is given by

$$x' = \Phi(t)c = \Phi(t)\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = e^{\lambda t}\begin{pmatrix} c_2 + c_3 t \\ c_1 + c_2 t + c_3\frac{t^2}{2} \\ c_1 + c_2(t-2) + c_3(\frac{t^2}{2} - 2t + 1) \end{pmatrix}$$

Example 2: Based on Exercise 5.5.16 in the textbook. Solve the system $x' = Ax$, where $A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & -2 & -3 \\ 2 & 3 & 4 \end{pmatrix}$, in two different ways.

Solution. The characteristic polynomial of A is

$$p(\lambda) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ -2 & -2-\lambda & -3 \\ 2 & 3 & 4-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} -2-\lambda & -3 \\ 3 & 4-\lambda \end{vmatrix} = (1-\lambda)((-2-\lambda)(4-\lambda)+9) = (1-\lambda)^3.$$

The matrix A has a single eigenvalue $\lambda = 1$, with algebraic multiplicity 3. To find the eigenvectors corresponding to the eigenvalue $\lambda = 1$ we solve the system $(A - I_3)v = 0$:

$$\begin{pmatrix} 0 & 0 & 0 \\ -2 & -3 & -3 \\ 2 & 3 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies 2a + 3b + 3c = 0 \implies c = -\frac{2}{3}a - b$$

$$\implies v = \begin{pmatrix} a \\ b \\ -\frac{2}{3}a - b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ -\frac{2}{3} \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$

where a and b are random constants. Because it's more convenient to work with integer numbers, we can also write

$$v = a \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$

where $a := a/3$ and b are any random constants. All eigenvectors belong to a plane in $\mathbb{R}^3$, spanned by the linearly independent eigenvectors

$$v = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$

hence the eigenvalue $\lambda = 1$ has geometric multiplicity 2.

We need to find one more generalized eigenvector u, starting from either v or w.

Starting from v: Find u such that $(A - I_3)u = v$. This condition implies

$$\begin{pmatrix} 0 & 0 & 0 \\ -2 & -3 & -3 \\ 2 & 3 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \implies 0a + 0b + 0c = 3 \implies 0 = 3,$$

which is a contradiction, hence there does not exist any generalized eigenvector u that solves $(A - I_3)u = v$. It means that $\{v\}$ is a chain of length 1.

Starting from w: Find u such that $(A - I_3)u = w$. This condition implies

$$\begin{pmatrix} 0 & 0 & 0 \\ -2 & -3 & -3 \\ 2 & 3 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \implies 2a + 3b + 3c = -1 \implies c = -\frac{1}{3} - \frac{2}{3}a - b$$

$$\implies v = \begin{pmatrix} a \\ b \\ -\frac{1}{3} - \frac{2}{3}a - b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ -\frac{2}{3} \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{3} \end{pmatrix}. \tag{9}$$

The first two terms can be ignored, as they are eigenvectors of the matrix A. A generalized eigenvector is given by $u = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{3} \end{pmatrix}$. Notice that we cannot scale u by any factor we want as this vector does not appear with a random constant in front in the formula given in Equation 9. We could pick another generalized eigenvector with integer entries by setting say $a = 3$, $b = 0$ in Equation 9, but we will choose to work with the generalized eigenvector that we have already selected.

The set $\{w, u\}$ is a chain of length 2. This is a maximal chain starting from the eigenvector w (if we tried to continue the chain by finding a vector z such that $(A - I_3)z = u$, we would get a contradiction).

Together, $\{v, w, u\}$ are a linearly independent set formed with eigenvectors and generalized eigenvectors of the matrix A.

Method 1: Three linearly independent solutions can be found by setting

$$\begin{aligned} x_1(t) &= e^t v = e^t \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \\ x_2(t) &= e^t w = e^t \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \\ x_3(t) &= e^t (u + tw) = e^t \begin{pmatrix} 0 \\ t \\ -\frac{1}{3} - t \end{pmatrix} \end{aligned}$$

The general solution is $x(t) = c_1 x_1(t) + c_2 x_2(t) + c_3 x_3(t)$, where c_1, c_2, c_3 are real numbers. In expanded form this gives:

$$x(t) = e^t \begin{pmatrix} 3c_1 \\ c_2 + c_3 t \\ -2c_1 + -c_2 + c_3(-\frac{1}{3} - t) \end{pmatrix}.$$

Method 2: Consider the matrix $S = [v, w, u] = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & -\frac{1}{3} \end{pmatrix}$.

Since A has a repeated eigenvalue $\lambda = 1$, with algebraic multiplicity 3, and geometric multiplicity 2, the Jordan canonical form J of A will contain two Jordan blocks, as follows: the first Jordan block is of size 1, corresponding to the chain formed by $\{v\}$, whereas the second Jordan block corresponds to the chain $\{w, u\}$ and has size 2. The Jordan canonical form of the matrix A is therefore

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Then $e^{Jt} = e^{\lambda t} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$, so a fundamental matrix for $x' = Ax$ is given by

$$\Phi(t) = Se^{Jt} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & -\frac{1}{3} \end{pmatrix} e^{\lambda t} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} = e^{\lambda t} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & t \\ -2 & -1 & -t - \frac{1}{3} \end{pmatrix}.$$

The general solution of $x' = Ax$ is given by

$$x' = \Phi(t)c = \Phi(t) \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = e^{\lambda t} \begin{pmatrix} 3c_1 \\ c_2 + c_3 t \\ -2c_1 - c_2 + c_3(-t - \frac{1}{3}) \end{pmatrix}.$$

MAT 303: Calculus IV with Applications

Homework 11 - Instructions

HW 11 contains only two exercises from Section 5.7 and several exercises from Section 6.2 of the textbook. Please read the lecture notes and browse the Mathematica tutorial before getting started. Please solve Exercises 5.7.24 and 5.7.31 completely, as follows:

1) Please compute the exponential of matrix A, given in the problem, both analytically and by using Mathematica. Analytically, this can be done as in the examples from the lecture notes:

   a) find the eigenvalues of the matrix, and their geometric and algebraic multiplicities. Write the Jordan canonical form J of A.

   b) Use the formula for the exponential of a Jordan block to compute e^{Jt}.

   c) Find a complete set of eigenvectors and generalized eigenvectors for A, and write them columnwise (in the proper order) in a matrix S, as in the Lecture Notes. At this moment, you know that $S^{-1}AS = J$.

   d) Compute $\Phi(t) = Se^{Jt}$ and $e^{At} = Se^{Jt}S^{-1}$. Both $\Phi(t)$ and e^{At} are fundamental matrices of the homogeneous system $x' = Ax$.

2) Find the solution of the nonhomogeneous system by using the method of Variation of Parameters. Two equivalent (but not identical ways) of writing the solution are:

$$\begin{aligned} x(t) &= e^{At}\int e^{-At}f(t)dt \\ x(t) &= \Phi(t)\int \Phi(t)^{-1}f(t)dt. \end{aligned}$$

   Here we take the most general antiderivative of the integrand, $\int g(t)dt = G(t) + c$. If we keep the constant of integration c, then $x(t)$ is the general solution of the nonhomogeneous system. If we ignore the constant of integration c, then we get a particular solution of the nonhomogeneous system. If we have an initial condition in the problem, like $x(t_0) = x_0$, then we can obtain the unique solution as

$$x(t) = e^{At}\left(x_0 + \int_{t_0}^{t} e^{-As}f(s)ds\right) = \Phi(t)\left(\Phi^{-1}(t_0)x_0 + \int_{t_0}^{t}\Phi(s)^{-1}f(s)ds\right).$$

   Use Mathematica to check the computations. You will need the functions MatrixExp[..] and Integrate[..], as in the tutorial. Matrix Multiplication in Mathematica is a dot "."

For the problems from Section 6.2, use Mathematica as instructed in the textbook.