MAT 303: Calculus IV with applications

Spring 2016

Syllabus

Instructor's Contact Details:

Instructor: Dr. Luigi Lombardi
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Office: Math Tower 3-120
Office hours: Tuesday & Thursday 1pm-2pm in Math Tower 3-120
Thursday 12pm-1pm in MLC
By appointment

Lecture (location and time):

Location: Library W4550
Time: Monday-Wednesday-Friday 10am-10:53am

Teaching Assistant's Contact Details:

Teaching Assistant: Zeyu Cao
Office: MLC (Mathematical Learning Center) in Math Tower S-Level
Office Hours: Friday 10am-11am and Tuesday & Thursday 5:30pm-6:30pm in MLC
E-mail: zeyu.cao@stonybrook.edu

Recitation Sections:

R01: Friday 12pm - 12:53pm in Library E4310
R02: Wednesday 12pm - 12:53pm in Library W4540

Textbook:

C. H. Edwards, D. E. Penney, D. T. Calvis "Differential equations and boundary value problems", 5th edition, Pearson Prentice Hall

Course Description:

This is a course about differential equations. We will study first-order differential equations, numerical methods, linear equations of higher-order, homogeneous and non-homogeneous equations, linear systems of differential equations, stability. A particular eye is given to techniques of solutions of differential equations and to the description of models coming from physics, chemistry, and economics.

Prerequisites:

Completion of one of the standard calculus sequences (either MAT 125-127 or MAT 131-132 or MAT 141-142) with a grade C or higher in MAT 127, 132 or 142 or AMS 161. Also, MAT 203/205 (Calculus III) and AMS 261/MAT 211 (Linear Algebra) are recommended. Informally, students should know integration and differentiation techniques and, desirably, be familiar with complex numbers and basic aspects of linear algebra.

Midterms:

Practice Final Exam

Solutions to Practice Final Exam Part I (Problems 1-4)
Solutions to Practice Final Exam Part II (Problems 5-8)
Solutions to Practice Final Exam Part III (Problems 9-11)
Solutions to Practice Final Exam Part IV (Problem 12)

Summary Topics

Practice Midterm II

Solutions to Practice Midterm II

Practice Midterm I

Solutions to Practice Midterm I

Final Exam:

Final Exam on Monday May 16th, 2016 at 8am - 10.45am: in Library W4550.

Homework:

There will be weekly homework assignments to be turned in to your TA at the beginning of your next recitation section. You are encouraged to work in groups, or to consult each other, or me, or the teaching assistant, in order to solve the homework set. However the writing-up of your homework should reflect your own understanding. It is very important that you work out each homework assignment as they will help you to solve the problems of the midterms and the final exam. Late homework will not be graded.

Assignments:

- **Assignment 1: 1.1:** 1, 4, 5, 32; **1.2:** 2, 7, 11, 14, 25, 27, 37 (Due date: February 12th)
- **Assignment 2: 1.3:** 12, 14, 28; **1.4:** 1, 4, 6, 23, 25, 27, 35, 47 (Due date: February 15th)

- **Assignment 3: 1.5:** 1, 3, 6, 17, 37; **1.6:** 1, 2, 13, 20, 31, 34 (Due date: February 22nd)
- **Assignment 4:** Solve all problems of the Practice Midterm I without the asterisks, and in addition 1 problem with an asterisk at your choice except for Problem 2. (Due date: February 29th)

- **Assignment 5: 2.1:** 10, 13, 21; **2.2:** 19 (Due date: March 9th/11th in Recitation Sections)
- **Assignment 6: 2.3:** 2; **3.1:** 1, 3, 22, 25, 33, 39, 31; **3.5:** 1 (Due date: March 23rd/25th in Recitation Sections)
- **Assignment 7: 3.2:** 7,8; **3.3:** 11, 13, 18, 27, 31; **3.5**: 1, 3, 4, 12 (Due date: March 30th/April 1st in Recitation Sections)
- **Assignment 8**: (Due date: April 13th/15th in Recitation Sections)
- **Assignment 9**: (Due date: April 20th/22th in Recitation Sections)
- **Assignment 10**: (Due date: April 29th in Recitation Sections)
- **Assignment 11**: (Due date: May 6th in Recitation Sections)

Midterms and Final Exam:

During the semester there will be two midterms to be taken in class during lecture hours. The final exams will occur in May. The dates of both the midterms and the final exam will be announced later on. Makeup exams will only be given in the event that circumstances beyond the students control do not allow the student to take the exams at the assigned times.

Course Grading:

20% Homework
20% Midterm 1
20% Midterm 2
40% Final Exam

Academic Integrity:

Each student must pursue his or her goals honestly and be personally accountable for all submitted work. Representing another person's work as your own is always wrong. Instructors are required to report any suspected instances of academic dishonesty to the Academic Judiciary. For more comprehensive information on academic integrity, including categories of academic dishonesty, see the academic judiciary web site at http://www.stonybrook.edu/cinncms/academic-integrity/index.html

Diary of Classes:

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Week	Material Covered	Assignment Homework	Additional Exercises	Notes
Week 1 (Jan. 25th-29th)	1.1: Differential Equations and Mathematical Models	**1.1:** 1, 4, 5, 32; **1.2:** 2, 7, 11, 14, 25, 27, 37 (due on Feb. 12th)	**1.1:** 7, 9, 43; **1.2:** 6, 10, 33, 35	
	1.2: Integrals as General and Particular Solutions			
Week 2 (Feb. 1st-	1.3: Slope fields and solutions curves	**1.3:** 12, 14, 28; **1.4:** 1, 4, 6, 23, 25, 27, 35, 47 (due on Feb.	**1.3:** 16, 30; **1.4:** 5, 10, 28,	

5th)		15th)	33, 51	
	1.4: Separable differential equations			Notes on the exponential equation and mathematical models
Week 3 (Feb. 8th-12th)	1.4: Exponential equation and Mathematical Models	**1.5:** 1, 3, 6, 17, 37; **1.6:** 1, 2, 13, 20, 31, 34 (due on Feb. 22nd)	**1.5:** 14, 17, 36, 38; **1.6:** 4, 16, 21, 59, 62	
	1.5: Linear first order differential equations			
Week 4 (Feb. 15th-19th)	1.5: Mixture problems and 1.6: Substitution methods	All problems of the Practice Midterm I without the asterisk, plus 1 problem at your choice with an asterisk except Problem 2 (due date Feb. 29th)	**1.2:** 35, 36; **1.3:** 30; **1.4:** 41, 51, 52; **1.5:** 22, 23, 24, 38; **1.6:** 15, 21, 36, 59	
	1.6: Substitution Methods - Homogeneous equations, Bernoulli equations			
	1.6: Exact Differential Equations			Examples of exact differential equations
Week 5 (Feb. 22th-26th)	2.1: Population Models: Logistic Equation	**2.1:** 10, 13, 21; **2.2:** 19 (due on March 9th/11th in Recitation Sections)	**2.1:** 2, 18, 30; **2.2:** 23	
	2.1: Logistic Equation and 2.2: Equilibrium solutions and stability			
	2.2: Bifurcation			
Week 6 (Feb. 29th-March 4th)	Review Midterm I			
	Midterm I			

	2.3: Acceleration-velocity models and 2.4: Euler Method			
Week 7 (March 7th-11th)	3.1: Introduction: Second-order linear equations	**2.3:** 2; **3.1**: 1, 3, 22, 25, 33, 39, 31; **3.5:** 1	**2.3:**4, 6; **3.1:** 21, 23, 36, 38	
	3.1: Introduction: Second-order linear equations			
	3.5: Undetermined Coefficients Method for homogeneous second-order linear equations			
Week 8 (March 14th-18th)	**Spring Break**			
Week 9 (March 21th-25th)	3.2: General solutions of linear equations	**3.2:** 7,8; **3.3:** 11, 13, 18, 27, 31; **3.5**: 1, 3, 4, 12 (due date March 30th / April 1st)		
	3.3: Homogeneous Equations with constant coefficients			
Week 10 (March 28th - April 1st)	3.5: Nonhomogeneous Equations	Practice Midterm II		
	Review Midterm 2			
Week 11 (April 4th - April 8th)	3.4: Mechanical Vibrations and Review Midterm 2			
	Midterm II (April 6th)			

	4.1: First-order systems	Assignment 8 (sections 3.4, 3.5, 4.1)		
Week 12 (April 11th - April 15th)	5.1: Matrices and linear systems			
	5.2: The eigenvalue method (distinct real solutions)	Assignment 9		
Week 13 (April 18th - April 22th)	5.2: The eigenvalue method (complex solutions)			
	5.5: The eigenvalue method (multiple solutions)			
Week 14 (April 25th - April 29th)	5.5: The eigenvalue method (multiple solutions)	Assignment 10 (due date April 29th)		
	5.6: Exponential matrix			
	5.7: Undetermined coefficients for linear systems			
	4.2: Elimination method for systems of differential equations (homogeneous and non-homogeneous)			
Week 15 (May 2nd - May 6th)	Review I	Assignment 11 (due date May 6th)		
	Review II			

	Review III			
	Final Exam on Monday May 16th at 8.00am			

MAT303: Calculus IV with applications
Practice Final Exam
Spring 2016

By no means this practice exam is going to be similar to the final. For a good preparation you should work out this practice final together with the previous midterms, practice midterms and homework. Moreover, with the help of the 'summary of the course', you should be able to understand on which topics you should do more practice (look up for problems on the textbook). Remember that in mathematics even though one knows the theory, by no means one is able to solve problems. The best way to learn Calculus IV, and math in general, is to solve plenty of problems in the right way. The more problems you will do, the more you will do good in exams.

Problem 1 Find the general solution of the following differential equations.

$$(1+x)\frac{dy}{dx} = 4y$$
$$xy' = 3y + x^3$$
$$x(x+y)y' = y(x-y)$$
$$y' = (4x+y)^2$$
$$y^2y' + 2xy^3 = 6x$$
$$(1 + ye^{xy})dx + (2y + xe^{xy})dy = 0$$

Problem 2 A 400-gal tank initially contains 100 gal of brine containing 50 lb of salt. Brine containing 1 lb of salt per gallon enters the tank at the rate of 5 gal/s, and the well-mixed brine in the tank flows out at the rate of 3 gal/s. How much salt will the tank contain when it is full of brine?

Problem 3 Consider the differential equation depending on the parameter h:

$$\frac{dy}{dx} = x^3(x^2-4) - h.$$

For $h = 0$ find the critical solutions and classify them as stable or unstable. What are the bifurcation values for h?

Problem 4 Find the general solution of the following second-order differential equations.

$$y^{(iv)} + 18y'' + 81y = 0$$
$$y'' + y' + y = \sin^2(x)$$
$$y^{(3)} - y = e^x + 7$$
$$y'' + 6y' + 13y = e^{-3x}\cos(2x)$$

Problem 5 Verify that $y_c(x) = c_1x + c_2x^{-1}$ is a complementary solution for

$$x^2y'' + xy' - y = 72x^5.$$

Use variation of parameters to find a particular solution of the equation (write first the equation in standard form). You can try other similar problems as in problems 58-59 of Section 3.5.

Problem 6 Suppose that in a mass-spring-dashpot system with $m = 25$, $c = 10$, and $k = 226$ is set in motion with $x(0) = 20$ and $x'(0) = 41$. Find the position function $x(t)$ and sketch a graph of the solution.

Problem 7 Use the method of elimination to find the general solution of the following systems.

$$a) \quad x' = x + 9y; \quad y' = -2x - 5y.$$
$$b) \quad x'' = -4x + \sin(t); \quad y'' = 4x - 8y.$$
$$c) \quad (D^2 + 1)x + D^2 y = 2e^{-t}; \quad (D^2 - 1)x + D^2 y = 0.$$

Problem 8 Use the eigenvalue method to find the general solution of the following systems.

$$a) \quad x_1' = x_1 + x_4, \quad x_2' = -x_2 - 2x_3, \quad x_3' = 2x_2 - x_3, \quad x_4' = x_4.$$
$$b) \quad x_1' = -\frac{1}{5}x_1 + \frac{1}{5}x_3; \quad x_2' = \frac{1}{5}x_1 - \frac{2}{5}x_2; \quad x_3' = \frac{2}{5}x_2 - \frac{1}{5}x_3.$$
$$c) \quad x_1' = x_1, \quad x_2' = x_1 + 3x_2 + x_3, \quad x_3' = -2x_1 - 4x_2 - x_3.$$

Problem 9 Problem 29 of section 5.2 on the textbook.

Problem 10 Use the exponential matrix to find a particular solution of the following initial value problem.

$$x_1' = 5x_1, \quad x_2' = 10x_1 + 5x_2, \quad x_3' = 20x_1 + 30x_2 + 5x_3, \quad x_1(0) = 4, \quad x_2(0) = 5, \quad x_3(0) = 6.$$

Problem 11 Use the method of undetermined coefficients for linear systems to find the general solution of the following systems

$$a) \quad x' = x + 2y + 3, \quad y' = 2x + y - 3$$
$$b) \quad x' = 4x + y + e^t, \quad y' = 6x - y - e^t.$$

Problem 12 A fish population $P(t)$ in a lake has constant birth rate of 4 and a death rate equal to P (the time t is measured in days). Moreover 3 fish are harvested each day. Write a model for this problem. Find then the critical points and classify them as stable or unstable. Find the expression for $P(t)$ in function of the initial data $P(0) = P_0$ and the limiting population. Sketch the solution curves corresponding to the different values of P_0.

Problem 13 Solve other problems you can find on the textbook. Moreover look up at the problems of the previous midterms and practice midterms.

Solutions Practice Final Exam

Problem 1

(a)

$$(1+x)\frac{dy}{dx} = 4y$$

$$\frac{dy}{dx} = \frac{4y}{1+x} = \frac{4\,y/x}{\frac{1}{x}+x/x} = \frac{4\,y/x}{\frac{1}{x}+1}.$$

Substitution: $v = y/x \quad y = vx$

$$\frac{dy}{dx} = \frac{dv}{dx}x + v$$

$$\frac{dv}{dx}x + v = \frac{4v}{\frac{1}{x}+1}$$

$$\frac{dv}{dx}x = \frac{4v}{\frac{1}{x}+1} - v = \frac{4vx}{1+x} - v = \frac{4vx - v - vx}{1+x} =$$

$$= \frac{3vx - v}{1+x} = \frac{3v(x-1)}{1+x}$$

Separable equation

$$\frac{1}{3v}\,dv = \frac{x-1}{x(x+1)}\,dx$$

Integrate:

$$\int \frac{1}{3v}\,dv = \int \frac{x-1}{x(x+1)}\,dx + C$$

$$\frac{1}{3}\ln|v|$$

To solve this use partial fraction technique.

(2)

$$\int \frac{x-1}{x(x+1)}\,dx$$

$$\frac{x-1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1)+Bx}{x(x+1)} = \frac{x(A+B)+A}{x(x+1)}$$

$$\begin{cases} A+B = 1 \\ A = -1 \end{cases} \longrightarrow \quad B = 1 - A = 1-(-1) = 2$$

$$\int \frac{x-1}{x(x+1)}\,dx = \int -\frac{1}{x}\,dx + \int \frac{2}{x+1}\,dx = -\ln|x| + 2\ln|x+1|$$

$$= -\ln|x| + \ln(x+1)^2$$

$$= \ln\left|\frac{(x+1)^2}{x}\right|$$

Hence:

$$\frac{1}{3}\ln|v| = \ln\left|\frac{(x+1)^2}{x}\right| + C$$

$$\ln|v| = 3\ln\left|\frac{(x+1)^2}{x}\right| + C$$

$$\ln|v| = \ln\left(\frac{(x+1)^6}{|x|^3}\right) + C$$

$$|v| = \frac{(x+1)^6}{|x|^3}e^C \qquad \text{But } v = y/x$$

$$\boxed{y = \pm\, x \frac{(x+1)^6}{|x|^3}e^C .}$$

b

$$x y' = 3y + x^3$$

$$x y' - 3y = x^3$$

$$y' - \frac{3}{x} y = x^2$$ Linear

$$\rho(x) = e^{\int -3/x \, dx} = e^{-3 \ln|x|} = |x|^{-3} = x^{-3}$$

$$y' x^{-3} - \frac{3}{x} x^{-3} y = x^2 x^{-3}$$

$$\frac{d}{dx}(x^{-3} y) = x^{-1}$$ Integrate:

$$x^{-3} y = \int x^{-1} \, dx + C$$

$$x^{-3} y = \ln|x| + C$$

$$\boxed{y = x^3 \ln|x| + C x^3}$$

c

$$x(x+y)\frac{dy}{dx} = y(x-y)$$

$$\frac{dy}{dx} = \frac{y(x-y)}{x(x+y)} = \frac{y}{x}\,\frac{x/x - y/x}{x/x + y/x} = \frac{y}{x}\,\frac{1-y/x}{1+y/x}$$ Homog.

Subst. $v = \frac{y}{x}$ $\quad y = vx \quad \frac{dy}{dx} = \frac{dv}{dx}x + v$

$$\frac{dv}{dx} + v = v\,\frac{1-v}{1+v}$$

$$\frac{dv}{dx} = v\,\frac{1-v}{1+v} - v = \frac{v - v^2 - \cancel{v} - v^2}{1+v} =$$

$$= \frac{-2v^2}{1+v}$$ Separable

$$\frac{1+v}{v^2}\,dv = -2\,dx$$ Integrate:

I solved this in class, see your notes for the last steps.

(d)

$$\frac{dy}{dx} = (4x+y)^2$$

subst. $v = 4x + y$

~~so~~ $y = v - 4x$

$$\frac{dy}{dx} = \frac{dv}{dx} - 4$$

$$\frac{dv}{dx} - 4 = v^2$$

$$\frac{dv}{dx} = v^2 + 4$$ Separable

$$\frac{dv}{v^2+4} = dx$$

$$\int \frac{dv}{v^2+4} = \int dx + C = x + C$$

$$\frac{1}{2} \operatorname{arctg}\left(\frac{v}{2}\right)$$

$$\operatorname{arctg}\left(\frac{v}{2}\right) = 2x + 2C$$

$$\frac{v}{2} = \tan(2x + 2C)$$

$v = 2\tan(2x + 2C)$ But $v = 4x + y$

$4x + y$

$$\boxed{y = 2\tan(2x + 2C) - 4x}$$

(e)

$$y^2 y' + 2x y^3 = 6x$$

$$\frac{dy}{dx} + 2xy = 6xy^{-2}$$ Bernoulli

$n = -2$

subst: $v = y^{1-n} = y^{1-(-2)} = y^3$

$$y = v^{1/3}$$

$$\frac{dy}{dx} = \frac{1}{3} v^{-2/3} \frac{dv}{dx}$$

$$\frac{1}{3} v^{-2/3} \frac{dv}{dx} + 2x v^{1/3} = 6x v^{-2/3}$$

Multiply by $v^{2/3}$

$$\frac{1}{3} \frac{dv}{dx} + 2xv = 6x$$

$$\frac{dv}{dx} + 6xv = 18x$$ Linear

$$\rho(x) = e^{\int 6x\,dx} = e^{3x^2}$$

$$\frac{dv}{dx} e^{3x^2} + 6x e^{3x^2} v = 18x e^{3x^2}$$

$$\frac{d}{dx}\left(e^{3x^2} v\right) = 18x e^{3x^2}$$ Integrate

$$e^{3x^2} v = \int 18x e^{3x^2}\,dx + C$$

$$e^{3x^2} v = 18 \int x e^{3x^2} dx + C =$$

subst. $\begin{bmatrix} u = 3x^2 \\ du = 6x\,dx \end{bmatrix}$

$$= 18 \frac{1}{6} \int 6x e^{3x^2} dx + C = 3 \int e^u du + C =$$

$$= 3 e^u + C = 3 e^{3x^2} + C.$$

Hence:

$$v = 3 + \frac{C}{e^{3x^2}} \qquad \text{But } v = Y^3$$

$$Y^3 = 3 + \frac{C}{e^{3x^2}}$$

$$\boxed{Y = \sqrt[3]{3 + \frac{C}{e^{3x^2}}}}$$

Problem 2

Tank of 400 gal

$V_0 = 100$ gal

Initial quantity of salt = 50 lb

$c_i = 1$

$r_i = 5$

$r_0 = 3$

$x(t)$ = quantity of salt at time t

$$c_0 = \frac{x}{V}$$

$$V = V_0 + (r_i - r_0)t = 100 + 2t$$

$$\frac{dx}{dt} = c_i r_i - c_0 r_0 = 5 - \frac{x}{100+2t}(3)$$

$$\frac{dx}{dt} + \frac{3}{100+2t}x = 5 \qquad \underline{\text{Linear}}$$

$$\rho(t) = e^{\int \frac{3}{100+2t}dt} = e^{3/2 \ln(100+2t)}$$

$$= (100+2t)^{3/2}$$

$$\frac{d}{dt}\left((100+2t)^{3/2} x\right) = 5(100+2t)^{3/2}$$

$$(100+2t)^{3/2}\, x = 5\int (100+2t)^{3/2}\, dt + C$$

$$= \frac{5}{2}\frac{(100+2t)^{3/2+1}}{\frac{3}{2}+1} + C$$

$$= \frac{5}{2}\frac{(100+2t)^{5/2}}{5/2} + C = (100+2t)^{5/2} + C.$$

Hence:

$$x(t) = \frac{(100+2t)^{5/2}}{(100+2t)^{3/2}} + \frac{C}{(100+2t)^{3/2}} =$$

$$= (100+2t) + \frac{C}{(100+2t)^{3/2}}$$

Initial condition : $x(0) = 50$

$$50 = x(0) = 100 + \frac{C}{100^{3/2}} = 100 + \frac{C}{1000}$$

$$50000 = 100\,000 + C$$

$$C = 100\,000 - 50\,000 = 50\,000$$

The tank is full when

$$V(t) = 100 + 2t = 400 \longrightarrow t = 150$$

Hence the quantity of salt when the tank is full is

$$x(150) = (100+300) + \frac{50\,000}{(100+300)^{3/2}} =$$

$$= 400 + \frac{50\,000}{400^{3/2}} =$$

$$= 400 + \frac{50\,000}{80\,000} = \boxed{400 + \frac{5}{8}}$$

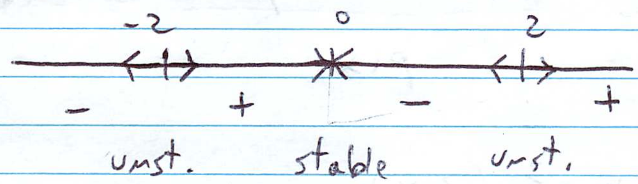
Problem 3

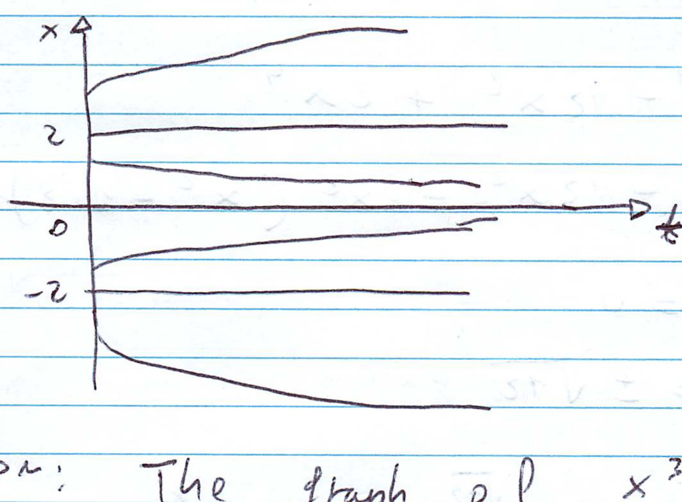
$$\frac{dy}{dx} = x^3(x^2-4) - h.$$

For $h=0$ find critical points and stability

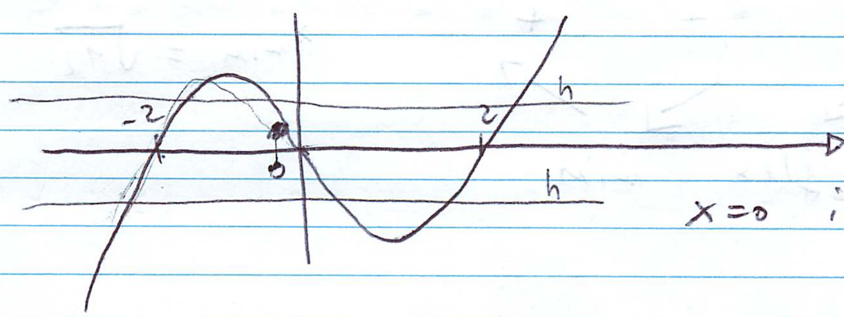
$$\frac{dy}{dx} = x^3(x^2-4)$$

$x^3(x^2-4)=0$ $\qquad x = 0, 2, -2$





Bifurcation: The graph of $x^3(x^2-4)$ is



$x=0$ is a flex.

Hence the number of solutions of

$$x^3(x^2-4)=+h$$

change when

$$h = p(x_{min}),\ p(x_{max})$$

where $p(x)=x^3(x^2-4)$

and x_{min}, x_{max} are the minimum and maximum points of $p(x)$.

$$p'(x)=3x^2(x^2-4)+x^3(2x)$$

$$=3x^4-12x^2+2x^4$$

$$=x^4-12x^2=x^2(x^2-12)=0$$

$$x=0$$

$$x=\pm\sqrt{12}$$

	$-\sqrt{12}$		0		$\sqrt{12}$	
$+$		$-$		$-$		$+$
$\nearrow$		$\searrow$		$\searrow$		$\nearrow$
	max		flex		min	

$x_{max}=-\sqrt{12}$

$x_{min}=\sqrt{12}$

$$h = p(-\sqrt{12}) = (-\sqrt{12})^3((-\sqrt{12})^2 - 4)$$

$$h = p(\sqrt{12}) = (\sqrt{12})^3((\sqrt{12})^2 - 4).$$

Problem 4

a) $$y^{(iv)} + 18y'' + 81y = 0$$

$$r^4 + 18r^2 + 81 = 0$$

$$(r^2 + 9)^2 = 0$$

$r = \pm 3i$ each of multiplicity 2

$$Y(x) = e^{0x}(c_1 \cos(3t) + c_2 \sin(3t)) +$$

$$+ x e^{0x}(c_3 \cos(3t) + c_4 \sin(3t))$$

$$= c_1 \cos(3t) + c_2 \sin(3t) + c_3 x \cos(3t) +$$

$$+ c_4 x \sin(3t).$$

(b)

$$y'' + y' + y = \sin^2 x$$

$$y(x) = y_c + y_p$$

<u>Find y_c</u>

$$r^2 + r + 1 = 0$$

$$r = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{3}{2} i$$

$$y_c(x) = e^{-1/2\, x} \left(c_1 \cos\left(\tfrac{3}{2} x\right) + c_2 \sin\left(3/2\, x\right)\right)$$

<u>Find y_p</u>

The derivatives of $\sin^2 x$ are

$$2 \sin x \cos x, \quad 2\cos^2 x - 2 \sin^2 x, \ldots$$

Hence they are linear combinations of

$$\sin^2 x, \quad \sin x \cos x, \quad \cos^2 x.$$

Our <u>guess</u> is then

$$y_p(x) = A \sin^2 x + B \sin x \cos x + C \cos^2 x.$$

$$Y_p'(x) = 2A\sin x\cos x + B\cos^2 x - B\sin^2 x$$
$$-2C\cos x\sin x$$

$$= (2A-2C)\sin x\cos x + B\cos^2 x - B\sin^2 x$$

$$Y_p''(x) = (2A-2C)\cos^2 x - (2A-2C)\sin^2 x$$
$$-2B\cos x\sin x - 2B\sin x\cos x$$

$$= (2A-2C)\cos^2 x + (2C-2A)\sin^2 x$$
$$-4B\sin x\cos x.$$

Plug-in:

$$(2A-2C)\cos^2 x + (2C-2A)\sin^2 x - 4B\sin x\cos x$$
$$+ (2A-2C)\sin x\cos x + B\cos^2 x - B\sin^2 x$$
$$+ A\sin^2 x + B\sin x\cos x + C\cos^2 x$$
$$=$$
$$\sin^2 x$$

Group factors on the LHS

$$\cos^2(x)\left(2A-2C+B+C\right) + \sin^2 x\left(2C-2A-B+A\right) +$$
$$+ \sin x\cos x\left(-4B+2A-2C+B\right) = \sin^2 x$$

Hence:

$$\begin{cases} 2A - C + B = 0 \\ 2C - A - B = 1 \\ 2A - 2C - 3B = 0 \end{cases}$$

Solve:

$R1 - R3 \longrightarrow C + 4B = 0 \longrightarrow C = -4B$

$$\begin{cases} 2A + 5B = 0 \\ -9B - A = 1 \end{cases} \longrightarrow A = -\tfrac{5}{2}B$$

$\longrightarrow 9B + A = -1$

$\downarrow$

$9B - \tfrac{5}{2}B = -1$

$\downarrow$

$\frac{13}{2}B = -1 \longrightarrow B = -\tfrac{2}{13}$

$A = -\tfrac{5}{2}\left(-\tfrac{2}{13}\right) = \tfrac{5}{13}$

$C = -4B = -4\left(-\tfrac{2}{13}\right) = \tfrac{8}{13}$

(c)

$$y''' - y = e^x + 7$$

$$y = y_c + y_p$$

Find y_c

$$r^3 - r = 0$$

$$r(r^2 - 1) = 0 \qquad r = 0, 1, -1 \qquad \text{mult. } 1.$$

$$y_c(x) = c_1 e^{0x} + c_2 e^x + c_3 e^{-x}$$

$$= c_1 + c_2 e^x + c_3 e^{-x}$$

Find y_p

Guess: $y_p = A e^x + B$

But this won't work because it is ~~already~~ of the same type of y_c. (take $c_1 = B$, $c_2 = A$, $c_3 = 0$).

Another guess: $y_p(x) = A x e^x + B x$

$$y_p' = A e^x + A x e^x + B$$

$$y_p'' = A e^x + A e^x + A x e^x$$

$$= 2A e^x + A x e^x$$

$$y_p''' = 2Ae^x + Ae^x + Axe^x$$
$$= 3Ae^x + Axe^x$$

Plug-in

$$3Ae^x + \cancel{Axe^x} - \cancel{Axe^x} - Bx = e^x + 7$$

$$\begin{cases} 3A = 1 \\ -B = 7 \end{cases} \rightarrow \boxed{\begin{aligned} A &= 1/3 \\ B &= -7 \end{aligned}}$$

1

Part II

Problem 5

$$x^2 y'' + x y' - y = 72 x^5$$

Verify that $y_c = c_1 x + \dfrac{c_2}{x}$ is a compliment ary solution.

$$y_c' = c_1 - \frac{c_2}{x^2} \qquad y_c'' = \frac{2c_2}{x^3}$$ → Plug into the equation.

$$x^2\left(\frac{2c_2}{x^3}\right) + x\left(c_1 - \frac{c_2}{x^2}\right) - c_1 x - \frac{c_2}{x} =$$

$$= \frac{2c_2}{x} + c_1 x - \frac{c_2}{x} - c_1 x - \frac{c_2}{x} = 0$$

$\boxed{\text{OK}}$

Variation of parameters

$$y_1 = x \qquad y_2 = \frac{1}{x} \qquad W = \det\begin{pmatrix} x & 1/x \\ 1 & -1/x^2 \end{pmatrix} = -\frac{2}{x}$$

Standard form $\quad y'' + \dfrac{y'}{x} - \dfrac{1}{x^2} y = 72x^3$

$f(x)$.

$$y_p(x) = -y_1 \int \frac{y_2 f}{W}\,dx + y_2 \int \frac{y_1 f}{W}\,dx =$$ →

$$= -x \int \frac{72x^2}{-2/x}\,dx + \frac{1}{x}\int \frac{72x^4}{-2/x}\,dx$$

$$= -x \int -36x^3\,dx + \frac{1}{x}\int -36x^5\,dx$$

$$= 36x\left(\frac{x^4}{4}\right) + \frac{1}{x}(-36)\frac{x^6}{6} =$$

$$= 9x^5 - 6x^5 = \boxed{3x^5}$$

Problem 6

$$25x'' + 10x' + 226x = 0$$

$$x(0) = 20$$

$$x'(0) = 41.$$

$$25r^2 + 10r + 226 = 0$$

$$r = -5 \pm \sqrt{25 - (25)(226)} =$$

$$= -5 \pm \sqrt{25(-225)} =$$

$$= -5 \pm 5\sqrt{-225} = -5 \pm 5(15)i$$

$$= -5 \pm 75i$$

(3)

$$x(t) = A e^{-5t} \cos(75t) + B e^{-5t} \sin(75t)$$

Impose initial conditions:

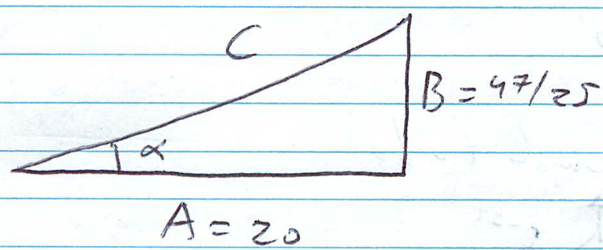
$$x'(t) = -5 A e^{-5t} \cos(75t) - A e^{-5t} \sin(75t)(75)$$
$$-5B e^{-5t} \sin(75t) + B e^{-5t} \cos(75t)(75)$$

$$\begin{cases} 20 = x(0) = A \\ 41 = x'(0) = -5A + 75B \end{cases}$$

$A = 20$ $\quad B = \dfrac{5A + 41}{75} = \dfrac{100 + 41}{75} = \dfrac{141}{75} = \dfrac{47}{25}$

$$x(t) = 20 e^{-5t} \cos(75t) + \frac{47}{25} \sin(75t)$$

standard form



$$C = \sqrt{400 + \frac{2209}{625}} = \sqrt{\frac{252209}{625}} = \frac{\sqrt{252209}}{25}$$

~~~~ As both $A>0$, $B>0$ are positive, α is simply given by

$$\alpha = \cancel{tan}^{-1}(B/A) =$$

$$= \tan^{-1}\left(\frac{47/25}{20}\right) =$$

$$= \tan^{-1}\left(\frac{47}{500}\right) = 0.094 \text{ rad.}$$

The standard form is given by

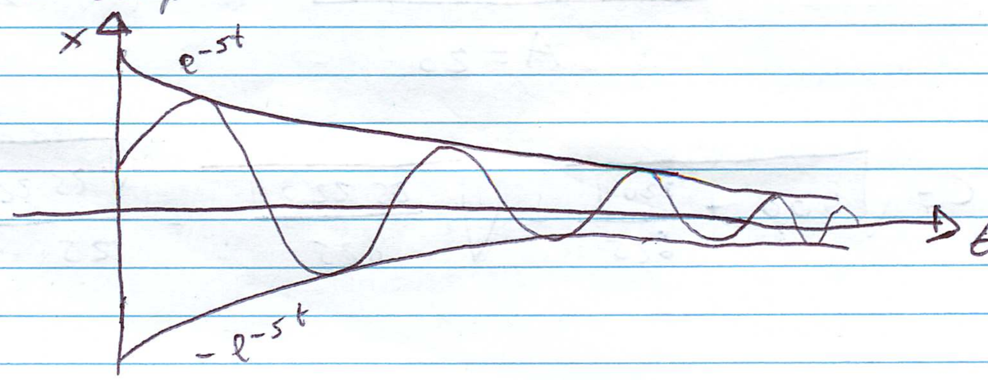
$$x(t) = C\, e^{-5t}\left(\cos(75t - 0.049)\right)$$

$$= \frac{\sqrt{252209}}{25}\, e^{-5t} \cos(75t - 0.049)$$

$\omega_0 = 75$ circular frequency.

$\alpha = 0.049$ phase

$\frac{\sqrt{252209}}{25} e^{-5t}$ = amplitube


~~~~

(5)

Problem 7

(a)

(b) $\begin{cases} x'' = -4x + \sin t \\ y'' = 4x - 8y \end{cases}$

$\begin{cases} x'' + 4x = \sin t \\ y'' + 8y - 4x = 0 \end{cases} \rightarrow \begin{cases} (D^2+4)\,x = \sin t \\ -4x + (D^2+8)\,y = 0 \end{cases}$

Eliminate x

Multiply R_1 by 4 and R_2 by (D^2+4). Then sum up the rows.

$\begin{cases} 4(D^2+4)\,x = 4\sin t \\ -4(D^2+4)\,x + (D^2+4)(D^2+8) = 0 \end{cases}$

$(D^2+4)(D^2+8)\,y = 4\sin t$

$y^{(iv)} + 12y'' + 32y = 4\sin t$

$y(x) = y_c + y_p$

Find y_c: $(r^2+4)(r^2+8) = 0$

$r = \pm 2i \qquad r = \pm\sqrt{8}\,i$

$y_c = c_1\cos(2t) + c_2\sin(2t)$
$\quad + c_3\cos(\sqrt{8}\,t) + c_4\sin(\sqrt{8}\,t)$

Find y_p

Guess: $y_p = A\cos t + B\sin t$

$y_p' = -A\sin t + B\cos t$

$y_p'' = -A\cos t - B\sin t$

$y_p''' = A\sin t - B\cos t$

$y_p^{(iv)} = A\cos t + B\sin t$

Plug-in.

$A\cos t + B\sin t - 12A\cos t - 12B\sin t$

$+ 32A\cos t + 32B\sin t = 4\sin t$

$\cos t(21A) + \sin t(21B) = 4\sin t$

$\begin{cases} 21A = 0 \\ 21B = 4 \end{cases} \rightarrow \boxed{\begin{array}{l} A = 0 \\ B = 4/21 \end{array}}$

$Y = c_1\cos(2t) + c_2\sin(2t) + c_3\cos(\sqrt{8}\,t) + c_4\sin(\sqrt{8}\,t)$

$+ 4/21\sin t.$

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Now find $x(t)$:

From the second equation $-4x + (D^2+8)y = 0$

we find $x = \frac{(D^2+8)y}{4} = y''/4 + 2y = \ldots$

c

$$\begin{cases} (D^2+1)x + D^2 y = 2e^{-t} \\ (D^2-1)x + D^2 y = 0 \end{cases}$$

Eliminate y : subtract the 2 rows.

$2x = 2e^{-t} \longrightarrow \boxed{x = e^{-t}}$

Plug $x = e^{-t}$ in the second row.

$(D^2-1)e^{-t} + D^2 y = 0$

$(e^{-t})'' - e^{-t} + y'' = 0$

$e^{-t} - e^{-t} + y'' = 0$

$y'' = 0$

$r^2 = 0 \rightarrow r = 0$ multipl. 2

$\boxed{y(t)} = c_1 e^{0t} + c_2 t e^{0t} = \boxed{c_1 + c_2 t}$

Problem 8

$$\begin{cases} x_1' = x_1 + x_4 \\ x_2' = -x_2 - 2x_3 \\ x_3' = 2x_2 - x_3 \\ x_4' = x_4 \end{cases}$$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{= A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 0 & 0 & 1 \\ 0 & -1-\lambda & -2 & 0 \\ 0 & 2 & -3-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix}$$

$$= -(1-\lambda) \det \begin{pmatrix} 1-\lambda & 0 & 0 \\ 1 & -1-\lambda & -2 \\ 0 & 2 & -3-\lambda \end{pmatrix}$$

$$= -(1-\lambda)(1-\lambda) \det \begin{pmatrix} -1-\lambda & -2 \\ 2 & -3-\lambda \end{pmatrix} =$$

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Problem 8

$$\begin{cases} x_1' = x_1 + x_4 \\ x_2' = -x_2 - 2x_3 \\ x_3' = 2x_2 - x_3 \\ x_4' = x_4 \end{cases}$$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\parallel \atop A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 0 & 0 & 1 \\ 0 & -1-\lambda & -2 & 0 \\ 0 & 2 & -1-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda) \det \begin{pmatrix} -1-\lambda & -2 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)(1-\lambda) \det \begin{pmatrix} -1-\lambda & -2 \\ 2 & -1-\lambda \end{pmatrix}$$

$$= (1-\lambda)^2 \left((-1-\lambda)^2 + 4 \right) = (1-\lambda)^2 \left(\lambda^2 + 2\lambda + 5 \right) = 0$$

$$\boxed{\lambda = 1 \quad \text{multipl.} \quad 2}$$

$$\lambda^2 + 2\lambda + 5 = 0$$

$$\boxed{\lambda = -1 \pm \sqrt{1-5} = -1 \pm 2i}$$

Eigenvectors for $\lambda = -1 \pm 2i$

Plug $\lambda = -1 + 2i$

Look for ~~ktla~~ $\underline{v} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

solutions of

$$\begin{pmatrix} 2-2i & 0 & 0 & 1 \\ 0 & 1-2i & -2 & 0 \\ 0 & 2 & 1-2i & 0 \\ 0 & 0 & 0 & 2-2i \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} (2-2i)a + d = 0 & \rightarrow a = 0 \\ (1-2i)b - 2c = 0 & \\ 2b + (1-2i)c = 0 & \rightarrow b = \dfrac{-(1-2i)c}{2} \\ (2-2i)d = 0 & \rightarrow d = 0 \end{cases}$$

Hence c is free

Can take $c = 1$.

Hence $b = -\frac{1}{2} + i$

An eigenvector is $\underline{v} = \begin{pmatrix} 0 \\ -1/2 + i \\ 1 \\ 0 \end{pmatrix}$

The solution associated to it is

$$\underline{x}(t) = \underline{v}\, e^{\lambda t} = \begin{pmatrix} 0 \\ -1/2 + i \\ 1 \\ 0 \end{pmatrix} e^{(-1+2i)t} =$$

$$= \begin{pmatrix} 0 \\ -1/2 + i \\ 1 \\ 0 \end{pmatrix} e^{-t} \left(\cos(2t) + i \sin(2t) \right)$$

From this we deduce 2 linearly indep. real solutions by taking the real and imaginary parts of $\underline{x}(t)$.

$$\boxed{\begin{aligned} \underline{x}_1(t) &= \operatorname{Re} \underline{x} = \begin{pmatrix} 0 \\ -1/2 \cos(2t) - \sin(2t) \\ \cos(2t) \\ 0 \end{pmatrix} e^{-t} \\ \underline{x}_2(t) &= \operatorname{Im} \underline{x} = \begin{pmatrix} 0 \\ -1/2 \sin(2t) + \cos(2t) \\ \sin(2t) \\ 0 \end{pmatrix} e^{-t} \end{aligned}}$$

Eigenvectors for $\lambda=1$

Plug $\lambda = 1$

Look for $\underline{v} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ ~~de~~ such that

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} d = 0 \\ -2b - 2c = 0 \quad \rightarrow \quad b = -c \\ 2b - 2c = 0 \quad \rightarrow \quad b = c \end{cases} \rightarrow \quad c = -c \quad \rightarrow \quad c = 0 \quad \rightarrow \quad b = 0$$

Hence a is free.

We get one eigenvector $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

But $\lambda = 1$ has multiplicity 2, so we want 2 eigenvectors.

We start over by looking for generalized eigenvectors.

The defect is $2 - 1 = 1$.

Hence we look for vectors $\underline{v}_2 \neq 0$ such that

$$(A - I)^2 \underline{v}_2 = 0$$

and $(A - \cancel{1} I)\, \underline{v}_2 = \underline{v}_1 \neq 0$.

$$(A - I)^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\begin{cases} 8c = 0 \\ 8b = 0 \end{cases} \rightarrow \boxed{\begin{matrix} c = 0 \\ b = 0 \end{matrix}}$ Hence a, d are free.

We get a vector by $\begin{matrix} a = 1 \\ d = 0 \end{matrix} \rightarrow \underline{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

~~Check:~~

$$\cancel{(A-I)\,\underline{v}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}$$

~~No! $\underline{v}_1$ we chose is not good.~~

~~We choose another $\underline{v}_1$ by $a=0$, $d=1$~~

$$\cancel{\underline{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}$$

Check:

$$(A-I)\,\underline{v}_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

No! The vector $\underline{v}_2$ we chose does not work.

Hence we choose another $\underline{v}_2$ by setting $a=0$, $d=1$. So we have $\underline{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Check:

$$(A-I)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\substack{\parallel \\ \underline{v}_1}} \neq \underline{0}$$

Therefore we have found 2 generalized eigenvectors.

$$\underline{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \underline{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The 2 linearly indep. solutions are:

$$\underline{x}_3(t) = \underline{v}_1 e^{\lambda t} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^t$$

$$\underline{x}_4(t) = \left(\underline{v}_1 t + \underline{v}_2\right) e^{\lambda t} = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right) e^t$$

$$= \begin{pmatrix} t \\ 0 \\ 0 \\ 1 \end{pmatrix} e^t$$

Putting everything together, the general solution of the system is given by

$$\boxed{\underline{x}(t) = c_1 \underline{x}_1(t) + c_2 \underline{x}_2(t) + c_3 \underline{x}_3(t) + c_4 \underline{x}_4(t)}.$$

(b)

(c) $\begin{cases} x_1' = x_1 \\ x_2' = x_1 + 3x_2 + x_3 \\ x_3' = -2x_1 - 4x_2 - x_3 \end{cases}$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & 1 \\ -2 & -4 & -1 \end{pmatrix}}_{\substack{\shortparallel \\ A}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 0 & 0 \\ 1 & 3-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{pmatrix} =$$

$$= (1-\lambda) \det \begin{pmatrix} 3-\lambda & 1 \\ -4 & -1-\lambda \end{pmatrix} =$$

$$= (1-\lambda) \Big((3-\lambda)(-1-\lambda) + 4 \Big) =$$

$$= (1-\lambda) \left(-3 - 3\lambda + \lambda + \lambda^2 + 4 \right) =$$

$$= (1-\lambda)(\lambda^2 - 2\lambda + 1) = (1-\lambda)(\lambda - 1)^2$$

$$= -(\lambda - 1)^3 = 0 \longrightarrow$$

$\lambda = 1$ wt. multiplicity 3.

Hence we expect 3 eigenvectors from $\lambda = 1$.

Plug $\lambda = 1$. Look for $\underline{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ such that:

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 1 \\ -2 & -4 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$a + 2b + c = 0$$

$$\cancel{-2a - 4b - 2c = 0}$$

$a = -2b - c$ Hence b and c are free

We obtain 2 eigenvectors, for instance $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

But they are not enough.

$\lambda = 1$ has defect $3 - 2 = 1$.

We use the technique of generalized eigenvectors.

Look for non-zero vectors $\underline{v}_2$ such that

$$(A-I)^2\,\underline{v}_2 = 0.$$

$$(A-I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 1 \\ -2 & -4 & -2 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 1 \\ -2 & -4 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence any non-zero vector is solution.

We choose for instance $\underline{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

<u>Check</u>: $(A-I)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 1 \\ -2 & -4 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}}_{= \underline{v}_1} \neq 0$

We got 2 generalized eigenvectors from the solution $\underline{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

~~To find the third generalized eigenvector $\underline{v}_3$ take another solution of $(A-I)^2\underline{v}_3 = 0$ such that $\underline{v}_1, \underline{v}_2, \underline{v}_3$ are linearly indep.~~

To find the third eigenvector $\underline{v}_3$, we take a linear combination of the original eigenvectors $\begin{pmatrix}-2\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}$

associated to $\lambda = 1$ which ~~is~~ such ~~&~~ that it is linearly independ. with $\underline{v}_1 = \begin{pmatrix}0\\1\\-2\end{pmatrix}$ and $\underline{v}_2 = \begin{pmatrix}1\\0\\0\end{pmatrix}$.

For instance we can take $\underline{v}_3 = \begin{pmatrix}-1\\0\\1\end{pmatrix}$.

In fact $\underline{v}_1, \underline{v}_2, \underline{v}_3$ are linearly indep. as $\det\begin{pmatrix}0 & 1 & -1\\1 & 0 & 0\\-2 & 0 & 1\end{pmatrix} = -1 \neq 0$.

The 3 linearly indep. solutions are given by,

$$\underline{x}_1(t) = \underline{v}_1 e^{\lambda t} = \begin{pmatrix}0\\1\\-2\end{pmatrix} e^t$$

$$\underline{x}_2(t) = (\underline{v}_1 t + \underline{v}_2) e^{\lambda t} = \begin{pmatrix}1\\t\\ \text{[illegible]}\end{pmatrix} e^t$$

$$\underline{x}_3(t) = \underline{v}_3 e^{\lambda t} = \begin{pmatrix}-1\\0\\1\end{pmatrix} e^t$$

The general solution is:

$$\underline{x}(t) = c_1 \underline{x}_1 + c_2 \underline{x}_2 + c_3 \underline{x}_3 =$$

$$= \begin{pmatrix} c_2 - c_3 \\ c_1 + c_2 t \\ -2c_1 - 2c_2 t + c_3 \end{pmatrix} e^t$$

Part III

Problem 9

$V_1 = 50$ gal $\quad V_2 = 25$ gal.

$r = 10$ gal/min $\quad x_1(0) = 15$

$x_2(0) = 0.$

$$\begin{cases} \dfrac{dx_1}{dt} = -k_1 x_1 + k_2 x_2 \\ \dfrac{dx_2}{dt} = k_1 x_1 - k_2 x_2 \end{cases}$$

$$k_1 = \frac{r}{V_1} = \frac{10}{50} = \frac{1}{5} \qquad k_2 = \frac{r}{V_2} = \frac{10}{25} = \frac{2}{5}$$

$$\begin{cases} \dfrac{dx_1}{dt} = -\dfrac{1}{5} x_1 + \dfrac{2}{5} x_2 \\ \dfrac{dx_2}{dt} = \dfrac{1}{5} x_1 - \dfrac{2}{5} x_2 \end{cases}$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \underbrace{\begin{pmatrix} -1/5 & 2/5 \\ 1/5 & -2/5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\det(A - \lambda I) = \det \begin{pmatrix} -1/5 - \lambda & 2/5 \\ \frac{1}{5} & -\frac{2}{5} - \lambda \end{pmatrix} =$$

$$= \left(-\frac{1}{5} - \lambda\right)\left(-\frac{2}{5} - \lambda\right) - \frac{2}{5}\frac{1}{5} =$$

$$= \frac{2}{25} + \frac{3}{5}\lambda + \lambda^2 - \frac{2}{25} = 0$$

$$25\lambda^2 + 15\lambda = 0$$

$$5\lambda(5\lambda + 3) = 0$$

$$\boxed{\lambda_1 = 0 \qquad \lambda_2 = -3/5 \qquad \text{both alg. multipl. } 1}$$

Eigenvector for $\lambda_1 = 0$.

Plug $\lambda = 0$, look for $\underline{v} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ such that $\underline{v} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{pmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & -2/5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \cancel{-\frac{1}{5}a + \frac{2}{5}b = 0} \\ \frac{1}{5}a - \frac{2}{5}b = 0 \end{cases} \longrightarrow \quad a = 2b$$

b is free

We get one eigenvector $\underline{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

and the solution $\boxed{\underline{x}_1(t) = \underline{v}_1 e^{0t} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$

Eigenvector for ~~...~~ $\lambda_2 = -\frac{3}{5}$

Plug $\lambda_2 = -3/5$ and look for a vector $\underline{v} = \begin{pmatrix} a \\ b \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ such that

$$\begin{pmatrix} 2/5 & 2/5 \\ 1/5 & 1/5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \cancel{2/5\, a + 2/5\, b = 0} \\ 1/5\, a + 1/5\, b = 0 \quad \rightarrow \quad a = -b \end{cases}$$

b is free

We get one eigenvector $\underline{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

and the solution

$$\underline{x}_2(t) = \underline{v}_2\, e^{\lambda_2 t} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3/5\, t}$$

The general solution is

$$\underline{x} = c_1\, \underline{x}_1 + c_2\, \underline{x}_2 = \begin{pmatrix} 2c_1 - c_2\, e^{-3/5 t} \\ c_1 + c_2\, e^{-3/5\, t} \end{pmatrix}$$

Initial conditions: (plug $t=0$)

$$\begin{cases} 2c_1 - c_2 = 15 \quad \rightarrow \quad c_2 = 2c_1 - 15 = -2c_2 - 15 \\ c_1 + c_2 = 0 \quad \rightarrow \quad c_1 = -c_2 \end{cases}$$

$$3c_2 = -15 \quad \longrightarrow \quad \boxed{c_2 = -15/3}$$

$$\boxed{c_1 = -c_2 = 15/3}$$

The particular solution is

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 10 + \frac{15}{3} e^{-3/5\,t} \\ 15/3 - \frac{15}{3} e^{-3/5\,t} \end{pmatrix}}$$

<u>Problem 10</u>

$$\begin{cases} x_1' = 5x_1 \\ x_2' = 10x_1 + 5x_2 \\ x_3' = 20x_1 + 30x_2 + 5x_3 \end{cases} \qquad \begin{cases} x_1(0) = 4 \\ x_2(0) = 5 \\ x_3(0) = 6 \end{cases}$$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \underbrace{\begin{pmatrix} 5 & 0 & 0 \\ 10 & 5 & 0 \\ 20 & 30 & 5 \end{pmatrix}}_{\substack{\parallel \\ A}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$A = \underbrace{\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}}_{\substack{\parallel \\ D}} + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 10 & 0 & 0 \\ 20 & 30 & 0 \end{pmatrix}}_{\substack{\parallel \\ B}}.$$

Want to find e^{At}.

Because $BD = DB$, we have

$$e^{At} = e^{(D+B)t} = e^{Dt}\, e^{Bt}$$

$$e^{Dt} = \begin{pmatrix} e^{5t} & 0 & 0 \\ 0 & e^{5t} & 0 \\ 0 & 0 & e^{5t} \end{pmatrix}$$ because D is diagonal.

Now we find e^{Bt}.

$$B^2 = \begin{pmatrix} 0 & 0 & 0 \\ 10 & 0 & 0 \\ 20 & 30 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 10 & 0 & 0 \\ 20 & 30 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 300 & 0 & 0 \end{pmatrix}$$

$$B^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 300 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 10 & 0 & 0 \\ 20 & 30 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$e^{Bt} = I + Bt + \frac{1}{2} B^2 t^2$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 10t & 0 & 0 \\ 20t & 30t & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 150t^2 & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 10t & 1 & 0 \\ 150t^2 + 20t & 30t & 1 \end{pmatrix}$$

Hence

$$e^{At} = \begin{pmatrix} e^{5t} & 0 & 0 \\ 0 & e^{5t} & 0 \\ 0 & 0 & e^{5t} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 10t & 1 & 0 \\ 150t^2 + 20t & 30t & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} e^{5t} & 0 & 0 \\ 10t\,e^{5t} & e^{5t} & 0 \\ (150t^2+20t)\cdot e^{5t} & (30t)\cdot(e^{5t}) & e^{5t} \end{pmatrix}$$

6. Particular solution of the system is

$$\underline{x}(t) = e^{At} \cdot \underline{x}_0 =$$

$$= \begin{pmatrix} e^{5t} & 0 & 0 \\ 10t\,e^{5t} & e^{5t} & 0 \\ (150t^2+20t)\cdot e^{5t} & (30t)\cdot e^{5t} & e^{5t} \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} =$$

$$= \begin{pmatrix} 4\,e^{5t} \\ (40t+5)\,e^{5t} \\ \left[4(150t^2+20t) + 150t + 6\right] e^{5t} \end{pmatrix}$$

Problem 12

$\beta(t) = 4 \qquad \delta(t) = P \qquad h = 3$

$$\frac{dP}{dt} = \big(\beta(t) - \delta(t)\big)P - h$$

$$\boxed{\frac{dP}{dt} = (4 - P)P - 3}$$

$\boxed{\text{Find CP}}$

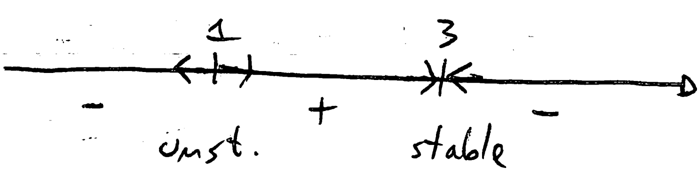
$$(4 - P)P - 3 = 0$$

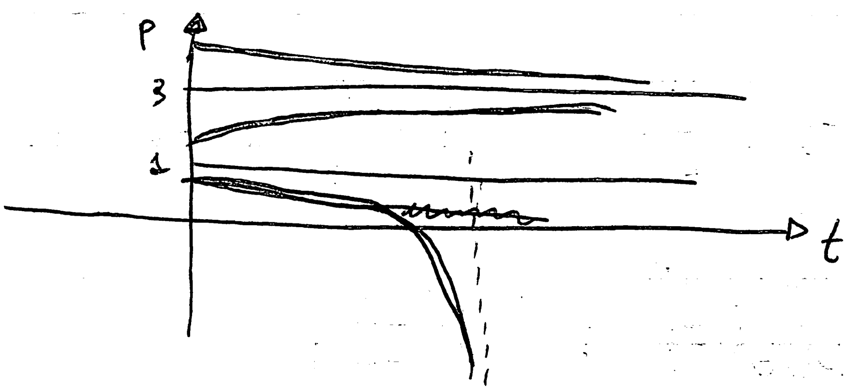
$$-P^2 + 4P - 3 = 0$$

$$P^2 - 4P + 3 = 0$$

$$(P-3)(P-1) = 0$$

$$P = 3 \qquad P = 1$$





Now we solve $\frac{dP}{dt} = (4-P)P - 3$

$$= -(P-3)(P-1)$$

$$\frac{dP}{(P-3)(P-1)} = -\,dt$$

$$\int \frac{dP}{(P-3)(P-1)} = \int -\,dt = -t + C$$

$$\frac{A}{P-3} + \frac{B}{P-1} = \frac{AP - A + BP - 3B}{(P-3)(P-1)} = \frac{P(A+B) - A - 3B}{(P-3)(P-1)}$$

$$\begin{cases} A + B = 0 \\ -A - 3B = 1 \end{cases} \rightarrow \quad A = -B$$

$$B = \frac{-1 - A}{3} = \frac{-1 + B}{3}$$

$$3B = -1 + B \rightarrow \boxed{\begin{array}{l} B = -1/2 \\ A = +1/2 \end{array}}$$

$$\int \frac{dP}{(P-3)(P-1)} = +\frac{1}{2}\int \frac{dP}{P-3} - \frac{1}{2}\int \frac{dP}{P-1} = +\frac{1}{2}\ln|P-3| - \frac{1}{2}\ln|P-1|$$

$$= \frac{1}{2}\ln\left|\frac{P-3}{P-1}\right|$$

Hence: $\frac{1}{2} \ln \left| \frac{P-3}{P-1} \right| = -t + C$

$$\ln \left| \frac{P-3}{P-1} \right| = -2t + 2C$$

$$\left| \frac{P-3}{P-1} \right| = e^{-2t} e^{2C}$$

$$\frac{P-3}{P-1} = \pm e^{-2t} e^{2C}$$

$$\frac{P-3}{P-1} = A e^{-2t} \qquad A = \pm e^{2C}$$

$$P-3 = (P-1) A e^{-2t} = AP e^{-2t} - A e^{-2t}$$

$$P(1 - A e^{-2t}) = 3 - A e^{-2t}$$

$$P = \frac{3 - A e^{-2t}}{1 - A e^{-2t}}$$

Initial condition: $P_0 = P(0) = \frac{3-A}{1-A}$

Solve for A:

$$(1-A) P_0 = 3 - A$$

$$A(1 - P_0) = 3 - P_0 \longrightarrow A = \frac{3 - P_0}{1 - P_0}$$

$$P(t) = \frac{3 - \frac{3-P_0}{1-P_0} e^{-2t}}{1 - \frac{3-P_0}{1-P_0} e^{-2t}} \cdot \frac{1-P_0}{1-P_0} = \frac{3(1-P_0) - (3-P_0) e^{-2t}}{(1-P_0) - (3-P_0) e^{-2t}}$$

Now we distinguish 3 cases:

CASE 1: $P_0 > 3$

Then $3 - P_0 < 0$ and $1 - P_0 < 0$

Hence $$P(t) = \frac{3(1-P_0) - (3-P_0)e^{-2t}}{(1-P_0) - (3-P_0)e^{-2t}} =$$

$$= \frac{-3(P_0 - 1) + (P_0 - 3)e^{-2t}}{-(P_0 - 1) + (P_0 - 3)e^{-2t}}$$

We check whether $P(t)$ explodes, namely if

$$-(P_0 - 1) + (P_0 - 3)e^{-2t} = 0 \quad \text{for some } t > 0$$

$$(P_0 - 3)e^{-2t} = (P_0 - 1)$$

$$e^{-2t} = \frac{P_0 - 1}{P_0 - 3}$$

$$-2t = \ln\left(\frac{P_0 - 1}{P_0 - 3}\right)$$

$$t = -\frac{1}{2} \underbrace{\ln\left(\frac{P_0 - 1}{P_0 - 3}\right)}_{\text{positive}}$$

Hence t is negative which impossible.

Hence $P(t)$ does not explode.

Hence $P(t)$ is a bounded population.

To find the limiting population,

we solve

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{-3(P_0-1) + (P_0-3)\overbrace{e^{-2t}}^{\to 0}}{-(P_0-1) + (P_0-3)\underbrace{e^{-2t}}_{0}} = \frac{-3(P_0-1)}{-(P_0-1)} = \boxed{3}$$

CASE 2 : $1 < P_0 < 3$

Also in this case $P(t)$ is bounded,

and the limiting population is 3.

CASE 3 : $0 < P_0 < 1$

Then $P_0 - 1 < 0$ and $P_0 - 3 < 0$

$$P(t) = \frac{3(1-P_0) - (3-P_0)\, e^{-2t}}{(1-P_0) - (3-P_0)\, e^{-2t}} = \qquad \text{where } \begin{matrix}(1-P_0) > 0 \\ (3-P_0) > 0\end{matrix}$$

$$= \frac{(3-P_0)\, e^{-2t} - 3(1-P_0)}{(3-P_0)\, e^{-2t} - (1-P_0)}$$

We check whether $P(t)$ explodes, namely

if $(3-P_0)\, e^{-2t} - (1-P_0) = 0$ for some $t > 0$.

$$e^{-2t} = \frac{1-P_0}{3-P_0} \quad \to \quad -2t = \ln\left(\frac{1-P_0}{3-P_0}\right)$$

$$t = -\frac{1}{2}\ln\left(\frac{1-P_0}{3-P_0}\right).$$

Now observe that

$$0 < \frac{1-P_0}{3-P_0} < 1 .$$

In fact $0 < \frac{1-P_0}{3-P_0}$ because $1-P_0 > 0$, $3-P_0 > 0$.

For the other inequality $\frac{1-P_0}{3-P_0} < 1$ we see that it is equivalent to:

$$1-P_0 < 3-P_0$$

$1 < 3$ which is true.

Then $\ln\left(\frac{1-P_0}{3-P_0}\right)$ is negative and

$t = -\frac{1}{2}\ln\left(\frac{1-P_0}{3-P_0}\right)$ is positive.

Hence $P(t)$ explodes when

$$t_{exp} = -\frac{1}{2}\ln\left(\frac{1-P_0}{3-P_0}\right).$$

Topics covered during the semester

MAT303: Calculus IV with applications

Part I

Chapter 1: First-order differential equations $\frac{dy}{dx} = f(x, y)$. Theorem of existence and uniqueness for first-order initial value problems. General Solution to $\frac{dy}{dx} = f(x)$ via integration. Velocity and acceleration of an object moving along a straight line. Separation of variables' technique $f(y)\frac{dy}{dx} = g(x)$. Population growth with constant birth and death rates $\frac{dP}{dt} = kP$. Linear first-order differential equations $\frac{dy}{dx} + P(x)y = Q(x)$. Mixture problems (tank problems). Substitution methods. Homogeneous equations $\frac{dy}{dx} = F(\frac{y}{x})$ (use the substitution $v = \frac{y}{x}$, $y = vx$). Bernoulli Equations $\frac{dy}{dx} + P(x)y = Q(x)y^n$ (use the substitution $v = y^{1-n}$). Exact differential equations $M(x, y)dx + N(x, y)dy = 0$, exactness and criterion of exactness. **Chapter 2**: Population models with non-constant birth and death rate $\frac{dP}{dt} = (\beta(t) - \delta(t))P$. Logistic equation $\frac{dP}{dt} = aP - bP^2 = kP(M - P)$ (where $k = b$ and $M = \frac{a}{b}$) and carrying capacity. Logistic equation with stocking and harvesting. Use of partial fraction technique to solve logistic equations. Explosion models and Extinction models. Equilibrium solutions and stability. Bifurcation. Acceleration-Velocity models with resistance.

Part II

Chapter 3: Homogeneous Second-Order linear equations $y'' + p(x)y' + q(x)y = 0$. Existence and uniqueness for linear equations with initial value data. Linear dependence of two functions and Wronskian. Wronskian theorem for the linear (in)dependence of two solution functions. The general solution for homogeneous second-order linear equations. Characteristic equation method to find two linearly independent solutions (distinct real roots, complex roots, multiple roots). Homogeneous nth-order linear equations $y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0$. Linear dependence of n functions on an interval and Wroskian. The useful trick: If the Wronskian $W(f_1, \ldots, f_n)$ of n functions is non-zero at one point of an interval I, then the n functions are linearly independent on I. General solution of nth-order homogeneous linear equations. Characteristic equation method to solve homogeneous linear equations with constant coefficients. The method of undetermined coefficients to solve non-homogeneous equations with constant coefficients. The method of variation of parameters to solve non-homogeneous equations with non-constant coefficients $y'' + P(x)y' + Q(x)y = f(x)$. Mechanical vibrations and standard form of solutions $x(t) = A(t)\cos(\omega t + \alpha)$.

Part III

Chapter 4: Simple 2×2 linear systems solvable by transforming the system into a second order linear equation with constant coefficients. Elimination method (homogeneous and non-homogeneous systems). **Chapter 5**: Matrices and determinants. Sum and product of matrices. Row reduction technique to solve linear systems. Definition of linear dependence of n vectors of scalars. Wronskian of n vectors and sufficient criterion for the linear dependence of n vectors of functions on an interval. General solution of linear systems of first-order linear equations $\underline{x}' = A\underline{x}$. Definition of eigenvalue and eigenvector. The eigenvalue method for the solution of linear homogeneous systems $\underline{x}' = A\underline{x}$ (distinct real roots, complex roots, multiple roots and generalized eigenvectors). Multiple tanks problems. Definition of exponential matrix. Use of the exponential matrix to solve systems of type $A = rI + (\text{nilpotent})$. Undetermined coefficients method to solve non-homogeneous linear systems $\underline{x}' = A\underline{x} + \underline{F}(t)$.

MAT303: Calculus IV with applications
Practice Midterm 2
Spring 2016

The second midterm is scheduled for Wednesday April 6th and it covers sections 2.1, 2.2, 2.3, 3.1, 3.2, 3.3 and 3.5 of the textbook. There will be two review sessions on Wednesday March 30th and Monday April 4th. You do not need to submit this practice midterm as an assignment homework, but you should know how to solve all the problems anyways. For further problems and exercises you can consult either the book or past midterms of previous MAT303 classes (see the link *http://www.math.stonybrook.edu/mathematics-departmentcourse-web-pages*).

Exercise 1: Consider an animal population $P(t)$ with constant death rate $\delta = 1/100$ and with birth rate β proportional to P. Suppose that $P(0) = 200$ and $P'(0) = 2$. When does explosion occur?

Exercise 2: Consider the following population (autonomous) model:

$$\frac{dP}{dt} = P^2 - 3P + 2.$$

Find the equilibrium solutions and classify them as stable or unstable. Sketch then two solutions curves for the initial conditions $P(0) = 1.9$ and $P(0) = 2.1$. Finally Solve the differential equation with the initial condition $P(0) = 3$ and find at what time the population explodes to infinity.

Exercise 3: Consider the following differential equation:

$$\frac{dx}{dt} = (x+2)(x-2)^2.$$

Find the equilibrium solutions, classify them as stable or unstable, and sketch the solution curves corresponding to the initial conditions $x(0) = x_0$, where x_0 can be either a negative or a positive real number.

Exercise 4: The differential equation

$$\frac{dx}{dt} = \frac{1}{100}x(x-5) + s$$

models a logistic population with stocking at rate s. Determine the bifurcation point and the equilibrium solutions as s varies. Finally classify the equilibrium solutions as stable or unstable, and sketch the solution curves for the value $s = 0$.

Exercise 5: Suppose that a body moves through a resisting medium with resistance proportional to v^α, so that

$$\frac{dv}{dt} = -kv^\alpha, \quad v(0) = v_0, \qquad k = \text{positive constant}$$

(a). If $\alpha = 1$, we have seen in a previous homework that the velocity function is $v(t) = v_0 e^{-kt}$ with a position function $x(t) = x_0 + \left(\frac{v_0}{k}\right)(1 - e^{-kt})$. Moreover we have seen that the body travels only a finite distance.

(b). Find the velocity and position functions for $\alpha = \frac{3}{2}$ and $\alpha = 3$. In which cases does the body travel a finite distance?

Exercise 6: Find the general solution for the following homogeneous linear differential equations:

$$y^{(3)} - 5y'' + 8y' - 4y = 0$$
$$y^{(4)} - y^{(3)} + y'' - 3y' - 6y = 0$$
$$y^{(5)} - y' = 0$$
$$y^{(4)} + 2y'' + y = 0.$$

Exercise 7: Find the general solution of the following nonhomogenous differential equations:

$$y'' - y' - 2y = 3x + 4$$
$$y'' + 2y' + 5y = e^x \sin(x)$$
$$y''' - 2y'' + y' = \sin(x)$$
$$y'' + y = \sin(x) + x\cos(x)$$
$$y'' - 4\,y = 2e^{2x}$$

Exercise 8: Use the Wronskian to check whether the following functions are linearly independent on the interval $(-1, 1)$:

$$f(x) = 1, \quad g(x) = \cos(x), \quad h(x) = \sin(x)$$
$$f(x) = x, \quad g(x) = e^x, \quad h(x) = xe^x$$

Practice Midterm 2

(1) $\beta(t) = kP$

$\delta(t) = \frac{1}{100}$

$$\frac{dP}{dt} = (\beta(t) - \delta(t))P = \left(kP - \frac{1}{100}\right)P$$

Impose initial conditions:

$P'(0) = 2, \quad P(0) = 200$

$$\frac{dP}{dt}\Big|_{t=0} = \left(kP(0) - \frac{1}{100}\right)P(0)$$

$$2 = \left(200k - \frac{1}{100}\right)200$$

$$2 = 40000k - 2$$

$$k = \frac{4}{40000} = \frac{1}{10^4}$$

Now solve $\frac{dP}{dt} = \left(\frac{1}{10^4}P - \frac{1}{100}\right)P = \frac{1}{100}\left(\frac{1}{100}P - 1\right)P$

Separable

$$\frac{dP}{\left(\frac{1}{100}P - 1\right)P} = \frac{1}{100}\,dt$$

$$\int \frac{dP}{\left(\frac{1}{100}P - 1\right)P} = \int \frac{1}{100}\,dt = \frac{t}{100} + C$$

Use partial fraction:

$$\frac{1}{\left(\frac{1}{100}P-1\right)P} = \frac{A}{\frac{1}{100}P-1} + \frac{B}{P} =$$

$$= \frac{AP + \frac{1}{100}PB - B}{\left(\frac{1}{100}P-1\right)P} = \frac{\left(A+\frac{1}{100}B\right)P - B}{\left(\frac{1}{100}P-1\right)P}$$

Equate the coeff.

$$\begin{cases} A + \frac{1}{100}B = 0 \\ -B = 1 \longrightarrow B = -1 \end{cases}$$

$$A = -\frac{1}{100}B = \frac{1}{100}$$

Hence $\displaystyle\int \frac{dP}{\left(\frac{1}{100}P-1\right)P} = \frac{1}{100}\int \frac{dP}{\frac{1}{100}P-1} - \int \frac{dP}{P} =$

$$= \frac{1}{100} \cdot 100 \cdot \ln\left|\frac{1}{100}P-1\right| - \ln|P|$$

$$= \ln\left|\frac{\frac{1}{100}P-1}{P}\right|$$

Putting together:

$$\ln\left|\frac{\frac{1}{100}P-1}{P}\right| = \frac{1}{100}t + C$$

$$\frac{\frac{1}{100}P-1}{P} = \pm e^{t/100} e^{C}$$

Set $A = \pm e^{c}$:

$$\frac{\frac{1}{100}P - 1}{P} = A e^{t/100}$$

$$\frac{1}{100}P - 1 = A P e^{t/100}$$

$$P - 100 = 100 A P e^{t/100}$$

$$P\left(1 - 100 A e^{t/100}\right) = 100$$

$$P = \frac{100}{1 - 100 A e^{t/100}}$$

Impose the initial condition
$P(0) = 200$

$$200 = \frac{100}{1 - 100A}$$

$$2 = \frac{1}{1 - 100A}$$

$$1 - 100A = \frac{1}{2}$$

$$-100A = -\frac{1}{2}$$

$$A = \frac{1}{200}$$

Hence

$$\boxed{P(t) = \frac{100}{1 - \frac{1}{2} e^{t/100}}}$$

Explosion happens when

$P(t) \to \infty$ for $t \to t_0$ for some finite time t_0

But $P(t)$ goes to ∞ when the denominator becomes zero.

Thus we Need to solve for t the equation:

$$1 - \frac{1}{2} e^{t/100} = 0$$

$$e^{t/100} = 2$$

$$t/100 = \ln(2)$$

$$\boxed{t = 100 \ln(2)}$$

(2)

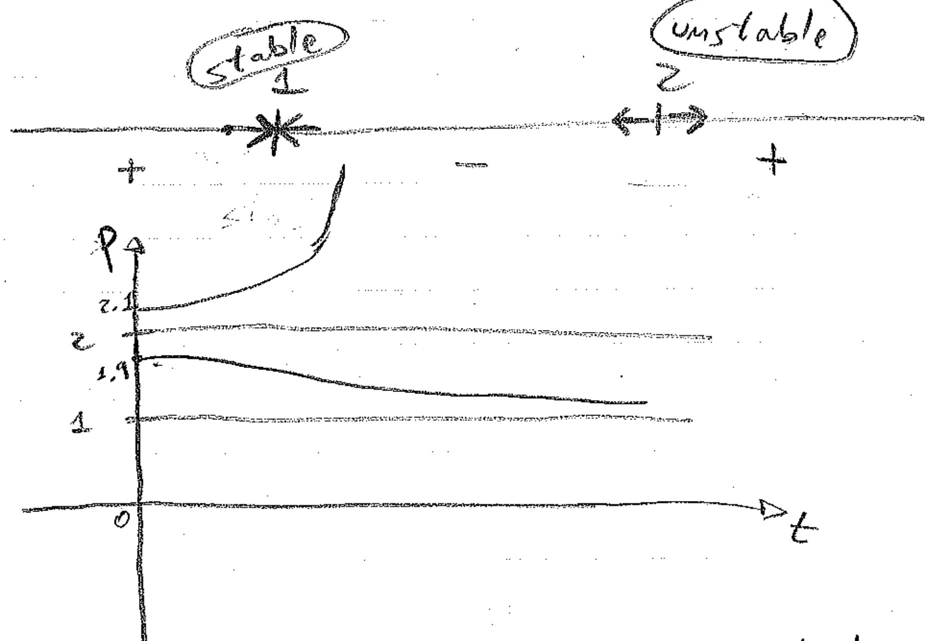
$$\frac{dP}{dt} = P^2 - 3P + 2$$

To find the equilibrium, we solve

$$P^2 - 3P + 2 = 0$$

$$(P-1)(P-2) = 0$$

$$P = 1, \quad P = 2$$



Solve the equation wt the initial condition

$$P(0) = 3.$$

$$\int \frac{dP}{(P-1)(P-2)} = \int dt + C = t + C$$

Partial fraction

$$\frac{1}{(P-1)(P-2)} = \frac{A}{P-1} + \frac{B}{P-2} = \frac{AP-2A+BP-B}{(P-1)(P-2)}$$

$$= \frac{(A+B)P-2A-B}{(P-1)(P-2)}$$

Equate the coeff.

$$\begin{cases} A+B=0 \longrightarrow A=-B \\ -2A-B=1 \end{cases}$$

$$2B-B=1 \longrightarrow B=1$$

$$A=-1$$

$$\int \frac{dP}{(P-1)(P-2)} = -\int \frac{dP}{P-1} + \int \frac{dP}{P-2} =$$

$$= -\ln|P-1| + \ln|P-2|$$

$$= \ln\left|\frac{P-2}{P-1}\right| = t + C$$

$$\frac{P-2}{P-1} = \pm e^t e^C = Ae^t$$

$$(P-2) = (P-1)Ae^t$$

$$P(1-Ae^t) = 2 - Ae^t$$

$$P = \frac{2-Ae^t}{1-Ae^t}$$

Impose initial condition $P(0) = 3$

$$3 = \frac{2 - A}{1 - A}$$

$$3 - 3A = 2 - A$$

$$-2A = -1$$

$$A = 1/2$$

The Population explodes when the denominator is zero:

$$1 - \frac{1}{2} e^t = 0$$

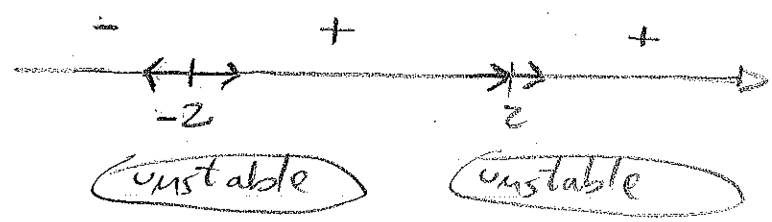
$$e^t = 2$$

$$t = \ln 2$$

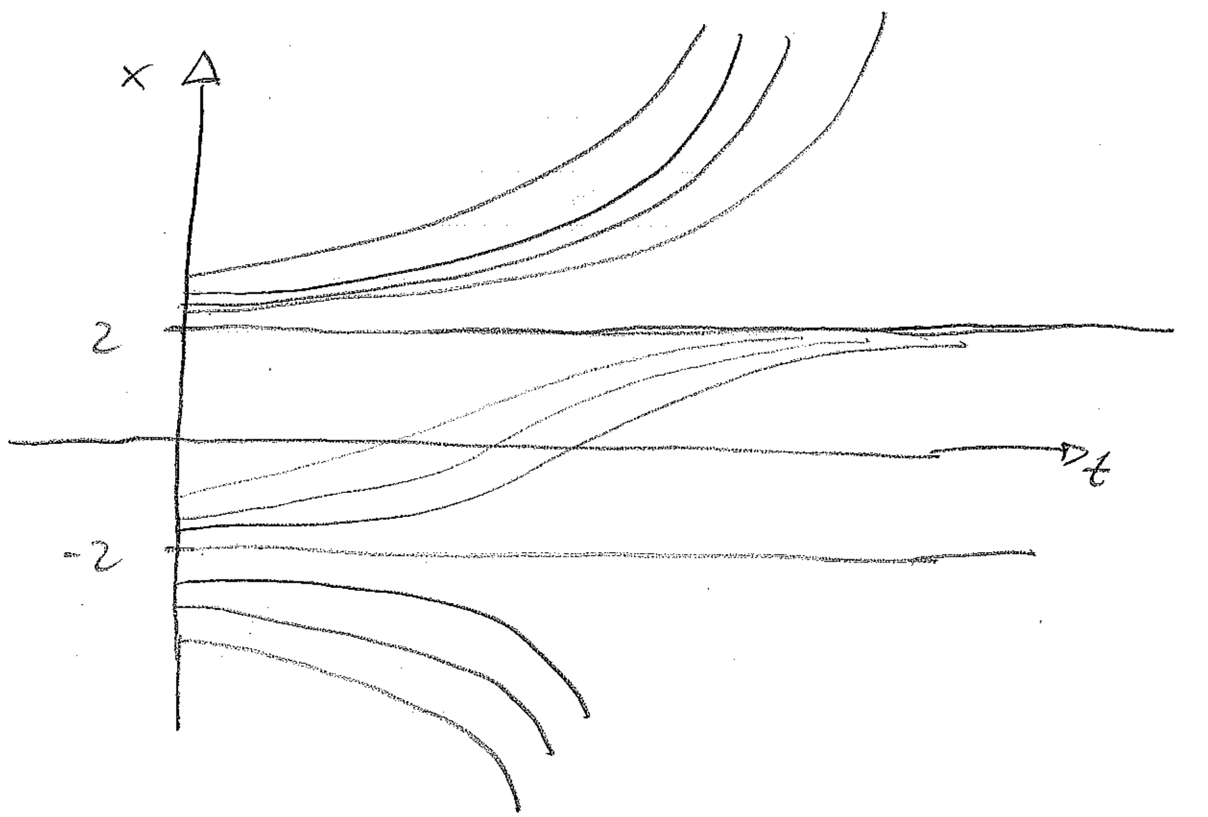
(3)

$$\frac{dx}{dt} = (x+2)(x-2)^2$$

Equilibrium : $x = -2$, $x = 2$



sketch of solutions with initial condition $x(0) = x_0$.



(4) $$\frac{dx}{dt} = \frac{1}{100}\, x(x-5) + s$$

$$\frac{1}{100}\, x(x-5) + s = 0$$

$$x^2 - 5x + 100s = 0$$

$$x = \frac{5 \pm \sqrt{25 - 400s}}{2}$$

# of Solutions vary according to the sign of the radicand.

To find the bifurcation values, set

$$25 - 400s = 0 \rightarrow s = \frac{25}{400} = \frac{1}{16}$$

Bifurcation value.

CASE 1: $\boxed{s < \frac{1}{16}}$

Then $25 - 400s > 0$ and there are 2 solutions

$$x = \frac{5 \pm 5\sqrt{1 - 16s}}{2} \leftarrow 2 \text{ equilibriums}$$

CASE 2: $s = \frac{1}{16}$ Then $25 - 400s = 0$ and there is only 1 solution

$$x = \frac{5 \pm 0}{2} = \frac{5}{2} \leftarrow \text{One Equilibrium}$$

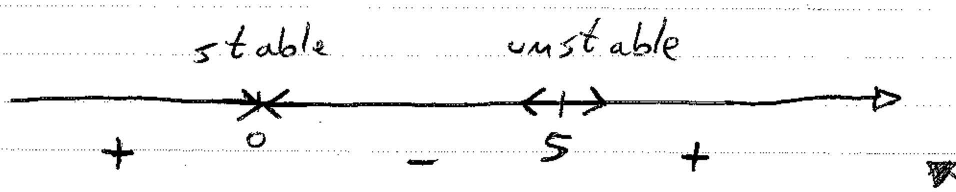
CASE 3: $s > \frac{1}{16}$

Then $25 - 400s < 0$ and hence there are no real solutions.

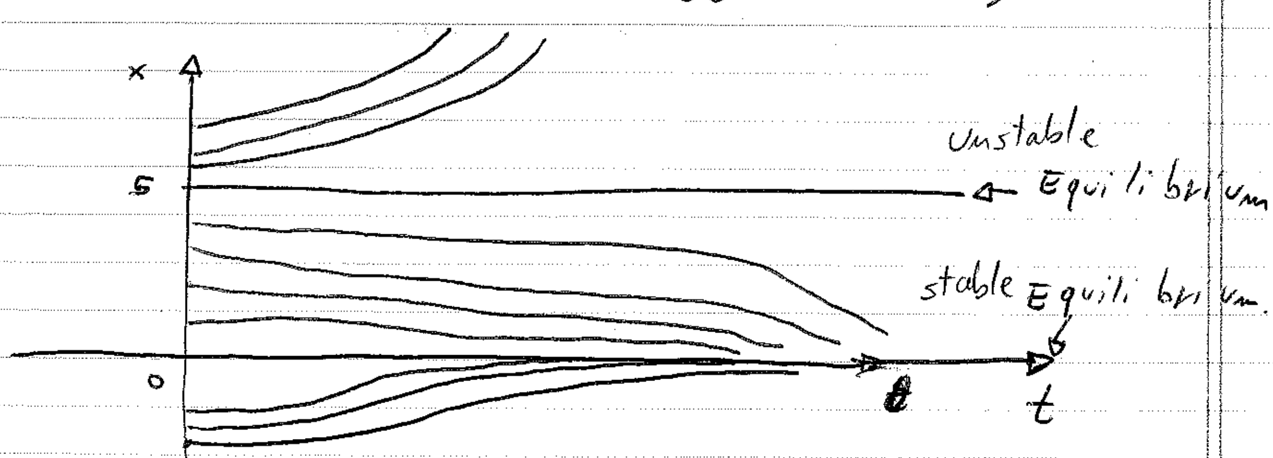
Thus there are no equilibria.

We now classify the equilibrium solutions for $s = 0$.

$$x = \frac{5 \pm \sqrt{25}}{2} = \frac{5 \pm 5}{2} = \begin{cases} \boxed{5} \\ \boxed{0} \end{cases} \quad 2 \text{ Equilibriums}$$



Study the sign of $\frac{1}{100} x(x-5)$.



(5)

$$\frac{dv}{dt} = -k v^{\alpha} \quad , \quad k > 0$$

$$\alpha = \frac{3}{2}$$

$$\frac{dv}{v^{3/2}} = -k\, dt$$

$$\int \frac{dv}{v^{3/2}} = -k \int dt = -kt + C$$

$$= \frac{v^{-3/2+1}}{-3/2+1} = \frac{v^{-1/2}}{-1/2} = \frac{-2}{\sqrt{v}}$$

$$-\frac{2}{\sqrt{v}} = -kt + C$$

$$\frac{2}{\sqrt{v}} = kt - C$$

$$\frac{\sqrt{v}}{2} = \frac{1}{kt - C}$$

$$\sqrt{v} = \frac{2}{kt - C} \qquad v = \left(\frac{2}{kt - C} \right)^2$$

Impose $v(0) = v_0$

$$v_0 = \left(\frac{2}{-C} \right)^2 = \frac{4}{C^2} \longrightarrow C^2 = \frac{4}{v_0}$$

$$C = \pm \frac{2}{\sqrt{v_0}}$$

We have to decide between $\pm \frac{2}{\sqrt{v_0}}$.

At some point we had the equation

$$\sqrt{v} = \frac{2}{kt - c}$$

Since $\sqrt{v}$ is positive, this means that $\frac{2}{kt-c}$ is positive.

~~As~~ This means that $kt - c > 0$ for all $t \geq 0$.

~~Thu~~ In particular for $t = 0$.

Hence $-c > 0$, and thus $c < 0$.

We see that c is negative and therefore $c = -\frac{2}{\sqrt{v_0}}$

The solution is

$$\boxed{v = \frac{4}{\left(kt + \frac{2}{\sqrt{v_0}}\right)^2}}$$

To find $x(t)$ we integrate:

$$x(t) = \int \frac{4}{\left(kt + \frac{2}{\sqrt{v_0}}\right)^2} dt + C$$

$$= \frac{4}{k} (-1) \frac{1}{kt + \frac{2}{\sqrt{v_0}}} + C$$

$$= \left(-\frac{4}{k}\right) \cdot \frac{1}{kt + \frac{2}{\sqrt{v_0}}} + C$$

Impose initial condition: $x(0) = x_0$

$$x_0 = \left(-\frac{4}{k}\right) \frac{\sqrt{v_0}}{2} + C$$

$$C = x_0 + \frac{4}{k} \frac{\sqrt{v_0}}{2} = x_0 + \frac{2}{k}\sqrt{v_0}$$

$$\boxed{x(t) = \left(-\frac{4}{k}\right) \frac{1}{kt + \frac{2}{\sqrt{v_0}}} + x_0 + \frac{2}{k}\sqrt{v_0}}$$

As:

$$\lim_{t \to \infty} x(t) = 0 + x_0 + \frac{2}{k}\sqrt{v_0} \quad \boxed{\text{is finite}}$$

we have that the particle travels a finite distance.

6

a $\boxed{y''' - 5y'' + 8y' - 4y = 0}$

$$r^3 - 5r^2 + 8r - 4 = 0$$

Need to find a root of the poly.

The rational roots of a polynomial divide the constant term, in this case -4.

Hence we start by trying all divisors of -4, whic are

$$\pm 1, \ \pm 2, \ \pm 4.$$

Try $+1$ $\quad (1)^3 - 5(1)^2 + 8(1) - 4 = 0$

The root $r=1$ satisfies the polyn.

$\Rightarrow (r-1)$ is a factor of the polym.

Now we can perform a long division

$$\begin{array}{r|l}
r^3 - 5r^2 + 8r - 4 & r - 1 \\
\hline
r^3 - r^2 & r^2 - 4r + 4 \\
\hline
'' \ -4r^2 + 8r - 4 & \\
-4r^2 + 4r & \\
\hline
'' \quad 4r - 4 & \\
4r - 4 & \\
\hline
'' \quad '' &
\end{array}$$

Hence $r^3 - 5r^2 + 8r - 4 = (r-1)(r^2 - 4r + 4) = 0$

$$r^2 - 4r + 4 = (r-2)^2$$

We have one solution $r = 1$ with mult $= 1$

and one root $r = 2$ with mult $= 2$

$$y(x) = c_1 e^x + c_2 e^{2x} + c_3 x e^{2x}$$

---

6 b $\boxed{y^{(4)} - y^{(3)} + y'' - 3y' - 6y = 0}$

$$r^4 - r^3 + r^2 - 3r - 6 = 0$$

Try to see if divisors of -6 are roots.

check : $\pm 1, \pm 2, \pm 3, \pm 6$

Try 1 : $1^4 - 1^3 + 1^2 - 3(1) - 6 \neq 0$

No!

Try -1 : $(-1)^4 - (-1)^3 + (-1)^2 - 3(-1) - 6$

$= 1 + 1 + 1 + 3 - 6 = 0$

Yes

Hence $r = -1$ is a root

and $(r+1)$ is a factor.

We can do a long division

$$\begin{array}{r|l} r^4 - r^3 + r^2 - 3r - 6 & r + 1 \\ \cline{2-2} r^4 + r^3 & r^3 - 2r^2 + 3r - 6 \\ \cline{1-1} " \quad -2r^3 + r^2 - 3r - 6 & \\ -2r^3 - 2r^2 & \\ \cline{1-1} " \quad 3r^2 - 3r - 6 & \\ 3r^2 + 3r & \\ \cline{1-1} " \quad -6r - 6 & \\ -6r - 6 & \\ \cline{1-1} " \quad " & \end{array}$$

Hence $r^4 - r^3 + r^2 - 3r - 6 = (r+1)(r^3 - 2r^2 + 3r - 6)$.

Now we look for a root of $r^3 - 2r^2 + 3r - 6$.

~~We look for~~ We try the divisors of -6

which are $\pm 1, \pm 2, \pm 3, \pm 6$.

Try 1 : $1 - 2 + 3 - 6 = -4 \neq 0$ No

Try -1 : $-1 - 2 - 3 - 6 \neq 0$ No

Try 2 : $8 - 8 + 6 - 6 = 0$ Yes

Hence $r = 2$ is a root and $(r-2)$ is a factor

Perform a long division

$$
\begin{array}{r|l}
r^3 - 2r^2 + 3r - 6 & r - 2 \\ \cline{2-2}
r^3 - 2r^2 & r^2 + 3 \\ \cline{1-1}
'' \quad '' \quad 3r - 6 & \\
3r - 6 & \\ \cline{1-1}
'' \quad '' &
\end{array}
$$

Hence $\quad r^3 - 2r^2 + 3r - 6 = (r-2)(r^2+3)$

To recap:

$$r^4 - r^3 + r^2 - 3r - 6 = (r+1)(r-2)(r^2+3) = 0$$

One root $r = -1$ wt. mult $= 1$

One root $r = 2$ wt. mult. $= 1$

One pair of complex conjugate roots

$$r = \pm\sqrt{-3} = \pm\sqrt{3}\, i$$

wt. mult. $= 1$.

$$y(x) = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{0x} \cos(\sqrt{3}\, x)$$
$$+ c_4 e^{0x} \sin(\sqrt{3}\, x).$$

6 e $\boxed{y^{(5)} - y' = 0}$

$r^5 - r = 0$

$r(r^4 - 1) = 0$

$r(r^2+1)(r^2-1) = 0$

Have $r = 0$ wt mult. 1

$r = \pm i$ wt mult. 1

$r = \pm 1$ wt. mult 1

$y(x) = c_1 + c_2 \cos(x) + c_3 \sin(x) +$
$+ c_4 e^x + c_5 e^{-x}$

6 d

$$y^{(4)} + 2y'' + y = 0$$

$$r^4 + 2r^2 + 1 = 0$$

$$(r^2 + 1)^2 = 0$$

We have $r = \pm i$ w/t. mult. $= 2$

$$y(x) = c_1 \cos x + c_2 \sin x + c_3 x \cos x + $$
$$+ c_4 x \sin x$$

7b

$$y'' + 2y' + 5y = e^x \sin x$$

$$r^2 + 2r + 5 = 0$$

$$r = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 4i$$

$$y_c(x) = c_1 e^{-x} \cos(4x) + c_2 e^{-x} \sin(4x)$$

Guess $y_p(x) = A e^x \cos x + B e^x \sin x$

$$y_p'(x) = A e^x \cos x - A e^x \sin x + B e^x \sin x + B e^x \cos x$$

$$= (A+B) e^x \cos x + (B-A) e^x \sin x$$

$$y_p''(x) = (A+B) e^x \cos x - (A+B) e^x \sin x + (B-A) e^x \sin x + (B-A) e^x \cos x$$

$$= 2B e^x \cos x - 2A e^x \sin x$$

Plug-in

$$2B e^x \cos x - 2A e^x \sin x + 2(A+B) e^x \cos x + 2(B-A) e^x \sin x + 5A e^x \cos x + 5B e^x \sin x$$

$$= e^x \sin x$$

$$(7A+4B)e^x\cos x+(-4A+7B)e^x\sin x=e^x\sin x$$

$$\begin{cases} 7A+4B=0 & \rightarrow\ A=-\frac{4}{7}B \\ -4A+7B=1 \end{cases}$$

$$\frac{16}{7}B+7B=1$$

$$16B+49B=7$$

$$\cancel{33}\,B=7$$

$$B=7/33$$

$$A=-\frac{4}{7}\left(\frac{7}{33}\right)=-\frac{4}{33}.$$

$$\boxed{y_p(x)=-\frac{4}{33}e^x\cos x+\frac{7}{33}e^x\sin x}$$

$y(x)=$ general $=y_c(x)+y_p(x)=$

$$=c_1e^{-x}\cos(4x)+c_2e^{-x}\sin(4x)-\frac{4}{33}e^x\cos x+\frac{7}{33}e^x\sin x.$$

7c

$$y'' + y = \sin x + x\cos x$$

Find $y_c(x)$.

$r^2 + 1 = 0 \qquad r = \pm i$

$$y_c(x) = c_1\cos(x) + c_2\sin(x)$$

Find $y_p(x)$:

We find $y_p(x)$ for $y'' + y = \sin x$ and $y'' + y = x\cos x$ independently.

Then $y_p(x)$ for $y'' + y = \sin x + x\cos x$ would just be the sum.

Find y_p for $y'' + y = \sin x$.

Guess: $y_p(x) = A\sin x + B\cos x$

$y_p'(x) = A\cos x - B\sin x$

$y_p''(x) = -A\sin x - B\cos x$

$$-A\sin x - B\cos x + A\sin x + B\cos x = 0$$

Does not work!

We change our guess in:

$$Y_p(x) = Ax \sin x + Bx \cos x$$

$$Y_p'(x) = A \sin x + Ax \cos x +$$
$$+ B \cos x - Bx \sin x$$

$$Y_p''(x) = A \cos x + A \cos x - Ax \sin x$$
$$- B \sin x - B \sin x - Bx \cos x$$
$$= 2A \cos x - 2B \sin x - Ax \sin x$$
$$- Bx \cos x$$

Plug-in:

$$2A \cos x - 2B \sin x - Ax \cancel{\sin x} - Bx \cancel{\cos x}$$
$$+ Ax \cancel{\sin x} + Bx \cancel{\cos x} = \sin x$$

$$\begin{cases} 2A = 0 \\ -2B = 1 \end{cases} \rightarrow \boxed{\begin{array}{l} A = 0 \\ B = -1/2 \end{array}}$$

$$Y_p(x) = -\frac{1}{2} x \cos x$$

Find y_p for $y'' + y = x\cos x$

Guess: $y_p(x) = (Ax + B)\cos x +$
$+ (Cx + D)\sin x$

$$y_p' = A\cos x - Ax\sin x - B\sin x + C\sin x + Cx\cos x + D\cos x$$

$$y_p'' = -A\sin x - A\sin x - Ax\cos x - B\cos x + C\cos x + C\cos x - Cx\sin x - D\sin x =$$
$$= -2A\sin x + 2C\cos x - Ax\cos x - Cx\sin x - B\cos x - D\sin x$$

Plug-in:

$$-2A\sin x + 2C\cos x - B\cos x - D\sin x - Ax\cos x - Cx\sin x + Ax\cos x + B\cos x + Cx\sin x + D\sin x = x\cos x.$$

Does not work! b/c

$-2A\sin x + 2C\cos x$

cannot be $x\cos x$.

We Try another guess:

$$y_p(x) = (Ax^2 + Bx)\cos x + (Cx^2 + Dx)\sin x$$

$$y_p'(x) = 2Ax\cos x - Ax^2\sin x + B\cos x - Bx\sin x$$
$$+ 2Cx\sin x + Cx^2\cos x + D\sin x + Dx\cos x$$

$$y_p'' = 2A\cos x - 2Ax\sin x - 2Ax\sin x - Ax^2\cos x$$
$$- B\sin x - B\sin x - Bx\cos x$$
$$+ 2C\sin x + 2Cx\cos x + 2Cx\cos x$$
$$- Cx^2\sin x + D\cos x + D\cos x - Dx\sin x$$

$$y_p'' + y_p = -4Ax\sin x + 4Cx\cos x$$
$$+ (2A + 2D)\cos x + (-2B + 2C)\sin x = x\cos x$$

$$\begin{cases} -4A = 0 & \to A = 0 \\ 4C = 1 & \to C = 1/4 \\ 2A + 2D = 0 & \to D = 0 \\ -2B + 2C = 0 & \to B = C = 1/4 \end{cases}$$

$$y_p(x) = \frac{1}{4}x\cos x + \frac{1}{4}x^2\sin x$$

Therefore y_p for $y'' + y = \sin x + x\cos x$

is

$$\boxed{\begin{aligned} y_p(x) &= -\frac{1}{2}x\cos x + \frac{1}{4}x\cos x + \frac{1}{4}x^2\sin x \\ &= -\frac{1}{4}x\cos x + \frac{1}{4}x^2\sin x \end{aligned}}$$

To conclude the general solution of the equation is:

$$\boxed{y(x) = c_1\cos x + c_2\sin x - \frac{1}{4}x\cos x + \frac{1}{4}x^2\sin x}$$

Exercise 8

$$f(x) = x \qquad g(x) = e^x \qquad h(x) = x e^x$$

$$W(f, g, h) = \det \begin{pmatrix} x & e^x & xe^x \\ 1 & e^x & xe^x + e^x \\ 0 & e^x & xe^x + 2e^x \end{pmatrix}$$

$$= x \det \begin{pmatrix} e^x & xe^x + e^x \\ e^x & xe^x + 2e^x \end{pmatrix} - (1) \det \begin{pmatrix} e^x & xe^x \\ e^x & xe^x + 2e^x \end{pmatrix}$$

$$= x \left(x e^{2x} + 2e^{2x} - x e^{2x} - e^{2x} \right) - \left(x e^{2x} + 2 e^{2x} - x e^{2x} \right)$$

$$= x \left(e^{2x} \right) - 2 e^{2x} = (x - 2) e^{2x}$$

There is at least a point of the interval $(-1, 1)$, for instance $x=0$, that makes $W \neq 0$.

In fact W in $x=0$ is $(0-2)e^0 = -2 \neq 0$

Hence f, g, h are linearly independent on $(-1, 1)$.

MAT303: Calculus IV with applications
Practice Midterm 1
Spring 2016

Midterm I is scheduled on March 2nd in class (Library W4550, 10am-10.53am). It will cover sections 1.1 through 1.6. of the book and it will consist of 5 problems. The use of calculators, books and notes is not allowed. Moreover every single answer you give to the problems should be well motivated so that the grader can easily understand your work. For a good preparation for the midterm you should work out and master *all* problems in this practice exam (even the problems with an asterisk). For further problems and exercises you can consult the book or past midterms of previous MAT303 classes (see the link *http://www.math.stonybrook.edu/mathematics-department-course-web-pages*).

Regarding Assignment 4, you will need to turn in *all* problems without the asterisk, plus 2 problems with an asterisk at your choice. Assignment 4 is due Monday February 29th.

Exercise 1: Find the general solution of the following differential equations:

(i).
$$\frac{dy}{dx} = xy - \frac{x}{y}$$

(ii).
$$\frac{dy}{dx} = \sqrt{x+y+1}$$

(iii).
$$(x+2y)\frac{dy}{dx} = y$$

(iv).
$$\frac{dy}{dx} = x - \frac{1}{x^2-2y}$$

Exercise 2*: Find the solution of the following initial value problems:

(i).
$$y\,\frac{dy}{dx} - y = \sqrt{x^2+y^2}; \qquad y(1) = 1$$

(ii).
$$x^2y + 2xy = 5y^4; \qquad y(1) = 0$$

Exercise 3: Find a general solution of the following linear first-order equations:

(i).
$$2x\frac{dy}{dx} - 3y = 9x^3$$

(ii).
$$x\frac{dy}{dx} + y = 3xy \qquad y(1) = 0$$

(iii).
$$x^3\,\frac{dy}{dx} = xy + 1; \qquad y(1) = 0$$

Exercise 4: Check the exactness of the following differential equations and find the solution.

(i).

$$(3x^2 + 2y^2)dx + (4xy + 6y^2)dy = 0$$

(ii).

$$\big(x + \arctan(y)\big)dx + \Big(\frac{x+y}{1+y^2}\Big)dy = 0$$

(iii).

$$\big(e^x \sin(y) + \tan(y)\big)dx + \big(e^x \cos(y) + x\sec^2(y)\big)dy = 0$$

Exercise 5: Check whether the hypothesis of the theorem of existence and uniqueness for initial value problems are satisfied for the following problem. If so, find the unique solution.

(i).

$$\frac{dy}{dx} = \frac{x-1}{y}; \qquad y(1) = 0$$

(ii).

$$\frac{dy}{dx} = \frac{x-1}{y}; \qquad y(0) = 1$$

(iii).

$$\frac{dy}{dx} = \sqrt[3]{y}; \qquad y(-1) = -1$$

(iv).

$$\frac{dy}{dx} = \sqrt{y}; \qquad y(-1) = -1.$$

(v). In general, for which y_0 does the initial value problem

$$\frac{dy}{dx} = \sqrt{y}; \qquad y(0) = y_0$$

admit a unique solution?

Exercise 6: A car travelling at 88 ft/s skids 176 ft after its brakes are suddenly applied. Under the assumption that the braking system provides constant deceleration, what is that deceleration?. For how long does the skid continue?

Exercise 7: The amount $A(t)$ of atmospheric pollutants in a certain mountain valley grows naturally (i.e. it grows according to the exponential equation $\frac{dA}{dt} = kA$ where k is a constant) and is tripling every 7.5 years. If the initial amount is 10 pu (pollutant units), write a formula for $A(t)$ giving the amount (in pu) present after t years. How long does it take for $A(t)$ to centuplicate?

Exercise 8*: A tank initially contains 60 gal of pure water. Brine containing 1 lb of salt per gallon enters the tank at 2 gal/min, and the (perfectly mixed) solution leaves the tank at 3 gal/min; thus the tank is empty after exactly 1 h. **(a)** Find the amount of salt in the tank after t minutes. **(b)** What is the maximum amount of salt ever in the tank?

Exercise 9* (Newton's law of cooling): According to Newton's law of cooling, the time rate of change of the temperature $T(t)$ of a body immersed in a medium of constant temperature A is proportional to the difference $A - T$. That is,

$$\frac{dT}{dt} = k\,(A - T)$$

where k is a positive constant. Use now these information to answer to the following problem.

A cake is removed from an oven at 210°F and left to cool at room temperature, which is 70°F. After 30 min the temperature of the cake is 140°F. When will it be 100°F?

Exercise 10*: By using partial fraction integration technique, find the solution of the following initial value problem:

$$\frac{dP}{dt} = \frac{kP(10-P)}{10}; \quad P(0) = 1$$

where $k > 0$ is a positive constant (you can assume that both t and P are positive variables). When t gets large, does $P(t)$ approach some fixed value? If yes, what is this value?

Practice Midterm I
Solutions

MAT 303 : Calculus IV

Problem 1 (i).

$$\frac{dy}{dx} = xy - \frac{x}{y} \qquad \text{Separable}$$

$$\frac{dy}{dx} = \frac{xy^2 - x}{y} = \frac{x(y^2-1)}{y}$$

$$\frac{y}{y^2-1}\,dy = x\,dx$$

$$\int \frac{y}{y^2-1}\,dy = \int x\,dx + C = \frac{x^2}{2} + C$$

$\int$ substitution

$u = y^2 - 1$

$du = 2y\,dy$

$$\int \frac{du/2}{u} = \frac{1}{2}\ln|u| = \frac{1}{2}\ln|y^2-1|$$

Hence $\frac{1}{2}\ln|y^2-1| = \frac{x^2}{2} + C$

$$\ln|y^2-1| = x^2 + 2C$$

$$|y^2-1| = e^{x^2+2C} = e^{x^2}e^{2C}$$

$$Y^2 - 1 = \pm e^{x^2} e^{2C}$$

Set $C' = \pm e^{2C}$ constant

$$Y^2 - 1 = C' e^{x^2}$$

$$Y^2 = C' e^{x^2} + 1$$

$$Y = \pm\sqrt{C' e^{x^2} + 1}$$

<u>Problem 1 (ii)</u>

$$\frac{dY}{dx} = \sqrt{x + Y + 1}$$

Use a substitution.

$$u^2 = x + Y + 1$$

$$Y = u^2 - x - 1$$

$$\frac{dY}{dx} = 2u\frac{du}{dx} - 1$$

Rewrite the equation:

$$2u\frac{du}{dx} - 1 = \sqrt{u^2} = u$$

$$2u\frac{du}{dx} = u + 1 \qquad \text{Separable}$$

$$\frac{2u}{u+1}\, du = dx$$

$$\int \frac{2u}{u+1}\,du = \int dx + C = x + C$$

$$= 2\int \frac{u}{u+1}\,du$$

$$= 2\int \frac{u+1-1}{u+1}\,du = 2\int du - 2\int \frac{1}{u+1}\,du =$$

$$= 2u - 2\ln|u+1|$$

Hence

$$2u - 2\ln|u+1| = x + C$$

But $u = \sqrt{x+y+1}$ and so

$$2\sqrt{x+y+1} - 2\ln\left|\sqrt{x+y+1}+1\right| = x + C$$

Leave this solution in implicit form.

Problem 1 (iii)

$$(x+2y)\frac{dy}{dx} = y$$

$$\frac{dy}{dx} = \frac{y}{x+2y} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \frac{y/x}{1+2\left(\frac{y}{x}\right)}$$

Homogeneous equation

Substitution: $v = \frac{y}{x}$

$$y = vx \quad \text{and} \quad \frac{dy}{dx} = \frac{dv}{dx}x + v$$

Rewrite the equation:

$$\frac{dv}{dx}x + v = \frac{v}{1+2v}$$

$$\frac{dv}{dx}x = \frac{v}{1+2v} - v = \frac{v - v - 2v^2}{1+2v}$$

Separable

$$\frac{1+2v}{v^2}\,dv = \frac{1}{x}\,dx$$

$$\int \frac{1+2u}{u^2}\,du = \int \frac{1}{x}\,dx + C = \ln|x| + C$$

$$\int \frac{1}{u^2}\,du + 2\int \frac{1}{u}\,du$$

$$-\frac{1}{u} + 2\ln|u|$$

Hence

$$-\frac{1}{u} + 2\ln|u| = \ln|x| + C$$

But $u = \frac{y}{x}$ and so

$$-\frac{x}{y} + 2\ln\left|\frac{y}{x}\right| = \ln|x| + C$$

We can leave this solution in implicit form.

Problem 1 (iv)

$$\frac{dY}{dx} = x - \frac{1}{x^2 - 2Y}$$

Substitution $u = x^2 - 2Y$

$$Y = \frac{1}{2}x^2 - \frac{1}{2}u$$

$$\frac{dY}{dx} = x - \frac{1}{2}\frac{du}{dx}$$

Rewrite the equation

$$\overbrace{\cancel{x} - \frac{1}{2}\frac{du}{dx}}^{dY/dx \,=} = \cancel{x} - \frac{1}{u}$$

$$\frac{1}{2}\frac{du}{dx} = \frac{1}{u} \qquad \text{Separable}$$

$$u\,du = 2\,dx$$

$$\int u\,du = \int 2\,dx + C$$

$$\frac{u^2}{2} = 2x + C$$

$$u^2 = 4x + 2C$$

$$u = \pm\sqrt{4x + 2C}$$

But $v = x^2 - 2y$, and so

$$x^2 - 2y = \pm\sqrt{4x + 2C}$$

$$-2y = -x^2 \pm \sqrt{4x+2C}$$

$$y = \frac{x^2}{2} \pm \frac{1}{2}\sqrt{4x+2C}$$

Problem 3 (i)

$$2x\frac{dy}{dx} - 3y = 9x^3$$

$$\frac{dy}{dx} - \frac{3}{2x}y = \frac{9}{2}x^2$$

Linear
$P(x) = -\frac{3}{2x}$
$Q(x) = \frac{9}{2}x^2$

$$\int P(x)\,dx = \int -\frac{3}{2x}\,dx$$

$$= -\frac{3}{2}\ln|x|$$

The integrating factor is

$$e^{\int P(x)\,dx} = e^{-3/2 \ln|x|} = |x|^{-3/2}$$

Now ~~the~~ because there is an absolute value, we have to distinguish 2 cases: $x \geq 0$ and $x < 0$.

We only work out the ~~one~~ case. $x \geq 0$.

Hence $|x| = x$ and

$$e^{\int P(x)\,dx} = x^{-3/2}$$

Multiply by $x^{-3/2}$:

$$x^{-3/2}\frac{dy}{dx} - x^{-3/2}\frac{3}{2x}y = x^{-3/2}\,\frac{9}{2}x^2$$

$$\frac{d}{dx}\left(x^{-3/2}y\right) = \frac{9}{2}x^{-3/2+2} = \frac{9}{2}x^{1/2}$$

Integrating

$$x^{-3/2}y = \int \frac{9}{2}x^{1/2} + C$$

$$= \frac{9}{2}\,\frac{x^{3/2}}{3/2} + C$$

$$= 3x^{3/2} + C$$

$$\boxed{\begin{aligned} y &= 3x^{3/2}\cdot x^{3/2} + Cx^{3/2} = \\ &= 3x^3 + Cx^{3/2} \end{aligned}}$$

Problem 3 (iii)

$$x \frac{dy}{dx} + y = 3xy \qquad y(1) = 0.$$

This is Linear.

First we rewrite it in standard form $\boxed{\frac{dy}{dx} + P(x)\,y = Q(x)}$

~~[illegible]~~ The RHS is a term with the y, so we move it to the LHS.

$$x \frac{dy}{dx} + y - 3xy = 0$$

$$x \frac{dy}{dx} + (1 - 3x)\,y = 0$$

$$\frac{dy}{dx} + \frac{1-3x}{x}\,y = 0 \quad \leftarrow \text{standard form}$$

$$P(x) = \frac{1-3x}{x}$$

$$Q(x) = 0$$

$$\int P(x)\,dx = \int \frac{1-3x}{x}\,dx = \int \frac{1}{x}\,dx - 3\int dx$$

$$= \ln|x| - 3x.$$

The integrating factor is

$$e^{\int P(x)\,dx} = e^{\ln|x| - 3x} = |x|\, e^{-3x}$$

Since the initial data $Y(1) = 0$ has $x_0 = 1 > 0$, we can assume that x is positive because we are looking for a solution near $x_0 = 1$.

Hence $|x| = x$ and

$$e^{\int P(x)\,dx} = x e^{-3x}$$

$$x e^{-3x} \frac{dy}{dx} + x e^{-3x} \frac{1-3x}{x} Y = 0$$

$$\frac{d}{dx}\left(x e^{-3x} Y\right) = 0$$

Integrating:

$$x e^{-3x} Y = \int 0\, dx + C = C$$

$$Y = \frac{C}{x e^{-3x}}$$

Impose the initial condition:

$x = 1, \; y = 0$

$$0 = \frac{c}{1} \longrightarrow c = 0$$

The particular solution is

$$y = 0$$

Problem 3 (iii)

$$x^3 \frac{dy}{dx} = xy + 1, \qquad y(1) = 0$$

Write this in the standard form

$$\boxed{\frac{dy}{dx} + P(x)\, y = Q(x)}$$

$$x^3 \frac{dy}{dx} - xy = 1$$

$$\frac{dy}{dx} - \frac{1}{x^2}\, y = \frac{1}{x^3}$$

Linear with

$P(x) = -\frac{1}{x^2}$

$Q(x) = \frac{1}{x^3}$

$$\int P(x)\,dx = \int -\frac{1}{x^2}\,dx =$$

$$= \frac{1}{x}$$

The integrating factor is

$$e^{\int P(x)\,dx} = e^{1/x}$$

$$e^{1/x}\frac{dy}{dx} - e^{1/x}\frac{1}{x^2}y = e^{1/x}\frac{1}{x^3}$$

$$\frac{d}{dx}\left(e^{1/x}y\right) = e^{1/x}\frac{1}{x^3}$$ ← Integrating

$$e^{1/x}y = \int \frac{1}{x^3}e^{1/x}dx + C$$

For the integral use a substitution:

$$u = \frac{1}{x}$$

$$du = -\frac{1}{x^2}\,du$$

Then $\int e^{1/x}\frac{1}{x^3}dx = \int e^{1/x}\left(-\frac{1}{x^2}\right)\left(-\frac{1}{x}\right)dx$

$$= \int e^u(-u)\,du = -\int u e^u\,du$$

By parts

$$= -ue^u + \int e^u = -ue^u + e^u$$

$$= -\frac{1}{x}e^{1/x} + e^{1/x}$$

$$\cancel{\frac{d}{dx}\left(e^{1/x} Y\right) = -\frac{1}{x} e^{1/x} + e^{1/x} + A}$$

~~Integrating~~

Hence:

$$e^{1/x} Y = -\frac{1}{x} e^{1/x} + e^{1/x} + C$$

Multiply by $e^{-1/x}$

$$Y = -\frac{1}{x} e^{1/x} e^{-1/x} + e^{1/x} e^{-1/x} + C e^{-1/x}$$

$$Y = -\frac{1}{x} + 1 + C e^{-1/x}$$

Impose initial data: $x = 1$, $Y = 0$

$$0 = -1 + 1 + C$$

$C = 0$

The particular solution is

$$Y = -\frac{1}{x} + 1$$

Problem 4 (ii)

$$\underbrace{\left(x + \arctan y\right)}_{M} dx + \underbrace{\left(\frac{x+y}{1+y^2}\right)}_{N} dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{1}{1+y^2}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{1+y^2}\right) + \frac{\partial}{\partial x}\left(\frac{y}{1+y^2}\right) =$$

$$= \frac{1}{1+y^2} + 0$$ 6

Hence $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and the equation is exact.

Then there exists $F(x,y)$ such that $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$

Integrating wrt x:

$$F(x,y) = \int M\,dx + \underbrace{g(y)}_{\text{function of } y}$$ 6

$$= \int (x + \arctan y)\, dx + g(y)$$

$$= \frac{x^2}{2} + x \arctan y + g(y)$$

$$\frac{\partial F}{\partial y} = \frac{x}{1+y^2} + g'(y) \overset{!}{=} N = \frac{x+y}{1+y^2}$$

Hence $g'(y) = \dfrac{x+y}{1+y^2} - \dfrac{x}{1+y^2} =$

$$= \frac{x + y - x}{1+y^2}$$

$$= \frac{y}{1+y^2}$$

$$g(y) = \int \frac{y}{1+2y^2}\, dy + C =$$

Use a subst. $u = 1 + 2y^2$

$du = 4y\, dy$

$$= \int \frac{du/4}{u} = \frac{1}{4}\int \frac{du}{u} = \frac{1}{4} \ln|u| + C =$$

$$= \frac{1}{4} \ln|y^2 + 1| + C$$

Then the solution is

$$F(x,y) = \frac{x^2}{2} + x \arctan y +$$
$$+ \frac{1}{4} \ln|y^2+1| + C = 0.$$

Problem 4 (iii)

$$\underbrace{(e^x \sin y + \tan y)}_{M} dx + \underbrace{(e^x \cos y + x \sec^2 y)}_{N} dy = 0$$

$$\frac{\partial M}{\partial y} = e^x \cos y + \sec^2 y$$

$$\frac{\partial N}{\partial x} = e^x \cos y + \sec^2 y$$

$=$ proves exactness

Therefore there exists a function $F(x, y)$ such that

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N$$

$$F(x, y) = \int M\,dx + g(y)$$

$$= \int (e^x \sin y + \tan y)\,dx + g(y)$$

$$= e^x \sin y + x \tan y + g(y)$$

$$\frac{\partial F}{\partial y} = e^x \cos y + x \sec^2 y + g'(y)$$

$$\stackrel{!}{=} N = e^x \cos y + x \sec^2 y$$

Hence $g'(y) = 0$ and $g(y) = \int 0\,dx + C$

$$= C$$

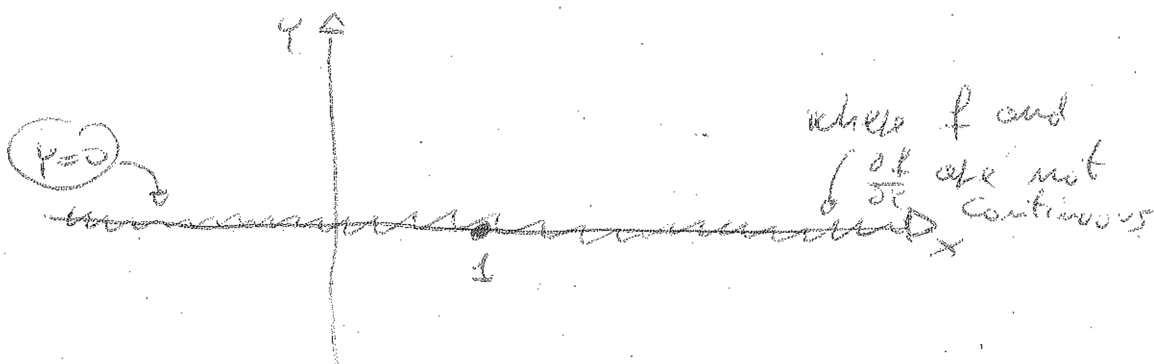
$$F(x, y) = e^x \sin y + x \tan y + C = 0$$

Problem 5 (i)

$$\frac{dy}{dx} = \frac{x-1}{y} \qquad \underset{x_0}{y(1)} = \underset{y_0}{0}$$

$f(x,y) = \dfrac{x-1}{y} \longrightarrow$ not continuous on $y=0$

$$\frac{\partial f}{\partial y} = (x-1)\left(-\frac{1}{y^2}\right)$$



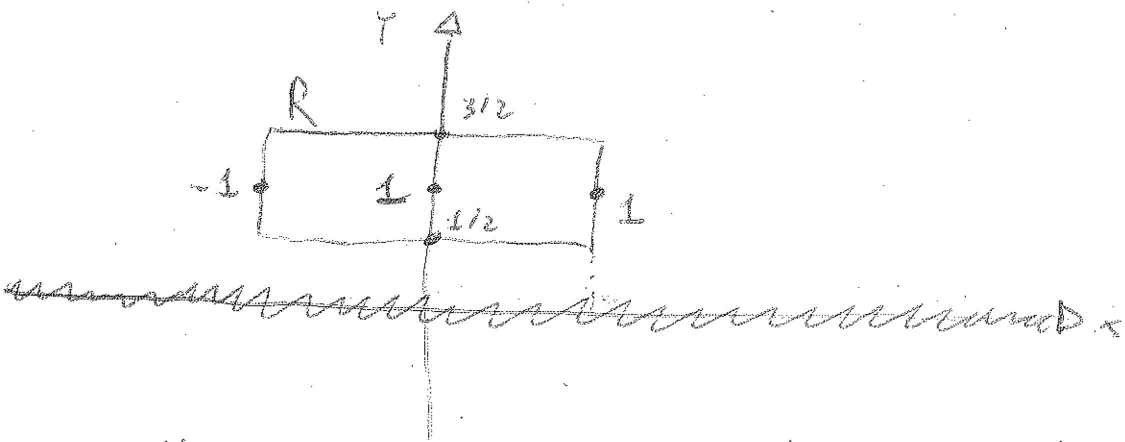
I can't find a rectangle containing (1, 0) and where both f and $\dfrac{\partial f}{\partial y}$ are continuous.

Problem 5 (ii)

$$\frac{dy}{dx} = \frac{x-1}{y} \qquad y(\underset{x_0}{\overset{\shortparallel}{0}}) = \underset{y_0}{\overset{\shortparallel}{1}}$$

$f(x,y) = \dfrac{x-1}{y} \longrightarrow$ not continuous on $y=0$

$\dfrac{\partial f}{\partial y} = (x-1)\left(-\dfrac{1}{y^2}\right) \nearrow$

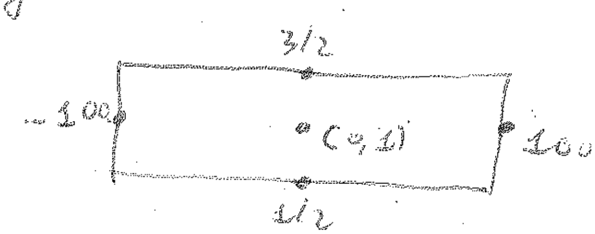


On the rectangle R both f and $\dfrac{\partial f}{\partial y}$ are continuous.

Hence there is a unique solution.

In fact any ~~the~~ symmetric rectangle containing $(0,1)$ and avoiding the x-axis works.

For instance



Problem 6

a = acceleration

$\frac{dv}{dt} = a$ gives

$$v(t) = at + v_0 = at + 88$$

$\frac{dv}{dt} = s$ (= position function) gives

$$s(t) = \frac{1}{2}at^2 + v_0 t + s_0$$

Can assume $s_0 = 0$ because we start counting time since the moment the brakes are applied.

The time at which the car stops is given by solving $v(t) = 0$

Hence $at + 88 = 0$

and $t = -\frac{88}{a}$

The car skids 176 ft it means that

$$s\left(-\frac{88}{a}\right) = 176$$

Hence $\frac{1}{2} a \frac{(-88)^2}{a^2} + 88\left(-\frac{88}{a}\right) = 176$

$$\frac{1}{2}\,\frac{88^2}{a} - \frac{88^2}{a} = 176$$

$$-\frac{1}{2}\,\frac{88^2}{a} = 176$$

$$a = -\frac{88^2}{2\cdot(176)} = -22 \text{ ft/s}^2$$

Problem 7

$$\begin{cases} \dfrac{dA}{dt} = kA \\ A(0) = 10 \end{cases}$$

~~A(t) tr~~ $\cancel{A(7.5) = 3\cdot A(0) = 3(10)}$

This is the exponential model
The solution is

$$A(t) = 10\,e^{kt}$$

$$A(7.5) = 3\cdot A(0) = 3(10) = 30$$

A(t) is tripling every 7.5 years

$$10e^{k(7.5)} = 30$$

$$e^{7.5k} = 3$$

$$(7.5)k = \ln 3$$

$$k = \frac{1}{7.5}\ln(3)$$

So $A(t) = 10 \cdot \left(3^{(1/7.5)t}\right)$.

How long does it take for $A(t)$ to centuplicate?

Need to solve $A(t) = 100 \cdot A(0)$

$$= 100(10)$$

$$= 1000$$

$$10 \cdot 3^{\left(\frac{1}{7.5}t\right)} = 1000$$

$$3^{\left(\frac{1}{7.5}t\right)} = 100$$

$$\frac{1}{7.5}t = \ln_3 100$$

$$t = (7.5)\left(\ln_3(100)\right)$$

Problem 9

$$\begin{cases} \dfrac{dT}{dt} = k(A - T) \\ \cancel{\dfrac{dT}{dt}}\ T(0) = 210 \end{cases}$$

$A = 70$

$$\begin{cases} \dfrac{dT}{dt} = k(70 - T) \\ T(0) = 210 \end{cases}$$

Solve this IVP

Separable

$$\frac{dT}{70-T} = k\,dt$$

$$\int \frac{dT}{70-T} = \int k\,dt + C$$

$$-\ln(T-70) = kt + C$$

$$\ln(T-70) = -kt - C$$

$$T - 70 = e^{-kt - c} = e^{-kt} e^{-c}$$

Let $c' = e^{-c}$

$$T = 70 + c' e^{-kt}$$

Initial condition $t = 0$ $T = 210$

$$210 = 70 + c' \rightsquigarrow c' = 140$$

So $T = 70 + 140 e^{-kt}$

(After 30 min the cake is at 140°F)

To find k need to solve

$$140 = T(30) = 70 + 140\, e^{(-30k}$$

$$70 = 140 e^{-30k}$$

$$e^{-30k} = \frac{1}{2}$$

$$-30k = \ln\left(\frac{1}{2}\right)$$

$$k = -\frac{1}{30} \ln\left(\frac{1}{2}\right)$$

When will the cake be 100°F?

Need to solve $T(t) = 100$

$$T(t) = 70 + 140\left(\frac{1}{2}\right)^{t/30} = 100$$

$$140\left(\frac{1}{2}\right)^{t/30} = 30$$

$$\left(\frac{1}{2}\right)^{t/30} = \frac{30}{140} = \frac{3}{14}$$

$$t/30 = \ln_{1/2}\left(\frac{3}{14}\right)$$

$$t = 30 \ln_{\frac{1}{2}}\left(\frac{3}{14}\right).$$

Problem 8

$V_0 = 60$

$c_{input} = 1$ [illegible]

$r_{input} = 2$ gal/min

$r_{out} = 3$ gal/min

After 60 minutes the tank is empty.

t = time in minutes

$x(t)$ = quantity of salt in the tank.

$$c_{out} = \frac{x}{V}$$

$$V = \text{volume} = V_0 + (r_{in} - r_{out})t$$
$$= 60 - t.$$

$$c_{out} = \frac{x}{60-t}$$

The equation to solve is:

$$\frac{dx}{dt} = c_{in} r_{in} - c_{out} r_{out} = 2 - \frac{3x}{60-t}$$

$$\frac{dx}{dt} + \frac{3x}{60-t} = 2$$

Linear with $\quad P(t) = \dfrac{3}{60-t}$

$\quad Q(t) = 2$

$$\int P(t)\,dt = 3\int \frac{dt}{60-t} = -3\ln(t-60)$$

$$e^{\int P(t)\,dt} = e^{-3\ln(t-60)} = (t-60)^{-3}$$

$$(t-60)^{-3}\frac{dx}{dx} + (t-60)^{-3}\cdot\frac{3x}{60-t} = (t-60)^{-3}\cdot 2$$

$$\frac{d}{dt}\left((t-60)^{-3}x\right) = \frac{2}{(t-60)^3} \qquad \text{Integrating}$$

$$(t-60)^{-3}x = \int \frac{2}{(t-60)^3}\,dt = 2\,\frac{(-2)}{(t-60)^2} + C$$

$$x = -4(t-60) + C(t-60)^3$$

Initial condition : $x(0) = 0$

$$0 = -4(-60) + C(-60)^3$$

$$C = \frac{-60\cdot 4}{(-60)^3} = \frac{4}{3600} = \frac{1}{900}$$

Hence

$$x(t) = -4(t-60) + \frac{1}{900}(t-60)^3$$

To find at what time there was the maximum quantity of salt, we need to find the max of the function $x(t)$.

$$\frac{dx}{dt} = -4 + \frac{3}{900}(t-60)^2 = 0$$

$$\frac{1}{300}(t-60)^2 = 4$$

$$(t-60)^2 = 4 \cdot 300 = 1200$$

$$t - 60 = \pm\sqrt{1200}$$

$$t = \pm\sqrt{1200} + 60$$

The solution with + is to be excluded.

Hence $t = 60 - \sqrt{1200}$ min.

Assignment 1: due date February 3rd-5th

Problems 1.1: 1, 4, 5: Verify by substitution that each given function is a solution of the given differential equation.

$$
\begin{aligned}
y' &= 3x^2;\ y = x^3 + 7\\
y'' &= 9y:\ y_1 = e^{3x},\ y_2 = e^{-3x}\\
y' &= y + 2e^{-x};\ y = e^x - e^{-x}
\end{aligned}
$$

Problem 1.1: 32: Write a differential equation that is a mathematical model of the situation described.

The time rate of a population P is proportional to the square root of P.

Problem 1.2: 2, 7: Solve the following IVP.

$$
\begin{aligned}
\frac{dy}{dx} &= (x-2)^2,\ y(2) = 1\\
\frac{dy}{dx} &= \frac{10}{x^2+1},\ y(0) = 0
\end{aligned}
$$

Problems 1.2: 11, 14: Find the position function $x(t)$ of a moving particle with the given acceleration $a(t)$, initial position $x_0 = x(0)$, and initial velocity $v_0 = v(0)$.

$$
\begin{aligned}
a(t) &= 50,\ v_0 = 10,\ x_0 = 20\\
a(t) &= 2t+1,\ v_0 = -7,\ x_0 = 4
\end{aligned}
$$

Problems 1.2: 25: The brakes of a car are applied when it is moving at 100 Km/h and provide a constant deceleration of 10 meters per second per second (m/s^2). How far does the car travel before coming to a stop?

Problem 1.2: 27: A ball is thrown straight downward from the top of a tall building. The initial speed of the ball is 10 m/s. It strikes the ground with a speed of 60 m/s. How tall is the building?

Problem 1.2: 37: At noon a car starts from rest at point A and proceeds at constant acceleration along a straight road toward point B. If the car reaches B at $12:50$ PM with a velocity of 60 mi/h, what is the distance from A to B?

Assignment 2: due date February 10th-12th

Problems 1.1: 12, 14: Verify whether the hypotheses of the Theorem on Existence and Uniqueness for initial value problems applies to the following problems. If so, determine a rectangle of the plane as in the statement of the theorem:

$$\frac{dy}{dx} = x\,ln(y);\ y(1) = 1$$
$$\frac{dy}{dx} = \sqrt[3]{y};\ y(0) = 0$$

Problem 1.1: 29:

Verify that if C is a constant, then the functions defined piecewise by

$$y(x) = \begin{cases} 0 & \text{for } x \le C \\ (x-C)^3 & \text{for } x > C \end{cases}$$

satisfies the differential equation

$$\frac{dy}{dx} = 3\,y^{2/3} \quad \text{for all } x.$$

Sketch a variety of such solution curves on the xy-plane. Discuss for which points (x_0, y_0) of the xy-plane the initial value problem

$$y(x) = \begin{cases} \frac{dy}{dx} = 3\,y^{2/3} \\ y(x_0) = y_0 \end{cases}$$

has either a unique solutions, no solutions, infinitely many solutions.

Problem 1.4: 1, 4, 6: Use the technique of separation of variables to solve the following differential equations:

$$\frac{dy}{dx} + 2\,x\,y = 0$$
$$(1+x)\,\frac{dy}{dx} = 4y$$
$$\frac{dy}{dx} = 3\,\sqrt{x\,y}$$

Problems 1.4: 23, 25, 27: Find a solution of the following initial value problems:

$$\frac{dy}{dx} + 1 = 2\,y;\ y(1) = 1$$
$$x\,\frac{dy}{dx} - y = 2\,x^2\,y;\ y(1) = 1$$
$$\frac{dy}{dx} = 6\,e^{2x-y};\ y(0) = 0$$

Problems 1.4: 35: Carbon extracted from an ancient skull contained only one-sixth as much ^{14}C as carbon extracted from present-day bone. How old is the skull?

Problem 1.4: 47: A certain piece of dubious information about phenylethylamine in the drinking water began to spread one day in a city with a population of $100,000$. Within a week, $10,000$ people had heard this rumor. Assume that the rate of increase of the number who have heard the rumor is proportional to the number who have not yet heard it. How long will it be until half the population of the city has heard the rumor?

Assignment 3: due date February 22nd

MAT303: Calculus IV

Problems 1.5: 1, 3, 6, 17: Find a general solution for the following equations, or a particular solution for the following initial value problems

$$
\begin{gathered}
y' + y = 2;\;\; y(0) = 0 \\
y' + 3y = 2xe^{-3x} \\
xy' + 5y = 7x^2;\;\; y(2) = 5 \\
(1+x)y' + y = \cos(x);\;\; y(0) = 1
\end{gathered}
$$

Problem 1.5: 37: A 400-gal tank initially contains 100 gal of brine containing 50 lb of salt. Brine containing 1 lb of salt per gallon enters the tank at the rate of 5 gal/s, and the well-mixed brine in the tank flows out at the rate of 3 gal/s. How much salt will the tank contain when it is full of brine?

Problem 1.6: 1, 3, 13, 20: Find a general solution of the following differential equations.

$$
\begin{gathered}
(x+y)y' = x - y \\
xy' = y + 2\sqrt{xy} \\
xy' = y + \sqrt{x^2+y^2} \\
y^2y' + 2xy^3 = 6x
\end{gathered}
$$

Problems 1.6: 31, 34: Verify that the given differential equations are exact; then solve them.

$$
\begin{gathered}
(2x+3y)dx + (3x+2y)dy = 0 \\
(2xy^2+3x^2)dx + (2x^2y+4y^3)dy = 0
\end{gathered}
$$

Assignment 4: due date February 29th

MAT303: Calculus IV

Problem 1: Solve all problems of the Practice Midterm I without the asterisk (i.e. Problems 1, 3, 4, 5, 6, 7).

Problem 2: Solve two problems from the Practice Midterm I with an asterisk at your choice (i.e. among Problems 2, 8, 9, 10).

Assignment 5: due date March 9th and 11th in Recitation Sections

MAT303: Calculus IV with Applications

Problem 2.1: 10. Suppose that the fish population $P(t)$ in a lake is attacked by a disease at time $t = 0$, with the result that the fish cease to reproduce (so that the birth rate is $\beta = 0$) and the death rate δ (deaths per week peer fish) is thereafter proportional to $1/\sqrt{P}$. If there were initially 900 fish in the lake and 441 were left after 6 weeks, how long did it take all the fish in the lake to die?

Problem 2.1: 13. Consider a prolific breed of rabbits whose birth and death rates, β and δ, are each proportional to the rabbit population $P = P(t)$, with $\beta > \delta$. **(a)** Show that

$$P(t) = \frac{P_0}{1 - k\,P_0 t}, \qquad k \text{ constant.}$$

Note that $P(t) \to +\infty$ as $t \to \frac{1}{kP_0}$. This is an explosion model (or doomsday). **(b)** Suppose that $P_0 = 6$ and that there are nine rabbits after ten months. When explosion occur?

Problem 2.1: 21. Suppose that the population $P(t)$ (measured in million) of a country satisfies the differential equation

$$\frac{dP}{dt} = k\,P\,(200 - P)$$

with k constant. Its population in 1940 was $P(0) = 100$ million and was then growing at the rate of 1 million per year. Predict this country's population for the year 2000.

Problem 2.2: 19. The differential equation

$$\frac{dx}{dt} = \frac{1}{10}\,x\,(10 - x) - h$$

models a logistic population with harvesting at rate h. Determine the dependence of the number of critical points on the parameter h and discuss whether or not the critical points are stable. Construct then a bifurcation diagram.

Assignment 6: due date March 23rd / 25th in Recitation Section

MAT303: Calculus IV

Problem 1 (§2.3 no. 2): Suppose that a body moves through a resisting medium with resistance proportional to its velocity v, so that $dv/dt = -kv$. (a) Show that its velocity and position at time t are given by

$$v(t) \;=\; v_0 e^{-kt}$$

and

$$x(t) \;=\; x_0 \;+\; \Big(\frac{v_0}{k}\Big)(1 - e^{-kt})$$

where x_0 and v_0 are the initial position and initial velocity of the body. (b) Conclude that the body travels only a finite distance, and find that distance.

Problem 2 (§3.1 no. 1 and 3): In the following problems, a homogeneous second-order linear differential equation, two functions y_1 and y_2, and a pair of initial conditions are given. First verify that y_1 and y_2 are solutions of the differential equation. Then find a particular solution of the form $y = c_1 y_1 + c_2 y_2$ that satisfies the given initial conditions.

$$(i). \qquad y'' - y = 0; \quad y_1(x) = e^x \quad y_2(x) = e^{-x}; \quad y(0) = 0, \quad y'(0) = 5$$

$$(ii). \qquad y'' + 4y = 0; \quad y_1(x) = \cos(2x), \quad y_2(x) = \sin(2x); \quad y(0) = 3, \quad y'(0) = 8.$$

Optional Problem 3 Use Wronskian's Theorem to check that the given solutions y_1 and y_2 of Problem 2 (i) and (ii) are linearly independent.

Problem 4 (§3.1 no. 22 and 25) Check whether the following pairs of functions are linearly dependent or not. (For this exercise it is not possible to use Wronskian's Theorem because we do not know a priori whether or not the given pairs of functions are solutions of a homogeneous differential equation.)

$$f(x) = 1 + x, \qquad g(x) = 1 + |x|$$
$$f(x) = e^x \sin(x), \qquad g(x) = e^x \cos(x).$$

Problem 5(§3.1 no. 33, 35, 39 and §3.5 no. 1) Find the general solution for the following second order differential equations.

$$y'' - 3y' + 2y = 0$$
$$4y'' + 4y' + y = 0$$
$$y'' + y' + y = 0$$
$$y'' + 16y = x^2 + 1.$$

Optional Problem (§3.1 no. 31) Show that $y_1 = \sin(x^2)$ and $y_2 = \cos(x^2)$ are linearly independent functions, but that their Wronskian vanishes at $x = 0$. Why does this imply that there is no differential equation of the form $y'' + p(x)y' + q(x)y = 0$, with both p and q continuous everywhere, having both y_1 and y_2 as solutions?

ASSIGNMENT 7 – DUE DATE MARCH 30TH / APRIL 1ST

Exercise 1 (§3.2 no. 7,8): Use the Wronskian to prove that the given functions are linearly independent on the indicated interval:

$$f(x) = 1, \quad g(x) = x, \quad h(x) = x^2; \quad I = \text{real line}$$
$$f(x) = e^x; \quad g(x) = e^{2x}; \quad h(x) = e^{3x}; \quad I = \text{real line}$$

Exercise 2 (§3.3 no. 11, 13, 18, 27, 31): Find the general solution for the following homogeneous equations.

$$y^{(4)} - 8y^{(3)} + 16y'' = 0$$
$$9y''' + 12y'' + 4y' = 0$$
$$y^{(4)} = 16y$$
$$y^{(3)} + 3y'' - 4y = 0$$
$$y''' + 3y'' + 4y' - 8y = 0.$$

Exercise 3 (§3.5 no. 1, 3, 4, 12): Find a particular solution y_P for the following non-homogeneous differential equations.

$$y'' + 16y = e^{3x}$$
$$y'' - y' - 6y = 2\sin(3x)$$
$$4y'' + 4y' + y = 3xe^x$$
$$y''' + y' = 2 - \sin(x)$$

Assignment 8: due date April 13th / 15th in Recitation Section

MAT303: Calculus IV with applications

For the first problems you can consult §3.4 of the textbook. For Problem 4 see §4.1 and for the last problem see §3.5.

Problem 1: Determine the period and the frequency of the motion of a 4-Kg mass on the end of a spring with spring constant 16 N/m.

Problem 2: Suppose that the mass in a mass-spring-dashpot system with $m = 10$, $c = 9$, and $k = 2$ is set in motion with $x(0) = 0$ and $x'(0) = 5$. **(A)**. Find the position function $x(t)$ and sketch a graph. **(B)**. Find how far the mass moves to the right before starting back toward the origin.

Problem 3: Consider the free mass-spring-dashpot system $x'' + 2x' + 5x = 0$. The object is released from still when the spring is compressed by 10 unit length. Solve the system as an initial value problem and classify the phenomenon. Write the solution in the standard form $x(t) = A(t)\cos(\omega\, t - \alpha)$.

Problem 4: Solve the systems:

$$x' = y, \quad y' = x$$
$$x' = -y, \quad y' = 10x - 7y; \quad x(0) = 2, \quad y(0) = -7$$

Problem 5: Use the variation of parameters formula

$$y_p(x) \; = \; -y_1(x) \int \frac{y_2(x) f(x)}{W(x)} dx \; + y_2(x) \int \frac{y_1(x) f(x)}{W(x)} dx$$

which gives the particular solution to an equation of type $y'' + P(x)y' + Q(x)y = f(x)$ (note that $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the associated homogeneous equation and $W(x)$ is the Wronskian of $y_1(x)$ and $y_2(x)$).

Use variation of parameters to find a particular solution of $y'' + y = \sec^3(x)$ (recall that $\sec(x) = \frac{1}{\cos(x)}$).

Assignment 9: due date April 20th / 22th in Recitation Section

MAT303: Calculus IV with applications

Problem 1: Consider the following matrices:

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 0 & 4 & 3 \\ -5 & 2 & 7 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & -3 & 2 \\ 1 & 4 & -3 \\ 2 & 5 & -1 \end{pmatrix}$$

Find (a) $7A + 4B$, (b) $3A - 5A$, (c) AB, (d) BA, (e) $A - tI$ (where I is the identity matrix of type 3×3).

Problem 2: Consider two general matrices of type 2×2

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

Show that

$$\det(AB) \;=\; \det(A)\det(B)$$

by computing the two sides of the equation separately. *In general the previous formula is true for any pair of matrices of type $n \times n$ and it goes under the name of "Binet's formula".*

Problem 3: Consider the following 3×3 linear system of first-order differential equations (derivatives are taken with respect to the independent variable t):

$$x_1' = -8x_1 - 11x_2 - 2x_3; \qquad x_2' = 6x_1 + 9x_2 + 2x_3; \qquad x_3' = -6x_1 - 6x_2 + x_3$$

and the following initial conditions

$$x_1(0) = 1; \quad x_2(0) = 2; \quad x_3(0) = 3.$$

(a) Write the system in matrix form. Consider now the following three vectors of functions

$$\mathbf{x}_1 = e^{-2t} \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$$

$$\mathbf{x}_2 = e^{t} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\mathbf{x}_3 = e^{3t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

(b) Show that the three vectors $\mathbf{x}_1$. $\mathbf{x}_2$, $\mathbf{x}_3$ are solutions of the system and show that they are linearly independent on the interval $(-\infty, +\infty)$. (c) Write the general solution of the system. (d)

Find the particular solution of the initial value problem satisfying the above initial conditions (use Row Reduction for this point).

Problem 4: Apply the eigenvalue method to find a general solution of the following linear system of type 2×2:

$$x_1' = 2x_1 + 3x_2; \qquad x_2' = 2x_1 + x_2$$

Problem 5: Apply the eigenvalue method to find a general solution of the following linear system of type 2×2:

$$x_1' = 4x_1 + x_2; \qquad x_2' = 6x_1 - x_2; \qquad .$$

Problem 6: Apply the eigenvalue method to find a general solution of the following linear system of type 3×3:

$$x_1' = 2x_1 + x_2 - x_3; \qquad x_2' = -4x_1 - 3x_2 - x_3; \qquad x_3' = 4x_1 + 4x_2 + 2x_3.$$

(Hint: Use long division (or synthetic division) to factor the characteristic equation. You should get $(\lambda - 1)(\lambda^2 + 4) = 0$).

Here there are a couple of extra problems you should work out to acquire familiarity with row reduction (do not turn these problem in).

Problem 7: Solve the linear system:

$$x_2 + x_3 = 5; \qquad 2x_1 + 2x_2 = 6; \qquad x_2 - x_3 = -1.$$

(Hint: After you have written the associated matrix of the system, you can see that it has a zero in the entry $(1, 1)$. This is bad because it doesn't allow us to proceed with the technique, however the solution to this issue is very simple. Just swap the first row with any other row that has a non-zero element in the first entry.)

Problem 8: Solve the linear system:

$$x - 3y + 5z = -10, \qquad 3x + y - 2z = 7, \qquad 2x + 4y + 3z = 3.$$

Assignment 10: due date April 29th

MAT303: Calculus IV

Solve the following problems that you can find on the textbook:

Edition 4
§5.4: 6, 7, 12, 13
§5.2: 11, 23
§5.5: 25, 27

Edition 5
§5.5: 6, 7, 12, 13
§5.2: 11, 23
§5.6: 25, 27

Assignment 11: due date May 6th

MAT303: Calculus IV

Solve the following problems that you can find on the textbook. Moreover don't forget to fill out the on-line evaluation.

Edition 4
§4.2: 2, 12, 8
§5.2: 27
§5.6: 5, 10

Edition 5
§4.2: 2, 12, 8
§5.2: 27
§5.7: 5, 10

THE EXPONENTIAL EQUATION AND MATHEMATICAL MODELS

1. The exponential Equation or Natural Growth Equation

We are going to consider an initial value problem that serves as a mathematical model for a wide range of natural phenomena. This model is called *exponential initial value problem* or *natural growth initial value problem*. It is defined as follows:

$$\begin{cases} \frac{dx}{dt} = k\,x \\ x(0) = x_0 \end{cases} \tag{1}$$

where k is a constant (it can be either positive or negative) and $x = x(t)$ is a function of t. In practice, the function $x(t)$ represents a certain quantity evolving during the time t, and x_0 represents that quantity at time $t = 0$. For instance you can think that $x(t)$ is the number of a certain population at time t, or the number of atoms of a certain radioactive isotope contained in a given sample. In particular, *we can suppose that both $x(t)$ and x_0 are positive since they represent physical quantities that cannot be negative.* The constant k depends on the given problem and usually we need to determine it.

1.1. **Solving the exponential initial value problem.** We aim to solve the initial value problem (1) with positive initial condition $x_0 > 0$. We will see that its solution is $x(t) = x_0\, e^{kt}$.

The differential equation in (1) is a separable differential equation. Hence we solve it by separating the variables. As we would like to divide by x the both sides of the equation, but we can not divide by a quantity that is zero, we first need to discuss the case $x(t) = 0$. However we can easily check that the function $x(t) = 0$ is a solution of the equation $\frac{dx}{dt} = kx$ (but not of the initial value problem as in (1) since we are assuming that $x(0) = x_0 > 0$). Now we suppose that $x \neq 0$ and hence divide the equation $\frac{dx}{dt} = kx$ by x. Hence we have

$$\frac{1}{x}\,\frac{dx}{dt} = k.$$

Now we multiply both sides of the equation by dt to have:

$$\frac{1}{x}\,dx = k\,dt.$$

We now integrate the above equation to find:

$$\int \frac{1}{x}\,dx = \int k\,dt + C$$

where C is an arbitrary constant. By solving the integrals we have that

$$\ln|x| = k\,t + C,$$

and by applying the exponential function to both sides, we have

$$|x| = e^{kt+C} = e^{kt}\, e^{C} = A\, e^{kt} \quad \text{where we set} \quad A = e^{C}.$$

Now we impose the initial condition $x(0) = x_0$. To this end it is enough to plug $t = 0$ and $x = x_0$ in the above equation. We have then (note that since x_0 is positive, we have $|x_0| = x_0$):

$$x_0 = |x_0| = A\, e^{k(0)} = A.$$

Hence the solution to the IVP (1) is

$$x(t) = x_0\, e^{kt}.$$

2. Population growth

Let $P(t)$ be a function that denotes the number of individuals of a certain population having constant birth rate β and constant death rate δ. Moreover let P_0 be the number of individuals at time $t = 0$. We will find a differential equation that predicts how the population $P(t)$ will vary during the time.

For the population P to have constant birth and death rates, it means that on an small interval of time Δt there approximately $\beta P \Delta t$ births and $\delta P \Delta t$ deaths occurring in the interval of time Δt. Hence the variation of the population ΔP is given by:

$$\Delta P = \beta P \Delta t - \delta P \Delta t = (\beta - \delta) P \Delta t.$$

Hence, by using the definition of the derivative as a limit, we have

$$\frac{dP}{dt} = \lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t} = \lim_{\Delta t \to 0} (\beta - \delta) P = (\beta - \delta) P = kP$$

where we set $k := \beta - \delta$.

Therefore the model that describes the evolution of a population P with constant birth and death rates and with P_0 individuals at time $t = 0$ is

$$\begin{cases} \frac{dP}{dt} = k\, P \\ P(0) = P_0 \end{cases} \tag{2}$$

where k is a constant. This is an exponential initial value problem as in §1, and hence its solution is given by the function:

$$P(t) = P_0\, e^{kt}.$$

2.1. **Exercise.** *World's total population was* 6 *billion in* 1999 *and was then increasing at a rate of* 212 *thousand persons per day throughout the year* 1999*. (a) What will be the world population in* 2050*? (b) How long will it take the world population to increase tenfold?*

We start by solving question (a). The model that represents our problem is

$$\begin{cases} \frac{dP}{dt} = k\, P \\ P(0) = P_0 \end{cases} \tag{3}$$

where t is the time measured in years, the time $t = 0$ corresponds to the year 1999, $P(t)$ denotes the number of persons at the year t, and $P_0 = 6$ billion is the population at time $t = 0$ (i.e. in the year 1999). First of all we need to determine the constant k.

The information that the world population is increasing by 212 thousand, or equivalently by 0.000212 billion, persons per day for the entire 1999, it means that at time $t = 0$ (or in 1999) the population increased by $(0.000212)(365.25) = 0.07743$ billion persons. This value is exactly the derivative of P at time $t = 0$, hence

$$\left(\frac{dP}{dt}\right)_{|t=0} = 0.07443.$$

We can then find the constant k by plugging $t = 0$ into the equation $\frac{dP}{dt} = kP$, which becomes

$$\left(\frac{dP}{dt}\right)_{|t=0} = k\,P(0)$$

and hence

$$0.07443 = k \cdot 6.$$

Solving for k we find

$$k = \frac{0.07443}{6} = 0.0129.$$

The solution of the initial value problem (3) is

$$P(t) = P_0\,e^{kt} = 6\,e^{(0.0129)t}.$$

To find the population in the year 2050, it is then enough to plug $t = 51$ in the previous equation:

$$P(51) = \text{population in 2050} = 6\,e^{(0.0129)(51)} = 11.58 \text{ billion}.$$

Now we solve question (b). To see at what time t the population increases tenfold, it is enough to solve for t the equation

$$P(t) = 10\,P_0.$$

Since $P(t) = 6\,e^{(0.0129)t}$ and $P_0 = 6$, we need to solve for t the following equation

$$6\,e^{(0.0129)t} = 60.$$

This gives $t = 178$ years that corresponds to the year 2177.

3. Radioactive Decay

Consider a sample of material that contains $N(t)$ atoms of a certain radioactive isotope at time t. It has been observed that a constant fraction of those atoms will spontaneously decay. Hence the sample behaves like a population with constant birth rate $\beta = 0$ and constant death rate δ, which we call in this situation k.

The model that describes $N(t)$ is given then by

$$\frac{dN}{dt} = (\beta - \delta)N = -kN$$

where $k = \delta > 0$ is a positive constant.

The associated initial value problem is

$$\begin{cases} \frac{dN}{dt} = -k\,N \\ N(0) = N_0 \end{cases} \tag{4}$$

where N_0 denotes the number of atoms at time $t = 0$. Hence N_0 is positive. To solve the previous IVP, we use the techniques of §1, hence we have:

$$N(t) = N_0\, e^{-kt}.$$

3.1. **Exercise.** *A specimen of charcoal turns out to contain* 63% *as much* ^{14}C *as a sample of present-day charcoal of equal mass. What is the age of the sample? Assume that the half-life of* ^{14}C *is* 5700 *years.*

The model that describes our problem is

$$\begin{cases} \frac{dN}{dt} = -k\,N \\ N(0) = N_0 \end{cases} \tag{5}$$

where $k > 0$ is a positive constant. The solution of this problem is

$$N(t) = N_0\, e^{-kt}.$$

First of all we determine the constant k. The information on the half-life, it simply means that

$$N(5700) = \frac{1}{2} N_0,$$

i.e. after 5700 years the amount of ^{14}C in the sample has halfed. We then have

$$N_0\, e^{-k(5700)} = \frac{1}{2} N_0$$

and we solve this equation for k. First of all we divide both sides by N_0, and then we apply the ln function to both sides. Hence one finds that

$$k = -\ln\left(\frac{1}{2}\right)\frac{1}{5700} = 0.0001216.$$

Now we answer to the question of the problem. Notice that it is just asking at what time t we have that

$$N(t) = 63\% \text{ of } N_0 = (.63) N_0.$$

Since $N(t) = N_0 e^{-(0.0001216)t}$, we need to solve for t the equation

$$N_0\, e^{-(0.0001216)t} = (.63)\, N_0.$$

This gives $t = 3800$ years.

/

Example (Exact differential Equation)

$$(6xy - y^3)\,dx + (4y + 3x^2 - 3xy^2)\,dy = 0$$

Is this form exact?

If so, solve it.

To check that the form is exact we only need to check the condition

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}.$$

In our case $M = 6xy - y^3$
$N = 4y + 3x^2 - 3xy^2$

Hence: $\dfrac{\partial M}{\partial y} = 6x - 3y^2$

$\qquad\qquad\;\, \parallel$

$\dfrac{\partial N}{\partial x} = 6x - 3y^2$

As they are equal to each other, ~~the~~
the form is exact

This means that there exists a function $F(x,y)$ in 2 variables such that $\dfrac{\partial F}{\partial x} = M$ and $\dfrac{\partial F}{\partial y} = N$.

We now find $F(x,y)$.

We start by integrating wrt x the equation $\frac{\partial F}{\partial x} = M$:

$$\int \frac{\partial}{\partial x} F \, dx = \int M \, dx + \underbrace{g(y)}$$

We need to add a function that depends on y because $F(x,y)$ is a function in 2 variables but we are integrating only wrt x. Note that $\frac{\partial}{\partial x} g(y) = 0$.

Hence:

$$F(x,y) = \int M \, dx + g(y) =$$

$$= \int (6xy - y^3) \, dx + g(y) =$$

$$= 6y \frac{x^2}{2} - y^3 x + g(y)$$

$$= 3x^2 y - y^3 x + g(y).$$

Now we need to find $g(y)$.

To this end we derive the previous equation wrt y:

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(3x^2 y - y^3 x + g(y)\right)$$

$$= 3x^2 - 3y^2 x + g'(y).$$

Now we impose the second condition $\frac{\partial F}{\partial y} = N = 4y + 3x^2 - 3xy^2$

Hence:

$$3x^2 - 3y^2 x + g'(y) =$$
$$= 4y + 3x^2 - 3xy^2$$

Solve for $g'(y)$:

$$g'(y) = 4y.$$

In order to find $g(y)$ we simply integrate wrt y.

$$\int g'(y)\,dy = \int 4y\,dy + \underset{\text{constant}}{C}$$

Hence $g(y) = 4\frac{y^2}{2} + C = 2y^2 + C.$

Finally $\boxed{F(x,y) = 3x^2 y - xy^3 + 2y^2 + C}$

Another Example

$$\left(x \ln(y)\right) dx + \left(\frac{x^2}{2y}\right) dy = 0$$

Assume $y>0$

Is it exact? If so solve it.

Here $\quad M = x \ln(y) \quad N = \dfrac{x^2}{2y}$

$$\frac{\partial M}{\partial y} = \frac{x}{y} \qquad \frac{\partial N}{\partial x} = \frac{1}{2y}(2x) = \frac{x}{y}.$$

They are equal to each other so the form is exact.

This means that there exists a function $F(x,y)$ such that

$$\begin{cases} \dfrac{\partial F}{\partial x} = M = x \ln(y) & (*) \\[2ex] \dfrac{\partial F}{\partial y} = N = \dfrac{x^2}{2y} & (**) \end{cases}$$

To find F we integrate equation $(*)$ wrt x:

$$\int \frac{\partial F}{\partial x} dx = \int x \ln(y)\, dx + g(y)$$

$$= \frac{x^2}{2} \ln(y) + g(y)$$

Hence $\quad F(x,y) = \dfrac{x^2}{2} \ln(y) + g(y) \qquad (***)$

Now ~~we~~ we need to find $g(y)$.
For this we impose the second condition $\frac{\partial F}{\partial y} = \frac{x^2}{2y}$.

So we find $\frac{\partial F}{\partial y}$ from $(***)$;

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^2}{2}\ln(y) + g(y)\right)$$

$$= \frac{x^2}{2}\frac{1}{y} + g'(y)$$

Hence we have

$$\frac{x^2}{2}\frac{1}{y} + g'(y) = \frac{x^2}{2y}$$

Hence $g'(y) = 0$

and by integrating wrt y:

$$\int g'(y)\,dy = \int 0\,dy + \underset{\uparrow \text{ constant.}}{C}$$

$$g(y) = 0 + C$$

Hence $\boxed{F(x,y) = \frac{x^2}{2}\ln(y) + C}$