MAT 303: Calculus IV with Applications
Spring 2015

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General Information

Differential equation is an equation relating an unknown function and its derivatives. Various scientific laws can be translated into differential equations. The course is dedicated to standard techniques for solving ordinary differential equations, including numerical methods, and their applications in different branches of science such as physics, biology, chemistry, economics and social sciences.

Instructor:
Artem Dudko, artem.dudko@stonybrook.edu
Lectures: MWF 10:00-10:53 (Library W4550)
Office hours: MW 11:00-11:53 (Math Tower 3114) and F 11:00-11:53 (Math Learning Center, Math Tower S-240A)

Teacing Assistant:
Aleksander Doan
Tutorials: R01 on F 12:00-12:53 (Library E4320) and R02 on W 12:00-12:53 (Library E4320)

Textbook: Edwards & Penney, Differential Equations with Boundary Value Problems: Computing and Modelling, Fourth Edition, Prentice Hall, Chapters 1-6. You can use other editions, but be aware that numeration of the exercises might be different.

Topics: an introduction to first order differential equations; phase plane analysis; numerical methods; higher order linear equations and systems; nonlinear phenomena.

Prerequisite is completion of one of the standard calculus sequences (either MAT 125-127 or MAT 131-132 or MAT 141-142) with a grade C or higher in MAT 127, 132 or 142 or AMS 161. Also, MAT 203/205 (Calculus III) and AMS 261/MAT 211 (Linear Algebra) are recommended. Informally, students should know integration and differentiation techniques and, desirably, be familiar with complex numbers and basic aspects of linear algebra.

Tests, quizzes, assignments:
Every week there will be either a homework assignment or a quiz (alternating). The quizzes (starting the second Quiz) will be written

on Mondays during the last 20 minutes of the class. You should hand in your assignments to the instructor during Monday class. The first homework assignment is due on Monday, February 16. No late assignments will be accepted.
Midterm Test I: Monday, March 2.
Midterm Test II: Monday, April 6.
Final Exam: Monday, May 18, 8:00AM-10:45AM, Library W4550.
Final Exam Review: Wendesday, May 13, 11am-1pm, Library W4550.
Last day of classes: Friday, May 8.

Course grade is computed by the following scheme:
Homework and Quizzes: 20%
Midterm Test I: 20%
Midterm Test II: 20%
Final Exam: 40%

Information for students with disabilities
If you have a physical, psychological, medical, or learning disability that may impact your course work, please contact Disability Support Services at (631) 632-6748 or http://studentaffairs.stonybrook.edu/dss/. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential.

Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to the following website:
http://www.sunysb.edu/ehs/fire/disabilities.shtml

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Syllabus

The following is a tentative schedule for MAT 303.

Week of	Section	Notes
Jan 26	1.1 Differential equations and mathematical models	
Feb 2	1.2 Integrals as general and particular solutions 1.3 Slope fields and solution curves	
Feb 9	1.4 Separable equations and applications 1.5 Linear first-order equations	
Feb 16	1.6 Substitution methods and exact equations	
Feb 23	2.1 Population models 2.2 Equilibrium solutions and stability	
Mar 2	2.3 Acceleration-Velocity models	Midterm 1 on M, March 2
Mar 9	2.4 Numerical approximation: Euler's method 3.1 Introduction: second-order linear equations	
Mar 23	3.2 General solutions of linear equations 3.3 Homogeneous equations. Constant coefficients	
Mar 30	3.4 Mechanical vibrations 3.5 Nonhomogeneous equations	
April 6	3.6 Forced oscillations and resonance	Midterm 2 on M, April 6
April 13	4.1 First-order systems and applications 5.1 Matrices and linear systems	
April 20	5.2 The eigenvalue method 5.3 Second-order systems and applications	
April 27	5.4 Multiple eigenvalue solutions 5.5 Matrix exponentials and linear systems	
May 4	6.1 Stability and the phase plane 6.2 Linear and almost linear systems	Final exam on May 18, 8-10:45AM

Recommended problems from the course book.

Section 1.1: 5, 7, 9, 27, 40, 45.
Section 1.2: 6, 7, 8, 24, 32, 38.
Section 1.3: 15, 16, 17.
Section 1.4: 6, 9, 17, 23, 25, 27, 31, 34, 44, 61.
Section 1.5: 7, 13, 16, 20, 36.
Section 1.6: 10, 15, 16, 30, 32, 37, 40.
Section 2.1: 4, 9, 17, 26.
Section 2.2: 4, 7.
Section 2.3: 1, 4, 10, 17, 20.
Section 2.4: 2, 4, 10.
Section 3.1: 4, 7, 14, 18, 20, 22, 25, 30, 31, 34, 41, 47, 48.
Section 3.2: 1, 3, 5, 8, 10, 13, 17, 27, 30, 33.
Section 3.3: 4,10, 16, 19, 22, 23, 24, 27, 31, 39, 42.

Nonhomogeneus equations: in each of the following question find by inspection a particular solution of the given differential equation, then find the general solution of the associated homogeneous equation and compose the general solution of the nonhomogeneous differential equation. Solve the initial value problem, if the initial conditions are given.

1. $y'' - 9y = \sin(2x),\ y(0) = 2,\ y'(0) = -2/13.$
2. $y'' + 4y = 3x - 1,\ y(0) = -1/4,\ y'(0) = 3/4.$
3. $y''' + 3y'' + 3y' + y = \exp(x).$

Section 3.4: 1, 2, 15, 17, 20.
Section 3.5: 1, 3, 9, 17, 25, 30.
Section 3.6: 1, 3, 5.
Section 4.1: 5, 7, 10, 16, 19, 24.
Section 4.2: 3, 5, 7, 9, 12, 14.
Section 5.1: 1, 2, 5, 12, 20, 22, 26.
Section 5.2: 4, 8, 11, 19, 22, 26.
Section 5.4: 2, 6, 7.
Section 6.2: solve the system, determine the type of the critical point, sketch the phase portrait for the questions 5, 7, 8, 9.

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Assignments

Assignment 1

Assignment 2

Assignment 3

Computational project (for extra credit, 6% of the course grade)

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Solutions

Quiz 1 solutions

Quiz 2 solutions

Quiz 3 solutions

Quiz 4 solutions

Quiz 5 solutions

Assignment 1 solutions

Assignment 2 solutions

Assignment 3 solutions

Midterm 1 solutions

Midterm 2 solutions

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Examples

Practice midterm II (1)

Practice midterm II (2)

Practice midterm II (1) solutions

Practice midterm II (2) solutions

Final exam review

MAT 303 Assignment 1.

Hand in to the instructor in class on Monday, February 16.

Problem 1. In each case verify by substitution that the function is a solution of the corresponding differential equation

$$1)\ y(x) = \sin(\tfrac{x}{2}) - 2\cos(\tfrac{x}{2}),\ \ 4y'' + y = 0,$$
$$2)\ y(x) = e^{x^2},\ \ y' = 2xy,$$
$$3)\ y(x) = \sqrt{x^2+1},\ (y')^2 = 1 - \tfrac{1}{y^2}.$$

Problem 2. Find the general solutions of the following differential equations:

$$1)\ \tfrac{dx}{dt} = 3t^2 + 2t - \cos(2t),\ \ 2)\ y' = x^2\sin(x^3).$$

Problem 3. Solve the initial value problems:

$$1)\ \tfrac{dy}{dt} = \tfrac{t}{t^2+1},\ \ y(0) = 5,\ \ 2)\ xy' = x^2 - 2,\ \ y(-1) = 0.$$

Problem 4. A car starting from rest reached the velocity 30 mi/h (44 ft/s) after traveling the distance of 44 ft. Assuming that the car had constant acceleration find this acceleration and the time which took the car to reach 60 mi/h.

Problem 5. Solve the following first order separable differential equations:

$$1)\ y' = x^2 y,\ \ y(2) = 1,\ \ 2)\ \tfrac{dx}{dt} = x + \tfrac{1}{x},$$

Problem 6. Among the following differential equations solve the one which is first order and separable

$$1)\ \frac{d^2x}{dt^2} = x^2t^2,\ \ 2)\ \frac{dy}{dt} = t^2 + y\sin t,$$
$$3)\ y' - 1 = xy + x + y,\ \ 4)\ (y')^2 = x^2 + y^2.$$

Problem 7. Show by substitution that the formula

$$y(x) = \tfrac{2}{1+Ce^x} - 1, \tag{1}$$

where C is a constant, gives *a general* solution of the differential equation

$$2y' = y^2 - 1.$$

Show that formula (1) is not *the general* solution of the given equation by finding a solution which is not described by (1).

MAT 303 Assignment 2.

Hand in to the instructor in class on Monday, March 23.

Problem 1. Find the escape velocity from the Jupiter's moon Europa given its mass $4.8 \cdot 10^{22}$ kg and radius $1,560$ km. The gravitational constant is equal to $G = 6.673 \cdot 10^{-11}$ $N \cdot (m/kg)^2$.

Problem 2. Suppose that a crossbow bolt is shot straight from the ground with initial velocity 28 m/s. Assume that
a) air resistance is proportional to the velocity of the bolt with the drag coefficient equal to 0.02;
b) air resistance is proportional to the square of the velocity with the drag coefficient 0.0003.
In each case find the maximal height the bolt will reach.

Problem 3. Suppose that a motorboat is moving at 20 m/s when its motor suddenly quits, and that 10 s later the boat has slowed to 10 m/s. Assume that the resistance it encounters is proportional to a) its velocity, b) the square of its velocity. How far will the boat coast after 2 minutes?

Problem 4. 1) Using Euler's method with step size a) $h = 0.25$, b) $h = 0.1$ to find approximate value $y(1)$ of the solution of the initial value problem

$$y' = x^2 - y, \quad y(0) = 1.$$

2) Solve the initial value problem from part 1). Find the error terms with the approximate solutions obtained in part 1).

Problem 5. Assume that a deer population satisfies the logistic equation

$$\frac{dP}{dt} = 0.0225P - 0.0003P^2$$

and initially there are 25 deers.
1) Approximate deer population $P(10)$ after 10 years using Euler's method with the step size $h = 1$.
2) Solve the equation of the population and find the exact value of $P(10)$. Compare with the result of 1) and calculate the error term.

MAT 303 Assignment 3.

Hand in to the instructor in class on Monday, April 20.

Problem 1. Find the general solution of the equation

$$y''' - 8y = x^2 - x + 1.$$

Problem 2. Solve the initial value problem

$$y'' - 3y' + 2y = e^{2x},\ y(0) = 0,\ y'(0) = 6.$$

Problem 3. Describe the motion of a body of mass m with initial position x_0 and initial velocity v_0 in a mass-spring-dashpot system with a spring constant k and damping constant c if
a) $m = 3,\ c = 10,\ k = 7,\ x_0 = 6,\ v_0 = 2;$
b) $m = 2,\ c = 8,\ k = 10,\ x_0 = 10,\ v_0 = 2.$

Problem 4. Describe the motion of a body of mass 1 kg in a mass-spring system with a spring constant $k = 4$ N/m and the external force $F(t) = 2\sin 3t$, if the body starts from rest (that is, $x_0 = v_0 = 0$). Sketch the graph of the position function.

MAT 303 Computational project (for extra credit).
Print your project and hand in to the instructor in class on Friday, May 1.

For your own personal computational project, let a, b, c be the three last digits of your student ID number. Set

$$p = 1 + 0.1a,\ q = 1 + 0.1b,\ r = 1 + 0.1c.$$

To do the project use a computer algebra system (like Maple or Mathematica).

Problem 1. Consider the differential equation

$$\tfrac{dy}{dx} = py - \tfrac{qx^2}{y},\quad y(0) = r. \tag{1}$$

(a) Using Euler's method with step size a) $h = 0.2$, b) $h = 0.05$, c) $h = 0.01$ find an approximate value of $y(1)$.

(b) Find the true solution of the initial value problem (1).

(c) Find the error terms between the approximate values of $y(1)$ calculated in (a) and the true value of $y(1)$. How does error term depend on h?

Problem 2. Consider the homogeneous differential equation with constant coefficients

$$py^{(3)} - 4qy' + ry = 0. \tag{2}$$

(a) Find the roots of the corresponding characteristic equation using a computer algebra system. Define the corresponding particular solutions y_1, y_2, y_3 of the equation (2).

(b) Consider the particular solution $Y(x) = y_1 + y_2 + y_3$ of (2). Plot $Y(x)$.

(c) Calculate the numbers $s_0 = Y(1), s_1 = Y'(1), s_2 = Y''(1)$. Solve the initial value problem

$$py^{(3)} - 4qy' + ry = 0, \quad y(1) = s_0, y'(1) = s_1, y''(1) = s_2.$$

Plot the solution and compare it with the function $Y(x)$.

Problem 3. Solve the system of linear differential equations

$$x' = x + py, y' = qx + y.$$

Draw the direction field corresponding to this system and a few solution curves, illustrating the behavior of the general solution.

Quiz 1.

Problem 1. Show that the function

$$y = (x + C)e^{-x}$$

is a solution of the differential equation

$$e^x(y' + y) = 1.$$

Find a solution satisfying to the initial condition $y(1) = 0$.

Problem 2. Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} = \sin(2t).$$

Problem 1

If $y = (x+C)e^{-x}$, then $y' = e^{-x} - (x+C)e^{-x}$

and :

$e^x(y'+y) = e^x(e^{-x} + (x+C)e^{-x}(-1) +$

$+ (x+C)e^{-x}) =$

$= 1 - (x+C) + (x+C) = 1$

so such y satisfies the equation.

If $y(1) = 0$, we get

$0 = y(1) = (1+C)e^{-1}$

so $C = -1$; and

$y(x) = (C-1)e^{-x}$

Problem 2

$$\frac{d^2x}{dt^2} = \sin 2t \quad \rightarrow \text{integrate}$$

$$\frac{dx}{dt} = \int \sin 2t \, dt = -\frac{1}{2}\cos 2t + C$$

$\rightarrow$ integrate again

$$x(t) = -\frac{1}{2}\int \cos 2t \, dt + \int C \, dt =$$

$$= -\frac{1}{4}\sin 2t + Ct + D$$

for any constants C and D.

Quiz 2.

Problem 1. Solve the initial value problem

$$y' = \frac{3y + x}{x}, \quad y(1) = \frac{3}{2}.$$

Problem 2. Find the general solution of the differential equation

$$yy'' = (y')^2.$$

Quiz 2 solutions

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Problem 1 First order linear ODE

$$y' = 3\frac{y}{x} + 1$$

$$y' - \frac{3}{x}y = 1$$

Integrating factor $\rho(x) = e^{\int -\frac{3}{x}dx} = e^{-3\ln x} = x^{-3}$

$$x^{-3}y' - \frac{3}{x}x^{-3}y = x^{-3}$$

$$\frac{d}{dx}(x^{-3}y) = x^{-3}$$

$$x^{-3}y = \int \frac{d}{dx}(x^{-3}y)\,dx = \int x^{-3}dx = -\frac{1}{2}x^{-2} + C$$

so $y(x) = -\frac{1}{2}x + Cx^3$

Initial condition :

$$\frac{3}{2} = y(1) = -\frac{1}{2} + C \implies C = 2$$

so $\underline{y(x) = -\frac{1}{2}x + 2x^3}$

Alternative solution: homogeneous equation

$$\frac{dy}{dx} = 3\left(\frac{y}{x}\right) + 1 = F\left(\frac{y}{x}\right)$$

Problem 2 Reducible second order ODE

$$y \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 \qquad (*)$$

We use substitution $u = \frac{dy}{dx}$ and find a differential equation for u as a function of y. Because $u = \frac{dy}{dx}$,

$$\frac{d^2y}{dx^2} = \underbrace{\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx}}_{\text{chain rule}} = \frac{du}{dy}\, u$$

so equation $(*)$ is equivalent to

$$y \frac{du}{dy} u = u^2$$

Assume $u \neq 0$ (if $u = 0$, then $y = \text{const.}$) and divide by u

$$y \frac{du}{dy} = u \qquad \text{separable equation}$$

$$\int \frac{du}{u} = \int \frac{dy}{y} \Rightarrow \ln u = \ln y + C$$

$$\Rightarrow \quad u = e^C y = Ay$$

Now, $\frac{dy}{dx} = u = Ay$

We solve for $y = y(x)$ by separating variables

$$\int \frac{dy}{y} = \int A\,dx = Ax + C$$

$$\ln y = Ax + C \Rightarrow y(x) = e^{Ax+C} = e^C e^{Ax}$$
$$= B e^{Ax}$$

$A, B =$ any constants.

Quiz 3 solutions

Problem 1

$$\begin{cases} \dfrac{dP}{dt} = 0.1\,P(t) - 0.0005\,P^2(t) = 0.1\,P(t)(1 - 0.005\,P(t)) \\ P(0) = 100 \end{cases}$$

separable equation

$$\int \frac{dP}{0.1\,P(1-0.005P)} = \int dt = t + C$$

$$10(\ln P - \ln(200-P)) = t + C$$

$$\ln\left(\frac{P}{200-P}\right) = \frac{1}{10}t + C' \qquad \left(C' = \frac{C}{10}\right)$$

$$\frac{P(t)}{200-P(t)} = e^{\frac{1}{10}t + C'}$$

For $t = 0$

$$\frac{P(0)}{200 - P(0)} = \frac{100}{200 - 100} = \frac{100}{100} = e^{C'}$$

$$\Rightarrow \quad C' = 0$$

and

$$\frac{P(t)}{200-P(t)} = e^{\frac{1}{10}t} \Rightarrow P(t) = e^{\frac{1}{10}t}(200 - P(t))$$

$$\Rightarrow \quad P(t) = \frac{200\,e^{\frac{1}{10}t}}{1 + e^{\frac{1}{10}t}} = \frac{200}{e^{-t/10} + 1}$$

so P is increasing and $\lim_{t\to\infty} P(t) = 200.$

Problem 2

$$\frac{dx}{dt} = (x^2-4)(x+3)^2$$

critical points

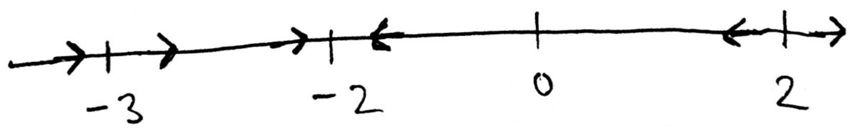
$x^2-4=0 \quad \Rightarrow \quad x=2,\ x=-2$

or

$x+3=0 \quad \Rightarrow \quad x=-3$

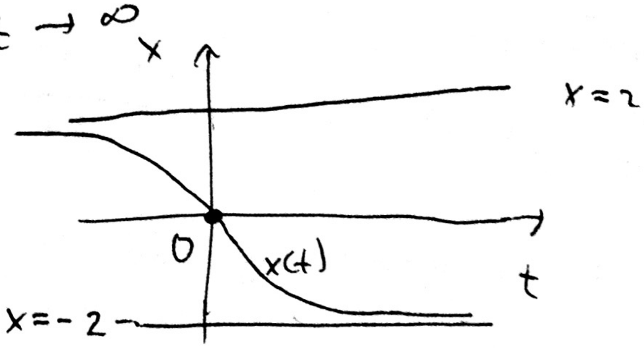
The function $(x^2-4)(x+3)^2$ is

positive for $x<-3$
positive for $x\in(-3,-2)$
negative for $x\in(-2,2)$
positive for $x>2$



Thus, $x=-3$ is semistable
$x=-2$ is stable
$x=2$ is unstable

If we start from $x(0)=0$, the solution $x(t)$ will asymptotically converge to the contant solution $x=-2$ when $t\to\infty$ and to $x=2$ when $t\to-\infty$



Quiz 4 solutions

Problem 1

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x(1+x) \end{vmatrix}$$

$$= e^{2x}(1+x) - e^{2x}x = e^{2x}$$

Problem 2

$$y_1'' - 2y_1' + y_1 = e^x - 2e^x + e^x = 0 \checkmark$$

$$y_2'' - 2y_2' + y_2 = (e^x + xe^x)' - 2(e^x + xe^x) +$$

$$+ xe^x = 2e^x + xe^x - 2(e^x + xe^x) + xe^x = 0 \checkmark$$

so y_1, y_2 are solutions.

We have $W(0) = e^0 = 1 \neq 0$ (see Problem 1)

so y_1, y_2 are linearly independent.

Thus, any solution y to the equation is a linear combination of y_1, y_2.

$$y(x) = C_1 y_1(x) + C_2 y_2(x) =$$
$$= C_1 e^x + C_2 x e^x$$
$$y'(x) = C_1 e^x + C_2 (e^x + x e^x)$$

so we have to find C_1, C_2 :

$$e = y(1) = C_1 e^1 + C_2 e$$
$$-e = y'(1) = C_1 e + 2 C_2 e$$

$$\begin{cases} C_1 + C_2 = 1 \\ C_1 + 2C_2 = -1 \end{cases} \Rightarrow \begin{array}{l} C_1 = \cancel{2}\ 3 \\ C_2 = -2 \end{array}$$

The solution is

$$y(x) = 3e^x - 2xe^x.$$

solution by Aleksander Doan
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Quiz 5 solutions

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Problem 1

$$\frac{dX_1}{dt} = \begin{bmatrix} (2e^{2t})' \\ (e^{2t})' \end{bmatrix} = \begin{bmatrix} 4e^{2t} \\ 2e^{2t} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} X_1 = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2e^{2t} \\ e^{2t} \end{bmatrix} = \begin{bmatrix} 4e^{2t} \\ 2e^{2t} \end{bmatrix}$$

so $X_1' = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} X_1$. Similarly X_2.

Problem 2

We compute the Wronskian

$$W(t) = \det \begin{bmatrix} 2e^{2t} & e^{-3t} \\ e^{2t} & -2e^{-3t} \end{bmatrix} = e^{2t}e^{-3t} \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}$$

$$= e^{-t}(-4-1) = -5e^{-t} \neq 0$$

since $W(t) \neq 0$ for $t \in (-\infty, \infty)$, X_1 and X_2 are linearly independent on the real line.

Problem 1

Straightforward calulcation

(1) $y(x) = \sin\frac{x}{2} - 2\cos\frac{x}{2}$

$y'(x) = \frac{1}{2}\cos\frac{x}{2} + \sin\frac{x}{2}$

$y''(x) = -\frac{1}{4}\sin\frac{x}{2} + \frac{1}{2}\cos\frac{x}{2}$

so $4y'' + y = (-\sin\frac{x}{2} + 2\cos\frac{x}{2}) + (\sin\frac{x}{2} - 2\cos\frac{x}{2}) = 0$

(2) and (3) similarly

Problem 2

1) $\frac{dx}{dt} = 3t^2 + 2t - \cos 2t$

$x(t) = \int \frac{dx}{dt}\,dt = \int (3t^2 + 2t - \cos 2t)\,dt$

$= t^3 + t^2 - \frac{1}{2}\sin 2t + C$

2) $\frac{dy}{dx} = x^2 \sin x^3$

$y(x) = \int \frac{dy}{dx}\,dx = \int x^2 \sin x^3\,dx = -\frac{1}{3}\cos x^3 + C$

Problem 3

1) $y(t) = \int \frac{dy}{dt}\, dt = \int \frac{t}{t^2+1}\, dt = \frac{1}{2}\log(t^2+1) + C$

$5 = y(0) = \frac{1}{2}\log 1 + C = C$

so $y(t) = \frac{1}{2}\log(t^2+1) + 5$

2) $y(x) = \int \frac{dy}{dx}\, dx = \int \frac{x^2-2}{x}\, dx = \frac{x^2}{2} - 2\log|x| + C$

$0 = y(-1) = \frac{1}{2} + C \Rightarrow C = -\frac{1}{2}$

Problem 4

$\frac{d^2x}{dt^2} = a$ acceleration

$\frac{dx}{dt} = \int \frac{d^2x}{dt^2}\, dx = at + v_0$ $\qquad v_0 = \frac{dx}{dt}(0)$

$x(t) = \int \frac{dx}{dt}\, dt = \frac{1}{2}at^2 + v_0 + x_0$ $\qquad x_0 = x(0)$

In our case $x_0 = 0,\ v_0 = 0$, so

$x(t) = \frac{1}{2}at^2, \quad v(t) = \frac{dx}{dt} = at$

For some t_0, $\qquad a = \cancel{88}\ 22\ ft/s^2$

$44\frac{ft}{s} = v(t_0) = at_0$

$44 ft = x(t_0) = \frac{1}{2}at_0^2$ $\Rightarrow$

and if

$60\frac{mi}{h} = 88\frac{ft}{s} = at$

then $t = \cancel{ls}\ 4s$

Problem 5

1) $\int \frac{dy}{y} = \int x^2 dx$

$\ln|y| = \frac{1}{3}x^3 + C$

$|y| = e^C e^{\frac{1}{3}x^3} \Rightarrow 1 = y(2) = e^C e^{\frac{1}{3}8}$

$\Rightarrow e^C = e^{-8/3}$

so $y(x) = e^{-8/3} e^{1/3 x^3}$

2) $\int \frac{dx}{x + \frac{1}{x}} = \int dt = t + C$

$\frac{1}{2} \ln(x^2+1) = t + C$

$\ln(x^2+1) = 2(t+C)$

$x^2 + 1 = e^{2(t+C)}$

$x = \pm\sqrt{e^{2(t+C)} - 1}$

Problem 6 $\quad y' = xy + x + y + 1 = (1+x)(1+y)$

so only (3) is first order and separable

$\int \frac{dy}{1+y} = \int (1+x)dx = \frac{1}{2}x^2 + x + C$

$$\ln|1+y| = \frac{1}{2}x^2 + x + C$$

$$|1+y| = e^{\frac{1}{2}x^2 + x + C}$$

so $y = \pm e^{\frac{1}{2}x^2 + x + C} - 1$

($\pm$ depending on the initial condition)

Problem 7

$$(*) \quad y(x) = \frac{2}{1+Ce^x} - 1 \qquad y^2(x) = \frac{4}{(1+Ce^x)^2} - \frac{4}{(1+Ce^x)} + 1$$

$$y'(x) = -\frac{2Ce^x}{(Ce^x+1)^2}$$

$$2y' - y^2 + 1 =$$

$$= -\frac{4Ce^x}{(Ce^x+1)^2} - \frac{4}{(1+Ce^x)^2} + \frac{4}{1+Ce^x} = \frac{-4-4Ce^x+4+4Ce^x}{(1+Ce^x)^2} = 0$$

$y(x) = -1$ satisfies the equation $2y' - y^2 + 1 = 0$

but is not of the form $(*)$ for any C.

solutions by Aleksander Doan

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Assignment 2 solutions

Problem 1

$$G = 6.673 \times 10^{-11} \frac{m^3}{kg \cdot s^2}$$

$$m = 4.8 \times 10^{22} \text{ kg}$$

$$r = 1560 \times 10^3 \text{ m}$$

$$v = \sqrt{\frac{2Gm}{r}} \simeq 2026.44 \frac{m}{s}$$

Problem 2

a) $\frac{dv}{dt} = -g - \rho v$ $\qquad v(0) = 28 \frac{m}{s}$

$\rho = 0.02$ 1/s

$g \simeq 9.81$ m/s²

$$v(t) = Ce^{-0.02t} - 490.5$$

$$28 = C - 490.5$$

$$\Rightarrow C = 518.5$$

$$v(t) = 518.5e^{-0.02t} - 490.5$$

$$h(t) = \int v(t)\,dt = -490.5t - 25925e^{-0.02t} + C$$

$$h(0) = 0 \Rightarrow C = 25925$$

$$h(t) = -490.5t - 25925e^{-0.02t} + 25925$$

$$v(t) = 0 \Rightarrow t = \cancel{2.96}\ 2.776$$

$$\cancel{h(2.96)} = h(2.776) \simeq 38.5 \text{ m}$$

b) $\frac{dv}{dt} = -g - \rho v^2$

$\rho = 0.0003$
$g = 9.81$
$v(0) = 28$

$v(t) = -\sqrt{\frac{g}{\rho}} \tan\left(\sqrt{g\rho}\,(c+t)\right)$

$= -180.831 \tan(0.0541(c+t))$

$v(0) = 28 \Rightarrow c \approx -2.833$

$v(t) = -180.831 \tan(\cancel{0.}\ 0.054t - 0.153)$

$v(t) = 0 \Rightarrow t = 2.833$

$\cancel{h(t) = \int v(t)\,dt =}$

$h(2.833) = \int_0^{2.833} v(t)\,dt \approx 39.349$

Problem 3

$v(0) = 20, \quad v(10) = 10$

a) $\frac{dv}{dt} = -pv \quad \Rightarrow \quad v(t) = v(0)\, e^{-pt}$

$10 = v(10) = 20\, e^{-10p}$

$\Rightarrow \quad p = \frac{\ln 2}{10} \approx 0.069$

so $\quad v(t) = 20\, e^{-0.069t}$

and $\quad x(t) = \int v(t)\,dt = -289.855\, e^{-0.069t} + C$

$x(0) = 0 \quad \Rightarrow \quad C = 289.855$

and $\quad x(120) \simeq 289.782 \text{ m}$

b) $\frac{dv}{dt} = -pv^2 \quad \Rightarrow \quad v(t) = \frac{1}{pt + c}$

$20 = v(0) = \frac{1}{c} \quad \Rightarrow \quad c = \frac{1}{20}$

$10 = v(10) = \frac{1}{10p + \frac{1}{20}} \quad \Rightarrow \quad p = \frac{1}{200}$

so $\quad v(t) = \frac{1}{\frac{t}{200} + \frac{1}{20}} = \frac{200}{t + 10}$

and $\quad x(t) = \int v(t)\,dt = 200 \ln(t+10) + C$

$0 = x(0) = 200 \ln 10 + C \quad \Rightarrow \quad C = -200 \ln 10$

$x(t) = 200 \ln\left(\frac{t+10}{10}\right)$

$x(120) \simeq 512.990 \text{ m}$

Problem 4

$$y' = x - y$$
$$y(0) = 1$$

1a) $h = 0.25$

$y'(0) = -1$
$y(0.25) = y(0) + hy'(0) = 1 - 0.25 = 0.75$
$y'(0.25) = 0.25 - y(0.25) = 0.25 - 0.75 = -0.5$

similarly

$y(0.5) = 0.6250$
$y(0.75) = 0.5938$
$y(1) = 0.6328$

1b) $h = 0.1$

$y(0.1) = 0.9$
$y(0.2) = 0.82$
$y(0.3) = 0.758$
$y(0.4) = 0.7122$
$y(0.5) = 0.6810$
$y(0.6) = 0.6629$
$y(0.7) = 0.6566$
$y(0.8) = 0.6609$
$y(0.9) = 0.6748$
$y(1) = 0.6974$

2) Actual solution (linear 1st order ODE)

$y(x) = Ce^{-x} + x - 1$
$1 = y(0) = C - 1 \Rightarrow C = 2$
$y(x) = 2e^{-x} + x - 1$
$y(1) = 2e^{-1} \approx 0.7358$

errors

$|0.7358 - 0.6328| = 0.103$
$|0.7358 - 0.6974| = 0.384$

Problem 5

$$\frac{dP}{dt} = 0.0225P - 0.003P^2$$

$P(0) = 25$

$P'(0) = 0.375$

$P(1) = P(0) + h\,P'(0) = 25 + 0.375 = 25.375$

$P'(1) = 0.378$

$P(2) = 25.375 + 0.378 = 25.753$

$P'(2) = 0.380$

$P(3) = 26.133$

$P'(3) = 0.383$

$P(4) = 26.516$

$P'(4) = 0.386$

$P(5) = 26.902$ $\quad P'(5) = 0.388$

$P(6) = 27.290$ $\quad P'(6) = 0.390$

$P(7) = 27.690$ $\quad P'(7) = 0.393$

$P(8) = 28.073$ $\quad P'(8) = 0.395$

$P(9) = 28.468$ $\quad P'(9) = 0.397$

$P(10) = 28.865$

Actual solution

$$\frac{dP}{dt} = 0.0003P(75 - P), \quad P(0) = 25$$

$$P(t) = \frac{1875}{25 + 50e^{-0.0225t}}$$

$P(10) \simeq 28.879$

so error is $|28.865 - 28.879| \simeq 0.014$

Assignement 3 solutions

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Problem 1

homogeneous equation $y''' - 8y = 0$

$t^3 - 8 = 0$

$t^3 = 8 \Rightarrow t_1 = 2$

$t_2 = 2e^{\frac{3\pi i}{3}} = -1 + \sqrt{3}i$

$t_3 = 2e^{9\pi i/3} = -1 - \sqrt{3}i$

so a general solution is

$y_h = C_1 e^{2x} + C_2 e^{-x} \cos(\sqrt{3}x) + C_3 e^{-x} \sin(\sqrt{3}x)$

particular solution

we look for a solution of the form

$y_p = ax^2 + bx + c$

since $y_p''' = 0$ we find

$-8(ax^2 + bx + c) = x^2 - x + 1$

so $y_p = -\frac{x^2}{8} + \frac{x}{8} - \frac{1}{8}$

and a general solution is

$y = y_h + y_p = C_1 e^{2x} + C_2 e^{-x} \cos\sqrt{3}x + C_3 e^{-x} \sin\sqrt{3}x + \left(-\frac{x^2}{8} + \frac{x}{8} - \frac{1}{8}\right)$

$\alpha_1 = \quad u_1 + i v_1$

$\alpha_2 =$

Problem 2

homogeneous solution: $t^2 - 3t + 2 = 0$

$\Rightarrow \quad t = \frac{3 \pm 1}{2} = 1$ and 2

general solution $y_h = C_1 e^x + C_2 e^{2x}$

particular solution: we are looking for y_p of the form $y_p = \cancel{x(ax+} \; axe^{2x}$

plugging to the equation we find

$$2ae^{2x} = e^{2x} \Rightarrow a = \frac{1}{2}$$

so $y_p = \frac{1}{2} x e^{2x}$

and general solution is $y = C_1 e^x + C_2 e^{2x} + \frac{1}{2} x e^{2x}$

$y' = C_1 e^x + 2C_1 e^{2x} + \frac{1}{2} e^{2x} + xe^{2x}$

$0 = y(0) = C_1 + C_2$

$6 = y'(0) = C_1 + 2C_2 + \frac{1}{2}$

so $C_2 = -\frac{1}{2}, \; C_1 = \frac{1}{2}$

$y = \frac{1}{2} e^x - \frac{1}{2} e^{2x} + \frac{1}{2} x e^{2x}$.

Problem 3 $\quad m\ddot{x} + c\dot{x} + kx = 0$

a) $m = 3,\ c = 10,\ k = 7$

characteristic polynomial

$$3t^2 + 10t + 7 = 0$$

$$t = \frac{-10 \pm \sqrt{100 - 4 \cdot 21}}{6} = \frac{-10 \pm 4}{6} = \begin{cases} -1 \\ -7/3 \end{cases}$$

a general solution is

$$x(t) = Ae^{-t} + Be^{-7/3 t}$$

initial conditions

$$6 = x(0) = A + B$$

$$2 = x'(0) = -A - 7/3\, B$$

$$8 = -\frac{4}{3} B \quad \Rightarrow B = -6$$

$$\Rightarrow \quad A = 6 - B = 12$$

so $\quad x(t) = 12e^{-t} - 6e^{-7/3 t}$

b) $m = 2,\ c = 8,\ k = 10$

characteristic polynomial

$$2t^2 + 8t + 10 = 0$$

$$t = \frac{-8 \pm \sqrt{64 - 80}}{4} = \frac{-8 \pm \sqrt{-16}}{4} = \begin{cases} -2 + i \\ -2 - i \end{cases}$$

general solution

$$x(t) = e^{-2t}(A\cos t + B\sin t)$$

Initial conditions :

$$10 = x(0) = A$$

$$2 = x'(0) = -2A + B$$

So $A = 10$, $B = 2 + 20 = 22$

$$x(t) = e^{-2t}(10\cos t + 22\sin t)$$

Problem 4 $\qquad \ddot{x} + 4x = 2\sin 3t$

homogeneous equation :

$$x(t) = C_1 \sin 2t + C_2 \cos 2t$$

non-homogenoous particular solution :

$$y_p = A\sin 3t + B\cos 3t$$

$$\dot{y}_p = 3A\cos 3t - 3B\sin 3t$$

$$\ddot{y}_p = -9A\sin 3t - 9B\cos 3t$$

$$\ddot{y}_p + 4y_p = 2\sin 3t \Rightarrow \begin{cases} -9A + 4A = 2 \\ -9B + 4B = 0 \end{cases}$$

$$A = -\frac{2}{5}, \quad B = 0$$

general solution

$$x(t) = C_1 \sin 2t + C_2 \cos 2t - \frac{2}{5}\sin 3t.$$

1.

Solve the initial value problem

$$xyy' = 2y^2 - 1, \quad y(1) = -1.$$

2 Separable. If $2y^2-1 \neq 0$:

2 $\dfrac{y\,dy}{2y^2-1} = \dfrac{dx}{x}$

4 $\displaystyle\int \frac{y\,dy}{2y^2-1} = \frac{1}{4}\int \frac{d(2y^2-1)}{2y^2-1} = \frac{1}{4}\ln|2y^2-1| = \ln|x|+C$

$\ln|2y^2-1| = 4\ln|x| + 4C$

4 $2y^2-1 = \pm e^{4C}\cdot x^4 = Ax^4$, A is any

2 ($A=0$ corresponds to $2y^2-1=0$).

Thus,

3 $y = \pm\sqrt{\frac{A}{2}x^4+\frac{1}{2}}$.

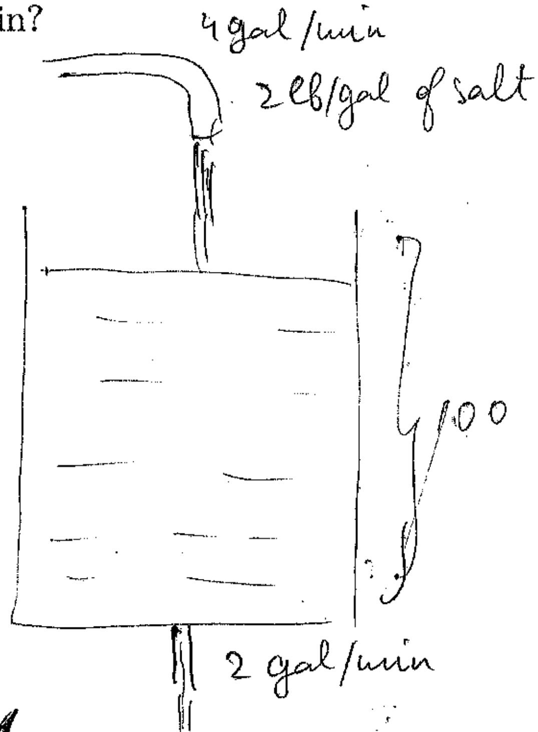
3 $y(1) = -1 \Rightarrow$ sign "$-$",

$1 = -\sqrt{\frac{A}{2}+\frac{1}{2}} \Rightarrow A = 1.$

Answer $y(x) = -\sqrt{\frac{x^4}{2}+\frac{1}{2}}$

2.

A tank initially contains 100 gal of pure water. A brine containing a solution of 2 lb/gal of salt flows into the tank at the rate 4 gal/min and the well stirred mixture flows out of the tank at the rate 2 gal/min. How much salt will be inside the tank after 50min?



The total amount of solution at time t:

$100+2t$ galons.

Let $x(t)$ be the amount of salt at time t.

Rate of change:

5 $$\frac{dx}{dt} = 8 - \frac{2x}{100+2t} = 8 - \frac{x}{50+t}$$

2 $$\frac{dx}{dt} + \frac{x}{50+t} = 8 \quad \text{Linear} \quad 1$$

3 $$\rho(t) = e^{\int \frac{1}{50+t}dt} = e^{\ln(50+t)} = 50+t$$

$$\frac{d}{dt}\left((50+t)x\right) = (50+t)\frac{dx}{dt} + x = 8(50+t) = 400+8t \quad 3$$

$$(50+t)x = \int (400+8t)\,dt = 400t + 4t^2 + C \quad 3$$

$$x = \frac{400t+4t^2+C}{50+t} \qquad x(0) = 0 \Rightarrow C = 0. \quad 2$$

$$x(50) = \frac{400\cdot 50 + 4\cdot 50^2}{50+50} = \frac{30000}{100} = 300 \quad 1$$

Answer 300 lb.

3.

Solve the differential equation

$$y' = x - \frac{1}{x^2 - 2y}.$$

4 substitution $u = x^2 - 2y$

6 $$\frac{du}{dx} = 2x - 2\frac{dy}{dx} = \cancel{2x} - 2\left(\cancel{x} - \frac{1}{x^2-2y}\right) = \frac{1}{u}$$

$$\boxed{\frac{du}{dx} = \frac{2}{u}}$$ separable

6 $$u\,du = 2dx$$

$$\int u\,du = \frac{u^2}{2} = \int dx = 2x + C$$

$$u = \pm\sqrt{4x + 2C} = \pm\sqrt{4x + A}$$

4 $$y = \frac{x^2 - u}{2} = \frac{x^2 \pm \sqrt{4x + A}}{2}$$

<u>Answer</u> $$y = \frac{x^2 \pm \sqrt{4x+A}}{2}$$

4.

Show that the following differential equation is exact; then solve it.

$$\underbrace{(y^2 \sin x + \cos x)}_{N} dx + \underbrace{(y^2 - 2y\cos x)}_{M} dy = 0.$$

5 $\frac{\partial M}{\partial x} = -2y(-\sin x) = 2y\sin x$

$\frac{\partial N}{\partial y} = 2y \sin x$ equal => exact.

Need to find F:

$\frac{\partial F}{\partial x} = N, \quad \frac{\partial F}{\partial y} = M$

2 $\frac{\partial F}{\partial x} = y^2 \sin x + \cos x \Rightarrow$

4 $F(x,y) = \int (y^2 \sin x + \cos x)\,dx =$

$= -y^2 \cos x + \sin x + C(y)$

4 $\frac{\partial F}{\partial y} = -2y\cos x + C'(y) = M = y^2 - 2y\cos x$

3 $\Rightarrow C'(y) = y^2, \quad C(y) = \frac{y^3}{3} + A.$

2 <u>answer</u> $\boxed{-y^2 \cos x + \sin x + \frac{y^3}{3} = B},$

where B is any constant

5.

In some population both the time birth rate and the time death rate are proportional to $1/P(t)$, where $P(t)$ is the size of the population. In 2000 the population was equal to one hundred. In 2010 the population was equal to two hundreds. What will be the size of this population in 2050?

5 $\dfrac{dP}{dt} = \beta - \delta = \dfrac{\beta_0}{P(t)} - \dfrac{\delta_0}{P(t)} = \dfrac{k}{P},$

where $k = \beta_0 - \delta_0$.

$P\,dP = k\,dt$

$\int P\,dP = \dfrac{P^2}{2} = \int k\,dt = kt + C$

5 $P = \sqrt{2kt + 2C} = \sqrt{k_1 t + C_1}$.

3 In 2000, $P(0) = 100 = \sqrt{C_1} \Rightarrow C_1 = 10^4$

3 In 2010: $P(10) = 200 = \sqrt{10k_1 + 10^4} \Rightarrow$

$10k_1 = 3\cdot 10^4$, $k_1 = 3\cdot 10^3$. Thus,

In 2050:

2 $P(t) = \sqrt{3\cdot 10^3 t + 10^4}$.

2 $P(50) = \sqrt{3\cdot 10^3\cdot 50 + 10^4} = \sqrt{16\cdot 10^4} = 400$

Answer 400

1.

Draw the phase diagram for the autonomous equation

$$\frac{dx}{dt} = x^4 - 2x^3 - 2x^2.$$

Determine the types of the critical points.

Critical points:

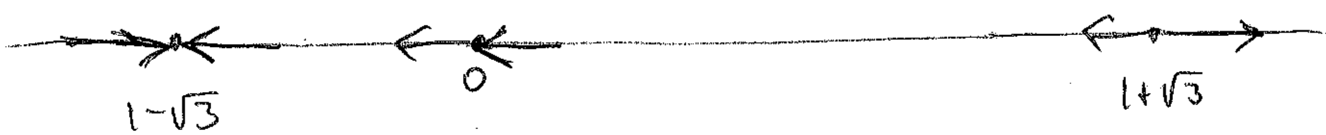
$x^4 - 2x^3 - 2x^2 = 0$

$x^2(x^2 - 2x - 2) = 0$

$x_1 = 0$

$x^2 - 2x - 2 = 0$: $\quad x_{2,3} = \dfrac{2 \pm \sqrt{4 + 4 \cdot 2}}{2} = 1 \pm \sqrt{3}$

stable | unstable (semistable) | unstable



$\frac{dx}{dt} = x^2(x^2-2x-2) > 0 \quad < 0 \quad < 0 \quad > 0$

2.

Consider the differential equation

$$\frac{dy}{dx} = \frac{x-1}{y+4}, \quad y(1) = 1.$$

Using Euler's method with step size $h = 0.5$ find approximate value of $y(2)$.

$x_0 = 1, \quad y_0 = 1$

$x_n = x_1 + n \cdot h$

$x_1 = 1.5, \quad x_2 = 2. \Rightarrow$

Need to find y_2

$y_{n+1} = y_n + h \cdot \frac{x_n - 1}{y_n + 4}$

$y_1 = 1 + 0.5 \cdot \frac{1-1}{1+4} = 1$

$y_2 = 1 + 0.5 \cdot \frac{1.5-1}{1+4} = 1 + 0.5 \cdot 0.1 = 1.05.$

Answer: 1.05

3.

Verify that the functions $y_1 = x, y_2 = x^3$ are solutions of the differential equation

$$x^2y'' - 3xy' + 3y = 0.$$

Solve the initial value problem

$$y(-1) = 1, y'(-1) = -2.$$

$x^2 y_1'' - 3x y_1' + 3y_1 = x^2 \cdot 0 - 3x \cdot 1 + 3x = 0$ $\Rightarrow$

$x^2 y_2'' - 3x y_2'' + 3y_2 = x^2 \cdot 6x - 3x \cdot 3x^2 + 3x^3 = 0$

y_1, y_2 are solutions

$y(x) = C_1 y_1 + C_2 y_2 = C_1 x + C_2 x^3$

$y(-1) = 1:$ $\quad -C_1 - C_2 = 1$ $\quad +$

$y'(-1) = -2:$ $\quad C_1 + 3C_2 = -2$

$2C_2 = -1, \; C_2 = -\frac{1}{2}, \; C_1 = -C_2 - 1 = -\frac{1}{2}.$

Answer: $y(x) = -\frac{1}{2} x - \frac{1}{2} x^3$

4.

Show directly (without using Wronskian) that the functions

$$2 + e^x - 3\sin x,\ 1 + 2e^x - 3\sin x,\ e^x - \sin x$$

are linearly dependent on the real line.

Need to find c_1, c_2, c_3 not all zero such that

$c_1(2+e^x-3\sin x)+c_2(1+2e^x-3\sin x)+c_3(e^x-\sin x)=0.$

Equivalently:

$(2c_1+c_2)+e^x(c_1+2c_2+c_3)+\sin x(-3c_1-3c_2-c_3)=0.$

$$\begin{array}{ll} R_1 & \left\{\begin{array}{l} 2c_1+c_2=0 \\ c_1+2c_2+c_3=0 \\ -3c_1-3c_2-c_3=0 \end{array}\right. \\ R_2 & \\ R_3 & \end{array}$$

R_1: $c_2=-2c_1$

R_2: $c_1+2(-2c_1)+c_3=0 \Rightarrow c_3=3c_1$

R_3: $-3c_1-3(-2c_1)-3c_1=0$, $0=0 \Rightarrow$ no conditions on c_1, c_1 is any

Take $c_1=1$, $c_2=-2$, $c_3=3$. Then:

$(2+e^x-3\sin x)-2(1+2e^x-3\sin x)+3(e^x-\sin x)=0$

$\Rightarrow$ these functions are linearly dependent on $\mathbb{R}$.

5.

Solve the initial value problem

$$y''' + 2y'' + y' = 0,\ y(0) = 2,\ y'(0) = -1,\ y''(0) = 0.$$

$r^3 + 2r^2 + r = 0$

$r(r+1)^2 = 0$, $r_1 = 0$, $r_2 = r_3 = -1$.

$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} + C_3 x e^{r_3 x} =$

$C_1 + C_2 e^{-x} + C_3 x e^{-x}$.

Then $y'(x) = -C_2 e^{-x} + C_3 e^{-x} - C_3 x e^{-x}$

$y''(x) = C_2 e^{-x} - 2C_3 e^{-x} + C_3 x e^{-x}$.

Initial conditions:

$y(0) = 2$: $\quad C_1 + C_2 = 2$

$y'(0) = -1$: $\quad -C_2 + C_3 = -1$

$y''(0) = 0$: $\quad C_2 - 2C_3 = 0$

$R_3 \Rightarrow$ $C_2 = 2C_3$

R_2: $-2C_3 + C_3 = -1$, $C_3 = 1$, $C_2 = 2$.

R_1: $C_1 = 2 - C_2 = 0$.

Answer: $y(x) = 2e^{-x} + xe^{-x}$

MAT 303 SPRING 2015 PRACTICE MIDTERM II (1)

1.

Suppose that a motorboat is moving at $4m/s$ when its motor suddenly quits, and that 10 seconds later the velocity of the boat is $2m/s$. Assume that the resistance motorboat encounters is proportional to its velocity. Find the velocity of the boat in 30 seconds after the motor has quit.

2.

Consider the differential equation

$$\frac{dy}{dx} = (x-2)y^2, \quad y(2) = 1.$$

Using Euler's method with step size $h = 0.2$ find approximate value of $y(2.4)$.

3.

Verify that the functions $y_1 = x^2, y_2 = \frac{1}{x}$ are solutions of the differential equation

$$x^2 y'' - 2y = 0.$$

Solve the initial value problem

$$y(1) = 5, y'(1) = -3.$$

4.

Using Wronskian show that the functions $y_1 = x^2, y_2 = \sin x, y_3 = \cos x$ are linearly independent on $\mathbf{R}$.

5.

Solve the initial value problem

$$y^{(3)} - 3y'' + 3y' - y = 0, \quad y(0) = 1, y'(0) = 1, y''(0) = 2.$$

MAT 303 SPRING 2015 PRACTICE MIDTERM II (2)

1.

Suppose that a motorboat is moving at $5m/s$ when its motor suddenly quits, and that 10 seconds later the velocity of the boat is $1m/s$. Assume that the resistance motorboat encounters is proportional to the cube of its velocity. Find the velocity of the boat 20 seconds after the motor has quit.

2.

Find all critical points of the autonomous differential equation

$$\frac{dx}{dt} = (e^x - 1)^2(x^2 - 4).$$

Determine their types (stable, unstable, or semistable). Draw the phase diagram.

3.

Verify that the functions $y_1 = x, y_2 = x \ln |x|$ are solutions of the differential equation

$$x^2 y'' - xy' + y = 0.$$

Solve the initial value problem

$$y(-e) = e, y'(-e) = -3.$$

4.

Find the general solution of the equation $y''' - 8y = 0$.

5.

Show directly (without using Wronskian) that the functions x^3+2x^2, x^2-1 and x^3-5x^2-2 are linearly independent on $\mathbb{R}$.

MAT 303 FALL 2014 PRACTICE MIDTERM II (1)

1. Suppose that a motorboat is moving at $4m/s$ when its motor suddenly quits, and that 10 seconds later the velocity of the boat is $2m/s$. Assume that the resistance motorboat encounters is proportional to its velocity. Find the velocity of the boat in 30 seconds after the motor has quit.

Solution. From the second Newton's law we get

$$\frac{dv}{dt} = a = F/m = kv,$$

where k is some constant. Solution of this differential equation is $v(t) = v_0 e^{kt}$. We have $v_0 = 4$ and $v(10) = 2$. Thus

$$e^{10k} = 2/4 = 1/2, \ k = -\frac{\ln 2}{10}.$$

We obtain:

$$v(30) = 4e^{30k} = 4e^{-3\ln 2} = 4/2^3 = 0.5.$$

Answer: 0.5 m/s.

2. Consider the differential equation

$$\tfrac{dy}{dx} = (x-2)y^2, \ \ y(2) = 1.$$

Using Euler's method with step size $h = 0.2$ find approximate value of $y(2.4)$.

Solution. We have: $x_0 = 2$, $y_0 = 1$, $x_n = x_0 + nh = 2 + 0.2n$, $2.4 = x_2$. The approximations y_n of $y(x_n)$ are defined by:

$$y_{n+1} = y_n + h(x_n - 2)y_n^2 = y_n + 0.04ny_n^2.$$

Thus, $y_1 = y_0 + 0 = 1, \ y_2 = y_1 + 0.04y_1^2 = 1 + 0.04 = 1.04.$

Answer: $y(2.4) \approx 1.04$.

3. Verify that the functions $y_1 = x^2, y_2 = \frac{1}{x}$ are solutions of the differential equation

$$x^2 y'' - 2y = 0.$$

Solve the initial value problem

$$y(1) = 5, y'(1) = -3.$$

Solution. We have: $y_1'' = 2,\ x^2 y_1'' - 2y_1 = 2x^2 - 2x^2 = 0,\ y_2'' = 2x^{-3},\ x^2 y_2'' - 2y_2 = 2x^{-1} - 2x^{-1} = 0$. Thus, x^2 and x^{-1} are solutions. By Principle of Superposition, for any c_1, c_2, $y(x) = c_1 x^2 + c_2 x^{-1}$ is a solution. We have: $y'(x) = 2c_1 x - c_2 x^{-2}$. Substituting $x = 1$ we get:

$$c_1 + c_2 = 5,\ 2c_1 - c_2 = -3,\ 3c_1 = 5 - 3 = 2,\ c_1 = \tfrac{2}{3},\ c_2 = 5 - \tfrac{2}{3} = 4\tfrac{1}{3}.$$

Answer: $y(x) = \frac{2}{3}x^2 + 4\frac{1}{3}x^{-1}$.

4. Using Wronskian show that the functions $y_1 = x^2, y_2 = \sin x, y_3 = \cos x$ are linearly independent on $\mathbf{R}$.

Solution:

$$W(x) = det \begin{bmatrix} x^2 & \sin x & \cos x \\ 2x & \cos x & -\sin x \\ 2 & -\sin x & -\cos x \end{bmatrix}.$$

To show that the function are linearly independent it is sufficient to find one point a such that $W(a) \neq 0$. Take point 0.

$$W(0) = det \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}.$$

Expanding by the first row we get

$$W(0) = 1 \cdot det \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = -2.$$

Therefore, the functions are linearly independent.

5. Solve the initial value problem

$$y^{(3)} - 3y'' + 3y' - y = 0, \quad y(0) = 1, y'(0) = 1, y''(0) = 2.$$

Solution. First, find the general solution. The characteristic equation is $r^3 - 3r^2 + 3r - 1 = 0$. Equivalently, $(r-1)^3 = 0$. Thus, $r_1 = 1$ is a root repeated 3 times. Therefore, the general solution is $y(x) = (c_1 + c_2 x + c_3 x^2)e^x$. We have:

$$y'(x) = (c_1 + c_2 + (c_2 + 2c_3)x + c_3 x^2)e^x,$$
$$y''(x) = (c_1 + 2c_2 + 2c_3 + (c_2 + 4c_3)x + c_3 x^2)e^x.$$

Substituting the initial condition, we obtain a system:

$$c_1 = 1, \ c_1 + c_2 = 1, \ c_1 + 2c_2 + 2c_3 = 2.$$

We get: $c_2 = 0, c_3 = \frac{1}{2}$.

Answer: $y(x) = (1 + \frac{x^2}{2})e^x$.

MAT 303 FALL 2012 MIDTERM II

1. Suppose that a motorboat is moving at $5m/s$ when its motor suddenly quits, and that 10 seconds later the velocity of the boat is $1m/s$. Assume that the resistance motorboat encounters is proportional to the cube of its velocity. Find the velocity of the boat in 20 seconds after the motor has quit.

Solution. Let $v(t)$ be the speed of the motorboat. From the conditions of the problem we get: $v(0) = 5, v(10) = 1$. The motorboat movement is influenced only by the water resistance, which is of the form $F = kv^3$ for some constant k. By the Newton's law, we get:

$$m\frac{dv}{dt} = ma = F = kv^3.$$

Thus, $\frac{dv}{dt} = cv^3$ for some constant $c = \frac{k}{m}$. Solving this separable equation, we get:

$$\frac{dv}{v^3} = cdt, \quad -\frac{1}{2v^2} = ct + b$$

for some constant b. Plugging $t = 0$ and $t = 10$, we get:

$$-\frac{1}{50} = b, \; -\frac{1}{2} = 10c + b, \; c = \frac{-1-2b}{20} = -\frac{24}{20 \cdot 25} = -\frac{6}{125}.$$

It follows that

$$v(t) = \frac{1}{\sqrt{-2ct-2b}} = \frac{1}{\sqrt{\frac{12}{125}t + \frac{1}{25}}}.$$

In particular,

$$v(20) = \frac{1}{\sqrt{\frac{12}{125} \cdot 20 + \frac{1}{25}}} = \frac{1}{\sqrt{\frac{49}{25}}} = \frac{5}{7}.$$

Answer: $\frac{5}{7}m/s$.

2. The critical points are solutions of $f(x) = 0$. $(e^x - 1)^2(x^2 - 4) = 0$ when $x = 0$ or ± 2. When $x < -2$ $x' = (e^x - 1)^2(x^2 - 4) > 0$, x increases; when $-2 < x < 0$ $x' < 0$, x decreases; when $0 < x < 2$ $x' < 0$, x decreases; when $x > 2$ $x' > 0$, x increases. From the phase

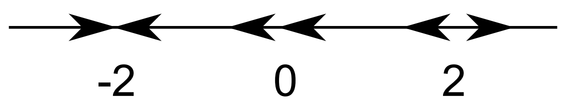


diagram we see that -2 is a stable critical point, 0 and 2 are unstable (0 is semistable).

3. We have:

$$x^2 y_1'' - x y_1' + y_1 = -x + x = 0, \;\; x^2 y_2'' - x y_2 + y_2 = \\ x^2\left(\tfrac{1}{x}\right) - x(\ln|x| + 1) + x\ln|x| = 0,$$

thus, x and $x \ln|x|$ are solutions. By the Principle of Superposition, $y(x) = c_1 x + c_2 x \ln|x|$ is a solution for any c_1, c_2. We have: $y'(x) = c_1 + c_2(\ln|x| + 1)$. Substituting the initial condition, we get:

$$-c_1 e + c_2 e = e, \; c_1 + 2c_2 = -3 \;\Rightarrow\; c_1 - c_2 = -1, \; 3c_2 = -2, c_2 = -\tfrac{2}{3}, c_1 = -\tfrac{5}{3}.$$

Answer: $y(x) = -\frac{5}{3}x - \frac{2}{3}x \ln|x|$.

4. The characteristic equation is: $r^3 - 8 = 0$. By the formula of the difference of cubes, it can be factored as: $(r - 2)(r^2 + 2r + 4) = 0$. Thus,

$$r_1 = 2, r_{2,3} = \frac{-2 \pm \sqrt{-12}}{2} = -1 \pm \sqrt{3}i.$$

The corresponding particular solutions are:

$$y_1 = e^{2x}, y_2 = e^{-x} \cos \sqrt{3}x, y_3 = e^{-x} \sin \sqrt{3}x.$$

Answer: $y(x) = c_1 e^{2x} + e^{-x}(c_2 \cos \sqrt{3}x + c_3 \sin \sqrt{3}x)$.

5. Assume that these functions are linearly dependent. Then there exists constants c_1, c_2, c_3 (not all equal to zero) such that

$$c_1(x^3 + 2x^2) + c_2(x^2 - 1) + c_3(x^3 - 5x^2 - 2) = 0$$

for all x. Gather coefficients in front of each power of x together. We get:

$$(c_1 + c_3)x^3 + (2c_1 + c_2 - 5c_3)x^2 + (-c_2 - 2c_3) = 0.$$

A polynomial is equal to zero everywhere only if all of its coefficients are zeros. Thus,

$$c_1 + c_3 = 0,\ 2c_1 + c_2 - 5c_3 = 0,\ -c_2 - 2c_3 = 0.$$

From the first and the last equations we obtain: $c_2 = -2c_3, c_1 = -c_3$. Substituting these formulas into the second equation we get: $-9c_3 = 0, c_3 = 0$, and so $c_1 = c_2 = 0$ as well. Thus, all the coefficients are zeros. There are no c_1, c_2, c_3 not all equal to zero satisfying the conditions, therefore, the functions are linearly independent.

MAT 303 Final exam review on systems of differential equations.

Solving a system by elimination

$x' = F(x, y, t)$ E1

$y' = G(x, y, t)$ E2

From E1 write y in terms of x, x', t, substitute in E2, solve for x, then find y from E1.

Example 1 $x' = y - tx$

$y' = -(t^2 + 2t)x + (t+2)y$

Solution E1 $\Rightarrow$ $y = x' + tx$ (*)

$y' = x'' + x + tx'$. Substitute in E2:

$x'' + x + tx' = -(t^2 + 2t)x + (t+2)(x' + tx)$.

Simplifying, we get:

$x'' - 2x' + x = 0$. $r^2 - 2r + 1 = 0$, $r_{1,2} = 1 \Rightarrow$

$x = e^t(c_1 t + c_2)$.

From (*): $y = x' + tx = e^t(c_1 t^2 + (c_1 + c_2)t + (c_1 + c$

Answer $x = e^t(c_1 t + c_2)$, $y = e^t(c_1 t^2 + (c_1 + c_2)t + c_1 + c$

Linear dependence of vector functions (2

$X_1, X_2, \ldots X_k$ are called linearly dependent on I if $\exists\, c_1, c_2, \ldots c_k$ not all equal to 0:

$c_1X_1(t)+c_2X_2(t)+\ldots+c_kX_k(t)=0$ on I. (*)

__Example__ 2. Show that $\begin{pmatrix} t+1 \\ \sin t \\ e^t-2 \end{pmatrix}, \begin{pmatrix} 2 \\ \cos t \\ e^t \end{pmatrix}, \begin{pmatrix} t-1 \\ \sin t-\cos t \\ -2 \end{pmatrix}$

are linearly dependent on $\mathbb{R}$

__Solution__ $X_3 = X_1 - X_2$, equivalently,

$1\cdot X_1 - 1\cdot X_2 - 1\cdot X_3 = 0 \Rightarrow \exists\, c_1=1, c_2=-1, c_3=-1$ satisfying the condition (*) $\Rightarrow$ linearly dependent.

__Example__ 3. Show that $\begin{pmatrix} 1 \\ t^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2t \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ -3 \\ 5 \end{pmatrix}$

are linearly independent on $[-1,1]$.

__Solution__ Assume that they are lin. dep.
Then $\exists\, c_1, c_2, c_3$: $c_1\begin{pmatrix} 1 \\ t^2 \\ 0 \end{pmatrix}+c_2\begin{pmatrix} 2 \\ 2t \\ 1 \end{pmatrix}+c_3\begin{pmatrix} 5 \\ -3 \\ 5 \end{pmatrix}=$

$=\begin{pmatrix} c_1+2c_2+5c_3 \\ c_1t^2+2c_2t-3c_3 \\ c_2+5c_3 \end{pmatrix}=0$ on $[-1,1]$. In particular,

$c_1t^2+2c_2t-3c_3=0$ a polynomial is identically zero on a segment $\Rightarrow$ all coefficients are zero

$\Rightarrow \quad c_1 = 2c_2 = -3c_3 = 0 \quad \Rightarrow c_1 = c_2 = c_3 = 0.$ (3

Thus, $\nexists$ c_1, c_2, c_3 not all of them equal to 0 such that (*) is true. Therefor, the vector functions are linearly independent on $[-1, 1]$

Wronskian of vector functions

$$X_1 = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix} \quad \ldots \quad X_n = \begin{pmatrix} x_{1n}(t) \\ x_{2n}(t) \\ \vdots \\ x_{nn}(t) \end{pmatrix}. \text{ Then}$$

$$W(t) = \left| X_1 \ X_2 \ \ldots X_n \right| = \begin{vmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \vdots & & & \vdots \\ x_{n1} & x_{n2} & \ldots & x_{nn} \end{vmatrix}$$

If $X_1, \ldots X_n$ are linearly dependent on I then $W(t) = 0$ on I.

Example 4. Show that $\begin{pmatrix} 1 \\ t^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2t \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ -3 \\ 5 \end{pmatrix}$ are lin. indep. on $[-1, 1]$ using Wronskean

Solution $W(t) = \begin{vmatrix} 1 & 2 & 5 \\ t^2 & 2t & -3 \\ 0 & 1 & 5 \end{vmatrix}$

$$W(0) = \begin{vmatrix} 1 & 2 & 5 \\ 0 & 0 & -3 \\ 0 & 1 & 5 \end{vmatrix} = -(-3) \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 3 \neq 0 \Rightarrow$$

linearly independent.

General solutions of homogeneous systems

Thm If $X_1, X_2, \dots X_n$ are n solutions of $X' = P(t)X$ linearly independent on I, where $\dim(X) = n$, then the general solution is

$$X(t) = c_1 X_1 + c_2 X_2 + \dots + c_n X_n$$

Solving Initial Value problem

Thm If $P(t)$ is a contin. on I $n\times n$ matrix function and $F(t)$ is a contin. on I dimension n vector function then $\forall a \in I$ and $\forall b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

the IVP
$$X' = P(t)X + F(t)$$
$$X(0) = b$$

has a unique solution.

Example 5 Solve
$$x' = y - tx$$
$$y' = -(t^2+2t)x + (t+2)y$$

$x(0) = 0,\ y(0) = 4.$

Solution $x = e^t(c_1 t + c_2),\ y = e^t(c_1 t^2 + (c_1+c_2)t + (c_1+c_2))$
(see example 1)

$0 = x(0) = c_2,\ c_2 = 0,\ y(0) = c_1 + c_2 \Rightarrow c_1 = 4.$

Answer $x(t) = 4te^t,\ y(t) = e^t(4t^2 + 4t + 4)$

Systems of homogeneous equations with constant coefficients

$$(1) \quad X' = AX, \qquad A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad a_{ij} \in \mathbb{R}.$$

Eigenvalue method

λ is an eigenvalue if $|A - \lambda I| = 0$.

V is an eigenvector corresponding to λ if

$AV = \lambda V$ (equivalently, $(A - \lambda I)V = 0$).

Then $X(t) = e^{\lambda t} V$ is a solution of (1)

To solve (1) find all eigenvalues and corresponding eigenvectors (or generalized eigenvectors), construct n linearly independent solutions $X_1, X_2, \dots X_n$. Then the general solution is

$X(t) = C_1 X_1 + C_2 X_2 + \dots + C_n X_n$.

Case 1 Distinct real eigenvalues.

If $p(\lambda) = |A - \lambda I| = 0$ has n distinct real roots $\lambda_1, \lambda_2, \dots \lambda_n$, and $V_1, V_2, \dots V_n$ are the corresponding eigenvectors then

$X(t) = C_1 e^{\lambda_1 t} V_1 + \dots + C_n e^{\lambda_n t} V_n$.

Example 6 (Distinct real roots)

$$x_1' = 3x_1 + x_2 + x_3$$
$$x_2' = -5x_1 - 3x_2 - x_3$$
$$x_3' = 5x_1 + 5x_2 + 3x_3$$

Solution

$$X' = \begin{bmatrix} 3 & 1 & 1 \\ -5 & -3 & -1 \\ 5 & 5 & 3 \end{bmatrix} X$$

$$p(\lambda) = \begin{vmatrix} 3-\lambda & 1 & 1 \\ -5 & -3-\lambda & -1 \\ 5 & 5 & 3-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} -3-\lambda & -1 \\ 5 & 3-\lambda \end{vmatrix} - \begin{vmatrix} -5 & -1 \\ 5 & 3-\lambda \end{vmatrix} + \begin{vmatrix} -5 & -3- \\ 5 & 5 \end{vmatrix}$$

$$= (3-\lambda)(\lambda^2-4) - (5\lambda-10) + (5\lambda-10) = (\lambda-2)(3-\lambda)(\lambda+2).$$

$$\lambda_1 = 2, \quad \lambda_2 = -2, \quad \lambda_3 = -3$$

1) $\lambda_1 = 2$

$$\begin{bmatrix} 1 & 1 & 1 \\ -5 & -5 & -1 \\ 5 & 5 & 1 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$$

$v_1 + v_2 + v_3 = 0$

$5v_1 + 5v_2 + v_3 = 0$

$\Downarrow$

$4(v_1+v_2) = 0 \Rightarrow v_1 = -v_2,$

$v_3 = 0.$

$$V_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad X_1(t) = e^{2t}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

2) $\lambda_2 = -2$

$$\begin{bmatrix} 5 & 1 & 1 \\ -5 & -1 & -1 \\ 5 & 5 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$5v_1 + v_2 + v_3 = 0$

$5v_1 + 5v_2 + 5v_3 = 0$

$4(v_2 + v_3) = 0 \Rightarrow$

$v_2 = -v_3, \quad v_1 = 0$

$$V_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad X_2(t) = e^{-2t}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

3) $\lambda_3 = 3$

$$\begin{bmatrix} 0 & 1 & 1 \\ -5 & -6 & -1 \\ 5 & 5 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$v_2 + v_3 = 0$

$5(v_1 + v_2) = 0 \Rightarrow$

$v_2 = -v_1, \quad v_3 = -v_2 = v_1$

$$V_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad X_3(t) = e^{3t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

The general solution is

$$X(t) = c_1 e^{2t}\begin{bmatrix}1\\-1\\0\end{bmatrix} + c_2 e^{-2t}\begin{bmatrix}0\\1\\-1\end{bmatrix} + c_3 e^{3t}\begin{bmatrix}1\\-1\\1\end{bmatrix} =$$

$$\begin{bmatrix} c_1 e^{2t} + c_3 e^{3t} \\ -c_1 e^{2t} + c_2 e^{-2t} - c_3 e^{3t} \\ -c_2 e^{-2t} + c_3 e^{3t} \end{bmatrix}$$

Answer: $x_1(t) = c_1 e^{2t} + c_3 e^{3t}$, $x_2(t) = -c_1 e^{2t} + c_2 e^{-2t} - c_3 e^{3t}$, $x_3(t) = -c_2 e^{-2t} + c_3 e^{3t}$

~~Example~~ Case 2 Complex eigenvalue

$\lambda = p + qi$. Eigenvector $V = a + bi$. Then $\bar{\lambda} = p - qi$ is also an eigenvalue with an eigenvector $\bar{V} = a - bi$. From complex solutions $e^{\lambda t}V$, $e^{\bar{\lambda} t}\bar{V}$ one can construct 2 real solutions

$$X_1(t) = e^{pt}(a\cos qt - b\sin qt)$$

$$X_2(t) = e^{pt}(b\cos qt + a\sin qt).$$

Notice that $X_1 = \operatorname{Re} e^{\lambda t}V$, $X_2 = \operatorname{Im} e^{\lambda t}V$.

Example 7 Solve $X' = \begin{bmatrix}1 & -2\\2 & 1\end{bmatrix}X$.

Solution $p(\lambda) = \begin{vmatrix}1-\lambda & -2\\2 & 1-\lambda\end{vmatrix} = \lambda^2 - 2\lambda + 5$

$$\lambda_{1,2} = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$$

To construct solutions you can take either λ_1 or λ_2 (
(the general solution will be the same)

$\lambda_1 = 1+2i$.

Eigenvector: $\begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \iff$

$2v_1 - 2iv_2 = 0 \iff v_1 = iv_2$

$$V = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$e^{\lambda t} V = e^{(1+2i)t}\begin{bmatrix} i \\ 1 \end{bmatrix} = e^t(\cos 2t + i\sin 2t)\begin{bmatrix} i \\ 1 \end{bmatrix} =$$

$$\begin{bmatrix} ie^t\cos 2t - e^t\sin 2t \\ e^t\cos 2t + ie^t\sin 2t \end{bmatrix} = e^t\begin{bmatrix} -\sin 2t \\ \cos 2t \end{bmatrix} + ie^t\begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix}$$

Thus, $X_1 = e^t\begin{bmatrix} -\sin 2t \\ \cos 2t \end{bmatrix}$, $X_2 = e^t\begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix}$

Answer $X(t) = c_1 e^t\begin{bmatrix} -\sin 2t \\ \cos 2t \end{bmatrix} + c_2 e^t\begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix}$

Case 3 Repeated eigenvalues.

If # linearly independent eigenvectors = multiplicity of the eigenvalue, then
$$X(t) = c_1 e^{\lambda_1 t}V_1 + c_2 e^{\lambda_2 t}V_2 + \dots + c_n e^{\lambda_n t}V_n.$$

If # linearly independent eigenvectors < multiplicity of the eigenvalue, then need to use generalized eigenvectors.

Definition A rank r generalized eigenvector associated to the eigenvalue λ is a vector V_r:
$(A-\lambda I)^r V_r = 0$, but $(A-\lambda I)^{r-1} V_r \neq 0$.

A length k chain of generalized eigenvectors $V_1, V_2, \dots V_k$ such that

$$(A-\lambda I) V_1 = 0$$
$$(A-\lambda I) V_2 = V_1$$
$$(A-\lambda I) V_3 = V_2$$
$$\vdots$$
$$(A-\lambda I) V_k = V_{k-1}.$$

- To construct a length k chain of generalized eigenvectors: find a rank k generalized eigenvector V_k: $(A-\lambda I)^k V_k = 0$, but $(A-\lambda I)^{k-1} V_k \neq$
set $V_{k-1} = (A-\lambda I) V_k$, $V_{k-2} = (A-\lambda I) V_{k-1}$, ...
$V_1 = (A-\lambda I) V_2$.
- If $k=2$ instead you can find first an eigenvector V_1 and solve $(A-\lambda I) V_2 = V_1$.

Solutions associated to generalized eigenvectors:

$$X_1(t) = V_1 e^{\lambda t}$$
$$X_2(t) = (V_1 t + V_2) e^{\lambda t}$$
$$X_3(t) = (\tfrac{1}{2} V_1 t^2 + V_2 t + V_3) e^{\lambda t}$$
$$\vdots$$
$$X_k(t) = \left(\frac{V_1 t^{k-1}}{(k-1)!} + \frac{V_2 t^{k-2}}{(k-2)!} + \dots + V_{k-1} t + V_k\right) e^{\lambda t}$$

Example 8

$$X' = \begin{bmatrix} -3 & 0 & -4 \\ -1 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix} X$$

Solution $\quad p(\lambda) = \begin{vmatrix} -3-\lambda & 0 & -4 \\ -1 & -1-\lambda & -1 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (-3-\lambda)(\lambda^2-1) - 4(\lambda+1) =$

$(\lambda+1)(-\lambda^2-2\lambda-1) = -(\lambda+1)^3 \Rightarrow \lambda = -1$ of mult. 3

Eigenvectors: $\qquad (A-\lambda I)V = 0$

$$\begin{bmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \qquad V = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

only one linearly independent eigenvector $\Rightarrow$ need to construct a chain of 3 generalized eigenvectors.

$$(A-\lambda I)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad (A-\lambda I)^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Need V_3: $\qquad (A-\lambda I)^3 V_3 = 0$, but $(A-\lambda I)^2 V_3 \neq 0$.

$V_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ works. Then $V_2 = (A-\lambda I)V_3 = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$

$V_1 = (A-\lambda I)V_2 = (A-\lambda I)^2 V_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

$X_1(t) = e^{-t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $X_2(t) = e^{-t}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right)$

$X_3(t) = e^{-t}\left(\frac{t^2}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$

The general solution is

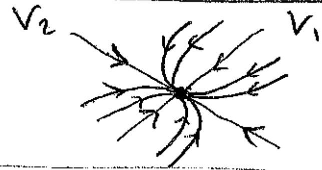
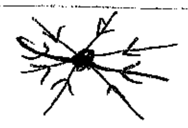
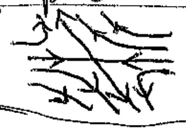
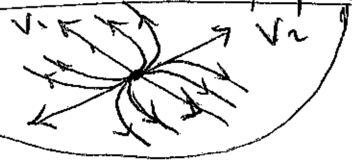
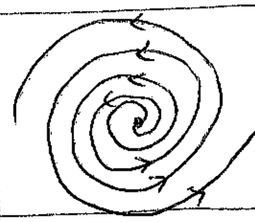
$$X(t) = c_1 X_1 + c_2 X_2 + c_3 X_3 =$$

$$c_1 e^{-t}\begin{bmatrix}0\\1\\0\end{bmatrix} + c_2 e^{-t}\left(\begin{bmatrix}0\\1\\0\end{bmatrix}t + \begin{bmatrix}-2\\-1\\1\end{bmatrix}\right) +$$

$$c_3 e^{-t}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\frac{t^2}{2} + \begin{bmatrix}-2\\-1\\1\end{bmatrix}t + \begin{bmatrix}0\\1\\0\end{bmatrix}\right)$$

<u>Answer</u> $$X(t) = e^{-t}\begin{bmatrix}-2c_2 + c_3\left(\frac{t^2}{2} - 2t\right)\\ c_1 + c_2(t-1) - c_3 t\\ c_2 + c_3 \cdot t\end{bmatrix}$$

<u>Critical points and phase portraits</u>

Eigenvalues	Type	phase portrait
$\lambda_1 < \lambda_2 < 0$	stable improper node	
$\lambda_1 = \lambda_2 < 0$	stable node (proper or improper)	 
$\lambda_1 < 0 < \lambda_2$	saddle point	
$\lambda_1 = \lambda_2 > 0$	Unstable node (proper or improper)	 
$\lambda_1 > \lambda_2 > 0$	Unstable improper node	
$\lambda_{1,2} = a \pm bi, (a<0)$	stable spiral point	
$\lambda_{1,2} = a \pm bi, (a>0)$	unstable spiral point	
$\lambda_{1,2} = \pm bi$	stable center	

Example 9 $X' = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} X$ $\boxed{\text{Drawing phase portraits}}$ (12

Solution $p(\lambda) = \begin{vmatrix} -1-\lambda & 1 \\ -2 & 1-\lambda \end{vmatrix} = \lambda^2 + 1,$

$\lambda_{1,2} = \pm i \Rightarrow$ stable center

$\lambda_1 = i$: $\begin{bmatrix} -1-i & 1 \\ -2 & 1-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$ $v_2 = (i+1) v_1$

$V = \begin{bmatrix} 1 \\ i+1 \end{bmatrix}$

$X(t) = e^{\lambda_1 t} V = (\cos t + i \sin t) \begin{bmatrix} 1 \\ 1+i \end{bmatrix} =$

$\begin{bmatrix} \cos t + i \sin t \\ \cos t - \sin t + i(\cos t + \sin t) \end{bmatrix}$

$X_1(t) = \begin{bmatrix} \cos t \\ \cos t - \sin t \end{bmatrix}$, $X_2(t) = \begin{bmatrix} \sin t \\ \cos t + \sin t \end{bmatrix}$

$X(t) = c_1 \begin{bmatrix} \cos t \\ \cos t - \sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ \cos t + \sin t \end{bmatrix}$

Trajectories are concentric ellipses.

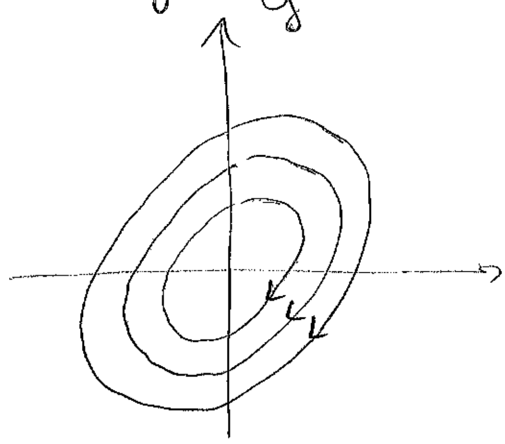
Consider $X_1(t) = \begin{bmatrix} \cos t \\ \cos t - \sin t \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$.

we have: $x^2 + (x-y)^2 = 1$

To see the direction of rotation take a sample point: $t=0$,

$X_1(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $X_1'(0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\Rightarrow$ clockwise



phase portrait

__Example 10__ $X' = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

__Solution__ We solved the equation in Example 7:

$\lambda = 1 \pm 2i \Rightarrow$ unstable spiral point

$$X(t) = c_1 e^t \begin{bmatrix} -\sin 2t \\ \cos 2t \end{bmatrix} + c_2 e^t \begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix}.$$

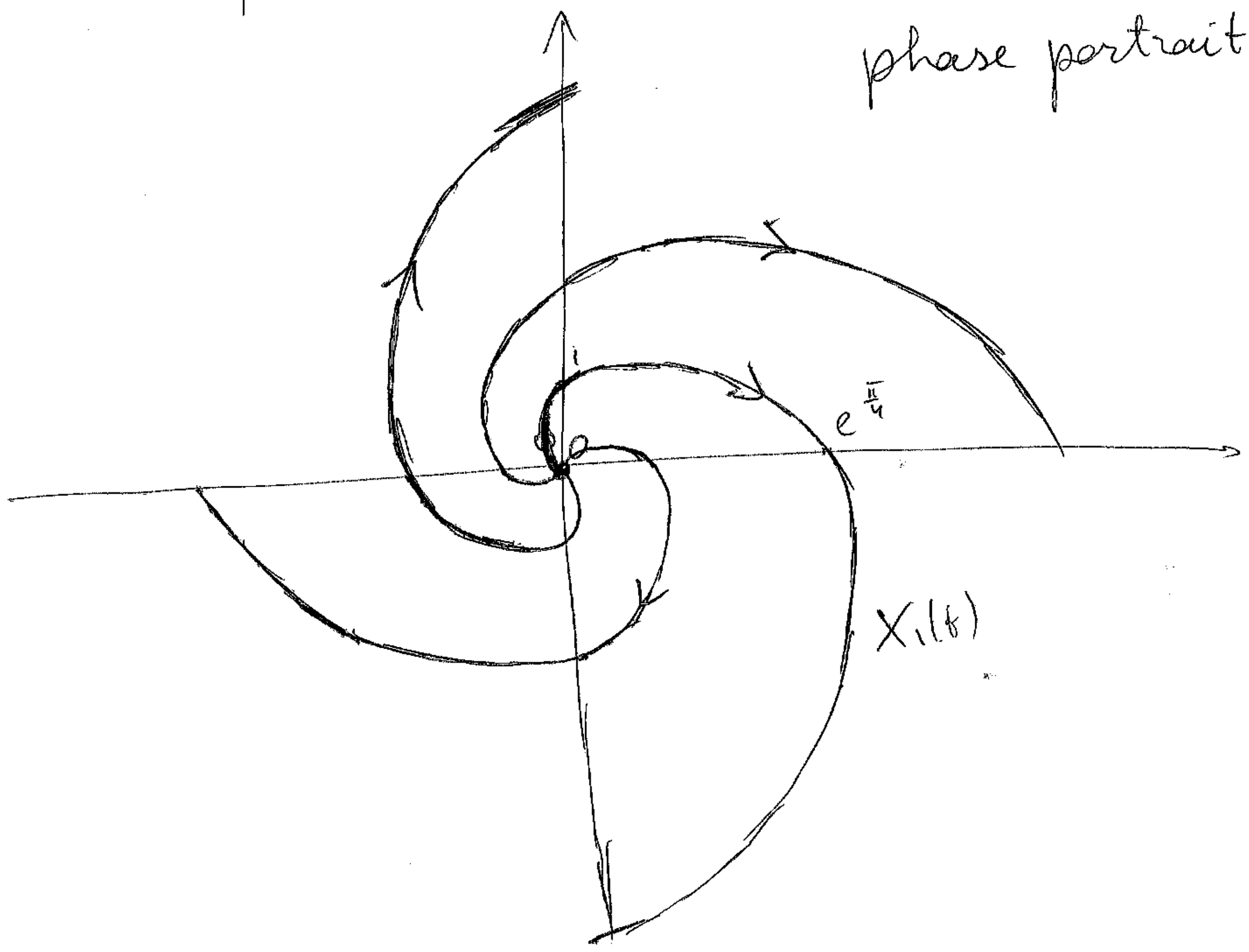
Trajectories are spirals. Consider

$$X_1(t) = e^t \begin{bmatrix} -\sin 2t \\ \cos 2t \end{bmatrix}$$

$$X_1(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad X_1\left(\tfrac{\pi}{4}\right) = e^{\frac{\pi}{4}} \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad X_1\left(\tfrac{\pi}{2}\right) = e^{\frac{\pi}{2}} \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

We see that the trajectory spirals clockwise escaping from the origin fast (by factor e^{π} when spiral goes around once).

phase portrait



Example 11 $X' = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$

Solution $p(\lambda) = \begin{vmatrix} 1-\lambda & 3 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 - 4$

$\lambda_{1,2} = \pm 2 \Rightarrow$ saddle point

1) $\lambda_1 = 2$: $\begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \qquad V_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

2) $\lambda_2 = -2$ $\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \qquad V_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$X(t) = c_1 e^{2t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$