MAT 303: Calculus IV with Applications
Spring 2014

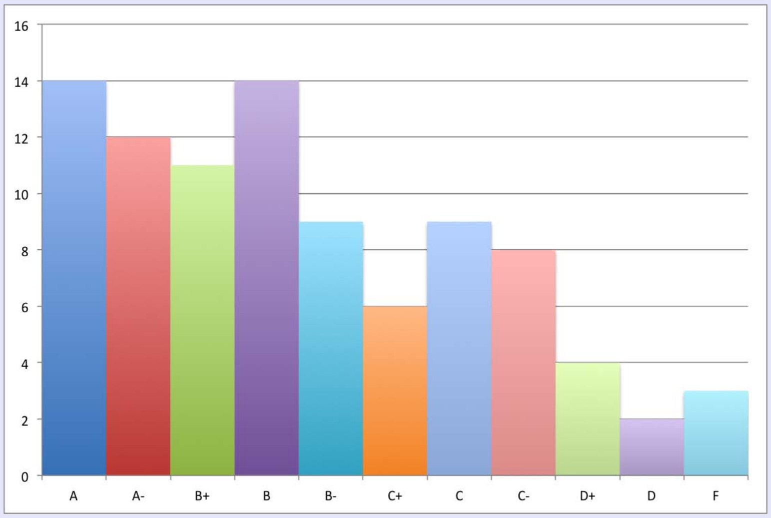
Final Exam: Monday, May 19, 8:00am-10:45am.

There will be 10 big problems with no more than 15 subproblems. The distribution of problems are (30% means 3 out of 10 big problems)
30%: 1st order equation
20%: higher order linear equation
30%: Linear system
20%: Series
2 or 3 of these problems will be related to models and will be similar to the problems in the homework/practice. So please review the models mentioned in the Review Sheet .

Check the Syllabus for Review Sheet , Old Exam and Old Practice . Please use the review sheet to guide you go over the materials we studied, and review your homework, midterms, quizzes. I will give solutions to the calculation problems contained in the review sheet.

The format of our exam will be like the Old Exam but with the above distribution of problems and with only the materials in the review sheet. You should also try to do the old practice exam and more problems from the textbook if you have energy.

Overall Grades:



Syllabus and homework

Differential equations are important in different branches of sciences and engineering. It's used to model the evolution process and dynamical systems.

Click here for the content and prerequisite for the course.

Textbook

Differential Equations and boundary value problems, computing and modeling, 4th edition, by C. Henry Edwards & David E. Penney

Schedule and location

Instructors

	Name	Office	Office hour	MLC hour	Email
Lecture	Dr. Chi Li	3-120	T/Th 1-2:30pm		chi.li@stonybrook.edu
R01&R02	Anant Atyam	2-105	Friday 1-2 pm	W/F 10:30-11:30 am	anant@math.sunysb.edu
R03	Chengjian Yao	S-240C	Monday 3-4 pm	Monday 4-6 pm	yao@math.sunysb.edu

Homework

Problem sets will be assigned weekly; check the syllabus webpage for the assignments. For the homework using Mathematica, you are encouraged to submit your work although it won't be graded. Finally, the lowest two homework grades will be dropped in the calculation of your overall grades.

Each homework is **due during your recitation class of the following week** (unless otherwises stipulated). **No late homework.** The recitation instructor will collect the homework and grade three of the problems.

Write the problem up carefully in your own words even if you have consulted the book for the final answer: always show your work. It is OK to discuss homework problems with other students. However, each student must write up the homework individually in his/her own words, rather than merely copying someone else's.

Quizzes

There are 5 quizzes. Each taking place in the last 15 minutes in Friday class. Usually there will be two problems. No make up quiz. Check the syllabus webpage for the schedule.

Grading

10% Homework
10% Quizzes
20% Midterm 1
20% Midterm 2
40% Final Exam

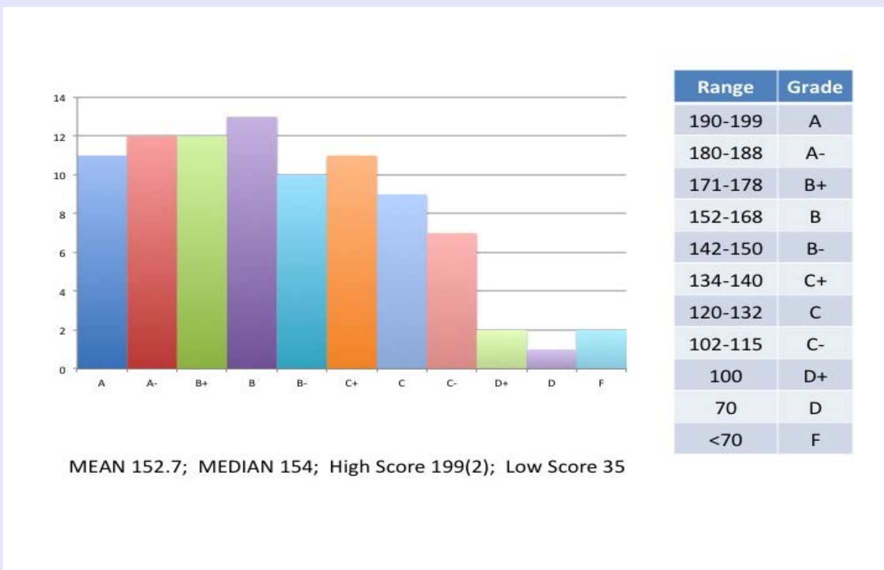
Computing Software: Mathematica

Mathematica is a powerful scientific computational and symbolical software. We will occasionally use the Mathematica to numerically solve the differential equation and visualize the solution. You can get a version from Stonybrook Softweb . Note that you can also use this software in many public computers through the campus.

Midterm exam

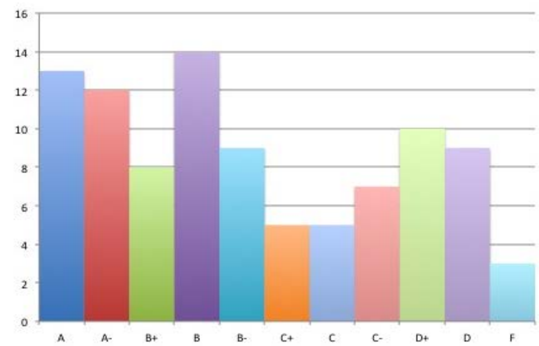
There are two midterms which are given in class.

Midterm 2



Midterm 1

The tentative curve for this midterm is shown in the picture. This is just to give you some idea of the distribution of the grades. The real curve will be made only after the final exam.



MEAN 140.5; MEDIAN 149; High Score 200(6); Low Score 10

Range	Grade
192-200	A
180-185	A-
172-178	B+
150-165	B
137-149	B-
128-131	C+
120-125	C
101-106	C-
84-97	D+
62-78	D
<62	F

Read and Take Notes

Read the relevant materials on the textbook both before and after the lecture. If you really want to master the course, it is wise to attempt or solve as many problems as you can in the relevant section of the book.

Help

A very useful resource is the **Math Learning Center (MLC)** located in room S240-A of the mathematics building basement. The Math Learning Center is open every day and most evenings. Check the schedule on the door.

Another useful resource are your teachers, whose office hours are listed above.

Stony Brook University expects students to maintain standards of personal integrity that are in harmony with the educational goals of the institution; to observe national, state, and local laws as well as University regulations; and to respect the rights, privileges, and property of other people. Faculty must notify the Office of Judicial Affairs of any disruptive behavior that interferes with their ability to teach, compromises the safety of the learning environment, or inhibits students' ability to learn.

DSS advisory: If you have a physical, psychiatric, medical, or learning disability that may affect your ability to carry out the assigned course work, please contact the office of Disabled Student Services (DSS), Humanities Building, room 133, telephone 632-6748/TDD. DSS will review your concerns and determine what accommodations may be necessary and appropriate. All information regarding any disability will be treated as strictly confidential.

Students who might require special evacuation procedures in the event of an emergency are urged to discuss their needs with both the instructor and DSS. For important related information, click here.

Final Review Sheets

1 Basic concepts

1. n-th order DE, existence and uniqueness, direction fields, general solutions, particular solutions, initial value problem, autonomous equation, equilibrium solution, stability of equilibrium solution
2. linear DE, homogeneous linear DE, non-homogeneous linear DE, basic solutions, complementary solutions, structure of general solution.
3. transformation to 1st order system, linear system, eigenvalues/eigenvectors, generalized eigenvectors, fundamental solution matrix, exponential of matrices
4. series solutions, radius of convergence, recurrence relations

2 1st order equation

Method: (a) Separable equations

$$\frac{dy}{dx} = \frac{1+\sqrt{y}}{1+\sqrt{x}}.$$

(b) Exact equation

$$(x^2 + \ln y)dx + \left(y^3 + \frac{x}{y}\right)dy = 0.$$

(c) Linear 1st order equation (integrating factor)

$$(1-x^2)y' + xy = 1.$$

(d) Substitution

- (Homogeneous)

$$y\frac{dy}{dx} - y = \sqrt{x^2+y^2}.$$

- (Bernoulli type)

$$(1+x)\frac{dy}{dx} + y = y^3.$$

Models: (a) Newton's Law of cooling: Notes, HW2 (6)

(b) Population model: Notes, HW4 (2,3), Practice1 (4), Mid1 (4).

(c) Acceleration-velocity model: HW5 (1), Practice1 (5)

3 Linear DE with constant coefficients

Method: (a) Homogeneous linear equations

$$y''' - 2y'' + y = 0.$$

(b) Non-homogeneous equations, particular solutions

i. Undetermined coefficients

$$y'' + 4y' + 4y = e^{-2x}.$$

ii. Variation of parameters

$$y'' + y = \frac{1}{\sin^2 x}.$$

Model: Mechanical vibration HW7 (9-11), HW9 (1-2), Practice2 (3), Midterm 2 (3)-(4).

(a) Free undamped/damped oscillation

(b) Forced undamped/damped oscillation, Resonance

4 Linear System

Method: (a) Elimination method HW10 (1)-(2), Quiz3 (1), HW11 (3a) (also the following three systems).

(b) Eigenvalue method

$$\begin{cases} x_1' = 4x_1 + 2x_2 \\ x_2' = -3x_1 - x_2 \\ x_3' = x_1 + x_2 + 2x_3 \end{cases}$$

(c) Complex eigenvalue

$$\begin{cases} x_1' = 7x_1 + x_2 \\ x_2' = -4x_1 + 3x_2 \end{cases}$$

(d) Multiple eigenvalues

$$\begin{cases} x_1' = x_1 - 4x_2 \\ x_2' = 4x_1 + 9x_2 \end{cases}$$

(e) Chain of generalized eigenvectors: Notes, HW11 (3b), HW12 (1).

(f) Exponential of matrices and initial value problems: HW12 (2), (3).

(g) Nonhomogeneous linear system

$$\begin{cases} x_1' = 2x_1 + 3x_2 + e^t \\ x_2' = 2x_1 + x_2 + e^{2t} \end{cases}$$

Model: (2-mass, 3-spring) system: HW9 (4), HW10 (2).

5 Power Series Solutions

Standard Mauclaurin series, Radius of convergence, series solutions, recurrence relations

1st order: $(2x+1)y' = y$.

2nd order: $y'' - 2xy' + 6y = 0$.

Syllabus

Week	Sections	Homework	Notes
Week 1: 1/27-1/31	**1.1:** Mathematical Models **1.2:** General and Particular Solutions **1.3:** Direction Fields	**Part I**: **1.1:** 43,44,45 **1.2:** 36 , 42 , 44 **1.3:** 22 , 28 Part II HW1 solution	Mathematica: Slope fields and stream lines
Week 2: 2/3-2/7	**1.4:** Separable Equations **1.5:** Linear 1st order equations	Homework 2 HW2 solution	Notes on separable equations Notes on Newton's law of cooling/heating Mathematica: Numerical calculation and Plot graphs Quiz 1 in Friday class (cancelled)
Week 3: 2/10-2/14	**1.6:** Substitution/exact equations	Homework 3 HW3 solution	Notes on population models
Week 4: 2/17-2/21	**2.1:** Population models **2.2:** Equilibrium solutions	Homework 4 HW4 solution	Mathematica: Interactive Manipulation and Piecewise defined function Quiz 2 in Friday class Quiz 2 solution
Week 5: 2/24-2/28	**2.3:** Acceleration-velocity models **2.4:** Numerical Method: Euler method Review for Midterm I	Homework 5 HW5 solution Practice Midterm I Practice Midterm I Solution	Mathematica: Numerical Solution of DE Implement Euler's method using Mathematica
Week 6: 3/3-3/7	**3.1:** 2nd order linear equations **Midterm I** **3.2:** General solutions of linear equations	Solution to midterm 1: Solution 1 , Solution 2 Homework 6 HW6 solution (by Anant Atyam)	Midterm I in class covering up to 2.4.
Week 7: 3/10-3/14	**3.3:** Homogeneous constant coefficient equations **3.4:** Mechanical vibrations	Homework 7 HW7 solution (by Chengjian Yao)	Mathematica: DSolve , Solve DE using Mathematica
3/17-3/21	Spring Break		
Week 8: 3/24-3/28	**3.5:** Nonhomogeneous equations, undetermined coefficients	Homework 8 Solution to Homework 8	A manual for finding particular solution using undetermined coefficients Quiz 3 in Friday class covering up to section 3.3
Week 9: 3/31-4/4	**3.5:** Variation of parameters **3.6:** Forced oscillations and resonance	Homework 9 Solution to Homework 9 Midterm 2 Practice Solution to Practice Midterm 2	Derivation of the formula in the variation of parameters Mathematica: Manipulate vibrations, beats and resonance
Week 10: 4/7-4/11	**4.1:** First order systems **4.2:** Method of Eliminations **Midterm II**	Solution to midterm 2: Solution 1 , Solution 2	Midterm II in class covering up to 4.1.
Week 11: 4/14-4/18	**5.1:** Matrices and linear systems **5.2:** Eigenvalue methods for homogeneous systems	Homework 10 Solution to Homework 10 (By Chengjian Yao (with correction/appendix))	Mathematica: Matrix Operations - Create matrix/vector - Eigensystem
Week 12: 4/21-4/25	**5.2:** Eigenvalue methods for homogeneous systems **5.4:** Multiple eigenvalues	Homework 11 Solution to Homework 11	Quiz 3 in Friday class covering section up to 5.2 Quiz 3 solution Classify the chains for 2*2 or 3*3 matrices
Week 13: 4/28-5/2	**5.5:** Matrix exponentials **5.6:** Nonhomogeneous linear system	Homework 12 Solution to Homework 12	
Week 14: 5/5-5/9	**8.1:** Power series **8.2:** Series solutions	Homework 13 Solution to Homework 13	Quiz 4 in Friday class covering up to section 5.5 Quiz 4 solution
Week 15: 5/5-5/9	Review of the course Reading period	Review sheet , Solutions to review exercises , Corrections Old Exam , Old Practice exam .	
Happy ending	**Final exam**	Monday, May 19, 8:00am-10:45am	Solution to the final exam ,

Final Exam

Name

Math 303 - Differential Equations with Applications

May 15, 2008

- No credit will be given for answers without mathematical or logical justification.
- You may leave answers in implicit form, when appropriate.
- Simplify answers as much as possible.
- No calculators, notes, or books.

Part I

1) (7 pts) Solve for x: $\frac{dx}{dt} = tx - \frac{t}{x}, \quad x(0) = 2.$

2) (8 pts) Find the general solution: $\frac{dy}{dx} = \frac{2x - y}{x + 6y}.$

3) (15 pts) Solution with a concentration of 0.1 lbs of salt per gallon pours into a tank at a rate of $\frac{2}{t+1}$ gallons per minute. Also, well-mixed solution leaves the tank at the same rate. How much salt is in the tank after 1 minute, if initially the tank contains 1 gallon of water mixed with 0.9 lb of salt?

4) The equation for a particle caught in the gravitational field of a body of mass M is

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2},$$

where $G \approx 6.67 \times 10^{-11} m^3 kg^{-1} s^{-2}$ and r is the distance to the center of mass of the body. A particle of dust is caught in the gravitational field of a small, spherically shaped asteroid of mass $\frac{16}{6.67} \times 10^{11} kg$ and radius $100m$.

a) (10 pts) Use the methods of differential equations to find $\frac{dr}{dt}$.

b) (10 pts) Initially the dust particle is motionless relative to the asteroid and $300m$ from its surface. With what speed will it strike the asteroid's surface?

Part II

5) Consider the differential equation

$$x'''' + 2x''' + 2x'' = 4 - 12t$$

a) (10 pts) Find the complimentary solution

b) (15 pts) Find the general solution.

c) (5 pts) Find the solution, given $x(0) = 0$, $x'(0) = 0$, $x''(0) = 8$, $x'''(0) = -6$.
(There's an easy way and a hard way.)

Part III

6) (5 pts) Compute e^{tA}, where $A = \begin{pmatrix} 0 & -1 & 6 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}$.

7) Consider the equation

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ t \end{pmatrix}.$$

a) (5 pts) If you use the method of undetermined coefficients, what would your 'guess' for $\mathbf{x_p}$ be?

b) (15 pts) Find $\mathbf{x_p}$.

Part IV

8) Consider the system $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}' = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

a) (10 pts) Find the eigenvalues of the matrix.

b) (20 pts) Find the system's general solution.

(continued from previous page)

c) (10 pts) Write down the fundamental matrix $\Phi(t)$, and compute $\Phi(0)^{-1}$
HINT: This particular $\Phi(0)$ should have some special properties. Before trying to compute $\Phi(0)^{-1}$, see what you get when you multiply $\Phi(0) \cdot \Phi(0)^T$.

d) (5 pts) Given $x(0) = (-1\ 1\ 0\ 2)^T$, find $x(t)$.

MAT303 Spring 2009

Practice Final

The actual final will consist of twelve problems with no more then two subproblems

You will be allowed to use calculators

Problem 1

Some of the given differential equations are separable and some are not. Solve those that are separable.

i $(1+x)ydx + xdy = 0$

ii $y' = y^{1/2}$

iii $y' + xy = 3$

iv $xy' - y\ln x = xy^2$

Problem 2

Some of the differential equations are linear and some are not. Determine those that are linear and give its integrating factor and solve them.

i $xy' + y = 3$

ii $xy' - y = 2x^2$

iii $y' - \frac{3}{x-1}y = (x-1)^4$

iv $y' + \frac{1}{\sin x}y - y^2 = 0$

v $xy' + y = x^5$

Problem 3

i) Is the equation exact? ii) If it is find the general solution. You may leave the answer in implicit form.

- $(3x - y)dx - (x + 3y)dy = 0$
- $(3x^2 - xy)dx - (x^2 + 3xy)dy = 0$, $x \neq 0$

Problem 4

Find the general solutions of differential equations (you may leave the answer in implicite form)

i $dy/dx = (x + y)/(2x - y)$

ii $dy/dx = xy + xy^4$

Problem 5

- The differential equation $dx/dt = \frac{1}{2}x(2-x)-h$ models a logistic population with harvesting at rate h. In the language of dynamics, one may say we are perturbing a logistic population by a constant h. So we usually want h to be small.

  i In terms of h, what are the equilibrium solutions?

  ii What are the stability of the solutions above? (Hint: Set $h = 0$, and then look at the stability there. This should tell you the stability of the solutions above.)

  iii What is the bifurcation point?

  iv Describe the stability of the bifurcation point. (Hint: Part ii)

  v For the problems below, set h to be the bifurcation point found in Part iv.

    a u is a solution with $u(2) = 5.5$. Compute $\lim_{t\to\infty} u(t)$.

    b u is a solution with $u(0) = 2$. Compute $\lim_{t\to\infty} u(t)$.

    c u is a solution with $u(0) = 1$. Using Eulers Method, approximate $u(2)$, using step size $\Delta x = 0.5$ to eight decimal places.

- Suppose that $-1 < a < 1$ is a constant parameter, and $y(t)$ satisfies the ODE

$$y^{'} = (a - y^2)(y - 2)$$

:

  i Find the equilibria, sketch the phase line, and determine the stability of the equilibria in each of the following cases:

    a $-1 < a < 0$;

    b $a = 0$

    c $0 < a < 1$.

  ii Suppose that $y(t)$ is the solution of the ODE that satisfies the initial condition $y(0) = 0$. What is the behavior of $y(t)$ as $t \to \infty$ in each of the cases ia,ib,ic.

Problem 6

Compute the general solution of each nonhomogenenous equation by the Method of Undetermined coefficients

i $y^{''} + y = \sin x$

ii $y^{''} - y^{'} - 2y = 2xe^{x} + x^2$

iii $y^{''} - 5y^{'} + 4y = e^{2x}\cos x + e^{2x}\sin x$

Problem 7

Use the method of variation of parameter to solve the following initial value problem

i $y^{''} - y^{'} - 2y = t^2e^{2t}$, $y(0) = 0, y^{'}(0) = 1$

ii $y^{''} + y = -2\sin t, y(0) = 1, y^{'}(0) = 1$

Problem 8

Use the

- Eulers Method
- Improved Eulers Method

with $h = 0.2$ to solve the initial value problems on $0 \leq x \leq 1$

i $y' = 3x + 2y \; y(0) = 1$

ii $y' = xy \; y(0) = 1$

Problem 9

- Solve the second-order linear equation

$$y^{''} + 5y^{'} + 6y = 0$$

  i by using characteristic equation,

  ii by transforming it into a system of 2 first-order equations.

Problem 10

Find the general solution of the system $\frac{dx}{dt} = Ax$ using the method of ellimination where

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$$

Problem 11

Find the general solution of the system $\frac{dx}{dt} = Ax$ using the method of eigenvalues where

i

$$A = \begin{pmatrix} -3 & 0 & -1 \\ 3 & 2 & 3 \\ 2 & 0 & 0 \end{pmatrix}$$

ii A is from Problem 10.

Problem 12

The motion of a mass on a spring can be described by the solution of the initial value problem $mu'' + cu' + ku = F(t)$, $u(0) = u_0, u'(0) = u_0'$. A mass weighing 8 lb stretches a spring 6 in. The mass is attached to a viscous damper with a damping constant of 2 lb-sec/ft., and it is acted on by an external force of $\cos 3t$ lb. The mass is displaced 2 in. downward and released.

i Formulate the initial valued problem describing the motion of the mass.

ii Solve the initial valued problem using either the Method of Undetermined Coefficients or Variation of Parameters.

Problem 13

Find the general solution of the system

$$\begin{aligned} (D^2+1)x - D^2 y &= 2e^{-t} \\ (D^2-1)x + D^2 y &= 0 \end{aligned} \tag{1}$$

As usual $D = d/dt$

Newton's law of cooling/heating

This law says that the rate of cooling/heating is proportional to the difference of the temperature of the object and cooling/heating source. Suppose the temperature of the cooling/heating source changes according to the function $A(t)$. Let $T = T(t)$ be the temperature of the object under consideration. Then we can write down the differential equation:

$$\frac{dT}{dt} = -k(T - A(t)).$$

k is called the cooling/heating constant. This is a 1st order linear differential equation:

$$\frac{dT}{dt} + kT = kA(t).$$

We can find the integrating factor:

$$F(t) = e^{\int k dt} = e^{kt}.$$

So it's easy to find the solution:

$$T(t) = e^{-kt} \int ke^{kt} A(t) dt. \tag{1}$$

Example: Let the cooling/heating constant be $k = 0.3$. Assume the initial temperature $T(0) = -20°\text{C}$. Assume the temperature of the source oscillates according to the function:

$$A(t) = 10 \sin\left(\frac{\pi}{12} t\right).$$

How does the temperature $T(t)$ of the object change?
Solution: By the above discussion, we just need to calculate:

$$T(t) = e^{-0.3t} \int 0.3e^{0.3t} 10 \sin(\pi t/12) dt. \tag{2}$$

To simplify the calculation a little bit, we can use the substitution: $u = \pi t/12$, the right hand side becomes:

$$T(u) = e^{-3.6u/\pi} \frac{36}{\pi} \int e^{3.6u/\pi} \sin(u) du \tag{3}$$

Now we can integrate by parts twice (let $a = 3.6/\pi$)

$$\begin{aligned} \int e^{au} \sin(u) du &= \int e^{au} d(-\cos u) = -e^{au} \cos u + a \int \cos(u) e^{au} du \\ = \quad & -e^{au} \cos u + a \int e^{au} d(\sin u) = -e^{au} \cos u + ae^{au} \sin u - a^2 \int e^{au} \sin u \end{aligned}$$

So we can solve:

$$\int e^{au}\sin(u)du = e^{au}\frac{a\sin u - \cos u}{1+a^2} + C.$$

Now we can substitute in to (3) to get

$$T(u) = \frac{36}{\pi}\left(\frac{a\sin u - \cos u}{1+a^2} + Ce^{-au}\right)$$

When $t = 0$, $u = 0$, so we can use the initial condition $T(0) = -20$ to get

$$-20 = T(0) = \frac{36}{\pi}\left(-\frac{1}{1+a^2} + C\right) \Longrightarrow C = \frac{1}{1+a^2} - \frac{5\pi}{9}.$$

Substitute $u = \pi t/12$ and $a = 3.6/\pi$ ($au = 0.3t$), then finally we get the particular solution:

$$\begin{aligned} T(t) &= \frac{36}{\pi}\left(\frac{\frac{3.6}{\pi}\sin\frac{\pi t}{12} - \cos\frac{\pi t}{12}}{1+(3.6/\pi)^2} + \left(\frac{1}{1+(3.6/\pi)^2} - \frac{5\pi}{9}\right)e^{-0.3t}\right) \\ &\approx 5.68\sin(0.26t) - 4.95\cos(0.26t) - 15.05e^{-0.3t}. \end{aligned}$$

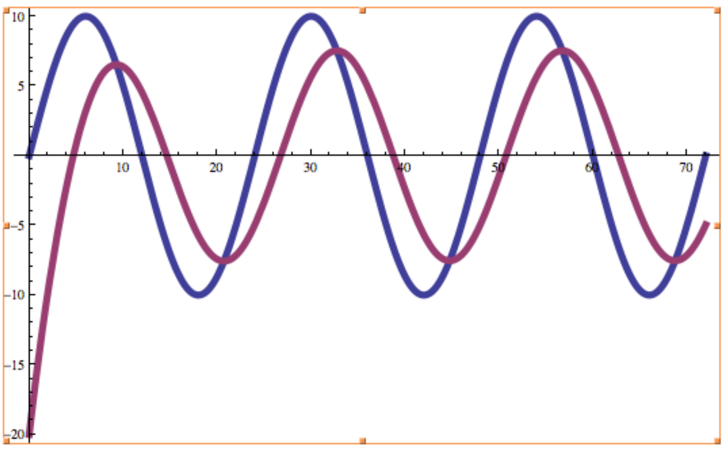
The term $-15.05e^{-0.3t}$ in the solution is called the "damped" part. The following is the plot of graphs using Mathematica. The blue curve is the temperature of the source, and the red curve is the temperature of the object. Also the first command (**T1**[**t**_] $= \ldots$) defines the function T1(t) to be plotted.

```
In[116]:= T1[t_] = 36 / Pi (3.6 / Pi * Sin[Pi t / 12] - Cos[Pi t / 12]) / (1 + (3.6 / Pi) ^2) + 36 / Pi * (1 / (1 + (3.6 / Pi) ^2) - 5 Pi / 9) Exp[-0.3 t]
```

Out[116]= $-15.046\, e^{-0.3\, t} + 4.95398 \left(-\mathrm{Cos}\left[\frac{\pi\, t}{12}\right] + 1.14592\, \mathrm{Sin}\left[\frac{\pi\, t}{12}\right]\right)$

```
In[125]:= Plot[{10 Sin[Pi t / 12], T1[t]}, {t, 0, 72}, PlotStyle → Thickness[0.01]]
```

Out[125]=



From the graphs, we see that the temperature of the object will also oscillate in the long term. But the amplitude of oscillation (red curve) is smaller than the amplitude of the source (blue curve). The oscillation of the temperature of the object lags behind the oscillation of the source.

We can actually calculate by using Mathematica. For this rewrite equality (2) such that it satisfies the initial condition:

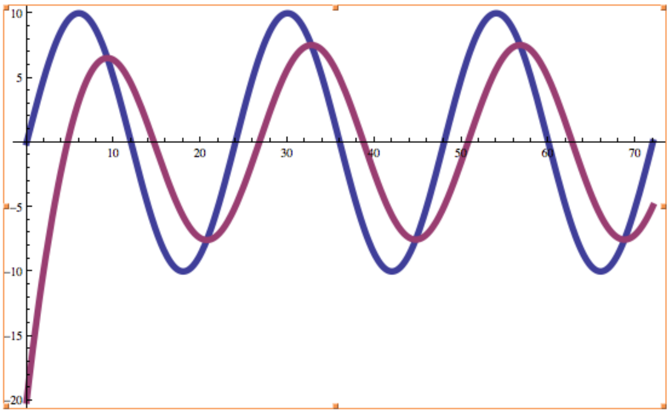
$$T(t) = e^{-0.3t}\left(\int_0^t 0.3e^{0.3x}10\sin(\pi x/12)dx - 20\right).$$

Then we can use Mathematica to integrate and plot the graphs:

```
In[118]:= T2[t_] = Exp[-0.3 t] * (Integrate[3 Exp[0.3 x] * Sin[Pi x / 12], {x, 0, t}] - 20)
Out[118]= e^-0.3 t (-20 + 3 ((1.65133 + 0. i) + e^0.3 t ((-1.65133 + 0. i) Cos[0.261799 t] + (1.89228 + 0. i) Sin[0.261799 t])))

In[119]:= Plot[{10 Sin[Pi t / 12], T2[t]}, {t, 0, 72}, PlotStyle → Thickness[0.01]]
```

Out[119]=



Homework 2

1. Solve the following differential equations:
   (a) $y' = \frac{x^2}{y(1+x^3)}$.
   (b) $y' + y^2 \sin x = 0$.
   (c) $y' = 1 + x + y^2 + xy^2$.
   (d) $xdx + ye^{-x}dy = 0$, $y(0) = 1$.

2. Solve the following differential equations:
   (a) $\frac{dy}{dx} + 2y = xe^{-2x}$.
   (b) $\frac{1}{x}\frac{dy}{dx} + y = 1$.
   (c) $x\frac{dy}{dx} + 2y = \sin x$, $y(\pi) = 1/\pi$.
   (d) $\frac{dy}{dx} - \frac{2}{1-x^2}y = 1 + x$, $y(0) = 1$.

3. Carbon extracted from and ancient skull contained only one-fifth as much ^{14}C as carbon extracted from present-day bone. How old is the skull?

4. A spherical tank of radius 4 ft is full of gasoline when a circular bottom hole with radius 2 in. is opened. How long will all the gasoline drain from the tank?

5. A tank initially contains 60 gal of pure water. Brine containing 1 lb of salt per gallon enters the tank at 1 gal/min, and the perfectly mixed solution leaves the tank at 3 gal/min. Thus the tank is empty after exactly 1/2 hour. Find the amount of salt in the tank after t minutes.

6. Assume the outdoor temperature changes according to the periodic function:
$$A(t) = 10\cos\left(\frac{\pi}{12}t\right).$$
   Assume the initial indoor temperature is $u(0) = 20°C$. Use Newton's law of heating/cooling to find the indoor temperature $u = u(t)$. Assume the cooling constant $k = 0.5$. Use Mathematica to draw the graph of $A(t)$ and $u(t)$. What conclusions do you get from the picture of graphs?

Population models

Notation:

- $P = P(t)$ population at time t;
- $\beta = \beta(P, t)$ birth rate;
- $\delta = \delta(P, t)$ death rate. The most general form of differential equation modeling the population is:

$$\frac{dP}{dt} = (\beta(P,t) - \delta(P,t))P(t), \quad P(0) = P_0. \tag{1}$$

Model 1: Logistic model In this model, $\beta = \beta_0 - kP$, $\delta = \delta_0$. β_0, k, δ_0 are constants. So the equation (1) becomes:

$$\frac{dP}{dt} = (\beta_0 - kP - \delta_0)P = kP(M - P), \quad P(0) = P_0. \tag{2}$$

Here $M = (\beta_0 - \delta_0)/k$. Equation (2) is called a logistic equation. It's a separable equation:

$$\frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right) dP = \frac{dP}{P(M-P)} = kdt$$

So if we integrate on both sides, we get:

$$\frac{1}{M} \ln \frac{P}{M-P} = kt + C_1 \Longrightarrow \frac{P}{M-P} = e^{MC_1} e^{kMt} = Ce^{kMt}.$$

Here $C = e^{MC_1}$ is a positive constant. We can determine it using the initial condition $P(0) = P_0$:

$$C = \frac{P_0}{M - P_0}.$$

Now we can solve $P = P(t)$ to get a general solution:

$$P(t) = \frac{MCe^{kMt}}{1 + Ce^{kMt}} = \frac{M}{1 + C^{-1}e^{-kMt}} = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}. \tag{3}$$

From the solution (3), we see that

- For any initial population $P_0 > 0$, we always have

$$\lim_{t \to +\infty} P(t) = M.$$

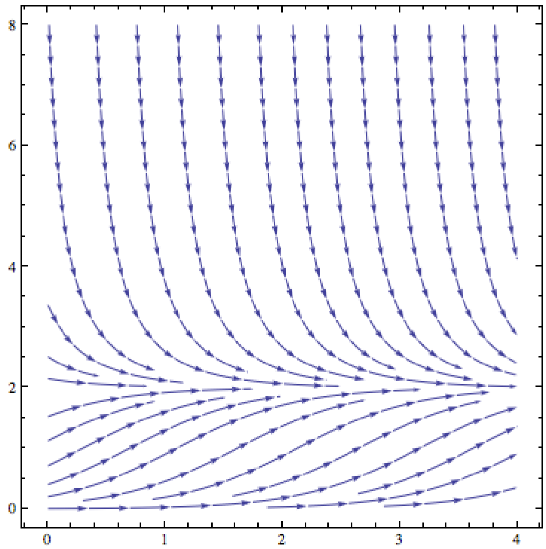
  M is called the **carrying capacity** in this logistic model.

- $P(t) \equiv M$ is a solution. This solution is called a **equilibrium solution**. It is a stable equilibrium.

We can use Mathematica to draw solution curves. For simplicity, assume $k = 1$, $M = 2$.

```
In[164]:= StreamPlot[{1, P (2 - P)}, {t, 0, 4}, {P, 0, 8}]
```

Out[164]=



Model 2: Doomsday-Extinction model In this model, $\beta = kP$, $\delta = \delta_0$ with k, δ_0 constants. So the equation (1) becomes

$$\frac{dP}{dt} = (kP - \delta)P = k(P - M)P, \quad P(0) = P_0. \tag{4}$$

Here $M = \delta/k$. Again this is a separable equation:

$$\frac{1}{M}\left(\frac{1}{P-M} - \frac{1}{P}\right) = \frac{dP}{(P-M)P} = kdt.$$

So we integrate both sides to get:

$$\frac{1}{M}\log\frac{P-M}{P} = kt + C_1 \Longrightarrow \frac{P-M}{P} = Ce^{Mkt}.$$

Here $C = e^{MC_1}$ is a positive constant. Substituting $P(0) = P_0$ we can determine C:

$$C = \frac{P_0 - M}{P_0}.$$

So we can solve $P = P(t)$ to get:

$$P(t) = \frac{M}{1 - Ce^{Mkt}} = \frac{MP_0}{P_0 - (P_0 - M)e^{Mkt}}. \tag{5}$$

From the solution (5) we see that:

- If $P_0 > M$, then the population will explode to infinity at time when the denominator becomes 0:

$$t_{doom} = \frac{1}{Mk} \log \frac{P_0}{P_0 - M}.$$

- If $P_0 < M$, then the population will decay exponentially to 0.

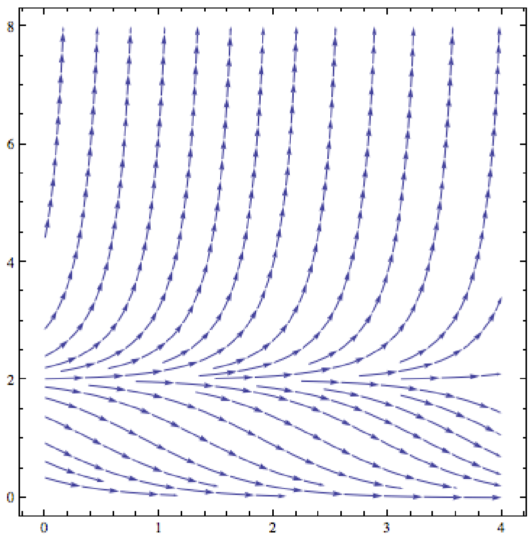
$$\lim_{t \to +\infty} P(t) = 0.$$

- $P(t) \equiv M$ is an **(unstable) equilibrium solution**.

Again we can use Mathematica to draw some streamlines to visualize the situation ($k = 1, M = 2$).

```
In[163]:= StreamPlot[{1, P (P - 2)}, {t, 0, 4}, {P, 0, 8}]
```

Out[163]=



Homework 4

1. First solve the equation $f(x) = 0$ to find the critical points of the given autonomous differential equation. Then determine whether each critical point is stable or unstable, and construct the corresponding phase diagram for the differential equation. Next, solve the differential equation explicitly for $x(t)$. Finally, use either the exact solution or computer generated solution curves to verify the stability.

   (a) $x' = x^2 - 4x + 3$.

   (b) $x' = x^2 - 4x + 5$.

   (c) $x' = -x^2 + 2x - 1$.

   (d) $x' = -x^2 + 2x + 3$.

2. The differential equation $\frac{dx}{dt} = \frac{1}{8}x(8-x) - h$ models a logistic population with harvesting at rate h. Determine the dependence of the number of critical points on the parameter h, and then construct a bifurcation diagram in the hc-plane. Use **Manipulate** in **Mathematica** to visualize the bifurcation process.

3. The differential equation $\frac{dx}{dt} = \frac{1}{8}x(x-8)+s$ models a explosion/extinction population with stocking at rate s. Determine the dependence of the number of critical points c on the parameter s and then construct a bifurcation diagram in the sc-plane. Use **Manipulate** in **Mathematica** to visualize the bifurcation process.

4. A woman bails out of an airplane, falls freely for 20 s, then opens her parachute. Assume the drag coefficient $\rho = 0.15$ without parachute and $\rho = 1.5$ with the parachute. Find the velocity $v(t)$ as a function of t and terminal velocity in the following two situations:

   (a) Assume the air resistance is $\rho v \ ft/s^2$.

   (b) Assume air resistance is $\rho v^2 \ ft/s^2$.

   Compare the two velocity functions by plotting their graphs (using **Mathematica**).

5. (*) (Taking account of the moon's gravitational field, re-solve the problem 1.2.42 in homework 1) A spacecraft is in free fall toward the surface of the moon at a speed of 1000 mi/h. Its retrorocket, when fired, provide a constant deceleration of 20,000 mi/h^2. At what height above the lunar surface should the astronauts fire the retrorockets to insure a soft touchdown? Note that you need to change the units by using the following data.

$G \approx 6.6726 \times 10^{-11} N \cdot (m/kg)^2,\ M_{moon} = 7.35 \times 10^{22} (kg),\ R_{moon} = 1740 km.$

6. (*) To what radius would the moon have to be compressed in order for it to become a *black hole* - the escape velocity from its surface equal to the velocity $c = 3 \times 10^8 m/s$ of light?

MIDTERM I PRACTICE PROBLEMS

(1)
$$xy' + 2y = x \cdot y^{1/2}.$$

(2)
$$\frac{dy}{dx} = -\frac{3x^2 + 2y^2}{4xy}.$$

(3)
$$y' = \sqrt{x + y}.$$

(4) (a) A logistic population model with harvesting is given by the differential equation:
$$\frac{dP}{dt} = 6P - P^2 - h.$$
Determine how the number of equilibrium solutions changes with h by drawing the bifurcation diagram on the hc-plane.

(b) Find the equilibrium solutions of the differential equation:
$$\frac{dP}{dt} = 6P - P^2 - 8.$$
Classify them as stable or unstable equilibrium solutions using the phase diagram. If the initial population is 1, what limit population will $P(t)$ approach?

(5) A woman bails out the plane at an altitude of $5000ft$ and immediately opens her parachute. Assume the air resistance is proportional to the velocity and the drag constant $\rho = 2$. What's her velocity at time t? What's her height at time t? The gravitational acceleration is $32ft/s^2$.

(6) Use Euler's method to approximate to the solution on the interval $[-0.3, 0]$ with step size -0.1. What value of $y(-0.3)$ do you get?
$$y'(x) = x + y^2, y(0) = 0.$$
Note that this is not Bernoulli equation.

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1. (30pts)

$$x^2y' + xy = \frac{\cos x}{y}, \quad y(\pi) = 1.$$

Method 1: This is a Bernoulli type equation.

$x^2 y y' + x y^2 = \cos x$. substitute $u = y^2 \Rightarrow u' = 2y \cdot y'$.

so. $x^2 \frac{u'}{2} + x \cdot u = \cos x \Rightarrow u' + \frac{2}{x} u = \frac{2\cos x}{x^2}$. This is 1st order linear.

Integrating factor $F(x) = e^{\int \frac{2}{x} dx} = e^{2\ln x} = x^2$.

so $(x^2 u)' = 2\cos x \Rightarrow x^2 u(x) = 2\sin x + C \Rightarrow x^2 y^2 = 2\sin x + C$.

$\xRightarrow{y(\pi)=1} 1^2 \cdot \pi^2 = 2 \cdot \sin \pi + C \Rightarrow C = \pi^2$. So $x^2 y^2 = 2\sin x + \pi^2$.

so $\boxed{y = \frac{\sqrt{2\sin x + \pi^2}}{x}}$

Method 2: $x^2 y \frac{dy}{dx} + (xy^2 - \cos x) = 0 \Rightarrow x^2 y\, dy + (xy^2 - \cos x) dx = 0$.

$\frac{\partial}{\partial x}(x^2 y) = 2xy = \frac{\partial}{\partial y}(xy^2 - \cos x) \Rightarrow$ Exact equation.

$\Phi_y = x^2 y + g'(y) = x^2 y \Rightarrow g'(y) = 0 \Rightarrow g(y) = \text{constant}$

$\Uparrow ②$

$$\begin{cases} \frac{\partial \Phi}{\partial y} = x^2 y & ② \\ \frac{\partial \Phi}{\partial x} = xy^2 - \cos x & ① \end{cases} \Rightarrow \Phi(x,y) = \frac{1}{2}x^2y^2 - \sin x + g(y).$$

so $\Phi(x,y) = \frac{1}{2}x^2y^2 - \sin x$, general solution: $\frac{1}{2}x^2y^2 - \sin x = C$.

$\xRightarrow{y(\pi)=1} C = \frac{1}{2}\pi^2 \Rightarrow y^2 = \frac{2\sin x + 2C}{x^2} = \frac{2\sin x + \pi^2}{x^2} \Rightarrow \boxed{y = \frac{\sqrt{2\sin x + \pi^2}}{x}}$

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2. **(30pts)**

$$(x+y)\cdot\frac{dy}{dx}=y,\quad y(1)=1.$$

Method 1: $\frac{dy}{dx}=\frac{y}{x+y}=\frac{\frac{y}{x}}{1+\frac{y}{x}}$. substitute $u=\frac{y}{x}\Rightarrow y=u\cdot x\Rightarrow y' \overset{\parallel}{} u'\cdot x+u$.

So $u'x+u=\frac{u}{1+u}\Rightarrow u'\cdot x=\frac{u}{1+u}-u=\frac{u-(u+u^2)}{1+u}=-\frac{u^2}{1+u}$.

$\Rightarrow \frac{1+u}{u^2}du=-\frac{dx}{x}\Rightarrow -\frac{1}{u}+\ln u=-\ln x+C$.

$\frac{1+u}{u^2}du = \left(\frac{1}{u^2}+\frac{1}{u}\right)du$

$\Downarrow$

$-\frac{x}{y}+\ln\frac{y}{x}=-\ln x+C$.

$\overset{y(1)=1}{\Longrightarrow} -\frac{1}{1}+\ln 1=-\ln 1+C\Rightarrow C=-1$. So:

$-\frac{x}{y}+\ln\frac{y}{x}=-\ln x-1\Rightarrow \ln\frac{y}{x}+\ln x+1=\frac{x}{y}\Rightarrow \boxed{x=y\ln y+y}$

$\ln\frac{y}{x}+\ln x+1 = \ln y+1$.

Method 2: $\frac{dx}{dy}=\frac{x+y}{y}=\frac{x}{y}+1$. $x=y\cdot v(y)$. $v=\frac{x}{y}\Rightarrow\frac{dx}{dy}=v'\cdot y+v(y)$.

$\Rightarrow v'\cdot y+v(y)=v+1\Rightarrow v'\cdot y=1\Rightarrow v'(y)=\frac{1}{y}\Rightarrow v(y)=\ln y+C$

$\Rightarrow \frac{x}{y}=\ln y+C \overset{y(1)=1}{\Longrightarrow} 1=\ln 1+C\Rightarrow C=1$.

So $\boxed{x=y\ln y+y}$

3. (30pts)

$$\frac{dy}{dx} = -1 + (2x+y)^2, \quad y(0) = 1.$$

Substitute $u = 2x + y \Rightarrow y = u - 2x \Rightarrow y' = u' - 2$.

So $u' - 2 = -1 + u^2 \Rightarrow u' = 1 + u^2$

$\Rightarrow \frac{du}{1+u^2} = dx \Rightarrow \tan^{-1}(u) = x + C$

$\tan^{-1}(u) = \tan^{-1}(2x+y)$.

$\xRightarrow{y(0)=1} \tan^{-1}(1) = C = \frac{\pi}{4}$.

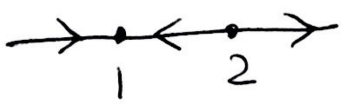
So. $\tan^{-1}(2x+y) = x + \frac{\pi}{4} \Rightarrow \boxed{y = \tan(x + \frac{\pi}{4}) - 2x}$

4. A explosion/extinction with stocking model of population is given by the following differential equation:

$$\frac{dP}{dt} = P^2 - 3P + 2.$$

4(a) (30pts): Find the equilibrium solutions and **classify** them as stable or unstable equilibrium solutions.

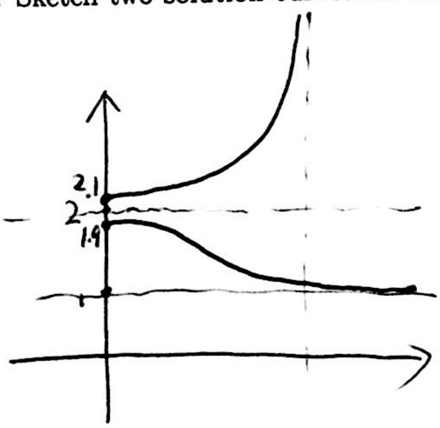
$0 = P^2 - 3P + 2 = (P-1)(P-2) \Rightarrow P=1$ or $P=2$



$P=1$ is stable.

$P=2$ is unstable.

4(b) (20pts): Sketch two solution curves for the initial conditions $P(0) = 1.9$ and $P(0) = 2.1$.



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4(c) (30pts) For the above population model:

$$\frac{dP}{dt} = P^2 - 3P + 2.$$

If $P(0) = 3$, find the doomsday time t_{doom} when the population explodes to infinity. (Hint: First solve for an explicit solution)

$$\frac{dP}{P^2-3P+2} = dt \Rightarrow \ln\frac{P-2}{P-1} = t + C_1 \Rightarrow \frac{P-2}{P-1} = C\cdot e^t$$

$$\| $$

$$\left(\frac{1}{P-2} - \frac{1}{P-1}\right)dP$$

$$\Downarrow P(0)=3$$

$$\frac{P-2}{P-1} = \frac{1}{2}e^t \Leftarrow \frac{1}{2} = C$$

so $P-2 = \frac{1}{2}e^t P - \frac{1}{2}e^t \Rightarrow \left(1-\frac{1}{2}e^t\right)P = 2 - \frac{1}{2}e^t$

so $P(t) = \dfrac{2-\frac{1}{2}e^t}{1-\frac{1}{2}e^t} = \dfrac{4-e^t}{2-e^t}$.

$P(t)$ explodes to infinity when the denominator goes to 0.

so. $2 - e^{t_{doom}} = 0 \Rightarrow t_{doom} = \ln 2$.

5. (30pts) Use Euler's method to approximate to the solution on the interval $[0, 0.3]$ with step size 0.1. What value of $y(0.3)$ do you get? Don't round off your answer.

$$y'(x) = x - y^2, y(0) = 0.$$

$h=0.1$ $\quad x_0=0, \quad y_0=0.$

$x_1=0.1, \quad y_1=y_0+f(x_0,y_0)\cdot h$

$=0+(0-0^2)\cdot 0.1=0.$

$x_2=0.2, \quad y_2=y_1+f(x_1,y_1)\cdot h$

$=0+(0.1-0^2)\cdot 0.1=0.01.$

$x_3=0.3, \quad y_3=y_2+f(x_2,y_2)\cdot h$

$=0.01+(0.2-(0.01)^2)\times 0.1$

$=0.01+0.02-0.00001$

$=0.02999.$

Homework 5

1. Suppose that a body moves through a resisting medium with resistance proportional to v^α, so that
$$\frac{dv}{dt} = -kv^\alpha, \quad v(0) = v_0.$$
   (a) If $\alpha = 1$, find the velocity and position as functions of t with parameters k and v_0. Does the body travel a finite or infinite distance?
   (b) If $\alpha > 1$, find the velocity and position as functions of t with parameters α, k and v_0. Does the body travel a finite or infinite distance?

2. Apply Euler's method twice to approximate to the actual solution on the interval $[0, \frac{1}{2}]$, first with step size $h = 0.25$, then with step size $h = 0.1$. Compare the three decimal-place values of the two approximations at $x = 1/2$ with the value $y(1/2)$ of the actual solution.
   (a) $y' = 2y$, $y(0) = 1$.
   (b) $y' = \frac{1}{\sqrt{1-x^2}}$, $y(0) = 0$.
   (c) $y' = x + y$, $y(0) = -1$.

3. Use **NDSolve** in Mathematica to get numerical solution, find the approximate value at the given point and plot the graphs:
   (a) $y' = x^2 + y^2$, $y(0) = 1$. Find $y(0.8)$. Try to find $y(1)$ and see what goes wrong.
   (b) $y' = x^3 + y^3$. $y(0) = 1$. Find $y(0.4)$. Can you find $y(0.5)$?

Homework 7

1. Are the three functions $\cos(2x)$, $2\cos^2(x)$, $5\sin^2(x)$ linearly independent or not?

2. Consider three functions $y_1(x) = x^2 + x + 1$, $y_2(x) = x + 1$, $y_3(x) = x - 1$. Are they linearly dependent or not?

3-8. Find the general solutions in problems 3 through 8.

   3. $y''' - 6y'' + 10y' = 0$.
   4. $y^{(4)} - y = 0$.
   5. $y^{(4)} + 2y'' + y = 0$.
   6. $y^{(4)} - 2y'' + y = 0$.
   7. $y^{(3)} + 2y'' + 2y' + y = 0$.
   8. $y^{(3)} - 2y' - 4y = 0$.

9-11. In problems 9 through 11, a mass is attached to both a spring (with given spring constant k) and a dashpot (with given damping constant c). The mass is set in motion with initial position x_0 and initial velocity v_0.

   (a) Find the position function $x(t)$ and determine whether the motion is overdamped, critically damped, or underdamped.

   (b) If it is underdamped, write the position function in the form $x(t) = C_1 e^{-pt}\cos(\omega_1 t - \alpha_1)$.

   (c) Find the undamped position function $u(t) = C_0\cos(\omega_0 t - \alpha_0)$ that would result if the mass on the spring were set in motion with the same initial position and velocity, but with the dashpot disconnected (so c=0).

   (d) Use Mathematica to plot the graphs that illustrate the effect of damping by comparing the graphs of $x(t)$ and $u(t)$.

   9. $m = 1, c = 4, k = 3$; $x_0 = 2$, $v_0 = -2$.
   10. $m = 1, c = 4, k = 4$; $x_0 = 2$, $v_0 = -2$.
   11. $m = 1, c = 4, k = 5$; $x_0 = 2$, $v_0 = -2$.

12 A body weighing 100 is oscillating attached to a spring and a dashpot. Its first two maximum displacements of 6 and 2 are observed to occur at times 1 and 2, respectively. Compute the damping constant and spring constant.

Homework 9

1. Consider a forced mass-spring system with equation $mx''+kx = F_0\sin(\omega t)$ with $m = 1$, $k = 25$, $F_0 = 50$, $x_0 = 0$, $v_0 = 0$. Solve the system for each ω. Manipulate the system using Mathematica with changing ω to visualize the **beats** and **resonance**.
   (a) $\omega = 4$.
   (b) $\omega = 5$.
   (c) $\omega = 6$.

2. Consider the forced mass-spring-dashpot system with equation $mx'' + cx' + kx = F_0\sin(\omega t)$ with $m = 1, c = 2, k = 50$, and $F_0 = 100$, and initial conditions $x(0) = 10$, $x'(0) = 10$.
   (a) Find the transient solution (i.e. general solution of the homogeneous differential equation).
   (b) If $\omega = 5$, what's the steady periodic solution? Write it in the standard form involving the amplitude, frequency and phase angle. Then write down the general solution to the initial value problem.
   (c) Find the amplitude $C(\omega)$ of steady periodic forced oscillations with frequency ω and find the practical resonance frequency ω. Using Mathematica to plot the graph of $C(\omega)$ and to manipulate the system using Mathematica with changing ω to visualize the **practical resonance**.

3. Transform the given differential equation or system into an equivalent system of first-order differential equations. Is the system linear or non-linear? If linear, is it homogeneous or non-homogeneous? Determine how many initial conditions are needed to determine a unique solution.
   (a) $mx'' + cx' + kx = f(t)$. $(x = x(t))$
   (b) $x''' - 6x'' + 10x' = x^2$. $(x = x(t))$
   (c) $x'' - x - y = 0, y'' + x + y = 0$. $(x = x(t), y = y(t))$
   (d) $x'' + x' - x - y = 0, y'' + y' + x + y = \cos(t)$. $(x = x(t), y = y(t))$

4. Exercise 24 in Section 4.1. Generalize it to the case of three objects and four springs.

Midterm II Practice Problems

1. Consider the non-homogeneous linear differential equation with constant coefficients:
$$y'' + 2y' + y = e^{-x} + xe^{x}. \tag{1}$$
   (a) Write down the general complementary solution.
   (b) Use the method of undetermined coefficients to find a particular solution to the equation (1).
   (c) Find the solution of (1) satisfying the initial condition:
$$y(0) = 0, \quad y'(0) = 0.$$

2. Use the variation of parameters to find a particular solution to the equation:
$$y'' + 25y = \sec(5x).$$

3. Consider a (forced) spring-mass-dashpot system: $x''(t) + cx'(t) + 4x(t) = F_0 \cos(2t)$. Suppose the object is released from still when the spring is stretched by 10 unit length.
   (a) If the dashpot is disconnected and $F_0 = 0$, solve the system and classify the phenomenon.
   (b) If the dashpot is disconnected and $F_0 = 16$, solve the system and classify the phenomenon.
   (c) If the damping constant $c = 2$ and $F_0 = 0$, solve the system and classify the phenomenon.
   (d) If the damping constant $c = 2$ and $F_0 = 16$, solve the system and classify the phenomenon.

4. Assume $x = x(t)$, $y = y(t)$ satisfy the following system of differential equations:
$$x'' + x' + y' + x + y = 0, y'' + y' + x' + y + x = 0.$$
   Transform this into an equivalent system of 1st order differential equations. Determine how many initial conditions are needed to determine a unique solution.

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1. (40pts) Consider the non-homogeneous linear differential equation:

$$y''(x) - 9y(x) = 2e^{3x}. \tag{1}$$

(a). Find the general **complementary** solution.

Associated homogeneous DE: $y''(x) - 9y(x) = 0$.

Characteristic eq.: $\lambda^2 - 9 = (\lambda-3)(\lambda+3) = 0$.

$\Rightarrow$

root	3	-3
mult.	1	1

$\Rightarrow y_1(x) = e^{3x}, \ y_2(x) = e^{-3x}$

$\Rightarrow y_c(x) = C_1 \cdot e^{3x} + C_2 \cdot e^{-3x}$ is the general complementary solution.

(b). Find a particular solution of the equation (1) using the method of undetermined coefficients.

$f(x) = 2 \cdot e^{3x} = P_m(x) \cdot e^{\mu x} \Rightarrow P_m(x) = 2 \ (m=0), \ \mu = 3 \ (k=1)$

so $y_p(x) = x \cdot A \cdot e^{3x} \Rightarrow y_p'(x) = A \cdot (e^{3x} + 3x \cdot e^{3x})$

$\Rightarrow y_p''(x) = A \cdot (3 \cdot e^{3x} + 3 \cdot e^{3x} + 9x \cdot e^{3x})$.

$\Rightarrow y_p''(x) - 9y_p(x) = 6A \cdot e^{3x} = 2 \cdot e^{3x} \Rightarrow A = \frac{1}{3}$.

so $y_p(x) = \frac{1}{3} x \cdot e^{3x}$

2. (40pts)

Use the method of variation of parameters to find a particular solution to the following linear equation:

$$y'' + y = \sec^3(x).$$

by using the following formula (the notations won't be explained in the exam)

$$y_p(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}dx.$$

Associated homogeneous DE: $y''+y=0 \Rightarrow$ Characteristic eq.:

$$\lambda^2+1=0$$

$$y_1(x)=\cos x,\quad y_2(x)=\sin x. \Leftarrow \quad \overset{\shortparallel}{(\lambda-i)(\lambda+i)}$$

Wronskian: $W(x)=W(y_1,y_2)(x)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}=1.$

$$\int \frac{y_2(x)f(x)}{W(x)}dx = \int \frac{\sin x\cdot \sec^3 x}{1}dx = \int \frac{\sin x}{\cos^3 x}dx = -\int \frac{d(\cos x)}{(\cos x)^3}.$$

$$\overset{u=\cos x}{=} -\int u^{-3}du = \frac{1}{2}\cdot u^{-2} = \frac{1}{2}\cdot \sec^2 x.$$

$$\int \frac{y_1(x)f(x)}{W(x)}dx = \int \frac{\cos x\cdot \sec^3 x}{1}dx = \int \sec^2 x\,dx = \tan(x).$$

plugging these into the formula we get:

$$y_p(x) = -\frac{1}{2}\cdot \sec^2 x\cdot \cos x + \sin x\cdot \tan x$$

$$= \boxed{-\frac{1}{2}\sec x + \sin x\cdot \tan x.}$$

3. (40pts) Consider the free mass-spring-dashpot system

$$x'' + 2x' + 5x = 0.$$

The object is released from still when the spring is **compressed** by 10 unit length. Solve the system as an initial value problem and classify the phenomenon. Write the solution in the standard form: $x(t) = A(t)\cos(\omega t - \alpha)$.

Initial value problem $\begin{cases} x''+2x'+5x=0 \\ x(0)=-10,\ x'(0)=0. \end{cases}$

$\Rightarrow \lambda^2+2\lambda+5=0 \Rightarrow \lambda = \dfrac{-2\pm\sqrt{2^2-4\times 5}}{2} = -1\pm 2i.$

$\Rightarrow x(t) = C_1\cdot x_1(t)+C_2\cdot x_2(t) = C_1\cdot e^{-t}\cos(2t)+C_2\cdot e^{-t}\cdot\sin(2t).$

$\Rightarrow \begin{cases} -10 = x(0) = C_1 \\ 0 = x'(0) = \left[-C_1 e^{-t}\cos(2t) - C_1\cdot e^{-t}\cdot 2\sin(2t) - C_2 e^{-t}\sin(2t) + C_2 e^{-t}\cdot 2\cos(2t)\right]_{t=0} \\ \quad = -C_1+2C_2 \end{cases}$

$\Rightarrow \begin{cases} C_1=-10 \\ C_2=\frac{C_1}{2}=-5 \end{cases}$ so:

$x(t) = -10\cdot e^{-t}\cdot\cos(2t) - 5\cdot e^{-t}\cdot\sin(2t) = -5e^{-t}\cdot[2\cdot\cos(2t)+\sin(2t)]$

$= -5\sqrt{5}\cdot\left(\frac{2}{\sqrt{5}}\cos(2t)+\frac{1}{\sqrt{5}}\cdot\sin(2t)\right)\cdot e^{-t}.$

$= -5\sqrt{5}\cdot e^{-t}\cdot\cos\left(2t-\tan^{-1}\left(\frac{1}{2}\right)\right).$

$\boxed{\begin{aligned} &= 5\sqrt{5}\cdot e^{-t}\cdot\cos\left(2t-\left(\tan^{-1}\left(\tfrac{1}{2}\right)+\pi\right)\right) \\ \text{or } &= 5\sqrt{5}\cdot e^{-t}\cdot\cos\left(2t-\left(\tan^{-1}\left(\tfrac{1}{2}\right)-\pi\right)\right). \end{aligned}}$

Since $(\cos(\beta\pm\pi) = -\cos\beta)$.

4. (40pts) Consider the undamped forced mass-spring system

$$x'' + x = 4\sin(\omega t).$$

At time $t = 0$, the object is at the equilibrium position with zero velocity. What value of ω will trigger the resonance? Solve the system for this value of ω.

The associated homogeneous DE: $x'' + x = 0$.

Characteristic eq.: $\lambda^2 + 1 = (\lambda - i)(\lambda + i) = 0 \Rightarrow$

root	i	$-i$
mult.	1	1

$\Rightarrow x_1(t) = \cos(t)$, $x_2(t) = \sin(t)$.

$\Rightarrow$ The natural frequency is $\omega_0 = 1$. There will be a resonance when the frequency of the external force coincides with the natural frequency i.e. when $\omega = \omega_0 = 1$. So we need to solve:

$$x''(t) + x = 4\sin t, \quad x(0) = 0, \quad x'(0) = 0.$$

$f(t) = 4\cdot\sin(t) = P_m(t)e^{at}\sin(bt) \Rightarrow P_m(t) = 4 \ (m=0)$. $a + bi = 0 + 1\cdot i = i \ (k=1)$

So $x_p(t) = t\cdot(A\cdot\cos t + B\cdot\sin t) \Rightarrow x_p'(t) = A\cos t + B\sin t + t(-A\sin t + B\cos t)$.

$\Rightarrow x_p''(t) = -2A\sin t + 2B\cos t + t\cdot(-A\cos t - B\sin t)$.

$\Rightarrow x_p'' + x_p = -2A\cdot\sin t + 2B\cdot\cos t = 4\sin t \Rightarrow A = -2, B = 0$.

$\Rightarrow x_p(t) = -2t\cdot\cos t \Rightarrow x(t) = C_1\cdot x_1(t) + C_2\cdot x_2(t) + x_p(t)$.

$= C_1\cdot\cos t + C_2\cdot\sin t - 2t\cdot\cos t$.

Using the initial conditions:

$0 = x(0) = C_1$, $0 = x'(0) = [-C_1\cdot\sin t + C_2\cos t - 2(\cos t - t\cdot\sin t)]_{t=0} = C_2 - 2$.

$\Rightarrow C_1 = 0$, $C_2 = 2$. So:

$$\boxed{x(t) = 2\cdot\sin t - 2t\cos t}$$

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5. (40pts) Assume $y = y(x)$. Transform the following equation into a system of first order differential equations:

$$y''' + y'' + y + x = 0.$$

Is it a linear or non-linear system? Is it homogeneous or non-homogenous? How many initial conditions are needed to determine a unique solution?

Let $y_1(x) = y(x)$, $y_2(x) = y'(x) = y_1'(x)$.

$y_3(x) = y''(x) = y_2'(x)$.

$\Rightarrow y_3'(x) = y'''(x) = -y''(x) - y(x) - x$

$= -y_3(x) - y_1(x) - x$.

$$\Rightarrow \begin{cases} y_1'(x) = y_2(x) \\ y_2'(x) = y_3(x) \\ y_3'(x) = -y_1(x) - y_3(x) - x. \end{cases}$$

This is a linear, non-homogeneous system.

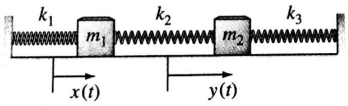
3 conditions are needed to determine a unique solution.

Homework 10

1. Use elimination method to solve the following systems
   (a) $x' = x - y,\ y' = -x + y,\ x(0) = 2,\ y(0) = 1.$
   (b) $x' = x + y,\ y' = -x + y,\ x(0) = 2,\ y(0) = 1.$

2. Consider the system of two masses and three springs shown in the figure.
   (a) Derive the equations of motion.

$$\begin{aligned} m_1 x'' &= -(k_1 + k_2)x + k_2 y, \\ m_2 y'' &= k_2 x - (k_2 + k_3)y. \end{aligned}$$



   (b) Assume $m_1 = 2,\ m_2 = 1,\ k_1 = 10,\ k_2 = 20,\ k_3 = 10$. Use the elimination method to solve the system. Find the natural frequencies of the mass-and-spring system and describe its natural modes of oscillation.
   (c) Transform the system in part (b) into a system of 1st order differential equations. Then write the transformed system into the form $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}$.

3. Let

$$\mathbf{A} = \begin{pmatrix} 1 & -4 \\ 1 & -3 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} -3 & 4 \\ -1 & 1 \end{pmatrix}, \quad \vec{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

   (a) Calculate the following expressions:

$$2\mathbf{A} - 3\mathbf{B}, \mathbf{AI}, \mathbf{IB}, \mathbf{A^2}, \mathbf{AB}, \mathbf{BA}, \mathbf{AC}, \mathbf{CA}, (\mathbf{AB})\mathbf{C}, \mathbf{A}(\mathbf{BC}), \mathbf{A}\vec{x}_1.$$

   (b) Answer the following questions: Is $\mathbf{AI} = \mathbf{A}$? Is $\mathbf{IB} = \mathbf{B}$? Is $\mathbf{AB} = \mathbf{BA}$? Is $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$? What's the relation between $\mathbf{A}$ and $\mathbf{C}$? What's the relation between $\mathbf{A}$ and $\vec{x}_1$? What properties do you expect to hold for matrix operations by these calculations and answer to above questions?
   (c) Find the eigenvalues and eigenvectors of $\mathbf{B}$.

4. Consider the general 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Verify that the inverse of $\mathbf{A}$ is given by:

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

5. Using the eigenvalue method to re-solve the questions in Problem 1.

1. $\begin{cases} x' = x - 2y & ① \\ y' = 2x + 5y & ② \end{cases}$

$① \Rightarrow 2y = -x' + x \Rightarrow y = -\frac{1}{2}x' + \frac{1}{2}x$

$y' = -\frac{1}{2}x'' + \frac{1}{2}x' = 2x + 5(-\frac{1}{2}x' + \frac{1}{2}x)$

$\Rightarrow x'' - 6x' + 9x = 0$

$\lambda^2 - 6\lambda + 9 = 0$

$(\lambda - 3)^2 = 0$

root	3
mult	2

$\boxed{x(t) = c_1 e^{3t} + c_2 t e^{3t}}$

$x'(t) = 3c_1 e^{3t} + c_2 e^{3t} + 3c_2 t e^{3t}.$

$y = -\frac{1}{2}x' + \frac{1}{2}x = \boxed{-c_1 e^{3t} - c_2 t e^{3t} - \frac{1}{2}c_2 e^{3t}}$ ✓

2a Let $A = \begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix}$

~~$\det(A - \lambda I) = \begin{pmatrix} 1-\lambda & -2 \\ 2 & 5-\lambda \end{pmatrix}$~~

$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -2 \\ 2 & 5-\lambda \end{vmatrix} = (1-\lambda)(5-\lambda) - (-2)(2) = 0$

$\Rightarrow \lambda^2 - 6\lambda + 9 = 0 \Rightarrow \boxed{\lambda = 3}$ with multiplicity 2.

when $\lambda = 3$

$(A - \lambda I) = \begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix}$

$(A - \lambda I)v = 0.$ Let $v = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

$\begin{cases} -2u_1 - 2u_2 = 0 \\ 2u_1 + 2u_2 = 0 \end{cases} \Rightarrow \begin{cases} u_1 = 1 \\ u_2 = -1 \end{cases}$

$\boxed{v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$ is an eigenvector associated with $\lambda = 3$

2b $A = \begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix}$

$$(A-3I) = \begin{pmatrix} 1-3 & -2 \\ 2 & 5-3 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix}$$

$$(A-3I)^2 = \begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4-4 & 4-4 \\ -4+4 & -4+4 \end{pmatrix} = 0. = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Homework 11

1. Consider the linear system with constant coefficient matrix $\mathbf{A}$.
$$\frac{d}{dt}\vec{x}(t) = \mathbf{A}\vec{x}(t).$$
   (a) Show that a vector valued function $\vec{x}(t) = \vec{v}(t)e^{\lambda t}$ is a solution to the above system if and only if $\vec{v}(t)$ satisfies the following identity:
$$\frac{d}{dt}\vec{v}(t) = (\mathbf{A} - \lambda\mathbf{I})\vec{v}(t). \tag{1}$$
   (b) Consider the following vector-valued polynomial function
$$\vec{v}(t) = \frac{\vec{v}_1 t^{k-1}}{(k-1)!} + \frac{\vec{v}_2 t^{k-2}}{(k-2)!} + \cdots + \frac{\vec{v}_{k-2} t^2}{2!} + \vec{v}_{k-1} t + \vec{v}_k.$$
   Show that this $\vec{v}(t)$ satisfies the identity (1) if and only if the chain of vectors $\{\vec{v}_1, \cdots, \vec{v}_k\}$ satisfies the following relations:
$$(\mathbf{A} - \lambda\mathbf{I})\vec{v}_k = \vec{v}_{k-1}, \quad (\mathbf{A} - \lambda\mathbf{I})\vec{v}_{k-1} = \vec{v}_{k-2}, \quad \cdots \quad (\mathbf{A} - \lambda\mathbf{I})\vec{v}_2 = \vec{v}_1, \quad (\mathbf{A} - \lambda\mathbf{I})\vec{v}_1 = 0.$$
   As a consequence, show that $\vec{v}_k$ is a generalized eigenvector:
$$(\mathbf{A} - \lambda\mathbf{I})^k\vec{v}_k = 0,$$

2. Apply the eigenvalue method to find the general solution of the following system. Then find the corresponding particular solution satisfying the initial conditions. Use the command **StreamPlot** of Mathematica to plot the direction field and typical solution curves for the system.
   (a) $x' = x + y$, $y' = -x + y$, $x(0) = 2$, $y(0) = 1$.
   (b) $x_1' = 3x_1 + 4x_2$, $x_2' = 5x_1 + 2x_2$, $x_1(0) = 2$, $x_2(0) = 1$.
   (c) $x_1' = 4x_1 + x_2$, $x_2' = -4x_1$, $x_1(0) = 2$, $x_2(0) = 1$.
   (d) $x_1' = 4x_1 + x_2$, $x_2' = -x_1 + 2x_2$, $x_1(0) = 2$, $x_2(0) = 1$.

3. Solve the following system:
   (a) Use elimination method to find the general solution of the following system.
$$\begin{cases} x_1' = 2x_1 + 2x_2 \\ x_2' = -x_2 + x_3 \\ x_3' = 2x_3. \end{cases}$$
   (b) Use eigenvalue method to find the general solution of the following system.
$$\begin{cases} x_1' = 3x_1 + x_2 \\ x_2' = -4x_1 - x_2 \\ x_3' = 4x_1 - 8x_2 - 2x_3 \end{cases}$$

4. Use Mathematica to find the eigenvalues and eigenvectors of the following matrix in order to find a general solution of the linear system $\vec{x}' = \mathbf{A}\vec{x}$.

$$\mathbf{A} = \begin{pmatrix} 15 & -21 & 24 & -6 & -22 \\ 0 & -2 & 8 & -12 & -4 \\ 10 & -19 & 21 & -4 & -18 \\ 0 & -14 & 16 & -4 & -8 \\ 20 & -30 & 30 & 0 & -30 \end{pmatrix}$$

Chains of generalized eigenvectors

Principle: For a **fixed** eigenvalue

- Number of Chains= Number of Eigenvectors=Multiplity-Defect;
- Sum of Length of Chains = Multiplicity of the Eigenvalue.

Caveat: Number of Eigenvectors, denoted by "# EVector" in the following charts, means the Number of **Linearly Independent** Eigenvectors.

- For 2×2 matrices, there are 3 possible cases:

  Case 1. Example (Jordan form): $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

EValue	Mult.	# EVector	Chain	Basic Solution
λ_1	1	1	$v_1 \to 0$	$e^{\lambda_1 t} v_1$
λ_2	1	1	$v_2 \to 0$	$e^{\lambda_2 t} v_2$

  Case 2. Example (Jordan form): $\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}$

EValue	Mult.	# EVector	Chain	Basic Solution
λ_1	2	1	$v_1 \to v_2 \to 0$	$e^{\lambda_2 t}(v_1 + v_2 t)$, $e^{\lambda_1 t} v_2$

  Case 3. (Happens **only** for $A = \lambda_1 \mathbf{I}$)

EValue	Mult.	# EVector	Chain	Basic Solution
λ_1	2	2	$v_1 \to 0$, $v_2 \to 0$	$e^{\lambda_1 t} v_1, e^{\lambda_1 t} v_2$.

- For 3×3 matrices, there are 6 possible cases:

  Case 1. Example (Jordan form): $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$

EValue	Mult.	# EVector	Chain	Basic Solution
λ_1	1	1	$v_1 \to 0$	$e^{\lambda_1 t} v_1$
λ_2	1	1	$v_2 \to 0$	$e^{\lambda_2 t} v_2$
λ_3	1	1	$v_3 \to 0$	$e^{\lambda_3 t} v_3$

  Case 2. Example (Jordan form): $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{pmatrix}$

EValue	Mult.	# EVector	Chain	Basic Solution
λ_1	1	1	$v_1 \to 0$	$e^{\lambda_1} v_1$
λ_2	2	1	$v_2 \to v_3 \to 0$	$e^{\lambda_2 t}(v_2 + v_3 t)$, $e^{\lambda_2 t} v_3$

Case 3. Example (Jordan form): $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$

EValue	Mult.	# EVector	Chain	Basic Solution
λ_1	1	1	$v_1 \to 0$	$e^{\lambda_1 t} v_1$
λ_2	2	2	$v_2 \to 0$	$e^{\lambda_2 t} v_2$
			$v_3 \to 0$	$e^{\lambda_2 t} v_3$

Case 4. Example (Jordan form): $\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}$

EValue	Mult.	# EVector	Chain	Basic Solution
λ_1	3	1	$v_1 \to v_2 \to v_3 \to 0$	$e^{\lambda_1 t}(v_1 + tv_2 + \frac{t^2}{2} v_3), e^{\lambda_1 t}(v_2 + tv_3), e^{\lambda_1 t} v_3$

Case 5. Example (Jordan form): $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}$

EValue	Mult.	# EVector	Chain	Basic Solution
λ_1	3	2	$v_1 \to 0$	$e^{\lambda_1 t} v_1$
			$v_2 \to v_3 \to 0$	$e^{\lambda_1 t}(v_2 + tv_3), e^{\lambda_1 t} v_3$

Case 6. (Happens **only** for $A = \lambda_1 \mathbf{I}$)

EValue	Mult.	# EVector	Chain	Basic Solution
λ_1	3	3	$v_1 \to 0,\ v_2 \to 0,\ v_3 \to 0$	$e^{\lambda_1 t} v_1, e^{\lambda_1 t} v_2, e^{\lambda_1 t} v_3.$

Lazier (rougher) way to write down basic solutions: For any fixed eigenvalue λ of multiplicity m. One can calculate the set of basic solutions as follows

1. Calculate $(A - \lambda I)^m$.
2. Find m **linearly independent** *generalized eigenvectors* $\{v_1, \cdots, v_m\}$. This means that:

$$(A - \lambda I)^m v_i = 0, \text{ for each } i = 1, \cdots, m.$$

3. Write down the m basic solutions for each v_i:

$$\begin{aligned} x_i(t) &= e^{\lambda t} \Big[v_i + t(A - \lambda I) v_i + \frac{t^2}{2!}(A - \lambda)^2 v_i + \cdots + \\ &\quad + \frac{t^{m-2}}{(m-2)!}(A - \lambda I)^{m-2} v_i + \frac{t^{m-1}}{(m-1)!}(A - \lambda I)^{m-1} v_i \Big]. \end{aligned}$$

Homework 12

1. Use the eigenvalue method to solve the system (a)-(d) following the steps:
   - Calculate the eigenvalues/eigenvectors.
   - Determine the number of chains and the length of each chain.
   - Try to find the chains by calculating the generalized eigenvectors.
   - Write down basic solutions and the general solutions. (You could use "lazier way" to do this.)

$$(\mathbf{a}).\begin{cases} x_1' = 4x_1 + x_2 \\ x_2' = -2x_1 + x_2 \\ x_3' = x_1 + x_2 + x_3 \end{cases} \qquad (\mathbf{b}).\begin{cases} x_1' = 4x_1 + x_2 \\ x_2' = -2x_1 + x_2 \\ x_3' = x_1 + x_2 + 3x_3 \end{cases}$$

$$(\mathbf{c}).\begin{cases} x_1' = 4x_1 + x_2 \\ x_2' = -x_1 + 2x_2 \\ x_3' = x_1 + x_2 + 3x_3 \end{cases} \qquad (\mathbf{d}).\begin{cases} x_1' = 4x_1 + x_2 \\ x_2' = -x_1 + 2x_2 \\ x_3' = -x_1 + x_2 + 3x_3 \end{cases}$$

2. Calculate the exponentials of following matrices using the definition of exponential.

$$(\mathbf{a}).\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (\mathbf{b}).\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \quad (\mathbf{c}).\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \quad (\mathbf{d}).\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

3. For each of the following matrix A, calculate e^{tA} using a fundamental solution matrix. Then use e^{tA} to calculate the solution to the initial value prolem:

$$\frac{d\vec{x}(t)}{dt} = \mathbf{A}\vec{x}(t), \quad \vec{x}(0) = \begin{pmatrix} -3 \\ 2 \end{pmatrix}.$$

$$(\mathbf{a}).\begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix}, \quad (\mathbf{b}).\begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}, \quad (\mathbf{c}).\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}.$$

4. Solve the nonhomogeneous system:

   (a)
$$\frac{d\vec{x}(t)}{dt} = \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix}\vec{x}(t) + \begin{pmatrix} e^t \\ e^{3t} \end{pmatrix}.$$

   (b)
$$\frac{d\vec{x}(t)}{dt} = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}\vec{x}(t) + \begin{pmatrix} e^t \\ e^{3t} \end{pmatrix}.$$

39. $xy' + y = 3x^2$ **40.** $(y')^2 + y^2 = 1$

41. $y' + y = e^x$ **42.** $y'' + y = 0$

43. **(a)** If k is a constant, show that a general (one-parameter) solution of the differential equation

$$\frac{dx}{dt} = kx^2$$

is given by $x(t) = 1/(C - kt)$, where C is an arbitrary constant.

(b) Determine by inspection a solution of the initial value problem $x' = kx^2$, $x(0) = 0$.

44. **(a)** Continuing Problem 43, assume that k is positive, and then sketch graphs of solutions of $x' = kx^2$ with several typical positive values of $x(0)$.

(b) How would these solutions differ if the constant k were negative?

45. Suppose a population P of rodents satisfies the differential equation $dP/dt = kP^2$. Initially, there are $P(0) = 2$ rodents, and their number is increasing at the rate of $dP/dt = 1$ rodent per month when there are $P = 10$ rodents. How long will it take for this population to grow to a hundred rodents? To a thousand? What's happening here?

46. Suppose the velocity v of a motorboat coasting in water satisfies the differential equation $dv/dt = kv^2$. The initial speed of the motorboat is $v(0) = 10$ meters per second (m/s), and v is decreasing at the rate of 1 m/s^2 when $v = 5$ m/s. How long does it take for the velocity of the boat to decrease to 1 m/s? To $\frac{1}{10}$m/s? When does the boat come to a stop?

47. In Example 7 we saw that $y(x) = 1/(C - x)$ defines a one-parameter family of solutions of the differential equation $dy/dx = y^2$. **(a)** Determine a value of C so that $y(10) = 10$. **(b)** Is there a value of C such that $y(0) = 0$? Can you nevertheless find by inspection a solution of $dy/dx = y^2$ such that $y(0) = 0$? **(c)** Figure 1.1.8 shows typical graphs of solutions of the form $y(x) = 1/(C - x)$.

32. Suppose that a car skids 15 m if it is moving at 50 km/h when the brakes are applied. Assuming that the car has the same constant deceleration, how far will it skid if it is moving at 100 km/h when the brakes are applied?

33. On the planet Gzyx, a ball dropped from a height of 20 ft hits the ground in 2 s. If a ball is dropped from the top of a 200-ft-tall building on Gzyx, how long will it take to hit the ground? With what speed will it hit?

34. A person can throw a ball straight upward from the surface of the earth to a maximum height of 144 ft. How high could this person throw the ball on the planet Gzyx of Problem 29?

35. A stone is dropped from rest at an initial height h above the surface of the earth. Show that the speed with which it strikes the ground is $v = \sqrt{2gh}$.

36. Suppose a woman has enough "spring" in her legs to jump (on earth) from the ground to a height of 2.25 feet. If she jumps straight upward with the same initial velocity on the moon—where the surface gravitational acceleration is (approximately) 5.3 ft/s^2—how high above the surface will she rise?

37. At noon a car starts from rest at point A and proceeds at constant acceleration along a straight road toward point B. If the car reaches B at 12:50 P.M. with a velocity of 60 mi/h, what is the distance from A to B?

38. At noon a car starts from rest at point A and proceeds with constant acceleration along a straight road toward point C, 35 miles away. If the constantly accelerated car arrives at C with a velocity of 60 mi/h, at what time does it arrive at C?

39. If $a = 0.5$ mi and $v_0 = 9$ mi/h as in Example 4, what must the swimmer's speed v_S be in order that he drifts only 1 mile downstream as he crosses the river?

40. Suppose that $a = 0.5$ mi, $v_0 = 9$ mi/h, and $v_S = 3$ mi/h as in Example 4, but that the velocity of the river is given by the fourth-degree function

$$v_R = v_0\left(1 - \frac{x^4}{a^4}\right)$$

38. At noon ... constant acceleration ... If the constant ... 35 miles away. If the ... at what time does it arrive ... C with a velocity of 60 mi/h, at what time does it arrive at C?

39. If $a = 0.5$ mi and $v_0 = 9$ mi/h as in Example 4, what must the swimmer's speed v_S be in order that he drifts only 1 mile downstream as he crosses the river?

40. Suppose that $a = 0.5$ mi, $v_0 = 9$ mi/h, and $v_S = 3$ mi/h as in Example 4, but that the velocity of the river is given by the fourth-degree function

$$v_R = v_0\left(1 - \frac{x^4}{a^4}\right)$$

rather than the quadratic function in Eq. (18). Now find how far downstream the swimmer drifts as he crosses the river.

41. A bomb is dropped from a helicopter hovering at an altitude of 800 feet above the ground. From the ground directly beneath the helicopter, a projectile is fired straight upward toward the bomb, exactly 2 seconds after the bomb is released. With what initial velocity should the projectile be fired, in order to hit the bomb at an altitude of exactly 400 feet?

42. A spacecraft is in free fall toward the surface of the moon at a speed of 1000 mph (mi/h). Its retrorockets, when fired, provide a constant deceleration of 20,000 mi/h^2. At what height above the lunar surface should the astronauts fire the retrorockets to insure a soft touchdown? (As in Example 2, ignore the moon's gravitational field.)

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t. How long
e projectile,

and how far will it have traveled by then?

44. A driver involved in an accident claims he was going only 25 mph. When police tested his car, they found that when its brakes were applied at 25 mph, the car skidded only 45 feet before coming to a stop. But the driver's skid marks at the accident scene measured 210 feet. Assuming the same (constant) deceleration, determine the speed he was actually traveling just prior to the accident.

ion Curves

a differential equation of the form

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

right-hand function $f(x, y)$ involves both the independent variable x and dent variable y. We might think of integrating both sides in (1) with re- , and hence write $y(x) = \int f(x, y(x))\,dx + C$. However, this approach ead to a solution of the differential equation, because the indicated integral he *unknown* function $y(x)$ itself, and therefore cannot be evaluated explic- ally, there exists *no* straightforward procedure by which a general differen- on can be solved explicitly. Indeed, the solutions of such a simple-looking al equation as $y' = x^2 + y^2$ cannot be expressed in terms of the ordinary y functions studied in calculus textbooks. Nevertheless, the graphical and l methods of this and later sections can be used to construct *approximate* of differential equations that suffice for many practical purposes.

Fields and Graphical Solutions

a simple geometric way to think about solutions of a given differential $y' = f(x, y)$. At each point (x, y) of the xy-plane, the value of $f(x, y)$

12. $\dfrac{dy}{dx} = x \ln y; \quad y(1) = 1$

13. $\dfrac{dy}{dx} = \sqrt[3]{y}; \quad y(0) = 1$

14. $\dfrac{dy}{dx} = \sqrt[3]{y}; \quad y(0) = 0$

15. $\dfrac{dy}{dx} = \sqrt{x - y}; \quad y(2) = 2$

16. $\dfrac{dy}{dx} = \sqrt{x - y}; \quad y(2) = 1$

17. $y\dfrac{dy}{dx} = x - 1; \quad y(0) = 1$

18. $y\dfrac{dy}{dx} = x - 1; \quad y(1) = 0$

19. $\dfrac{dy}{dx} = \ln(1 + y^2); \quad y(0) = 0$

20. $\dfrac{dy}{dx} = x^2 - y^2; \quad y(0) = 1$

In Problems 21 and 22, first use the method of Example 2 to construct a slope field for the given differential equation. Then sketch the solution curve corresponding to the given initial condition. Finally, use this solution curve to estimate the desired value of the solution $y(x)$.

21. $y' = x + y, \quad y(0) = 0; \quad y(-4) = ?$

22. $y' = y - x, \quad y(4) = 0; \quad y(-4) = ?$

$$\frac{dP}{dt} = 0.0225P - 0.0003P^2.$$

Construct a slope field and appropriate solution curve to answer the following questions: If there are 25 deer at time $t = 0$ and t is measured in months, how long will it take the number of deer to double? What will be the limiting deer population?

The next seven problems illustrate the fact that, if the hypotheses of Theorem 1 are not satisfied, then the initial value problem $y' = f(x, y)$, $y(a) = b$ may have either no solutions, finitely many solutions, or infinitely many solutions.

27. **(a)** Verify that if c is a constant, then the function defined piecewise by

$$y(x) = \begin{cases} 0 & \text{for } x \leqq c, \\ (x-c)^2 & \text{for } x > c \end{cases}$$

satisfies the differential equation $y' = 2\sqrt{y}$ for all x (including the point $x = c$). Construct a figure illustrating the fact that the initial value problem $y' = 2\sqrt{y}$, $y(0) = 0$ has infinitely many different solutions. **(b)** For what values of b does the initial value problem $y' = 2\sqrt{y}$, $y(0) = b$ have (i) no solution, (ii) a unique solution that is defined for all x?

28. Verify that if k is a constant, then the function $y(x) \equiv kx$ satisfies the differential equation $xy' = y$ for all x. Construct a slope field and several of these straight line solution curves. Then determine (in terms of a and b) how many different solutions the initial value problem $xy' = y$, $y(a) = b$ has—one, none, or infinitely many.

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Part II of Homework 1

1. Verify the following function is a general solution of the differential equation. Then find the constants C_1 and C_2 under the initial conditions:

   (a)
   $$y = C_1 e^{3x} + C_2 e^{-3x}, \quad y'' - 9y = 0;$$
   Initial conditions:
   $$y(0) = 0, \quad y'(0) = -3.$$

   (b)
   $$y = C_1 \cos(3x) + C_2 \sin(3x), \quad y'' + 9y = 0.$$
   Initial conditions:
   $$y(\pi/6) = 1, \quad y'(\pi/6) = -1.$$

2. Solve the following differential equations with initial conditions:
   $$y''' = x, \quad y(0) = a_0, y'(0) = a_1, y''(0) = a_2.$$
   Here a_0, a_1, a_2 are constants.

3. Does Theorem 1 guarantee local existence and local uniqueness of the solution to the initial value problem?

   (a) $y' = x \cdot y^{2/3}, \quad y(0) = 0.$

   (b) $y' = x^{2/3} \cdot y, \quad y(0) = 0.$

4. Use **Mathematica** to generate the slope fields of the following differential equations, and also streamlines passing through $(-1,-1), (-2,-2), (-2,2)$.

   (a) $y' = -y - \sin x.$

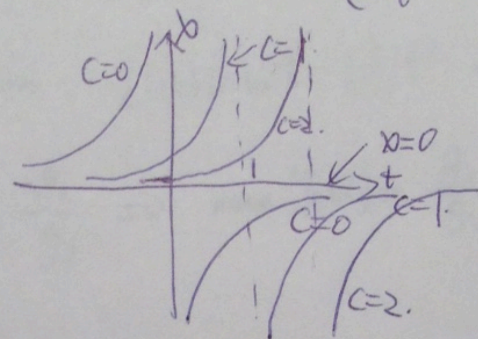
   (b) $y' = -x^2 + \sin y.$

   (c) $y' = x^2 - y.$

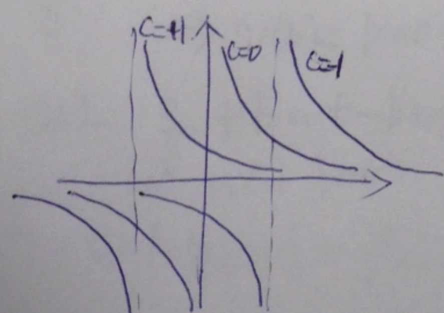
43. (a). $x(t) = \frac{1}{C-kt}$. $\frac{d}{dt}x(t) = \frac{k}{(C-kt)^2} = k \cdot x(t)^2$.

(b). A solution satisfying $x(0)=0$ is $x(t)=0$. Note that this is not included in the general solution in (a).

44. (a) Assume $k=1$ $x(t) = \frac{1}{C-t}$



$k=-1 \Rightarrow x(t) = \frac{1}{C+t}$



45. By 43. $P(t) = \frac{1}{C-kt}$. $P(0) = 2 = \frac{1}{C} \Rightarrow C = \frac{1}{2}$.

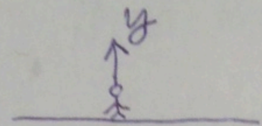
$\frac{dP}{dt}\left(= \frac{k}{(C-kt)^2}\right) = k \cdot P^2$. when $P=10$: $1 = \frac{dP}{dt} = k(10)^2 \Rightarrow k = \frac{1}{100} = 0.01$.

so $P(t) = \frac{1}{\frac{1}{2} - \frac{1}{100}t} = \frac{100}{50-t}$. $\Rightarrow t(P) = 50 - \frac{100}{P}$.

$t(100) = 49$ months. $t(1000) = 50 - \frac{1}{10} = 49.9$ months

Note that $\lim_{P\to\infty} t(P) = 50$. $P(t)$ is only defined when $t<50$.

36.



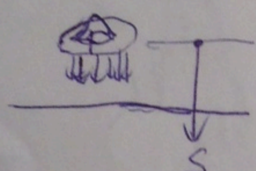
On the earth: $\frac{dy_1}{dt} = v(t)$. $\frac{dv}{dt} = -g_1$, $v(0) = v_0 \Rightarrow v(t) = v_0 - g_1 t$.

$y_1(t) = v_0 t - \frac{1}{2} g_1 t^2$. $\max y_1(t) = y_1\left(\frac{v_0}{g_1}\right) = \frac{v_0^2}{2g_1} = 2.25$

Similarly on the moon, $y_2(t) = v_0 t - \frac{1}{2} g_2 t^2$ and $\max y_2(t) = \frac{v_0^2}{2g_2}$.

So. $\frac{\max y_2(t)}{\max y_1(t)} = \frac{g_1}{g_2} \Rightarrow \max y_2(t) = \frac{g_1}{g_2} \cdot \max y_1(t) \approx \frac{32}{5.3} \times 2.25 \approx 13.58\,ft$.

42.



$\frac{ds}{dt} = v(t)$. $\frac{dv}{dt} = a = -20000$, $v(0) = 1000$.

$\Rightarrow v(t) = 1000 - 20000t = v_0 + at$.

$s(t) = \underset{\underset{0}{\parallel}}{S_0} + 1000t - 10000t^2$. $v(t_1) = 0 \Rightarrow t_1 = \frac{1}{20} = -\frac{v_0}{a}$.

$= v_0 t + \frac{1}{2} a t^2$.

$s(t_1) = v_0 \cdot \left(-\frac{v_0}{a}\right) + \frac{1}{2} a \cdot \frac{v_0^2}{a^2} = -\frac{v_0^2}{2a} = +\frac{(1000)^2}{2 \times 20000} = \frac{10^6}{4 \times 10^4} = 25\,mi$.

44. $y_i(t) = v_i t - \frac{1}{2} a t^2 \Rightarrow \max y_1 = \frac{v_1^2}{2a} = 45$.

$\max y_2 = \frac{v_2^2}{2a} = 210$.

So $\frac{\max y_1}{\max y_2} = \frac{v_1^2}{v_2^2} \Rightarrow v_2 = v_1 \sqrt{\frac{\max y_2}{\max y_1}} = 25 \sqrt{\frac{210}{45}}$

$\approx 54\,mi/h$.

1.3.22. $y' = y - x$.

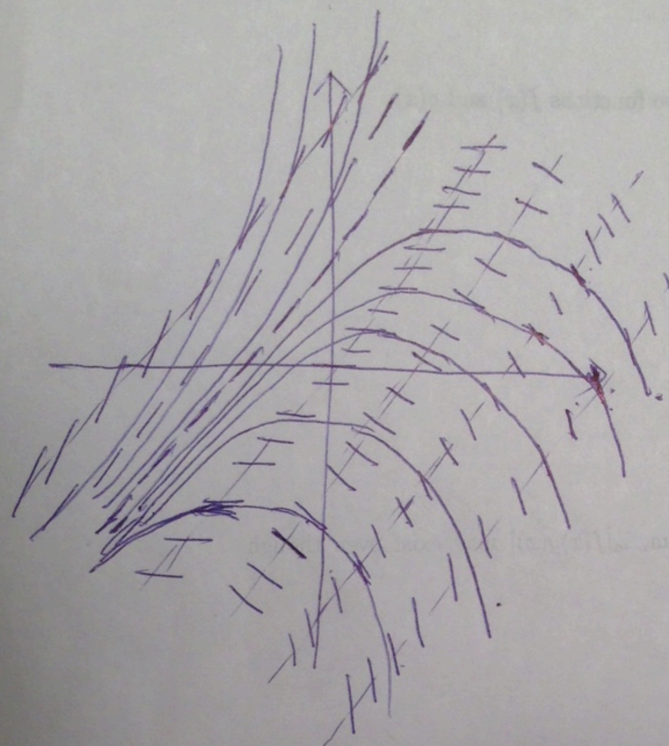
$k=0$. $y-x=0$

$k=1$ $y-x=1$

$k=-1$ $y-x=-1$

$k=2$. $y-x=2$

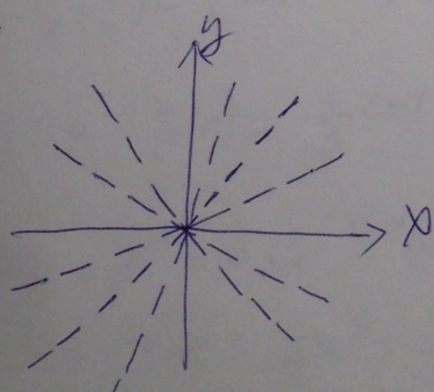
$\vdots$



$y = x+1$ is an asymptote for all solution curves.

$y(-4) \approx -3$.

28.



$y(a)=b$.

(i) $a \neq 0$: unique solution

(ii) $a=0$ and $b=0$: infinitely many solution.

(iii) $a=0$. $b \neq 0$: no solution.

Part II: 1. (a). $y'' = 9c_1 e^{3x} + 9c_2 e^{-3x} = 9y$.

$y(0) = 0 \Rightarrow c_1 + c_2 = 0$.

$y'(0) = 0 \Rightarrow 3c_1 - 3c_2 = -3$ $\Rightarrow c_1 = -\frac{1}{2},\ c_2 = \frac{1}{2}$.

$(y_1' = 3c_1 \cdot e^{3x} - 3c_2 \cdot e^{3x})$ so the particular solution satisfying the initial condition is $y(x) = -\frac{1}{2} e^{3x} + \frac{1}{2} e^{-3x}$.

(b). $y' = -3c_1 \sin(3x) + 3c_2 \cdot \cos(3x)$

$y'' = -9 \cdot c_1 \cos(3x) - 9c_2 \cdot \sin(3x) = -9 \cdot y$.

$y(\frac{\pi}{6}) = 1 \Rightarrow c_1 \cdot 0 + c_2 \cdot 1 = 1 \Rightarrow c_2 = 1$.

$y'(\frac{\pi}{6}) = -1 \Rightarrow -3 \cdot c_1 = -1 \Rightarrow c_1 = \frac{1}{3}$.

so $y(x) = \frac{1}{3} \cdot \cos(3x) + \sin(3x)$.

2. $y''' = x \Rightarrow y''(x) = \frac{1}{2}x^2 + y''(0) = \frac{1}{2}x^2 + a_2$

$\Rightarrow y'(x) = \frac{1}{6}x^3 + a_2 x + y'(0) = \frac{1}{6}x^3 + a_2 x + a_1$

$\Rightarrow y(x) = \frac{1}{24}x^4 + \frac{1}{2}a_2 x^2 + a_1 x + y(0)$

$= \frac{1}{24}x^4 + \frac{1}{2}a_2 x^2 + a_1 x + a_0$.

Selected solutions

Part II: 3 Does Theorem 1 guarantee local existence and local uniqueness of the solution to the initial value problem?

(a) $y' = x \cdot y^{2/3}, \quad y(0) = 0.$

Solution: Let $F(x,y) = x \cdot y^{2/3}$. $F(x,y)$ is continuous around the point $(0,0)$. So there exists a solution passing through $(0,0)$.

Now

$$F_y = \frac{\partial F}{\partial y} = x\frac{2}{3}y^{-1/3} = \frac{2x}{3y^{1/3}}.$$

F_y is not continuous at point $(0,0)$. It's not even defined at point $(0,0)$. So the Theorem 1 does not guarantee the solution to be unique.

If we solve the equation using separating variable method, we get:

$$\frac{dy}{y^{2/3}} = xdx \implies 3y^{1/3} = \frac{1}{2}x^2 + C_1 \implies y = \left(\frac{1}{6}x^2 + C\right)^3.$$

$y(0) = 0 \Rightarrow C = 0$. So we get a solution $y = (x^2/6)^3 = \frac{x^6}{216}$. But this is only one solution for the differential equation. The other solution is $y \equiv 0$. The reason that we miss this solution is that we divided $y^{2/3}$ on both sides of the original differential equation. So we were assuming $y \neq 0$.

(b) $y' = x^{2/3} \cdot y, \quad y(0) = 0.$

Solution: Let $F(x,y) = x^{2/3} \cdot y$. Because $F(x,y)$ continuous around $(0,0)$, there exists a solution passing through $(0,0)$. $F_y = x^{2/3}$ is also continuous around $(0,0)$, so the solution passing thourgh $(0,0)$ is unique by Theorem 1.

If we solve the equation using spearing variable method, we get:

$$\frac{dy}{y} = x^{2/3}dx \implies \ln y = \frac{3}{5}x^{5/3} + C_1 \implies y = Ce^{\frac{3}{5}x^{5/3}}.$$

$y(0) = 0 \Rightarrow C = 0$. So we get the unique solution: $y \equiv 0$. In this special example, we don't miss $y \equiv 0$.

But you can try to solve

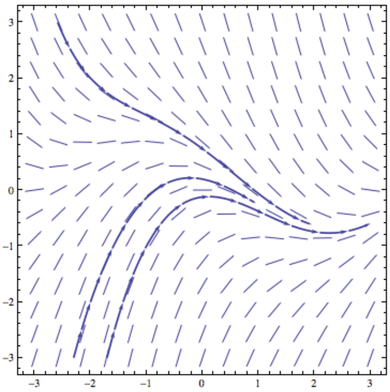
$$y' = y^2, y(0) = 0,$$

to see that you could miss the $y \equiv 0$ solution.

4

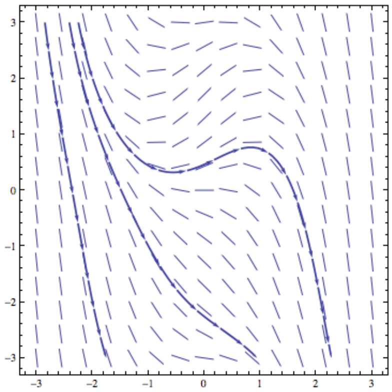
```
In[138]:= VectorPlot[{1, -y - Sin[x]}, {x, -3, 3}, {y, -3, 3}, VectorScale → {Tiny, Automatic, None}, VectorStyle → Arrowheads[0],
          StreamPoints → {{-1, -1}, {-2, -2}, {-2, 2}}, StreamStyle → Thickness[0.005]]
```

Out[138]=



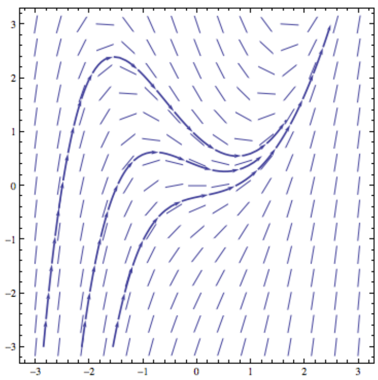
```
In[139]:= VectorPlot[{1, -x^2 + Sin[y]}, {x, -3, 3}, {y, -3, 3}, VectorScale → {Tiny, Automatic, None}, VectorStyle → Arrowheads[0],
          StreamPoints → {{-1, -1}, {-2, -2}, {-2, 2}}, StreamStyle → Thickness[0.005]]
```

Out[139]=



```
VectorPlot[{1, x^2 - y}, {x, -3, 3}, {y, -3, 3}, VectorScale → {Tiny, Automatic, None}, VectorStyle → Arrowheads[0],
 StreamPoints→{{-1,-1},{-2,-2},{-2,2}},StreamStyle→Thickness[0.005]]
```

Out[140]=



Mathematica 1: Slope fields and stream lines

Example: Use Mathematica to draw the slope field of the following differential equation:

$$\frac{dy}{dx} = -y^2 + 2\cos(x).$$

We first input the following command and press **Shift+Enter** to get a figure of vector fields 1.

VectorPlot[{1, -y^2+2Cos[x]},{x,-3,3}, {y,-3,3}]

Pay attention to the following **input rules of commands**: *(a): the upper cases*, and *(b): the type of brackets.*

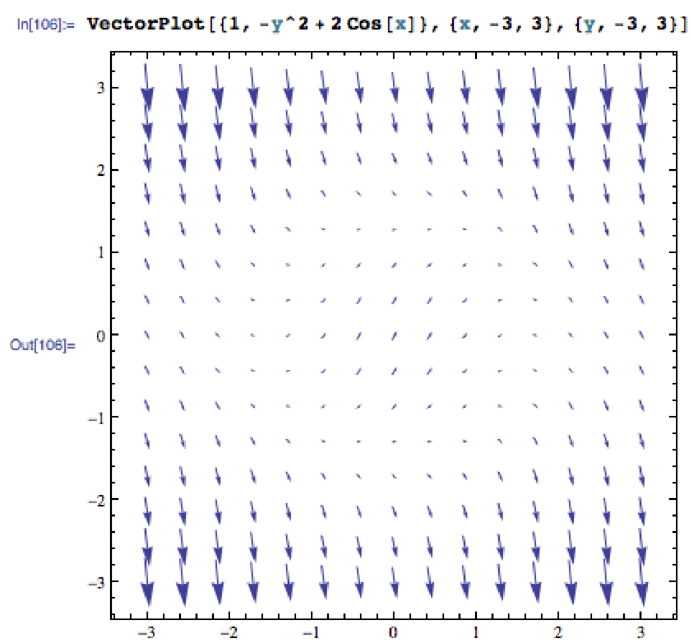


Figure 1: Vector field

The vectors were scaled according to their lengths, to uniformize the lengths and remove the arrow heads, we can input:

VectorPlot[{1, -y^2+2Cos[x]},{x,-3,3}, {y,-3,3},
VectorScale -> {Tiny, Automatic, None}, VectorStyle ->Arrowheads[0]]

Press Shift+Enter to get figure 2 (slope fields):

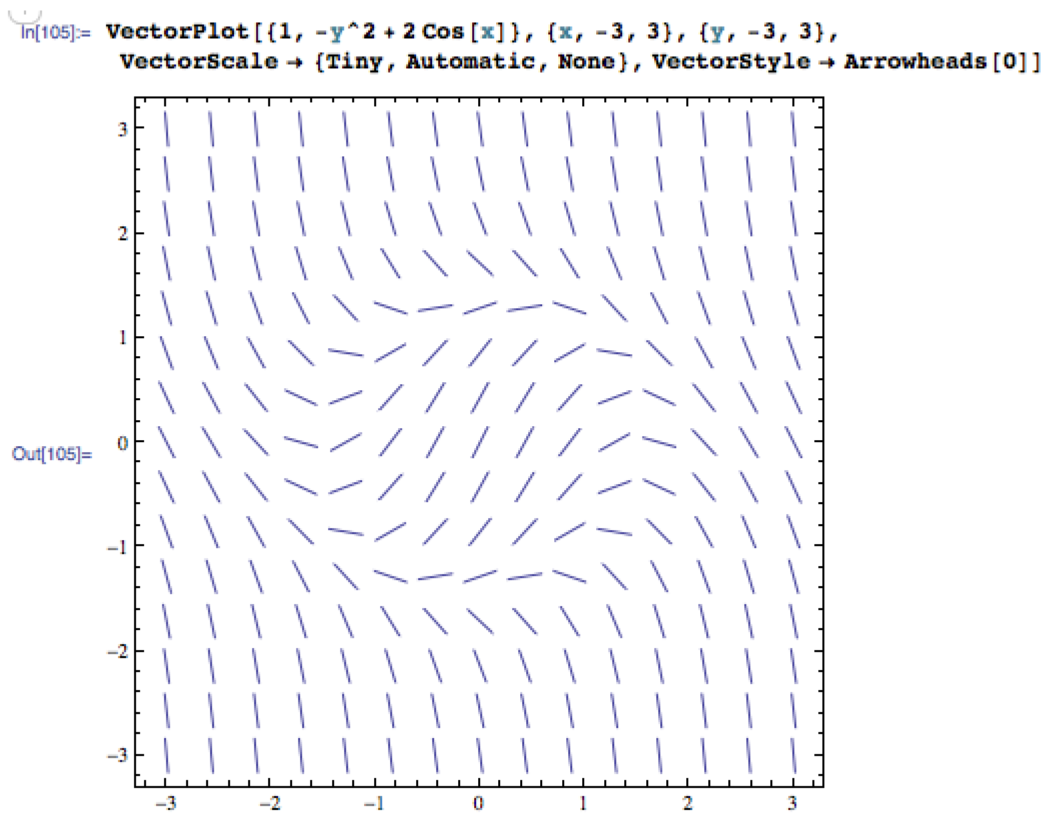


Figure 2: Slope field

To get specific stream lines, just add the stream points. Here we add three stream points: $(-2,-2)$, $(-2,-0.5)$, $(-2,0)$:

VectorPlot[{1, -y^2+2Cos[x]},{x,-3,3}, {y,-3,3},
VectorScale->{Tiny, Automatic, None}, VectorStyle->Arrowheads[0],
StreamPoints-> {{-2,-2},{-2,-0.5},{-2,0}}]

```
In[111]:= VectorPlot[{1, -y^2 + 2 Cos[x]}, {x, -3, 3}, {y, -3, 3},
  VectorScale → {Tiny, Automatic, None}, VectorStyle → Arrowheads[0],
  StreamPoints → {{-2, -2}, {-2, -0.5}, {-2, 0}}, StreamStyle → Thickness[0.01]]
```

Out[111]=

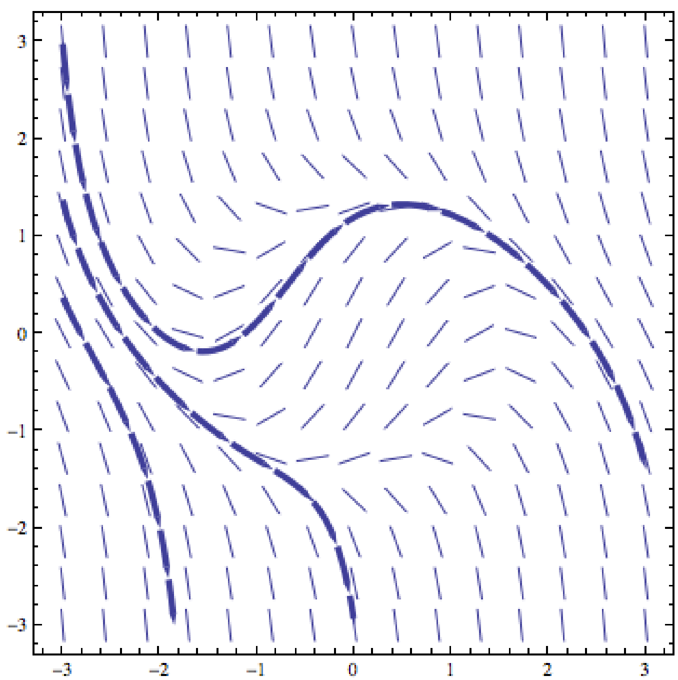


Figure 3: Slope field with stream lines

Note that we have added the optional property **StreamStyle** to specify the thickness of streamlines.

We could also plot lots of stream lines using the commands:

StreamPlot[{1, -y^2+2Cos[x]},{x,-3,3}, {y,-3,3}]

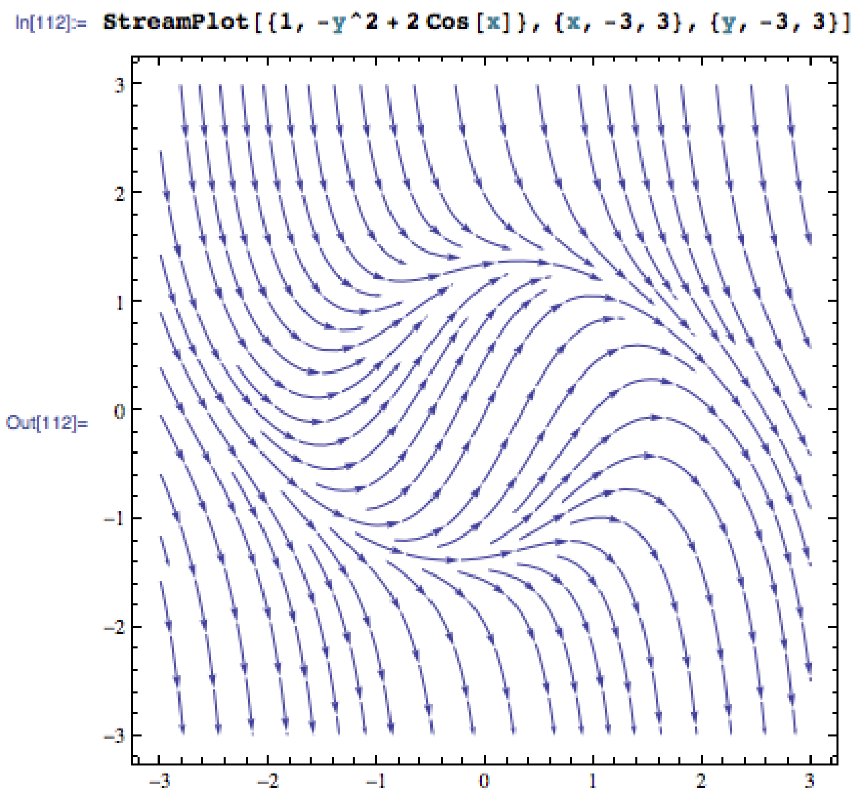


Figure 4: Stream field

1.(a). $y\,dy = \dfrac{x^2\,dx}{1+x^3} \Rightarrow \underbrace{\int y\,dy}_{=\frac{1}{2}y^2+C} = \underbrace{\int \dfrac{x^2\,dx}{1+x^3}}_{=\frac{1}{3}\int \frac{d(1+x^3)}{1+x^3} = \frac{1}{3}\ln(1+x^3)}$

so.

$$y^2 = \frac{2}{3}\ln(1+x^3) + C.$$

(b). $\dfrac{dy}{y^2} = -\sin x\,dx \xRightarrow{\text{integrate}} -\dfrac{1}{y} = \cos x + C \Rightarrow y = \dfrac{1}{C-\cos x}$

(c) $y' = (1+x)(1+y^2) \Rightarrow \dfrac{dy}{1+y^2} = (1+x)\,dx \Rightarrow \tan^{-1} y = x + \frac{1}{2}x^2 + C.$

$\Rightarrow y = \tan\left(x + \frac{1}{2}x^2 + C\right).$

(d). $e^x x\,dx = -y\,dy \xRightarrow{\text{integrate}} x\cdot e^x - e^x + C = -\frac{1}{2}y^2$

substitute $y(0)=1 \Rightarrow -1 + C = -\frac{1}{2} \Rightarrow C = 1/2$ so. the particular solution

is $y^2 = -2xe^x + 2e^x - 1$ or $y = \sqrt{-2x\cdot e^x + 2e^x - 1}$.

2. (a). Integrating factor $F(x)=e^{2x}$. so.

$$(e^{2x}y)'=x \Rightarrow e^{2x}y=\frac{1}{2}x^2+C \Rightarrow y=\frac{1}{2}x^2e^{-2x}+Ce^{-2x}.$$

standard form:

(b). $y'+xy=x$. Integrating factor $F(x)=e^{\int x\,dx}=e^{\frac{1}{2}x^2}$.

so

$$(e^{\frac{1}{2}x^2}y)'=x\cdot e^{\frac{1}{2}x^2} \Rightarrow e^{\frac{1}{2}x^2}y=\int x e^{\frac{1}{2}x^2}dx=\int e^{\frac{1}{2}x^2}d(\tfrac{1}{2}x^2)$$

$$=e^{\frac{1}{2}x^2}+C$$

so $y=1+C\cdot e^{-\frac{1}{2}x^2}$

(c) $y'+\frac{2}{x}y=\frac{\sin x}{x}$. Integrating factor $F(x)=e^{\int\frac{2}{x}dx}=e^{2\ln x}=x^2$

so $(x^2y)'=x\cdot\sin x \Rightarrow x^2y=\int x\sin x\,dx=\int x\,d(-\cos x)=-x\cdot\cos x+\int\cos x\,dx$

$$=-x\cos x+\sin x+C.$$

substitute $(x,y)=(\pi,\frac{1}{\pi})$, we get

$$\pi=-\pi\cos\pi+\sin\pi+C=\pi+C\Rightarrow C=0.$$

so. $y=-\frac{\cos x}{x}+\frac{\sin x}{x^2}.$

2. (d). $F(x) = e^{\int \frac{2}{1-x^2}dx}$. $\quad -\int \frac{2}{1-x^2}dx = -\int\left(\frac{1}{1+x}+\frac{1}{1-x}\right)dx = -\ln\frac{1+x}{1-x}$.

$$= e^{-\ln\frac{1+x}{1-x}} = \frac{1-x}{1+x}. \quad \text{So.}$$

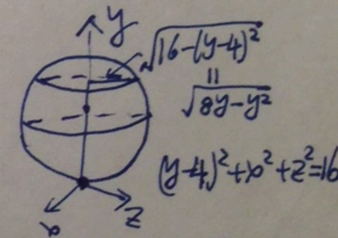
$$\left(\frac{1-x}{1+x}y\right)' = 1-x \Rightarrow \frac{1-x}{1+x}y = x - \frac{1}{2}x^2 + C.$$

substitute $(0,1) \Rightarrow 1 = C$ so. $\frac{1-x}{1+x}y = x-\frac{1}{2}x^2+1$.

so $y = \frac{1+x}{1-x}\left(x-\frac{1}{2}x^2+1\right)$.

3. $\frac{dN}{dt} = -kN \Rightarrow N(t) = N_0 \cdot e^{-kt}$

$$N_0e^{-kt} = N(t) = \frac{1}{5}N_0 \Rightarrow t = \frac{\ln 5}{k} \approx \frac{\ln 5}{0.0001216} \approx 13235.5 \text{ years.}$$



4. $\begin{cases} A(y)\frac{dy}{dt} = -k\cdot\sqrt{y} \\ y(0) = 8. \end{cases}$ $\quad k = a\cdot\sqrt{2g}$

$A(y) = \pi(8y-y^2)$. $0 \le y \le 8$

$$(8y^{\frac{1}{2}} - y^{\frac{3}{2}})dy = -\frac{k}{\pi}\cdot dt \Rightarrow \frac{16}{3}y^{\frac{3}{2}} - \frac{2}{5}y^{\frac{5}{2}} = -\frac{k}{\pi}t + C.$$

substitute $(t,y) = (0,8)$ to get $C = \frac{16}{3}\times 8^{\frac{3}{2}} - \frac{2}{5}8^{\frac{5}{2}} = \frac{1}{15}\times 2^{\frac{19}{2}}$.

when $y=0$, $t_1 = \frac{\pi C}{k} = \frac{\pi C}{a\sqrt{2g}} = \frac{\pi C}{\pi \cdot r_0^2\sqrt{2g}} = \frac{C}{r_0^2\sqrt{2g}}$ $\quad r_0 = 2\text{in.} = \frac{1}{6}\text{ft.}$

$g = 32\text{ft}/s^2$

$$t_1 \approx \frac{1}{15}\times 2^{\frac{19}{2}}\cdot 36 \times \frac{1}{\sqrt{2\times 32}} = 217.223 \text{ sec}$$

5. Let's derive a general formula (as in class)

$$\begin{cases} \dfrac{dx}{dt} = r_i C_i - r_o C_o = r_i C_i - r_o \dfrac{x}{V_o + (r_i - r_o)t} \\ x(0) = x_o \end{cases}$$

Integrating factor: $F(t) = e^{\int \frac{r_o \, dt}{V_o + (r_i - r_o)t}} = e^{\frac{r_o}{r_i - r_o} \ln (V_o + (r_i - r_o)t)}$

$$= (V_o + (r_i - r_o)t)^{\frac{r_o}{r_i - r_o}}$$

Multiplying the integrating factor, we get:

$$(F(t) \cdot x(t))' = r_i \cdot C_i \cdot (V + (r_i - r_o)t)^{\frac{r_o}{r_i - r_o}}$$ Integrate w.r.t. t:

$$(V + (r_i - r_o)t)^{\frac{r_o}{r_i - r_o}} x(t) = \cancel{r_i} C_i \cdot \frac{\cancel{r_i - r_o}}{\cancel{r_i (r_i - r_o)}} \cdot (V_o + (r_i - r_o)t)^{\frac{\cancel{r_o + (r_i - r_o)}}{r_i - r_o}} + C$$

So $$x(t) = C_i \cdot (V_o + (r_i - r_o)t)^{\frac{r_i}{r_i - r_o} - \frac{r_o}{r_i - r_o}} + C \cdot (V + (r_i - r_o)t)^{-\frac{r_o}{r_i - r_o}}$$

$$= C_i (V_o + (r_i - r_o)t) + C \cdot (V_o + (r_i - r_o)t)^{-\frac{r_o}{r_i - r_o}}.$$

Substitute $x(0) = x_o = C_i \cdot V_o + C \cdot (V_o)^{-\frac{r_o}{r_i - r_o}} \Rightarrow C = V_o^{\frac{r_o}{r_i - r_o}} (x_o - C_i \cdot V_o)$.

So we get the particular solution:

$$x(t) = C_i \cdot (V_o + (r_i - r_o)t) + V_o^{\frac{r_o}{r_i - r_o}} (x_o - C_i \cdot V_o)(V_o + (r_i - r_o)t)^{-\frac{r_o}{r_i - r_o}}.$$

Now we can substitute the data: $V_o = 60$, $r_i = 1$, $C_i = 1$, $r_o = 3$, $x_o = 0$.

So. we get

$$x(t) = (60 - 2t) + 60^{-\frac{3}{2}} \cdot (-60) \cdot (60 - 2t)^{+\frac{3}{2}}$$

$$= (60 - 2t) - 60^{-\frac{1}{2}} (60 - 2t)^{+\frac{3}{2}} \qquad 0 \le t \le 30.$$

6. $\begin{cases} \frac{du}{dt} = -0.5\left(u - 10\cdot\cos\left(\frac{\pi}{12}t\right)\right) \\ u(0) = 20. \end{cases}$

standard form: $\frac{du}{dt} + 0.5u = 5\cos\left(\frac{\pi}{12}t\right)$. $F(t) = e^{0.5t}$

$\Rightarrow \left(e^{0.5t}u\right)' = 5e^{0.5t}\cos\left(\frac{\pi}{12}t\right)$

$\Rightarrow e^{0.5t}u = 5\int e^{0.5t}\cos\left(\frac{\pi}{12}t\right)dt$. we calculate the right hand side:

$\int e^{0.5t}\cos\left(\frac{\pi}{12}t\right)dt = 2\int\cos\left(\frac{\pi}{12}t\right)d\left(e^{0.5t}\right) = 2e^{0.5t}\cos\left(\frac{\pi}{12}t\right) + 2\cdot\int e^{0.5t}\sin\left(\frac{\pi}{12}t\right)\frac{\pi}{12}dt$

$= 2\cdot e^{0.5t}\cos\left(\frac{\pi}{12}t\right) + \frac{\pi}{6}\cdot\int\sin\left(\frac{\pi}{12}t\right)d\left(e^{0.5t}\right)\cdot 2$

$= 2\cdot e^{0.5t}\cos\left(\frac{\pi}{12}t\right) + \frac{\pi}{3}\cdot e^{0.5t}\sin\left(\frac{\pi}{12}t\right) - \frac{\pi}{3}\int e^{0.5t}\cdot\cos\left(\frac{\pi}{12}t\right)\cdot\frac{\pi}{12}dt$

$\Rightarrow \int e^{0.5t}\cos\left(\frac{\pi}{12}t\right)dt = \frac{1}{1+\frac{\pi^2}{36}}\cdot\left(2\cos\left(\frac{\pi}{12}t\right) + \frac{\pi}{3}\sin\left(\frac{\pi}{12}t\right)\right)\cdot e^{0.5t} + C_1$

so. $e^{0.5t}u = \frac{5}{1+\frac{\pi^2}{36}}\left(2\cos\left(\frac{\pi}{12}t\right) + \frac{\pi}{3}\sin\left(\frac{\pi}{12}t\right)\right)e^{0.5t} + C$

substitute $u(0) = 20 \Rightarrow 20 = \frac{10}{1+\frac{\pi^2}{36}} + C \Rightarrow C = 20 - \frac{10}{1+\frac{\pi^2}{36}}$

so. $u(t) = \frac{5}{1+\frac{\pi^2}{36}}\left(2\cos\left(\frac{\pi}{12}t\right) + \frac{\pi}{3}\sin\left(\frac{\pi}{12}t\right)\right) + \left(20 - \frac{10}{1+\frac{\pi^2}{36}}\right)\cdot e^{-0.5t}$

$\approx 7.848\cos\left(\frac{\pi}{12}t\right) + 4.109\sin\left(\frac{\pi}{12}t\right) + 12.152e^{-0.5t}$.

Homework 2

From the above calculation, we get the function of temperature:

$$
\begin{aligned}
u(t) &= \frac{5}{1+\frac{\pi^2}{36}}\left(2\cos\frac{\pi t}{12}+\frac{\pi}{3}\sin\frac{\pi t}{12}\right)+10\left(2-\frac{1}{1+\frac{\pi^2}{36}}\right)e^{-0.5t} \\
&\approx 7.85\cos\frac{\pi t}{12}+4.11\sin\frac{\pi t}{12}+12.15e^{-0.5t} \\
&\approx 12.15e^{-0.5t}+8.86\cos\left(\frac{\pi}{12}(t-1.84)\right). \quad \text{(why?)}
\end{aligned}
$$

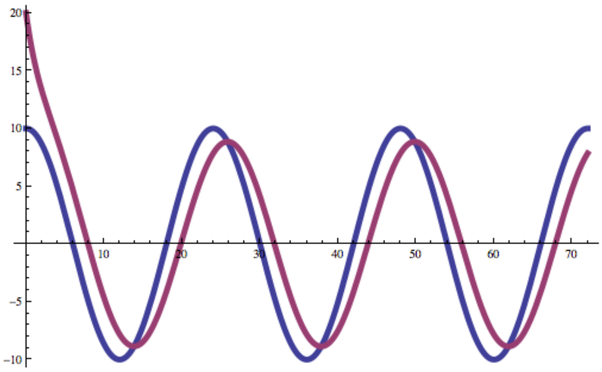
So we have the following conclusion. The "damped" term $12.15e^{-0.5t}$ decays exponentially to 0. The indoor temperature will also oscillate in the long term. But the amplitude of oscillation is approximately 8.86, which is smaller than the amplitude of the outdoor temperature (10). The oscillation of indoor temperature lags behind the oscillation of the outdoor temperature by approximately 1.84 hours. These can also be seen from the graphs below.

```
In[41]:= T1[t_] = 5 / (1 + Pi^2 / 36) (2 Cos[Pi t / 12] + Pi / 3 Sin[Pi t / 12]) + (20 - 10 / (1 + Pi^2 / 36)) Exp[-0.5 t]
```

Out[41]= $e^{-0.5\,t}\left(20-\frac{10}{1+\frac{\pi^2}{36}}\right)+\frac{5\left(2\,\mathrm{Cos}\left[\frac{\pi t}{12}\right]+\frac{1}{3}\pi\,\mathrm{Sin}\left[\frac{\pi t}{12}\right]\right)}{1+\frac{\pi^2}{36}}$

```
In[39]:= Plot[{10 Cos[Pi t / 12], T1[t]}, {t, 0, 72}, PlotStyle → Thickness[0.01]]
```

Out[39]=



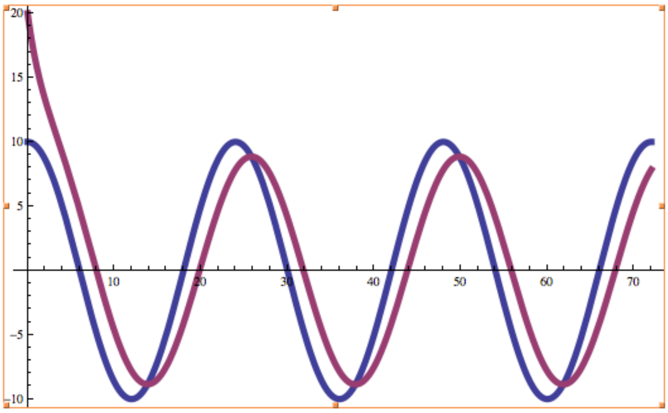
We can also integrate using mathematica directly:

```
In[29]:= T2[t_] = Exp[-0.5 t] * (Integrate[5 Exp[0.5 x] * Cos[Pi x / 12], {x, 0, t}] + 20)

Out[29]= e^(-0.5 t) (20 + 5 ((-1.56967 + 0. i) + e^(0.5 t) ((1.56967 + 0. i) Cos[0.261799 t] + (0.821876 + 0. i) Sin[0.261799 t])))

In[40]:= Plot[{10 Cos[Pi t / 12], T2[t]}, {t, 0, 72}, PlotStyle → Thickness[0.01]]
```

Out[40]=



Separable equations and applications

1. Solve the following differential equation:

$$e^{-x}\frac{dy}{dx} = (1+y^2)\sin x; \quad y(0) = 1.$$

Solution: Step 1: First separate variables:

$$\frac{dy}{1+y^2} = e^x \sin x dx.$$

Step 2: Integrate on both sides, that is, integrate the left (resp. right) hand side with respect to y (resp. x) variable:

$$\arctan(y) = \int e^x \sin x dx.$$

Question: How to integrate the right hand side?

Answer: integration by parts twice:

$$\begin{aligned}
\int e^x \sin x dx &= -\int e^x d(\cos x) = -e^x \cos x + \int (\cos x) e^x dx \\
&= -e^x \cos x + \int e^x d(\sin x) \\
&= -e^x \cos x + e^x \sin x - \int e^x \sin x dx.
\end{aligned}$$

So we get:

$$\int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x) + C.$$

So we get the **implicit solutions**:

$$\arctan y = \frac{1}{2} e^x (\sin x - \cos x) + C.$$

We can solve $y = y(x)$ to get **explicit solutions**:

$$y(x) = \tan\left[\frac{e^x}{2}(\sin x - \cos x) + C\right].$$

Step 3: Use initial condition to determine the constant C:

$$y(0) = \tan\left[-\frac{1}{2} + C\right] = 1 \Longrightarrow C = \frac{\pi}{4} + \frac{1}{2}.$$

So finally we get the particular solution:

$$y(x) = \tan\left[\frac{e^x}{2}(\sin x - \cos x) + \frac{\pi}{4} + \frac{1}{2}\right].$$

2. **Application 1: Radioactive decay and radiocarbon dating**

   The radioactive isotope decays exponentially:

$$\frac{dN}{dt} = -kN.$$

   k is some decay constant. Solve this separable equation to get

$$N(t) = N_0 e^{-kt}, \ N(0) = N_0.$$

   Question: What is the half life τ?

   Answer:

$$N(\tau) = N(0)/2 \Longrightarrow \tau = \frac{\ln 2}{k}.$$

   For example, for radioactive isotope ^{14}C,

$$k \approx 0.0001216 \Longleftrightarrow \tau \approx 5700 \ \text{ years}.$$

   Example (Exercise 1.4.36): Carbon taken from a purported relic of the time of Christ contained 4.6×10^{10} atoms of ^{14}C per gram. Carbon extracted from a present-day specimen of the same substance contained 5.0×10^{10} atoms of ^{14}C per gram. Compute the approximate age of the relic. What is your opinion as to its authenticity.

   Solution: From above, we know that the amount of ^{14}C per gram after t years is:

$$N(t) = N_0 e^{-kt}, \text{ with } N_0 = 5 \times 10^{10}, k = 0.0001216.$$

   We can find inverse function

$$t = -\frac{1}{k} \log \frac{N(t)}{N_0}.$$

   So we can calculate the age is approximately:

$$t_1 = -\frac{1}{0.0001216} \log \frac{4.6 \times 10^{10}}{5.0 \times 10^{10}} \approx 685.7 \text{ years}.$$

   (You can use *Mathematica* to calculate: **N[-Log[4.6/5]/0.0001216]**)

   By the time when Christ lived, this relic is not authentic.

3. **Application 2 (Torricelli's Law):** Notations:

   $V(t)$: the volume of water at time t;

   $y(t)$: the height level of water surface at time t.

   $A(y)$: the area of the slice at height y.

   a: area of the hole at bottom;

   g: gravitational acceleration ≈ 32 $\mathbf{ft/s^2} \approx 9.8 \ m/s^2$.

   Torricelli's law says that the velocity of water exiting through the hole is:

$$v = \sqrt{2gy}.$$

So the volume decreases according to the following differential equation:

$$\frac{dV(t)}{dt} = -a\sqrt{2gy}.$$

From geometry, we have:

$$\frac{dV}{dy} = A(y).$$

Using the chain rule for the volume function: $V(t) = V(y(t))$, we get

$$\frac{dV}{dt} = \frac{dV}{dy}\frac{dy}{dt} = A(y)\frac{dy}{dt}.$$

We get (let $k = a\sqrt{2g}$)

$$A(y)\frac{dy}{dt} = -k\sqrt{y}. \tag{1}$$

Equation (1) is a separable equation.

Example (Exercise 1.4.59): A water tank has the shape obtained by revolving the parabola $x^2 = by$ around the y-axis. A circular plug at the bottom of the tank is removed at 12 noon, when the depth of water is 4 ft. At 1 P.M. the depth of the water is 1 ft.

(a) Find the depth $y(t)$ of water remaining after t **hours**.

(b) When will the tank be empty?

(c) If the in initial radius of the top surface of the water is 2 ft, what is the radius of the circular hole in the bottom?

Solution:

(a) Since the tank is rotationally symmetric, the area $A(y) = \pi x(y)^2 = \pi by$. So the equation (1) becomes:

$$\pi by\frac{dy}{dt} = -k\sqrt{y}, \quad y(0) = 4, y(1) = 1.$$

Solve this equation using separable variable method:

$$\sqrt{y}dy = -\frac{k}{\pi b}dt \quad \Longrightarrow \quad \frac{2}{3}y^{3/2} = -\frac{k}{\pi b}t + C_1.$$

So we get a general solution:

$$y = \left(C - \frac{3k}{2\pi b}t\right)^{2/3}.$$

Use the boundary conditions:

$$C^{2/3} = 4, \left(C - \frac{3k}{2\pi b}\right)^{2/3} = 1 \Longrightarrow C = 8, \frac{3k}{2\pi b} = 7.$$

So we get the particular solution:

$$y(t) = (8 - 7t)^{2/3}.$$

(b) The tank is empty when the height level is 0:

$$y(t) = 0 \Longrightarrow t = 8/7\ h = 1\ h + 8\ min + 34\ sec.(why?)$$

So the tank will be empty at 1:08:34 P.M..

(c) (the change of units is tricky) By assumption, $2^2 = x(4)^2 = b \times 4$. So $b = 1$. So

$$k = \frac{14\pi b}{3} = \frac{14\pi}{3}.$$

Because $k = a\sqrt{2g} = \pi r^2\sqrt{2g}$. To calculate we need **change the units:**

$$g \approx 32\ \mathbf{ft/s^2} = 32\frac{ft}{(h/(3600))^2} = 32 \times (3600)^2\ \frac{\mathbf{ft}}{\mathbf{h^2}}.$$

So

$$\begin{aligned} r &= \sqrt{\frac{k}{\pi\sqrt{2g}}} = \sqrt{\frac{14}{3\sqrt{2g}}} \approx \sqrt{\frac{14}{3\sqrt{2 \times 32 \times (3600)^2}}} = \frac{1}{60}\sqrt{\frac{7}{12}} \\ &\approx 0.0127\ ft = 0.153\ in. \end{aligned}$$

(You can use *Mathematica* to calculate: **N[Sqrt[7/12]/60*12]**)

Homework 3

1. $2xy^3dx + 3x^2y^2dy = 0$.
2. $(t^2+1)\cos u du + 2t \sin u dt = 0$.
3. $(2x-y)y' = 2y - x$.
4. $xy' = y + \sqrt{x^2+y^2}$.
5. $y' = x^3y^3 - xy$.
6. $x^2y' + 2xy = 5y^3$
7. $y' = \cos(x-y)$.
8. $(-x+e^y)y' = xe^{-y} + 1$.
9. $xy'' + y' = x\cos x + \sin x$.
10. $y^2y'' + y' = 0$.
11. Suppose that the population $P(t)$ of a country satisfies the differential equation

$$\frac{dP}{dt} = kP(300-P), \qquad \text{(Note that: 300=300 million)}$$

with k constant. Its population in 1950 was 200 million and was then growing at the rate of 1 million per year. Predict this country's population for the year 2000. What's the limiting population?

12. A population $P(t)$ of small rodents has birth rate $\beta = 0.002P$ (births per month per rodent) and constant death rate δ. If $P(0) = 100$ and $P'(0) = 4$, how long will it take this population to double to 200 rodents?

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Name	Last modified	Size	Description
Parent Directory		-	
1-2.JPG	2014-02-22 19:01	414K	
3.JPG	2014-02-22 19:01	381K	
4.JPG	2014-02-22 19:01	424K	
5.JPG	2014-02-22 19:01	389K	
6.JPG	2014-02-22 19:01	345K	
7.JPG	2014-02-22 19:01	369K	
8.JPG	2014-02-22 19:01	430K	
9-10.JPG	2014-02-22 19:01	443K	
9-Correction.JPG	2014-03-02 18:24	1.3M	
11.JPG	2014-02-22 19:01	397K	
12.JPG	2014-02-22 19:01	414K	

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1(b)-1(c).jpg	2014-02-28 17:56	1.3M	
1(d).jpg	2014-02-28 17:56	1.3M	
2-3.jpg	2014-02-28 17:56	1.4M	
4(a).jpg	2014-02-28 17:56	1.4M	
4(b).jpg	2014-02-28 17:56	1.4M	
5.JPG	2014-02-28 20:56	1.6M	
6.jpg	2014-02-28 17:56	1.4M	
Mathematica-bifurcation.nb	2014-02-28 21:07	6.0K	
Mathematica-parachute.nb	2014-02-28 20:57	45K	

1. $\frac{dy}{dx} = 4x^3y - y, \quad y(1) = 1.$

$$\frac{dy}{dx} = (4x^3-1)y \Rightarrow \frac{dy}{y} = (4x^3-1)dx \Rightarrow \ln y = x^4 - x + C_1$$

$$\Downarrow y(1)=1$$

$$C_1 = 0 \Leftarrow 0 = 1-1+C_1$$

So $\boxed{y(x) = e^{x^4-x}}$

2. $(x^2+1)y' + 2xy = x, \quad y(0) = 1.$

Write into standard form: $y' + \frac{2x}{x^2+1}y = \frac{x}{x^2+1}.$

Integrating factor: $F(x) = e^{\int \frac{2x}{x^2+1}dx} = e^{\ln(x^2+1)} = x^2+1.$

So. $((x^2+1)y)' = x \Rightarrow (x^2+1)y = \frac{1}{2}x^2 + C \overset{y(0)=1}{\Longrightarrow} 1 = C.$

So: $(x^2+1)y = \frac{1}{2}x^2 + 1 \Rightarrow \boxed{y = \frac{\frac{1}{2}x^2+1}{x^2+1}}$

3. $(x+y)dx + (x-y)dy = 0.$

$\frac{\partial}{\partial y}(x+y) = 1 = \frac{\partial}{\partial x}(x-y) \Rightarrow$ exact equation.

$$\begin{cases} \frac{\partial F}{\partial x} = x+y & ① \\ \frac{\partial F}{\partial y} = x-y & ② \end{cases}$$

① $\Rightarrow F(x,y) = \frac{1}{2}x^2 + xy + g(y) \overset{②}{\Longrightarrow} x + g'(y) = x - y$

$$\Downarrow$$

$$g(y) = -\frac{1}{2}y^2. \Leftarrow g'(y) = -y$$

So $F(x,y) = \frac{1}{2}x^2 + xy - \frac{1}{2}y^2$. A general solution is:

$$\boxed{\frac{1}{2}x^2 + xy - \frac{1}{2}y^2 = C}$$

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Name	Last modified	Size	Description
Parent Directory		-	
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1(b).jpg	2014-02-28 17:57	1.6M	
2(a).jpg	2014-02-28 17:57	1.4M	
2(b).jpg	2014-02-28 17:57	1.2M	
2(c).jpg	2014-02-28 17:57	1.3M	
Euler/	2014-03-03 14:07	-	
Mathematica-NDSolve.nb	2014-02-28 23:05	32K	

1. $xy' + 2y = x \cdot y^{\frac{1}{2}}$ (This is a Bernoulli type equation)

$xy' \cdot y^{-\frac{1}{2}} + 2y^{+\frac{1}{2}} = x$ substitute $u = y^{+\frac{1}{2}} \Rightarrow u' = +\frac{1}{2} y^{-\frac{1}{2}} y'$.

$\Downarrow$

so $2x \cdot u' + 2u = x \Rightarrow u' + \frac{1}{x} u = \frac{1}{2}$ $y^{-\frac{1}{2}} y' = 2u'$.

Integrating factor $F(x) = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$.

$\Rightarrow (xu)' = \frac{1}{2} x \Rightarrow x u(x) = \frac{1}{4} x^2 + C \Rightarrow \underset{y^{\frac{1}{2}}}{\underset{\parallel}{u(x)}} = \frac{1}{4} x + C \cdot x^{-1}$.

$\Rightarrow y(x) = \left(\frac{1}{4} x + C \cdot x^{-1}\right)^2$.

2. $\frac{dy}{dx} = -\frac{3x^2 + 2y^2}{4xy}$. (There are two method to solve this)

! Method 1: only apply when the DE is exact

First write it in the differential form:

$\underset{N}{\underset{\parallel}{(4xy)}} dy + \underset{M}{\underset{\parallel}{(3x^2 + 2y^2)}} dx = 0$. (Warning: This is not always exact. For this problem it is a coincidence)

Test whether this is exact or not:

$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(4xy) = 4y, = \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(3x^2 + 2y^2) = 4y$

So this is an exact equation. We try to find $F(x,y)$ s.t.

$$\begin{cases} \frac{\partial F}{\partial x} = 3x^2 + 2y^2 \quad ① \Rightarrow F(x,y) = x^3 + 2xy^2 + g(y) \\ \frac{\partial F}{\partial y} = 4xy \quad ② \qquad 4xy = \frac{\partial F}{\partial y} = 4y + g'(y) \Rightarrow g'(y) = 0 \Rightarrow g(y) = \text{const} \end{cases}$$

$\Downarrow ②$

So. $F(x,y) = x^3 + 2xy^2$. A general solution: $\boxed{x^3 + 2xy^2 = C}$

Method 2: $\frac{dy}{dx} = -\frac{3x^2+2y^2}{4xy} = -\frac{3+2\left(\frac{y}{x}\right)^2}{4\frac{y}{x}}$.

This is a homogeneous equation. Substitute: $u = \frac{y}{x}$

$\Rightarrow y = u \cdot x \Rightarrow y' = u' \cdot x + u$. so we get:

$$u' \cdot x + u = -\frac{3+2u^2}{4u} \Rightarrow u' \cdot x = -\frac{3+2u^2}{4u} - u = -\frac{3+2u^2+4u^2}{4u} = -\frac{3+6u^2}{4u}.$$

$$\Rightarrow \frac{4u}{3+6u^2} du = -\frac{dx}{x}$$

$$\Rightarrow \int \frac{4u}{3+6u^2} du = -\ln x + C_1$$

$v = 3+6u^2$, $dv = 12u\,du$.

$$\int \frac{1}{v} \cdot \frac{dv}{3} = \frac{1}{3} \ln v$$

$$\Rightarrow \ln(3+6u^2) = -3\ln x + C_2$$

$$\Downarrow$$

$$3+6u^2 = x^{-3} \cdot C_3$$

$$3+6 \cdot \left(\frac{y}{x}\right)^2 = \frac{C_3}{x^3}$$

$$x^3+2xy^2 = \frac{C_3}{3} \Leftarrow$$

so $\boxed{x^3+2xy^2 = C}$

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3. $y' = \sqrt{x+y}$. Substitute $u = x+y \Rightarrow y = u - x \Rightarrow y' = u' - 1$

So $u' - 1 = \sqrt{u} \Rightarrow \dfrac{du}{\sqrt{u}+1} = dx \Rightarrow \int \dfrac{du}{\sqrt{u}+1} = x + C$.

$$\frac{du}{dx} = u' = \sqrt{u}+1$$

To integrate $\int \dfrac{du}{\sqrt{u}+1}$, substitute $v = \sqrt{u}+1 \Rightarrow u = (v-1)^2 \Rightarrow du = 2(v-1)\,dv$.

so $\int \dfrac{du}{\sqrt{u}+1} = \int \dfrac{2(v-1)\,dv}{v} = 2\int (1 - v^{-1})\,dv = 2(v - \ln v)$.

$$= 2(\sqrt{u}+1 - \ln(\sqrt{u}+1)).$$

so. $2(\sqrt{u}+1 - \ln(\sqrt{u}+1)) = x + C$

substitute back $u = y + x$:

$$\boxed{2\left(\sqrt{x+y}+1 - \ln\left(\sqrt{x+y}+1\right)\right) = x + C.}$$

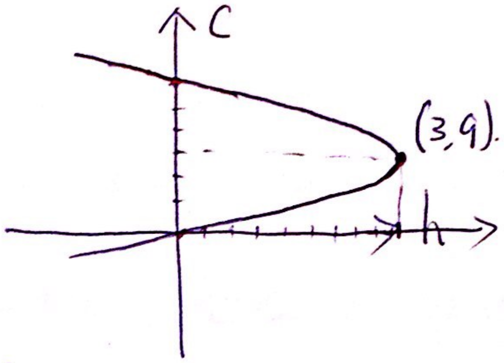
$$\Downarrow$$

$$\sqrt{x+y} - \ln\left(\sqrt{x+y}+1\right) = \frac{x}{2} + \tilde{C}. \qquad \left(\tilde{C} = \frac{C}{2} - 1\right).$$

(Implicit solution).

4. (a): The bifurcation diagram is the graph of:

$$6c - c^2 - h = 0 \Leftrightarrow h = 6c - c^2 = -c^2 + 6c - 9 + 9 = -(c-3)^2 + 9.$$



$\left(c = 3 \pm \sqrt{9-h} \right)$.

so if $h > 9$: no equilibrium solutions.

$h = 9$: 1 equilibrium solution.

$h < 9$: 2 equilibrium solution.

We can also determine the number of equilibrium solutions by looking at the formula for the roots of the quadratic equation:

$$6c - c^2 - h = 0 \Leftrightarrow c^2 - 6c + h = 0.$$

$$\Rightarrow \quad c = \frac{6 \pm \sqrt{36-4h}}{2} = 3 \pm \sqrt{9-h}.$$

so. if $h > 9$: no real roots $\Leftrightarrow$ no equilibrium sol

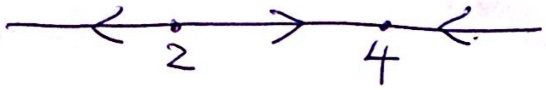
$h = 9$: 1 real root $c = 3 \Leftrightarrow$ 1 equilibrium sol.

$h < 9$: 2 real roots. $\Leftrightarrow$ 2 equilibrium sols

4.(b). $\frac{dP}{dt} = 6P - P^2 - 8 = -(P^2 - 6P + 8) = -(P-2)(P-4)$.

So there are 2 equilibrium solutions: $P(t) = 2$ and $P(t) = 4$.

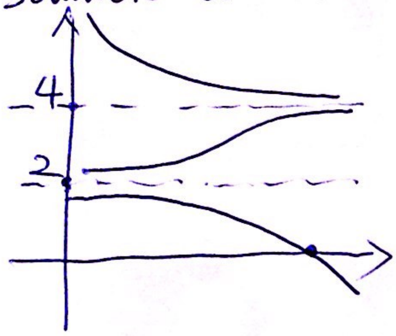
phase diagram:



So $P=2$ is unstable. $P=4$ is stable.

If $P(0) = 1 < 2$, the solution will approach $-\infty$ as time goes to $+\infty$.
So the population will become 0 in a finite time period.

Solution curves.



Let's find the extinction time by solving the DE explicitly. (The following kind of computation is required in the exam!)

$$\begin{cases} \frac{dP}{dt} = -(P-2)(P-4) \\ P(0) = 1 \end{cases} \Rightarrow \frac{dP}{(P-2)(P-4)} = -dt.$$

$$\frac{1}{(P-2)(P-4)} = \frac{1}{2}\left(\frac{1}{P-4} - \frac{1}{P-2}\right).$$

$$\Downarrow$$

$$\frac{1-4}{1-2} = C\cdot e^0 \overset{P(0)=1}{\Longleftarrow} \frac{P-4}{P-2} = C\cdot e^{-2t} \Longleftarrow \frac{1}{2}\ln\frac{P-4}{P-2} = -t + C_1.$$

$$\frac{1-4}{1-2} = 3, \quad C\cdot e^0 = C \quad \Rightarrow \quad 3 = C$$

So. $\frac{P(t)-4}{P(t)-2} = 3\cdot e^{-2t} \Rightarrow P(t) - 4 = 3\cdot e^{-2t}P(t) - 6\cdot e^{-2t}$

$$\Downarrow$$

$$P(t) = \frac{4 - 6\cdot e^{-2t}}{1 - 3\cdot e^{-2t}}.$$

At extinction time T, $P(T) = 0$.

$$\Rightarrow \underbrace{\frac{0-4}{0-2}}_{=2} = 3\cdot e^{-2T} \Rightarrow \boxed{T = \frac{\ln\frac{2}{3}}{-2} = \frac{\ln 3 - \ln 2}{2}}$$

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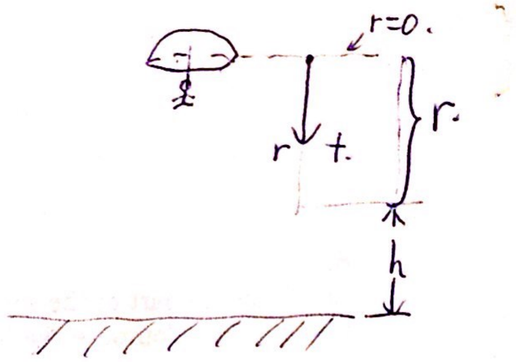
5.



$$\begin{cases} \frac{dv}{dt} = g - \rho v = 32 - 2v. \\ v(0) = 0. \end{cases}$$

$(c = e^{-C_1})$

$$\Rightarrow \frac{dv}{16-v} = 2\,dt \Rightarrow -\ln(16-v) = 2t + C_1 \Rightarrow 16 - v = C \cdot e^{-2t}$$

$\Downarrow v(0)=0.$

$16 = C.$

So. $v(t) = 16 - C e^{-2t} = 16(1 - e^{-2t}).$

Then we find height:

$$\begin{cases} \frac{dr}{dt} = v(t) = 16 - 16 \cdot e^{-2t} \Rightarrow r(t) = \int (16 - 16e^{-2t})\,dt \\ r(0) = 0. \end{cases}$$

$$= 16t + 8 \cdot e^{-2t} + C_2$$

$\Downarrow r(0)=0.$

$0 = 0 + 8 + C_2 \Rightarrow C_2 = -8.$

So, $r(t) = 16t + 8 \cdot e^{-2t} \cancel{+} - 8. = 16t + 8 \cdot e^{-2t} - 8.$

The height: $h(t) = 5000 - r(t) = \cancel{4992 - 16t + 8.}$
$(5008 - 16t - 8 \cdot e^{-2t})\,ft.$

6: $y'(x) = x + y^2$, $y(0) = 0$

$h = -0.1$: $x_0 = 0$, $y_0 = 0$.

$x_1 = -0.1$, $y_1 = y_0 + f(x_0, y_0) \cdot h$

$= 0 + (0 + 0^2) \times (-0.1) = 0$.

$x_2 = -0.2$, $y_2 = y_1 + f(x_1, y_1) \times h$

$= 0 + (-0.1 + 0^2) \times (-0.1) = 0.01$

$x_3 = -0.3$, $y_3 = y_2 + f(x_2, y_2) \times h$

$= 0.01 + (-0.2 + (0.01)^2) \times (-0.1)$

$= 0.01 + (-0.2 + 0.0001) \times (-0.1)$

$= 0.01 + 0.02 - 0.00001$

$= 0.03 - 0.00001$

$= 0.02999$.

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1. (30pts)

$$2xy' + y = \frac{\cos x}{y}, \quad y(\pi) = 1.$$

Method 1: This is a Bernoulli type equation.

$2xyy' + y^2 = \cos x$. substitute $u = y^2 \Rightarrow u' = 2y \cdot y'$.

so $x \cdot u' + u = \cos x \Rightarrow u' + \frac{1}{x}u = \frac{\cos x}{x}$ This is a 1st order linear.

Integrating factor $F(x) = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$.

so $(xu)' = \cos x \Rightarrow x\,u(x) = \sin x + C \Rightarrow x \cdot y^2 = \sin x + C$.

$\xRightarrow{y(\pi)=1} \pi \cdot 1^2 = \sin\pi + C \Rightarrow C = \pi$. so $xy^2 = \sin x + \pi$

so $\boxed{y = \sqrt{\frac{\sin x + \pi}{x}}}$

Method 2: $2xy\frac{dy}{dx} + (y^2 - \cos x) = 0 \Rightarrow (y^2 - \cos x)dx + 2xy\,dy = 0$.

$\frac{\partial}{\partial y}(y^2 - \cos x) = 2y = \frac{\partial}{\partial x}(2xy) \Rightarrow$ exact equation. So we solve:

$$\begin{cases} \frac{\partial\Phi}{\partial x} = y^2 - \cos x \quad ① \Rightarrow \Phi(x,y) = xy^2 - \sin x + g(y) \\ \frac{\partial\Phi}{\partial y} = 2xy \quad ② \qquad \frac{\partial\Phi}{\partial y} = 2xy + g'(y) = 2xy \Rightarrow g'(y) = 0 \Rightarrow g(y) = \text{Constant}. \end{cases}$$

so. $\Phi(x,y) = xy^2 - \sin x$. A general solution: $xy^2 - \sin x = C$

$\xRightarrow{y(\pi)=1} C = \pi \Rightarrow y^2 = \frac{\sin x + C}{x} = \frac{\sin x + \pi}{x} \Rightarrow \boxed{y = \sqrt{\frac{\sin x + \pi}{x}}}$

2. **(30pts)**

$$(x-y)\cdot\frac{dy}{dx}=y,\quad y(1)=1.$$

Method 1: $\frac{dy}{dx}=\frac{y}{x-y}=\frac{\frac{y}{x}}{1-\frac{y}{x}}$ substitute $u=\frac{y}{x}\Rightarrow y=u\cdot x$

$\Downarrow$

$y'=u'\cdot x+u.$

so. $u'\cdot x+u=\frac{u}{1-u}\Rightarrow u'\cdot x=\frac{u}{1-u}-u=\frac{u-(u-u^2)}{1-u}=\frac{u^2}{1-u}$

$\Rightarrow \frac{1-u}{u^2}du=\frac{dx}{x}\Rightarrow -\frac{1}{u}-\ln u=\ln x+C$

$\frac{1-u}{u^2}du = \left(\frac{1}{u^2}-\frac{1}{u}\right)du$

$\Downarrow$

$-\frac{x}{y}-\ln\frac{y}{x}=\ln x+C.$

$\xRightarrow{y(1)=1} -1-\ln 1=\ln 1+C\Rightarrow C=-1.$ so.

$-\frac{x}{y}-\ln\frac{y}{x}=\ln x-1\Rightarrow \frac{x}{y}=-\ln\frac{y}{x}-\ln x+1=-\ln y+1$

$\Rightarrow \boxed{x=-y\ln y+y}.$

Method 2: $\frac{dx}{dy}=\frac{x-y}{y}=\frac{x}{y}-1$ substitute $v(y)=\frac{x}{y}\Rightarrow x(y)=y\cdot v(y)$

$\Downarrow$

$x'(y)=y\cdot v'+v.$

so. $v'\cdot y+v=v-1\Rightarrow v'(y)=-\frac{1}{y}\Rightarrow v(y)=-\ln y+C.$

$\Rightarrow \frac{x}{y}=-\ln y+C\xRightarrow{y(1)=1} 1=-\ln 1+C\Rightarrow C=1$

so. $\boxed{x=-y\ln y+y}$

3. (30pts)

$$\frac{dy}{dx} = -2 + (3x+y)^2, \quad y(0) = 1.$$

Substitute $u = 3x + y \Rightarrow y = u - 3x \Rightarrow y' = u' - 3.$

So $u' - 3 = -2 + u^2 \Rightarrow u' = 1 + u^2.$

$$\Rightarrow \frac{du}{1+u^2} = dx \Rightarrow \tan^{-1}(u) = x + C$$

$$\tan^{-1}(u) = \tan^{-1}(3x+y)$$

$$\xRightarrow{y(0)=1} \tan^{-1}(1) = C = \frac{\pi}{4}$$

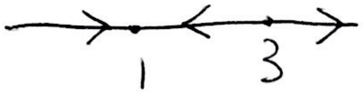
So $\tan^{-1}(3x+y) = x + \frac{\pi}{4} \Rightarrow y = \tan\left(x + \frac{\pi}{4}\right) - 3x.$

3. A explosion/extinction with stocking model of population is given by the following differential equation:

$$\frac{dP}{dt} = P^2 - 4P + 3.$$

4(a) (30pts): Find the equilibrium solutions and **classify** them as stable or unstable equilibrium solutions.

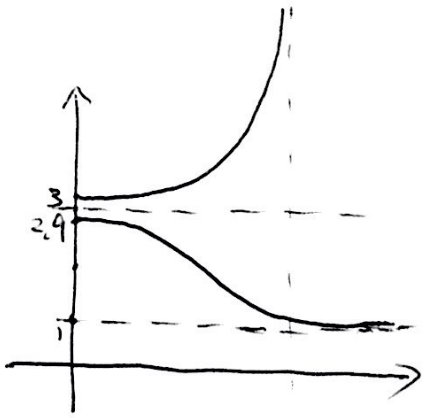
$0 = P^2 - 4P + 3 = (P-1)(P-3) \Rightarrow P=1$ or $P=3$.



$P=1$ is stable

$P=3$ is unstable

4(b) (20pts): Sketch two solution curves for the initial conditions $P(0) = 2.9$ and $P(0) = 3.1$.



4(c) (30pts) For the above population model:

$$\frac{dP}{dt} = P^2 - 4P + 3.$$

If $P(0) = 4$, find the doomsday time t_{doom} when the population explodes to infinity.
(Hint: First solve for an explicit solution)

$$\frac{dP}{P^2-4P+3} = dt \Rightarrow \frac{1}{2}\ln\frac{P-3}{P-1} = t + C_1 \Rightarrow \frac{P-3}{P-1} = C\cdot e^{2t}$$

$$\frac{1}{P^2-4P+3} = \frac{1}{2}\left(\frac{1}{P-3} - \frac{1}{P-1}\right)$$

$$\Downarrow \; P(0)=4$$

$$\frac{P-3}{P-1} = \frac{1}{3}e^{2t} \quad \Leftarrow \frac{1}{3} = C$$

so $3P - 9 = e^{2t}P - e^{2t} \Rightarrow (3-e^{2t})P = 9 - e^{2t}$

so $P(t) = \dfrac{9-e^{2t}}{3-e^{2t}}$.

$P(t)$ explodes to infinity when the denominator goes to 0.

so $3 - e^{2t_{doom}} = 0 \Rightarrow t_{doom} = \frac{1}{2}\ln 3$.

5. (30pts) Use Euler's method to approximate to the solution on the interval $[0, 0.3]$ with step size 0.1. What value of $y(0.3)$ do you get? Don't round off your answer.

$$y'(x) = -x + y^2, y(0) = 0.$$

$h = 0.1$ $\quad x_0 = 0, \quad y_0 = 0$

$x_1 = 0.1, \quad y_1 = y_0 + f(x_0, y_0) \cdot h$

$= 0 + (-0 + 0^2) \cdot 0.1 = 0.$

$x_2 = 0.2, \quad y_2 = y_1 + f(x_1, y_1) \cdot h$

$= 0 + (-0.1 + 0^2) \times 0.1 = -0.01.$

$x_3 = 0.3, \quad y_3 = y_2 + f(x_2, y_2) \cdot h$

$= -0.01 + (-0.2 + (-0.01)^2) \times 0.1$

$= -0.01 - 0.02 + 0.00001$

$= -(0.03 - 0.00001)$

$= -0.02999.$

Homework 6

1. Consider the 2nd order linear differential equation:
$$y'' - 3y' + 2y = 2x^2 + 1. \tag{1}$$
   (a) Write down the associated homogeneous equation.
   (b) Solve for the general solution of the homogeneous equation.
   (c) Find a particular solution of (1) the form $y(x) = Ax^2 + Bx + C$.
   (d) What's the general solution of the equation (1)?
   (e) Find the particular solution of (1) satisfying the initial conditions:
$$y(0) = 0, y'(0) = 0.$$

2. Consider the 2nd order linear differential equation:
$$y'' + 2y' + y = 2\cos x - 2\sin x. \tag{2}$$
   (a) Write down the associated homogeneous equation.
   (b) Solve for the general solution of the homogeneous equation.
   (c) Find a particular solution of (2) of the form $y(x) = A\cos x + B\sin x$.
   (d) What's the general solution of the equation (2)?
   (e) Find the particular solution of (2) satisfying the initial conditions:
$$y(0) = 0, y'(0) = 0.$$

3. Consider the Euler's equation:
$$x^2y'' - 2xy' + y = 0. \tag{3}$$
   (a) Find the solutions of the form x^α.
   (b) What's the general solution of (3)?
   (c) Find the solution satisfying the initial condition:
$$y(1) = 2, y'(1) = 3.$$
   (d) Substitute $v = \ln x$. What differential equation for $y = y(v)$ do you get? Solve it to find the general solution of (3). Do you get the same answer as (b)?

4. Consider the Euler's equation:
$$x^2y'' - 3xy' + 4y = 0. \tag{4}$$

(a) Find the solutions of the form x^α.

(b) Find the solution of the form $x^\alpha \ln x$.

(c) What's the general solution of (4)?

(d) Find the solution satisfying the initial condition:

$$y(1) = 2, y'(1) = 3.$$

(e) Substitute $v = \ln x$. What differential equation for $y = y(v)$ do you get? Solve it to find the general solution of (4). Do you get the same answer as (b)?

5. Calculate the Wronskian of

(a) $y_1(x) = e^{r_1 x}, y_2(x) = e^{r_2 x}$.

(b) $y_1(x) = e^{rx}, y_2(x) = xe^{rx}$.

(c) $y_1(x) = e^{ax}\cos(bx), y_2(x) = e^{ax}\sin(bx)$.

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Homework 7.

1. By the trigonometric Identity $\cos 2x = \cos^2 x - \sin^2 x$,

get $\underline{\cos 2x} = \frac{1}{2} \cdot \underline{2\cos^2 x} + (-\frac{1}{5}) \cdot \underline{5\sin^2 x}$

therefore, the three functions $\{\cos 2x,\ 2\cos^2 x,\ 5\sin^2 x\}$ are not linearly independent.

$\triangle$

2. Calculate the Wronskian of y_1, y_2 and y_3.

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} x^2+x+1 & x+1 & x-1 \\ 2x+1 & 1 & 1 \\ 2 & 0 & 0 \end{vmatrix}$$

$$= 2 \begin{vmatrix} x+1 & x-1 \\ 1 & 1 \end{vmatrix} = 2\left((x+1) - (x-1)\right) = 4$$

Therefore, y_1, y_2, y_3 are linearly independent.

$\triangle$

3. Characteristic equation: $\lambda^3 - 6\lambda^2 + 10\lambda = \lambda(\lambda^2 - 6\lambda + 10) = 0$

$\lambda = 0,\ 3+i,\ 3-i.$

General solution: $C_1 e^{0\cdot x} + C_2 e^{3x}\cos x + C_3 e^{3x}\sin x$

$= C_1 + C_2 e^{3x}\cos x + C_3 e^{3x}\sin x$

$\triangle$

4. $\lambda^4 - 1 = 0 \Rightarrow (\lambda^2+1)(\lambda+1)(\lambda-1) = 0$. Four Roots $\lambda = 1, -1, i, -i$.

General Solutions: $C_1 e^x + C_2 e^{-x} + C_3\cos x + C_4 \sin x$

$\triangle$

5. $\lambda^4 + 2\lambda^2 + 1 = (\lambda^2+1)^2 = 0$ $\lambda = i$ (multiplicity 2)

$-i$ (multiplicity 2)

General solutions: $C_1 \cos x + C_2 \cdot x \cos x + C_3 \sin x + C_4 \cdot x \sin x$. △

6. $\lambda^4 - 2\lambda^2 + 1 = (\lambda^2 - 1)^2 = 0$ $\lambda = 1$ (multiplicity 2)

-1 (multipliaty 2).

General solutions: $C_1 e^x + C_2 x e^x + C_3 e^{-x} + C_4 x e^{-x}$. △

7. $\lambda^3 + 2\lambda^2 + 2\lambda + 1 = 0$

$(\lambda^3 + \lambda^2) + (\lambda^2 + 2\lambda + 1) = \lambda^2(\lambda+1) + (\lambda+1)^2 = (\lambda+1)(\lambda^2+\lambda+1) = 0$

$\lambda = -1$ or $\dfrac{-1 \pm \sqrt{3}i}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$.

General solutions: $C_1 e^{-x} + C_2 e^{-\frac{1}{2}x} \cos \frac{\sqrt{3}}{2} x + C_3 e^{-\frac{1}{2}x} \sin \frac{\sqrt{3}}{2} x$. △

8. $\lambda^3 - 2\lambda - 4 = 0$

By observing directly, $\lambda = 2$ is one root of this equation, therefore $(\lambda - 2)$ is a factor of $\lambda^3 - 2\lambda - 4$, then we could do "Long Division".

$$
\begin{array}{r|l}
 & \lambda^2 + 2\lambda + 2 \\
\hline
\lambda - 2 & \lambda^3 - 2\lambda - 4 \\
 & \lambda^3 - 2\lambda^2 \\
\hline
 & 2\lambda^2 - 2\lambda - 4 \\
 & 2\lambda^2 - 4\lambda \\
\hline
 & 2\lambda - 4 \\
 & 2\lambda - 4 \\
\hline
 & 0
\end{array}
$$

$\Rightarrow \lambda^3 - 2\lambda - 4 = (\lambda - 2)(\lambda^2 + 2\lambda + 2) = 0$

$\Rightarrow \lambda = 2$ or $-1 \pm i$.

General solutions $C_1 e^{2x} + C_2 e^{-x} \cos x + C_3 e^{-x} \sin x$ △.

9-11. (a) 9(a).

$c^2 - 4mk = 4^2 - 12 > 0 \Rightarrow$ Overdamped.

$$\begin{cases} x'' + 4x' + 3x = 0 \\ x(0) = 2, \quad v(0) = x'(0) = -2. \end{cases}$$

Roots for the characteristic equation $\lambda^2 + 4\lambda + 3 = 0$ is

$$\lambda = -1, \text{ or } -3.$$

$$x(t) = Ae^{-t} + Be^{-3t}$$

By the initial condition:

$$\begin{cases} x(0) = A + B = 2 \\ x'(0) = \left(-Ae^{-t} - 3Be^{-3t}\right)\Big|_{t=0} = -A - 3B = -2 \end{cases}$$

$\Rightarrow B = 0, \quad A = 2.$

$$x(t) = 2e^{-t}$$

10(a). $c^2 - 4mk = 4^2 - 4 \cdot 4 = 0 \Rightarrow$ Critically damped.

$$\begin{cases} x'' + 4x' + 4 = 0 \\ x(0) = 2, \quad x'(0) = -2 \end{cases}$$

Roots for the characteristic equation $\lambda^2 + 4\lambda + 4 = 0$ is

$\lambda = -2$ (with multiplicity 2).

$$x(t) = (A + Bt)e^{-2t}.$$

By the initial condition

$$\begin{cases} x(0) = A = 2 \\ x'(0) = Be^{-2t} - 2(A + Bt)e^{-2t}\Big|_{t=0} = B - 2(A + 0) = -2 \end{cases}$$

$\Rightarrow \begin{cases} A = 2 \\ B = 2 \end{cases} \Rightarrow x(t) = 2(1+t)e^{-2t}$

11 (a). $c^2 - 4mk = 4^2 - 4\cdot 5 = -4 < 0$ Under damped.

$$\begin{cases} x'' + 4x' + 5x = 0 \\ x(0) = 2, \quad x'(0) = -2 \end{cases}$$

Roots for the characteristic equation $\lambda^2 + 4\lambda + 5 = 0$ is

$$\lambda = -2 \pm i$$

$$x(t) = A e^{-2t} \cos t + B e^{-2t} \sin t.$$

By the initial condition:

$$\begin{cases} x(0) = A = 2 \\ x'(0) = -2Ae^{-2t}\cos t - Ae^{-2t}\sin t - 2Be^{-2t}\sin t + Be^{-2t}\cos t \Big|_{t=0} \end{cases}$$

$$= -2A + B = -2$$

$$\Rightarrow A = 2, \quad B = 2.$$

$$x(t) = \underline{2e^{-2t}(\cos t + \sin t)}.$$

b). Problem 11. is underdamped, we'd try to write $\cos t + \sin t$ in the form of $\cos(t-\alpha)$.

$$\cos t + \sin t = \sqrt{1^2+1^2}\,.\left(\cos t \cdot \frac{\sqrt{2}}{2} + \sin t \cdot \frac{\sqrt{2}}{2}\right)$$

$$= \sqrt{2} \quad \cos t \cos\frac{\pi}{4} + \sin t \sin\frac{\pi}{4}$$

$$= \sqrt{2} \cos(t - \pi/4).$$

$$\Rightarrow x(t) = 2\sqrt{2}\, e^{-2t} \cos\left(t - \frac{\pi}{4}\right).$$

c). Undamped Vibration is Harmonic Oscillation:

$$\begin{cases} x'' + 5x = 0 \\ x(0) = 2, \quad x'(0) = -2. \end{cases}$$

Continued.

General Solution

$$x(t) = A\cos\sqrt{5}t + B\sin\sqrt{5}t.$$

$$\begin{cases} x(0) = A = 2 \\ x'(0) = -\sqrt{5}A\sin\sqrt{5}t + \sqrt{5}B\cos\sqrt{5}t\Big|_{t=0} = \sqrt{5}B = -2 \end{cases}$$

$$\Rightarrow \quad A = 2, \quad B = \frac{-2}{\sqrt{5}}$$

~~x(t) =~~

Solution $\quad u(t) = 2\cos\sqrt{5}t - \frac{2}{\sqrt{5}}\sin\sqrt{5}t$

$$= \sqrt{4+\frac{4}{5}}\left(\cos\sqrt{5}t \cdot \frac{2}{2\sqrt{\frac{6}{5}}} + \sin\sqrt{5}t \cdot \frac{-\frac{2}{\sqrt{5}}}{2\sqrt{\frac{6}{5}}}\right)$$

$$= 2\sqrt{\frac{6}{5}}\left(\cos\sqrt{5}t \cdot \sqrt{\frac{5}{6}} + \sin\sqrt{5}t \cdot \left(-\sqrt{\frac{1}{6}}\right)\right)$$

$$= 2\sqrt{\frac{6}{5}}\cos(\sqrt{5}t - \alpha)$$

For $\begin{cases} \cos\alpha = \sqrt{\frac{5}{6}} \\ \sin\alpha = -\sqrt{\frac{1}{6}} \end{cases}$ $\Rightarrow$ α is in the second quadrant.

$$\alpha = -\tan^{-1}\frac{1}{\sqrt{5}} + \pi \doteq 2.72$$

$$\Rightarrow \quad u(t) = 2\sqrt{\frac{6}{5}}\cos(\sqrt{5}t - 2.72)$$

(d) ______

12. Since two maximum displacement appears, it's an underdamped system.

The Solution function has the form

$$x(t) = A e^{pt}\cos(\omega_1 t - \alpha)$$

$$\begin{cases} x(1) = 6 = Ae^{p} \\ x(2) = 2 = Ae^{2p} \end{cases} \Rightarrow \quad e^{p} = \frac{1}{3}, \quad p = -\ln 3 \qquad \text{Frequence } \omega_1 = \frac{2\pi}{2-1} = 2\pi$$

$\Rightarrow$ $-\ln 3 \pm 2\pi i$ are the two roots of equation $100\lambda^2 + c\lambda + k = 0$ $\Rightarrow \begin{cases} c = 200\ln 3 \\ k = 100((\ln 3)^2 + 4\pi^2) \end{cases}$

Homework 8

1. For each of equations (a)-(g), find a particular solution y_p by setting up the appropriate form of a particular solution y_p and determining the values of the coefficients. Then write down the general solution to the corresponding equation.

   (a) $y'' + 4y' + 3y = e^x$.

   (b) $y'' + 4y' + 3y = e^x + 4e^{-3x}$.

   (c) $y'' + 4y' + 4y = 8x^2 + 8\cos(2x)$.

   (d) $y'' + 4y' + 4y = 2e^{-2x}$.

   (e) $y'' + 4y = \cos(x) + \sin(x)$.

   (f) $y'' + 4y = \cos(2x) + \sin(2x)$.

   (g) $y'' - 2y' + 2y = xe^x$.

2. For the following equations

   (i) Set up the appropriate form of a particular solution y_p, but do not determine the values of the coefficients.

   (ii) On the other hand, use **variation of parameters** to find a particular solution y_p. Note that you have calculated the needed Wronskian in homework 6.

   (a) $y'' + 4y' + 3y = 8xe^{-x}$.

   (b) $y'' + 4y' + 4y = 6xe^{-2x}$.

   (c) $y'' - 2y' + 2y = e^x \cos(x)$.

1. (a): $y''+4y'+3y=e^x$.

associated homogeneous DE: $y''+4y'+3y=0$.

Characteristic equation: $\lambda^2+4\lambda+3=(\lambda+1)(\lambda+3)=0$.

There are two roots:

root	-1	-3
mult.	1	1

We get 2 basic solutions to the homogeneous DE:

$$y_1(x)=e^{-x}, \quad y_2(x)=e^{-3x}.$$

The non-homogeneous term $f(x)=e^x=1\cdot e^x=P_m(x)e^{\mu x}$.

so $P_m(x)=1$. (degree $m=0$). $\mu=1$. (multiplicity 0).

so we consider particular solution of the form $y_p(x)=x^k Q_m(x)e^{\mu x}$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\parallel}{x^0\cdot C\cdot e^{x}}=C\cdot e^x$.

$y_p'=C\cdot e^x=y_p''$ so:

$$y_p''+4y'+3y=C\cdot e^x+4C\cdot e^x+3\cdot Ce^x=8C\cdot e^x=e^x$$

$$\Rightarrow 8C=1 \Rightarrow C=\frac{1}{8} \Rightarrow y_p(x)=\frac{1}{8}\cdot e^x.$$

The general solution to the non-homogeneous DE:

$$y(x)=y_c(x)+y_p(x)=C_1\cdot y_1(x)+C_2\,y_2(x)+y_p(x)$$

$$=C_1\cdot e^{-x}+C_2\cdot e^{-3x}+\frac{1}{8}\cdot e^x.$$

(b). $y'' + 4y' + 3y = e^{x} + 4e^{-3x}$.

In (a) we have found the complementary solution: $y_c(x) = C_1 \cdot e^{-x} + C_2 \cdot e^{-3x}$
and a particular solution to $y'' + 4y' + 3y = e^{x}$.

Now we just need to find a particular solution to $\underbrace{y'' + 4y' + 3y = 4e^{-3x}}_{f_2(x)}$

Now $f_2(x) = 4 \cdot e^{-3x} = P_m(x) \cdot e^{\mu x}$ with $P_m(x) = 4$ ($m=0$)

so $\tilde{y}_p(x) = x^k \cdot Q_m(x) \cdot e^{\mu x}$ $\qquad \mu = -3$ ($k=1$)

root	-1	-3
mult	1	1

$= x \cdot C \cdot e^{-3x}$.

$\tilde{y}_p'(x) = C \cdot (e^{-3x} - 3xe^{-3x}) = e^{-3x} \cdot C \cdot (1-3x)$, $\tilde{y}_p''(x) = C \cdot (e^{-3x}(-3) - 3 \cdot e^{-3x} + 9xe^{-3x}) = e^{-3x} \cdot C \cdot (-6+9x)$.

so. $4e^{-3x} = e^{-3x} \cdot C \cdot (-6 + 9x + 4 \cdot (1-3x) + 3x) = -2C \cdot e^{-3x}$

$\Rightarrow -2C = 4 \Rightarrow C = -2 \Rightarrow \tilde{y}_p(x) = -2x \cdot e^{-3x}$

so the general solution to the original non-homogeneous DE
$\Rightarrow$

$$y(x) = y_c(x) + y_p(x) + \tilde{y}_p(x)$$
$$= C_1 e^{-x} + C_2 \cdot e^{-3x} + \frac{1}{8} e^{x} - 2x e^{-3x}.$$

(c). $y''+4y'+4y=8x^2+8\cos(2x)$

homogeneous DE: $y''+4y'+4y=0$.

Characteristic polynomial: $\lambda^2+4\lambda+4=(\lambda+2)^2=0 \Rightarrow \lambda=-2$

$\Rightarrow y_1(x)=e^{-2x},\ y_2(x)=x\cdot e^{-2x}$.

root	-2
mult.	2

First term $f_1(x)=8x^2=P_m(x)\cdot e^{\mu x} \Rightarrow P_m(x)=8x^2\ (m=2)$. $\mu=0.\ (k=0)$.

so $y_p(x)=x^k\cdot Q_m(x)\cdot e^{\mu x}=x^0\cdot(Ax^2+Bx+C)\,e^{0\cdot x}=Ax^2+Bx+C$.

$y_p'=2Ax+B,\ y_p''=2A \Rightarrow 8x^2=2A+4(2Ax+B)+4(Ax^2+Bx+C)$

$= 4Ax^2+(8A+4B)x+(2A+4B+4C)$

so we get $\begin{cases}4A=8\\ 8A+4B=0\\ 2A+4B+4C=0\end{cases} \Rightarrow \begin{cases}A=2\\ B=-4\\ C=3\end{cases} \Rightarrow y_p(x)=2x^2-4x+3$.

The other term:

$f(x)=8\cos(2x)=e^{ax}(A_m(x)\cos(bx))$ with $a=0$. $A_m(x)=8\ (m=0)$.

$a+bi=0+2i=2i\ (k=0)$.

so $\tilde{y}_p(x)=x^k\cdot e^{ax}(C_m(x)\cdot\cos(bx)+D_m(x)\sin(bx))$

$=x^0\cdot e^{0x}(C\cdot\cos(2x)+D\cdot\sin(2x))=C\cdot\cos(2x)+D\sin(2x)$.

$\tilde{y}_p'(x)=-2C\cdot\sin(2x)+2D\cos(2x),\ \tilde{y}_p''(x)=-4C\cdot\cos(2x)-4D\cdot\sin(2x)$.

So. $8\cdot\cos(2x)=(-4C\cdot\cos(2x)-4D\sin(2x))+4\cdot(-2C\cdot\sin(2x)+2D\cos(2x))$
$+4(C\cdot\cos(2x)+D\cdot\sin(2x))$

$=\cos(2x)\underbrace{(-4C+8D+4C)}_{8D}+\sin(2x)\underbrace{(-4D-8C+4D)}_{-8C}$

$\Rightarrow \begin{cases}8=8D\\ 0=-8C\end{cases} \Rightarrow \begin{cases}D=1\\ C=0\end{cases} \Rightarrow \tilde{y}_p(x)=\sin(2x)$ so $y(x)=y_c(x)+y_p(x)+\tilde{y}_p(x)$

$=\boxed{C_1e^{-2x}+C_2xe^{-2x}+(2x^2-4x+3)+\sin(2x)}$

(d). $y''+4y'+4y=2\cdot e^{-2x}$

By (c), we have. $\begin{array}{c|c} \text{root} & -2 \\ \hline \text{mult} & 2 \end{array}$. $\quad y_1(x)=e^{-2x},\ y_2(x)=x\cdot e^{-2x}$

Now $f(x)=2\cdot e^{-2x}=P_m(x)\cdot e^{\mu x} \Rightarrow P_m(x)=2\ (m=0).$
$\mu=-2\ (k=2).$

so $y_p(x)=x^k Q_m(x)\cdot e^{\mu x}=x^2\cdot A\cdot e^{-2x}$

$\Rightarrow y_p'(x)=A\left(2x\cdot e^{-2x}-2x^2 e^{-2x}\right)=2A\cdot(x-x^2)e^{-2x}.$

$y_p''(x)=2A\left((1-2x)\cdot e^{-2x}-2(x-x^2)e^{-2x}\right)=2A\cdot(1-4x+2x^2)e^{-2x}.$

so $2\cdot e^{-2x}=\left(2A(1-4x+2x^2)+8A(x-x^2)+4A\cdot x^2\right)\cdot e^{-2x}$

$\qquad \parallel$

$A\cdot(2-\cancel{8x}+4x^2+\cancel{8x}-8x^2+4x^2)=2A$

$\Rightarrow 2A=2 \Rightarrow A=1 \Rightarrow y_p(x)=x^2\cdot e^{-2x}.$

so $y(x)=y_c(x)+y_p(x)=C_1 e^{-2x}+C_2\cdot xe^{-2x}+x^2e^{-2x}$

(e) $y'' + 4y = \cos(x) + \sin(x)$.

homogeneous DE: $y'' + 4y = 0$. Characteristic eq.: $\lambda^2 + 4 = 0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$

so we get: $\qquad$ $(\lambda - 2i)(\lambda + 2i)$.

root	$2i$	$-2i$
mult	1	1

$\Rightarrow \quad y_1(x) = \cos(2x), \quad y_2(x) = \sin(2x)$.

$f(x) = \cos(x) + \sin(x) = e^{ax}(A_m(x)\cdot\cos(bx) + B_m(x)\sin(bx))$.

$\Rightarrow \quad A_m(x) = 1 = B_m(x) \quad (m=0)$. $\quad a + bi = 0 + i = i \quad (k=0)$

(not a root)

so $y_p(x) = x^k \cdot e^{ax}(C_m(x)\cos(bx) + D_m(x)\sin(bx))$

$= x^0 \cdot e^{0\cdot x}(C\cdot\cos(x) + D\cdot\sin(x)) = C\cdot\cos(x) + D\sin(x)$.

$y_p'(x) = -C\cdot\sin(x) + D\cdot\cos(x)$. $\quad y_p''(x) = -C\cdot\cos(x) - D\cdot\sin(x)$.

so $\cos x + \sin x = (-C\cdot\cos(x) - D\sin(x)) + 4(C\cdot\cos(x) + D\sin(x))$

$= 3C\cdot\cos x + 3D\cdot\sin(x)$

$\Rightarrow \begin{cases} 1 = 3C \\ 1 = 3D \end{cases} \Rightarrow C = D = \frac{1}{3} \Rightarrow y_p(x) = \frac{1}{3}(\cos x + \sin x)$

so $y(x) = y_c(x) + y_p(x) = C_1\cdot\cos(2x) + C_2\cdot\sin(2x) + \frac{1}{3}(\cos x + \sin x)$.

(f). $y''+4y=\cos(2x)+\sin(2x)$.

In (e), we got

root	$2i$	$-2i$
mult	1	1

$y_1(x)=\cos(2x)$, $y_2(x)=\sin(2x)$.

Now $f(x)=\cos(2x)+\sin(2x)=e^{ax}\left(A_m(x)\cdot\cos(bx)+B_m(x)\sin(bx)\right)$.

$\Rightarrow A_m(x)=1=B_m(x)$ $(m=0)$, $a+bi=0+2i=2i$ $(k=1)$.

so $y_p(x)=x^k e^{ax}\left(C_m(x)\cos(bx)+D_m(x)\sin(bx)\right)=x^1e^{0\cdot x}\left(C\cos(2x)+D\cdot\sin(2x)\right)$

$= x\cdot\left(C\cdot\cos(2x)+D\cdot\sin(2x)\right)$

$y_p'(x)=C\cdot\cos(2x)+D\cdot\sin(2x)+x\cdot\left(-2C\cdot\sin(2x)+2D\cos(2x)\right)$

$y_p''(x)=-2C\cdot\sin(2x)+2D\cos(2x)+\left(-2C\cdot\sin(2x)+2D\cos(2x)\right)+x\cdot\left(-4C\cdot\cos(2x)-4D\sin(2x)\right)$

$=\cos(2x)\cdot\left(\underbrace{2D+2D}_{4D}\right)+\sin(2x)\cdot\left(\underbrace{-2C-2C}_{-4C}\right)-4x\left(\cos(2x)+\sin(2x)\right)$

so $\cos(2x)+\sin(2x)=y_p''+4y_p'=4D\cdot\cos(2x)-4C\cdot\sin(2x)-4x\left(\cancel{\cos(2x)+\sin(2x)}\right)$
$+4x\left(\cancel{\cos(2x)+\sin(2x)}\right)$

$\Rightarrow \begin{cases}4D=1\\-4C=1\end{cases}\Rightarrow D=\frac{1}{4},\ C=-\frac{1}{4}$

$\Rightarrow y_p(x)=x\left(\frac{-1}{4}\cos(2x)+\frac{1}{4}\sin(2x)\right)=\frac{x}{4}\cdot\left(-\cos(2x)+\sin(2x)\right)$.

$\Rightarrow y(x)=y_c(x)+y_p(x)$

$=C_1\cdot\cos(2x)+C_2\cdot\sin(2x)+\frac{x}{4}\left(-\cos(2x)+\sin(2x)\right)$.

(g). $y'' - 2y' + 2y = x \cdot e^x$.

homogeneous eq.: $y'' - 2y' + 2y = 0$ characteristic eq.: $\lambda^2 - 2\lambda + 2 = 0$

$$\Downarrow$$

root	$1+i$	$1-i$
mult	1	1

$y_1(x) = e^x \cdot \cos x$. $\Leftarrow$ $\lambda = \frac{2 \pm \sqrt{(-2)^2 - 4\times 2}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$

$y_2(x) = e^x \cdot \sin x$.

$f(x) = x \cdot e^x = P_m(x) \cdot e^{\mu x} \Rightarrow P_m(x) = x \ (m=1)$, $\mu = 1 \ (k=0)$.
(not a root).

so $y_p(x) = x^k \cdot Q_m(x) e^{\mu x} = x^0 \cdot (Ax+B) \cdot e^x = (Ax+B) \cdot e^x$.

$\Rightarrow y_p'(x) = A \cdot e^x + (Ax+B) e^x = (Ax + (A+B)) e^x$.

$y_p''(x) = A \cdot e^x + (Ax + (A+B)) e^x = (Ax + (2A+B)) \cdot e^x$.

so $x \cdot e^x = (Ax + (2A+B)) \cdot e^x - 2 \cdot (Ax + (A+B)) e^x + 2(Ax+B) \cdot e^x$

$$\| $$

$e^x \cdot (Ax + (2A+B-2A-2B+2B)) = e^x \cdot (Ax+B)$

$\Rightarrow A = 1$, $B = 0$. so $y_p(x) = x \cdot e^x$.

$\Rightarrow y(x) = y_c(x) + y_p(x)$

$= C_1 \cdot e^x \cos x + C_2 \cdot e^x \sin x + x \cdot e^x$.

2. (a) $y''+4y'+3y=8x e^{-x}$.

~~homogeneous~~ By 1(a), we get

root	-1	-3
mult	1	1

$y_1(x)=e^{-x}$
$y_2(x)=e^{-3x}$

$f(x)=8x\cdot e^{-x}=P_m(x)\cdot e^{\mu x} \Rightarrow P_m(x)=8x\ (m=1),\quad \mu=-1\ (k=1).$

so (i) $y_p(x)=x^k Q_m(x) e^{\mu x}=x^1(Ax+B)\cdot e^{-x}=(Ax^2+Bx)e^{-x}$.

(ii). We need Wronskian: $W(x)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} e^{-x} & e^{-3x} \\ -e^{-x} & -3e^{-3x} \end{vmatrix}=-3e^{-4x}-(-e^{-4x}) = -2\cdot e^{-4x}$.

so

$$\tilde{y}_p(x)=-y_1(x)\int\frac{y_2(x)f(x)}{W(x)}dx+y_2(x)\int\frac{y_1(x)f(x)}{W(x)}dx$$

$$=-e^{-x}\int\frac{e^{-3x}\cdot 8xe^{-x}}{-2\cdot e^{-4x}}dx+e^{-3x}\int\frac{e^{-x}\cdot 8xe^{-x}}{-2\cdot e^{-4x}}dx$$

$$=4e^{-x}\int x\,dx-4e^{-3x}\int x\cdot e^{2x}dx$$

$$=2x^2e^{-x}-4e^{-3x}\int x\cdot d(e^{2x})\frac{1}{2}=2x^2e^{-x}-2xe^{-x}+2e^{-3x}\int e^{2x}dx$$

$$2x^2e^{-x}-2e^{-3x}\left(x\cdot e^{2x}-\int e^{2x}dx\right)$$

$$=\boxed{(2x^2-2x)\cdot e^{-x}+e^{-x}}$$

From this, we also know that the particular-solution in (i) is

$$y_p(x)=(Ax^2+Bx)\cdot e^{-x}=(2x^2-2x)\cdot e^{-x}\ \left(=\tilde{y}_p(x)-\underline{e^{-x}}\right)$$

sol. of the homogeneous eq.

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2.(b) $y''+4y'+4y=6x\cdot e^{-2x}$.

By 1.(c), we get $\begin{array}{c|c} \text{root} & -2 \\ \hline \text{mult} & 2 \end{array}$ $y_1(x)=e^{-2x}, y_2(x)=x\cdot e^{-2x}$

$f(x)=6x\cdot e^{-2x}=P_m(x)\cdot e^{\mu x} \Rightarrow P_m(x)=6x\ (m=1),\ \mu=-2\ (k=2)$.

so (i) $y_p(x)=x^k\cdot Q_m(x)\cdot e^{\mu x}=x^2\cdot(Ax+B)e^{-2x}$

(ii) The Wronskian $W(y_1,y_2)(x)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} e^{-2x} & x\cdot e^{-2x} \\ -2\cdot e^{-2x} & e^{-2x}-2x e^{-2x} \end{vmatrix}=e^{-4x}$

so $\tilde{y}_p(x)=-y_1(x)\cdot\int\frac{y_2(x)f(x)}{W(y_1,y_2)}dx+y_2(x)\cdot\int\frac{y_1(x)f(x)}{W(y_1,y_2)}dx$

$=-e^{-2x}\cdot\int\frac{xe^{-2x}\cdot 6x\cdot e^{-2x}}{e^{-4x}}dx+x\cdot e^{-2x}\int\frac{e^{-2x}6x\cdot e^{-2x}}{e^{-4x}}dx$

$=-e^{-2x}\cdot\int 6x^2dx+x\cdot e^{-2x}\int 6x\,dx$

$=-2x^3\cdot e^{-2x}+3x\cdot e^{-2x}\cdot x^2=\boxed{x^3\cdot e^{-2x}}$

The particular solution in (i) is the same as $\tilde{y}_p(x)$:

$y_p(x)=x^2(Ax+B)e^{-2x}=x^3\cdot e^{-2x}$. $\begin{pmatrix} A=1 \\ B=0 \end{pmatrix}$.

2.(c). $y''-2y'+2y=e^{x}\cdot\cos(x)$.

By 1(g), we know that

root	$1+i$	$1-i$
mult	1	1

$y_1(x)=e^{x}\cos x$
$y_2(x)=e^{x}\sin x$.

$f(x)=e^{x}\cdot\cos x=e^{ax}\cdot A_m(x)\cos(bx) \Rightarrow A_m(x)=1.(m=0)$.
$a+bi=1+i \quad (k=1)$.

so

(i) $y_p(x)=x^k e^{ax}\left(C_m(x)\cos(bx)+D_m(x)\sin(bx)\right)$

$=x^1\cdot e^{x}\left((\cancel{S_1x+}S_0)\cos(x)+(T_0)\cdot\sin(x)\right)=x\cdot e^{x}\cdot(S_0\cdot\cos x+T_0\cdot\sin x)$.

(ii). Wronskian: $W(x)=\begin{vmatrix} e^{x}\cos x & e^{x}\sin x \\ e^{x}\cos x-e^{x}\sin x & e^{x}\sin x+e^{x}\cos x\end{vmatrix}=e^{x}\cos x(e^{x}\sin x+e^{x}\cos x)-e^{x}\sin x\cdot e^{x}(\cos x-\sin x)$

$=e^{2x}(\cos^2 x+\sin^2 x)=e^{2x}$.

so $\tilde{y}_p(x)=-e^{x}\cdot\cos x\cdot\int\frac{e^{x}\sin x\cdot e^{x}\cos x}{e^{2x}}dx+e^{x}\cdot\sin x\int\frac{e^{x}\cos x\, e^{x}\cos x}{e^{2x}}dx$

$=-e^{x}\cdot\cos x\cdot\int\underbrace{\sin x\cdot\cos x}_{=\frac{1}{2}\sin(2x)}dx+e^{x}\sin x\int\underbrace{\cos^2 x}_{=\frac{1+\cos(2x)}{2}}dx$

$=-(e^{x}\cancel{\cos x})\cancel{\tfrac{1}{2}}\,\frac{1}{2}\left(-\cos(2x)\right)+e^{x}\cdot\sin x\,\frac{1}{2}\cdot\left(x+\frac{1}{2}\sin(2x)\right)$

$=\frac{1}{4}e^{x}\cdot\left(\cos x\cdot\cos(2x)+\sin x\cdot\sin(2x)\right)+\frac{1}{2}x\cdot e^{x}\cdot\sin x$

$=\boxed{\frac{1}{4}e^{x}\cdot\cos x+\frac{1}{2}x\cdot e^{x}\cdot\sin x}$

$\Rightarrow$ The particular solution in (i) is

$y_p(x)=\tilde{y}_p(x)-\underbrace{\frac{1}{4}e^{x}\cos x}_{\text{solution of the homogeneous DE}}=\frac{1}{2}x\cdot e^{x}\cdot\sin x \quad \left(=x\cdot e^{x}(S_0\cos x+T_0\sin x)\right.$

i.e. $S_0=0$,
$T_0=\frac{1}{2}$.

1. (a)&(c) $\begin{cases} x''(t)+25x(t)=50\sin(\omega t). \\ x(0)=0,\quad x'(0)=0. \end{cases}$ $\boxed{\omega\neq 5.}$

homogeneous DE: $x''+25x=0 \Rightarrow \lambda^2+25=0=(\lambda-5i)(\lambda+5i)$.

$\Rightarrow$

root	$5i$	$-5i$
mult	1	1

$\underbrace{\cancel{y_1(x)}}_{x_1(t)}=\cos(5t)$, $\underbrace{\cancel{y_2(x)}}_{x_2(t)}=\sin(5t)$.

$\cancel{f}(t)=50\cdot\sin(\omega t)=e^{0t}(A_m(t)\cdot\sin(\omega t)) \Rightarrow A_m(t)=50\ (m=0)$.
$a+\omega i=0+\omega i=\omega i\ (k=0$ since $\omega\neq 5)$

so. $x_p(t)=t^k e^{at}(C_m(t)\cos(\omega t)+D_m(t)\sin(\omega t))=t^0\cdot e^{0t}(C\cdot\cos(\omega t)+D\cdot\sin(\omega t))$ $= C\cdot\cos(\omega t)+D\cdot\sin(\omega t)$.

$\Rightarrow x_p'(t)=-C\cdot\sin(\omega t)\cdot\omega+D\cos(\omega t)\cdot\omega$

$x_p''(t)=-C\cdot\omega^2\cdot\cos(\omega t)-D\omega^2\cdot\sin(\omega t).=-\omega^2\cdot(C\cdot\cos(\omega t)+D\sin(\omega t))$

$\Rightarrow \underbrace{x_p''(t)+25x_p(t)}_{=50\sin(\omega t)}=\underbrace{-\omega^2\cdot(C\cdot\cos(\omega t)+D\sin(\omega t))+25(C\cdot\cos(\omega t)+D\sin(\omega t))}_{=(25-\omega^2)\cdot C\cdot\cos(\omega t)+(25-\omega^2)D\cdot\sin(\omega t).}$

$\Rightarrow \begin{cases} 0=(25-\omega^2)\cdot C \\ 50=(25-\omega^2)D \end{cases} \Rightarrow \begin{cases} C=0 \\ D=\frac{50}{25-\omega^2} \end{cases} \Rightarrow \frac{50}{25-\omega^2}\sin(\omega t)=x_p(t).$

$\Rightarrow x(t)=C_1\cdot x_1(t)+C_2x_2(t)+x_p(t)=C_1\cos(5t)+C_2\cdot\sin(5t)+\frac{50}{25-\omega^2}\sin(\omega t)$

$\Rightarrow 0=x(0)=C_1,\ 0=x'(0)=\left[-C_1\cdot\sin(5t)\cdot 5+C_2\cdot\cos(5t)\cdot 5+\frac{50}{25-\omega^2}\cos(\omega t)\cdot\omega\right]\Big|_{t=0} = 5C_2+\frac{50\omega}{25-\omega^2}$

$\Rightarrow C_1=0,\ C_2=-\frac{10\omega}{25-\omega^2}$

$\Rightarrow x(t)=-\frac{10\omega}{25-\omega^2}\sin(5t)+\frac{50}{25-\omega^2}\sin(\omega t)=\boxed{\frac{10}{25-\omega^2}(-\omega\cdot\sin(5t)+5\sin(\omega t))}$ $\omega\neq 5$

$\omega=4$: $x(t)=\frac{10}{9}(5\cdot\sin(4t)-4\sin(5t))$; $\omega=6$: $x(t)=\frac{10}{11}(6\sin(5t)-5\sin(6t))$

1. (b). $\begin{cases} x''(t)+25x(t)=50\cdot\sin(5t). \\ x(0)=0,\ x'(0)=0. \end{cases}$

From (a), we know that. $\begin{array}{c|c|c} \text{root} & 5i & -5i \\ \hline \text{mult} & 1 & 1 \end{array}$. $x_1(t)=\cos(5t),\ x_2(t)=\sin(5t)$

$f(t)=50\cdot\sin(5t)=e^{at}\cdot(A_m(t)\sin(bt)) \Rightarrow A_m(t)=50\ (m=0).$
$a+bi=0+5i=5i\ (k=1).$

so $x_p(t)=t^{k}\overset{e^{at}}{}(C_m(t)\cos(bt)+D_m(t)\sin(bt))=t^1\cdot e^{0\cdot t}\cdot(C\cdot\cos(5t)+D\cdot\sin(5t)).$

$\Rightarrow x_p'(t)=C\cdot\cos(5t)+D\cdot\sin(5t)+t\cdot(-C\cdot\sin(5t)\cdot5+D\cos(5t)\cdot5).\quad t\cdot(\overset{\shortparallel}{C}\cdot\cos(5t)+D\cdot\sin(5t))$

$\Rightarrow x_p''(t)=+10(-C\cdot\sin(5t)+D\cos(5t))-25t\cdot(C\cdot\cos(5t)+D\sin(5t)).$

$\Rightarrow x_p''(t)+25x_p(t)=10\cdot(-C\cdot\sin(5t)+D\cos(5t)) \Rightarrow \begin{cases}-10C=50. \\ 10D=0\end{cases}$

$\overset{\shortparallel}{50\sin(5t)}$

$\Downarrow$

so $x_p(t)=t\cdot(-5\cdot\cos(5t))=-5t\cdot\cos(5t).\quad \begin{cases}C=-5 \\ D=0.\end{cases}$

$\Rightarrow x(t)=C_1\cdot\cos(5t)+C_2\cdot\sin(5t)-5t\cdot\cos(5t).$

$\Rightarrow 0=x(0)=C_1,\ 0=x'(0)=\left[-C_1\cdot\sin(5t)\cdot5+C_2\cos(5t)\cdot5-5\cos(5t)+5t\sin(5t)\right]_{t=0}$

$\Rightarrow C_1=0,\ C_2=1.\quad \overset{\shortparallel}{5C_2-5}$

$\Rightarrow x(t)=\boxed{\sin(5t)-5t\cdot\cos(5t).} \Rightarrow \left(\begin{array}{c}\text{Maximum amplitude} \\ \overset{\mathrm{VI}}{5t-1} \xrightarrow{t\to+\infty} +\infty.\end{array} \Rightarrow \text{Resonance}\right)$

Note that the solutions in 1(a) and 1(b) are related by taking limit:

$\lim_{\omega\to5}\frac{10}{25-\omega^2}(-\omega\cdot\sin(5t)+5\cdot\sin(\omega t)) \overset{\text{L'Hospital}}{=\!=} \lim_{\omega\to5}\frac{10}{-2\omega}(-\sin(5t)+5\cos(\omega t)\cdot t)$

(From 1(a)) $\quad = \sin(5t)-5t\cdot\cos(5t)$ ($\leftarrow$ From 1(b))

$$(\lambda+1)^2+7^2 = (\lambda^2+2\lambda+1)+49 = \lambda^2+2\lambda+50=0$$

2. (a)&(b). $\begin{cases} x''+2x'+50x=100\cdot\sin(\omega t). \\ x(0)=10,\ x'(0)=10. \end{cases}$

homogeneous DE: $x''+2x'+50x=0$. $\Rightarrow$ Characteristic eq.: $\lambda^2+2\lambda+50=0$

$\Rightarrow 1+\lambda=\pm 7i \Longrightarrow$

root	$-1+7i$	$-1-7i$
mult	1	1

$x_1(t)=e^{-t}\cos(7t)$

$x_2(t)=e^{-t}\sin(7t)$.

So the complementary solution is (transient). $\underline{x_c(t)=C_1\cdot e^{-t}\cos(7t)+C_2\cdot e^{-t}\sin(7t)}$.

$f(t)=100\cdot\sin(\omega t)=e^{at}\cdot A_m(t)\sin(\omega t)\Rightarrow A_m(t)=100\ (m=0)$. $a+i\omega=0+i\omega=\omega i$ ($k=0$ not a root)

so $x_p(t)=x^k\cdot e^{at}(C_m(t)\cos(\omega t)+D_m\sin(\omega t))=x^0\cdot e^{0t}\cdot(C\cdot\cos(\omega t)+D\cdot\sin(\omega t)) = C\cdot\cos(\omega t)+D\sin(\omega t)$

$\Rightarrow x_p'(t)=-C\omega\sin(\omega t)+D\omega\cdot\cos(\omega t)$.

$\underline{x_p''(t)=-C\omega^2\cos(\omega t)-D\omega^2\sin(\omega t)}\Rightarrow x_p''(t)+2x_p'(t)+50x_p(t)=100\cdot\sin(\omega t)=$

$\underline{\qquad}\ \ (-C\cdot\omega\sin(\omega t)+D\cdot\omega\cos(\omega t))+50(C\cos(\omega t)+D\sin(\omega t))$

$=\omega^2\cdot(-C\cos(\omega t)-D\sin(\omega t))+2(-C\cdot\omega\sin(\omega t)+D\cdot\omega\cos(\omega t))+50(C\cos(\omega t)+D\sin(\omega t))$

$=\cos(\omega t)\cdot(-\omega^2\cdot C+2D\cdot\omega+50\cdot C)+\sin(\omega t)\cdot(-\omega^2\cdot D-2C\cdot\omega+50\cdot D)$.

$\Rightarrow\begin{cases}(50-\omega^2)C+2D\omega=0. & ① \\ -2\omega C\quad+(50-\omega^2)D=100. & ②\end{cases}\Longrightarrow\begin{cases} C=\dfrac{-200\cdot\omega}{(50-\omega^2)^2+4\omega^2} & \left(\Leftarrow \begin{matrix}①\times(50-\omega^2)\\ - \\ ②\times(2\omega)\end{matrix}\right) \\ D=\dfrac{100\cdot(50-\omega^2)}{(50-\omega^2)^2+4\omega^2} & \left(\Leftarrow\begin{matrix}①\times 2\omega\\ + \\ ②\times(50-\omega^2)\end{matrix}\right)\end{cases}$

$\xrightarrow{\omega=5}\begin{cases}C=-\dfrac{40}{29}\\ D=\dfrac{100}{29}\end{cases}\Rightarrow x(t)=C_1e^{-t}\cos(7t)+C_2e^{-t}\sin(7t)+\underbrace{\left(-\frac{40}{29}\cos(5t)+\frac{100}{29}\sin(5t)\right)}_{x_p(t)}$.

Use initial conditions:

$\boxed{10=x(0)=C_1-\frac{40}{29},\ 10=x'(0)=-C_1+7C_2+\frac{500}{29}}$

$\Rightarrow C_1=\frac{330}{29},\ C_2=\frac{120}{29\times 7}$

$x_p(t) = \frac{20}{29}(-2\cos 5t+5\sin(5t)) = \frac{20}{\sqrt{29}}\cos\left(5t-\left(\pi-\tan^{-1}\left(\frac{5}{2}\right)\right)\right)$

so finally — transient part. — Steady Periodic part.

$$\boxed{x(t)=\frac{10e^{-t}}{29}\left(33\cos(7t)+\frac{12}{7}\sin(7t)\right)+\frac{20}{\sqrt{29}}\cos\left[5t-\left(\pi-\tan^{-1}\left(\frac{5}{2}\right)\right)\right]}$$

2.(c). In (a) & (b), we have obtained the steady periodic oscillation for any ω to be.

$$x_p(t) = C\cdot\cos(\omega t) + D\cdot\sin(\omega t)$$

$$= -\frac{200\omega}{(50-\omega^2)^2+4\omega^2}\cos(\omega t) + \frac{100(50-\omega^2)}{(50-\omega^2)^2+4\omega^2}\sin(\omega t).$$

So the amplitude is:

$$A(\omega) = \text{Amplitude} = \sqrt{C^2+D^2} = \frac{100}{(50-\omega^2)^2+4\omega^2}\sqrt{(2\omega)^2+(50-\omega^2)^2}$$

$$= \frac{100}{\sqrt{(50-\omega^2)^2+4\omega^2}}$$

The denominator^2

$$\underset{\uparrow}{\overset{\shortparallel}{(50-\omega^2)^2+4\omega^2}} = 5\omega^4-100\omega^2+2500 = 5\cdot(\omega^4-20\omega^2+100)+2000$$

$$= 5\cdot(\omega^2-10)^2+2000.$$

obtains minimum when $\omega=\sqrt{10}$.

So $A(\omega)$ obtains the maximum when $\boxed{\omega=\sqrt{10}}$ which is the practical resonance frequency.

3. (a): $mx'' + c\cdot x' + kx = f(t)$. $x = x(t)$.

Let $x_1 = x$, $x_2 = x' = x_1'$ $\Rightarrow x_2' = x'' = \frac{1}{m}\cdot(-c\cdot x' - k\cdot x + f(t))$

$$\Rightarrow \begin{cases} x_1' = x_2 \\ x_2' = \frac{1}{m}\cdot(-k\cdot x_1 - c\cdot x_2) + \frac{1}{m} f(t). \end{cases}$$

This is Linear system, non-homogeneous if $f(t) \neq 0$. Because there are 2 unknown functions and 2 1st order DE, we need 2 initial conditions to determine a unique solution.

(b). $x''' - 6x'' + 10x' = x^2$. $x = x(t)$.

Let. $x_1 = x$, $x_2 = x' = x_1'$, $x_3 = x'' = x_2'$.

$$\Rightarrow x_3' = x''' = 6x'' - 10x' + x^2 = x_1^2 - 10x_2 + 6\cdot x_3.$$

$$\Rightarrow \begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_1^2 - 10x_2 + 6x_3 \end{cases}$$

This is a non-linear system.

3 initial conditions are needed to determine a unique solution.

3.(c). $x'' - x - y = 0$, $y'' + x + y = 0$. $x = x(t)$, $y = y(t)$

Let $x_1 = x$, $x_2 = x' = x_1'$. $y_1 = y$, $y_2 = y' = y_1'$.

$\Rightarrow$ $x_2' = x'' = x + y = x_1 + y_1$

$y_2' = y'' = -x - y = -x_1 - y_1$

$\Rightarrow$ $\begin{cases} x_1' = x_2 \\ x_2' = x_1 + y_1 \\ y_1' = y_2 \\ y_2' = -x_1 - y_1 \end{cases}$

This is a homogeneous linear system.

4 initial conditions are needed to determine a unique solution

---

3(d). $x'' + x' - x - y = 0$, $y'' + y' + x + y = \cos(t)$. $(x = x(t), y = y(t))$

Let $x_1 = x$, $x_2 = x' = x_1'$, $y_1 = y$, $y_2 = y' = y_1'$

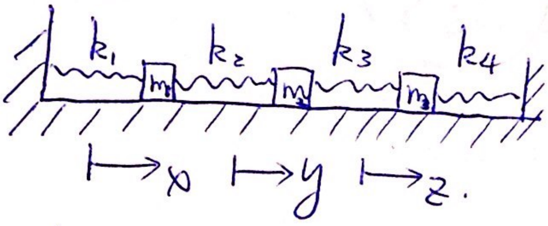
$\Rightarrow$ $x_2' = x'' = -x' + x + y = -x_2 + x_1 + y_1$

$y_2' = y'' = -y' - x - y + \cos(t) = -y_2 - x_1 - y_1 + \cos(t)$

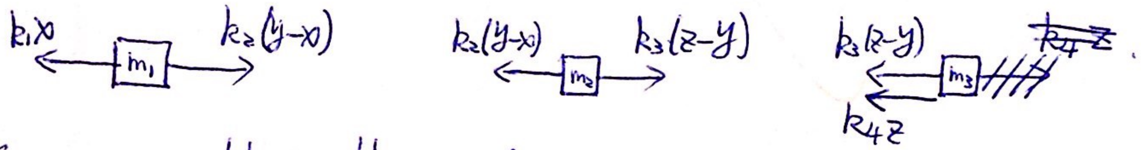
$\Rightarrow$ $\begin{cases} x_1' = x_2 \\ x_2' = x_1 - x_2 + y_1 \\ y_1' = y_2 \\ y_2' = -x_1 - y_1 - y_2 + \cos t. \end{cases}$

This is a non-homogeneous linear system. 4 initial conditions are needed to determine a unique solution.

4. Three objects case:



The "free body diagrams":



so we obtain the system:

$$\begin{cases} m_1 \cdot x'' = -k_1 x + k_2(y-x) = -(k_1+k_2)x + k_2 y \\ m_2 y'' = -k_2(y-x) + k_3(z-y) = k_2 x - (k_2+k_3)y + k_3 z \\ m_3 z'' = -k_3(z-y) \underline{\cancel{+}} k_4 z = k_3 y - (k_3+k_4) z. \end{cases}$$

1. $y'' + 2y' + y = e^{-x} + xe^{x}$.

(a). the general complementary solution is the general solution to the associated homogeneous equation:

$$y'' + 2y' + y = 0.$$

The characteristic equation: $\lambda^2 + 2\lambda + 1 = (\lambda+1)^2 = 0$.

so we get

root	-1
mult	2

$y_1(x) = e^{-x}$, $y_2(x) = xe^{-x}$.

$\Rightarrow$ the general complementary solution $y_c(x) = C_1 \cdot e^{-x} + C_2 \cdot xe^{-x}$.

(b). Let $f(x) = e^{-x}$, $F(x) = x\cdot e^{x}$, so that $e^{-x} + x\cdot e^{x} = f(x) + F(x)$

For $f(x) = e^{-x} = P_m(x)\cdot e^{\mu x} \Rightarrow P_m(x) = 1$ $(m=0)$, $\mu = -1$ $(k=2)$.

so $y_p(x) = x^k \cdot Q_m(x)\cdot e^{\mu x} = x^2 \cdot A\cdot e^{-x} \Rightarrow y_p'(x) = A\cdot(2x e^{-x} - x^2 e^{-x})$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \; A\cdot(2x - x^2)\cdot e^{-x}$

$\Rightarrow y_p''(x) = A\cdot((2-2x)\cdot e^{-x} - (2x-x^2)e^{-x}) = A\cdot(\overset{(2-4x+x^2)}{\cancel{2-2x}})\cdot e^{-x}$.

So. $y_p'' + 2y_p' + y_p = A\cdot(\overset{2-4x+x^2}{\cancel{2-2x}})e^{-x} + 2\cdot A\cdot(2x - x^2)\cdot e^{-x} + x^2\cdot A\cdot e^{-x}$

$\quad \|$

$\quad e^{-x} \qquad = A\cdot(\underset{2-4x+x^2}{\cancel{2-2x}} + 4x - 2x^2 + x^2)e^{-x} = 2A\cdot e^{-x}$.

$\Rightarrow 2A = 1 \Rightarrow A = \frac{1}{2} \Rightarrow y_p(x) = \frac{1}{2}x^2 e^{-x}$ is a particular solution

to the non-homogeneous DE $y'' + 2y' + y = e^{-x}$.

Next we consider the other non-homogeneous term $F(x) = x\cdot e^{x}$:

$\tilde{f}(x) = x \cdot e^{x} = P_m(x) \cdot e^{\mu x} \Rightarrow P_m(x) = x \ (m=1).\ \mu = 1 \ (k=0).$

so $\tilde{y}_p(x) = x^k \cdot Q_m(x) e^{\mu x} = x^0 \cdot (C \cdot x + D) \cdot e^{1 \cdot x} = (Cx+D)e^{x}$

$\Rightarrow \tilde{y}_p'(x) = C \cdot e^{x} + (Cx+D)e^{x} = (Cx + (C+D)) \cdot e^{x}$

$\tilde{y}_p''(x) = C \cdot e^{x} + (Cx + (C+D))e^{x} = (Cx + (2C+D))e^{x}.$

so. $\tilde{y}_p''(x) + 2\tilde{y}_p'(x) + \tilde{y}_p(x) = (Cx + (2C+D)) \cdot e^{x} + 2(Cx + (C+D))e^{x} + (Cx+D)e^{x}$

$x \cdot e^{x} = (4C \cdot x + (4C+4D)) \cdot e^{x}$

$\Rightarrow \begin{cases} 1 = 4C \\ 0 = 4C + 4D \end{cases} \Rightarrow \begin{cases} C = \frac{1}{4} \\ D = -\frac{1}{4} \end{cases} \Rightarrow \tilde{y}_p(x) = \left(\frac{1}{4}x - \frac{1}{4}\right)e^{x}$

is a particular solution to the non-homogeneous DE:

$$y''(x) + 2y' + y = x \cdot e^{x} \left(= \tilde{f}(x)\right).$$

So a particular solution to the original non-homogeneous DE is given by.

$$u(x) = y_p(x) + \tilde{y}_p(x) = \frac{1}{2}x^2 \cdot e^{-x} + \frac{1}{4}(x-1)e^{x}.$$

1.(c). By 1(a) and 1(b), we get that the general solution to the original non-homogeneous DE is

$$y(x) = C_1 \cdot y_1(x) + C_2\, y_2(x) + u(x)$$
$$= C_1 e^{-x} + C_2 \cdot x e^{-x} + \frac{1}{2}x^2 e^{-x} + \frac{1}{4}(x-1)\cdot e^{x}.$$

Now we use the initial conditions:

$$0 = y(0) = C_1 - \frac{1}{4}; \quad 0 = y'(0) = \Big[-C_1 e^{-x} + C_2 \cdot e^{-x} - C_2 \cdot x \cdot e^{-x} + 2x \cdot e^{-x} - \frac{1}{2}x^2 \cdot e^{-x} + \frac{1}{4}\cdot e^{x} + \frac{1}{4}(x-1)e^{x}\Big]\Big|_{x=0}$$

So. $C_1 = \frac{1}{4}$, $C_2 = \frac{1}{4}$. $\Leftarrow$

$$\| \quad -C_1 + C_2 + \frac{1}{4} - \frac{1}{4} = -C_1 + C_2.$$

So we get the solution to the initial value problem to be:

$$y(x) = \frac{1}{4}\cdot e^{-x} + \frac{1}{4} x e^{-x} + \frac{1}{2}x^2 \cdot e^{-x} + \frac{1}{4}(x-1)e^{x}.$$
$$= \frac{1}{4} e^{-x}\cdot(1 + x + 2x^2) + \frac{1}{4}\cdot(x-1)e^{x}.$$

2. $y'' + 25y = \sec(5x)$.

The associated homogeneous DE: $y'' + 25y = 0$.

Characteristic equation $\lambda^2 + 25\lambda = (\lambda - 5i)(\lambda + 5i) = 0$

$\Rightarrow$

root	$5i$	$-5i$
mult.	1	1

$\Rightarrow y_1(x) = \cos(5x)$, $y_2(x) = \sin(5x)$.

The Wronskian: $W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(5x) & \sin(5x) \\ -5\cdot\sin(5x) & 5\cos(5x) \end{vmatrix} = 5$

$= \cos(5x)\cdot 5\cos(5x) - \sin(5x)(-5\sin(5x))$

By the method of variation of parameters

$$y_p(x) = -y_1(x)\cdot\int \frac{y_2(x)f(x)}{W(y_1,y_2)}\,dx + y_2(x)\int \frac{y_1(x)f(x)}{W(y_1,y_2)}\,dx$$

$$= -\cos(5x)\cdot\int \frac{\sin(5x)\cdot\sec(5x)}{5}\,dx + \sin(5x)\cdot\int \frac{\cos(5x)\cdot\sec(5x)}{5}\,dx$$

$$= -\frac{\cos(5x)}{5}\cdot\int \frac{\sin(5x)}{\cos(5x)}\,dx + \frac{\sin(5x)}{5}\int 1\cdot dx$$

$$= -\frac{\cos(5x)}{5}\left(\int \frac{d(\cos(5x))}{\cos(5x)}\cdot\left(-\frac{1}{5}\right)\right) + \frac{\sin(5x)}{5}\cdot x$$

(or $u = \cos(5x)$ as substitution)

$$= \frac{\cos(5x)}{25}\cdot\ln|\cos(5x)| + \frac{1}{5}x\cdot\sin(5x).$$

3.(a). $x''(t) + C\cdot x'(t) + 4x(t) = F_0 \cdot \cos(2t)$

By assumption, $C=0$, $F_0=0$, initial position $=10$ ("stretched by 10 unit length") initial velocity $=0$ ("from still").

so $\begin{cases} x''(t)+4x(t)=0 \\ x(0)=10,\ x'(0) \end{cases}$ Characteristic equation: $\lambda^2+4=(\lambda-2i)(\lambda+2i)=0$

$\Longrightarrow$

root	$2i$	$-2i$
mult.	1	1

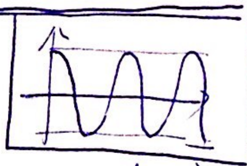
$x_1(t)=\cos(2t)$, $x_2(t)=\sin(2t)$.

So the general solution is $x(t)=C_1\cdot\cos(2t)+C_2\cdot\sin(2t)$.

Using the initial conditions: $10=x(0)=C_1$, $0=x'(0)=\underbrace{[-C_1 2\sin(2t)+C_2 2\cos(2t)]\Big|_{t=0}}_{= 2\cdot C_2}$

so $C_1=10$, $C_2=0 \Rightarrow \boxed{x(t)=10\cdot\cos(2t)}$

This is a $\boxed{\text{harmonic Oscillation}}$



(b). By assumption, we consider the initial value problem:

$\begin{cases} x''(t)+4x(t)=16\cdot\cos(2t). \\ x(0)=10,\ x'(0)=0 \end{cases}$ $f(t)=16\cos(2t)=e^{at}A_m(t)\cos(bt)$
$\Downarrow$
$A_m(t)=16$ $(m=0)$, $a+bi=0+2i=2i$ $(k=1)$

$\Rightarrow x_p(t)=t^k\cdot e^{at}\left(C_m(t)\cos(bt)+D_m(t)\sin(bt)\right)=t\cdot\left(C\cdot\cos(2t)+D\cdot\sin(2t)\right)$.

$\Rightarrow x_p''(t)= t''\cdot(\cdots)+2t'\cdot(-C\cdot 2\sin(2t)+D\cdot 2\cos(2t))+t\cdot(-4C\cos(2t)-4D\cdot\sin(2t))$.

$=\cos(2t)\cdot 4D-\sin(2t)\,4C+4t\cdot(-C\cdot\cos(2t)-D\cdot\sin(2t))$

$\Rightarrow \underbrace{x_p''(t)+4x_p(t)}_{=16\cdot\cos(2t)}=\cos(2t)\cdot 4D-\sin(2t)\cdot 4C+\cancel{4t\cdot(-C\cos(2t)-D\sin(2t))+4t(C\cos(2t)+D\sin(2t))}$

$=4D\cdot\cos(2t)-4C\cdot\sin(2t)$

$\Rightarrow \begin{cases} 16=4D \\ 0=-4C \end{cases} \Rightarrow \begin{cases} D=4 \\ C=0 \end{cases} \Rightarrow x_p(t)=4t\cdot\sin(2t)$.

So the general solution to the non homogeneous DE is

$x(t)=C_1\cdot\cos(2t)+C_2\cdot\sin(2t)+4t\cdot\sin(2t)$. Next, we use the initial conditions:

$10 = x(0) = C_1$, $\quad 0 = x'(0) = \left[-C_1 \cdot 2\sin(2t) + C_2 \cdot 2\cos(2t) + 4\cdot \sin(2t) + 4t\cdot 2\cos(2t)\right]\Big|_{t=0} = 2C_2$

$\Rightarrow C_1 = 10, \quad C_2 = 0$

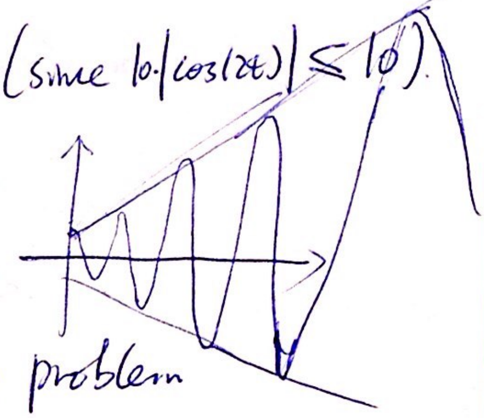
$$\Rightarrow \boxed{x(t) = 10\cdot\cos(2t) + 4t\cdot\sin(2t).}$$

The maximum amplitude $\geq 4t - 10$. (since $10\cdot|\cos(2t)| \leq 10$).

$\downarrow t \to +\infty$.

$+\infty$

So we have the $\boxed{\text{resonance}}$



(c). By assumption, we consider the initial value problem

$$\begin{cases} x''(t) + 2x'(t) + 4x(t) = 0. \\ x(0) = 10, \ x'(0) = 0 \end{cases} \Rightarrow \lambda^2 + 2\lambda + 4 = 0 \Rightarrow \lambda = \frac{-2 \pm \sqrt{2^2 - 4\times 4}}{2} = \frac{-2\pm\sqrt{-12}}{2} = \frac{-2\pm 2\sqrt{3}i}{2} = -1 \pm \sqrt{3}i$$

$\Rightarrow x_1(t) = e^{-t}\cdot\cos(\sqrt{3}t)$, $x_2(t) = e^{-t}\sin(\sqrt{3}t)$.

$\Rightarrow x(t) = C_1\cdot e^{-t}\cos(\sqrt{3}t) + C_2\cdot e^{-t}\cdot\sin(\sqrt{3}t)$ is the general Solution.

Using initial conditions:

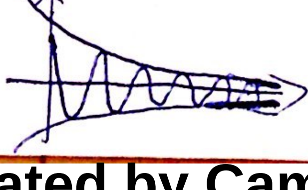
$10 = x(0) = C_1$, $\quad 0 = x'(0) = \left[-C_1 e^{-t}\cos(\sqrt{3}t) - C_1 e^{-t}\sin(\sqrt{3}t)\cdot\sqrt{3} - C_2 e^{-t}\sin(\sqrt{3}t) + C_2 e^{-t}\cos(\sqrt{3}t)\cdot\sqrt{3}\right]\Big|_{t=0} = -C_1 + \sqrt{3}C_2$

So: $C_1 = 10$, $C_2 = \frac{10}{\sqrt{3}}$

$$\Rightarrow x(t) = 10\cdot e^{-t}\cos(\sqrt{3}t) + \frac{10}{\sqrt{3}}e^{-t}\cdot\sin(\sqrt{3}t) = \frac{10}{\sqrt{3}}e^{-t}\left(\sqrt{3}\cos(\sqrt{3}t) + \sin(\sqrt{3}t)\right)$$

$$= \frac{20}{\sqrt{3}}e^{-t}\Big(\underbrace{\frac{\sqrt{3}}{2}}_{\cos(\frac{\pi}{6})}\cos(\sqrt{3}t) + \underbrace{\frac{1}{2}}_{\sin(\frac{\pi}{6})}\sin(\sqrt{3}t)\Big) = \boxed{\frac{20}{\sqrt{3}}e^{-t}\cos\left(\sqrt{3}t - \frac{\pi}{6}\right)}.$$

This is $\boxed{\text{under-damped oscillation}}$



3(d). By 3(c) we get. $\begin{cases} x''(t)+2x'(t)+4x(t)=16\cos(2t) \\ x(0)=10,\ x'(0)=0. \end{cases}$

root	$-1+\sqrt{3}i$	$-1-\sqrt{3}i$
mult.	1	1

$x_1(t)=e^{-t}\cdot\cos(\sqrt{3}t)$, $x_2(t)=e^{-t}\sin(\sqrt{3}t)$.

$f(t)=16\cdot\cos(2t)=e^{at}\cdot A_m(t)\cos(bt) \Rightarrow A_m(t)=16\ (m=0)$
$a+bi=0+2i=2i\ (k=0)$

so $x_p(t)=x^k e^{at}\left(C_m(t)\cos(bt)+D_m\sin(bt)\right)= C\cdot\cos(2t)+D\cdot\sin(2t)$.

$x_p'(t)=-2\cdot C\sin(2t)+2D\cdot\cos(2t)$, $x_p''(t)=-4C\cdot\cos(2t)-4D\sin(2t)$.

$\Rightarrow x_p''+2x_p'+4x_p=-4C\cdot\cos(2t)-4D\sin(2t)+2\cdot(-2C\cdot\sin(2t)+2D\cos(2t))$
$+\ 4(C\cdot\cos(2t)+D\cdot\sin(2t))$

$\underset{16\cos(2t)}{\shortparallel}$ $\qquad \shortparallel$

$\cos(2t)\cdot(\underbrace{-4C+4D+4C}_{4D})+\sin(2t)\cdot(\underbrace{-4D-4C+4D}_{-4C})$.

$\begin{cases} 4D=16 \\ -4C=0 \end{cases} \Leftarrow$

so $\begin{cases} D=4 \\ C=0 \end{cases} \Rightarrow x_p(t)=4\cdot\sin(2t)$.

so the general solution: $x(t)=C_1e^{-t}\cos(\sqrt{3}t)+C_2e^{-t}\sin(\sqrt{3}t)+4\sin(2t)$.

$10=x(0)=C_1$, $0=x'(0)=\left[-C_1\cdot e^{-t}\cos(\sqrt{3}t)-C_1e^{-t}\cdot\sin(\sqrt{3}t)\cdot\sqrt{3}-C_2e^{-t}\cdot\sin(\sqrt{3}t)+C_2e^{-t}\cos(\sqrt{3}t)\sqrt{3}+8\cdot\cos(2t)\right]_{t=0}$

$\shortparallel$

$-C_1+\sqrt{3}C_2+8$

$\Rightarrow \begin{cases} C_1=10 \\ C_2=\dfrac{C_1-8}{\sqrt{3}}=\dfrac{2}{\sqrt{3}}. \end{cases}$

so $\boxed{x(t)=\underbrace{10e^{-t}\cos(\sqrt{3}t)+\frac{2}{\sqrt{3}}e^{-t}\sin(\sqrt{3}t)}+4\sin(2t)}$

steady periodic solution
$x_{sp}=4\sin(2t)$.

transient solution
$x_{tr}=e^{-t}(10\cdot\cos(\sqrt{3}t)+2\sin(\sqrt{3}t))$

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4. $x''(t) + x' + y' + x + y = 0, \quad y'' + y' + x' + y + x = 0.$

Let $\quad x_1 = x, \quad x_2 = x' = x_1', \quad y_1 = y, \quad y_2 = y' = y_1'$

$$\Rightarrow \quad x_2' = x'' = -x' - y' - x - y = -x_2 - y_2 - x_1 - y_1$$

$$y_2' = y'' = -y' - x' - y - x = -y_2 - x_2 - y_1 - x_1$$

so we get a system:

$$\begin{cases} x_1' = x_2 \\ x_2' = -x_1 - x_2 - y_1 - y_2 \\ y_1' = y_2 \\ y_2' = -x_1 - x_2 - y_1 - y_2. \end{cases}$$

We need 4 initial conditions to determine a unique solution.

!!! WRITE YOUR NAME, STUDENT ID BELOW !!!

NAME : ID :

1. (40pts) Consider the non-homogeneous linear differential equation:

$$y''(x) - 4y(x) = 2e^{2x}. \tag{1}$$

(a). Find the general **complementary** solution.

Associated homogeneous DE: $y''(x) - 4y(x) = 0$.

Characteristic eq.: $\lambda^2 - 4 = (\lambda+2)\cdot(\lambda-2) = 0$

$\Rightarrow$

root	-2	2
mult	1	1

$\Rightarrow y_1(x) = e^{-2x}, \ y_2(x) = e^{2x}$

$\Rightarrow y_c(x) = C_1 \cdot e^{-2x} + C_2 \cdot e^{2x}$ is the general complementary solution.

(b). Find a particular solution of the equation (1) using the method of undetermined coefficients.

$f(x) = 2e^{2x} = P_m(x) \cdot e^{\mu x} \Rightarrow P_m(x) = 2 \ (m=0), \ \mu = 2 \ (k=1)$

So $y_p(x) = x \cdot (A \cdot e^{2x}) \Rightarrow y_p'(x) = A \cdot (e^{2x} + 2x \cdot e^{2x})$

$\Rightarrow y_p''(x) = A \cdot (2 \cdot e^{2x} + 2 \cdot e^{2x} + 4x \cdot e^{2x})$

$\Rightarrow y_p''(x) - 4y_p(x) = 4A \cdot e^{2x} = 2 \cdot e^{2x} \Rightarrow A = \frac{1}{2}.$

So $y_p(x) = \frac{1}{2} \cdot x \cdot e^{2x}.$

2. (40pts)

Use the method of variation of parameters to find a particular solution to the following linear equation:

$$y'' + y = \sec^3(x).$$

by using the following formula (the notations won't be explained in the exam)

$$y_p(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}dx.$$

Associated homogeneous DE: $y'' + y = 0 \Rightarrow$ Characteristic eq.: $\lambda^2 + 1 = 0$

$$y_1(x) = \cos x, \quad y_2(x) = \sin x. \Leftarrow (\lambda - i)(\lambda + i)$$

Wronskian: $W(x) = W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$

$$\int \frac{y_2(x)f(x)}{W(x)}dx = \int \frac{\sin x \cdot \sec^3 x}{1}dx = \int \frac{\sin x}{\cos^3 x}dx = -\int \frac{d(\cos x)}{(\cos x)^3}.$$

$$\overset{u=\cos x}{=\!=} -\int u^{-3}du = \frac{1}{2}\cdot u^{-2} = \frac{1}{2}\cdot \sec^2 x.$$

$$\int \frac{y_1(x)f(x)}{W(x)}dx = \int \frac{\cos x \cdot \sec^3 x}{1}dx = \int \sec^2 x\, dx = \tan(x).$$

plugging these into the formula we get:

$$y_p(x) = -\frac{1}{2}\cdot \sec^2 x \cdot \cos x + \sin x \cdot \tan x$$

$$= \boxed{-\frac{1}{2}\sec x + \sin x \cdot \tan x.}$$

3. (40pts) Consider the free mass-spring-dashpot system

$$x'' + 2x' + 10x = 0.$$

The object is released from still when the spring is **compressed** by 10 unit length. Solve the system as an initial value problem and classify the phenomenon. Write the solution in the standard form: $x(t) = A(t)\cos(\omega t - \alpha)$.

Initial value problem: $\begin{cases} x'' + 2x' + 10x = 0 \\ x(0) = -10, \ x'(0) = 0. \end{cases}$

$\Rightarrow \lambda^2 + 2\lambda + 10 = 0 \Rightarrow \lambda = \dfrac{-2 \pm \sqrt{2^2 - 4\times 10}}{2} = -1 \pm 3i.$

$\Rightarrow x(t) = C_1 \cdot x_1(t) + C_2 \cdot x_2(t) = C_1 \cdot e^{-t}\cos(3t) + C_2 \cdot e^{-t} \cdot \sin(3t).$

$\Rightarrow \begin{cases} -10 = x(0) = C_1. \\ 0 = x'(0) = \left[-C_1 \cdot e^{-t} \cdot \cos(3t) - C_1 e^{-t} \cdot 3\sin(3t) - C_2 \cdot e^{-t} \cdot \sin(3t) + C_2 \cdot e^{-t} 3 \cdot \cos(3t)\right]_{t=0} = -C_1 + 3C_2 \end{cases}$

$\Rightarrow \begin{cases} C_1 = -10 \\ C_2 = \dfrac{C_1}{3} = -\dfrac{10}{3}. \end{cases}$ so

$$x(t) = -10 \cdot e^{-t} \cdot \cos(3t) - \frac{10}{3} e^{-t} \sin(3t) = -\frac{10}{3} \cdot e^{-t} \cdot \left(3\cos(3t) + \sin(3t)\right).$$

$$= -\frac{10}{3} e^{-t} \cdot \sqrt{10} \cdot \left(\frac{3}{\sqrt{10}}\cos(3t) + \frac{1}{\sqrt{10}}\sin(3t)\right).$$

$$= -\frac{10\sqrt{10}}{3} e^{-t} \cdot \cos\left(3t - \tan^{-1}\left(\tfrac{1}{3}\right)\right).$$

$$= \boxed{\frac{10\sqrt{10}}{3} e^{-t} \cdot \cos\left(3t - \left(\tan^{-1}\left(\tfrac{1}{3}\right) + \pi\right)\right)} \qquad \left(\cos(\beta \pm \pi) = -\cos\beta\right)$$

or $$= \boxed{\frac{10\sqrt{10}}{3} e^{-t} \cos\left(3t - \left(\tan^{-1}\left(\tfrac{1}{3}\right) - \pi\right)\right).}$$

4. (40pts) Consider the undamped forced mass-spring system

$$x'' + x = 4\sin(\omega t).$$

At time $t = 0$, the object is at the equilibrium position with zero velocity. What value of ω will trigger the resonance? Solve the system for this value of ω.

The associated homogeneous DE: $x'' + x = 0$.

Characteristic eq.: $\lambda^2 + 1 = (\lambda - i)(\lambda + i) = 0 \Rightarrow$

root	i	$-i$
mult.	1	1

$\Rightarrow x_1(t) = \cos(t),\quad x_2(t) = \sin(t)$.

$\Rightarrow$ The natural frequency is $\omega_0 = 1$. There will be a resonance when the frequency of the external force coincides with the natural frequency i.e. when $\omega = \omega_0 = 1$. So we need to solve:

$$x''(t) + x = 4\sin t. \qquad x(0) = 0,\ x'(0) = 0.$$

$f(t) = 4\cdot\sin(t) = P_m(t)e^{at}\sin(bt) \Rightarrow P_m(t) = 4\ (m=0).\ a+bi = 0 + 1\cdot i = i\ (k=1)$

So $x_p(t) = t\cdot(A\cdot\cos t + B\cdot\sin t) \Rightarrow x_p'(t) = A\cos t + B\sin t + t(-A\sin t + B\cos t)$.

$\Rightarrow x_p''(t) = -2A\sin t + 2B\cos t + t\cdot(-A\cos t - B\sin t)$.

$\Rightarrow x_p'' + x_p = -2A\cdot\sin t + 2B\cdot\cos t = 4\sin t \Rightarrow A = -2,\ B = 0.$

$\Rightarrow x_p(t) = -2t\cdot\cos t \Rightarrow x(t) = C_1\cdot x_1(t) + C_2\cdot x_2(t) + x_p(t).$

$= C_1\cdot\cos t + C_2\cdot\sin t - 2t\cdot\cos t.$

Using the initial conditions:

$0 = x(0) = C_1,\quad 0 = x'(0) = [-C_1\sin t + C_2\cos t - 2(\cos t - t\cdot\sin t)]_{t=0} = C_2 - 2.$

$\Rightarrow C_1 = 0,\ C_2 = 2.$ So:

$$\boxed{x(t) = 2\cdot\sin t - 2t\cos t}$$

5. **(40pts)** Assume $y = y(x)$. Transform the following equation into a system of first order differential equations:

$$y''' + y' + y + x = 0.$$

Is it a linear or non-linear system? Is it homogeneous or non-homogenous? How many initial conditions are needed to determine a unique solution?

Let $y_1 = y$, $y_2 = y'(x) = y_1'$

$y_3(x) = y''(x) = y_2'(x)$

$\Rightarrow y_3'(x) = y'''(x) = -y' - y - x$

$= -y_2 - y_1 - x$

$$\Rightarrow \begin{cases} y_1'(x) = y_2(x) \\ y_2'(x) = y_3(x) \\ y_3'(x) = -y_1(x) - y_2(x) - x. \end{cases}$$

This is a linear, non-homogeneous system.

3 conditions are needed to determine a unique solution.

2.(b).

$$\begin{cases} m_1 x'' = -(k_1+k_2)x + k_2 y \\ m_2 y'' = \quad k_2 x - (k_2+k_3) y. \end{cases}$$

$m_1 = 2,\ m_2 = 1,\ k_1 = 10,\ k_2 = 20,\ k_3 = 10.$

$$\begin{cases} 2x'' = -30x + 20y. \quad ① \\ y'' = 20x - 30y \quad ② \end{cases} \Rightarrow y = \frac{1}{10}(x'' + 15x).$$

$\Downarrow ②$

$$\frac{1}{10}(x^{(4)} + 15x'') = 20 \cdot x - \frac{30}{10}(x'' + 15x).$$

$$\Rightarrow x^{(4)} + 15x'' = 200x - 30 \cdot x'' - 450x \Rightarrow x^{(4)} + 45x'' + 250x = 0.$$

(Note that we can also use notation of differential operators:

$$\begin{cases} 2(D^2+15)x - 20y = 0. \\ -20x + (D^2+30)y = 0 \end{cases} \Leftrightarrow \begin{pmatrix} D^2+15 & -10 \\ -20 & D^2+30 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\Rightarrow \left((D^2+15)\cdot(D^2+30) - 200\right) \cancel{y}\, x = (D^4 + 45D^2 + 250)x = x^{(4)} + 45x'' + 250x = 0.$$

Characteristic eq.: $\lambda^4 + 45\lambda^2 + 250 = 0 \Rightarrow \lambda^2 = \dfrac{-45 \pm \sqrt{45^2 - 4\times 250}}{2} = \dfrac{-45 \pm 5\sqrt{41}}{2} < 0.$

$\Rightarrow \lambda_1 = ai,\ \lambda_2 = -ai,\ \lambda_3 = bi,\ \lambda_4 = -bi.$ with

$$a = \sqrt{\frac{45+5\sqrt{41}}{2}} = \sqrt{\frac{5}{2}(9+\sqrt{41})}, \quad b = \sqrt{\frac{45-5\sqrt{41}}{2}} = \sqrt{\frac{5}{2}(9-\sqrt{41})}.$$

$$\Rightarrow x(t) = C_1 \cdot \cos(at) + C_2 \cdot \sin(at) + C_3 \cdot \cos(bt) + C_4 \cdot \sin(bt)$$

$$\Rightarrow y(t) = \frac{1}{10}(x'' + 15x) = \frac{1}{10}(15 - a^2)C_1 \cos(at) + \frac{C_2}{10}(15-a^2)\sin(at) + \frac{C_3}{10}(15-b^2)\cos(bt) + \frac{C_4}{10}(15-b^2)\sin(bt).$$

$$\Rightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} \cos(at) \\ \frac{1}{10}(15-a^2)\cos(at) \end{pmatrix} + C_2 \cdot \begin{pmatrix} \sin(at). \\ \frac{1}{10}(15-a^2)\sin(at) \end{pmatrix}$$

$$15 - a^2 = 15 - \frac{5}{2}(9+\sqrt{41}) = \frac{5}{2}(-3-\sqrt{41}) < 0.$$

Oscillate in the opposite direction

$$+ C_3 \begin{pmatrix} \cos(bt) \\ \frac{15-b^2}{10}\cos(bt) \end{pmatrix} + C_4 \cdot \begin{pmatrix} \sin(bt) \\ \frac{15-b^2}{10}\sin(bt) \end{pmatrix}$$

$$15 - b^2 = 15 - \frac{5}{2}(9-\sqrt{41}) = \frac{5}{2}(-3+\sqrt{41}) > 0$$

Oscillate in the same direction

Two natural frequencies: a, b.

$a = \sqrt{\frac{5}{2}(9+\sqrt{41})}$, $b = \sqrt{\frac{5}{2}(9-\sqrt{41})}$

5. (a). $x' = x - y$. $y' = -x + y$, $x(0) = 2$, $y(0) = 1$.

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \quad A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

characteristic polynomial: $\det(A - \lambda I) = \begin{pmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 1 = \lambda^2 - 2\lambda = \lambda(\lambda - 2)$

$\Rightarrow$ $\lambda = 0$, or $\lambda = 2$.

$\lambda = 0$: $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = 0 \Leftrightarrow \begin{cases} a_1 - b_1 = 0 \\ \cancel{-a_1 + b_1 = 0} \end{cases} \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector associated with 0.

$\lambda = 2$: $\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = 0 \Leftrightarrow \begin{cases} -a_2 - b_2 = 0 \\ \cancel{-a_2 - b_2 = 0} \end{cases} \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector associated with 2.

So the general solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{\lambda_1 t} \vec{v}_1 + C_2 e^{\lambda_2 t} \vec{v}_2 = C_1 e^{0 \cdot t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \cdot e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} C_1 + C_2 \cdot e^{2t} \\ C_1 - C_2 \cdot e^{2t} \end{pmatrix}.$$

initial conditions:

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} C_1 + C_2 \\ C_1 - C_2 \end{pmatrix} \Rightarrow \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \end{pmatrix}.$$

So the solution to the initial value problem is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \frac{3}{2} + \frac{1}{2} e^{2t} \\ \frac{3}{2} - \frac{1}{2} e^{2t} \end{pmatrix}.$$

1. (a)

$$\begin{cases} x' = x - y & x(0) = 2 \\ y' = -x + y & y(0) = 1 \end{cases}$$

$$\begin{cases} (D-1)x + y = 0 \\ -x + (D-1)y = 0 \end{cases} \qquad \left((D-1)^2 + 1\right) y = 0$$

$$(D^2 - 2D)\, y = 0$$

$$\begin{cases} y = c_1 + c_2 e^{2t} \\ x = y - y' = (c_1 + c_2 e^{2t}) - 2c_2 e^{2t} = c_1 - c_2 e^{2t} \end{cases}$$

$$\begin{cases} c_1 + c_2 = 1 \\ c_1 - c_2 = 2 \end{cases} \Rightarrow \begin{cases} c_1 = \frac{3}{2} \\ c_2 = -\frac{1}{2} \end{cases}$$

$$x(t) = \frac{3}{2} + \frac{1}{2} e^{2t} \quad ; \quad y(t) = \frac{3}{2} - \frac{1}{2} e^{2t}.$$

(b)

$$\begin{cases} x' = x + y & x(0) = 2 \\ y' = -x + y & y(0) = 1 \end{cases}$$

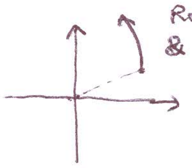
$$\begin{cases} (D-1)x - y = 0 \\ x + (D-1)y = 0 \end{cases} \qquad \left((D-1)^2 + 1\right) y = 0$$

$$\Rightarrow \quad y = c_1 e^t \cos t + c_2 e^t \sin t$$

$$x = y - y' = c_1 e^t \cos t + c_2 e^t \sin t - \left(c_1 e^t \cos t - c_1 e^t \sin t + c_2 e^t \sin t + c_2 e^t \cos t\right)$$

$$= c_1 e^t \sin t - c_2 e^t \cos t$$

$$\begin{cases} -c_2 = 2 \\ c_1 = 1 \end{cases}$$



Rotation & Stretching. $\Rightarrow$

$$\begin{cases} x(t) = e^t \sin t + 2e^t \cos t \\ y(t) = e^t \cos t - 2e^t \sin t \end{cases} \Rightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^t \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

2. By the Newton's Motion Law.

$$\begin{cases} m_1 x'' = -k_1 x + k_2 (y - x) = -(k_1 + k_2) x + k_2 y \\ m_2 y'' = -k_2 (y - x) - k_3 y = k_2 x - (k_2 + k_3) y \end{cases}$$

(a) $m_1 = 2$, $m_2 = 1$, $k_1 = 10$, $k_2 = 20$, $k_3 = 10$.

$$\begin{cases} 2x'' = -30x + 10y \\ y'' = 20x - 30y \end{cases} \Rightarrow \begin{cases} (D^2 + 15)x - 5y = 0 & \text{①} \\ -20x + (D^2 + 30)y = 0 & \text{②} \end{cases}$$

$$(D^2 + 15)\,② + 20 \cdot ① \Rightarrow \left((D^2+15)(D^2+30) - 100\right) y = (D^4 + 45D^2 + 350)\, y = 0 \Rightarrow (D^2 + 10)(D^2 + 35)\, y = 0$$

$\Rightarrow$ Characteristic Roots are $\pm\sqrt{10}\, i$, $\pm\sqrt{35}\, i$

Continued: General solution is

$$\begin{cases} y = c_1 \cos\sqrt{10}\,t + c_2 \sin\sqrt{10}\,t + c_3 \cos\sqrt{35}\,t + c_4 \sin\sqrt{35}\,t \\ x = \frac{1}{20}(y'' + 30y) = c_1 \cos\sqrt{10}\,t + c_2 \sin\sqrt{10}\,t - \frac{1}{4} c_3 \cos\sqrt{35}\,t - \frac{1}{4} c_4 \sin\sqrt{35}\,t . \end{cases}$$

Natural Frequence is $\sqrt{10}$ and $\sqrt{35}$.

(c). Set $u = x'$, $v = y'$

$$\Rightarrow \begin{cases} x' = u \\ y' = v \\ u' = -15x + 5y \\ v' = 20x - 30y \end{cases} \qquad A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -15 & 5 & 0 & 0 \\ 20 & -30 & 0 & 0 \end{pmatrix}, \qquad \frac{d}{dt}\vec{X} = A\vec{X}, \quad \vec{X} = \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix}.$$

---

3. (a)

$$2A - 3B = \begin{pmatrix} 2 & -8 \\ 2 & -6 \end{pmatrix} - \begin{pmatrix} 3 & 6 \\ -3 & 12 \end{pmatrix} = \begin{pmatrix} -1 & -14 \\ 5 & -18 \end{pmatrix}$$

$$AI = \begin{pmatrix} 1 & -4 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 1 & -3 \end{pmatrix} = A.$$

$$IB = B$$

$$A^2 = \begin{pmatrix} 1 & -4 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 1 & -4 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} -3 & 8 \\ -2 & 5 \end{pmatrix}.$$

$$AB = \begin{pmatrix} 1 & -4 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 5 & -14 \\ 4 & -10 \end{pmatrix}, \quad BA = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 1 & -4 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 3 & -10 \\ 3 & -8 \end{pmatrix}$$

$$AC = \begin{pmatrix} 1 & -4 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} -3 & 4 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad CA = \begin{pmatrix} -3 & 4 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -4 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$(AB)C = \begin{pmatrix} 5 & -14 \\ 4 & -10 \end{pmatrix}\begin{pmatrix} -3 & 4 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 6 \\ -2 & 6 \end{pmatrix}, \quad A(BC) = \begin{pmatrix} 1 & -4 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} -5 & 6 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 6 \\ -2 & 6 \end{pmatrix}.$$

$$A\vec{x}_1 = \begin{pmatrix} 1 & -4 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} = -\vec{x}_1$$

(b) $AI = A$, $IB = B$. $AB \neq BA$, $(AB)C = A(BC)$, $C = A^{-1}$.

$\vec{x}_1$ is the eigenvector of A with eigenvalue -1.

Operator on matrix: Associative ; Non-commutative in general.

(c). $|\lambda I - B| = \begin{vmatrix} \lambda-1 & -2 \\ 1 & \lambda-4 \end{vmatrix} = (\lambda-1)(\lambda-4) + 2 = (\lambda^2 - 5\lambda + 6) = (\lambda-2)(\lambda-3) \Rightarrow \lambda = 2$ or 3.

4. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$A \cdot \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}.$$

5. (a). $\vec{X} = \begin{pmatrix} x \\ y \end{pmatrix}$. $\quad \frac{d\vec{X}}{dt} = A\vec{X}$, $\quad A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

Eigenvalues are $0, \quad 2$

$u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\quad u_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$$AU = U\Lambda$$

$$\frac{d}{dt}\vec{X} = U\Lambda U^{-1}\vec{X}$$

$$\Rightarrow \frac{d}{dt}(U^{-1}\vec{X}) = \Lambda(U^{-1}\vec{X})$$

Set $\vec{Y} = U^{-1}\vec{X}$, $\Rightarrow \begin{cases} \frac{d\vec{Y}}{dt} = \Lambda\vec{Y} \\ \vec{Y}(0) = U^{-1}\vec{X}(0) \end{cases}$

$$\Rightarrow \vec{Y} = e^{t\Lambda} = \Lambda_t \vec{Y}(0), \quad \Lambda_t = \begin{pmatrix} e^{t\lambda_1} & \\ & e^{t\lambda_2} \end{pmatrix}$$

$$\Rightarrow \vec{X}(t) = U\Lambda_t U^{-1}\vec{X}(0).$$

Here $U = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\quad U^{-1} = \frac{1}{-2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$

$$\Rightarrow \vec{X}(t) = -\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & \\ & e^{2t} \end{pmatrix}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} + \frac{1}{2}e^{2t} \\ \frac{3}{2} - \frac{1}{2}e^{2t} \end{pmatrix}.$$

(b). $\vec{X} = \begin{pmatrix} x \\ y \end{pmatrix}$. $\quad \frac{d\vec{X}}{dt} = A\vec{X}$, $\quad A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & -1 \\ 1 & \lambda-1 \end{vmatrix} = (\lambda-1)^2+1 \Rightarrow \lambda = 1 \pm i.$$

- $\lambda = 1+i$, $\quad \lambda I - A = \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \Rightarrow \vec{u}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$
- $\lambda = 1-i$, $\quad \lambda I - A = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \Rightarrow \vec{u}_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$

$\Rightarrow U = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$, $\quad U^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$

Solution $\vec{X}(t) = U\Lambda_t U^{-1}\vec{X}(0) = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} e^{(1+i)t} & 0 \\ 0 & e^{(1-i)t} \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = e^t\begin{pmatrix} 2\cos t + \sin t \\ \cos t - 2\sin t \end{pmatrix}.$

1、 (a). $\vec{x}(t) = \vec{v}(t)e^{\lambda t} \Rightarrow \frac{d\vec{x}}{dt} = \frac{d\vec{v}}{dt} e^{\lambda t} + \vec{v}(t) \cdot \lambda \cdot e^{\lambda t}$

so $\frac{d\vec{x}}{dt} = A \cdot \vec{x}(t) \Leftrightarrow \frac{d\vec{v}}{dt} e^{\lambda t} + \lambda \vec{v}(t) \cdot e^{\lambda t} = A \cdot \vec{v}(t) e^{\lambda t}$

$$\Leftrightarrow \frac{d\vec{v}}{dt} + \lambda \vec{v}(t) = A\vec{v}(t)$$

$$\Leftrightarrow \frac{d\vec{v}}{dt} = (A - \lambda I)\vec{v}(t).$$

(b). $\vec{v}(t) = \frac{\vec{v}_1 t^{k-1}}{(k-1)!} + \frac{\vec{v}_2 t^{k-2}}{(k-2)!} + \cdots + \frac{\vec{v}_{k-2} t^2}{2!} + \vec{v}_{k-1} t + \vec{v}_k$

$$\Rightarrow \frac{d\vec{v}}{dt} = \frac{\vec{v}_1 \cdot t^{k-2}}{(k-2)!} + \frac{\vec{v}_2 \cdot t^{k-3}}{(k-3)!} + \cdots + \frac{\vec{v}_{k-2} \cdot t}{1} + \vec{v}_{k-1}.$$

so $\frac{d\vec{v}}{dt} = (A - \lambda I)\vec{v}(t) \Leftrightarrow \frac{\vec{v}_1 \cdot t^{k-2}}{(k-2)!} + \frac{\vec{v}_2 \cdot t^{k-3}}{(k-3)!} + \cdots + \frac{\vec{v}_{k-2} t}{1} + \vec{v}_{k-1}$

$$\| $$

$$\frac{(A-\lambda I)\vec{v}_1 \, t^{k-1}}{(k-1)!} + \frac{(A-\lambda I)\vec{v}_2 t^{k-2}}{(k-2)!} + \frac{(A-\lambda I)\vec{v}_3 t^{k-3}}{(k-3)!} + \cdots + \frac{(A-\lambda I)\vec{v}_{k-1} t}{1} + (A-\lambda I)\vec{v}_k$$

$\Leftrightarrow$. $(A-\lambda I)\vec{v}_1 = 0$, $(A-\lambda I)\vec{v}_2 = \vec{v}_1$, $(A-\lambda I)\vec{v}_3 = \vec{v}_2$, $\cdots$,

$(A-\lambda I)\vec{v}_{k-1} = \vec{v}_{k-2}$, $(A-\lambda I)\vec{v}_k = \vec{v}_{k-1}$

$$\Leftrightarrow \quad \vec{v}_k \xrightarrow{A-\lambda I} \vec{v}_{k-1} \xrightarrow{A-\lambda I} \cdots \xrightarrow{A-\lambda I} \vec{v}_2 \xrightarrow{A-\lambda I} \vec{v}_1 \xrightarrow{A-\lambda I} 0.$$

2. (a). $x' = x + y,\ y' = -x + y,\ x(0) = 2,\ y(0) = 1$

$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. characteristic polynomial:

$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 2 = 0$

$\Rightarrow \lambda = \frac{2 \pm \sqrt{(-2)^2 - 8}}{2} = 1 \pm i$. (eigenvalues).

Eigenvector: $\lambda_1 = 1 + i$. $\quad (A - \lambda_1 I) v_1 = 0 \iff \begin{cases} -i \cdot a_1 + b_1 = 0 \\ \cancel{-a_1 - i b_1 = 0} \end{cases}$

$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$

$\Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$.

Taking the conjugation, we get $\vec{v}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ is an eigenvector associated with $\lambda_2 = 1 - i$.

So basic solutions:

$\vec{X}_1(t) = e^{\lambda_1 t} \vec{v}_1 = e^{(1+i)t} \begin{pmatrix} 1 \\ i \end{pmatrix} = e^t (\cos t + i \sin t) \begin{pmatrix} 1 \\ i \end{pmatrix} = e^t \cdot \begin{pmatrix} \cos t + i \sin t \\ -\sin t + i \cos t \end{pmatrix}$.

Extracting the real and imaginary part we get 2 basic REAL Solutions.

$\vec{u}_1(t) = \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = \begin{pmatrix} e^t \cos t \\ -e^t \sin t \end{pmatrix}, \quad \vec{u}_2(t) = \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} e^t \sin t \\ -e^t \cos t \end{pmatrix}$.

General solution: $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \cdot \vec{u}_1(t) + C_2 \vec{u}_2(t)$.

Fundamental matrix: $\Phi(t) = (\vec{u}_1(t)\ \vec{u}_2(t)) = e^t \cdot \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$.

$\Rightarrow \Phi(0) = I$. $\Rightarrow e^{tA} = \Phi(t) \cdot \Phi(0)^{-1} = e^t \cdot \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$.

$\Rightarrow$ solution to the initial value problem is:

$e^{tA} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = e^t \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \boxed{e^t \cdot \begin{pmatrix} 2\cos t + \sin t \\ -2\sin t + \cos t \end{pmatrix}}$

(b). $x_1' = 3x_1 + 4x_2$, $x_2' = 5x_1 + 2x_2$, $x_1(0) = 2$, $x_2(0) = 1$.

$$A = \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix} \Rightarrow \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 4 \\ 5 & 2-\lambda \end{vmatrix} = \lambda^2 - 5\lambda - 14 = (\lambda - 7)\cdot(\lambda + 2).$$

$$\lambda_1 = 7: \quad \begin{pmatrix} -4 & 4 \\ 5 & -5 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = 0 \Leftrightarrow 4(-a_1 + b_1) = 0 \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$\lambda_2 = -2: \quad \begin{pmatrix} 5 & 4 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = 0 \Leftrightarrow 5a_2 + 4b_2 = 0 \Rightarrow \vec{v}_2 = \begin{pmatrix} 4 \\ -5 \end{pmatrix}.$$

$$\Rightarrow \vec{x}_1(t) = e^{7t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{x}_2(t) = e^{-2t}\begin{pmatrix} 4 \\ -5 \end{pmatrix}.$$

$\Rightarrow$ General solution $\boxed{\vec{x}(t) = c_1 \cdot e^{7t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \cdot e^{-2t}\begin{pmatrix} 4 \\ -5 \end{pmatrix}}$

Fundamental matrix $\Phi(t) = \begin{pmatrix} e^{7t} & 4\cdot e^{-2t} \\ e^{7t} & -5 e^{-2t} \end{pmatrix} \Rightarrow \Phi(0) = \begin{pmatrix} 1 & 4 \\ 1 & -5 \end{pmatrix}$

$$\Rightarrow \Phi(0)^{-1} = \frac{1}{1\times(-5) - 4\times 1}\begin{pmatrix} -5 & -4 \\ -1 & 1 \end{pmatrix} = -\frac{1}{9}\begin{pmatrix} -5 & -4 \\ -1 & 1 \end{pmatrix}.$$

$$\Rightarrow e^{tA} = \Phi(t)\,\Phi(0)^{-1} = -\frac{1}{9}\begin{pmatrix} e^{7t} & 4e^{-2t} \\ e^{7t} & -5\cdot e^{-2t} \end{pmatrix}\begin{pmatrix} -5 & -4 \\ -1 & 1 \end{pmatrix} = -\frac{1}{9}\begin{pmatrix} -5e^{7t} - 4e^{-2t} & -4e^{7t} + 4e^{-2t} \\ -5e^{7t} + 5e^{-2t} & -4e^{7t} - 5e^{-2t} \end{pmatrix}$$

$\Rightarrow$ solution to the initial value problem:

$$\vec{x}(t) = e^{tA}\cdot\vec{x}(0) = -\frac{1}{9}\cdot\begin{pmatrix} -5\cdot e^{7t} - 4e^{-2t} & -4e^{7t} + 4e^{-2t} \\ -5e^{7t} + 5e^{-2t} & -4e^{7t} - 5e^{-2t} \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

$$= \boxed{-\frac{1}{9}\cdot\begin{pmatrix} -14 e^{7t} - 4\cdot e^{-2t} \\ -14 e^{7t} + 5\cdot e^{-2t} \end{pmatrix}} = \boxed{\frac{14}{9} e^{7t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{9} e^{-2t}\begin{pmatrix} 4 \\ -5 \end{pmatrix}}$$

$$= \begin{pmatrix} \frac{14}{9} e^{7t} + \frac{4}{9} e^{-2t} \\ \frac{14}{9} e^{7t} - \frac{5}{9} e^{-2t} \end{pmatrix}$$

We can also use more direct way to solve

$$\begin{cases} c_1 + 4c_2 = 2 \\ c_1 - 5c_2 = 1 \end{cases} \Rightarrow \begin{cases} c_1 = \frac{14}{9} \\ c_2 = \frac{1}{9}. \end{cases}$$

(c) $x_1' = 4x_1 + x_2$, $x_2' = -4x_1$, $x_1(0) = 2$, $x_2(0) = 1$.

$$A = \begin{pmatrix} 4 & 1 \\ -4 & 0 \end{pmatrix} \Rightarrow \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 1 \\ -4 & -\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0$$

$\Longrightarrow \lambda = 2$, mult. 2. $\quad (A - \lambda I)\vec{w} = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = 0.$

$\Longleftrightarrow \begin{cases} 2a_1 + b_1 = 0 \\ -4a_1 - 2b_1 = 0 \end{cases} \Rightarrow \vec{w} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}. \Rightarrow$ We are in the

Situation:

EVal	mult	#EVector	Chain
2	2	1	$\vec{v}_1 \to \vec{v}_2 \to 0$

$\vec{v}_1$ satisfies $(A - 2I)^2 \vec{v}_1 = 0.$

$\Longrightarrow \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}^2 \vec{v}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \vec{v}_1 = 0.$ Choose $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow$

$\vec{v}_2 = (A - 2I)\vec{v}_1 = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} (= 2\vec{w})$ so we get the chain

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 2 \\ -4 \end{pmatrix} \to 0 \Rightarrow \quad \vec{x}_1(t) = (\vec{v}_1 + \vec{v}_2 t) e^{\lambda t} = \begin{pmatrix} 1 + 2t \\ -4t \end{pmatrix} \cdot e^{2t}.$

$\quad \begin{matrix} \| \\ \vec{v}_1 \end{matrix} \qquad \begin{matrix} \| \\ \vec{v}_2 \end{matrix} \qquad \vec{x}_2(t) = \vec{v}_2 e^{\lambda t} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} \cdot e^{2t}.$

General solution:

$$\vec{x}(t) = C_1 \vec{x}_1(t) + C_2 \vec{x}_2(t) = C_1 \begin{pmatrix} 1 + 2t \\ -4t \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 2 \\ -4 \end{pmatrix} e^{2t}.$$

initial conditions: $\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \vec{x}(0) = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 2 \\ -4 \end{pmatrix}. \Rightarrow \begin{cases} C_1 = \frac{5}{2} \\ C_2 = -\frac{1}{4} \end{cases}$

$\Rightarrow$ solution to the initial value problem

$$\boxed{\vec{x}(t) = \frac{5}{2} \begin{pmatrix} 1 + 2t \\ -4t \end{pmatrix} e^{2t} - \frac{1}{4} \begin{pmatrix} 2 \\ -4 \end{pmatrix} e^{2t} = \begin{pmatrix} (2 + 5t) e^{2t} \\ (-10t + 1) e^{2t} \end{pmatrix}.}$$

(d) $x_1' = 4x_1 + x_2$, $x_2' = -x_1 + 2x_2$, $x_1(0) = 2$, $x_2(0) = 1$.

$$A = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix} \Rightarrow \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 9 = \underset{\overset{\parallel}{0}.}{(\lambda-3)^2}$$

$\Rightarrow \lambda = 3$, mult. $= 2$. We are in the situation.

EValue	mult	# EVector	Chain
3	2	1	$\vec{v}_1 \to \vec{v}_2 \to 0$

One can calculate.

$$(A - 3I)^2 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Choose $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_2 = (A - 3I)\vec{v}_1 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$\Rightarrow$ chain: $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ -1 \end{pmatrix} \to 0$

$$\Rightarrow \vec{x}_1(t) = (\vec{v}_1 + \vec{v}_2 t) e^{\lambda t} = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} t\right) e^{3t} = \begin{pmatrix} 1+t \\ -t \end{pmatrix} e^{3t}.$$

$$\vec{x}_2(t) = \vec{v}_2 e^{\lambda t} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t}$$

$\Rightarrow$ general solution $\vec{x}(t) = c_1 \begin{pmatrix} 1+t \\ -t \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t}$

initial conditions: $\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \vec{x}(0) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \begin{cases} c_1 = 3 \\ c_2 = -1 \end{cases}$

$\Rightarrow$ solution to the initial value problem

$$\vec{x}(t) = 3\begin{pmatrix} 1+t \\ -t \end{pmatrix} e^{3t} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} = \begin{pmatrix} (2+3t)e^{3t} \\ (1-3t)e^{3t} \end{pmatrix}.$$

3. (a). $\begin{cases} x_1' = 2x_1 + 2x_2 & ① \\ x_2' = -x_2 + x_3 & ② \\ x_3' = 2x_3 & ③ \end{cases}$

$$③ \Rightarrow x_3 = C_1 \cdot e^{2t} \xRightarrow{②} x_2' + x_2 = C_1 \cdot e^{2t}$$

$$\Rightarrow (e^t x_2)' = C_1 \cdot e^{3t} \Rightarrow e^t x_2(t) = \frac{1}{3} C_1 e^{3t} + C_2$$

$$\Rightarrow x_2(t) = \frac{1}{3} C_1 \cdot e^{2t} + C_2 \cdot e^{-t}$$

$$\xRightarrow{①} x_1' - 2x_1 = \frac{2}{3} C_1 \cdot e^{2t} + 2C_2 \cdot e^{-t}$$

$$\Rightarrow (e^{-2t} x_1)' = \frac{2}{3} C_1 + 2C_2 \cdot e^{-3t}$$

$$\Rightarrow e^{-2t} x_1 = \frac{2}{3} C_1 t - \frac{2}{3} C_2 \cdot e^{-3t} + C_3.$$

$$\Rightarrow x_1(t) = \frac{2}{3} C_1 t e^{2t} - \frac{2}{3} C_2 \cdot e^{-t} + C_3 \cdot e^{2t}.$$

So $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} \frac{2}{3} C_1 t e^{2t} - \frac{2}{3} C_2 \cdot e^{-t} + C_3 \cdot e^{2t} \\ \frac{1}{3} C_1 \cdot e^{2t} + C_2 \cdot e^{-t} \\ C_1 \cdot e^{2t} \end{pmatrix}$

$$= C_1 \begin{pmatrix} \frac{2}{3} t e^{2t} \\ \frac{1}{3} e^{2t} \\ e^{2t} \end{pmatrix} + C_2 \begin{pmatrix} -\frac{2}{3} e^{-t} \\ e^{-t} \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} e^{2t} \\ 0 \\ 0 \end{pmatrix}.$$

$$\begin{cases} x_1' = 3x_1 + x_2 \\ x_2' = -4x_1 - x_2 \\ x_3' = 4x_1 - 8x_2 - 2x_3 \end{cases} \iff \frac{d\vec{x}}{dt} = \begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ 4 & -8 & -2 \end{pmatrix} \vec{x}(t). \quad \vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

$A = \begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ 4 & -8 & -2 \end{pmatrix}$ We will first use the more canonical way to determine the number of chains and the length of each chain. Then we calculate the chains and write down the basic solutions.

1. Calculate characteristic polynomial and eigenvalues.

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 1 & 0 \\ -4 & -1-\lambda & 0 \\ 4 & -8 & -2-\lambda \end{vmatrix} = (-2-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ -4 & -1-\lambda \end{vmatrix} = -(\lambda+2)(\lambda-1)^2$$

so the eigenvalues: $\lambda_1 = -2$, mult 1; $\lambda_2 = 1$, mult. $= 2$.

Then calculate the eigenvectors:

$\lambda_1 = 2$:

$$(A-\lambda_1 I)\vec{v}_1 = \begin{pmatrix} 5 & 1 & 0 \\ -4 & 1 & 0 \\ 4 & -8 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = 0 \iff \begin{cases} 5a_1 + b_1 = 0 \\ -4a_1 + b_1 = 0 \\ 4a_1 - 8b_1 = 0 \end{cases}$$

$\Rightarrow a_1 = b_1 = 0 \Rightarrow \vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector associated with $\lambda_1 = -2$.

$\lambda_2 = 1$:

$$(A-\lambda_2 I)\vec{w} = \begin{pmatrix} 2 & 1 & 0 \\ -4 & -2 & 0 \\ 4 & -8 & -3 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = 0 \iff \begin{cases} 2a_2 + b_2 = 0 \\ \cancel{-4a_2 - 2b_2 = 0} \\ 4a_2 - 8b_2 - 3c_2 = 0 \end{cases}$$

$\Rightarrow$ only 1 eigenvector. For example choosing $a_2 = 3, \Rightarrow b_2 = -6 \Rightarrow c_2 = 20$.

By these calculations, we know that we are in the following situation:

EValue	mult	#EVector	Chain
$\lambda_1 = -2$	1	1	$v_1 \to 0$
$\lambda_2 = 1$	2	1	$v_2 \to v_3 \to 0$.

Then we can choose $v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \to 0$.

We still need to calculate the chain $v_2 \to v_3 \to 0$.
v_2 satisfies

$$\underbrace{(A-\lambda_2 I)^2}_{\parallel} v_2 = (A-\lambda_2 I) v_3 = 0.$$

$$\begin{pmatrix} 2 & 1 & 0 \\ -4 & -2 & 0 \\ 4 & -8 & -3 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ -4 & -2 & 0 \\ 4 & -8 & -3 \end{pmatrix} v_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 28 & 44 & 9 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = 0.$$

$\Leftrightarrow 28a_2 + 44b_2 + 9c_2 = 0$. We can choose (try).

$a_2 = 9,\ b_2 = 0,\ c_2 = -28.$

$$0 \xleftarrow{(A-\lambda I)} \underbrace{6 \cdot \begin{pmatrix} 3 \\ -6 \\ 20 \end{pmatrix}}_{6\cdot \vec{w}} = \begin{pmatrix} 18 \\ -36 \\ 120 \end{pmatrix} \xleftarrow{\begin{pmatrix} 2 & 1 & 0 \\ -4 & -2 & 0 \\ 4 & -8 & -3 \end{pmatrix}} \begin{pmatrix} 9 \\ 0 \\ -28 \end{pmatrix}$$

So luckily we get a chain of length 2:

$$\underbrace{\begin{pmatrix} 9 \\ 0 \\ -28 \end{pmatrix}}_{v_2} \longrightarrow \underbrace{\begin{pmatrix} 18 \\ -36 \\ 120 \end{pmatrix}}_{v_3} \longrightarrow 0$$

so finally we can write down the basic solutions.

$$\vec{x}_1(t) = e^{\lambda_1 t} \cdot \vec{v}_1 = e^{-2t}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$\vec{x}_2(t) = e^{\lambda_2 t}(\vec{v}_2 + \vec{v}_3 t) = e^{t} \cdot \begin{pmatrix} 9+18t \\ -36t \\ -28+120t \end{pmatrix}$$

$$\vec{x}_3(t) = e^{\lambda_2 t}\,\vec{v}_3 = e^{t}\begin{pmatrix} 18 \\ -36 \\ 120 \end{pmatrix}$$

```
In[22]:= StreamPlot[{x + y, -x + y}, {x, -3, 3}, {y, -3, 3}, StreamPoints → {{{{2, 1}, Thickness[0.01]}, Automatic}}]
```

Out[22]=

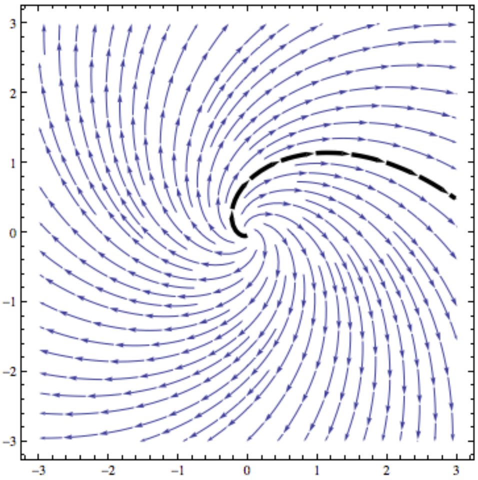


Figure 1: 3(a)

```
In[28]:= StreamPlot[{3 x + 4 y, 5 x + 2 y}, {x, -3, 3}, {y, -3, 3}, StreamPoints → {{{{2, 1}, Thickness[0.01]}, Automatic}}]
```

Out[28]=

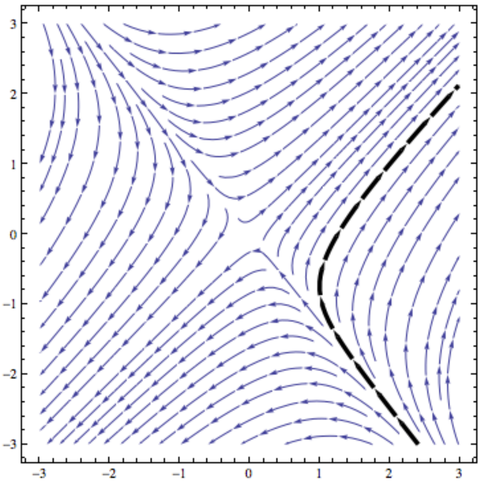


Figure 2: 3(b)

```
In[29]:= StreamPlot[{4 x + y, -4 x}, {x, -3, 3}, {y, -3, 3}, StreamPoints → {{{{2, 1}, Thickness[0.01]}, Automatic}}]
```

Out[29]=

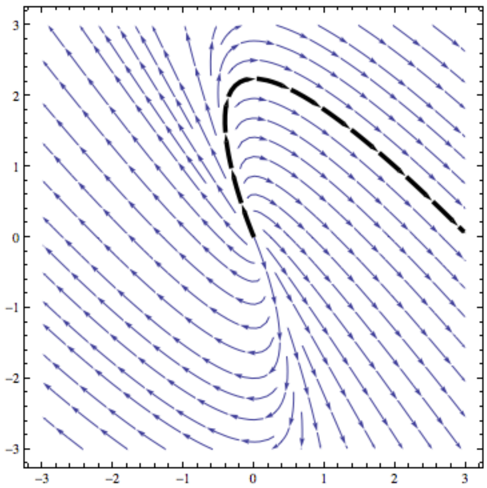


Figure 3: 3(c)

```
In[30]:= StreamPlot[{4 x + y, -x + 2 y}, {x, -3, 3}, {y, -3, 3}, StreamPoints → {{{{2, 1}, Thickness[0.01]}, Automatic}}]
```

Out[30]=

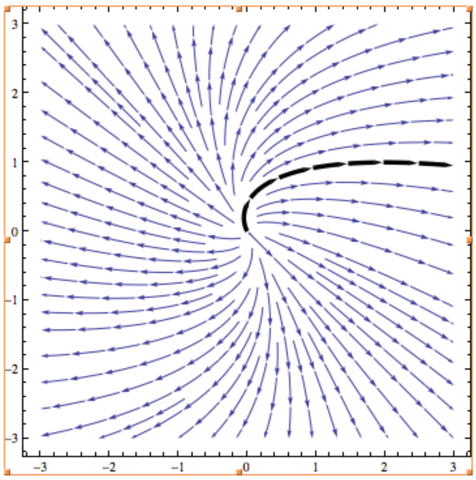


Figure 4: 3(d)

4. Using Mathematica we find the eigenvalues/eigenvectors of A:

```
In[36]:= A = {{15, -21, 24, -6, -22}, {0, -2, 8, -12, -4}, {10, -19, 21, -4, -18}, {0, -14, 16, -4, -8}, {20, -30, 30, 0, -30}}
Out[36]= {{15, -21, 24, -6, -22}, {0, -2, 8, -12, -4}, {10, -19, 21, -4, -18}, {0, -14, 16, -4, -8}, {20, -30, 30, 0, -30}}

In[43]:= A // MatrixForm
Out[43]//MatrixForm=
```

$$\begin{pmatrix} 15 & -21 & 24 & -6 & -22 \\ 0 & -2 & 8 & -12 & -4 \\ 10 & -19 & 21 & -4 & -18 \\ 0 & -14 & 16 & -4 & -8 \\ 20 & -30 & 30 & 0 & -30 \end{pmatrix}$$

```
In[40]:= Eigensystem[A]
Out[40]= {{-10, 10, -5, 5, 0}, {{1, 1, 1, 1, 1}, {1, -1, 1, 1, 2}, {1, 0, 1, 0, 2}, {2, 0, 1, 0, 2}, {0, 2, 2, 1, 0}}}
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So we get $\lambda_1 = -10$, $\lambda_2 = 10$, $\lambda_3 = -5$, $\lambda_4 = 5$, $\lambda_5 = 0$. The corresponding eigenvectors are:

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \quad v_5 = \begin{pmatrix} 0 \\ 2 \\ 2 \\ 1 \\ 0 \end{pmatrix}.$$

So the general solution to the system is:

$$\vec{x}(t) = C_1 e^{-10t} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + C_2 e^{10t} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 2 \end{pmatrix} + C_3 e^{-5t} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 2 \end{pmatrix} + C_4 e^{5t} \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 2 \end{pmatrix} + C_5 \begin{pmatrix} 0 \\ 2 \\ 2 \\ 1 \\ 0 \end{pmatrix}.$$

1.(a). $\begin{cases} x_1' = 4x_1 + x_2 \\ x_2' = -2x_1 + x_2 \\ x_3' = x_1 + x_2 + x_3 \end{cases} \iff \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

Characteristic polynomial $\det\begin{pmatrix} 4-\lambda & 1 & 0 \\ -2 & 1-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{pmatrix} = (1-\lambda)(\lambda^2-5\lambda+6) = -(\lambda-1)(\lambda-2)(\lambda-3)$.

So there are 3 eigenvalues: $\lambda_1=1$, $\lambda_2=2$, $\lambda_3=3$.

$\lambda_1=1$: $\begin{pmatrix} 3 & 1 & 0 \\ -2 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = 0 \iff \begin{cases} 3a_1+b_1=0 \\ -2a_1=0 \\ a_1+b_1=0 \end{cases} \Rightarrow a_1=b_1=0 \Rightarrow \vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

$\lambda_2=2$: $\begin{pmatrix} 2 & 1 & 0 \\ -2 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = 0 \iff \begin{cases} 2a_2+b_2=0 \\ \cancel{-2a_2-b_2=0} \\ a_2+b_2-c_2=0 \end{cases} \xRightarrow{\text{choose } a_2=1} \vec{v}_2 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$.

$\lambda_3=3$: $\begin{pmatrix} 1 & 1 & 0 \\ -2 & -2 & 0 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = 0 \iff \begin{cases} a_3+b_3=0 \\ \cancel{-2a_3-2b_3=0} \\ a_3+b_3-2c_3=0 \end{cases} \xRightarrow{a_3=1} \vec{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

3 eigenvectors. $\iff$ 3 chains of length 1: $\vec{v}_1 \to 0$, $\vec{v}_2 \to 0$, $\vec{v}_3 \to 0$

basic solutions: $\vec{x}_1(t) = e^t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\vec{x}_2(t) = e^{2t}\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$, $\vec{x}_3(t) = e^{3t}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

General solution:

$$\vec{x}(t) = C_1 \cdot e^t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + C_2 \cdot e^{2t}\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} + C_3 \cdot e^{3t}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

(b). $\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ characteristic polynomial:

$\det(A-\lambda I) = (3-\lambda)\cdot(\lambda-2)(\lambda-3)$.

$\Rightarrow \lambda_1 = 2$ with mult. 1, and $\lambda_2 = 3$ with mult. 2.

eigenvector: $\lambda_1 = 2$:

$$\begin{pmatrix} 2 & 1 & 0 \\ -2 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = 0 \Leftrightarrow \begin{cases} 2a_1 + b_1 = 0 \\ \cancel{-2a_1 - b_1 = 0} \\ a_1 + b_1 + c_1 = 0 \end{cases} \xRightarrow{a_1 = 1} \vec{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}. \longrightarrow 0.$$

$\lambda_2 = 3$:

$$\begin{pmatrix} 1 & 1 & 0 \\ -2 & -2 & 0 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = 0 \Leftrightarrow \begin{cases} a_2 + b_2 = 0 \\ \cancel{-2a_2 - 2b_2 = 0} \\ \cancel{a_2 + b_2 = 0} \end{cases} \Longrightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \rightarrow 0$$

$$\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow 0.$$

Basic solutions:

$$\vec{x}_1(t) = e^{2t}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \quad \vec{x}_2(t) = e^{3t}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \vec{x}_3(t) = e^{3t}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(c). $\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 0 \\ -1 & 2 & 0 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. $\lambda_1 = 3$, mult $= 3$.

$\Uparrow$

Characteristic polynomial:

$$\det(A-\lambda I) = (3-\lambda)\cdot\left[(4-\lambda)(2-\lambda) - 1\cdot(-1)\right] = -(\lambda-3)(\lambda^2-6\lambda+9) = -(\lambda-3)^3 = 0$$

Eigenvector:

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = 0 \Leftrightarrow \begin{cases} a_1 + b_1 = 0 \\ \cancel{a_1 + b_1 = 0} \\ \cancel{a_1 + b_1 = 0} \end{cases} \Longrightarrow \quad \vec{w}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \vec{W}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

So we are in the following case:

EValue	mult.	#EVector	Chain
3	3	2	$v_1 \to 0$ $v_2 \to v_3 \to 0$

To find $\vec{v}_2$, we calculate

$$(A-3I)^2 = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Because $(A-3I)^2\vec{v}_2 = 0$, we can choose any $\vec{v}_2$ that generate a chain of length 2.

For example. $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_3 = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \left(= \vec{w}_1 + \vec{w}_2\right) \neq 0$

We can choose $\vec{v}_1 = \vec{w}_1$ (or $\vec{w}_2$). so we get 2 chains:

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \longrightarrow 0 \quad , \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \to 0$$

$\vec{x}_2(t)$ $\quad$ $\vec{x}_3(t)$.

$//$ $\quad$ $//$

Basic solutions: $\vec{x}_1(t) = e^{3t}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$. $e^{3t}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right)$, $e^{3t}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$,

General solution:

$$\vec{x}(t) = C_1 e^{3t}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + C_2 e^{3t}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right] + C_3 e^{3t}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

(d) $\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 0 \\ -1 & 2 & 0 \\ -1 & 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Characteristic polynomial:

$\det(A-\lambda I) = -(\lambda-3)^3 = 0$

$\Rightarrow \lambda_1 = 3$, mult. $= 3$. Eigenvector: $(A-3I)\vec{w} = 0$.

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = 0 \Leftrightarrow \begin{cases} a_1 + b_1 = 0 \\ \cancel{-a_1 - b_1 = 0} \\ -a_1 + b_1 = 0 \end{cases} \Rightarrow a_1 = b_1 = 0 \Rightarrow \vec{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

so we are in the following case:

EValue	mult.	#Vector	Chain
3	3	1	$v_1 \to v_2 \to v_3 \to 0$.

$\vec{v}_1$ satisfies $(A-3I)^3 \vec{v}_1 = 0$. So we calculate:

$$(A-3I)^3 = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & -2 & 0 \end{pmatrix}\cdot\begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So we can choose any $\vec{v}_1$ as long as it generates a chain of length 3.

Try $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_2 = (A-3I)\vec{v}_1 = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$.

$\Rightarrow \underset{(A-3I)\vec{v}_2}{\underset{\parallel}{\vec{v}_3}} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} = 2\cdot\vec{w}_1 \neq 0$. So we get a

chain of length 3: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \to 0$.

General solutions:

$$\boxed{\vec{x}(t) = C_1\cdot e^{3t}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + \frac{t^2}{2}\begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}\right) + C_2\cdot e^{3t}\left(\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}\right) + C_3\cdot e^{3t}\begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}.}$$

2. (a) $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = A$

$$e^A = I + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} + \begin{pmatrix} \frac{\lambda_1^2}{2!} & 0 \\ 0 & \frac{\lambda_2^2}{2!} \end{pmatrix} + \begin{pmatrix} \frac{\lambda_1^3}{3!} & 0 \\ 0 & \frac{\lambda_2^3}{3!} \end{pmatrix} + \cdots$$

$$\| \qquad\qquad\qquad\qquad\qquad \|$$

$$I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots \qquad \begin{pmatrix} 1+\lambda_1+\frac{\lambda_1^2}{2!}+\frac{\lambda_1^3}{3!}+\cdots & 0 \\ 0 & 1+\lambda_2+\frac{\lambda_2^2}{2!}+\frac{\lambda_2^3}{3!}+\cdots \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}.$$

(b). $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = B + C.$

$$e^C = e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} = I + C + \frac{C^2}{2!} + \cdots = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \frac{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2}{2!} + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0 + 0 + \cdots = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

$$\Rightarrow e^A = e^B \cdot e^C = \begin{pmatrix} e^\lambda & 0 \\ 0 & e^\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^\lambda & e^\lambda \\ 0 & e^\lambda \end{pmatrix}.$$

(c). $$e^{\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} + \frac{\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}^2}{2!} + \frac{\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}^3}{3!} + \frac{\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}^4}{4!} + \frac{\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}^5}{5!} + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} + \frac{\begin{pmatrix} -a^2 & 0 \\ 0 & -a^2 \end{pmatrix}}{2!} + \frac{\begin{pmatrix} 0 & -a^3 \\ a^3 & 0 \end{pmatrix}}{3!} + \frac{\begin{pmatrix} a^4 & 0 \\ 0 & a^4 \end{pmatrix}}{4!} + \frac{\begin{pmatrix} 0 & a^5 \\ -a^5 & 0 \end{pmatrix}}{5!} + \cdots$$

$$= \begin{pmatrix} 1-\frac{a^2}{2!}+\frac{a^4}{4!}-\frac{a^6}{6!}+\cdots & a-\frac{a^3}{3!}+\frac{a^5}{5!}-\cdots \\ -a+\frac{a^3}{3!}-\frac{a^5}{5!}-\cdots & 1-\frac{a^2}{2!}+\frac{a^4}{4!}-\cdots \end{pmatrix} = \begin{pmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{pmatrix}.$$

because: $\cos a = 1 - \frac{a^2}{2!} + \frac{a^4}{4!} - \frac{a^6}{6!} + \frac{a^8}{8!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{a^{2n}}{(2n)!}$.

$\sin a = a - \frac{a^3}{3!} + \frac{a^5}{5!} - \frac{a^7}{7!} + \frac{a^9}{9!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{a^{2n+1}}{(2n+1)!}$.

(d). $e^{\begin{pmatrix}\lambda&1&0\\0&\lambda&1\\0&0&\lambda\end{pmatrix}} = e^{\lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}} = e^{\lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}}\cdot e^{\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}}$.

$$= \begin{pmatrix}e^\lambda&0&0\\0&e^\lambda&0\\0&0&e^\lambda\end{pmatrix}\left[\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}+\frac{1}{2!}\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}^2+\frac{1}{3!}\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}^3+\cdots\right]$$

$$= \begin{pmatrix}e^\lambda&0&0\\0&e^\lambda&0\\0&0&e^\lambda\end{pmatrix}\cdot\left[\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}+\frac{1}{2}\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}+0+0+\cdots\right]$$

$$= \begin{pmatrix}e^\lambda&0&0\\0&e^\lambda&0\\0&0&e^\lambda\end{pmatrix}\cdot\begin{pmatrix}1&1&\frac{1}{2}\\0&1&1\\0&0&1\end{pmatrix} = \begin{pmatrix}e^\lambda&e^\lambda&\frac{1}{2}e^\lambda\\0&e^\lambda&e^\lambda\\0&0&e^\lambda\end{pmatrix}.$$

3. (a) $A=\begin{pmatrix}4&1\\-2&1\end{pmatrix}$. characteristic polynomial.

$\det(A-\lambda I) = \begin{vmatrix}4-\lambda&1\\-2&1-\lambda\end{vmatrix} = \lambda^2-5\lambda+6 = (\lambda-2)(\lambda-3)$

eigenvalue: $\lambda_1=2$, $\lambda_2=3$.

$\lambda_1=2$, $\begin{pmatrix}2&1\\-2&-1\end{pmatrix}\cdot\begin{pmatrix}a_1\\b_1\end{pmatrix}=0 \Leftrightarrow \begin{cases}2a_1+b_1=0\\ \cancel{-2a_1-b_1=0}\end{cases} \xRightarrow{a_1=1} \vec{v}_1=\begin{pmatrix}1\\-2\end{pmatrix}. \Rightarrow \vec{x}_1(t)=e^{2t}\begin{pmatrix}1\\-2\end{pmatrix}$

$\lambda_2=3$ $\begin{pmatrix}1&1\\-2&-2\end{pmatrix}\cdot\begin{pmatrix}a_2\\b_2\end{pmatrix}=0 \Leftrightarrow \begin{cases}a_2+b_2=0\\ \cancel{-2a_2-2b_2=0}\end{cases} \xRightarrow{a_2=1} \vec{v}_2=\begin{pmatrix}1\\-1\end{pmatrix}. \Rightarrow \vec{x}_2(t)=e^{3t}\begin{pmatrix}1\\-1\end{pmatrix}$

Fundamental matrix: $\Phi(t)=\begin{pmatrix}e^{2t}&e^{3t}\\-2e^{2t}&-e^{3t}\end{pmatrix}$. $\Phi(0)=\begin{pmatrix}1&1\\-2&-1\end{pmatrix}$. $|\Phi(0)| = 1\times(-1)-1\times(-2)=1$.

$\Rightarrow \Phi(0)^{-1}=\frac{1}{1}\begin{pmatrix}-1&-1\\2&1\end{pmatrix} \Rightarrow e^{tA}=\Phi(t)\cdot\Phi(0)^{-1}=\begin{pmatrix}e^{2t}&e^{3t}\\-2e^{2t}&-e^{3t}\end{pmatrix}\cdot\begin{pmatrix}-1&-1\\2&1\end{pmatrix}$

$= \boxed{\begin{pmatrix}-e^{2t}+2e^{3t}&-e^{2t}+e^{3t}\\+2e^{2t}-2e^{3t}&2e^{2t}-e^{3t}\end{pmatrix}}$

initial value problem:

$\vec{x}(t)=e^{tA}\vec{x}_0=e^{tA}\begin{pmatrix}-3\\2\end{pmatrix}=\begin{pmatrix}-3(-e^{2t}+2e^{3t})+2(-e^{2t}+e^{3t})\\-3(+2e^{2t}-2e^{3t})+2(2e^{2t}-e^{3t})\end{pmatrix}.$

$$\vec{x}(t) = e^{tA}\vec{x}_0 = \begin{pmatrix} e^{2t} - 4e^{3t} \\ -\cancel{4}\,2e^{2t} + 4e^{3t} \end{pmatrix} = \begin{pmatrix} e^{2t} - 4e^{3t} \\ -2e^{2t} + 4e^{3t} \end{pmatrix}.$$

(b). $\begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix} = A$. Characteristic polynomial.

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2.$$

Eigenvalue $\lambda_1 = 3$, mult. $= 2$.

Eigenvector $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \cdot \vec{w}_1 = 0 \Rightarrow \vec{w}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. $\rightsquigarrow$ $\vec{v}_1 \to \vec{v}_2 \to 0$.

$$(A - 3I)^2 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \vec{v}_2 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \vec{w}_1$$

chain: $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ -1 \end{pmatrix} \to 0$. $\Rightarrow \vec{x}_1(t) = e^{3t}\cdot\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\cdot\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$, $\vec{x}_2(t) = e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$$\Rightarrow \Phi(t) = (\vec{x}_1(t), \vec{x}_2(t)) = \begin{pmatrix} e^{3t}\cdot(1+t) & e^{3t} \\ e^{3t}(-t) & -e^{3t} \end{pmatrix} \Rightarrow \Phi(0) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow \Phi(0)^{-1} = \frac{1}{-1}\cdot\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow e^{tA} = \Phi(t)\cdot\Phi(0)^{-1} = \begin{pmatrix} e^{3t}(1+t) & e^{3t} \\ -e^{3t}\cdot t & -e^{3t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} e^{3t}\cdot(1+t) & e^{3t}(1+t) - e^{3t} \\ -e^{3t}\cdot t & -e^{3t}\cdot t + e^{3t} \end{pmatrix}.$$

$$= \begin{pmatrix} (1+t)\cdot e^{3t} & t\cdot e^{3t} \\ -t\cdot e^{3t} & (1-t)\cdot e^{3t} \end{pmatrix}.$$

initial value problem:

$$\vec{x}(t) = e^{tA}\cdot\vec{x}_0 = \begin{pmatrix} (1+t)e^{3t} & t\cdot e^{3t} \\ -t\cdot e^{3t} & (1-t)e^{3t} \end{pmatrix}\cdot\begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -3(1+t)e^{3t} + 2t\cdot e^{3t} \\ 3t\cdot e^{3t} + 2(1-t)e^{3t} \end{pmatrix}$$

$$= \begin{pmatrix} (-3-t)e^{3t} \\ (2+t)e^{3t} \end{pmatrix}.$$

(c). $A=\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$. Characteristic polynomial:

$$\det(A-\lambda I)=\begin{vmatrix} -\lambda & a \\ -a & -\lambda \end{vmatrix}=\lambda^2+a^2=0$$

$\Rightarrow \lambda_1=ai,\ \lambda_2=-ai.$

$\lambda_1=ai$: $\begin{pmatrix} -ai & a \\ -a & -ai \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=0 \Leftrightarrow \begin{cases} -ai\cdot u_1+au_2=0 \\ -au_1-ai\cdot u_2=0 \end{cases} \xRightarrow{u_1=1} \begin{pmatrix} 1 \\ i \end{pmatrix}=\vec{v_1}$

$\Rightarrow \lambda_2=-ai,\ \vec{v_2}=\begin{pmatrix} 1 \\ -i \end{pmatrix}$. $\Rightarrow \vec{x_1}(t)=e^{(ai)t}\begin{pmatrix} 1 \\ i \end{pmatrix},\ \vec{x_2}(t)=e^{(-ai)t}\begin{pmatrix} 1 \\ -i \end{pmatrix}$.

$\Rightarrow \Phi(t)=\begin{pmatrix} e^{(ai)t} & e^{-(ai)t} \\ i\cdot e^{(ai)t} & -i\cdot e^{-(ai)t} \end{pmatrix} \Rightarrow \Phi(0)=\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix},\ |\Phi(0)|=-2i$

$\Rightarrow \Phi(0)^{-1}=\frac{1}{-2i}\cdot\begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix}$

$\Rightarrow e^{tA}=\Phi(t)\Phi(0)^{-1}=\begin{pmatrix} e^{(ai)t} & e^{-(ai)t} \\ i\cdot e^{(ai)t} & -i\cdot e^{-(ai)t} \end{pmatrix}\cdot\frac{1}{-2i}\begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix}$

$$=-\frac{1}{2i}\cdot\begin{pmatrix} -i\cdot e^{(ai)t}-i\cdot e^{-(ai)t} & -e^{(ai)t}+e^{-(ai)t} \\ i(-i)\cdot e^{(ai)t}+(-i)(-i)\cdot e^{-(ai)t} & -i\cdot e^{(ai)t}-i\cdot e^{-(ai)t} \end{pmatrix}$$

$$=\begin{pmatrix} \frac{1}{2}\cdot(e^{(ai)t}+e^{-(ai)t}) & \frac{1}{2i}(e^{(ai)t}-e^{-(ai)t}) \\ -\frac{1}{2i}(e^{(ai)t}-e^{-(ai)t}) & \frac{1}{2}(e^{(ai)t}+e^{-(ai)t}) \end{pmatrix}.$$

$$=\begin{pmatrix} \cos(at) & \sin(at) \\ -\sin(at) & \cos(at) \end{pmatrix}.$$

4. (a) $\dfrac{d\vec{x}}{dt} = \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix} \vec{x}(t) + \begin{pmatrix} e^{t} \\ e^{3t} \end{pmatrix}.$

From 3(a), we got $\Phi(t) = \begin{pmatrix} e^{2t} & e^{3t} \\ -2e^{2t} & -e^{3t} \end{pmatrix} \Rightarrow |\Phi(t)| = e^{2t}(-e^{3t}) - e^{3t}\cdot(-2e^{2t})$

$= e^{5t}.$

$\Rightarrow \Phi(t)^{-1} = \dfrac{1}{e^{5t}} \begin{pmatrix} -e^{3t} & -e^{3t} \\ 2e^{2t} & e^{2t} \end{pmatrix}$

We use the following formula (variation of parameters) to find a particular solution:

$$\vec{x}_p(t) = \Phi(t) \int \Phi(t)^{-1} \vec{f}(t)\, dt.$$

$$\Phi(t)^{-1} \vec{f}(t) = \begin{pmatrix} -e^{-2t} & -e^{-2t} \\ 2e^{-3t} & e^{-3t} \end{pmatrix} \cdot \begin{pmatrix} e^{t} \\ e^{3t} \end{pmatrix} = \begin{pmatrix} -e^{-t} - e^{t} \\ 2\cdot e^{-2t} + 1 \end{pmatrix}.$$

$$\Rightarrow \int \Phi(t) f(t)^{-1}\, dt = \int \begin{pmatrix} -e^{-t} - e^{t} \\ 2\cdot e^{-2t} + 1 \end{pmatrix} dt = \begin{pmatrix} e^{-t} - e^{t} \\ -e^{-2t} + t \end{pmatrix}.$$

$$\Rightarrow \vec{x}_p(t) = \begin{pmatrix} e^{2t} & e^{3t} \\ -2e^{2t} & -e^{3t} \end{pmatrix} \cdot \begin{pmatrix} e^{-t} - e^{t} \\ -e^{-2t} + t \end{pmatrix} = \begin{pmatrix} e^{t} - e^{3t} - e^{t} + te^{3t} \\ -2\cdot e^{t} + 2\cdot e^{3t} + e^{t} - t\cdot e^{3t} \end{pmatrix}$$

$$= \begin{pmatrix} (t-1)\cdot e^{3t} \\ -e^{t} + (2-t)\cdot e^{3t} \end{pmatrix}.$$

the general solution is

$$\vec{x}(t) = c_1 \cdot \vec{x}_1(t) + c_2 \cdot \vec{x}_2(t) + \vec{x}_p(t)$$

$$= c_1 \cdot \begin{pmatrix} e^{2t} \\ -2e^{2t} \end{pmatrix} + c_2 \cdot \begin{pmatrix} e^{3t} \\ -e^{3t} \end{pmatrix} + \begin{pmatrix} (t-1)\, e^{3t} \\ -e^{t} + (2-t)\, e^{3t} \end{pmatrix}.$$

(b). $\frac{d}{dt}\vec{x}(t) = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix} \vec{x}(t) + \begin{pmatrix} e^t \\ e^{3t} \end{pmatrix}$

From 3(b), we get $\Phi(t) = e^{3t}\begin{pmatrix} 1+t & 1 \\ -t & -1 \end{pmatrix}$. $|\Phi(t)| = e^{6t}\cdot(-1-t+t) = -e^{6t}$

$$\Rightarrow \Phi(t)^{-1} = -\frac{1}{e^{6t}}\cdot e^{3t}\cdot\begin{pmatrix} -1 & -1 \\ t & 1+t \end{pmatrix} = -e^{-3t}\begin{pmatrix} -1 & -1 \\ t & 1+t \end{pmatrix}.$$

$$\Rightarrow \Phi(t)^{-1}\vec{f}(t) = -e^{-3t}\begin{pmatrix} -1 & -1 \\ t & 1+t \end{pmatrix}\cdot\begin{pmatrix} e^t \\ e^{3t} \end{pmatrix} = -e^{-3t}\cdot\begin{pmatrix} -e^t - e^{3t} \\ t\cdot e^t + (1+t)e^{3t} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-2t} + 1 \\ -t\cdot e^{-2t} - (1+t) \end{pmatrix}.$$

$$\Rightarrow \int \Phi(t)^{-1}\vec{f}(t)\,dt = \int \begin{pmatrix} e^{-2t}+1 \\ -te^{-2t} - (1+t) \end{pmatrix} dt = \begin{pmatrix} -\frac{1}{2}e^{-2t} + t \\ -\int te^{-2t}dt - t - \frac{1}{2}t^2 \end{pmatrix}.$$

$$\left(\int te^{-2t}dt = \int t\, d(e^{-2t})\cdot\left(-\tfrac{1}{2}\right) = -\frac{1}{2}te^{-2t} + \frac{1}{2}\int e^{-2t}dt = -\frac{1}{2}te^{-2t} - \frac{1}{4}\cdot e^{-2t}\right)$$

$$= \begin{pmatrix} -\frac{1}{2}e^{-2t} + t \\ +\frac{1}{2}te^{-2t} + \frac{1}{4}\cdot e^{-2t} - t - \frac{1}{2}t^2 \end{pmatrix}.$$

$$\Rightarrow \vec{x}_p(t) = \begin{pmatrix} e^{3t}(1+t) & e^{3t} \\ -t\cdot e^{3t} & -e^{3t} \end{pmatrix}\cdot\begin{pmatrix} -\frac{1}{2}e^{-2t} + t \\ +\frac{1}{2}te^{-2t} + \frac{1}{4}\cdot e^{-2t} - t - \frac{1}{2}t^2 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2}(1+t)e^t + t(1+t)e^{3t} + \left(\frac{1}{2}t + \frac{1}{4}\right)e^t - \left(t + \frac{1}{2}t^2\right)e^{3t} \\ \frac{1}{2}t\cdot e^t - t^2e^{3t} - \left(\frac{1}{2}t + \frac{1}{4}\right)e^t + \left(t + \frac{1}{2}t^2\right)e^{3t} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{4}\cdot e^t + \frac{1}{2}t^2e^{3t} \\ -\frac{1}{4}\cdot e^t + \left(t - \frac{1}{2}t^2\right)e^{3t} \end{pmatrix}.$$

General solution: $\vec{x}(t) = C_1\cdot\begin{pmatrix} (1+t)e^{3t} \\ -t\cdot e^{3t} \end{pmatrix} + C_2\cdot\begin{pmatrix} e^{3t} \\ -e^{3t} \end{pmatrix} + \begin{pmatrix} -\frac{1}{4}e^t + \frac{1}{2}t^2e^{3t} \\ -\frac{1}{4}e^t + \left(t - \frac{1}{2}t^2\right)e^{3t} \end{pmatrix}.$

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Homework 13

1. Find a power series solution of the given differential equation. Determine the radius of convergence of the resulting series, and identify the series solution in terms of familiar elementary functions.

   (a) $(x-3)y' + y = 0$.

   (b) $(1-x^2)y' + 2xy = 0$.

   (c) $y'' - 4y = 0$.

   (d) $y'' + y = x$.

2. For equations (a)-(e),

   (i) Find general solutions in powers of x of the differential equations. State the recurrence relation and the guaranteed radius of convergence in each case.

   (ii) Use power series to solve the initial value problem $y(0) = 0$, $y'(0) = 1$.

   (a) $(1-x)y'' + y = 0$.

   (b) $(x^2-1)y'' + 6xy' + 12y = 0$.

   (c) $y'' - 2xy' + 6y = 0$. (Hermite equation)

   (d) $y'' - 2xy' + 8y = 0$. (Hermite equation)

   (e) $y'' = xy$. (Airy equation)

1.(a) $(x-3)y' + y = 0$.

$y = \sum_{n=0}^{\infty} c_n \cdot x^n \Rightarrow y' = \sum_{n=0}^{\infty} c_n \cdot n \cdot x^{n-1}$ so

$$\underset{0}{\overset{\parallel}{(x-3)y' + y}} = (x-3) \sum_{n=0}^{\infty} c_n \cdot n \cdot x^{n-1} + \sum_{n=0}^{\infty} c_n \cdot x^n = \sum_{n=0}^{\infty} c_n \cdot n \cdot x^n - 3\sum_{n=0}^{\infty} c_n \cdot n \cdot x^{n-1} + \sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{n=0}^{\infty} c_n \cdot (n+1) \cdot x^n - 3 \sum_{m=0}^{\infty} c_{m+1} \cdot (m+1) \cdot x^m$$

$$= \sum_{n=0}^{\infty} (n+1) \cdot (c_n - 3 \cdot c_{n+1}) \cdot x^n$$

$$\boxed{\text{radius of convergence } R = \lim_{n \to \infty} \left| \frac{c_n}{c_{n+1}} \right| = 3.}$$

$$\Rightarrow c_{n+1} = \frac{c_n}{3} \Rightarrow c_n = \left(\frac{1}{3}\right)^n \cdot c_0.$$

$$\Rightarrow y = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n c_0 \cdot x^n = c_0 \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n = c_0 \cdot \frac{1}{1-\frac{x}{3}} = \frac{3c_0}{3-x}.$$

(b) $(1-x^2)y' + 2x \cdot y = 0$. $y = \sum_{n=0}^{\infty} c_n \cdot x^n \Rightarrow y' = \sum_{n=0}^{\infty} c_n \cdot n \cdot x^{n-1}$

$$\underset{0}{\overset{\parallel}{(1-x^2) \cdot y' + 2xy}} = (1-x^2) \sum_{n=0}^{\infty} c_n n x^{n-1} + 2x \sum_{n=0}^{\infty} c_n \cdot x^n = \sum_{n=0}^{\infty} c_n \cdot n \cdot x^{n-1} - \sum_{n=0}^{\infty} c_n \cdot n \, x^{n+1} + 2 \sum_{n=0}^{\infty} c_n \cdot x^{n+1}$$

$$= \sum_{m=0}^{\infty} c_{m+1} (m+1) x^m + \sum_{n=0}^{\infty} (2-n) \cdot c_n \, x^{n+1}$$

$$= \sum_{n=0}^{\infty} c_{n+1} (n+1) \cdot x^n + \sum_{n=1}^{\infty} (2-(n-1)) \, c_{n-1} \, x^n$$

$$= c_1 1 x^0 + \sum_{n=1}^{\infty} \left(c_{n+1}(n+1) + (3-n) \, c_{n-1} \right) x^n.$$

$$\Rightarrow c_1 = 0, \text{ and } c_{n+1} = \frac{n-3}{n+1} \cdot c_{n-1} \text{ for } n \geq 1$$

$\Rightarrow$ • $c_{2m+1} = \frac{2m-3}{2m+1} \cdot c_{2m-1} \xRightarrow{c_1 = 0} c_{2m+1} = 0$.

• $c_{2m+2} = \frac{2m-2}{2m+2} \, c_{2m} \Rightarrow c_2 = -c_0 \ (m=0)$. $c_4 = \frac{2-2}{2+2} = 0 \Rightarrow c_6 = c_8 = c_{10} = \cdots = 0$.

$\Rightarrow y(x) = c_0 + c_2 \cdot x^2 = c_0 - c_0 \cdot x^2 = c_0(1-x^2)$ radius of convergence $+\infty$.

1.(c) $y''-4y=0$ $\quad y=\sum_{n=0}^{\infty} C_n\cdot x^n \Rightarrow y'=\sum_{n=0}^{\infty} C_n\cdot n\cdot x^{n-1}$

$$\Rightarrow y''=\sum_{n=0}^{\infty} C_n\cdot n(n-1)\cdot x^{n-2} \Rightarrow y''-4y=\sum_{n=0}^{\infty} C_n\cdot n(n-1)\cdot x^{n-2}-4\sum_{n=0}^{\infty} C_n\cdot x^n$$

$$\overset{\parallel}{0} \qquad =\sum_{m=0}^{\infty} C_{m+2}\cdot(m+2)\cdot(m+1)\cdot x^m-4\sum_{n=0}^{\infty} C_n\cdot x^n$$

$$\boxed{C_{n+2}=\frac{4\cdot C_n}{(n+2)(n+1)}} \Longleftarrow \quad =\sum_{n=0}^{\infty}\left(C_{n+2}\cdot(n+2)(n+1)-4\cdot C_n\right)x^n$$

$2+n=2m+1$: $\quad C_{2m+1}=\frac{4\cdot C_{2m-1}}{(2m+1)\cdot(2m)}=\frac{4}{(2m+1)(2m)}\frac{4}{(2m-1)(2m-2)}\cdot C_{2m-3}=\cdots$

$$=\frac{4}{(2m+1)(2m)}\cdot\frac{4}{(2m-1)(2m-2)}\cdots\frac{4}{3\cdot 2}\cdot C_1$$

$$=\frac{4^m}{(2m+1)!}C_1.$$

$n+2=2m$: $\quad C_{2m}=\frac{4}{(2m)\cdot(2m-1)}\cdot C_{2m-2}=\frac{4}{(2m)(2m-1)}\cdot\frac{4}{(2m-2)(2m-3)}\cdot C_{2m-4}=\cdots$

$$=\frac{4}{(2m)(2m-1)}\frac{4}{(2m-2)(2m-3)}\cdots\frac{4}{2\cdot 1}\cdot C_0=\frac{4^m}{(2m)!}\cdot C_0.$$

so. $\quad y(x)=C_0\cdot\sum_{m=0}^{\infty}\frac{4^m}{(2m)!}\cdot x^{2m}+C_1\cdot\sum_{m=0}^{\infty}\frac{4^m}{(2m+1)!}\cdot x^{2m+1}$

$$=C_0\cdot\sum_{m=0}^{\infty}\frac{(2x)^{2m}}{(2m)!}+\frac{C_1}{2}\sum_{m=0}^{\infty}\frac{(2x)^{2m+1}}{(2m+1)!}$$

$$=C_0\cdot\frac{e^{2x}+e^{-2x}}{2}+\frac{C_1}{2}\cdot\frac{e^{2x}-e^{-2x}}{2}$$

$$=\left(\frac{1}{2}C_0+\frac{C_1}{4}\right)e^{2x}+\left(\frac{1}{2}C_0-\frac{C_1}{4}\right)e^{-2x}$$

Radius of convergence $\parallel$ $+\infty$

$\left(\lim_{n\to+\infty}\left|\frac{C_n}{C_{n+2}}\right|=+\infty\right)$

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1(d) $y'' + y = x$. $y = \sum_{n=0}^{\infty} C_n \cdot x^n$, $y' = \sum_{n=0}^{\infty} C_n \cdot n \cdot x^{n-1}$.

$$y'' = \sum_{n=0}^{\infty} C_n \cdot n(n-1) \cdot x^{n-2} = \sum_{m=0}^{\infty} C_{m+2}(m+2)(m+1) \cdot x^m = \sum_{n=0}^{\infty} C_{n+2}(n+2)(n+1)x^n.$$

$$\Rightarrow x = y'' + y = \sum_{n=0}^{\infty} \left(C_n + C_{n+2}(n+2)\cdot(n+1)\right) x^n.$$

when $n = 2m$: $C_{2m+2}(2m+2)(2m+1) + C_{2m} = 0 \Rightarrow C_{2m+2} = -\frac{C_{2m}}{(2m+2)(2m+1)}$.

$$\Rightarrow C_{2m} = -\frac{C_{2m-2}}{(2m)\cdot(2m-1)} = (-1)^2 \frac{C_{2m-4}}{(2m)(2m-1)(2m-2)(2m-3)} = \cdots$$

$$= (-1)^m \cdot \frac{1}{(2m)!} \cdot C_0.$$

when $n = 2m+1$: $C_{2m+3}(2m+3)(2m+2) + C_{2m+1} = \begin{cases} 1 & m=0 \\ 0 & m \geq 1. \end{cases}$

- $m=0$: $C_3 \cdot 6 + C_1 = 1 \Rightarrow C_1 = 1 - 6C_3$.
- $m \geq 1$: $C_{2m+3} = -\frac{1}{(2m+3)(2m+2)} \cdot C_{2m+1} = (-1)^2 \frac{1}{(2m+3)(2m+2)} \cdot \frac{1}{(2m+1)\,2m} C_{2m-1}$

$$= \cdots = (-1)^m \frac{1}{(2m+3)(2m+2)\cdots 5 \cdot 4} \cdot C_3.$$

$$\Rightarrow y(x) = \sum_{m=0}^{\infty} (-1)^m \frac{C_0}{(2m)!} x^{2m} + \sum_{m \geq 0} (-1)^m \frac{6C_3}{(2m+3)!} x^{2m+3} + (1-6C_3) \cdot x$$

$$= C_0 \cdot \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{(2m)!} - 6C_3 \cdot \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!} + x$$

$$= C_0 \cdot \cos x - 6C_3 \cdot \sin x + x.$$

$\boxed{\text{radius of convergence} = +\infty}$ $\left(\lim_{n\to\infty} \left|\frac{C_n}{C_{n+2}}\right| = +\infty\right)$.

2. (a) $(1-x)y''+y=0$.

$$y=\sum_{n=0}^{\infty} C_n \cdot x^n \Rightarrow y'=\sum_{n=0}^{\infty} C_n \cdot n \cdot x^{n-1} \Rightarrow y''=\sum_{n=0}^{\infty} C_n \cdot n(n-1) \cdot x^{n-2}$$

$$\Rightarrow \underset{0}{\underset{\|}{(1-x)y''+y}} = (1-x)\cdot\sum_{n=0}^{\infty} C_n \cdot n(n-1)\cdot x^{n-2} + \sum_{n=0}^{\infty} C_n \cdot x^n$$

$$= \sum_{n=0}^{\infty} C_n \cdot n(n-1)\cdot x^{n-2} - \sum_{n=0}^{\infty} C_n \cdot n(n-1)\cdot x^{n-1} + \sum_{n=0}^{\infty} C_n \cdot x^n$$

$$= \sum_{n=0}^{\infty} C_{n+2}\cdot(n+2)(n+1)\cdot x^n - \sum_{n=0}^{\infty} C_{n+1}(n+1)\cdot n \cdot x^n + \sum_{n=0}^{\infty} C_n \cdot x^n$$

$$\Rightarrow \quad C_{n+2}(n+2)(n+1) - C_{n+1}(n+1)\cdot n + C_n = 0$$

$$\Rightarrow \boxed{C_{n+2} = \frac{C_{n+1}(n+1)\cdot n - C_n}{(n+2)(n+1)}}$$ recurrence relation. (3-terms)

• $y_1(0)=0,\ y_1'(0)=1 \Rightarrow C_0=0,\ C_1=1$.

$$C_2=\frac{C_1\cdot 1\cdot 0 - C_0}{2\cdot 1}=0, \quad C_3=\frac{C_2\cdot 2\cdot 1 - C_1}{3\times 2}=-\frac{1}{6}.$$

$$C_4=\frac{C_3\cdot 3\cdot 2 - C_2}{4\times 3}=-\frac{1}{12}, \quad C_5=\frac{C_4\cdot 4\times 3 - C_3}{5\times 4}=-\frac{1}{24}.$$

$\ldots$

$$\Rightarrow y_1(x)= x-\frac{1}{6}x^3-\frac{1}{12}x^4-\frac{1}{24}x^5+\cdots$$

To find Guaranteed radius of convergence, transform the equation into the standard form: $y''+\frac{1}{1-x}y=0$. so $x=1$ is a singular point.

$$\Rightarrow R \geq \text{dist}(\underset{\substack{\uparrow\\ \text{center}}}{0}, 1)=1$$

the other basic solution

$y_2(0)=1,\ y_2'(0)=0 \Rightarrow C_0=1,\ C_1=0.$

$$C_2=\frac{C_1\cdot 1\cdot 0-C_0}{2\times 1}=-\frac{1}{2},\quad C_3=\frac{C_2\cdot 2\cdot 1-C_1}{3\times 2}=-\frac{1}{6}.$$

$$C_4=\frac{C_3\times 3\times 2-C_2}{4\times 3}=\frac{-1-(-\frac{1}{2})}{12}=-\frac{1}{24}$$

$$C_5=\frac{C_4\times 4\times 3-C_3}{5\times 4}=\frac{-\frac{1}{2}-(-\frac{1}{6})}{20}=\frac{-\frac{1}{3}}{20}=-\frac{1}{60}.$$

$$\Rightarrow y_2(x)=1+0\cdot x-\frac{1}{2}x^2-\frac{1}{6}x^3-\frac{1}{24}x^4-\frac{1}{60}x^5+\dots$$

General solution:

$$y(x)=C_1\cdot y_1(x)+C_0\cdot y_2(x)$$

$$=C_1\left(x-\frac{1}{6}x^3-\frac{1}{12}x^4-\frac{1}{24}x^5+\dots\right)$$

$$+C_0\cdot\left(1-\frac{1}{2}x^2-\frac{1}{6}x^3-\frac{1}{24}x^4-\frac{1}{60}x^5+\dots\right).$$

$y_1(x)$ satisfies $y_1(0)=0,\ y_1'(0)=1.$

2(b). $(x^2-1)y''+6xy'+12y=0$

$y=\sum_{n=0}^{\infty} C_n x^n \Rightarrow y'=\sum_{n=0}^{\infty} C_n\cdot n\cdot x^{n-1}$, $y''=\sum_{n=0}^{\infty} C_n\cdot n(n-1)\cdot x^{n-2}$.

$(x^2-1)y''+6xy'+12y=(x^2-1)\cdot\sum_{n=0}^{\infty} C_n\, n(n-1)\, x^{n-2}+6x\sum_{n=0}^{\infty} C_n\cdot n\cdot x^{n-1}+12\sum_{n=0}^{\infty} C_n\cdot x^n$

$=\sum_{n=0}^{\infty} C_n\cdot n(n-1)\cdot x^n-\sum_{n=0}^{\infty} C_n\cdot n(n-1)x^{n-2}+6\sum_{n=0}^{\infty} C_n\cdot n\cdot x^n+12\sum_{n=0}^{\infty} C_n\cdot x^n$

$=\sum_{n=0}^{\infty} C_n\cdot x^n\cdot\left(n(n-1)+6n+12\right)-\sum_{n=0}^{\infty} C_{n+2}(n+2)\cdot(n+1)\cdot x^n=0$

$\Rightarrow\quad C_n\cdot\left(n^2+5n+12\right)-C_{n+2}(n+2)\cdot(n+1)=0$

$\Rightarrow\quad \boxed{C_{n+2}=\frac{n^2+5n+12}{(n+2)(n+1)}C_n}$ recurrence relation.

even terms: $C_2=\frac{12}{2}C_0=6C_0$, $C_4=\frac{26}{4\times3}C_2=\frac{13}{6}C_2=13C_0$.

$C_6=\frac{4^2+5\times4+12}{6\times5}C_4=\frac{48}{6\times5}C_4=\frac{8}{5}C_4=\frac{104}{5}C_0$.

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odd terms: $C_3=\frac{1^2+5\times1+12}{3\times2}C_1=3C_1$, $C_5=\frac{3^2+5\times3+12}{5\times4}C_3=\frac{36}{20}C_3=\frac{9}{5}C_3=\frac{27}{5}C_1$.

$C_7=\frac{5^2+5\times5+12}{7\times6}C_5=\frac{62}{42}C_5=\frac{31}{21}\times\frac{27}{5}C_1=\frac{279}{35}C_1$.

so $y(x)=C_0\left(1+6x^2+13x^4+\frac{104}{5}x^6+\cdots\right)$
$+C_1\left(x+3x^3+\frac{27}{5}x^5+\frac{279}{35}x^7+\cdots\right)$.

standard form

$y''+\frac{6x}{x^2-1}y'+\frac{12}{x^2-1}y=0$

Guaranteed radius of convergence

$\text{dist}(0,\pm1)=1$.

singular points.

$\lim_{n\to\infty}\left|\frac{C_n}{C_{n+2}}\right|=1\Rightarrow$ Radius of convergence $=1$.

2. (c)-(d). $y''-2xy'+2\lambda y$.

$y=\sum_{n=0}^{\infty} C_n x^n$, $y'=\sum_{n=0}^{\infty} C_n \cdot n \cdot x^{n-1}$, $y''=\sum_{n=0}^{\infty} C_n \cdot n(n-1)\cdot x^{n-2}$.

$$y''-2x\cdot y'+2\lambda y=\sum_{n=0}^{\infty} C_n\cdot n(n-1)x^{n-2}-2x\cdot\sum_{n=0}^{\infty} C_n\cdot n\cdot x^{n-1}+2\lambda\cdot\sum_{n=0}^{\infty} C_n\cdot x^n$$

$$=\sum_{n=0}^{\infty} C_{n+2}(n+2)(n+1)x^n+\sum_{n=0}^{\infty} C_n\cdot x^n\cdot(-2n+2\lambda)$$

$$\Rightarrow \quad C_{n+2}(n+2)(n+1)-2C_n(n-\lambda)=0 \Rightarrow \boxed{C_{n+2}=\frac{2(n-\lambda)}{(n+2)(n+1)}C_n}.$$

(c) $\lambda=3$ $\quad \boxed{C_{n+2}=\frac{2(n-3)}{(n+2)(n+1)}C_n}$ reccurence relation.

odd terms: $C_3=\frac{2\times(1-3)}{3\times 2}C_1=-\frac{2}{3}C_1$, $C_5=\frac{2\times(3-3)}{(3+2)\times(3+1)}C_3=0$.

$\Rightarrow C_7=C_9=C_{11}=\cdots=0$

$\Rightarrow \boxed{y_1(x)=x-\frac{2}{3}x^3. \leftarrow \text{satisfies } y_1(0)=0,\ y_1'(0)=1}$

even terms: $C_2=\frac{2\times(-3)}{2\times 1}C_0=-3C_0$, $C_4=\frac{2\times(-1)}{4\times 3}C_2=-\frac{C_2}{6}=\frac{1}{2}C_0$

$C_6=\frac{2\times 1}{6\times 5}C_4=\frac{1}{15}C_4=\frac{1}{30}C_0$, $C_8=\frac{2\times 3}{8\times 7}C_6=\frac{3}{28}C_6=\frac{1}{280}C_0$.

$\Rightarrow y_2(x)=1-3x^2+\frac{1}{2}x^4+\frac{1}{30}x^6+\frac{1}{280}x^8+\cdots$

$$\Rightarrow \boxed{y(x)=C_1\cdot\left(x-\frac{2}{3}x^3\right)+C_0\cdot\left(1-3x^2+\frac{1}{2}x^4+\frac{1}{30}x^6+\frac{1}{280}x^8+\cdots\right)}.$$

$\lim_{n\to\infty}\left|\frac{C_n}{C_{n+2}}\right|=\lim_{n\to\infty}\left|\frac{(n+2)(n+1)}{2(n-3)}\right|=\infty$ = radius of convergence.

(no singular points).

(d) $\lambda = 4$. $\quad C_{n+2} = \dfrac{2(n-4)}{(n+2)(n+1)} C_n$.

odd terms:

$$C_3 = \frac{2\times(1-4)}{(1+2)\times(1+1)} C_1 = \frac{-6}{6} C_1 = -C_1.$$

$$C_5 = \frac{2\times(3-4)}{5\times 4} C_3 = -\frac{1}{10} C_3 = \frac{1}{10} C_1.$$

$$C_7 = \frac{2\times(5-4)}{7\times 6} C_5 = \frac{1}{21} C_5 = \frac{1}{210} C_1$$

$$\Rightarrow y_1(x) = x - x^3 + \frac{1}{10}x^5 + \frac{1}{210}x^7 + \cdots$$

satisfies $y_1(0) = 0$, $y_1'(0) = 1$, $y_1'' - 2xy_1' + 8y_1 = 0$.

even terms:

$$C_2 = \frac{2\times(0-4)}{2\times 1} C_0 = -4C_0.$$

$$C_4 = \frac{2\times(2-4)}{4\times 3} C_2 = -\frac{1}{3} C_2 = \frac{4}{3} C_0.$$

$$C_6 = \frac{2\times(4-4)}{6\times 5} C_4 = 0 \Rightarrow C_8 = C_{10} = C_{12} = \cdots = 0.$$

$$\Rightarrow y_2(x) = 1 - 4x^2 + \frac{4}{3}x^4 \quad (\text{satisfies } y_2(0) = 1, y_2'(0) = 0).$$

$\Rightarrow$ general solution

$$y(x) = C_0 \cdot \left(1 - 4x^2 + \frac{4}{3}x^4\right) + C_1 \cdot \left(x - x^3 + \frac{1}{10}x^5 + \frac{1}{210}x^7 + \cdots\right).$$

$$\lim_{n\to\infty} \left|\frac{C_n}{C_{n+2}}\right| = \lim_{n\to\infty} \left|\frac{(n+2)(n+1)}{2(n-4)}\right| = \infty = \text{radius of convergence}$$

(no singular points).

2(e). $y'' = xy$ $\quad y = \sum_{n=0}^{\infty} C_n \cdot x^n$, $\quad y' = \sum_{n=0}^{\infty} C_n \cdot n \cdot x^{n-1}$

$y'' = \sum_{n=0}^{\infty} C_n \cdot n(n-1) \cdot x^{n-2}$

$$y'' - xy = \sum_{n=0}^{\infty} C_n \cdot n(n-1)\, x^{n-2} - x \cdot \sum_{n=0}^{\infty} C_n \cdot x^n$$

$$= \sum_{n=0}^{\infty} C_{n+2}(n+2)(n+1) \cdot x^n - \sum_{n=0}^{\infty} C_n \cdot x^{n+1}$$

$$= \sum_{n=0}^{\infty} C_{n+2}(n+2)(n+1)\, x^n - \sum_{n=1}^{\infty} C_{n-1} \cdot x^n.$$

$$\Rightarrow = C_2 \cdot 2 \cdot 1 \cdot x^0 + \sum_{n=1}^{\infty} \left(C_{n+2}(n+2)(n+1) - C_{n-1} \right) x^n \cdot = 0$$

$$\Rightarrow C_2 = 0. \qquad C_{n+2} = \frac{C_{n-1}}{(n+2)(n+1)}$$

- $C_0 = 1 \Rightarrow C_3 = \frac{C_0}{3\times 2} = \frac{C_0}{6} \Rightarrow C_6 = \frac{C_3}{6\times 5} = \frac{1}{30} \times \frac{1}{6} C_0 = \frac{1}{180} C_0$

$$\Rightarrow C_9 = \frac{C_6}{9\times 8} = \frac{1}{72} \times \frac{1}{180} \times C_0 = \frac{1}{12960} C_0$$

$$\Rightarrow \boxed{y_1(x) = 1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \frac{1}{12960}x^9 + \cdots}$$

- $C_1 = 1 \Rightarrow C_4 = \frac{C_1}{4\times 3} = \frac{1}{12} C_1 \Rightarrow C_7 = \frac{C_4}{7\times 6} = \frac{1}{42} \times \frac{1}{12} C_1 = \frac{1}{504} C_1.$

$$\Rightarrow C_{10} = \frac{C_7}{10\times 9} = \frac{1}{90} \times \frac{1}{504} C_1 = \frac{1}{45360} C_1$$

$$\boxed{\Rightarrow y_2(x) = x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \frac{1}{45360}x^{10} + \cdots}$$

satisfies $y_2(0) = 0$, $y_2'(0) = 1$

general solution $y(x) = C_0\left(1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \frac{1}{12960}x^9 + \cdots\right)$

$$+ C_1 \cdot \left(x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \frac{1}{45360}x^{10} + \cdots\right).$$

$\lim_{n\to\infty} \left| \frac{C_{n-1}}{C_{n+2}} \right| = \lim_{n\to\infty} (n+2)(n+1) = \infty =$ radius of convergence (no singular points)

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$. Characteristic polynomial:

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 9 = (\lambda-3)^2.$$

$\Rightarrow \lambda = 3$, mult. $= 2$. $\Rightarrow$ we are in the situation

EValue	mult.	#EVector	chains
3	2	1	$\vec{v}_1 \to \vec{v}_2 \to 0$.

$\vec{v}_1$ satisfies $(A-3I)^2\vec{v}_1 = 0$.

$$(A-3I)^2 = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\cdot\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

so $\vec{v}_1$ can be chosen to be any vector as long as $\vec{v}_2 = (A-3I)\vec{v}_1 \neq 0$.

try $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_2 = (A-3I)\vec{v}_1 = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \neq 0.$

so we get the chain: $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} -1 \\ -1 \end{pmatrix} \to 0.$

basic solutions: $\vec{x}_1(t) = e^{3t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ -1 \end{pmatrix}\right)$, $\vec{x}_2(t) = e^{3t}\begin{pmatrix} -1 \\ -1 \end{pmatrix}$

General solution: $\vec{x}(t) = C_1 e^{3t}\begin{pmatrix} 1-t \\ -t \end{pmatrix} + C_2 e^{3t}\begin{pmatrix} -1 \\ -1 \end{pmatrix}$

initial condition:

$$\vec{x}(0) = C_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} C_1 - C_2 = 0 \\ -C_2 = 1 \end{cases}$$

$$\Downarrow$$

$$C_1 = C_2 = -1.$$

$$\Rightarrow \vec{x}(t) = -e^{3t}\begin{pmatrix} 1-t \\ -t \end{pmatrix} - e^{3t}\begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

$$= \boxed{\begin{pmatrix} t\cdot e^{3t} \\ (1+t)e^{3t} \end{pmatrix}}$$

2 (a) $\frac{dy}{dx} = \frac{1+\sqrt{y}}{1+\sqrt{x}} \Rightarrow \frac{dy}{1+\sqrt{y}} = \frac{dx}{1+\sqrt{x}}$

$$\int \frac{dy}{1+\sqrt{y}} \overset{u=\sqrt{y}}{=\!=} \int \frac{2u\,du}{1+u} = \int \frac{(2u+2)-2}{1+u}du = \int\left(2-\frac{2}{1+u}\right)du$$

$y=u^2$, $dy=2u\,du$

$$= 2u - 2\ln(1+u) = 2\sqrt{y} - 2\ln(1+\sqrt{y}).$$

So we get $2\sqrt{y}-2\ln(1+\sqrt{y}) = 2\sqrt{x} - 2\ln(1+\sqrt{x}) + C.$

$$\Rightarrow \sqrt{y} - \ln(1+\sqrt{y}) - \sqrt{x} + \ln(1+\sqrt{x}) = C.$$

(b) $(x^2+\ln y)dx + (y^3+\frac{x}{y})dy = 0.$ (*)

(*).

$\frac{\partial}{\partial y}(x^2+\ln y) = \frac{1}{y} = \frac{\partial}{\partial x}(y^3+\frac{x}{y}) \Rightarrow$ exact equation $\Rightarrow$ find the potential function

$$\begin{cases} \frac{\partial F}{\partial x} = x^2+\ln y & ① \\ \frac{\partial F}{\partial y} = y^3 + \frac{x}{y} & ② \end{cases}$$

$\Rightarrow F(x,y) = \frac{1}{3}x^3 + x\cdot\ln y$ $(=\int(x^2+\ln y)dx)$ $+g(y).$

$\Downarrow$

$\frac{\partial F}{\partial y} = \frac{x}{y} + g'(y) \overset{②}{=} y^3+\frac{x}{y} \Rightarrow g'(y) = y^3.$

$\Rightarrow g(y) = \frac{1}{4}y^4$. So $F(x,y) = \frac{1}{3}x^3 + x\cdot\ln y + \frac{1}{4}y^4.$

implicit solution : $\frac{1}{3}x^3 + x\cdot\ln y + \frac{1}{4}y^4 = C.$

(*). (c). $(1-x^2)y' + xy = 1 \Longrightarrow$ standard form $y' + \frac{x}{1-x^2}y = \frac{1}{1-x^2}.$

integrating factor $F(x) = e^{\int\frac{x}{1-x^2}dx} = e^{-\frac{1}{2}\int\frac{d(1-x^2)}{1-x^2}} = e^{-\frac{1}{2}\ln(1-x^2)} = (1-x^2)^{-\frac{1}{2}}$

$$\Rightarrow \left((1-x^2)^{-\frac{1}{2}}y\right)' = \frac{1}{\left(\sqrt{1-x^2}\right)^3} = \frac{1}{(1-x^2)^{\frac{3}{2}}}$$

$$\Rightarrow (1-x^2)^{-\frac{1}{2}}y = \int\frac{dx}{(1-x^2)^{\frac{3}{2}}} \overset{x=\sin\theta}{\Longrightarrow} \int\frac{\cos\theta\cdot d\theta}{(1-\sin^2\theta)^{\frac{3}{2}}} = \int\frac{\cos\theta\, d\theta}{\cos^3\theta} \rightarrow$$

$$= \int \sec^2\theta \, d\theta = \tan\theta + C = \frac{\sin\theta}{\cos\theta} + C = \frac{x}{\sqrt{1-x^2}} + C.$$

So. $(1-x^2)^{-\frac{1}{2}} y = x \cdot (1-x^2)^{-\frac{1}{2}} + C$

$$\Rightarrow \boxed{y = x + C\sqrt{1-x^2}.}$$

(d) $y\frac{dy}{dx} - y = \sqrt{x^2+y^2} \Rightarrow \frac{dy}{dx} = \frac{\sqrt{x^2+y^2}+y}{y} = \sqrt{\left(\frac{x}{y}\right)^2+1} + 1.$

Substitute $u = \frac{x}{y} \Rightarrow y = \frac{x}{u} \Rightarrow y' = \frac{1}{u} - \frac{x}{u^2} \cdot u'$

$$\Rightarrow \frac{1}{u} - \frac{x}{u^2} u' = \sqrt{1+u^2} + 1 \Rightarrow -\frac{x}{u^2}\frac{du}{dx} = \sqrt{1+u^2} + 1 - \frac{1}{u}$$

$$\Rightarrow \frac{du}{u^2\left(\sqrt{1+u^2}+1-\frac{1}{u}\right)} = -\frac{dx}{x}$$

$$\mathbin{/\!/}$$

$$\frac{du}{u^2\sqrt{1+u^2}+u^2-u}.$$

$$\Rightarrow \int \frac{du}{u^2\sqrt{1+u^2}+u^2-u} = -\ln x + C.$$

$\uparrow$

hard to integrate

(d) Bernoulli $(1+x)\,y' + y = y^3 \Rightarrow (1+x)\,y^{-3}y' + y^{-2} = 1$

Substitute $u = y^{-2} \Rightarrow u' = -2y^{-3}\cdot y' \Rightarrow y^{-3}y' = -\frac{u'}{2}$

$\Rightarrow (1+x)\cdot\frac{u'}{-2} + u = 1 \Longrightarrow$ Standard form $u' - \frac{2}{1+x}u = -\frac{2}{1+x}$.

Integrating factor $F(x) = e^{-\int\frac{2}{1+x}dx} = e^{-2\ln(1+x)} = \frac{1}{(1+x)^2}$

$$\Rightarrow \left(\frac{u}{(1+x)^2}\right)' = -\frac{2}{(1+x)^3} \Longrightarrow \frac{u(x)}{(1+x)^2} = \underbrace{-2\cdot\frac{1}{-3+1}\cdot(1+x)^{-3+1} + C}_{=\,(1+x)^{-2}+C}$$

$$\Rightarrow \underbrace{u(x)}_{=\frac{1}{y(x)^2}} = 1 + C\cdot(1+x)^2 \Rightarrow \boxed{y(x) = \pm\frac{1}{\sqrt{1+C(1+x)^2}}}$$

---

3 (a). $y''' - 2y'' + y = 0$. (*)

(*) Characteristic polynomial: $\lambda^3 - 2\lambda + 1 = 0$

$= \lambda^3 - \lambda - \lambda + 1 = \lambda(\lambda^2-1) - (\lambda-1)$

$= \lambda(\lambda+1)(\lambda-1) - (\lambda-1) = (\lambda-1)\cdot(\lambda^2+\lambda-1)$.

roots: $\lambda_1 = 1$.

$\lambda_2 = \frac{-1+\sqrt{1-4(-1)}}{2} = \frac{-1+\sqrt{5}}{2}$

$\lambda_3 = \frac{-1-\sqrt{5}}{2}$

$\Rightarrow y_1(x) = e^x$, $y_2(x) = e^{\left(\frac{-1+\sqrt{5}}{2}\right)x}$, $y_3(x) = e^{\left(\frac{-1-\sqrt{5}}{2}\right)x}$.

General solution:

$y(x) = C_1\cdot e^x + C_2\cdot e^{\left(\frac{-1+\sqrt{5}}{2}\right)x} + C_3\cdot e^{\left(\frac{-1-\sqrt{5}}{2}\right)x}$.

3(b) i. $y''+4y'+4y=e^{-2x}$. (*) method

associated ~~characteristic polynomial~~ $y''+4y'+4y=0$.

(homogeneous equation:

Characteristic polynomial: $\lambda^2+4\lambda+4=0 \Rightarrow \lambda=-2$, mult.$=2$.

$\Rightarrow y_1(x)=e^{-2x},\ y_2(x)=x\cdot e^{-2x} \Rightarrow y_c(x)=C_1 e^{-2x}+C_2\cdot xe^{-2x}$

complementary solution

Undetermined coefficient:

$f(x)=e^{-2x}=A_m(x)e^{\mu x}$ $\quad A_m(x)=1\cdot(m=0)$, $\mu=-2$, mult.$=2=k$.

so $y_p(x)=x^2\cdot A\cdot e^{-2x} \Rightarrow y_p'=A\cdot(2x\cdot e^{-2x}-2x^2e^{-2x})$

$=2A\,(x-x^2)\,e^{-2x}$

$\Rightarrow y_p''(x)=2A\left[(1-2x)e^{-2x}+(x-x^2)e^{-2x}\cdot(-2)\right]=2A\cdot e^{-2x}\cdot\left[(1-2x)-2x+2x^2\right]$

$\parallel$

$2A\cdot e^{-2x}\cdot[2x^2-4x+1]$.

$\Rightarrow y_p''+4y_p'+4y_p=A\cdot e^{-2x}\cdot\left[4x^2-8x+2+(2x-2x^2)4+4x^2\right]$

$\parallel$

$e^{-2x} \qquad =2\cdot A\,e^{-2x} \Rightarrow A=\frac{1}{2}$.

$\Rightarrow y_p(x)=\frac{1}{2}x^2e^{-2x}$

$\Rightarrow y(x)=y_c(x)+y_p(x)=C_1e^{-2x}+C_2\cdot xe^{-2x}+\frac{1}{2}x^2e^{-2x}$.

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3(b) ii $\quad y'' + y = \frac{1}{\sin^2 x}$

associated homogeneous DE: $y'' + y = 0. \Rightarrow \lambda^2 + 1 = 0$

$\Rightarrow \lambda_1 = i, \lambda_2 = -i$

$\Rightarrow \quad y_1(x) = \cos x, \; y_2(x) = \sin x.$

Variation of parameters:

$$y_p(x) = -y_1(x) \int \frac{y_2(x) f(x)}{W(x)} dx + y_2(x) \int \frac{y_1(x) f(x)}{W(x)} dx.$$

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x - \sin x(-\sin x) = 1.$$

so. $$y_p(x) = -\cos x \int \frac{\sin x \cdot \frac{1}{\sin^2 x}}{1} dx + \sin x \int \frac{\cos x \cdot \frac{1}{\sin^2 x}}{1} dx$$

$$= -\cos x \int \frac{-d(\cos x)}{1 - \cos^2 x} + \sin x \int \frac{d(\sin x)}{\sin^2 x}$$

$$= \cos x \int \frac{du}{1-u^2} + \sin x \cdot \left(-\frac{1}{\sin x}\right)$$

$(u = \cos x)$.

$\frac{1}{1-u^2} = \left(\frac{1}{1+u} + \frac{1}{1-u}\right)\frac{1}{2}$

$$= \cos x \cdot \frac{1}{2} \ln \frac{1+\cos x}{1-\cos x} - 1.$$

$\int \frac{du}{1-u^2} = \frac{1}{2} \ln \frac{1+u}{1-u}.$

So $$\boxed{\begin{aligned} y(x) &= y_c(x) + y_p(x) \\ &= C_1 \cdot \cos x + C_2 \cdot \sin x + \frac{\cos x}{2} \ln \frac{1+\cos x}{1-\cos x} - 1 \end{aligned}}$$

$$= C_1 \cdot \cos x + C_2 \sin x + \cos x \cdot \ln \frac{\cos \frac{x}{2}}{\sin \frac{x}{2}} - 1.$$

4 (b). (*)
$$\begin{cases} x_1' = 4x_1 + 2x_2 \\ x_2' = -3x_1 - x_2 \\ x_3' = x_1 + x_2 + 2x_3 \end{cases} \Leftrightarrow \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 4 & 2 & 0 \\ -3 & -1 & 0 \\ 1 & 1 & 2 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Characteristic polynomial:

$$\det(A-\lambda I) = \begin{vmatrix} 4-\lambda & 2 & 0 \\ -3 & -1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = (2-\lambda)\cdot\begin{vmatrix} 4-\lambda & 2 \\ -3 & -1-\lambda \end{vmatrix} = \underbrace{(2-\lambda)\cdot[\lambda^2-3\lambda+2)}_{-(\lambda-2)^2(\lambda-1).}$$

$\Rightarrow \lambda_1 = 1$, mult. $=1$; $\lambda_2 = 2$, mult. $=2$.

Eigenvector: $\lambda_1 = 1$:
$$\begin{pmatrix} 3 & 2 & 0 \\ -3 & -2 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = 0 \Leftrightarrow \begin{cases} 3a_1 + 2b_1 = 0 \\ \cancel{-3a_1 - 2b_1 = 0} \\ a_1 + b_1 + c_1 = 0 \end{cases} \xRightarrow{a_1 = 2} \underbrace{\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}}_{\vec{v}_1}.$$

$\lambda_2 = 2$:
$$\begin{pmatrix} 2 & 2 & 0 \\ -3 & -3 & 0 \\ 1 & 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = 0 \Leftrightarrow \begin{cases} \cancel{2a_2 + 2b_2 = 0} \\ \cancel{-3a_2 - 3b_2 = 0} \\ a_2 + b_2 = 0 \end{cases} \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \ \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

So. basic solutions:

$$\vec{x}_1(t) = e^{\lambda_1 t}\vec{v}_1 = e^t\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}. \quad \vec{x}_2(t) = e^{\lambda_2 t}\vec{v}_2 = e^{2t}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$
$$\vec{x}_3(t) = e^{\lambda_2 t}\vec{v}_3 = e^{2t}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

General Solution:

$$\vec{x}(t) = C_1\cdot e^t\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + C_3\cdot e^{2t}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Note that we are in the situation:

EValue	mult.	# EVector	Chains
1	1	1	$v_1 \to 0$
2	2	2	$v_2 \to 0$ $v_3 \to 0$.

4.(c). (*) $\begin{cases} x_1' = 7x_1 + x_2 \\ x_2' = -4x_1 + 3x_2 \end{cases} \Leftrightarrow \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$

Characteristic polynomial : $\det(A-\lambda I) = \begin{vmatrix} 7-\lambda & 1 \\ -4 & 3-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 25 = (\lambda-5)^2.$

We need to find a chain of length 2: $\vec{v}_1 \to \vec{v}_2 \to 0$

$\vec{v}_1$ satisfies $(A-5I)^2\vec{v}_1 = 0$ $\quad (A-5I)^2 = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\cdot\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$

Choose $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_2 = (A-5I)\vec{v}_1 = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} \neq 0.$

So. $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 2 \\ -4 \end{pmatrix} \to 0.$

$\Rightarrow \vec{x}_1(t) = \cancel{e^{5t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$, $\vec{x}_2(t) = e^{5t}\cdot\begin{pmatrix} 2 \\ -4 \end{pmatrix}$ are 2 basic solutions.

$e^{5t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ -4 \end{pmatrix}\right).$

General solution:

$$\vec{x}(t) = C_1\cdot e^{5t}\begin{pmatrix} 1+2t \\ -4t \end{pmatrix} + C_2 e^{5t}\begin{pmatrix} 2 \\ -4 \end{pmatrix}.$$

4(d). (*) $\begin{cases} x_1' = x_1 - 4x_2 \\ x_2' = 4x_1 + 9x_2 \end{cases} \iff \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 4 & 9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

$A = \begin{pmatrix} 1 & -4 \\ 4 & 9 \end{pmatrix}$ Characteristic polynomial

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -4 \\ 4 & 9-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 25 = (\lambda - 5)^2.$$

$\Rightarrow \lambda = 5$, mult. $= 2$. $\Rightarrow$ chain of length 2. $\vec{v}_1 \to \vec{v}_2 \to 0$

$\vec{v}_1$ satisfies $(A - 5I)^2 \vec{v}_1 = 0$. $(A-5I)^2 = \begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix} \cdot \begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow$ can choose $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_2 = \begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \end{pmatrix}$

so we get the chain $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} -4 \\ 4 \end{pmatrix} \to 0$.

$\Rightarrow$ basic solutions:

$$\vec{x}_1(t) = e^{5t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -4 \\ 4 \end{pmatrix}\right), \quad \vec{x}_2(t) = e^{5t}\begin{pmatrix} -4 \\ 4 \end{pmatrix}.$$

General solution:

$$\vec{x}(t) = C_1 e^{5t}\left(\begin{pmatrix} 1-4t \\ 4t \end{pmatrix}\right) + C_2 e^{5t}\begin{pmatrix} -4 \\ 4 \end{pmatrix}.$$

4 (9). $(*)$ $\begin{cases} x_1' = 2x_1 + 3x_2 + e^t \\ x_2' = 2x_1 + x_2 + e^{2t} \end{cases} \Longleftrightarrow \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} e^t \\ e^{2t} \end{pmatrix}$.

associated homogeneous system: $\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

characteristic polynomial: $\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 4 = (\lambda-4)(\lambda+1)$.

$\Rightarrow \lambda_1 = 4,\ \lambda_2 = -1$. Find the eigenvectors:

$\lambda_1 = 4$. $\begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = 0 \Longleftrightarrow \begin{cases} -2a_1 + 3b_1 = 0 \\ \cancel{2a_1 - 3b_1 = 0} \end{cases} \xRightarrow{a_1 = 3} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \vec{v}_1$

$\lambda_2 = -1$. $\begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = 0 \Longleftrightarrow \begin{cases} 3a_2 + 3b_2 = 0 \\ \cancel{2a_2 + 2b_2 = 0} \end{cases} \xRightarrow{a_2 = +1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \vec{v}_2$.

basic solution $\vec{x}_1(t) = e^{4t}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\vec{x}_2(t) = e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Fundamental matrix $\Phi(t) = (\vec{x}_1(t), \vec{x}_2(t)) = \begin{pmatrix} 3e^{4t} & e^{-t} \\ 2e^{4t} & -e^{-t} \end{pmatrix} \Rightarrow |\Phi(t)| = e^{3t}$.

$\Rightarrow \Phi(t)^{-1} = \frac{1}{-5\cdot e^{3t}}\begin{pmatrix} -e^{-t} & -e^{-t} \\ -2e^{4t} & 3e^{4t} \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -e^{-4t} & -e^{-4t} \\ -2e^{t} & 3e^{t} \end{pmatrix}$. $\quad \begin{matrix} (3\times(-1) - 1\times 2) \\ \parallel \\ -5\cdot e^{3t} \end{matrix}$.

$\Rightarrow \Phi(t)^{-1}\vec{f}(t) = +\frac{1}{5}\begin{pmatrix} e^{-4t} & e^{-4t} \\ 2e^{t} & -3e^{t} \end{pmatrix}\begin{pmatrix} e^t \\ e^{2t} \end{pmatrix} = \frac{1}{5}\begin{pmatrix} e^{-3t} + e^{-2t} \\ 2e^{2t} - 3e^{3t} \end{pmatrix}$.

$\Rightarrow \int \Phi(t)^{-1}\vec{f}(t)\,dt = \frac{1}{5}\begin{pmatrix} -\frac{1}{3}e^{-3t} - \frac{1}{2}e^{-2t} \\ e^{2t} - e^{3t} \end{pmatrix}$

$\Rightarrow \vec{x}_p(t) = \Phi(t)\int \Phi(t)^{-1}\vec{f}(t) = \begin{pmatrix} 3e^{4t} & e^{-t} \\ 2e^{4t} & -e^{-t} \end{pmatrix}\cdot\frac{1}{5}\begin{pmatrix} -\frac{1}{3}e^{-3t} - \frac{1}{2}e^{-2t} \\ e^{2t} - e^{3t} \end{pmatrix}$.

$= \frac{1}{5}\begin{pmatrix} \cancel{-e^{t}} - \frac{3}{2}e^{2t} \cancel{+e^{t}} - e^{2t} \\ -\frac{2}{3}e^{t} \cancel{-e^{2t}} - e^{t} \cancel{+e^{2t}} \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -\frac{5}{2}e^{2t} \\ -\frac{5}{3}e^{t} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}e^{2t} \\ -\frac{1}{3}e^{t} \end{pmatrix}$

5(i) $(2x+1)y' = y$.

$$y = \sum_{n=0}^{\infty} C_n \cdot x^n \Rightarrow y' = \sum_{n=0}^{\infty} C_n \cdot n \cdot x^{n-1}$$

$$0 = (2x+1)y' - y = (2x+1)\sum_{n=0}^{\infty} C_n \cdot n \cdot x^{n-1} - \sum_{n=0}^{\infty} C_n \cdot x^n$$

$$= 2\sum_{n=0}^{\infty} C_n \cdot n \cdot x^n + \sum_{n=0}^{\infty} C_{n+1}(n+1)\cdot x^n - \sum_{n=0}^{\infty} C_n \cdot x^n$$

$$= \sum_{n=0}^{\infty} C_n \cdot (2n-1)\cdot x^n + \sum_{n=0}^{\infty} C_{n+1}\cdot(n+1).$$

$\Rightarrow$ Recurrence relation: $C_{n+1} = -\frac{2n-1}{n+1} C_n = \frac{1-2n}{n+1} C_n = 2 \cdot \frac{\frac{1}{2}-n}{n+1} C_n$

Radius of convergence: $R = \lim_{n\to\infty} \left|\frac{C_n}{C_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{n+1}{2n-1}\right| = \frac{1}{2}$.

$$C_n = 2 \cdot \frac{\frac{1}{2}-(n-1)}{n} C_{n-1} = 2^2 \cdot \frac{\left(\frac{1}{2}-(n-1)\right)\left(\frac{1}{2}-(n-2)\right)}{n\cdot(n-1)} C_{n-2} = 2^3 \cdot \frac{\left(\frac{1}{2}-(n-1)\right)\left(\frac{1}{2}-(n-2)\right)\left(\frac{1}{2}-(n-3)\right)}{n(n-1)\cdot(n-2)} C_{n-3}$$

$$= \cdots 2^n \cdot \frac{\left(\frac{1}{2}-(n-1)\right)\left(\frac{1}{2}-(n-2)\right)\cdots\left(\frac{1}{2}-0\right)}{n\cdot(n-1)\cdots 1} \cdot C_0 = 2^n \binom{\frac{1}{2}}{n} \cdot C_0$$

$$\Rightarrow \quad y(x) = \sum_{n=0}^{\infty} C_0 \cdot \binom{\frac{1}{2}}{n} \cdot 2^n \cdot x^n = C_0 \cdot \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \cdot (2x)^n = C_0 (1+2x)^{\frac{1}{2}}.$$

Try a simpler exercise: $\boxed{(2x+1)y' + 2y(x) = 0. \quad (*)}$

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(X). 5.(ii) $y'' - 2xy' + 6y = 0$

$$y = \sum_{n=0}^{\infty} C_n \cdot x^n \Rightarrow y' = \sum_{n=0}^{\infty} C_n \cdot n \cdot x^{n-1}, \quad y'' = \sum_{n=0}^{\infty} C_n \cdot n(n-1) x^{n-2}.$$

$$0 = y'' - 2xy' + 6y = \sum_{n=0}^{\infty} C_n \cdot n(n-1) x^{n-2} - 2x \sum_{n=0}^{\infty} C_n \cdot n \cdot x^{n-1} + 6 \sum_{n=0}^{\infty} C_n \cdot x^n.$$

$$= \sum_{n=0}^{\infty} C_{n+2}(n+2)(n+1) \cdot x^n - \sum_{n=0}^{\infty} C_n \cdot (2n-6) \cdot x^n.$$

$$\Rightarrow \boxed{C_{n+2} = \frac{2(n-3)}{(n+1)(n+2)} C_n \quad \underline{\text{Recurrence relation}}}$$

$$\lim_{n\to\infty} \left| \frac{C_n}{C_{n+2}} \right| = \infty$$

Radius of Convergence.

even terms: $\underline{y(0) = 1, \ y'(0) = 0} \Leftrightarrow C_0 = 1, \ C_1 = 0.$

$$\Rightarrow C_2 = \frac{2\times(-3)}{2\times 1} C_0 = -3, \quad C_4 = \frac{2\times(2-3)}{4\times 3} \cdot C_2 = -\frac{1}{6}\times(-3) = \frac{1}{2}.$$

$$C_6 = \frac{2\times(4-3)}{6\times 5} C_4 = \frac{1}{15}\times\frac{1}{2} = \frac{1}{30}, \quad C_8 = \frac{2\times(6-3)}{8\times 7} C_6 = \frac{3}{28}\times\frac{1}{30} = \frac{1}{280}$$

$$\Rightarrow y_1(x) = 1 - 3x^2 + \frac{1}{2}x^4 + \frac{1}{30}x^6 + \frac{1}{280}x^8 + \cdots$$

odd terms: $y(0) = 0, \ y'(0) = 1 \Leftrightarrow C_0 = 0, \ C_1 = 1$

$$\Rightarrow C_3 = \frac{2\times(1-3)}{3\times 2} C_1 = -\frac{2}{3}, \quad C_5 = \frac{2\times(3-3)}{5\times 3} C_3 = 0 \Rightarrow C_7 = C_9 = \cdots = 0.$$

$$\Rightarrow y_2(x) = x - \frac{2}{3}x^3.$$

General solution:

$$y(x) = C_0 \cdot y_1(x) + C_1 \cdot y_2(x) = C_0\left(1 - 3x^2 + \frac{1}{2}x^4 + \frac{1}{30}x^6 + \frac{1}{280}x^8 + \cdots\right) + C_1\left(x - \frac{2}{3}x^3\right).$$

(*) 3(a) $y''' - 2y'' + y = 0$ (*).

characteristic polynomial: $\lambda^3 - 2\lambda^2 + 1 = 0$

$$\lambda^3 - 2\lambda^2 + 1 = \lambda^3 - \lambda^2 - (\lambda^2 - 1) = \lambda^2(\lambda - 1) - (\lambda + 1)(\lambda - 1) = (\lambda - 1)\cdot(\lambda^2 - \lambda - 1).$$

roots: $\lambda_1 = 1.$

$$\lambda_2 = \frac{1 + \sqrt{(-1)^2 - 4\times(-1)}}{2} = \frac{1 + \sqrt{5}}{2}$$

$$\lambda_3 = \frac{1 - \sqrt{5}}{2}$$

$\Rightarrow y_1(x) = e^x,\ y_2(x) = e^{\left(\frac{1+\sqrt{5}}{2}\right)x},\ y_3(x) = e^{\left(\frac{1-\sqrt{5}}{2}\right)x}.$

general solution:

$$y(x) = C_1 \cdot e^x + C_2 \cdot e^{\frac{1+\sqrt{5}}{2}x} + C_3 \cdot e^{\frac{1-\sqrt{5}}{2}\cdot x}$$

!!! WRITE YOUR N

NAME : Yi Ren

1. (50pts) Solve the following 1st order differential equation

$$(1+x^2)\frac{dy}{dx} + 4xy = 2, \quad y(0) = 0.$$

S-L $\quad y' + \frac{4x}{1+x^2} y = \frac{2}{1+x^2}$

$F(x) = e^{\int \frac{4x}{1+x^2} dx} = e^{2\ln(1+x^2)} = (1+x^2)^2$

$y = F(x)^{-1} \int F(x) Q(x)$

$= (1+x^2)^{-2} \int (1+x^2)^2 \cdot \frac{2}{1+x^2}$

$= (1+x^2)^{-2} (2x + \frac{2}{3}x^3 + C)$

$= \frac{2x + \frac{2}{3}x^3 + C}{(1+x^2)^2}$

$y(0) = 0 \Rightarrow C = 0$

$y = \frac{2x + \frac{2}{3}x^3}{(1+x^2)^2}$

50

2. (50pts) Solve the following 1st order differential equation:

$$(x+2y)dx + (2x-y)dy = 0, \quad y(0) = 1.$$

sol

$$\frac{\partial(x+2y)}{\partial y} = \frac{\partial(2x-y)}{\partial x} = 2.$$

So It is exact.

$$\begin{cases} \frac{F(x,y)}{\partial x} = x+2y \\ \frac{F(x,y)}{\partial y} = 2x-y \end{cases}$$

$\Rightarrow$ $F(x,y) = \frac{1}{2}x^2 + 2yx - \frac{1}{2}y^2 = C.$

$y(0) = 1$

$\Rightarrow$. $-\frac{1}{2} = C$.

$\Rightarrow \frac{1}{2}x^2 + 2yx - \frac{1}{2}y^2 = -\frac{1}{2}$

50

3. (50pts) A free moving body enters a resisting medium with resistance proportional to $v^{1/2}$ where v denotes the velocity function. Assume the drag coefficient is 1. Then v satisfies the following differential equation.

$$\frac{dv}{dt} = -v^{1/2}.$$

(a): Suppose the initial velocity is $v(0) = 16$, find the velocity as a function of t.

$$\frac{dV}{dt} = -V^{\frac{1}{2}} \Rightarrow \frac{dV}{V^{\frac{1}{2}}} = -1dt$$

$$\Rightarrow \int V^{-\frac{1}{2}}\,dV = \int -1dt \Rightarrow 2V^{\frac{1}{2}} = -t + C. \quad (V \geq 0)$$

$$\Rightarrow V = \frac{(-t+C)^2}{4} \quad (V \geq 0)$$

$$V(0) = 16 \Rightarrow \frac{C^2}{4} = 16 \Rightarrow C = 8$$

$$V = \cancel{\frac{(-t+8)^2}{4}} \; \frac{(8-t)^2}{4}$$

Because when $V=0$, $\frac{dV}{dt} = 0$, the body ~~is still~~ stop travelling.

~~So~~ ~~$t \leq 8$~~.

so $t \leq 8 \quad V = \frac{(8-t)^2}{4}$

$t > 8 \quad V = 0.$

(b): Does the body travel a finite or infinite distance? If it's a finite distance, how far does the body travel?

$$\frac{dx}{dt} = \frac{(8-t)^2}{4}$$

$$\int dx = \int \frac{(8-t)^2}{4}\, dt$$

$$\Rightarrow x = \cancel{16x}\ 16t - 2t^2 + \frac{t^3}{12} + C$$

suppose $x(0) = 0$. $\Rightarrow$ $C = 0$.

$$x(8) = \frac{128}{3}.$$

The body travel a finite distance which is $\frac{128}{3}$

$\frac{64 - 16t + t^2}{4}$

$16 - 4t + \frac{t^2}{4}$

$16t - 2t^2 + \frac{t^3}{12}$

$128 - 128 +$

8×8

$64 \times 8 = 512$

$512 / 12$

$\frac{128}{3}$

4. (50pts) (a): Find the general solution to the following equation:

$$y''' + y'' - 2y = 0.$$

sol $\lambda^3+\lambda^2-2=0$

$\Rightarrow \lambda^3-1+\lambda^2-1=0 \Rightarrow (\lambda-1)(\lambda^2+\lambda+1)+(\lambda-1)(\lambda+1)=0$

$\Rightarrow (\lambda-1)(\lambda^2+2\lambda+2)=0$

$\Rightarrow \lambda_1=1 \quad \lambda_2=-1+i \quad \lambda_3=-1-i$

general solution $y=C_1e^{x}+C_2e^{-x}\cos x+C_3e^{-x}\sin x$

25

(b): Find a particular solution to the following equation using the method of undetermined coefficients:

$$y''' + y'' - 2y = 4x^2.$$

Then write down the general solution to this equation.

sol $y_c = C_1 e^{x} + C_2 e^{-x}\cos x + C_3 e^{-x}\sin x$

$y_p = x^0 e^0 (Ax^2 + Bx + C)$

$y_p' = 2Ax + B$

$y_p'' = 2A$

$y_p''' = 0$

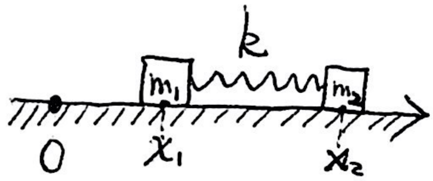
$0 + 2A - 2Ax^2 - 2Bx - 2C = 4x^2.$

$\Rightarrow \begin{cases} -2A = 4 \\ -2Bx = 0 \\ 2A - 2C = 0 \end{cases} \Rightarrow \begin{cases} A = \cancel{4}\ -2 \\ C = \cancel{2}\ -2 \end{cases}$

$y_g = C_1 e^{x} + C_2 e^{-x}\cos x + C_3 e^{-x}\sin x - 2x^2 - 2$

25

5. (50pts) (a). Assuming that there is no friction force, derive equations of motion for the following mass-spring system by drawing the free-body diagram:



$$\begin{cases} m_1x_1'' = -kx_1 + kx_2 \\ m_2x_2'' = kx_1 - kx_2. \end{cases}$$

(b). Assume $m_1 = m_2 = 1$ and $k = 1$. Use the elimination method to solve the system. Note that you should get a translating mode (linear part) and oscillating mode (trigonometric part) for the system.

sol $\begin{cases} x_1'' = -x_1 + x_2 & (1) \\ x_2'' = x_1 - x_2. & (2) \end{cases}$

(1) $\Rightarrow$ $x_2 = x_1'' + x_1$

$x_2'' = x_1^{(4)} + x_1'' = x_1 - x_2 = x_1 - (x_1'' + x_1).$

$\Rightarrow$. $x_1^{(4)} + 2x_1'' = 0.$ $\Rightarrow$ $\lambda^4 + 2\lambda^2 = 0.$

$\Rightarrow$ $\lambda^2(\lambda^2+2) = 0$ $\quad \lambda_1 = 0$ mult $= 2.$

$\lambda_2 = \sqrt{2}i.$

$\lambda_3 = -\sqrt{2}i$

$x_1 = C_1 + C_2 t + C_3 \cos\sqrt{2}t + C_4 \sin\sqrt{2}t$

$x_1' = C_2 + \sqrt{2}C_3(-\sin\sqrt{2}t) + \sqrt{2}C_4 \cos\sqrt{2}t$

$x_1'' = -2C_3 \cos\sqrt{2}t - 2C_4 \sin\sqrt{2}t$

$$x_2 = x_1'' + x_1 = -2C_3\cos\sqrt{2}t - 2C_4\sin\sqrt{2}t$$
$$+ C_1 + C_2t + C_3\cos\sqrt{2}t + C_4\sin\sqrt{2}t$$
$$= C_1 + C_2t - C_3\cos\sqrt{2}t - C_4\sin\sqrt{2}t$$

$$\vec{x}(t) = C_1\begin{pmatrix}1\\1\end{pmatrix} + C_2\begin{pmatrix}t\\t\end{pmatrix} + C_3\begin{pmatrix}\cos\sqrt{2}t\\-\cos\sqrt{2}t\end{pmatrix} + C_4\begin{pmatrix}\sin\sqrt{2}t\\-\sin\sqrt{2}t\end{pmatrix}$$

6. (50pts) Use the eigenvalue method to solve the initial value problem.

50

$$\begin{cases} x_1' = 2x_1 - 3x_2 \\ x_2' = x_1 - 2x_2 \end{cases}, \quad x_1(0) = -2, x_2(0) = 0.$$

Sol $\begin{cases} x_1' = 2x_1 - 3x_2 \\ x_2' = x_1 - 2x_2 \end{cases} \Rightarrow \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A\begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

$\det(A - \lambda I) = 0 \Rightarrow \det\begin{pmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{pmatrix} = 0.$

$\Rightarrow (2-\lambda)(-2-\lambda) + 3 = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow$ ~~$\lambda_1 = i$ mult~~ ~~$\lambda_2 = -i$ mult~~

$\lambda = 1$ mult 1

$\lambda = -1$ mult 1

~~$\lambda = 1 \quad (A - \lambda I)^2 \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} -2 & 6 \\ -2 & 6 \end{pmatrix}$~~

~~$(A - \lambda I)^2 \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \vec{0} \Rightarrow \begin{pmatrix} -2 & 6 \\ -2 & 6 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = 0 \Rightarrow \begin{cases} -2a_1 + 6b_1 = 0 \\ -2a_1 + 6b_1 = 0 \end{cases} \Rightarrow v_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$~~

~~$(A - \lambda I)\vec{V_1} = \vec{V_2} \Rightarrow \begin{pmatrix} 1 & -3 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$~~

$\lambda = 1 \quad (A - I)\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \vec{0} \Rightarrow \begin{pmatrix} 1 & -3 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = 0 \Rightarrow \begin{cases} a_1 - 3b_1 = 0 \\ a_1 - 3b_1 = 0 \end{cases} \Rightarrow \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\lambda = -1 \quad (A + I)\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \vec{0} \Rightarrow \begin{pmatrix} 3 & -3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = 0 \Rightarrow \begin{cases} 3a_2 - 3b_2 = 0 \\ a_2 - b_2 = 0 \end{cases} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\vec{x}(t) = c_1 e^t \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{cases} x_1(0) = -2 \\ x_2(0) = 0 \end{cases} \Rightarrow \begin{cases} 3c_1 + c_2 = -2 \\ c_1 + c_2 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = -1 \\ c_2 = 1 \end{cases}$

$\vec{x}(t) = -e^t \begin{pmatrix} 3 \\ 1 \end{pmatrix} + e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

7: Find a particular solution to the system

$$\begin{cases} x_1' = 2x_1 - 3x_2 \\ x_2' = x_1 - 2x_2 + 2e^t \end{cases}$$

Hint: You can use the result from Problem 6 to form the fundamental matrix $\Phi(t)$ and the following formula (notations won't be explained)

$$\vec{x}_p(t) = \Phi(t) \int \Phi(t)^{-1} \vec{f}(t)dt.$$

Sol $\Phi(t) = \begin{pmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{pmatrix}$ $\quad \vec{f}(t) = \begin{pmatrix} 0 \\ 2e^t \end{pmatrix}$

$$\Phi(t)^{-1} = \frac{1}{\det \Phi(t)} \begin{pmatrix} e^{-t} & -e^{-t} \\ -e^t & 3e^t \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{-t} & -e^{-t} \\ -e^t & 3e^t \end{pmatrix}.$$

$$\vec{X}_p(t) = \Phi(t) \int \Phi(t)^{-1} \vec{f}(t)\,dt$$

$$= \begin{pmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{pmatrix} \int \frac{1}{2} \cdot \begin{pmatrix} e^{-t} & -e^{-t} \\ -e^t & 3e^t \end{pmatrix} \begin{pmatrix} 0 \\ 2e^t \end{pmatrix}.$$

$$= \begin{pmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{pmatrix} \begin{pmatrix} -t \\ \frac{3}{2}e^{2t} \end{pmatrix}.$$

$$= \begin{pmatrix} -3te^t + \frac{3}{2}e^t \\ -te^t + \frac{3}{2}e^t \end{pmatrix}$$

50

8. (50pts) Solve the system:

$$\begin{cases} x_1' = 2x_1 - 3x_2 \\ x_2' = x_1 - 2x_2 \\ x_3' = x_1 + x_2 - x_3 \end{cases}$$

(Hint: there is a chain of length 2)

Sol
$$\begin{cases} x_1' = 2x_1 - 3x_2 \\ x_2' = x_1 - 2x_2 \\ x_3' = x_1 + x_2 - x_3 \end{cases} \Rightarrow \frac{d}{dt}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}2&-3&0\\1&-2&0\\1&1&-1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

$$\det\begin{pmatrix}2-\lambda&-3&0\\1&-2-\lambda&0\\1&1&-1-\lambda\end{pmatrix} = 0 \Rightarrow (-1-\lambda)\left((2-\lambda)(-2-\lambda)+3\right) = 0$$

$$\Rightarrow (\lambda+1)^2(1-\lambda) = 0 \Rightarrow \begin{matrix}\lambda=-1 \text{ mult } 2\\ \lambda=1 \text{ mult } 1\end{matrix}$$

$$\lambda=-1 \quad (A+I)^2 = \begin{pmatrix}6&-6&0\\2&-2&0\\4&-4&0\end{pmatrix} \quad (A+I)^2\begin{pmatrix}a_1\\b_1\\c_1\end{pmatrix} = \vec{0} \Rightarrow \begin{cases}6a_1-6b_1=0\\2a_1-2b_1=0\\4a_1-4b_1=0\end{cases}$$

we choose $V_1 \begin{pmatrix}1\\1\\0\end{pmatrix}$ $(A+I)\vec{V_1} = \vec{V_2} \Rightarrow \begin{pmatrix}3&-3&0\\1&-1&0\\1&1&0\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\2\end{pmatrix}$

$$\lambda=1 \quad (A-I)\begin{pmatrix}a_3\\b_3\end{pmatrix} = \vec{0} \Rightarrow \begin{pmatrix}1&-3&0\\1&-3&0\\1&1&-2\end{pmatrix}\begin{pmatrix}a_3\\b_3\\c_3\end{pmatrix} = \vec{0}$$

$$\Rightarrow \begin{cases}a_3-3b_3=0\\a_3-3b_3=0\\a_3+b_3-2c_3=0\end{cases} \Rightarrow \begin{pmatrix}3\\1\\2\end{pmatrix}$$

$$\vec{X}(t) = C_1e^{t}\begin{pmatrix}3\\1\\2\end{pmatrix} + C_2e^{-t}\begin{pmatrix}1\\1\\2t\end{pmatrix} + C_3e^{-t}\begin{pmatrix}0\\0\\2\end{pmatrix}$$

9. (50pts)

This problem is to find the series solution to the equation:

$$(2x-3)y' + 2y = 0.$$

(a): What's the recurrence relation? What's the radius of convergence?

$y = \sum_{n=0}^{+\infty} C_n x^n \quad y' = \sum_{n=0}^{+\infty} C_n \cdot n \cdot x^{n-1}$

$(2x-3)\sum_{n=0}^{+\infty} C_n \cdot n \cdot x^{n-1} + 2\sum_{n=0}^{+\infty} C_n x^n = 0.$

$\Rightarrow \cancel{2C_n}\ 2\sum_{n=0}^{+\infty} C_n \cdot n \cdot x - 3\sum_{n=0}^{+\infty} C_{n+1}\cdot(n+1)x^n + 2\sum_{n=0}^{+\infty} C_n \cdot x^n = 0$

$\Rightarrow \sum_{n=0}^{+\infty} \left((2n+2)C_n - 3(n+1)C_{n+1}\right)x^n = 0$

$\Rightarrow C_{n+1} = \cancel{\frac{2n+2}{3(n+1)}}\ \frac{2}{3}\ \frac{2n+2}{3(n+1)}C_n = \frac{2}{3}C_n.$

recurrence relation is $C_{n+1} = \frac{2}{3}C_n$

radius of convergence is $\lim_{n\to+\infty}\left|\frac{C_n}{C_{n+1}}\right| = \frac{3}{2}$

(b): Use geometric series to express the solution as an elementary function.

$C_{n+1} = \frac{2}{3}C_n \Rightarrow C_n = (\frac{2}{3})^n C_0.$

$y = \sum_{n=0}^{+\infty} C_n \cdot x^n = \sum_{n=0}^{+\infty} C_0 (\frac{2x}{3})^n = C_0 \frac{1-(\frac{2}{3}x)^n}{1-\frac{2}{3}x}$

$= \frac{3C_0}{3-2x}.$

10. (50pts)

This problem is to find the series solution to the following differential equation:

$$(x^2-1)y''-7xy'+16y=0.$$

(a): Find the recurrence relation. What's the (guaranteed) radius of convergence?

sol ~~$y=\sum^{+\infty}$~~ $y=\sum_{n=0}^{+\infty} C_n x^n$ $\quad y'=\sum_{n=0}^{+\infty} C_n n x^{n-1}$

$y''=\sum_{n=0}^{+\infty} C_n n(n-1)x^{n-2}$

$(x^2-1)y''-7xy'+16y=0$

$\Rightarrow \sum_{n=0}^{+\infty} C_n n(n-1)x^n-\sum_{n=0}^{+\infty} C_{n+2}(n+2)(n+1)x^n-7\sum_{n=0}^{+\infty} C_n\cdot n x^n+16\sum_{n=0}^{+\infty} C_n x^n=0$

$\Rightarrow C_n(n-1)n-7nC_n+16C_n=C_{n+2}(n+2)(n+1)$

$C_{n+2}=\frac{(n-4)^2}{(n+2)(n+1)}C_n$ it is recurrence relation

$\lim_{n\to\infty}\left|\frac{C_n}{C_{n+2}}\right| \lim_{n\to\infty}\left|\frac{(n+2)(n+1)}{(n-4)^2}\right|=1$ it is radius of convergence.

(b): Use series method to find the solution $y(x)$ satisfying $y(0) = 1$ and $y'(0) = 0$. (Hint: the answer is a polynomial)

$y(0)=1 \Rightarrow C_0 = 1 \quad y'(0) = 0 \Rightarrow C_1 = 0$

odd items: $C_3 = \frac{9}{6} = \frac{3}{2} C_1 = 0.$

$C_5 = \frac{1}{20} C_3 = 0$.

$\cdots\cdots$

$C_{2m+1} = 0.$

even items: $C_2 = \cancel{4} \; 8 C_0 = 8$

$C_4 = \frac{1}{3} C_2 = \frac{8}{3}$

$C_6 = 0 \cdot C_4 = 0$

$C_8 = 0$

$\cdots\cdots$

$C_{2m} = 0$

$y = 1 + 8x^2 + \frac{8}{3} x^4$