MAT 303: Calculus IV with Applications

Fall 2007
Department of Mathematics
SUNY at Stony Brook

Differential equations are the language in which the laws of physics are expressed. This course will introduce basic methods for solving ordinary differential equations. There is a particular emphasis on linear differential equations with constant coefficients and systems of differential equations. Numerous applications in the physical, biological, and social sciences will be discussed.

Instructor: Dr. Corbett Redden
Math Tower 3-114. Phone: 632-8261. email: *redden at math dot sunysb dot edu*
Office Hours: Wednesday 12:50-2:20p, Thursday 9:30-11:00a, or drop-in, or by appointment.

Recitation instructor and grader: Andrew Stimpson
Math. Tower 3-101, e-mail: *stimpson at math dot sunysb dot edu*
Office hours ???,

Homework: Working homework problems is the only way to really learn the material. While you are encouraged to work with others, you must write up all solutions on your own. Homework sets will usually be collected in class on Fridays. No late homework will be accepted, but the lowest homework grade will be dropped. If you miss a homework assignment, you should still work out the problems on your own. Also, you are encouraged to read the corresponding section of the text book *before* attending each lecture.

Exams:

- Midterm 1: Wednesday, October 10 (in class). Review Sheet, Review Sheet Answers, Midterm 1 Solutions
- Midterm 2: Wednesday, November 14 (in class). Review Sheet (Answers). Midterm 2 Solutions
- Final Exam: Monday, December 17, 2-4:30p (in classroom Harriman 116). Review (Answers)
- Final Exam Review with Stimpson: Sunday Dec. 16 3p Library N3063

These dates are firm, and make-ups will only be given in the case of unforseeable circumstances beyond the student's control. In such a case, the student should contact the instructor as soon as possible.

Class schedule:

LEC 1	MWF	11:45am-12:40pm	Harriman Hall	116	Corbett Redden
R01	Tu	9:50am-10:45am	Physics	P117	Andrew Stimpson
R02	Th	2:20pm- 3:15pm	Library	N3063	Andrew Stimpson

Textbook:: *Differential Equations and Boundary Value Problems: Computing and Modeling* (4th Edition), by Edwards & Penney. Pearson/Prentice Hall, 2008. Though we use the 4th edition, it appears that the homework problems are identical to the 3rd edition.

Prerequisites: The completion of one of the standard calculus sequences (MAT 125-127, MAT 131-132, or MAT 141-142) with the grade C or higher in MAT 127 or 132 or 142 or AMS 161. The course will rely *heavily* on material covered in the standard calculus sequences. Familiarity with complex numbers and the basic concepts of linear algebra will be important, so the 200-level courses MAT 203/205 (Calculus III) and/or AMS 261/MAT 211 (Linear Algebra) are strongly recommended.

Course Grade: Midterm 1 25%, Midterm 2 25%, Final Exam 35%, Homework 15%

MLC: The Math Learning Center is located in Math Tower S-240A and offers free help to any student requesting it. It also provides a locale for students wishing to form study groups.

Disabilities: If you have a physical, psychological, medical or learning disability that may impact your course work, please contact Disability Support Services, ECC (Educational Communications Center) Building, room 128, (631) 632-6748. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential. Students requiring emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information, go to the following web site: http://www.www.ehs.stonybrook.edu/fire/disabilities.shtml

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Starred (*) problems are optional problems (not to be turned in) for those students wishing to challenge themselves.

Week	Section	Notes	Homework
9/5-9/7	1.1, 1.2, 1.4	Due Monday 9/17	**§1.1**: 3, 9, 14, 17, 27, 33, 34 **§1.2**: 2, 6, 7, 10, 15, 16, 25, 35 **§1.4**: 2, 3, 5, 17, 21, 25, 27, 33, 37
9/10-9/12	1.3, 1.4	§1.3 (3,6,9) due Monday 9/17	**§1.3**:3, 6, 9 (you may trace on top of book's slope fields)
9/17-9/21	1.5, 1.6	Due Fri. 9/21	**§1.3**:12-14 **§1.5**:3, 6, 13, 16, 27, 33
9/24-9/28	1.6, 2.1-2.2	Due Fri. 9/28	**§1.6**: 3, 5, 10, 17, 26, 29, 31, 33, 37, 43, 44, 46
10/1-10/5	2.3, 2.4	Due Fri. 10/5	**§2.1**:5, 10, 21 **§2.2**:1, 2, 3, 7, 21 **§2.3**:2, 4, 10, 19, 27
10/8-10/12	3.1, 3.2	Mid 1 10/10 (Review, and Review Answers, Midterm 1 Solutions)	
10/15-10/19	3.2, 3.3	Due Monday, 10/22 (b/c of late posting)	**§3.1**: 1, 5, 13, 18, 24, 25, 31, 37, 39, 43, *27, *28, *29
10/22-10/26	3.4, 3.5	Due Fri 10/26	**§3.2**: 3, 5, 7, 8, 15, 17, *27, *28, *32 **§3.3**: 2, 8, 11, 14, 21, 24, 33, 39, *43, *49, *50
10/29-11/2	3.5, 3.6, 4.1	Due Fri 11/2	**§3.4**: 3, 14, 17, 24, 25, *14, *23 **§3.5**: 3, 9, 13, 16, 33, 34, 47
11/5-11/9	5.1	Due Fri. 11/9	**§3.6**: 2, 6, 11, 15, 20, *23, *27 **§3.7**: 2, *7, *11 **§4.1**: 1, 5, 12, 13 (no graphing), *24, *29
11/12-11/16	5.2	Mid 2 11/14 Review (Answers) Midterm 2 Solutions	§5.1: 5, 6, 12, 14, 22, 26, 31, 35 Due Mon 11/19
11/19-11/21	5.3, 5.4	Due Fri 11/30 (no graphing)	**§5.2**: 1, 3, 7, 8, 11, 21 **§5.4**: 3, 4, 10, 11, 15
11/26-11/30	5.4, 5.5	Due Fri 12/7	**§5.5**:2, 3, 7, 17, 22, 27 **§5.2**:28, 29
12/3-12/7	6.1, 6.2	Due Fri 12/14	**§5.3:** 3, 4 **§6.1**: 4, 5 **§6.2**: 1, 2, 3, 7, 9, 19, 24 (no graphing)

12/10-12/14	6.1, 6.2, Review	Last HW Due Fri Final Exam Monday 12/17 2p (same room as lecture) Review, (Answers) Review with Stimpson Sunday Dec. 16 3p Library N3063	

Math 303: Fall 2007
Midterm 1: Review Sheet

No calculators, notes, or books will be allowed on the mid-term. The exam will consist of around 7 questions and last 55 minutes. It is essential that you clearly and neatly show all work in order to receive full or partial credit on the problems. Remember that your goal should not just be to arrive at the correct answer; you should convince the grader that you arrived at your answer by a correct (and followable) method.

For Midterm 1, you may be asked to:

- Write a differential equation which models a situation in the natural world.
- Solve a first-order differential equation. This includes separable, linear, homogeneous, and exact equations. You should know how to find a general solution and solve an initial value problem.
- Reduce certain second-order equations to first-order equations.
- Find equilibrium solutions and their stability. You should understand their relationship to a physical situation which the equation models.
- Determine the existence and uniqueness of solutions for a first-order equation.

The best way to study is by working problems, both old and new. Please review your old homeworks, including any comments, and work out new problems as well. Here are a few sample problems, with similar problems from the book referenced.

1. (§1.1: 1-12) Show that the function y satisfies the differential equation

$$y' + 2y = 0; y = 3e^{-2x}.$$

2. (§1.1: 27-36, §2.1, §2.3) Write a differential equation that describes the following situation: The acceleration dv/dt of Lamborghini is proportional to the difference between 250 km/h and the velocity of the car. Also, determine any equilibrium solutions and their stability, as well as what they mean in this physical situation. Solve the differential equation.

3. (§1.4: 1-18) Solve the initial value problem:

$$\frac{dy}{dx} = 3x^2(y^2+1); \quad y(0) = 1.$$

4. (§1.5: 1-25) Solve the equation

$$2xy' + y = 10\sqrt{x}.$$

5. (§1.6: 1-15) Solve the equation

$$x^2 y' = xy + y^2.$$

6. (§1.6: 16-18, 26-30) Solve the equation

$$xe^y y' = 2(e^y + x^3 e^{2x}).$$

7. (§1.6: 31-42) Is the following equation exact? If so, solve it.

$$\left(x^3 + \frac{y}{x}\right) dx + \left(y^2 + \ln x\right) dy = 0.$$

8. (§1.6: 43-54) Reduce the following second-order equation to a first-order equation:

$$yy'' + (y')^2 = yy'.$$

9. (§2.1: 1-8) Solve the separable equation by using partial fractions

$$\frac{dx}{dt} = 7x(x - 13), \quad x(0) = 17.$$

10. (§2.1: 9-31) The time rate of change of an alligator population P in a swamp is proportional to the square of P. The swamp contained a dozen alligators in 1988, two dozen in 1998. When will there be four dozen alligators in the swamp? What happens thereafter?

11. (§2.2: 1-18) Find equilibrium solutions and their stability.

$$\frac{dx}{dt} = 3x - x^2.$$

12. (§2.3) Suppose that a motorboat is moving at 40 ft/s when its motor suddenly quits, and that 10 s later the boat has slowed to 20 ft/s. Assuming that the resistance it encounters while coasting is proportional to its velocity, how far will the boat coast in all?

13. (§2.4, but not computationally intensive) Using Euler's method with a step size of .1, estimate $y(.2)$ in the solution to

$$y' = -y, \quad y(0) = 1.$$

1. $y = 3e^{-2x}$. Thus, $y' = -6e^{-2x}$.

$$y' + 2y = -6e^{-2x} + 2 \cdot 3e^{-2x} = 0.$$

2. $\frac{dv}{dt} = k(250 - v)$. (where $k > 0$ for physical reasons. If this were not so, the car would be able to accelerate to an indefinitely large velocity). Equilibrium solutions occur when $0 = k(250 - v)$, and hence $v = 250$. By plugging in values of v to dv/dt, we see that $dv/dt > 0$ when $v < 250$ and $dv/dt < 0$ when $v > 250$. Therefore, the equilibrium solution is stable. This means that as time increases, the car's velocity approaches 250 km/h.

3. This is a separable equation.

$$\begin{aligned}
\int \frac{dy}{y^2+1} &= \int 3x^2 dx \\
\tan^{-1}(y) &= x^3 + C \\
y &= \tan(x^3 + C)
\end{aligned}$$

$$\begin{aligned}
1 &= \tan(0 + C) \\
C &= \pi/4
\end{aligned}$$

$$y(x) = \tan(x^3 + \frac{\pi}{4})$$

Of course, because of the periodicity of tan, one could have picked other values for C, such as $\pi/4 + 2\pi$.

4. This is a linear equation. To find the proper integrating factor, we should first put it in standard form.

$$y' + \frac{1}{2x}y = 5x^{-1/2}.$$

Then, multiply both sides by the integrating factor

$$e^{\int \frac{1}{2x} dx} = e^{\frac{1}{2}\ln x} = e^{\ln x^{1/2}} = x^{\frac{1}{2}}.$$

Therefore, we have that

$$\begin{aligned}
x^{1/2}y' + \frac{1}{2}x^{-1/2}y &= 5 \\
\frac{d}{dx}\left(x^{1/2}y\right) &= \frac{d}{dx}(5x) \\
x^{1/2}y &= 5x + C \\
y &= 5\sqrt{x} + \frac{C}{\sqrt{x}}
\end{aligned}$$

5. This equation is homogeneous, and we use the substitution

$$u = y/x, \quad y = ux, \quad \frac{dy}{dx} = \frac{du}{dx}x + u.$$

It is usually a good idea (though not necessary) to first divide by x^n where n is the total degree of each term that appears.

$$\begin{aligned}
x^2\frac{dy}{dx} &= xy + y^2 \\
\frac{dy}{dx} &= \frac{y}{x} + (\frac{y}{x})^2 \\
\frac{du}{dx}x + u &= u + u^2 \\
\frac{du}{dx}x &= u^2 \\
\frac{du}{u^2} &= \frac{dx}{x} \\
-\frac{1}{u} &= \ln|x| + C \\
-\frac{x}{y} &= \ln|x| + C \\
y &= \frac{-x}{\ln|x| + C}
\end{aligned}$$

6. We use substitution

$$u = e^y, \quad u' = e^y y'.$$

Substituting gives us a linear equation

$$\begin{aligned}
xe^y y' &= 2(e^y + x^3e^{2x}) \\
xu' &= 2u + 2x^3e^{2x} \\
u' - \frac{2}{x}u &= 2x^2e^{2x} \\
x^{-2}\left(u' - 2x^{-1}\right) &= x^{-2}2x^2e^{2x} \\
\frac{d}{dx}\left(x^{-2}u\right) &= \frac{d}{dx}\left(\int 2e^{2x}dx\right) \\
x^{-2}u &= e^{2x} + C \\
x^{-2}e^y &= e^{2x} + C \\
y &= \ln(x^2e^{2x} + Cx^2)
\end{aligned}$$

7. We first check exactness by showing that

$$\frac{\partial}{\partial y}\left(x^3 + \frac{y}{x}\right) = \frac{1}{x} = \frac{\partial}{\partial x}\left(y^2 + \ln x\right).$$

Therefore, this equation is equivalent to $\frac{d}{dx}F(x,y)=0$. Solving for F, we find that

$$F(x,y)=\int(x^3+\frac{y}{x})dx=\frac{1}{4}x^4+y\ln x+g(y).$$

$$\begin{aligned}y^2+\ln x=\frac{\partial F}{\partial y}&=\ln x+g'(y)\\ g'(y)&=y^2\\ g(y)&=\frac{1}{3}y^3\end{aligned}$$

Therefore, the final solution is $F=C$, which is

$$\frac{1}{4}x^4+y\ln x+\frac{1}{3}y^3=C.$$

8. Since there is no x dependence in the equation, we use the substitution

$$v=\frac{dy}{dx},\quad d^2ydx^2=\frac{dv}{dx}=\frac{dv}{dy}\frac{dy}{dx}=\frac{dv}{dy}v.$$

Substituting, we get a (linear) first-order equation

$$\begin{aligned}yy''+(y')^2&=yy'\\ y\frac{dv}{dy}v+v^2&=yv\\ \frac{dv}{dy}+\frac{1}{y}v&=1\end{aligned}$$

9. We have a separable equation

$$\frac{dx}{7x(x-13)}=dt$$

which we will solve by partial fractions.

$$\begin{aligned}\frac{1}{7x(x-13)}&=\frac{A}{7x}+\frac{B}{x-13}\\ A(x-13)+B(7x)&=1\end{aligned}$$

$$\begin{aligned}-13A+0&=1\\ (A+7B)x&=0x\end{aligned}$$

$$A=\frac{-1}{13},\quad B=-\frac{A}{7}=\frac{1}{13\cdot 7}$$

Using this, we can now perform in the integral

$$
\begin{aligned}
\int \frac{dx}{7x(x-13)} &= \int dt \\
\int \frac{1}{7\cdot 13}\left(\frac{-1}{x} + \frac{1}{x-13}\right) &= \int dt \\
\frac{1}{91}\left(-\ln|x| + \ln|x-13|\right) &= t + c \\
\ln|\frac{x-13}{x}| &= 91t + c \\
\frac{x-13}{x} &= Ce^{91t} \\
x &= \frac{13}{1-Ce^{91t}}
\end{aligned}
$$

$$
\begin{aligned}
17 &= \frac{13}{1-C} \\
C &= \frac{4}{17}
\end{aligned}
$$

$$
\begin{aligned}
x &= \frac{13}{1-\frac{4}{17}e^{91t}} \\
x &= \frac{221}{17-4e^{91t}}
\end{aligned}
$$

10. $\frac{dP}{dt} = kP^2$ is a separable equation, and has solution

$$P = \frac{1}{C-kt}.$$

Letting t be the number of years after 1988, we have that $P(0) = 12, P(10) = 24$, and plugging these in we find that $C = \frac{1}{12}, k = \frac{1}{240}$, giving us

$$P = \frac{240}{20-t}.$$

Setting $P = 48$, we find that $t = 15$, and we see that P has a vertical assymptote at $t = 20$, which means the population explodes around 2008 (which is *next year*... uh-oh....)

11. Equilibrium solutions occur when $3x - x^2 = 0$, which is $x = 0, 3$. The derivative of x is negative when $x < 0$, positive for $0 < x < 3$, and negative for $x > 3$ (this can be seen because $\frac{dx}{dt} = 3x - x^2$). Therefore, $x = 0$ is an unstable equilibrium, and $x = 3$ is a stable equilibrium.

12. $\frac{dv}{dt} = -kv, \quad v(0) = 40, v(10) = 20, x(0) = x_0$. After solving the differential equation and determining the constants, we have

$$\begin{aligned} \frac{dx}{dt} &= v(t) = 40e^{-.1 \ln 2t} \\ x(t) &= \frac{-40}{.1 \ln 2} e^{-.1 \ln 2t} + C = \frac{-400}{\ln 2} e^{-.1 \ln 2t} + C \\ x(t) &= \frac{400}{\ln 2} \left(1 - e^{-.1 \ln 2t}\right) + x_0 \end{aligned}$$

Therefore, $\lim_{t\to\infty} x(t) = \frac{400}{\ln 2} + x_0$, so the boat coasts for $\frac{400}{\ln 2}$ ft (about 577 ft).

13. Using Euler's method with step-size .1, we get $y(.2) \approx .81$.

i	x_i	y_i	y_{i+1} = -.1$y_i + y_i$
0	0	1	1-.1 = .9
1	.1	.9	.9 - .1· .9 = .9 - .09 = .81
2	.2	.81	

1. (10 points) Show that $y = x^4 e^x$ is a solution to the differential equation

$$y' = \left(\frac{4}{x} + 1\right) y.$$

$y' = 4x^3 e^x + x^4 e^x$
$(\frac{4}{x} + 1)y = \frac{4}{x} x^4 e^x + x^4 e^x = 4x^3 e^x + x^4 e^x.$
Therefore, we have a y such that $y' = (4/x + 1)y$.

2. (10 points) Find the general solution to

$$\frac{dy}{dx} + 2xy = x.$$

This is a linear equation, so we multiply by the integrating factor

$$e^{\int 2x dx} = e^{x^2},$$

resulting in

$$\begin{aligned}
e^{x^2} \frac{dy}{dx} + e^{x^2} 2xy &= e^{x^2} x \\
\frac{d}{dx}\left(e^{x^2} y\right) &= \frac{d}{dx}\left(\int e^{x^2} x dx\right) = \frac{d}{dx}\left(\frac{1}{2} e^{x^2}\right) \\
e^{x^2} y &= \frac{1}{2} e^{x^2} + C \\
y &= \frac{1}{2} + C e^{-x^2}
\end{aligned}$$

3. Newton's law of cooling states that the time rate of change of the temperature $T(t)$ of a body is proportional to the difference between T and the temperature A of the surrounding medium.

a. (10 points) Write a differential equation expressing Newton's law of cooling.

$$\frac{dT}{dt} = k(A - T) = -k(T - A), \qquad (k > 0).$$

Note the sign of k for physical reasons. If an object is cooler than the surrounding temperature, the temperature will increase. Hence, $\frac{dT}{dt} > 0$ if $T < A$.

b. (10 points) Find all equilibrium solutions and determine their stability. Setting $0 = \frac{dT}{dt} = k(A - T)$ gives us $T = A$ (note that k is a constant here, not a variable depending on t, and hence $k = 0$ is not considered an equilibrium solution). To check the stability, we see that $\frac{dT}{dt} > 0$ for $T < A$ and $\frac{dT}{dt} < 0$ for $T > A$. This implies that $T = A$ is a stable equilibrium (draw a phase diagram).

c. (5 points) What do the equilibrium solution(s) in part (b) tell us about the temperature of the object?
As time increases, the temperature of the object will tend towards the temperature A of the surrounding medium, regardless of what the starting temperature of the object was.

4. (25 points) Suppose an object slides along the ground, and the only force acting on it is friction. Let $x(t)$ be the horizontal displacement in feet of the object after t seconds, and suppose that $v = \frac{dx}{dt}$ satisfies the equation

$$\frac{dv}{dt} = -kv.$$

The object's velocity is 25 ft/s after 0 seconds, and the velocity is $25e^{-2}$ ft/s after 1 seconds.

a. What is the velocity at time t?

$$\begin{aligned} \frac{dv}{dt} &= -kv \\ \int \frac{dv}{v} &= \int -kdt \\ \ln|v| &= -kt + c \\ v &= Ce^{-kt} \end{aligned}$$

Plugging in our two data points for velocity gives us $v(0) = 25 = C$ and $25e^{-2} = v(1) = 25e^{-k}$, hence $k = 2$ and

$$v(t) = 25e^{-2t}.$$

b. How far has the object travelled after t seconds?

$$\begin{aligned} \frac{dx}{dt} &= 25e^{-2t} \\ x &= \int 25e^{-2t}dt = -\frac{25}{2}e^{-2t} + C \end{aligned}$$

$$x_0 = -\frac{25}{2}e^{0} + C$$

Since we are considering x as displacement, this means we set $x_0 = 0$ and have

$$x(t) = -\frac{25}{2}e^{-2t} + \frac{25}{2}.$$

c. After a large amount of time, approximately how far will the object have travelled?

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty}\left(-\frac{25}{2}e^{-2t} + \frac{25}{2}\right) = 0 + \frac{25}{2}.$$

Hence, for large values of t, the distance travelled is approximately $\frac{25}{2}$ ft.

5. (10 points) Solve the initial value problem

$$2xyy' = x^2 + 2y^2, \quad y(1) = 2, \quad (x > 0).$$

This is a homogeneous equation (you can also use the substitution of $u = y^2$ and obtain a linear equation). Using

$$u = \frac{y}{x}, \quad y = ux, \quad \frac{dy}{dx} = \frac{du}{dx}x + u,$$

we get a separable equation by

$$\begin{aligned}
2\frac{y}{x}\frac{dy}{dx} &= 1 + 2\left(\frac{y}{x}\right)^2 \\
2u(\frac{du}{dx}x + u) &= 1 + 2u^2 \\
2xu\frac{du}{dx} &= 1 \\
\int 2udu &= \int \frac{dx}{x} \\
= \frac{y^2}{x^2} = u^2 &= \ln x + C \\
y^2 &= x^2(\ln x + C) \\
\\
2^2 &= 1^2(\ln 1 + C) = C \\
\\
y^2 &= x^2(\ln x + 4) \\
y &= x\sqrt{\ln x + 4}
\end{aligned}$$

Notice that because of our initial condition, we see that $y > 0$ and hence we take the positive square root.

6. (10 points) Show that

$$\left(2x + e^{x^2y}2xy\right)dx + \left(e^{x^2y}x^2 + \cos y\right)dy = 0$$

is exact, and find a general solution.

To check exactness, show that

$$\frac{\partial}{\partial y}\left(2x + e^{x^2y}2xy\right) = \frac{\partial}{\partial x}\left(e^{x^2y}x^2 + \cos y\right),$$

both of which are equal to $e^{x^2y}2x^3y + e^{x^2y}2x$. Since the equation is exact, we will obtain a solution of the form $F(x, y) = C$ where

$$F = \int (2x + e^{x^2y}2xy)dx + g(y) = x^2 + e^{x^2y} + g(y)$$

$$\begin{aligned} \frac{\partial F}{\partial y} &= e^{x^2y}x^2 + \cos y \\ g'(y) &= \cos y \\ g(y) &= \sin y \end{aligned}$$

$$F(x, y) = x^2 + e^{x^2y} + \sin y = C.$$

7. (10 points) The motion of a mass m on a spring is governed by Hooke's Law, which states that the restoring force of the spring is proportional to the displacement x of the mass from its equilibrium position. If no other forces act, the motion of the mass is governed by the second-order differential equation

$$\frac{d^2x}{dt^2} + \left(\frac{k}{m}\right) x = 0,$$

(where $k > 0$ is a constant). Using substitution, reduce this second-order equation to a first-order equation. You do not need to solve the equation.

Use the substitution

$$p = \frac{dx}{dt}, \quad d^2xdt^2 = \frac{dp}{dt} = \frac{dp}{dx}\frac{dx}{dt} = \frac{dp}{dx}p,$$

to give the separable equation

$$\frac{dp}{dx}p + \left(\frac{k}{m}\right) x = 0.$$

Math 303 Fall 2007 Midterm 2 Review

On the midterm, you may have one $8\frac{1}{2}$ x 11 sheet of paper (one-side) with formulas, notes, and examples (or favorite recipes or ...). There are no calculators allowed, and you may not use any other material, including books, homework assignments, or class notes. All questions will be partial credit. It will cover 3.1-3.7 and 4.1 with the emphasis on 3.1-3.6. Notice that there is some overlap in the following problems.

1. (3.1: 20-26, 3.2: 1-12) Show whether the following sets of sets of functions are linearly dependent or independent:
a. $\{\cos 2x, \sin 2x\}$
b. $\{\cos^2 x, \sin^2 x\}$
c. $\{\cos^2 x, \sin^2 x, 4\}$.

2. (3.1: 1-16, 3.2: 13-20) Verify that the following functions are solutions to the given differential equation. Solve the initial value problem. (Bonus: What does the Wronskian of your solutions have to do with solving the initial value problem?)
a. $x^2y'' + 2xy' - 6y = 0; y_1 = x^2, y_2 = x^{-3}; y(1) = 3, y'(1) = 1$
b. $y^{(3)} - 6y'' + 11y' - 6y = 0; y_1 = e^x, y_2 = e^{2x}, y_3 = e^{3x};$
$y(0) = 0, y'(0) = 1, y''(0) = 3.$

3. (3.1: 33-48, 3.3: 1-20, 33-36) Find general solutions to the following homogeneous equations.
a. $3y' - y = 0$
b. $y'' + 2y' - 15y = 0$
c. $9y^{(3)} + 12y'' + 4y' = 0$
d. $y^{(4)} - 8y'' + 16y = 0$

4. (3.1: 1-16, 3.3: 21-26) Solve the following initial value problems.
a. $3y' - y = 0; y(0) = 5$
b. $y'' + 2y' - 15y = 0; y(0) = 0, y'(0) = 4$
c. $9y^{(3)} + 12y'' + 4y' = 0; y(0) = 0, y'(0) = 1, y''(0) = \frac{10}{3}$

5. Show that the function $y = \sin x$ satisfies the equation

$$yy'' - (y')^2 = -1.$$

Does it then follow that $y = 3\sin x$ is also a solution? Why or why not?

6. (3.2: 21-24, 3.5: 1-20, 31-40) Find a general solution to the following nonhomogeneous equations and/or solve the initial value problem.
a. $y'' + y = 3x; y(0) = 2, y'(0) = -2$
b. $y'' + 9y = 2\cos 3x + 3\sin 3x$

7. (3.5:21-30) Set up the appropriate form of a particular solution y_p, but do not determine the values of the coefficients.
a. $y'' - 2y' + 2y = e^x \sin x$

8. (3.4: 24-33) Show that if a mass-spring-dashpot system with no external force is underdamped (i.e. $c^2 < 4km$), then the mass passes through the equilibrium position an infinite number of times.

9. (3.4: 15-21) Find the position function for the following mass-spring-dashpot system. What happens to the position for large time?
a. $m = \frac{1}{2}, c = 3, k = 4; x_0 = 2, v_0 = 0$

10. (3.4: 1-4) What is the amplitude and period for the undamped mass-spring system with $m = 2, k = 8, x_0 = 3, v_0 = 8$?

11. (3.6: 1-14)a. Find the position function for the undamped mass-spring system with external force $4e^{-t}$ given by the equation

$$x'' + x = 4e^{-t}; x_0 = 3, v_0 = -2.$$

b. Find the amplitude of the steady-periodic solution to

$$mx'' + kx = F_0 \cos(\omega t).$$

12. (3.6: 15-18) In the following mass-spring-dashpot systems with external force $F_0 \cos(\omega t)$, is there practical resonance for some $\omega > 0$? If so, at what frequency ω will this occur? (Hint: The formula for the amplitude of the steady-periodic solution is

$$C(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}.$$

a. $2x'' + \sqrt{2}x' + 5x = F_0 \cos(\omega t)$.

13. (4.1: 1-10) Transform the mass-spring-dashpot equation $mx''+cx'+kx = 0$ into a system of first-order equations.

Math 303 Fall 2007 Midterm 2 Review Answers

1. (3.1: 20-26, 3.2: 1-12) Show whether the following sets of sets of functions are linearly dependent or independent:
a. $\{\cos 2x, \sin 2x\}$
The Wronskian

$$W = W(\cos 2x, \sin 2x) = \begin{vmatrix} \cos 2x & \sin 2x \\ (\cos 2x)' & (\sin 2x)' \end{vmatrix}$$
$$= \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x = 1$$

Since $W \neq 0$, the functions $\cos 2x$ and $\sin 2x$ are linearly independent.

b. $\{\cos^2 x, \sin^2 x\}$

$$W = \begin{vmatrix} \cos^2 x & \sin^2 x \\ -2\cos x \sin x & 2\sin x \cos x \end{vmatrix} = 2\sin x \cos x$$

The Wronskian W is not identically 0 (plug in for a sample value of x like $\pi/4$), and therefore the pair of functions is linearly independent.

c. $\{\cos^2 x, \sin^2 x, 4\}$.
The relation

$$4\cos^2 x + 4\sin^2 x + (-1)\cdot 4 = 0$$

shows the linear dependence of the three given functions.

2. (3.1: 1-16, 3.2: 13-20) Verify that the following functions are solutions to the given differential equation. Solve the initial value problem. (Bonus: What does the Wronskian of your solutions have to do with solving the initial value problem?)
a. $x^2y'' + 2xy' - 6y = 0; y_1 = x^2, y_2 = x^{-3}; y(1) = 3, y'(1) = 1$

$$x^2y_1'' + 2xy_1' - 6y = x^2 2 + 2x2x - 6x^2 = 0.$$
$$x^2y_2'' + 2xy_2' - 6y = x^2(12x^{-5}) + 2x(-3x^4) - 6x^{-3} = 0$$

$$\begin{cases} c_1y_1(1) + c_2y_2(1) = 3 \\ c_1y_1'(1) + c_2y_2'(1) = 1 \end{cases} \qquad \begin{cases} c_1 + c_2 = 3 \\ 2c_1 - 3c_2 = 1 \end{cases} \quad \begin{cases} c_1 = 1 \\ c_2 = 1 \end{cases}$$

Therefore, $y = c_1y_1 + c_2y_2 = 2x^2 + x^{-3}$.

b. $y^{(3)} - 6y'' + 11y' - 6y = 0; y_1 = e^x, y_2 = e^{2x}, y_3 = e^{3x};$
$y(0) = 0, y'(0) = 1, y''(0) = 3$.

Verifying that y_1, y_2, y_3 are solutions is done by plugging in the functions and obtaining. For y_3 this looks like

$$e^{3x}(3^3 - 6 \cdot 3^2 + 1 \cdot 3 - 6 \cdot 3) = 0.$$

The solution $y = -e^x + e^{2x}$ is obtained by solving

$$\begin{cases} c_1 + c_2 + c_3 = 0 \\ c_1 + 2c_2 + 3c_3 = 1 \\ c_1 + 4c_2 + 9c_3 = 3 \end{cases}$$

3. (3.1: 33-48, 3.3: 1-20, 33-36) Find general solutions to the following homogeneous equations.
In each of the following, we first solve the characteristic equation and use the roots to construct the general solution.
a. $3y' - y = 0$
$3r - 1 = 0$. $r = \frac{1}{3}$. $y = c_1e^{\frac{1}{3}x}$.
b. $y'' + 2y' - 15y = 0$
$r^2 + 2r - 15 = 0$. $r = -5, 3$. $y = c_1e^{-5x} + c_xe^{3x}$.
c. $9y^{(3)} + 12y'' + 4y' = 0$
$9r^3 + 12r^2 + 4r = r(3r + 2)^2$. $y = c_1 + (c_2 + c_3x)e^{-2/3x}$.
d. $y^{(4)} - 8y'' + 16y = 0$
$r^4 - 8r^2 + 16 = (r^2 - 4)^2 = 0$. $y = (c_1 + c_2x)e^{-2x} + (c_3 + c_4x)e^{2x}$.

4. (3.1: 1-16, 3.3: 21-26) Solve the following initial value problems.
We found the general solutions in question 3.
a. $3y' - y = 0; y(0) = 5$
$y = 5e^{1/3x}$.
b. $y'' + 2y' - 15y = 0; y(0) = 0, y'(0) = 4$
$y = \frac{1}{2}e^{-5x} - \frac{1}{2}e^{3x}$.
c. $9y^{(3)} + 12y'' + 4y' = 0; y(0) = 0, y'(0) = 1, y''(0) = \frac{10}{3}$
$y = \frac{21}{2} + (\frac{-21}{2} + 8x)e^{-2/3x}$.

5. Show that the function $y = \sin x$ satisfies the equation

$$yy'' - (y')^2 = -1.$$

Does it then follow that $y = 3\sin x$ is also a solution? Why or why not?
The original differential equation is **not linear**, so it does not follow that $y = 3\sin x$ is a solution. In fact, it is easily verified that $y = 3\sin x$ is not a solution.

6. (3.2: 21-24, 3.5: 1-20, 31-40) Find a general solution to the following nonhomogeneous equations and/or solve the initial value problem.
a. $y'' + y = 3x; y(0) = 2, y'(0) = -2$
First, we solve the complimentary homogeneous equation $y'' + y = 0$, giving us $y_c = c_1 \cos x + c_2 \sin x$. Then, we guess that a particular solution of the non-homogeneous equation will be of the form $A3x$ plus terms coming from the derivatives (a constant). Because none of these appear in the solution to the homogeneous equation, we make the guess $y_p = Ax + B$ and plugging in find $y_p = 3x$.

b. $y'' + 9y = 2\cos 3x + 3\sin 3x$
First, $y_c = c_1 \cos 3x + c_2 \sin 3x$. For our guess at y_p, we take the generalized right hand side plus derivatives, which will look like $A\cos 3x + B\sin 3x$ and multiply it by the lowest power of x so that it does not appear in the complimentary solution. This gives $y_p = Ax\cos 3x + Bx\sin 3x$, and after plugging in, we solve

$$-6A\sin 3x + 6B\cos 3x = 2\cos 3x + 3\sin 3x$$

to give us $y_p = -\frac{1}{2}x\cos 3x + \frac{1}{3}\sin 3x$, and hence $y = y_p + y_c = -\frac{1}{2}x\cos 3x + \frac{1}{3}\sin 3x + c_1\cos 3x + c_2\sin 3x$.

7. (3.5:21-30) Set up the appropriate form of a particular solution y_p, but do not determine the values of the coefficients.
a. $y'' - 2y' + 2y = e^x \sin x$
First, $y_c = e^x(c_1 \cos x + c_2 \sin x)$. Initially, we would guess

$$y_i = Ae^x \sin x + Be^x \cos x,$$

but we notice that that is a solution to the complimentary equation, so we multiply by a power of x to eliminate such duplication, giving us

$$y_p = xe^x(A\sin x + B\cos x).$$

8. (3.4: 24-33) Show that if a mass-spring-dashpot system with no external force is underdamped (i.e. $c^2 < 4km$), then the mass passes through the equilibrium position an infinite number of times.
Solving $mx'' + cx' + kx = 0$ with $c^2 < 4mk$ gives a solution of the form

$$x(t) = e^{-pt}(c_1 \cos \omega_1 t + c_2 \sin \omega_1 t).$$

Finding when the mass passes through the equilibrium position is equivalent to solving $x = 0$, and $c_1 \cos \omega_1 t + c_2 \sin \omega_1 t = 0$ for an infinite number of $t > 0$.

9. (3.4: 15-21) Find the position function for the following mass-spring-dashpot system. What happens to the position for large time?
a. $m = \frac{1}{2}, c = 3, k = 4; x_0 = 2, v_0 = 0$
$\frac{1}{2}x'' + 3x' + 4x = 0$. $x(t) = c_1 e^{-2t} + c_2 e^{-4t}$.

$$\begin{cases} c_1 + c_2 = 2 \\ -2c_1 - 4c_2 = 0 \end{cases} \qquad \begin{cases} c_1 = 4 \\ c_2 = -2 \end{cases}$$

$x(t) = 4e^{-2t} - 2e^{-4t}$, and $\lim_{t\to\infty} x(t) = 0$, so the mass' position is approximately at equilibrium for large time.

10. (3.4: 1-4) What is the amplitude and period for the undamped mass-spring system with $m = 2, k = 8$ and $x_0 = 3, v_0 = 8$?
$2x'' + 8x = 0$. $x(t) = A\cos 2t + B \sin 2t$. $x(t) = 3\cos 2t + 4 \sin 2t$. Written like this, we see that the amplitude will be $\sqrt{3^2 + 4^2} = 5$, and the period will be $\frac{2\pi}{2} = \pi$. This follows explicitly from

$$\begin{aligned} x &= 3\cos 2t + 4 \sin 2t \\ &= 5(\frac{3}{5}\cos 2t + \frac{4}{5}\sin 2t) \\ &= 5\cos(2t - \alpha) \end{aligned}$$

where $\cos\alpha = \frac{3}{5}, \sin\alpha = \frac{4}{5}$. (The last step above is given by a trig identity for $\cos(a - b)$.

11. (3.6: 1-14)a. Find the position function for the undamped mass-spring system with external force $4e^{-t}$ given by the equation

$$x'' + x = 4e^{-t}; x_0 = 3, v_0 = -2.$$

$x = 2e^{-t} + c_1 \cos t + c_2 \sin t = -2e^{-t} + \cos t$.

b. Find the amplitude of the steady-periodic solution to

$$mx'' + kx = F_0 \cos(\omega t).$$

Assuming $\omega \neq \sqrt{\frac{k}{m}}$, we find

$$x_s p(t) = x_p(t) = \frac{F_0}{k - m\omega^2} \cos \omega t = \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t.$$

Therefore, the amplitude is the coefficient $\frac{F_0/m}{\omega_0^2 - \omega^2}$.

12. (3.6: 15-18) In the following mass-spring-dashpot systems with external force $F_0 \cos(\omega t)$, is there practical resonance for some $\omega > 0$? If so, at what frequency ω will this occur? (Hint: The formula for the amplitude of the steady-periodic solution is

$$C(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}.$$

a. $2x'' + \sqrt{2}x' + 5x = F_0 \cos(\omega t)$. The amplitude will be

$$C(\omega) = \frac{F_0}{\sqrt{(5 - 2\omega^2)^2 + (\sqrt{2}\omega)^2}} = \frac{F_0}{\sqrt{4\omega^4 - 18\omega^2 + 25}}.$$

To maximize this, we take the derivative and set it equal to zero, giving us

$$\begin{aligned} 0 =& C'(\omega) = \frac{-1/2 F_0}{(4\omega^4 - 18\omega^2 + 25)^{3/2}} \left(16\omega^3 - 36\omega\right) \\ 0 =& 4\omega(4\omega^2 - 9) \\ \omega =& 0, -\frac{3}{2}, \frac{3}{2} \end{aligned}$$

We normally assume $\omega > 0$ (this is convention; notice that $\cos(-\omega t) = \cos(\omega t)$), and we can see that $\omega = \frac{3}{2}$ is the global maximum, giving us the value where practical resonance occurs.

13. Transform the mass-spring-dashpot equation $mx'' + cx' + kx = 0$ into a system of first-order equations.
Let $y = x'$. Then

$$\begin{cases} y = x' \\ my' + cy + kx = 0 \end{cases}$$

Math 303: Midterm 2 Answers

November 14, 2007

1. The functions $y_1 = e^{2x}$ and $y_2 = e^{3x}$ are solutions to the differential equation

$$y'' - 5y' + 6y = 0.$$

   a. (10 points) Verify that y_1 and y_2 are linearly independent.

$$W(y_1, y_2) = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = e^{5x} \neq 0$$

   b. (10 points) Find a solution satisfying the initial conditions $y(0) = 1$ and $y'(0) = 4$.

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{2x} + c_2 e^{3x}$$
$$\begin{cases} y(0) = c_1 + c_2 = 1 \\ y'(0) = 2c_1 + 3c_2 = 4 \end{cases}$$
$$c_1 = -1, c_2 = 2, \quad y = -e^{2x} + 2e^{3x}$$

2. (15 points) Find a specific solution to the equation

$$y'' - 2y' + 5y = \cos x$$

First, set up the general form of a particular solution

$$\begin{aligned} y_p &= A\cos x + B\sin x \quad \text{Notice no overlap with comp. solution} \\ y_p' &= -A\sin x + B\cos x \\ y_p'' &= -A\cos x - B\sin x \end{aligned}$$

Plugging in, we see

$$y'' - 2y' + 5y = \cos x(-A - 2B + 5A) + \sin x(-B + 2A + 5B) = \cos x$$
$$\begin{cases} 4A - 2B = 1 \\ 2A + 4B = 0 \end{cases} \implies \begin{cases} A = \frac{1}{5} \\ B = -\frac{1}{10} \end{cases}$$
$$y_p = \frac{1}{5}\cos x - \frac{1}{10}\sin x$$

Since the solution the complimentary homogeneous equation $y'' - 2y' + 5y = 0$ is of the form

$$y_c = e^x(c_1 \cos 2x + c_2 \sin 2x),$$

then any equation of the form

$$y = \frac{1}{5}\cos x - \frac{1}{10}\sin x + e^x(c_1 \cos 2x + c_2 \sin 2x)$$

for some values c_1, c_2 is also a specific solution.
3. Find general solutions to the following equations.

a. (10 points) $y^{(3)} - 6y'' + 9y' = 0$

$$\begin{aligned} r^3 - 6r^2 + 9r = r(r-3)^2 &= 0 \\ r &= 0, 3, 3 \\ y &= c_1 + (c_2 + c_3 x)e^{3x} \end{aligned}$$

b. (15 points) $y^{(4)} - y'' = 3x$.
Note that the complimentary solution y_c is not a solution to the above equation, but instead a solution to $y^{(4)} - y = 0$. A particular solution y_p must be added to it to have a solution of the above equation.

$$\begin{aligned} y &= y_c + y_p \\ y_c :& r^4 - r^2 = r^2(r+1)(r-1) = 0 \\ r &= 0, 0, -1, 1 \\ y_c &= c_1 + c_2 x + c_3 e^{-x} + c_4 e^x \\ \\ y_i &= Ax + B \quad \text{Initial guess} \\ y_p &= x^2(Ax + B) = Ax^3 + Bx^2 \quad \text{Remove overlap with } y_c \\ y_p^{(4)} - y'' &= -6Ax - B = 3x \\ A &= -\frac{1}{2}, B = 0 \\ y_p &= -\frac{1}{2}x^3 \\ \\ y &= -\frac{1}{2}x^3 + c_1 + c_2 x + c_3 e^{-x} + c_4 e^x \end{aligned}$$

4. A mass of $m = 1$ kg is attached to a large spring with spring constant $k = 9$ (N/m). It is set in motion with initial position $x_0 = 2$ (m) from the equilibrium position and initial velocity $v_0 = -6$ (m/s).

a. (15 points) If the mass-spring system is undamped (no resistance) and free (no external force), then what is the amplitude and period of the position function?

$$\begin{aligned} mx'' + kx &= 0 \\ x'' + 9x &= 0 \end{aligned}$$

$$\begin{aligned} r^2 &= 9 = 0 \\ r &= \pm 3i \end{aligned}$$

$$\begin{aligned} x(t) &= A\cos 3t + B\sin 3t \\ x'(t) &= -3A\sin 3t + 3B\cos 3t \\ &\begin{cases} A = x_0 = 2 \\ 3B = v_0 = -6 \end{cases} \\ x(t) &= 2\cos 3t - 2\sin 3t \end{aligned}$$

For $x(t)$ of the form above, the amplitude will be $\sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$.
The period is $\frac{2\pi}{3}$.
This can also be seen by transforming the solution to

$$x(t) = 2\sqrt{2}\cos(3t + \frac{\pi}{4}.)$$

b. (10 points) Suppose a dashpot with damping constant $c = 2$ (Ns/m) is connected to the spring (still with no external force). What is the pseudo-period of the position function?

$$x'' + 2x' + 9x = 0$$

$$\begin{aligned} r^2 + 2r + 9 &= 0 \\ r &= -1 \pm i2\sqrt{2} \end{aligned}$$

$$x(t) = e^{-t}(c_1\cos 2\sqrt{2}t + c_2\sin 2\sqrt{2}t)$$

The pseudo-period of the above solution is then $\frac{2\pi}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}}$.

c. (10 points) Suppose that the mass-spring-dashpot system from part (b) is acted on by an external force of $F_0 \cos(\omega t)$ (Newtons) for some ω. Then, the steady-periodic solution will be of the form

$$x_{sp}(t) = C(\omega)\cos\left(\omega t - \alpha(\omega)\right),$$

where

$$C(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \qquad \alpha(\omega) = \begin{cases} \tan^{-1} \frac{c\omega}{k - m\omega^2} & k > m\omega^2 \\ \pi/2 & k = m\omega^2 \\ \pi + \tan^{-1} \frac{c\omega}{k - m\omega^2} & k < m\omega^2 \end{cases}$$

For what value(s) of $\omega > 0$, if any, will the system exhibit practical resonance?

To find practical resonance, we find what value of ω maximizes the amplitude function $C(\omega)$. To do this, we maximize $C(\omega)$ by taking the derivative and setting it equal to zero.

$$\begin{aligned} C(\omega) &= \frac{F_0}{\sqrt{(9 - \omega^2)^2 + (2\omega)^2}} \\ &= \frac{F_0}{\sqrt{\omega^4 - 14\omega^2 + 81}} \end{aligned}$$

$$\begin{aligned} C'(\omega) &= \frac{-1/2F_0}{(\omega^4 - 14\omega^2 + 81)^{3/2}}\left(4\omega^3 - 28\omega\right) \\ 0 &= 4\omega^3 - 28\omega = 4\omega(\omega^2 - 7) \\ \omega &= 0, -\sqrt{7}, \sqrt{7} \\ \omega &= \sqrt{7} \end{aligned}$$

The values $\omega = 0, \pm\sqrt{7}$ all give critical points of the function $C(\omega)$. However, we always consider $\omega \geq 0$, and by looking at a sign chart, we see that $\omega = \sqrt{7}$ gives us the global maximum for $C(\omega)$. Therefore, practical resonance occurs when $\omega = \sqrt{7}$.

5. (5 points) Transform the following 3rd-order equation into a system of linear equations

$$y^{(3)} - 5y'' + 14y' + 2y = 1.$$

$$\begin{cases} y' = y_1 \\ y_1' = y_2 \\ y_2' = -2y - 14y_1 + 5y_2 + 1 \end{cases}$$

Math 303 Fall 2007 Final Exam Review

The final exam will be cumulative with a slight emphasis on the material covered since the second midterm. The best way to review is to work through the first two midterms and homework-style problems.

On the final, you may have one $8\frac{1}{2}$ x 11 sheet of paper (one-side) with formulas, notes, and examples. The cheat-sheet should be made by you and not merely copied from another student. There are no calculators allowed, and you may not use any other material, including books, homework assignments, or class notes. All questions will be partial credit.

The review sheets for Midterms 1-2 along with the following problems provide a good idea of what kind of questions to expect.

1. a. Show directly (by plugging in solution $x(t)$) that $x(t) = e^{\lambda t}$ is a solution to $ax'' + bx' + cx = 0$ if and only if λ is a solution to $a\lambda^2 + b\lambda + c = 0$.

b. Show directly that $\overrightarrow{x}(t) = \overrightarrow{v}e^{\lambda t}$ is a solution to $\overrightarrow{x}(t)' = A\overrightarrow{x}(t)$ if and only if $\overrightarrow{v}$ is an eigenvector of A with eigenvalue λ.

c. Suppose that $\overrightarrow{v_1}$ is an eigenvector of A with eigenvalue λ. If $\overrightarrow{x}(t) = (\overrightarrow{v_1}t + \overrightarrow{v_2})e^{\lambda t}$ is a solution to $\overrightarrow{x}(t)' = A\overrightarrow{x}(t)$, then determine an explicit relationship between $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ (derive the equation for a chain of generalized eigenvectors).

2. Solve the second-order linear equation

$$x'' + 5x' + 6x = 0$$

(a) by using the methods from Chapter 3 (use characteristic equation)
(b) by transforming it into a system of 2 first-order equations and using the techniques of Chapter 5.

3. Find 2 linearly independent solutions to

$$\overrightarrow{x}' = \begin{bmatrix} 3 & -1 \\ 5 & 3 \end{bmatrix} \overrightarrow{x}.$$

Show that they are linearly independent. Write a general solution.

4. Solve the initial value problem

$$\begin{cases} \frac{dx}{dt} = 4x + 2y \\ \frac{dy}{dt} = 3x - y \end{cases} ; x(0) = 3, y(0) = -2.$$

5. Find general solutions and write a fundamental matrix solution to $\overrightarrow{x}' = A\overrightarrow{x}$ for the following matrices

$$A = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \\ -5 & -3 & -7 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 0 & 0 \\ -7 & 9 & 7 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 1 \\ -2 & -4 & -1 \end{bmatrix}.$$

6. Calculate e^{At} and use this to solve the initial value problem $\overrightarrow{x}' = A\overrightarrow{x}$ with $\overrightarrow{x}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, where

$$A = \begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}.$$

7. Suppose that we have a two brine tank system with constant flow rate of 10 gal/min going into and out of both tanks. If the volume of tank I is 50gal and that of tank II is 25 gal, find the amount of salt in both tanks as a function of time, assuming that the original amount in tank I is 15 gal and that of tank II is 0 gal.

8. Find equilibrium solutions and determine their stability in the systems:

$$\begin{cases} \frac{dx}{dt} = -2x + y \\ \frac{dy}{dt} = x - 2y \end{cases}, \begin{cases} \frac{dx}{dt} = 1 - y^2 \\ \frac{dy}{dt} = x + 2y \end{cases}$$

Math 303 Fall 2007 Final Exam Review Answers

1. These can all be seen by directly substituting the solution into the differential equation. For instance, for part (b), we have that

$$\overrightarrow{x}' = \lambda \overrightarrow{v} e^{\lambda t}, \quad A\overrightarrow{x} = (A\overrightarrow{v})e^{\lambda t}.$$

Therefore (since $e^{\lambda t} \neq 0$), $\overrightarrow{x}(t) = \overrightarrow{v} e^{\lambda t}$ is a solution if and only if $A\overrightarrow{v} = \lambda \overrightarrow{v}$, which is the definition for an eigenvector and eigenvalue of A.

2. $x(t) = c_1 e^{-2t} + c_2 e^{-3t}$. Notice that the characteristic equation from part (a) is the same polynomial as the eigenvalue equation in part (b). In part (b), you use $x' = y$ and solve

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \mathbf{x},$$

which gives you the solution

$$\mathbf{x}(t) = \begin{bmatrix} c_1 e^{-2t} + c_2 e^{-3t} \\ -2c_1 e^{-2t} - 3c_2 e^{-3t} \end{bmatrix}.$$

The top row is the solution to the original equation, and the second row $(y(t))$ is $\frac{dx}{dt}$).

3. **I meant for this problem to be

$$\overrightarrow{x}' = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} \overrightarrow{x}.$$

Notice the negative sign in front of the 3 (it makes things much nicer). However, the answer to the problem, as stated, is....

$$\mathbf{x_1} = e^{3t} \begin{bmatrix} \cos \sqrt{5}t \\ \sqrt{5} \sin \sqrt{5}t \end{bmatrix}, \mathbf{x_2} = e^{3t} \begin{bmatrix} \sin \sqrt{5}t \\ -\sqrt{5} \cos \sqrt{5}t \end{bmatrix}.$$

Linear independence is shown using the Wronskian,

$$W(\mathbf{x_1}, \mathbf{x_2}) = e^{3t} \begin{vmatrix} \cos \sqrt{5}t & \sin \sqrt{5}t \\ \sqrt{5} \sin \sqrt{5}t & -\sqrt{5} \cos \sqrt{5}t \end{vmatrix} = \sqrt{5} e^{3t}$$

4. The general solution

$$\overrightarrow{x}(t) = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t}$$

is worked out in Example 1 of 5.2. The solution to the initial value problem is

$$\begin{cases} x(t) = e^{-2t} + 2e^{5t} \\ y(t) = -3e^{-2t} + e^{5t} \end{cases}.$$

5. Please note that there are multiple equivalent ways of writing solutions on these problems (depending on the basis vectors one chooses for eigenspaces).

$$A = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} \Rightarrow \overrightarrow{x}(t) = e^{4t} \begin{bmatrix} c_1 \cos 3t - c_2 \sin 3t \\ c_1 \sin 3t + c_2 \cos 3t \end{bmatrix} \quad \text{(5.2 Ex.3)}$$

$$A = \begin{bmatrix} 2 & 1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{bmatrix} \Rightarrow \Phi(t) = \begin{bmatrix} e^t & 2\cos 2t - \sin 2t & \cos 2t + 2\sin 2t \\ -e^t & -3\cos 2t + \sin 2t & \cos 2t - 3\sin 2t \\ & 3\cos 2t + \sin 2t & 3\sin 2t - \cos 2t \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 2 \\ -5 & -3 & -7 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow \overrightarrow{x}(t) = c_1 \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -2t+1 \\ -2t-5 \\ 2t+1 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} -t^2+t+1 \\ -t^2-5t \\ t^2+t \end{bmatrix} e^{-t}$$

(5.4 Ex. 4)

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -7 & 9 & 7 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow \Phi(t) = \begin{bmatrix} e^{2t} & e^{2t} & 0 \\ e^{2t} & 0 & e^{9t} \\ 0 & e^{2t} & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 1 \\ -2 & -4 & -1 \end{bmatrix} \Rightarrow \overrightarrow{x}(t) = \begin{bmatrix} -c_1 - 2c_2 + c_3 \\ c_2 + c_3 t \\ c_1 - 2c_3 t \end{bmatrix} e^t$$

6.

$$e^{\begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix} t} = e^{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} t} e^{\begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} t} = \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 5t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{2t} & 5te^{2t} \\ 0 & e^{2t} \end{bmatrix}$$

$$e^{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} t} = \begin{bmatrix} e^{t} & 0 & 2te^{t} \\ 0 & e^{t} & -3te^{t} \\ 0 & 0 & e^{t} \end{bmatrix}$$

7. Letting $x_1(t), x_2(t)$ be the amount of salt in tanks 1 and 2, respectively, we obtain the system of differential equations

$$\begin{cases} \frac{dx_1}{dt} = -\frac{10}{50} x_1 \\ \frac{dx_2}{dt} = -\frac{10}{50} x_1 - \frac{10}{25} x_2 \end{cases}$$

Solving this, we obtain

$$\mathbf{x} = \begin{bmatrix} c_1 e^{-1/5t} \\ c_1 e^{-1/5t} + c_2 e^{-2/5t} \end{bmatrix} = \begin{bmatrix} 15e^{-1/5t} \\ 15e^{-1/5t} - 15e^{-2/5t} \end{bmatrix}$$

8.

$\begin{cases} \frac{dx}{dt} = -2x + y \\ \frac{dy}{dt} = x - 2y \end{cases}$ is linear, and has only one equilibrium solution, located at $(0,0)$.

The eigenvalues of $\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ are $\lambda = -1, -3$, which are both negative, so the equilibrium is stable (in fact, it is an asymptotically stable nodal sink).

$\begin{cases} \frac{dx}{dt} = 1 - y^2 \\ \frac{dy}{dt} = x + 2y \end{cases}$ has two equilibrium solutions, located at $(-2,1)$ and $(2,-1)$.

The Jacobean at (-2,1) is $\begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix}$ and has eigenvalues $\lambda = 1 \pm i$. The real parts of the eigenvalue are positive, so the equilibrium solution is unstable (in fact, it is an unstable spiral source). The Jacobean at (1,-1) is $\begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}$ with eigenvalues $\lambda = 1 \pm \sqrt{3}$, one of which is positive, and the other negative. Therefore, that equilibrium point is also unstable (and in fact, is an unstable saddle point).