MAT 260: Problem Solving in Mathematics

Office hours Fall 2012: Tuesdays 4pm-5pm in the Math Learning Center, Tuesdays 5pm-6pm and Wednesdays 2pm-3pm in my office.

The last class on December 3 will be held in room P143.

The last two classes we will discuss the following problems. Try to solve not only your assignment but all of them.

Homework4: Here are the problems from the test we had.

We will have a **practice test for Putnam** on **Saturday, 13** at 1.00pm room P131 in the Math Tower.

Homework3: Read the following article. Here is a link to the IMO probems website.

Homework2: Keep on reading about inequalities. The following recaps and explains in more details the last lecture. Use this approach to solve some of the inequalities from the practice problems. Finish the solution of the last problem from here and solve at least one other problem from this list. **Due date: September, 17**

Inequalities:
I suggest the following reading material. You are encouraged to look for other sources.

1. This is the first thing to read and do the exercises

2. List of useful inequalities with some examples of applications

3. This could also be an interesting reading. Has a lot of good problems.

4. This is an excellent book. Highly recommended.

Homework1: Please try to read as much as possible from the links above and solve as many problems as you can. In particular, solve problems from Section 2.1 from the first source and submit solutions for 3 of them of your choice. I would prefer if you do that in LaTeX and submit to my email. **Due date: September, 10**

Recommended reading:

This is a wonderful book with problems in Linear Algerbra. We will talk about some of the topics later on.

Here is a selection of problems from Putnam exam. Let us try to solve all of them.

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1. In a town every two residents who are not friends have a friend in common, and no one is a friend of everyone else. Let us number the residents from 1 to n and let a_i be the number of friends of the i-th resident. Suppose that $\sum_{i=1}^{n} a_i^2 = n^2 - n$. Let k be the smallest number of residents (at least three) who can be seated at a round table in such a way that any two neighbors are friends. Determine all possible values of k.

2. Denote by V the real vector space of all real polynomials in one variable, and let $P : V \to R$ be a linear map. Suppose that for all $f, g \in V$ with $P(fg) = 0$ we have $P(f) = 0$ or $P(g) = 0$. Prove that there exist real numbers x_0, c such that $P(f) = cf(x_0)$ for all $f \in V$.

3. Let a, b be two integers and suppose that n is a positive integer for which the set
$$\mathbb{Z} \setminus \{ax^n + by^n | x, y \in \mathbb{Z}\}$$
is finite. Prove that $n = 1$.

4. How many nonzero coefficients can a polynomial $P(z)$ have if its coefficients are integers and $|P(z)| \leq 2$ for any complex number z of absolute value 1?

5. Suppose that for a function $f : \mathbb{R} \to \mathbb{R}$ and real numbers $a < b$ one has $f(x) = 0$ for all $x \in (a, b)$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$ if
$$\sum_{k=0}^{p-1} f\left(y + \frac{k}{p}\right) = 0$$
for every prime number p and every real number y.

6. For each positive integer k, find the smallest number n_k for which there exist real $n_k \times n_k$ matrices $A_1, A_2, ..., A_k$ such that all of the following conditions hold:

   (a) $A_1^2 = A_2^2 = \ldots = A_k^2 = 0$,

   (b) $A_i A_j = A_j A_i$ for all $1 \leq i, j \leq k$, and

   (c) $A_1 A_2 \ldots A_k \neq 0$.

7. Does there exist a finite group G with a normal subgroup H such that $|\mathrm{Aut}\, H| > |\mathrm{Aut}\, G|$?

1. Let $n > 1$ be an integer. Find all sequences $a_1, a_2, ..., a_{n^2+n}$ satisfying the following conditions:

   (a) $a_i \in 0, 1$ for all $1 \leq i \leq n^2 + n$;

   (b) $a_{i+1} + a_{i+2} + \ldots + a_{i+n} < a_{i+n+1} + a_{i+n+2} + \cdots + a_{i+2n}$ for all $0 \leq i \leq n^2 - n$.

2. Let $b, n > 1$ be integers. Suppose that for each $k > 1$ there exists an integer a_k such that $b - a_k^n$ is divisible by k. Prove that $b = A^n$ for some integer A.

3. Suppose that A and B are square matrices such that $AB - BA = A$. Prove that A is nilpotent, i.e. there exists k such that $A^k = 0$.

4. Let S_n be the set of all sums $\Sigma_{k=1}^{n} x_k$, where $n \geq 2, 0 \leq x_1, x_2, \ldots, x_n \leq \pi/2$ and $\Sigma_{k=1}^{n} \sin x_k = 1$.

   a) Show that S_n is a closed interval.

   b) Let l_n be the length of S_n. Find $\lim_{n\to\infty} l_n$.

5. Let X be an arbitrary set, let f be a one-to-one mapping from X onto itself. Prove that there exist mappings $g_1; g_2 : X \to X$ such that $f = g_1 \circ g_2$ and $g_1 \circ g_1 = \ id = g_2 \circ g_2$, where id denotes the identity mapping on X

6. Find all functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$f(x + f(y)) = f(x + y) + f(y)$$

   for all $x, y \in \mathbb{R}_+$.

Putnam Calculus Questions

2011

A2 Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be sequences of positive real numbers such that $a_1 = b_1 = 1$ and $b_n = b_{n-1}a_n - 2$ for $n = 2, 3, \ldots$. Assume that the sequence (b_j) is bounded. Prove that

$$S = \sum_{n=1}^{\infty} \frac{1}{a_1 \ldots a_n}$$

converges, and evaluate S.

A3 Find a real number c and a positive number L for which

$$\lim_{r\to\infty} \frac{r^c \int_0^{\pi/2} x^r \sin x \, dx}{\int_0^{\pi/2} x^r \cos x \, dx} = L.$$

A5 Let $F : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable functions with the following properties:

- $F(u, u) = 0$ for every $u \in \mathbb{R}$;
- for every $x \in \mathbb{R}$, $g(x) > 0$ and $x^2 g(x) \leq 1$;
- for every $(u, v) \in \mathbb{R}^2$, the vector $\nabla F(u, v)$ is either **0** or parallel to the vector $\langle g(u), -g(v) \rangle$.

Prove that there exists a constant C such that for every $n \geq 2$ and any $x_1, \ldots, x_{n+1} \in \mathbb{R}$, we have

$$\min_{i \neq j} |F(x_i, x_j)| \leq \frac{C}{n}.$$

A6 Let G be an abelian group with n elements, and let

$$\{g_1 = e, g_2, \ldots, g_k\} \subsetneq G$$

be a (not necessarily minimal) set of distinct generators of G. A special die, which randomly selects one of the elements $g_1, g_2, \ldots, g_k$ with equal probability, is rolled m times and the selected elements are multiplied to produce an element $g \in G$. Prove that there exists a real number $b \in (0, 1)$ such that

$$\lim_{m\to\infty} \frac{1}{b^{2m}} \sum_{x \in G} \left(\text{Prob}(g = x) - \frac{1}{n} \right)^2$$

is positive and finite.

B3 Let f and g be (real-valued) functions defined on an open interval containing 0, with g nonzero and continuous at 0. If fg and f/g are differentiable at 0, must f be differentiable at 0?

B5 Let $a_1, a_2, \ldots$ be real numbers. Suppose that there is a constant A such that for all n,

$$\int_{-\infty}^{\infty} \left(\sum_{i=1}^{n} \frac{1}{1 + (x - a_i)^2} \right)^2 dx \leq An.$$

Prove there is a constant $B > 0$ such that for all n,

$$\sum_{i,j=1}^{n} (1 + (a_i - a_j)^2) \geq Bn^3.$$

2010

A2 Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

for all real numbers x and all positive integers n.

A3 Suppose that the function $h : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives and satisfies the equation

$$h(x,y) = a\frac{\partial h}{\partial x}(x,y) + b\frac{\partial h}{\partial y}(x,y)$$

for some constants a, b. Prove that if there is a constant M such that $|h(x,y)| \le M$ for all $(x,y) \in \mathbb{R}^2$, then h is identically zero.

A6 Let $f : [0,\infty) \to \mathbb{R}$ be a strictly decreasing continuous function such that $\lim_{x\to\infty} f(x) = 0$. Prove that $\int_0^\infty \frac{f(x)-f(x+1)}{f(x)}\, dx$ diverges.

B5 Is there a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(f(x))$ for all x?