MAT 211: Introduction to Linear Algebra Spring 2018

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Welcome to MAT 211 (Lecture 4)

Textbook: Linear Algebra with Applications, by Otto Bretscher (5th edition).

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Homework

Homework assignments will be posted **here** and on BlackBoard. Homework will be assigned for each week and should be handed in during your Wednesday lecture of the following week (unless otherwise stated in your homework assignment).

Exams and Grading

There will be two midterms, and a final exam (dates **here**), whose weights in the overall grade are listed below.

15% Homework

25% Midterm 1

25% Midterm 2

35% Final Exam (cumulative)

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General Information

Information for students with disabilities
If you have a physical, psychological, medical, or learning disability that may impact your course work, please contact Disability Support Services at (631) 632-6748 or http://studentaffairs.stonybrook.edu/dss/. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential.

Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to the following website: http://www.sunysb.edu/ehs/fire/disabilities.shtml

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Syllabus and Weekly Plan

Week of	Topics
Jan 22	1.1 Intro to linear systems 1.2 Gauss–Jordan, RREF
Jan 29	1.3 Solutions of linear systems 2.1 **Linear Transformations**
Feb 5	1.3 Matrix Algebra 2.2 Linear Transformations in Geometry
Feb 12	2.3 Matrix multiplication 2.4 Inverse of a linear transformation
Feb 19	3.1 Image and Kernel 3.2 Subspaces of Euclidean spaces
Feb 26	3.2 Bases and linear independence Midterm I, Wed. Feb 28
March 5	3.3 Dimension 3.4 Coordinates

March 12	Spring Break!
March 19	3.4 Coordinates 4.3 Matrix of a linear transformation (general version)
March 26	4.3 Matrix of a linear transformation (general version) 5.1 Orthonormality
April 2	5.2 Gram–Schmidt and QR Factorization 5.3 Orthogonal Matrices
April 9	Midterm, Mon. April 9 6.1 Determinants 6.2 Properties of Determinants
April 16	6.3 More on Determinants; Cramer's Rule 7.1 Diagonalization
April 23	7.2 Eigenvalues 7.3 Eigenvectors
April 30	Final Review
May 7	Final Exam Wednesday, May 9, 8:00am-10:45am

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Homework

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Exams

Midterm I: Wed. Feb 28 (in class)

Practice Midterm 1

Solutions to Practice Midterm 1

Midterm II: Mon. April 9 (in class)

Practice Midterm 2

Solutions to Practice Midterm 2

Final Exam: Wed. May 9

Time: 8:00am-10:45am

Location: Engineering 143

Before we start a formal discussion of linear spaces/operators, let us restrict our attention ~~to~~ a concrete example; the simplest

linear space $\mathbb{R}^n$.

Note: Understanding an example well enough makes the study of an abstract concept much simpler.

Definition ($\mathbb{R}^n$):

The real Euclidean space of dimension n is defined as the cartesian product:

$$\{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in \mathbb{R}\}$$

In other words, $\mathbb{R}^n$ consists of all n-tuples of real numbers. (we'll sometimes call them vectors).
There's a natural notion of addition on $\mathbb{R}^n$:

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

One can also define a <u>scalar multiplication.</u> For any $c \in \mathbb{R}$ and $(x_1, \ldots, x_n) \in \mathbb{R}^n$, one defines:

$$c(x_1, \ldots, x_n) = (cx_1, \ldots, cx_n).$$

<u>Definition (Linear maps/transformations on $\mathbb{R}^n$):</u>

A map $T: \mathbb{R}^n \to \mathbb{R}^m$ is called linear if it satisfies

$$T(au + bv) = aT(u) + bT(v),$$

where $a, b \in \mathbb{R}$, and $u, v \in \mathbb{R}^n$.

<u>Example:</u>

1) Let A be an $m \times n$ real matrix; i.e. A has m rows and n columns

In particular:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Any element of $\mathbb{R}^n$ is of the form:

$$u = (x_1, \dots, x_n).$$

We define a linear map $T: \mathbb{R}^n \to \mathbb{R}^m$ by:

$$T\big((x_1, \dots, x_n)\big) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= Au$$

This is just the usual mult. of a matrix and a column vector

$$= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \text{—} \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}$$

clearly, $T\big((x_1, \dots, x_n)\big) \in \mathbb{R}^m$.

Also, matrix multiplication satisfies:

$$A(au + bv) = aAu + bAv,$$

for any $a, b \in \mathbb{R}$, $u, v \in \mathbb{R}^n$

Therefore, T is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$.

2) Reflection in $\mathbb{R}^2$ (wrt a line).

3) Rotation in $\mathbb{R}^2$ (wrt the origin),

4) Scaling.

We'll see in class (geometrically) why these define linear maps.

Example (1) had a concrete algebraic description in terms of a matrix. One can ask whether every linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ has a similar representation.
We'll now proceed to answer this question affirmatively. Let's start with the notion of a basis.

In $\mathbb{R}^n$, the vectors

$$e_1 = (1, 0, \ldots, 0),\ e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$$

play a special role.

Indeed, any vector $(x_1, x_2, \ldots x_n)$ in $\mathbb{R}^n$ can be written as a linear combination of $e_1, \ldots, e_n$:

$$(x_1, \ldots, x_n) = x_1 (1, 0, \ldots, 0) + \cdots + x_n (0, \ldots, 0, 1)$$

$$= x_1 e_1 + x_2 e_2 + \cdots + x_n e_n.$$

we'll see later that $\{e_1, \ldots, e_n\}$ is a basis of $\mathbb{R}^n$.

A simple yet crucial observation:

Since the vectors $\{e_1, \ldots, e_n\}$ span/generate all of $\mathbb{R}^n$, it is enough to understand the action of a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ on the vectors $\{e_1, \ldots, e_n\}$.

More precisely, any element of $\mathbb{R}^n$ can be written as $x_1 e_1 + \cdots + x_n e_n$, for some real numbers $x_1, \ldots, x_n$.

Then, $T(x_1 e_1 + \cdots + x_n e_n)$

$$= x_1 T(e_1) + \cdots + x_n T(e_n).$$

Thus, the action of T on $\{e_1, \ldots, e_n\}$ determines the action of T on all of $\mathbb{R}^n$.

Theorem: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then there exists an $m \times n$ real matrix A such that for any $u = (x_1, \dots, x_n) \in \mathbb{R}^n$, we've

$$Tu = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Proof: Note that $T(e_1), T(e_2), \dots, T(e_n)$ are elements of $\mathbb{R}^m$ and we can think of them as column vectors.
Let A be ~~a~~ the $m \times n$ matrix whose columns ~~are~~ are:

$$T(e_1), \dots, T(e_n);$$

i.e. the i-th column of A is $T(e_i)$.

Now, a direct computation shows that

$$Ae_i = A \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow \text{ith row} = \text{The i-th column of } A.$$

Therefore, we've: $Ae_i = T(e_i)$, for $i = 1, \dots, n$.

By linearity, this yields:

$T(x_1 e_1 + \cdots + x_n e_n)$

$= x_1 T(e_1) + \cdots + x_n T(e_n)$

$= x_1 Ae_1 + \cdots + x_n Ae_n$

$= A(x_1 e_1 + \cdots + x_n e_n).$

$= A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$ ⊠

Remark: The ~~gene~~ matrix A (as above) is called the matrix of the linear map T with respect to $\{e_1, \ldots, e_n\}$.

Composition of linear maps and multiplication of matrices:

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ and $S: \mathbb{R}^m \to \mathbb{R}^k$ be two linear maps with corresponding matrices A and B respectively.

Then the composition
$S \circ T: \mathbb{R}^n \to \mathbb{R}^k$ is a map satisfying

$$
\begin{aligned}
& S \circ T(au+bv) \\
&= S\big(T(au+bv)\big) = S\big(aT(u) + bT(v)\big) \\
&= aS\big(T(u)\big) + bS\big(T(v)\big) \\
&= a(S\circ T)(u) + b(S \circ T)(v),
\end{aligned}
$$

for all $a, b \in \mathbb{R}$, $u, v \in \mathbb{R}^n$.

Thus, the composition $S \circ T$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^k$.

<u>What is the matrix of $S \circ T$?</u>

Recall that $e_1^n = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, e_n^n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

span $\mathbb{R}^n$.

Now, by definition of A, we've:

$$T(e_i^n) = A e_i^n = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix} = a_{1i} e_1^m + \cdots + a_{mi} e_m^m$$

Again, $(S \circ T)(e_i) = S(T(e_i))$

$$= S(a_{1i} e_1^m + \cdots + a_{mi} e_m^m)$$

$$= a_{1i} S(e_1^m) + \cdots + a_{mi} S(e_m^m)$$

$$= a_{1i}\, B\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix} + \cdots + a_{mi}\, B\begin{pmatrix}0\\0\\ \vdots\\0\\1\end{pmatrix}$$ [By definition of B]

$$= a_{1i}\begin{pmatrix}b_{11}\\ \vdots\\ b_{K1}\end{pmatrix} + \cdots + a_{mi}\begin{pmatrix}b_{1m}\\ \vdots\\ b_{Km}\end{pmatrix}$$

$$= \begin{pmatrix} a_{1i}b_{11} + \cdots + a_{mi}b_{1m} \\ \cdots \\ a_{1i}b_{K1} + \cdots + a_{mi}b_{Km}\end{pmatrix}$$

$$= \text{i-th column of } \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \cdots & & \cdots \\ b_{K1} & \cdots & b_{Km}\end{pmatrix}\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & & \cdots \\ a_{m1} & \cdots & a_{mn}\end{pmatrix}$$

$$\Rightarrow (S\circ T)(e_i^n) = \text{i-th column of } BA$$

$$\Rightarrow (S\circ T)(e_i^n) = (BA)(e_i^n) = (BA)\begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix} \to \text{i-th row}$$

Thus, the matrix of $S \circ T : \mathbb{R}^n \to \mathbb{R}^k$ is the $k \times n$ matrix BA.

Hence, the composition of two linear maps is given by the product of the corresponding matrices.

Note: This is the real reason why matrices are multiplied the way they are.

Definition:

- Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear.

Then, $\text{Image}(T) := \{Tu : u \in \mathbb{R}^n\}$.

- $\text{Domain}(T) = \mathbb{R}^n$.
- T is called onto iff

$$\text{Image}(T) = \mathbb{R}^m.$$

- T is called one-to-one iff

$$T(u) = T(v) \Rightarrow u = v.$$

Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be one-to-one and onto. Such a map is called bijective. In this case, there exists an inverse

$T^{-1}: \mathbb{R}^n \to \mathbb{R}^n$ of T. In fact,

$$T \circ T^{-1} = T^{-1} \circ T = Id.$$

Let $T(u) = u'$, $T(v) = v'$.
Then, by definition, $T^{-1}(u') = u$,
$T^{-1}(v') = v$.
Also, by linearity of T,

$$T(au+bv) = aT(u) + bT(v) = au' + bv'.$$

Therefore,

$$T^{-1}(au'+bv')$$
$$= au+bv = aT^{-1}(u') + bT^{-1}(v'),$$
for all $a, b \in \mathbb{R}$, $u', v' \in \mathbb{R}^n$

Hence, $T^{-1}: \mathbb{R}^n \to \mathbb{R}^n$ is also linear.

Let B be the matrix of A.
Since, $T \circ T^{-1} = T^{-1} \circ T = Id$, ~~we have observed~~ and the matrices of $T \circ T^{-1}$, $T^{-1} \circ T$ and Id are AB, BA and Id_n respectively, we've:

$$AB = BA = Id_n,$$ where Id_n is the $n \times n$ identity matrix

Thus, the matrix of T^{-1} is $B = A^{-1}$.

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Problem Set 1

MAT 211: Linear Algebra

Homework Problems

1.1. *Elimination Method.* (10 points)

Solve the following system of linear equations using the method of elimination:

$$\begin{aligned} x + 2y + 3z &= 1, \\ 2x + 4y + 7z &= 2, \\ 3x + 7y + 11z &= 8. \end{aligned}$$

1.2. *Dependence on a Parameter.* (20 points)

Consider the linear system

$$\begin{aligned} x + y - z &= -2, \\ 3x - 5y + 13z &= 18, \\ x - 2y + 5z &= k, \end{aligned}$$

where k is an arbitrary real number.

(a) For which value(s) of k does this system have one or infinitely many solutions?

(b) For each value of k you found in the previous part, how many solutions does the system have?

(c) Find all solutions for each value of k obtained in the first part.

1.3. *Application to Geometry.* (15 points)

Find a, b, and c such that the ellipse $ax^2 + bxy + cy^2 = 1$ passes through the points $(1, 2)$, $(2, 2)$, and $(3, 1)$.

1.4. *Gauss-Jordan.* (10+10 points)

Solve the following systems of linear equations using Gauss-Jordan elimination (i.e. write down the augmented matrix, and put it in RREF):

(a) $3x + 11y + 19z = -2,$
$7x + 23y + 39z = 10,$
$-4x - 3y - 2z = 6.$

(b) $x_1 + 2x_3 + 4x_4 = -8,$
$x_2 - 3x_3 - x_4 = 6,$
$3x_1 + 4x_2 - 6x_3 + 8x_4 = 0,$
$-x_2 + 3x_3 + 4x_4 = -12.$

1.5. *(Bonus problem) Integral Solutions.* (10 points)

Consider the system

$2x + y = C,$

$3y + z = C,$

$x + 4z = C,$

where C is a constant. Find the smallest positive integer C such that x, y, and z are all integers.

Due Date: Wednesday, February 7.

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Problem Set 2

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Homework Problems

Recall that $\mathbb{R}^n$ is the collection of all column vectors (or coordinate vectors) of size n; i.e.

$$\mathbb{R}^n = \{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : x_1, x_2, \cdots, x_n \in \mathbb{R}\}$$

.

The coordinate vectors $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ can be thought of as building blocks of $\mathbb{R}^n$ (the vector e_k has a 1 at the k-th position and 0 everywhere else). Indeed, we can write any coordinate vector $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ as $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1e_1 + x_2e_2 + \cdots + x_ne_n$.

A map $T : \mathbb{R}^n \to \mathbb{R}^m$ is called *linear* if

(a) $T(u + v) = T(u) + T(v)$, for all $u, v \in \mathbb{R}^n$, and

(b) $T(cu) = cT(u)$, for all $c \in \mathbb{R}$ and $u \in \mathbb{R}^n$.

We saw that a map $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear (according to the definition above) if and only if there exists an $m \times n$ matrix A such that $T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = A\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. Moreover, the columns of A are $T(e_1), T(e_2), \cdots$, and $T(e_n)$. The matrix A is called the matrix of the linear map T.

2.1. *Matrix of Linear Maps.* (15+15 points)

(a) Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be defined as $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 9x_1 + 3x_2 - 3x_3 \\ 2x_1 - 9x_2 + x_3 \\ 4x_1 - 9x_2 - 2x_3 \\ 5x_1 + x_2 + 5x_3 \end{bmatrix}$. Is T a linear map? If so, find the matrix of T.

(b) Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - x_2\begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix}$. Is this transformation linear? If so, find its matrix.

2.2. *Orthogonal Projection onto a Line.* (10 points)

Let L be the line in $\mathbb{R}^3$ that consists of all scalar multiples of the vector $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Find the orthogonal projection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto L.

2.3. *Reflection about a Line.* (10 points)

Let L be the line in $\mathbb{R}^3$ that consists of all scalar multiples of the vector $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Find the reflection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ about L.

2.4. *Rotation as a Linear Map.* (10 points)

Find the rotation matrix that transforms $\begin{bmatrix} 0 \\ 5 \end{bmatrix}$ to $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

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Problem Set 3

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Homework Problems

3.1. *Matrix Multiplication.* (10 points)

Compute the following matrix product.

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

3.2. *Commuting Matrices.* (10 points)

Find all 3×3 matrices A such that $AB = BA$, where $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

3.3. *Geometric Interpretation of Matrices.* (5 points)

Find a 2×2 matrix A such that $A^5 = \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix}$.

3.4. *Computing The Inverse of a Matrix.* (10+10 points)

Decide whether the following matrices are invertible. If they are, find the inverse.

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

3.5. *Conditions for Invertibility.* (10 points)

For which values of the constants a, b, and c is the following matrix invertible?

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

3.6. *Invertible Transformations.* (15 points)

Which of the following linear transformations $T : \mathbb{R}^3 \to \mathbb{R}^3$ are invertible? Find the inverse if it exists.

(a) Reflection about a plane,

(b) Orthogonal projection onto a plane,

(c) Scaling by a factor of 5 [i.e., $T(u) = 5u$, for all vectors u in $\mathbb{R}^3$].

3.7. *(Bonus problem) Classifying Linear Transformations of Order Two.* (15 points)

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T^2 = \mathrm{Id}$. Prove that one of the following conditions holds.

- T is the identity transformation,
- T preserves every straight line through the origin (i.e. T maps every straight line through $(0, 0)$ to itself),
- T fixes exactly two straight lines through the origin.

Due Date: Wednesday, February 21.

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Problem Set 4

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Homework Problems

4.1. *Subspace or Not?* (5+5 points)

Decide whether the following subsets of $\mathbb{R}^3$ are linear subspaces.

(a) $V = \{(x, y, z) \in \mathbb{R}^3 : x = 2y + 3z\}$.

(b) $V = \{(x, y, z) \in \mathbb{R}^3 : x \leq y \leq z\}$.

4.2. *Image and Kernel of a Linear Map.* (20 points)

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map such that $T(1, 0, 0) = (1, 2, 4)$, $T(1, 0, 1) = (1, 3, -1)$, $T(1, 1, 1) = (6, 17, -1)$.

(a) Find the matrix of T.

(b) Find a minimal set of generators for the image and kernel of T.

4.3. *Composition of Linear Transformations.* (10 points)

Write the matrix representing a linear transformation that first rotates vectors by 90 degrees counter-clockwise, and then projects them onto the line $y = 2x$.

4.4. *(Bonus Problem) Containment of subspaces.* (10 points)

Let W_1, W_2 and W_3 be linear subspaces of $\mathbb{R}^n$ such that W_1 is contained in $W_2 \cup W_3$. Show that W_1 is either contained in W_2, or contained in W_3.

Due Date: Wednesday, March 7.

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Problem Set 5

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Homework Problems

5.1. *Linear Independence-I* (5+5 points)

Decide whether the following sets of vectors are linearly independent.

(a) $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\3\\6\end{bmatrix}.$

(b) $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}1\\4\\7\\10\end{bmatrix}.$

5.2. *Linear Independence-II.* (10 points)

Suppose that u_1, u_2 and u_3 are linearly independent vectors in $\mathbb{R}^n$. Show that the vectors $u_1 + u_2$, $u_2 + u_3$ and $u_3 + u_1$ are also linearly independent.

5.3. *Coordinates of Vectors.* (10 points)

Show that $\mathcal{B} = \{ \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \}$ is a basis of $\mathbb{R}^3$. What are the coordinates of the vector $\begin{bmatrix}x\\y\\z\end{bmatrix}$ with respect to the ordered basis $\mathcal{B}$?

5.4. *Basis and Dimension.* (10+15 points)

(a) For which value(s) of the constant k do the vectors

$$\begin{bmatrix}1\\0\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\0\\3\end{bmatrix}, \begin{bmatrix}0\\0\\1\\4\end{bmatrix}, \begin{bmatrix}2\\3\\4\\k\end{bmatrix}$$

form a basis of $\mathbb{R}^4$?

(b) Find a basis of the subspace W of $\mathbb{R}^5$ defined below:

$$W = \{ \begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} \in \mathbb{R}^5 : 2x_1 - x_2 + \frac{4}{3}x_3 - x_4 = 0, x_1 + \frac{2}{3}x_3 - x_5 = 0, 9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 = 0 \}.$$

What is the dimension of W?

5.5. *Matrix of a Linear Map.* (10 points)

Find the matrix B of the linear transformation $T(\vec{u}) = A\vec{u}$ with respect to the basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, where

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 2 \\ 3 & -9 & 6 \end{bmatrix},$$

and

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}.$$

5.6. *Finding Basis with Prescribed Properties.* (10 points)

Find a basis $\mathcal{B}$ of $\mathbb{R}^2$ such that the coordinates of the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ with respect to $\mathcal{B}$ are $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ respectively.

Due Date: Wednesday, March 28.

Stony Brook University
Sabyasachi Mukherjee

Spring 2018

Problem Set 6

MAT 211: Linear Algebra

Homework Problems

6.1. *Orthogonal Complement* (8+7 points)

(a) Find a basis of the subspace W of $\mathbb{R}^4$ defined below:

$$W = \{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 0, x_1 - x_2 + x_3 - x_4 = 0, x_1 + 5x_2 + x_3 + 5x_4 = 0\}.$$

What is the dimension of W?

(b) Find a basis of the orthogonal complement of W.

6.2. *Orthogonal Matrix.* (10 points)

Find a, b, c, d such that the matrix $\begin{bmatrix} a & b & 2/3 \\ 1/(3\sqrt{2}) & c & 2/3 \\ -4/(3\sqrt{2}) & 0 & d \end{bmatrix}$ is orthogonal.

6.3. *Gram-Schmidt Orthonormalization.* (8+7 points)

(a) Apply Gram-Schmidt orthonormalization on the vectors $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 18 \\ 0 \\ 0 \end{bmatrix}$.

(b) Using the result of part (a), find the QR factorization of the matrix $\begin{bmatrix} 2 & -2 & 18 \\ 2 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$.

6.4. *Orthogonal Projection.* (10 points)

Find the orthogonal projection of $\begin{bmatrix} 49 \\ 49 \\ 49 \end{bmatrix}$ onto the subspace of $\mathbb{R}^3$ spanned by $\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}$.

Due Date: Wednesday, April 11.

Stony Brook University
Sabyasachi Mukherjee

Spring 2018

Problem Set 7

MAT 211: Linear Algebra

Homework Problems

7.1. *Determinant and Invertibility* (10 points)

Decide whether the following matrix is invertible by computing its determinant:

$$\begin{bmatrix} 2 & 3 & 0 & 2 \\ 4 & 3 & 2 & 1 \\ 6 & 0 & 0 & 3 \\ 7 & 0 & 0 & 4 \end{bmatrix}.$$

7.2. *Determinant of Orthogonal Matrices.* (5 points)

If A is an orthogonal matrix, what are the possible values of $\det(A)$?

7.3. *Determinants of a Special Type of Matrices.* (10 points)

Let P_n be the $n \times n$ matrix whose entries are all ones, except for zeros directly below the main diagonal; for example,

$$P_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

Find the determinant of P_n.

7.4. *Eigenvector of The Square of a Matrix.* (10 points)

Let $\vec{v}$ be an eigenvector of a matrix A with associated eigenvalue λ. Show that $\vec{v}$ is an eigenvector of A^2 as well. What is the corresponding eigenvalue?

7.5. *Finding Eigenvalues.* (15 points)

Find all eigenvalues of the matrix

$$\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}.$$

7.6. *(Bonus Problem) Invariant Line.* (10 points)

Let A be a 3×3 matrix of real numbers. Show that there exists a line L (in $\mathbb{R}^3$) passing through the origin such that $A(L) \subset L$.

Due Date: Wednesday, April 25.

Stony Brook University
Sabyasachi Mukherjee

Spring 2018

Problem Set 8

MAT 211: Linear Algebra

Homework Problems

8.1. *Diagonalizable or Not?* (15+5 points)

(a) Find the eigenvalues and corresponding eigenspaces of the matrix

$$A = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 0 & 1 \\ -4 & 0 & 3 \end{bmatrix}.$$

(b) Write down the algebraic and geometric multiplicities of the eigenvalues of A. Is the matrix A diagonalizable?

8.2. *Diagonalization and Its Applications.* (15+5+5+10+10 points)

(a) Find the eigenvalues and corresponding eigenspaces of the matrix

$$A = \begin{bmatrix} 0 & 0 & 1/4 \\ 1 & 0 & -13/8 \\ 0 & 1 & 11/4 \end{bmatrix}.$$

(b) Write down the algebraic and geometric multiplicities of the eigenvalues, and argue that A is diagonalizable.

(c) Write down a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of $\mathbb{R}^3$ consisting of eigenvectors of A. Using this, find an invertible matrix S such that $S^{-1}AS$ is a diagonal matrix.

(d) Find the coordinates of $\begin{bmatrix} 5 \\ -20 \\ 14 \end{bmatrix}$ with respect to the basis $\mathcal{B}$; i.e. write $\begin{bmatrix} 5 \\ -20 \\ 14 \end{bmatrix}$ as a linear combination of $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$.

(e) Compute $A^{1000} \begin{bmatrix} 5 \\ -20 \\ 14 \end{bmatrix}$.

8.3. *(Bonus question) Dynamics of Linear Maps* (10 points)

Let A be an $n \times n$ diagonalizable matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$ such that $|\lambda_1| > |\lambda_i|$, for $i = 2, \cdots, n$. Moreover, assume that $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis of $\mathbb{R}^n$ consisting of eigenvectors of A with $A(\vec{v}_i) = \lambda_i \vec{v}_i$, for $i = 1, \cdots, n$. Finally, let $\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$, for some scalars $c_1, \cdots, c_n$. Prove that $\frac{1}{\lambda_1^n} A^n(\vec{v})$ converges to the vector $c_1\vec{v}_1$ as $n \to +\infty$.

Due Date: Wednesday, May 02.

Practice Midterm 1

Problem 1. Solve the following systems using augmented matrices. State whether the solution is unique, there are no solutions, or whether there are infinitely many. If the solution is unique give it. If there infinitely many give the solution parametrically, namely in terms of the free variables.

$$\begin{cases} x_1 \qquad\quad - x_3 = 8 \\ 2x_1 + 2x_2 + 9x_3 = 7 \\ \qquad\quad x_2 + 5x_3 = -2 \end{cases}$$

$$\begin{cases} 3x_1 - \ \ 4x_2 + 2x_3 = 0 \\ -9x_1 + 12x_2 - 6x_3 = 0 \\ -6x_1 + \ \ 8x_2 - 4x_3 = 0 \end{cases}$$

Problem 2. Discuss the number of solutions of the following systems depending on the real parameter k. Moreover when the solution is unique, or there are infinitely many solutions, write all the solutions in parametric form.

$$\begin{bmatrix} x_1 & + & 2x_2 & - & x_3 & + & kx_4 & = & 1 \\ -2x_1 & + & x_2 & + & 2x_3 & - & x_4 & = & 2 \\ 4x_1 & + & 3x_2 & - & 4x_3 & + & 3x_4 & = & 0 \end{bmatrix}$$

$$\begin{bmatrix} & & y & + & z & = & k \\ x & & & + & z & = & k \\ x & + & y & & & = & k \end{bmatrix}$$

Problem 3. Say for which values of the real parameter a the following matrix is invertible:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & a & 2 \\ 0 & 0 & a^2 - 3a \end{pmatrix}.$$

Then set $a = 1$ and find the inverse.

Problem 4. a) Write the matrix representing a linear transformation that rotates vectors of $\mathbf{R}^2$ by 30 degrees counterclockwise.

b) Write the matrix representing a linear transformation that reflects vectors of $\mathbf{R}^2$ about the line $y = 2x$.

c) Write the matrix representing a linear transformation that first rotates vectors by 30 degrees counterclockwise, and then reflects them about the line $y = 2x$.

d) Find the vector obtained by first reflecting $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ about the line $y = 2x$, and then rotating it by 30 degrees counterclockwise.

Problem 5. Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be the orthogonal projection onto the line $x - 2y = 0$ followed by a counterclockwise rotation by 45 degrees. Find the matrix A that represents T. Is A invertible? Show on a picture the kernel and the image of T.

Problem 6. (Orthogonal projection in $\mathbf{R}^3$.) Recall that the orthogonal projection of a vector $\vec{x}$ in $\mathbf{R}^3$ onto a line L of $\mathbf{R}^3$ is defined as $\mathrm{proj}_L(\vec{x}) = (\vec{x} \cdot \vec{u})\,\vec{u}$, where $\vec{u}$ is a unit vector parallel to L. Alternatively, if instead of a unit vector $\vec{u}$ we have an arbitrary non-zero vector $\vec{w}$ parallel to L, the projection of $\vec{x}$ onto L is defined as

$$\mathrm{proj}_L(\vec{x}) \ = \ \left(\frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\right)\vec{w}.$$

Let now $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ be the orthogonal projection onto the line L spanned by the vector $\vec{w} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$.

a) Write the matrix A that represents T.

b) Find the orthogonal projection $\vec{r}$ of the vector $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ onto L.

c) Find all vectors in $\mathbf{R}^3$ that are perpendicular to $\vec{w}$ and $\vec{r}$. Write them in parametric form (namely in terms of free variables).

Problem 7. (Orthogonal Projections onto a plane of $\mathbf{R}^3$.) The orthogonal projection $\mathrm{proj}_V(\vec{x})$ of a vector $\vec{x}$ in $\mathbf{R}^3$ onto a plane V in $\mathbf{R}^3$ of equation $ax_1 + bx_2 + cx_3 = 0$ is given by the formula:

$$\mathrm{proj}_V(\vec{x}) \;=\; \vec{x} - \left(\frac{\vec{x}\cdot\vec{r}}{\vec{r}\cdot\vec{r}}\right)\vec{r}, \qquad \text{where } \vec{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Note that the 'dot' in the previous formula denotes the dot product of vectors in $\mathbf{R}^3$.
a) Write the matrix that represents the linear transformation proj_V.
b) Find the orthogonal projection of $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ onto the plane $x_2 - x_1 + x_3 = 0$ in $\mathbf{R}^3$.

Problem 8. Consider the following linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} \;=\; \begin{pmatrix} x+y-z \\ -y+z \\ -2x-2y+2z \end{pmatrix}.$$

a) Find the matrix A that represents T.
b) Write the kernel of T as a span of a minimal set of generators.
c) Write the image of T as a span of a minimal set of generators.

Problem 9. Consider the following matrix:

$$A = \begin{pmatrix} 2 & -2 & -4 \\ -3 & -5 & -2 \\ 4 & -2 & -6 \end{pmatrix}.$$

a) Write the image of A as a span of a minimal set of generators.
b) Write the kernel of A as a span of a minimal set of generators.

Problem 10. Consider the following matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 & 3 \\ 1 & 2 & 4 & 3 \\ 1 & 2 & 5 & 3 \end{pmatrix}.$$

a) Write the image of A as a span of a minimal set of generators.
b) Write the kernel of A as a span of a minimal set of generators.

Problem 11. (Rotations in $\mathbf{R}^3$.) Consider $\mathbf{R}^3$ with coordinates (x, y, z). The matrix $R_x(\theta)$ that represents the linear transformation $T_{x,\theta}\colon \mathbf{R}^3 \to \mathbf{R}^3$ rotating vectors in $\mathbf{R}^3$ by θ degrees counterclockwise about the x-axis is:

$$R_x(\theta) \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Similarly we can define $R_y(\theta)$ and $R_z(\theta)$ which are the matrices that rotate vectors by θ degrees counterclockwise about the y- and z-axis, respectively:

$$R_y(\theta) \;=\; \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}, \qquad R_z(\theta) \;=\; \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

a) Find the vector obtained by rotating $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ first by 90 degrees counterclockwise about the x-axis, and then by rotating it by 180 degrees counterclockwise about the z-axis.

b) Find the vector obtained by rotating $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ first by 30 degrees counterclockwise about the y-axis, and then by rotating it by 45 degrees counterclockwise about the z-axis.

MAT 211: INTRODUCTION TO LINEAR ALGEBRA

Answer Keys to the Practice Midterm 1

If you find any mistake in the following answer keys, please do let me know via email. The instructor is not responsible of any possible mistake in these notes.

Problem 1: a) $x = 3, \quad y = 23, \quad z = -5.$

b)

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 4/3 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2/3 \\ 0 \\ 1 \end{pmatrix}$$

where t and s are free variables. You can also say that the space of solutions is the span of $\begin{pmatrix} 4/3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2/3 \\ 0 \\ 1 \end{pmatrix}$, which is a plane in $\mathbf{R}^3$.

Problem 2: a) If $k = 1$ the solutions are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -3/5 \\ -1/5 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -3/5 \\ 4/5 \\ 0 \\ 0 \end{pmatrix}$$

where t and s are free variables. In this case there are ∞^2-many solutions. The solutions form a plane in $\mathbf{R}^4$ (not passing thorough the origin).

If $k \neq 1$, the solutions are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -3/5 \\ 4/5 \\ 0 \\ 0 \end{pmatrix}$$

where t is a free variable. In this case there are ∞^1-many solutions. The solutions form a line, not passing through the origin, in $\mathbf{R}^4$.

b) For any value of k there is only one solution $x = y = z = k/2$.

Problem 3:The matrix A is not invertible only when either $a = 0$ or $a = 3$. If $a = 1$, the inverse of A is

$$\begin{pmatrix} 1 & -2 & -1/2 \\ 0 & 1 & 1 \\ 0 & 0 & -1/2 \end{pmatrix}$$

Problem 4: a) $\begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$

b) $\begin{pmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{pmatrix}$

c) $\begin{pmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} -3\sqrt{3} + 4 & 3 + 4\sqrt{3} \\ 4\sqrt{3} + 3 & -4 + 3\sqrt{3} \end{pmatrix}.$

d) $\frac{1}{10} \begin{pmatrix} \sqrt{3} - 7 \\ 7\sqrt{3} + 1 \end{pmatrix}$

Problem 5:

$A = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \begin{pmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} \sqrt{2} & \sqrt{2}/2 \\ 3\sqrt{2} & 3\sqrt{2}/2 \end{pmatrix}$. The matrix A is not invertible. $\mathrm{Ker}(T) = \mathrm{span}\{\begin{pmatrix} -1 \\ 2 \end{pmatrix}\}$. $\mathrm{Im}(T) = \mathrm{span}\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}\}$.

Problem 6:

a)

$$A = \frac{1}{9}\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix}.$$

b) $\vec{r} = \frac{2}{3}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$.

c) All vectors of the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t\begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

where t and s are free variables are perpendicular to both $\vec{r}$ and $\vec{w}$. Therefore the vectors perpendicular to both $\vec{r}$ and $\vec{w}$ form a plane in $\mathbf{R}^3$.

Problem 7:

a)

$$\frac{1}{a^2+b^2+c^2}\begin{pmatrix} b^2+c^2 & -ab & -ac \\ -ab & a^2+c^2 & -bc \\ -ac & -bc & a^2+b^2 \end{pmatrix}.$$

b) $\frac{1}{3}\begin{pmatrix} 7 \\ 2 \\ 5 \end{pmatrix}$

Problem 8: a)

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -2 & -2 & 2 \end{pmatrix}.$$

b) $\mathrm{Im}(T) = \mathrm{span}\{\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}\}$.

c) $\mathrm{Ker}(T) = \mathrm{span}\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\}$.

Problem 9: a) $\mathrm{Im}(A) = \mathrm{span}\{\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ -5 \\ -2 \end{pmatrix}\}$.

b) $\mathrm{Ker}(A) = \mathrm{span}\{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\}$.

Problem 10: a) $\mathrm{Im}(A) = \mathrm{span}\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}\}$.

b) $\mathrm{Ker}(A) = \mathrm{span}\{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \end{pmatrix}\}$.

Problem 11: a)

$$R_z(180)R_x(90)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

b)

$$R_z(45)R_y(30)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} \sqrt{6}-\sqrt{2} \\ \sqrt{6}+3\sqrt{2} \\ 2\sqrt{3}-2 \end{pmatrix}.$$

Practice Problems for Midterm II

Problem 1: Find a basis $\mathcal{B}$ for the following subspace of $\mathbf{R}^4$

$$U = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\}.$$

Find the dimension of U. Find the $\mathcal{B}$-coordinates of the vector $\vec{w} = \begin{pmatrix} 7 \\ 3 \\ 3 \\ 3 \end{pmatrix}$.

Problem 2: Let $(\vec{u}, \vec{v}, \vec{w})$ be a basis of $\mathbf{R}^3$. Say for which values of the real parameter k the following vectors form a basis of $\mathbf{R}^3$:

$$\vec{u} + \vec{v} + \vec{w}, \qquad \vec{u} - \vec{v} + \vec{w}, \qquad \vec{u} + k\vec{v} + k^2\vec{w}.$$

Problem 3: Find a basis of the subspace U in $\mathbf{R}^4$ defined by the equations $x_1 + 2x_2 - 3x_3 + x_4 = 0$ and $2x_1 - x_3 - 2x_4 = 0$. Find moreover a basis of the orthogonal complement $U^\perp$ of U (in other words find a basis of the subspace of $\mathbf{R}^4$ consisting of all vectors perpendicular to U).

Problem 4:

Let T be the linear operator on R^3 defined by

$$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3).$$

(a) What is the matrix of T in the standard ordered basis for R^3?

(b) What is the matrix of T in the ordered basis

$$\{\alpha_1, \alpha_2, \alpha_3\}$$

where $\alpha_1 = (1, 0, 1)$, $\alpha_2 = (-1, 2, 1)$, and $\alpha_3 = (2, 1, 1)$?

Problem 5. **(1)**. Find an orthonormal basis for the plane in $\mathbf{R}^4$ spanned by the vectors $(1, 1, 1, 1)$ and $(1, 9, -5, 3)$. **(2)**. Find an orthonormal basis of $\mathbf{R}^3$ starting from the vectors $(1, 1, 1)$, $(1, 0, 1)$ and $(0, 1, -1)$. **(3)**. Find an orthonormal basis for the plane in $\mathbf{R}^3$ defined by $x + y + z = 0$ (find first a basis for the plane).

Problem 6:

Find an orthogonal matrix of the form

$$\begin{bmatrix} 2/3 & 1/\sqrt{2} & a \\ 2/3 & -1/\sqrt{2} & b \\ 1/3 & 0 & c \end{bmatrix}.$$

Problem 7:

Find the orthogonal projection of

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

onto the subspace of $\mathbb{R}^4$ spanned by

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

1) $$U = \text{span}\left\{ \underset{c_1}{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}}, \underset{c_2}{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}}, \underset{c_3}{\begin{pmatrix}5\\2\\2\\2\end{pmatrix}}, \underset{c_4}{\begin{pmatrix}1\\2\\3\\4\end{pmatrix}} \right\}.$$

To find a basis for U, we need to find redundant vectors amongst the four vectors above. To do so, we'll reduce the following matrix to its RREF.

$$\begin{pmatrix} 1 & 1 & 5 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 4 \end{pmatrix}$$

$$\xrightarrow[\substack{R_1 - R_2 \mapsto R_1 \\ R_4 - R_2 \mapsto R_4}]{R_3 - R_2 \mapsto R_3} \begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$\xrightarrow[\substack{R_2 - 2R_3 \mapsto R_2 \\ R_4 - 2R_3 \mapsto R_4}]{R_1 + R_3 \mapsto R_1} \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The RREF ~~of~~ has three independent columns (first, second and fourth), and one redundant column (third).

$\boxed{\text{In fact, } C_3 = 3C_1 + 2C_2}$

Therefore, the columns $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\\4\end{pmatrix}$

are linearly independent and they span U.

Thus, a basis of U is

$$\beta = \left\{ \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\\4\end{pmatrix} \right\}.$$

So, $\dim(U) = 3$.

Let, $$\begin{pmatrix}7\\3\\3\\3\end{pmatrix} = a_1\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + a_2\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + a_3\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$$

(3)

$$\Rightarrow \begin{pmatrix} 7 \\ 3 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 + a_3 \\ a_2 + 2a_3 \\ a_2 + 3a_3 \\ a_2 + 4a_3 \end{pmatrix}$$

$\Rightarrow a_3 = 0, \quad a_2 = 3, \quad a_1 = 4$

So, $\begin{pmatrix} 7 \\ 3 \\ 3 \\ 3 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$.

So, the β-coordinates of $\begin{pmatrix} 7 \\ 3 \\ 3 \\ 3 \end{pmatrix}$

is $\begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$.

2) $\{\vec{u}, \vec{v}, \vec{w}\}$ is a basis of $\mathbb{R}^3$.

Let $a_1, a_2, a_3 \in \mathbb{R}$ be such that

$$a_1(\vec{u}+\vec{v}+\vec{w}) + a_2(\vec{u}-\vec{v}+\vec{w}) + a_3(\vec{u}+k\vec{v}+k^2\vec{w}) = \vec{0}$$

$$\Rightarrow (a_1+a_2+a_3)\vec{u} + (a_1-a_2+ka_3)\vec{v} + (a_1+a_2+k^2a_3)\vec{w} = \vec{0} \rightarrow (1)$$

Since $\{\vec{u}, \vec{v}, \vec{w}\}$ is a basis of $\mathbb{R}^3$, they are linearly independent. So, (1) implies that:

$$a_1 + a_2 + a_3 = 0$$

$$a_1 - a_2 + ka_3 = 0$$

$$a_1 + a_2 + k^2 a_3 = 0$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & k \\ 1 & 1 & k^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow (2)$$

By definition of linear independence, the vectors $\{\vec{u}+\vec{v}+\vec{w},\ \vec{u}-\vec{v}+\vec{w},\ \vec{u}+k\vec{v}+k^2\vec{w}\}$ are linearly independent if and only if

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

But this means that the vectors are L.I. iff $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is the only solution of (2). ~~Hence, the~~; i.e the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & k \\ 1 & 1 & k^2 \end{pmatrix}$$ is invertible

Now,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & k \\ 1 & 1 & k^2 \end{pmatrix}$$

$$\xrightarrow[R_3 - R_1 \mapsto R_3]{R_2 - R_1 \mapsto R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & k-1 \\ 0 & 0 & k^2-1 \end{pmatrix}$$

$$\xrightarrow{\left(-\frac{1}{2}\right)R_2 \mapsto R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{1-k}{2} \\ 0 & 0 & k^2-1 \end{pmatrix}$$

$$\xrightarrow{R_1 - R_2 \mapsto R_1} \begin{pmatrix} 1 & 0 & \frac{k+1}{2} \\ 0 & 1 & \frac{1-k}{2} \\ 0 & 0 & k^2-1 \end{pmatrix}$$

It's now easy to see that the RREF of $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & k \\ 1 & 1 & k^2 \end{pmatrix}$ is the identity matrix if and only if $\underline{k^2-1 \neq 0}$.

Thus, $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & k \\ 1 & 1 & k^2 \end{pmatrix}$ is invertible

if and only if $\underline{k^2 \neq 1}$.

Hence, $\{\vec{u}+\vec{v}+\vec{w},\ \vec{u}-\vec{v}+\vec{w},\ \vec{u}+k\vec{v}+k^2\vec{w}\}$ is a linearly independent set iff

$$\underline{k^2 \neq 1} \ ; \text{ i.e. } \underline{k \neq \pm 1}.$$

Therefore, the vectors are linearly independent precisely when $k \neq \pm 1$.

---

3)

$$U = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 : \begin{array}{l} x_1 + 2x_2 - 3x_3 + x_4 = 0, \\ 2x_1 - x_3 - 2x_4 = 0 \end{array} \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 : \begin{pmatrix} 1 & 2 & -3 & 1 \\ 2 & 0 & -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

To find solutions of

$$\begin{pmatrix} 1 & 2 & -3 & 1 \\ 2 & 0 & -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

we reduce the following augmented matrix to its RREF.

$$\left(\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 0 \\ 2 & 0 & -1 & -2 & 0 \end{array}\right)$$

$$\xrightarrow{R_2 - 2R_1 \mapsto R_2} \left(\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 0 \\ 0 & -4 & 5 & -4 & 0 \end{array}\right)$$

$$\xrightarrow{(R_2)\times(-\frac{1}{4}) \mapsto R_2} \left(\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 0 \\ 0 & 1 & -\frac{5}{4} & 1 & 0 \end{array}\right)$$

$$\xrightarrow{R_1 - 2R_2 \mapsto R_1} \left(\begin{array}{cccc|c} 1 & 0 & -\frac{1}{2} & -1 & 0 \\ 0 & 1 & -\frac{5}{4} & 1 & 0 \end{array}\right)$$

It follows that x_3 and x_4 are free variables (as they don't have leading ones).

We set $x_3 = s$, $x_4 = t$.

Then, $x_1 - \frac{x_3}{2} - x_4 = 0$

$$\Rightarrow x_1 = \frac{s}{2} + t, \quad \text{and}$$

$$x_2 - \frac{5x_3}{4} + x_4 = 0$$

$$\Rightarrow x_2 = \frac{5s}{4} - t$$

So, $U = \left\{ \begin{pmatrix} \frac{s}{2} + t \\ \frac{5s}{4} - t \\ s \\ t \end{pmatrix} : s, t \in \mathbb{R} \right\}$

$$= \left\{ s \begin{pmatrix} 1/2 \\ 5/4 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} : s, t \in \mathbb{R} \right\}$$

$$= \text{Span}\left\{ \begin{pmatrix} 1/2 \\ 5/4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

The two vectors above are clearly linearly independent.

So, a basis of U is

$$\beta = \left\{ \begin{pmatrix} 1/2 \\ 5/4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

By definition,

$U^{\perp}$ = The orthogonal complement of U

$= \{ \vec{v} \in \mathbb{R}^4 : \vec{v} \cdot \vec{u} = 0 \text{ for all } \vec{u} \in U \}$

" = Vectors that are orthogonal to all vectors in U "

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \begin{pmatrix} 1/2 \\ 5/4 \\ 1 \\ 0 \end{pmatrix} = 0, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} = 0 \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 : \begin{array}{l} \frac{x_1}{2} + \frac{5x_2}{4} + x_3 = 0 \\ x_1 - x_2 + x_4 = 0 \end{array} \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 : \begin{array}{l} x_1 - x_2 + x_4 = 0, \\ 2x_1 + 5x_2 + 4x_3 = 0 \end{array} \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 : \begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & 5 & 4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

To find a basis of $U^{\perp}$, we solve the above system of equations.

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 2 & 5 & 4 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_2 - 2R_1 \mapsto R_2} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 0 & 7 & 4 & -2 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{7}R_2 \mapsto R_2} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & \frac{4}{7} & -\frac{2}{7} & 0 \end{array}\right)$$

$$\xrightarrow{R_1 + R_2 \mapsto R_1} \left(\begin{array}{cccc|c} 1 & 0 & 4/7 & 5/7 & 0 \\ 0 & 1 & \frac{4}{7} & -\frac{2}{7} & 0 \end{array}\right)$$

Again, x_3 and x_4 are free variables.
so, we set $x_3 = s$, $x_4 = t$.

Now,

$$x_1 + \frac{4}{7}x_3 + \frac{5}{7}x_4 = 0$$

$$\Rightarrow x_1 = -\frac{4}{7}s - \frac{5}{7}t,$$

and $$x_2 + \frac{4}{7}x_3 - \frac{2}{7}x_4 = 0$$

$$\Rightarrow x_2 = -\frac{4}{7}s + \frac{2}{7}t.$$

Hence, $$U^{\perp} = \left\{ \begin{pmatrix} -\frac{4}{7}s - \frac{5}{7}t \\ -\frac{4}{7}s + \frac{2}{7}t \\ s \\ t \end{pmatrix} : s, t \in \mathbb{R} \right\}$$

$$= \left\{ s\begin{pmatrix} -4/7 \\ -4/7 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -5/7 \\ 2/7 \\ 0 \\ 1 \end{pmatrix} : s, t \in \mathbb{R} \right\}$$

$$= \text{span}\left\{ \begin{pmatrix} -4/7 \\ -4/7 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5/7 \\ 2/7 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Hence, a basis of $U^{\perp}$ is $\left\{ \begin{pmatrix} -4/7 \\ -4/7 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5/7 \\ 2/7 \\ 0 \\ 1 \end{pmatrix} \right\}$.

6) A matrix M is orthogonal if and only if its column vectors form an orthonormal set.

We have

$$M = \begin{pmatrix} 2/3 & 1/\sqrt{2} & a \\ 2/3 & -1/\sqrt{2} & b \\ 1/3 & 0 & c \end{pmatrix}.$$

Since M is orthogonal, the set of column vectors

$$\left\{ \begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\}$$

is an orthonormal set.

Hence, $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 1$

$\Rightarrow \underline{a^2 + b^2 + c^2 = 1} \rightarrow (i)$

$$\begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$\Rightarrow \frac{2a}{3} + \frac{2b}{3} + \frac{c}{3} = 0$$

$$\Rightarrow 2a + 2b + c = 0$$

$$\Rightarrow \underline{c = -2(a+b)} \rightarrow \text{(ii)}$$

and,

$$\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$\Rightarrow \frac{a}{\sqrt{2}} - \frac{b}{\sqrt{2}} = 0$$

$$\Rightarrow \underline{a = b} \rightarrow \text{(iii)}$$

putting (iii) in (ii), we get

$$c = -2(2a) = -4a$$

$$\Rightarrow \underline{c = -4a} \rightarrow \text{(iv)}$$

Putting (iii) and (iv) in (i), we get

$$a^2 + a^2 + (-4a)^2 = 1$$

$$\Rightarrow 18a^2 = 1$$

$$\Rightarrow a^2 = \frac{1}{18} \Rightarrow a = \pm \frac{1}{3\sqrt{2}}.$$

Let us choose $a = \frac{1}{3\sqrt{2}}$.

Then, $b = a = \frac{1}{3\sqrt{2}}$, and

$$c = -4a = \frac{-4}{3\sqrt{2}}.$$

Therefore,
$$\begin{cases} a = \frac{1}{3\sqrt{2}}, \\ b = \frac{1}{3\sqrt{2}}, \\ c = -\frac{4}{3\sqrt{2}} \end{cases}$$

makes the matrix orthogonal. $\otimes$

7) Define

$$W = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

Note that the three vectors spanning W are mutually orthogonal. We can find an orthonormal basis of W simply by turning each of them into a unit vector. Thus, an orthonormal basis of W is given by

$$\left\{ \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{pmatrix} \right\}.$$

By the formula for ortho. projection, we've

$$\text{Proj}_W \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \right) \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} +$$

$$\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{pmatrix} \right) \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{pmatrix} +$$

$$\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{pmatrix} \right) \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ \frac{1}{2} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1/2 \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 \\ 1/4 \\ -1/4 \\ \frac{1}{4} \end{pmatrix} .$$

X

4) $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined as

a) $T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} 3x_1 + x_3 \\ -2x_1 + x_2 \\ -x_1 + 2x_2 + 4x_3 \end{pmatrix}$

So, $T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (-2)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + (-1)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$,

$T\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$,

$T\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

So, the matrix of T in the standard basis is

$$A = \begin{pmatrix} 3 & 0 & 1 \\ -2 & 1 & 0 \\ -1 & 2 & 4 \end{pmatrix}.$$

b). The change of basis matrix of $\{\alpha_1, \alpha_2, \alpha_3\}$ in terms of the standard basis is

$$S = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Hence, the matrix of T wrt the basis $\{\alpha_1, \alpha_2, \alpha_3\}$ is $\underline{S^{-1}AS}$. ⊗ (finish the computation)