MAT 211: Introduction to Linear Algebra (Lecture 01)

Fall 2017

Important: This webpage is intended solely to students of MAT 211 Lecture 01. All other students of MAT 211 should contact their own instructor for further information regarding the course.

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Lecture (location and time):
Location: Library E4320
Time: Monday, Wednesday, Friday 12:00pm - 12:53pm

Textbook:
(Required) Otto Bretscher "Linear Algebra with Applications", 5th or 4th edition, Pearson Prentice Hall
(Optional) Student solutions manual for Linear Algebra with Applications, Otto Bretscher, 5th edition

Course Description:
Introduction to the theory of linear algebra with some interesting applications; vectors, vector spaces, bases and dimension, applications to geometry, linear transformations and rank, eigenvalues and eigenvectors, determinants and inner products. It may not be taken for credit in addition to AMS 210.

Homework:
Homework will be assigned every week and posted in BlackBoard. Late assignments cannot be accepted. Homework that appears to be copied from someone else will receive a grade of 0 and may result in charges of academic dishonesty.

Midterms and Final Exam:
Midterm 1: October 6th, in class
Midterm 2: November 10th, in class

Final Exam: December 13th, 11:15am - 1:45pm, Room: Engineering 143

Course Grading:
20% Midterm 1
20% Midterm 2
40% Final Exam
20% Homework

Academic Integrity:
Each student must pursue his or her goals honestly and be personally accountable for all submitted work. Representing another person's work as your own is always wrong. Instructors are required to report any suspected instances of academic dishonesty to the Academic Judiciary. For more comprehensive information on academic integrity, including categories of

academic dishonesty, see the academic judiciary web site at
http://www.stonybrook.edu/cinncms/academic-integrity/index.html

Homework 1 – Due Date: Wednesday September 6th in class

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Reading: Section 1.1 of the textbook. For any possible question about the homework please visit me during my office hours, or write to me an email.

Problem 1: For each of the following system, write the associated matrix, use the elimination process to find the solutions of the system, and specify whether there are no solutions, one solution, or infinitely many solutions.

$$\begin{cases} x + 2y = 1 \\ 2x + 3y = 1 \end{cases} \qquad \begin{cases} x + 2y + 3z = 8 \\ x + 3y + 3z = 10 \\ x + 2y + 4z = 9 \end{cases} \qquad \begin{cases} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 0 \\ 7x + 8y + 10z = 0 \end{cases}$$

Problem 2: Find the solutions of the following upper triangular system with 3 equation and 5 variables. Note that here there is no need to perform the elimination process!

$$\begin{cases} x_1 + 2x_2 - x_3 + 4x_5 = -3 \\ x_2 + 3x_3 + 7x_4 = 5 \\ x_4 - x_5 = 2 \end{cases}$$

Problem 3: Find the solutions of the following 2×2 systems. Then represent your solutions graphically, as intersection of lines in the x-y-plane.

$$\begin{cases} x - 2y = 3 \\ 2x - 4y = 8 \end{cases} \qquad \begin{cases} x - 2y = 3 \\ 4x - 8y = 16 \end{cases}$$

Problem 4: Consider the following linear system:

$$\begin{cases} x + y - z = -2 \\ 3x - 5y + 13z = 18 \\ x - 2y + 5z = k \end{cases}$$

where k is an arbitrary real number. Write the associated matrix of the system. Start performing the elimination process, and say then for which value(s) of k does this system have one, none, or infinitely many solutions. Moreover write down all the solutions of the system according to the different values of k.

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Homework 2 – Due date 09/13/17

Reading: Sections 1.2 and 1.3 of the textbook.

Problems: *Textbook Edition 5.* Section 1.2: problems 1, 5, 6, 7, 18, 32, 47
Section 1.3: problems 1, 2, 27

Textbook Edition 4. Section 1.2: problems 1, 5, 6, 7, 18, 30, 45
Section 1.3: problems 1, 2, 27

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Homework 3 – Due date: Friday Sept. 22nd, 2017

Reading: Appendix A: vectors, operations on vectors, dot product, length, perpendicular vectors
Section 2.3: Matrix multiplication, Theorem 2.3.3 and Theorem 2.3.4
Section 2.4: Inverse Matrix, Theorem 2.4.3 and Theorem 2.4.5

Problem 1: Solve the following problems of the textbook.

Edition 5. Section **1.2**: problems 36, 37
Section **2.3**: problems 2, 7, 11, 12, 35, 65
Section **2.4**: problems 3, 6, 11, 29

Edition 4. Section **1.2**: problems 34, 35
Section **2.3**: problems 4, 7, 10, 11, 35, 65
Section **2.4**: problems 3, 4, 11, 29

Homework 4 – Due Date: Friday September 29th in class

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Edition 5. **Section 1.3**: problem 9
Section 2.1: problems 1, 2, 3, 4, 5, 6, 7, 14a, 16, 20, 22, 32, 43a, 43b, 44, 46.
True or False at page 107: questions 3, 5, 7, 9, 10, 11.

Edition 4. **Section 1.3**: problem 9.
Section 2.1: problems 1, 2, 3, 4, 5, 6, 7, 14a, 16, 20, 22, 32, 43a, 43b, 44, 46.
True or False at page 98: questions 3, 5, 6, 9, 19, 26.

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Homework 5 – Due date 10/13/17

Read: Section 3.1. Section 1.3 (only Definition 1.3.9 and Example 13).

Important definitions. I recall here some important definitions we have seen in class.

i). A vector $\vec{v}$ in $\mathbf{R}^n$ is a linear combination of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ in $\mathbf{R}^n$ if there exist scalars $c_1, c_2, \ldots, c_k$ such that

$$\vec{v} \;=\; c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k.$$

ii). We say that a vector $\vec{v}_i$ in a set of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ is redundant if $\vec{v}_i$ is a linear combination of the other vectors (namely of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_k$). We have seen in class that in order to understand which vectors are non-redundant we need to use the Gauss–Jordan elimination.

Problem 1: i). Write the vector $\begin{pmatrix}7\\8\\9\end{pmatrix}$ as a linear combination of $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\begin{pmatrix}4\\5\\6\end{pmatrix}$, if possible.

ii). Write the vector $\begin{pmatrix}7\\8\\9\end{pmatrix}$ as linear combination of $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\begin{pmatrix}2\\4\\6\end{pmatrix}$, if possible.

Problem 2: In the following sets of vectors, say which vectors are non-redundant. Moreover write the redundant vectors as linear combination of the others.

$$(i).\quad \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\0\\-1\end{pmatrix} \qquad\qquad (ii).\quad \begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}2\\-1\\4\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$(iii).\quad \begin{pmatrix}1\\2\\-1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\-1\\1\\1\end{pmatrix}.$$

Problem 3: Consider the following subset $U = \operatorname{span}\{\begin{pmatrix}1\\2\\-1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\2\\-1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\4\\-2\\1\\1\end{pmatrix}\}$ of $\mathbf{R}^5$.

i). Write U as a span of a minimal set of generators.

ii). Say whether the vector $\begin{pmatrix}0\\2\\2\\0\\1\end{pmatrix}$ is in U.

Problem 4: Consider the following vectors $\vec{v}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix}0\\2\\3\end{pmatrix}$ in $\mathbf{R}^3$.

i). Show that $\vec{v}_1$ and $\vec{v}_2$ are non-redundant vectors.

ii). Say whether the vector $\vec{v} = \begin{pmatrix}3\\7\\9\end{pmatrix}$ is in $\operatorname{span}\{\vec{v}_1, \vec{v}_2\}$.

iii). Choose a vector $\vec{v}_3$ in $\mathbf{R}^3$ so that the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are non-redundant.

iv). Write $\begin{pmatrix}1\\0\\0\end{pmatrix}$ as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

v). Write $\begin{pmatrix}0\\1\\0\end{pmatrix}$ as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

Problem 5: i). Write three vectors in $\mathbf{R}^3$ such that their span consists of a line passing through the origin.

ii). Write three vectors in $\mathbf{R}^3$ such that their span consists of a plane passing through the origin.

iii). Write three vectors in $\mathbf{R}^3$ such that their span is equal to $\mathbf{R}^3$ itself.

iv). Is it possible to write 4 distinct non-redundant vectors in $\mathbf{R}^3$? Explain why yes or why not.

Problem 6: Use your own words to answer to the following questions:

- Why is the image of the orthogonal projection in $\mathbf{R}^2$ onto a line L passing thorough the origin equal to L?
- Why is the kernel of the orthogonal projection in $\mathbf{R}^2$ onto a line L passing thorough the origin equal the line perpendicular to L?

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Homework 6 – Due date 10/20/17

Read: Section 3.2, Section 3.3 and Theorem 2.1.3 in section 2.1. Keywords: Linear independence of vectors, non-trivial relations, subspaces of $\mathbf{R}^n$, bases of a subspace, spans kernels and images are subspaces, linear transformations preserve the sum of vectors, and the scalar multiplication of a scalar and a vector.

Problem 1: Consider the following vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ in $\mathbf{R}^2$.

i) Write $\vec{v}_3$ as linear combination of $\vec{v}_1$ and $\vec{v}_2$.
ii) Write a non-trivial relation between $\vec{v}_1, \vec{v}_2, \vec{v}_3$.
iii) Are $\vec{v}_1, \vec{v}_2, \vec{v}_3$ linearly independent? Do they span $\mathbf{R}^2$? Do they form a basis of $\mathbf{R}^2$?

Problem 2: Repeat the first 2 points of Problem 1 with the following vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix}$. Then answer to the following questions: Are $\vec{v}_1, \vec{v}_2, \vec{v}_3$ linearly independent? Do they span $\mathbf{R}^3$? Do they form a basis of $\mathbf{R}^3$?

Problem 3: Prove (or explains with your own words) why we can find at most n linearly independent vectors in $\mathbf{R}^n$. (*Hint:* The answer to this problem is very similar to that of Problem 5 (iv) in Homework 4.)

Problem 4: Make a sketch of the following subset $W = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \text{ s.t. } xy \leq 0 \right\}$ consisting of the union of the second and fourth quadrants, including the axes. Show that W is not closed under addition, by writing down two vectors that belong to W, but such that their sum is not in W.

Problem 5: Repeat Problem 4 with the following subset $W = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \text{ s.t. } x^2 - y^2 = 0 \right\}$. (*Hint*: Note that $x^2 - y^2 = (x + y)(x - y) = 0$, so that W is simply the union of the two lines $y = x$ and $y = -x$. Make a sketch of W!)

Problem 6: Prove that the image of a linear transformation $T\colon \mathbf{R}^m \to \mathbf{R}^n$ is a linear subspace of $\mathbf{R}^n$. You can read the solution from your textbook (Edition 5: Theorem 3.1.4 at page 114; Edition 4: Theorem 3.1.4 at page 105).

Problem 7: Find a basis for the following subset $U = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \text{ in } \mathbf{R}^4 \text{ s.t. } x_2 + x_3 + x_4 = 0 \text{ and } x_3 - x_4 = 0 \right\}$ of $\mathbf{R}^4$. Write then U as kernel of some matrix.

Problem 8: Consider the following subset

$$U = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \text{ s.t. } \begin{cases} y_1 = x_1 + 2x_2 + 3x_3 \\ y_2 = 4x_1 + 5x_2 + 6x_3 \\ y_3 = 7x_1 + 8x_2 + 9x_3 \end{cases} \text{ where } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ is in } \mathbf{R}^3 \right\}.$$

Convince yourself that U is nothing else than the image of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. In this way we have proved that U is a subspace of $\mathbf{R}^3$ as the image of any linear transformation is a subspace. Find a basis of U.

Problem 9: Find a basis of the subspace in $\mathbf{R}^4$ defined by $2x_1 - x_2 + 2x_3 + 4x_4 = 0$.

Problem 10: Consider the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. Is $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ in the image of A? Is $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ in the kernel of A? Perform your calculations and justify your answers.

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Homework 7 – Due date 10/27/17

Read: Section 3.3, Section 3.4.

Problem 1: Consider the following vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 3 \\ -5 \\ -2 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 1 \\ -5 \\ -10 \end{pmatrix}$ in $\mathbf{R}^3$. Do they form a basis of $\mathbf{R}^3$?

Problem 2: Find the dimension of $\text{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -5 \\ k \end{pmatrix} \right\}$ as k varies in $\mathbf{R}$. In other words say for which values of k the dimension of the span is 0, 1, 2, or 3.

Problem 3: Find bases and dimensions for the kernel and image of the following matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & -2 \\ -1 & 1 & 2 \end{pmatrix}.$$

Convince yourself that the Rank-Nullity Theorem works.

Problem 4: Find the dimensions for the kernel and image of the following matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 1 & -1 & -2 & 0 \\ -1 & 1 & 2 & k \end{pmatrix}$$

as k varies in $\mathbf{R}$. (*Hint:* First find the dimension of the image and then use the Rank-Nullity Theorem to find the dimension of the kernel.)

Problem 5: Find a basis of the subspace in $\mathbf{R}^3$ consisting of all vectors perpendicular to $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$. What it its dimension?

Problem 6: Consider the subspaces $U = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$, $V = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix} \right\}$, and $W = \text{span}\left\{ \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \right\}$. Find bases and dimensions of U, V and W. Is $U = V$? Is $U = W$? Is $V = W$? (*Hint*: Two subspaces defined as $\text{span}\{v_1, v_2\}$ and $\text{span}\{w_1, w_2\}$ are equal if and only if v_1 and v_2 are linear combinations of w_1, w_2, and w_1 and w_2 are linear combinations of v_1, v_2.)

Problem 7: Solve the following problems in your textbook:
Edition 5. Problems number 6, 7, 8, 20 (only part a) and c)), 21 (only part a) and c)) in page 159.
Edition 4. Problems number 6, 7, 8, 21 (only part a) and c)), 22 (only part a) and c)) in page 147.

Problem 8: Consider a basis $\mathcal{B}$ of $\mathbf{R}^3$ consisting of vectors $\mathcal{B} = \left(\vec{u}, \vec{v}, \vec{w}\right)$. Say whether the following sets of vectors are linearly independent or not:

$i).\left(\vec{u} + \vec{v}, \vec{v}, \vec{v} + \vec{w}\right)$ $\quad ii).\left(\vec{u} - \vec{w}, \vec{v} - \vec{u}, \vec{w} - \vec{v}\right)$ $\quad iii).\left(\vec{u} - \vec{v} + \vec{w}, \vec{v} - \vec{u} - 3\vec{w}, \vec{u} - \vec{v} - \vec{w}\right)$.

(*Hint*: Find the coordinates of the vectors in $i)$, $ii)$, $iii)$ with respect to the basis $\mathcal{B}$. Then recall that the linear independence of vectors can be checked by calculating the rank of the matrix having the vectors as columns.)

Problem 9: Find a basis $\mathcal{B}$ of $\mathbf{R}^3$ so that the $\mathcal{B}$-matrix B of the reflection in $\mathbf{R}^3$ about the plane defined by the equation $x + y + z = 0$ is diagonal. (*Hint*: Think geometrically. We have solved this problem in class for the orthogonal projection onto a plane.)

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Homework 8 – Due date 11/03/17

Read: In this homework set, Problems 1, 2, 3, 4, 5, 6 are a review problems. Whereas Problem 7 is about the new material on Vector Spaces. You should read Sections 3.4, 4.1, and 4.2.

Problem 1: Consider the linear transformation T from $\mathbf{R}^5$ to $\mathbf{R}^5$:

$$T\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix} = \begin{pmatrix}x_2+3x_3-x_5\\x_1+7x_3\\x_1+7x_3-x_5\\x_1+7x_3\\-x_5\end{pmatrix}.$$

a) Write the matrix A that represents T with respect to the standard basis of $\mathbf{R}^5$ (Namely A satisfies $T(\vec{x}) = A\vec{x}$).
b) Find a basis for the image of T. Find a basis for the kernel of T. Find the dimensions of the image and of the kernel.
c) Find the matrix B that represents T with respect to the basis $\mathcal{B} = \left(\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\\1\end{pmatrix}\right)$. Find the $\mathcal{B}$-coordinates of the image under T of the vector $(2, 2, 3, 3, 0)$ (you should think of this vector as a column vector) using the matrix B.

Problem 2: Recall that two matrices A and B are *similar* if there exists an invertible matrix S such that $SB = AS$. Show that the matrices $\begin{pmatrix}1&0\\0&0\end{pmatrix}$ and $\begin{pmatrix}0&0\\0&1\end{pmatrix}$ are similar by finding the matrix S that realizes the similarity. (*Hint*: See Example 6 in Section 3.4.)

Problem 3: Let P be the plane in $\mathbf{R}^3$ defined by the equation $x - y - z = 0$, and consider the linear transformation T from $\mathbf{R}^3$ to $\mathbf{R}^3$ defined as $T(\vec{x}) = \text{proj}_P(\vec{x}) - 2\text{ref}_P(\vec{x})$. Find a basis $\mathcal{B}$ of $\mathbf{R}^3$ such that the matrix B that represents T with respect to the basis $\mathcal{B}$ is diagonal. Afterwards find the matrix A that represents T with respect the standard basis by using the formula $A = SBS^{-1}$. (*Hint*: See the notes "Hints for Homework 8" in BB.)

Problem 4: Let L be the line in $\mathbf{R}^3$ defined by the equations $x-y-z = 0$ and $y+2z = 0$. Also consider the linear transformation T from $\mathbf{R}^3$ to $\mathbf{R}^3$ defined as $T(\vec{x}) = \text{proj}_L\big(\text{ref}_L(2\vec{x})\big)$ (this means that first we multiply $\vec{x}$ by 2, then we find its reflection about L, and then we project the resulting vector orthogonally onto L). Find a basis $\mathcal{B}$ of $\mathbf{R}^3$ such that the matrix B that represents T with respect to the basis $\mathcal{B}$ is diagonal. (*Hint*: In this case a smart choice of the basis $\mathcal{B}$ would be that consisting of one vector spanning L, and two other linearly independent vectors, both perpendicular to L.)

Problem 5: (**Example of an Orthogonal Complement of a Subspace**). Consider the subspace U of $\mathbf{R}^4$ defined by the equations $x_1 - x_2 + x_3 - x_4 = 0$ and $x_2 - 2x_4 - 2x_3 = 0$. Find a basis of U. Then find a basis of $U^\perp$. We define $U^\perp$ to be the subspace of $\mathbf{R}^4$ consisting of vectors of $\mathbf{R}^4$ that are perpendicular to all vectors of U. Equivalently, if we have a basis $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)$ of U, then $U^\perp$ consists of all vectors of $\mathbf{R}^4$ that are perpendicular to $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$.

Problem 6: Let $\vec{u}, \vec{v}, \vec{w}$ be a basis of $\mathbf{R}^3$. Determine the dimensions of the following subspaces:

i). $\text{span}\{\vec{w}-\vec{u}, \vec{w}-\vec{v}, \vec{w}-2\vec{u}\}$, ii). $\text{span}\{\vec{u}-\vec{v}, \vec{v}-\vec{w}, -\vec{u}+\vec{w}\}$ (*Hint*: See the notes "Hints for Homework 8" in BB).

Problem 7: This problem consists of several independent fun subproblems. You may want to read Sections 4.1 and 4.2.

(1). Show that the subset W consisting of 2×2 matrices that commute with the matrix $A = \begin{pmatrix}0&1\\2&3\end{pmatrix}$ is a subspace of $\mathbf{R}^{2\times 2}$. Find a basis of W and its dimension.

(2). Show that the subset W consisting of 2×2 matrices S such that

$$\begin{pmatrix}3&2\\4&5\end{pmatrix} S = S$$

is a subspace of $\mathbf{R}^{2\times 2}$. Find a basis of W and its dimension.

(3). The *trace* of a 2×2 matrix is defined as

$$\text{trace}\begin{pmatrix}a&b\\c&d\end{pmatrix} = a+d$$

(in other words, the trace of a matrix is the sum of the elements along the main diagonal). Show that the subset W consisting of 2×2 matrices with trace equal to zero is a subspace of $\mathbf{R}^{2\times 2}$. Find a basis of W and its dimension.

(4). Show that the subset W of P_3 consisting of polynomials $p(x)$ of degree at most 3 such that $p(0) = 0$ is a subspace of P_3. Find a basis of W and its dimension.

(5). Show that the subset W of P_3 consisting of polynomials $p(x)$ of degree at most 3 such that $p'(0) = 0$ is a subspace of P_3 (the prime sign denotes the first derivative). Find a basis of W and its dimension.

(6). Show that the subset W of P_3 consisting of polynomials $p(x)$ of degree at most 3 such that $\int_{-1}^{1} p(x)dx = 0$ is a subspace of P_3. Find a basis of W and its dimension.

(7). Solve Problems 6, 9, 23, 25 of Section 4.2 in your textbook.

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Homework 9 – Due date 11/17/17

Read: Sections 6.2 and 6.3.

Problem 1: Find the determinant of the following matrices:

$$A = \begin{pmatrix} -2 & 9 \\ 4 & -5 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 0 & 4 \\ -1 & -2 & 3 \\ 0 & 6 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -1 & 0 & -1 \\ 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ -3 & 0 & -3 & 0 \end{pmatrix}.$$

Recall that Sarrus' formula is valid only for 3×3 matrices. While using Laplace's expansion don't forget to use the matrix of signs.

Problem 2: In this problem we will calculate the determinant of diagonal matrices. We start with an example. Use Laplace's expansion to compute the determinant of

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

where a, b, c, d are scalars. Then convince yourself that the determinant of any diagonal matrix (of any order) equals the product of the entries along the main diagonal.

Problem 3: In this problem we will calculate the determinant of upper/lower-triangular matrices. We start with an example. Use Laplace's expansion in order to compute the determinant of

$$\begin{pmatrix} 2 & 1 & 3 \\ 0 & -3 & 6 \\ 0 & 0 & 2 \end{pmatrix}.$$

Convince yourself that the determinant of any upper/lower-triangular matrix equals the product of the entries along the main diagonal.

Problem 4: Use Cramer's rule to find the solution of the following 3×3 system: $x + 2z = 9$, $2y + z = 8$, $4x - 3y = -2$.

Problem 5: We learned in class that Gauss-Jordan's elimination can be used to find the inverse of an invertible matrix. There is another technique to find the inverse of a matrix. It is called the *adjoint method* and uses determinants. Read Theorem 6.3.9 in the textbook, or watch a couple of times the video `https://www.youtube.com/watch?v=YvjkPF6C_LI`

Then find the inverse of the matrix B of Problem 1. Check that $\det(B^{-1}) = \frac{1}{\det(B)}$. In fact this formula is true for any invertible matrix B. Another interesting formula is Binet's theorem: *If A and B are two $n \times n$ matrices, then* $\det(AB) = \det(A)\det(B)$.

Problem 6: The determinant is a good tool to find the rank of a matrix. For instance, an $n \times n$ matrix has rank equal to n if and only if its determinant is non-zero.

In class we have seen that we can push this statement a little further in order to understand the rank of a 3×2 matrix A (so that its rank is at most 2). Then we said that:

(i). $\text{rank}(A) = 2$ if and only if there exists a 2×2 submatrix of A with non-zero determinant.
(ii). $\text{rank}(A) \leq 1$ if and only if every 2×2 submatrix of A has determinant equal to zero.
(iii). $\text{rank}(A) = 1$ if and only if every 2×2 submatrix of A has determinant equal to zero, and there exists a 1×1 submatrix whose determinant is non-zero.
(iv). $\text{rank}(A) = 0$ if and only if A is the zero-matrix.

(Note that a 1×1 matrix is nothing else than a scalar. Its determinant is the scalar itself.)

Let now A be a 4×3 matrix, so that its rank is at most 3. In the following list, connect each statement denoted by a Greek letter, to a statement denoted by a number.

(α). $\operatorname{rank}(A) = 3$
(β). $\operatorname{rank}(A) \leq 2$
(γ). $\operatorname{rank}(A) = 2$
(δ). $\operatorname{rank}(A) \leq 1$
(ε). $\operatorname{rank}(A) = 1$
(ζ). $\operatorname{rank}(A) = 0$

(1). Every 2×2 submatrix of A has determinant equal to zero.
(2). Every 2×2 submatrix of A has determinant equal to zero, and there exists a 1×1 submatrix with non-zero determinant.
(3). Every 3×3 submatrix of A has determinant equal to zero, and there exists a 2×2 submatrix with non-zero determinant.
(4). Every 3×3 submatrix of A has determinant equal to zero.
(5). A is the zero matrix.
(6). There exists a 3×3 submatrix of A with non-zero determinant.

Problem 7: Find the equation of the plane in $\mathbf{R}^3$ spanned by the vectors $\begin{pmatrix} -2 \\ -4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}$. (*Hint*: We solved a similar problem in class.)

Problem 8: Write the matrices of the following transformations: 1) the orthogonal projection in $\mathbf{R}^2$ onto a line L spanned by the unit vector $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$; 2) the reflection in $\mathbf{R}^2$ about the line L spanned by the unit vector $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$; 3) the counterclockwise rotation in $\mathbf{R}^2$ by θ degrees. Find their determinants and say which transformation is an isomorphism (you may want to re-read section 2.2).

Optional Problem 9: Find the equations of the line in $\mathbf{R}^3$ spanned by the vector $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

Optional Problem 10 (Cross product in $\mathbf{R}^3$): Denote by $\vec{e}_1, \vec{e}_2, \vec{e}_3$ the standard basis of $\mathbf{R}^3$. We define the *cross product* of two vectors $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ in $\mathbf{R}^3$ as the vector $\vec{v} \times \vec{w}$ in $\mathbf{R}^3$ defined by:

$$\vec{v} \times \vec{w} \;=\; \det \begin{pmatrix} \vec{e}_1 & \vec{e}_1 & \vec{e}_1 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}.$$

For instance $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \det \begin{pmatrix} \vec{e}_1 & \vec{e}_1 & \vec{e}_1 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = -3\vec{e}_1 + 6\vec{e}_2 - 3\vec{e}_3 = \begin{pmatrix} -3 \\ 6 \\ -3 \end{pmatrix}$.

Warning: Do not confuse the cross product with the dot product. They are two different operations. In fact the result of the dot product is a scalar, that of the cross product is a vector in $\mathbf{R}^3$.

(1). Expand the determinant in the definition of the cross product (by using either Sarrus or Laplace) in order to find the coordinates of $\vec{v} \times \vec{w}$ with respect to the standard basis $\vec{e}_1, \vec{e}_2, \vec{e}_3$.
(2). Prove that the cross product is anticommutative, namely that $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$ for all $\vec{v}, \vec{w}$ in $\mathbf{R}^3$.
(3). Compute the following vectors: $\vec{e}_1 \times \vec{e}_2$, $\vec{e}_2 \times \vec{e}_3$, $\vec{e}_3 \times \vec{e}_1$, and $\vec{e}_1 \times \vec{e}_3$.
(4). Compute $\vec{v} \times \vec{v}$ for any vector $\vec{v}$ in $\mathbf{R}^3$.
(5). Show that if $\vec{v}$ is parallel to $\vec{w}$ (namely $\vec{v} = k\vec{w}$ for some scalar k), then $\vec{v} \times \vec{w} = \vec{0}$.
(6). Show that if $\vec{v} \times \vec{w} = \vec{0}$, then $\vec{v}$ is parallel to $\vec{w}$.
(7). We conclude that the cross product of two vectors is the zero vector if and only if the vectors are parallel.
(8). Show that $\vec{v} \times \vec{w}$ is perpendicular to both $\vec{v}$ and $\vec{w}$.

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Homework 10 – Due date November 29th

Read. "Formulas about base change" in BB. "Exercises with solutions about diagonalization" in BB. Sections 4.3, 7.2, 7.3 and Example 2 in 7.4 of the Textbook.

Problem 1. For each matrix find its eigenvalues together with their algebraic multiplicities, find bases of the eigenspaces associated to each eigenvalue, find the geometric multiplicity of each eigenvalue. Then say whether the matrix is diagonalizable and specify which criterion you use to answer to this question. If a matrix A is diagonalizable, then write a diagonal matrix B and an invertible matrix S such that $B = S^{-1}AS$ (do not perform this product, unless you want to check that calculations are correct).

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & -1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

(*Hint:* For the first two matrices you can check that your work is correct by looking at the file "Exercises with solutions about diagonalization" in BB. For the third matrix, recall that the determinant of an upper-triangular matrix is the product of the entries along the main diagonal.)

Problem 2. Consider the linear function $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ defined as $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x - 4y \\ 2x - 1y \end{pmatrix}$. Write the matrix A that represents the linear function T with respect to the standard basis of $\mathbf{R}^2$. Find a basis $\mathcal{B}$ of $\mathbf{R}^2$ such that the matrix B that represents T with respect to $\mathcal{B}$ is diagonal. In other words you want to diagonalize A.

Problem 3. Say whether $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ is diagonalizable. If not say why.

Problem 4. Write the matrix that represents the rotation in $\mathbf{R}^2$ by 90 degrees counterclockwise with respect to the standard basis. Find its eigenvalues. Do you notice anything strange? Could you provide a geometric explanation?

Problem 5. **(i)**. Make a picture of a linear function that is an isomorphism. **(ii)**. Make a picture of a linear function that is not an isomorphism. **(iii)**. Is is true that the determinant of a matrix is equal to the determinant of its rref? If yes prove it, if not give an example (*Hint:* The answer is negative!).

Problem 6. For Problems 6 and 7 you may want to read the notes "Formulas about base change" I posted in BB.

Check that $\mathcal{B} = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right)$ is a basis of $\mathbf{R}^2$. Write the matrix of change of basis $S_{\mathcal{B}\to\mathcal{E}}$ from the basis $\mathcal{B}$ to the standard basis $\mathcal{E}$ of $\mathbf{R}^2$. Calculate the standard coordinates of a vector knowing that its $\mathcal{B}$-coordinates are $\begin{pmatrix} -4/5 \\ 3/5 \end{pmatrix}$ (you should use the matrix $S_{\mathcal{B}\to\mathcal{E}}$). Compute the $\mathcal{B}$-coordinates of $\begin{pmatrix} -12 \\ 7 \end{pmatrix}$ (you should use the matrix $S^{-1}_{\mathcal{B}\to\mathcal{E}}$).

Now consider the linear function $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ defined as $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x-y \\ -x+2y \end{pmatrix}$. Write the matrix A that represents T with respect to the standard basis $\mathcal{E}$ of $\mathbf{R}^2$. Write the matrix B that represents T with respect to the basis $\mathcal{B}$. What relation should A and B satisfy? (Do not perform any product.) Find the $\mathcal{B}$-coordinates of $\begin{pmatrix} 33 \\ 101 \end{pmatrix}$. Find the $\mathcal{B}$-coordinates of $T\begin{pmatrix} 33 \\ 101 \end{pmatrix}$. Find the standard coordinates of $T\begin{pmatrix} 33 \\ 101 \end{pmatrix}$. Is T an isomorphism?

Problem 7. In class we showed that $\mathcal{B} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$ is a basis of $\mathbf{R}^{2\times 2}$, the set of 2×2 matrices with real coefficients. Hence the dimension of $\mathbf{R}^{2\times 2}$ is 4. Show that the following set of matrices $\mathcal{C} = \left(\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \right)$ form a basis of $\mathbf{R}^{2\times 2}$ (you should first find the $\mathcal{B}$-coordinates of the matrices of the set $\mathcal{C}$). Write the matrix of change of basis $S_{\mathcal{C}\to\mathcal{B}}$. Find the matrix whose $\mathcal{C}$-coordinates are $\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$.

Now consider the linear function $T\colon \mathbf{R}^{2\times 2} \to \mathbf{R}^{2\times 2}$ defined as $T(M) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} M$. Write the matrix $C_{\mathcal{B}}$ that represents T with respect to the basis $\mathcal{B}$. Write the matrix $C_{\mathcal{C}}$ that represents T with respect to the basis $\mathcal{C}$. What is the relation between $C_{\mathcal{C}}$ and $C_{\mathcal{B}}$? Find bases for the kernel and image of T (you can work out these bases either using $C_{\mathcal{B}}$ or $C_{\mathcal{C}}$). Is T an isomorphism?

Find the $\mathcal{C}$-coordinates of $\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$. Find the $\mathcal{C}$-coordinates of $T\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$. Find the $\mathcal{B}$-coordinates of $T\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$. Find $T\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$.

Problem 8 (Optional). Recall that the formula of the orthogonal projection in $\mathbf{R}^3$ onto a plane V of equation $ax+by+cz=0$ is

$$\operatorname{proj}_V(\vec{x}) \;=\; \vec{x} - \left(\frac{\vec{x}\cdot\vec{r}}{\vec{r}\cdot\vec{r}}\right)\vec{r}, \qquad \text{where} \qquad \vec{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

(as usual the dot denotes the dot product). Write the matrix that represents with respect to the standard basis of $\mathbf{R}^3$ the transformation $T(\vec{x}) = -3\operatorname{proj}_V(\vec{x})$, where V is the plane of equation $x+y+z=0$ (you should find $A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$). Find a basis $\mathcal{B}$ of $\mathbf{R}^3$ so that the matrix B that represents T with respect to $\mathcal{B}$ is diagonal. Is $\mathcal{B}$ formed by 2 vectors spanning the plane V and one other perpendicular to V? Find bases for the kernel and image of T.

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Practice Midterm 1 – Midterm 1 is scheduled for October 6th

You are very welcome to solve and discuss the problems of this worksheet with your classmates. These problems need not to be turned in. As I will review most of these problems in the following classes, I will not write their solutions. In any case you are very welcome to visit me during office hours, or by appointment, to get help on the problems. Alternatively I can also advise via email. The midterm will consists of 5 problems and it will cover Chapters 1 and 2, and Section 3.1.

Problem 1. Solve the following systems using augmented matrices. State whether the solution is unique, there are no solutions, or whether there are infinitely many. If the solution is unique give it. If there infinitely many give the solution parametrically, namely in terms of the free variables.

$$\begin{cases} x_1 \qquad\quad - \ x_3 = 8 \\ 2x_1 + 2x_2 + 9x_3 = 7 \\ \qquad\ \ x_2 + 5x_3 = -2 \end{cases}$$

$$\begin{cases} 3x_1 - \ \ 4x_2 + 2x_3 = 0 \\ -9x_1 + 12x_2 - 6x_3 = 0 \\ -6x_1 + \ \ 8x_2 - 4x_3 = 0 \end{cases}$$

Problem 2. Discuss the number of solutions of the following systems depending on the real parameter k. Moreover when the solution is unique, or there are infinitely many solutions, write all the solutions in parametric form.

$$\left[\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & - & x_3 & + & kx_4 & = & 1 \\ -2x_1 & + & x_2 & + & 2x_3 & - & x_4 & = & 2 \\ 4x_1 & + & 3x_2 & - & 4x_3 & + & 3x_4 & = & 0 \end{array}\right]$$

$$\left[\begin{array}{ccccccc} & & y & + & z & = & k \\ x & & & + & z & = & k \\ x & + & y & & & = & k \end{array}\right]$$

Problem 3. Say for which values of the real parameter a the following matrix is invertible:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & a & 2 \\ 0 & 0 & a^2 - 3a \end{pmatrix}.$$

Then set $a = 1$ and find the inverse.

Problem 4. a) Write the matrix representing a linear transformation that rotates vectors of $\mathbf{R}^2$ by 30 degrees counterclockwise.

b) Write the matrix representing a linear transformation that reflects vectors of $\mathbf{R}^2$ about the line $y = 2x$.

c) Write the matrix representing a linear transformation that first rotates vectors by 30 degrees counterclockwise, and then reflects them about the line $y = 2x$.

d) Find the vector obtained by first reflecting $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ about the line $y = 2x$, and then rotating it by 30 degrees counterclockwise.

Problem 5. Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be the orthogonal projection onto the line $x - 2y = 0$ followed by a counterclockwise rotation by 45 degrees. Find the matrix A that represents T. Is A invertible? Show on a picture the kernel and the image of T.

Problem 6. (Orthogonal projection in $\mathbf{R}^3$.) Recall that the orthogonal projection of a vector $\vec{x}$ in $\mathbf{R}^3$ onto a line L of $\mathbf{R}^3$ is defined as $\mathrm{proj}_L(\vec{x}) = (\vec{x} \cdot \vec{u})\,\vec{u}$, where $\vec{u}$ is a unit vector parallel to L. Alternatively, if instead of a unit vector $\vec{u}$ we have an arbitrary non-zero vector $\vec{w}$ parallel to L, the projection of $\vec{x}$ onto L is defined as

$$\mathrm{proj}_L(\vec{x}) \ = \ \left(\frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\right)\vec{w}.$$

Let now $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ be the orthogonal projection onto the line L spanned by the vector $\vec{w} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$.

a) Write the matrix A that represents T.

b) Find the orthogonal projection $\vec{r}$ of the vector $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ onto L.

c) Find all vectors in $\mathbf{R}^3$ that are perpendicular to $\vec{w}$ and $\vec{r}$. Write them in parametric form (namely in terms of free variables).

Problem 7. (Orthogonal Projections onto a plane of $\mathbf{R}^3$.) The orthogonal projection $\mathrm{proj}_V(\vec{x})$ of a vector $\vec{x}$ in $\mathbf{R}^3$ onto a plane V in $\mathbf{R}^3$ of equation $ax_1 + bx_2 + cx_3 = 0$ is given by the formula:

$$\mathrm{proj}_V(\vec{x}) \;=\; \vec{x} - \left(\frac{\vec{x}\cdot\vec{r}}{\vec{r}\cdot\vec{r}}\right)\vec{r}, \qquad \text{where } \vec{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Note that the 'dot' in the previous formula denotes the dot product of vectors in $\mathbf{R}^3$.
a) Write the matrix that represents the linear transformation proj_V.
b) Find the orthogonal projection of $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ onto the plane $x_2 - x_1 + x_3 = 0$ in $\mathbf{R}^3$.

Problem 8. Consider the following linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} \;=\; \begin{pmatrix} x+y-z \\ -y+z \\ -2x-2y+2z \end{pmatrix}.$$

a) Find the matrix A that represents T.
b) Write the kernel of T as a span of a minimal set of generators.
c) Write the image of T as a span of a minimal set of generators.

Problem 9. Consider the following matrix:

$$A = \begin{pmatrix} 2 & -2 & -4 \\ -3 & -5 & -2 \\ 4 & -2 & -6 \end{pmatrix}.$$

a) Write the image of A as a span of a minimal set of generators.
b) Write the kernel of A as a span of a minimal set of generators.

Problem 10. Consider the following matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 & 3 \\ 1 & 2 & 4 & 3 \\ 1 & 2 & 5 & 3 \end{pmatrix}.$$

a) Write the image of A as a span of a minimal set of generators.
b) Write the kernel of A as a span of a minimal set of generators.

Problem 11. (Rotations in $\mathbf{R}^3$.) Consider $\mathbf{R}^3$ with coordinates (x, y, z). The matrix $R_x(\theta)$ that represents the linear transformation $T_{x,\theta}\colon \mathbf{R}^3 \to \mathbf{R}^3$ rotating vectors in $\mathbf{R}^3$ by θ degrees counterclockwise about the x-axis is:

$$R_x(\theta) \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Similarly we can define $R_y(\theta)$ and $R_z(\theta)$ which are the matrices that rotate vectors by θ degrees counterclockwise about the y- and z-axis, respectively:

$$R_y(\theta) \;=\; \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}, \quad R_z(\theta) \;=\; \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

a) Find the vector obtained by rotating $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ first by 90 degrees counterclockwise about the x-axis, and then by rotating it by 180 degrees counterclockwise about the z-axis.

b) Find the vector obtained by rotating $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ first by 30 degrees counterclockwise about the y-axis, and then by rotating it by 45 degrees counterclockwise about the z-axis.

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Answer Keys to the Practice Midterm 1
Fall Semester

If you find any mistake in the following answer keys, please do let me know via email. The instructor is not responsible of any possible mistake in these notes.

Problem 1: a) $x = 3, \quad y = 23, \quad z = -5.$

b)

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 4/3 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2/3 \\ 0 \\ 1 \end{pmatrix}$$

where t and s are free variables. You can also say that the space of solutions is the span of $\begin{pmatrix} 4/3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2/3 \\ 0 \\ 1 \end{pmatrix}$, which is a plane in $\mathbf{R}^3$.

Problem 2: a) If $k = 1$ the solutions are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -3/5 \\ -1/5 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -3/5 \\ 4/5 \\ 0 \\ 0 \end{pmatrix}$$

where t and s are free variables. In this case there are ∞^2-many solutions. The solutions form a plane in $\mathbf{R}^4$ (not passing thorough the origin).

If $k \neq 1$, the solutions are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -3/5 \\ 4/5 \\ 0 \\ 0 \end{pmatrix}$$

where t is a free variable. In this case there are ∞^1-many solutions. The solutions form a line, not passing through the origin, in $\mathbf{R}^4$.

b) For any value of k there is only one solution $x = y = z = k/2$.

Problem 3:The matrix A is not invertible only when either $a = 0$ or $a = 3$. If $a = 1$, the inverse of A is

$$\begin{pmatrix} 1 & -2 & -1/2 \\ 0 & 1 & 1 \\ 0 & 0 & -1/2 \end{pmatrix}$$

Problem 4: a) $\begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$

b) $\begin{pmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{pmatrix}$

c) $\begin{pmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} -3\sqrt{3} + 4 & 3 + 4\sqrt{3} \\ 4\sqrt{3} + 3 & -4 + 3\sqrt{3} \end{pmatrix}.$

d) $\frac{1}{10} \begin{pmatrix} \sqrt{3} - 7 \\ 7\sqrt{3} + 1 \end{pmatrix}$

Problem 5:

$A = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \begin{pmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} \sqrt{2} & \sqrt{2}/2 \\ 3\sqrt{2} & 3\sqrt{2}/2 \end{pmatrix}$. The matrix A is not invertible. $\mathrm{Ker}(T) = \mathrm{span}\{\begin{pmatrix} -1 \\ 2 \end{pmatrix}\}$. $\mathrm{Im}(T) = \mathrm{span}\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}\}$.

Problem 6:

a)

$$A = \frac{1}{9}\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix}.$$

b) $\vec{r} = \frac{2}{3}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$.

c) All vectors of the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t\begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

where t and s are free variables are perpendicular to both $\vec{r}$ and $\vec{w}$. Therefore the vectors perpendicular to both $\vec{r}$ and $\vec{w}$ form a plane in $\mathbf{R}^3$.

Problem 7:

a)

$$\frac{1}{a^2+b^2+c^2}\begin{pmatrix} b^2+c^2 & -ab & -ac \\ -ab & a^2+c^2 & -bc \\ -ac & -bc & a^2+b^2 \end{pmatrix}.$$

b) $\frac{1}{3}\begin{pmatrix} 7 \\ 2 \\ 5 \end{pmatrix}$

Problem 8: a)

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -2 & -2 & 2 \end{pmatrix}.$$

b) $\mathrm{Im}(T) = \mathrm{span}\{\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}\}$.

c) $\mathrm{Ker}(T) = \mathrm{span}\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\}$.

Problem 9: a) $\mathrm{Im}(A) = \mathrm{span}\{\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ -5 \\ -2 \end{pmatrix}\}$.

b) $\mathrm{Ker}(A) = \mathrm{span}\{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\}$.

Problem 10: a) $\mathrm{Im}(A) = \mathrm{span}\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}\}$.

b) $\mathrm{Ker}(A) = \mathrm{span}\{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \end{pmatrix}\}$.

Problem 11: a)

$$R_z(180)R_x(90)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

b)

$$R_z(45)R_y(30)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} \sqrt{6}-\sqrt{2} \\ \sqrt{6}+3\sqrt{2} \\ 2\sqrt{3}-2 \end{pmatrix}.$$

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Practice Midterm 2

General information. The second midterm is scheduled for Friday November 10th. The midterm will cover the entire chapter 3, and sections 4.1, 4.2 of the textbook. The midterm will consist of 5 problems. There will be no homework assignment due next week, so that you will have enough time to solve all problems of this worksheet. Most of the problems of this worksheet are very similar to problems of previous homework, whose solutions have been posted in BB. You are encouraged to solve more similar problems from the textbook.

Problem 1: (1). Show that the subset U_2 consisting of all upper-triangular matrices of type 2×2 is a subspace of $\mathbf{R}^{2\times 2}$. Find a basis and the dimension of U_2. Finally define an isomorphism between U_2 and $\mathbf{R}^{\dim U_2}$. Write the inverse of the isomorphism you have found.

(2). Show that the subset W consisting of all 2×2 matrices that commute with $\begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}$ is a subspace of $\mathbf{R}^{2\times 2}$. Find a basis and the dimension of W.

Problem 2: Show that the subset W consisting of all even polynomials of degree at most 3 is a subspace of P_3 (recall that a polynomial $p(x)$ is *even* if $p(x) = p(-x)$ for all values of x). Find a basis and the dimension of W.

Problem 3: Find a basis of the image and kernel of each of the following linear functions. Then say whether a function T is an isomorphism or not.

$$(1).\quad T\colon \mathbf{R}^{2\times 2} \to \mathbf{R}^{2\times 2}, \qquad T(M) = M \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}.$$

$$(2).\quad T\colon P_3 \to P_3, \qquad T(f(x)) = f'(x) - f''(x).$$

$$(3).\quad T\colon P_2 \to P_2, \qquad T(f(x)) = f''(x) - 2f(x).$$

$$(4).\quad T\colon P_2 \to \mathbf{R}^{2\times 2}, \qquad T(f(x)) = \begin{pmatrix} f(0) & f(1) \\ -f(0) & f'(\frac{1}{2}) \end{pmatrix}.$$

Problem 4: Find a basis of the subspace U in $\mathbf{R}^4$ defined by the equations $x_1 + 2x_2 - 3x_3 + x_4 = 0$ and $2x_1 - x_3 - 2x_4 = 0$. Find moreover a basis of the orthogonal complement $U^\perp$ of U (in other words find a basis of the subspace of $\mathbf{R}^4$ consisting of all vectors perpendicular to U).

Problem 5: Consider the plane P in $\mathbf{R}^3$ defined by the equation $2x - y - 2z = 0$. Find a basis $\mathcal{B}$ of $\mathbf{R}^3$ such that the matrix B that represents the transformation $T(\vec{x}) = 2\text{proj}_P(\vec{x}) - 3\,\text{ref}_P(2\vec{x})$ with respect to the basis $\mathcal{B}$ is diagonal. Is T an isomorphism?

Problem 6: Find a basis $\mathcal{B}$ for the following subspace of $\mathbf{R}^4$

$$U = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\}.$$

Find the dimension of U. Find the $\mathcal{B}$-coordinates of the vector $\vec{w} = \begin{pmatrix} 7 \\ 3 \\ 3 \\ 3 \end{pmatrix}$.

Problem 7: Let $(\vec{u}, \vec{v}, \vec{w})$ be a basis of $\mathbf{R}^3$. Say for which values of the real parameter k the following vectors form a basis of $\mathbf{R}^3$:

$$\vec{u} + \vec{v} + \vec{w}, \qquad \vec{u} - \vec{v} + \vec{w}, \qquad \vec{u} + k\vec{v} + k^2\vec{w}.$$

Problem 8: Answer True or False.
(1). There exists a linear transformation T from $\mathbf{R}^5$ to $\mathbf{R}^5$ such that both $\dim \text{Ker}(T)$ and $\dim \text{Im}(T)$ are even numbers.
(2). If a set of vectors $v_1, \ldots, v_k$ span a subspace V of $\mathbf{R}^n$, then they also form a basis of V.
(3). If four vectors of $\mathbf{R}^n$ span a subspace U of dimension four, then they also form a basis of U.
(4). If a subspace U of $\mathbf{R}^n$ is contained into another subspace V and $\dim U = \dim V$, then $U = V$.

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Answer Keys to Practice Midterm 2

Problem 1. (1). A basis of U_2 is $\mathcal{B} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$. The dimension of U_2 is 3. An isomorphism between U_2 and $\mathbf{R}^3$ is given by $L\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. The inverse is given by $L^{-1}\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix} \right) = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$.

(2). A basis of W is $\mathcal{B} = \left(\begin{pmatrix} -3 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$. The dimension of W is 2.

Problem 2. A basis of the subspace of even polynomials of degree at most 3 is $\mathcal{B} = \big(1, x^2\big)$. The dimension is 2.

Problem 3. (1). A basis of the image of T is $\mathcal{B} = \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 3 & 2 \end{pmatrix} \right)$. The dimension of the image is 4. The dimension of the kernel is $\dim \operatorname{Ker}(T) = \dim \mathbf{R}^{2\times 2} - \dim \operatorname{Im}(T) = 4 - 4 = 0$. Therefore the empty set is a basis of the kernel. The linear function T is an isomorphism.

(2). A basis of the image of T is $\mathcal{B} = \big(1, 2x-2, 3x^2-6x\big)$. Its dimension is 3. A basis of the kernel of T is $\mathcal{C} = \big(1\big)$. Its dimension is 1. The linear function T is not an isomorphism.

(3). A basis of the image is $\mathcal{B} = \big(-2, -2x, 2-2x^2\big)$. The dimension of the image is 3. Therefore the dimension of the kernel is $\dim \operatorname{Ker}(T) = 3 - 3 = 0$. It follows that $\operatorname{Ker}(T) = \{0\}$. The linear function T is an isomorphism.

(4). A basis of the image of T is $\mathcal{B} = \left(\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right)$. Its dimension is 2. A basis of the kernel of T is $\mathcal{C} = \big(x^2 - x\big)$. Its dimension is 1. The linear function T is not an isomorphism.

Problem 4. A basis of U is $\left(\begin{pmatrix} 1/2 \\ 5/4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right)$. A basis of $U^\perp$ is $\left(\begin{pmatrix} 1 \\ 0 \\ -1/2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -5/4 \\ 1 \end{pmatrix} \right)$. Both U and $U^\perp$ have dimension 2.

Problem 5. One possible basis of $\mathbf{R}^3$ is $\mathcal{B} = \left(\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \right)$. The diagonal matrix B that represents T with respect to $\mathcal{B}$ is $B = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$. The linear transformation T is an isomorphism because B is invertible (its rank is 3).

Problem 6. A basis of the span is $\mathcal{B} = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right)$. Its dimension is 3. The vector $\begin{pmatrix} 7 \\ 3 \\ 3 \\ 3 \end{pmatrix}$ can be written as a linear combination of the vectors of the basis $\mathcal{B}$ as follows

$$\begin{pmatrix} 7 \\ 3 \\ 3 \\ 3 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

Therefore its $\mathcal{B}$-coordinates are $\left[\begin{pmatrix} 7 \\ 3 \\ 3 \\ 3 \end{pmatrix} \right]_{\mathcal{B}} = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$.

Problem 7. The given vectors form a basis of $\mathbf{R}^3$ if $k \neq 1$ and $k \neq -1$.

Problem 8. F, F, T, T.

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Practice Final Exam

Problem 1. For each of the following matrices

$$A = \begin{pmatrix} 1 & 4 \\ 4 & 7 \end{pmatrix}, \qquad A = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

find an invertible matrix S and a diagonal matrix B such that $B = S^{-1}AS$, in case the matrix A is diagonalizable.

Problem 2. For which values of the real parameter k is the matrix $A = \begin{pmatrix} 2 & 1 & 2 \\ k & 2 & k-3 \\ 1 & -1 & 1 \end{pmatrix}$ diagonalizable? (*Hint:* The characteristic polynomial of A is $p(\lambda) = |A - \lambda I| = -(\lambda+1)(\lambda-3)^2$).

Problem 3. **(1)**. Find an orthonormal basis for the plane in $\mathbf{R}^4$ spanned by the vectors $(1, 1, 1, 1)$ and $(1, 9, -5, 3)$. **(2)**. Find an orthonormal basis of $\mathbf{R}^3$ starting from the vectors $(1, 1, 1)$, $(1, 0, 1)$ and $(0, 1, -1)$. **(3)**. Find an orthonormal basis for the plane in $\mathbf{R}^3$ defined by $x + y + z = 0$ (find first a basis for the plane).

Problem 4. **(1)**. A matrix A is called *skew-symmetric* if $A^T = -A$. Find a basis for the subspace C_2 of $\mathbf{R}^{2\times 2}$ consisting of all 2×2 skew-symmetric matrices. **(2)**. A matrix A is called *symmetric* if $A^T = A$. Find a basis for the subspace S_2 of $\mathbf{R}^{2\times 2}$ consisting of all symmetric 2×2 matrices. **(3)**. Are C_2 and S_2 isomorphic as vector spaces?

Problem 5. In this problem we denote by U_2 the set of all 2×2 upper-triangular matrices. Note that U_2 has dimension 3. Consider the linear function $T\colon U_2 \to U_2$ defined by $T(M) = M \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. **(1)**. Write the 3×3 matrix $C_{\mathcal{B}}$ representing T with respect to the basis $\mathcal{B} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$. **(2)**. Write the 3×3 matrix $C_{\mathcal{C}}$ representing T with respect to the basis $\mathcal{C} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$. **(3)**. Find bases and dimensions for the kernel and image of T. Say whether T is an isomorphism.

Problem 6. Recall that the columns of an orthogonal matrix form an orthonormal basis. **(1)**. Find an orthogonal matrix of type $\begin{pmatrix} 2/3 & 2/3 & 1/3 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ a & b & c \end{pmatrix}$. **(2)**. Find an orthogonal matrix of type $\begin{pmatrix} a & b & 2/3 \\ 1/(3\sqrt{2}) & c & 2/3 \\ -4/(3\sqrt{2}) & 0 & d \end{pmatrix}$.

Problem 7. Consider the following data:

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 2 & 1 & 3 \\ 4 & 1 & 0 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

(1). Find the solutions of the system $A\vec{x} = \vec{b}$. **(2)**. Find A^{-1}. Find the product $A^{-1}\vec{b}$. **(3)**. Is the solution-vector of the first point equal to the product $A^{-1}\vec{b}$ of the second point? Can you motivate why?

Problem 8. Consider the matrix $A = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & k \end{pmatrix}$ depending on the real parameter k. **(1).** Find the determinant of A and say for which values of k is the matrix A invertible. **(2)**. Find the dimensions of the kernel and the image of A as k varies. **(3)**. For $k = 4$ find the eigenvalues of A. Say whether A is diagonalizable for $k = 4$ (do not calculate eigenvectors).

Problem 9. Consider the linear subspace of $\mathbf{R}^4$ defined as:

$$V = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 6 \\ 4 \\ 6 \\ 4 \end{pmatrix} \right\}.$$

(1). Find a basis of V. Find a basis of $V^{\perp}$. **(2).** Find an orthonormal basis for V. **(3).** Write the formula of the orthogonal projection in $\mathbf{R}^4$ onto V. Write the matrix P that represents proj_V with respect to the standard basis of $\mathbf{R}^4$, namelu $\mathrm{proj}_V(\vec{x})$. **(4).** Find a basis of the kernel of P, and check that $\ker(P) = V^{\perp}$.

Problem 10. Consider a *linear* transformation $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ satisfying $T\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $T\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. **(1)**. Find the matrix B that represents T with respect to the basis $\mathcal{B} = \left(\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right)$. **(2)**. Find the matrix A that represents T with respect to the standard basis $\mathcal{E}$ of $\mathbf{R}^2$.

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Solutions Practice Final Exam

Problem 1. (1). $B = \begin{pmatrix} 9 & 0 \\ 0 & -1 \end{pmatrix}$, $S = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$. **(2).** $B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $S = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Problem 2. The matrix is diagonalizable only if $k = 1$. In fact in this case the algebraic multiplicity of each eigenvalue equals its geometric multiplicity.

Problem 3. (1). $u_1 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $u_2 = \frac{1}{10}\begin{pmatrix} -1 \\ 7 \\ -7 \\ 1 \end{pmatrix}$.

(2). $u_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $u_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, $u_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. **(3).** $u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $u_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$.

Problem 4. A basis of C_2 is formed by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. A basis of S_2 is formed by the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The subspaces C_2 and S_2 are not isomorphic because they have different dimensions.

Problem 5. (1). $B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. **(2).** $C = \begin{pmatrix} 1/2 & 0 & -1/2 \\ 1 & 1 & 1 \\ -1/2 & 0 & 1/2 \end{pmatrix}$.

(3). A basis of the image of T is formed by the matrices $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. A basis of the kernel of T is formed by the matrix $\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$.

(4). T is not an isomorphism because the dimension of the kernel is not equal to zero (or because the determinant of B is not zero).

Problem 6. The columns of an orthogonal matrix form an orthonormal basis. Hence if $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are the columns of an orthogonal matrix, then they satisfy the following relations:

$$\vec{v}_1 \cdot \vec{v}_1 = 1,\ \vec{v}_2 \cdot \vec{v}_2 = 1,\ \vec{v}_3 \cdot \vec{v}_3 = 1,\ \vec{v}_1 \cdot \vec{v}_2 = 0.\ \vec{v}_1 \cdot \vec{v}_3 = 0,\ \vec{v}_2 \cdot \vec{v}_3 = 0.$$

In other words each column has length one, and any two distinct columns are perpendicular. One possible way to solving the conditions, is to first solve the first three in order to obtain two possible values of a, two of b, and two of c. Then with a trial-and-error process we can find which values of a, b, c satisfy also the remaining three conditions. **(1).** $\begin{pmatrix} 2/3 & 2/3 & 1/3 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/(3\sqrt{2}) & 1/(3\sqrt{2}) & -2\sqrt{2}/3 \end{pmatrix}$. **(2).** $\begin{pmatrix} 1/(3\sqrt{2}) & -1/\sqrt{2} & 2/3 \\ 1/(3\sqrt{2}) & 1/\sqrt{2} & 2/3 \\ -4/(3\sqrt{2}) & 0 & 1/3 \end{pmatrix}$.

Problem 7. The solution of the system is $(x, y, z) = (-5, 22, -4)$. The inverse of A is $\begin{pmatrix} -3 & 1 & -1 \\ 12 & -4 & 5 \\ -2 & 1 & -1 \end{pmatrix}$.

Problem 8. **(1).** $|A| = -4k + 16$. The matrix is invertible for $k \neq 4$, and it is not invertible for $k = 4$. **(2).** We can row-reduce the matrix A to $\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & k-4 \end{pmatrix}$. We see that if $k = 4$, then the rank of A is 3. Moreover if $k \neq 4$, then the rank of A is 4. As the dimension of the image of A is equal to the rank of A, we deduce that the image of A has dimension 4 for $k \neq 4$, and 3 for $k = 4$. By the Rank-Nullity Theorem the kernel of A has dimension 0 for $k \neq 4$, and dimension 1 for $k = 4$. **(3).** The eigenvalues of A are $0, -2, 2, 5$. As we have 4 distinct eigenvalues, the matrix A is diagonalizable.

Problem 9. **(1).** A basis of V is formed by the vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 6 \\ 4 \\ 6 \\ 4 \end{pmatrix}$. A basis of the orthogonal complement $V^{\perp}$ is formed by the vectors $\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$. **(2).** An orthonormal basis of V is formed by the vectors $\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ and $\frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$. **(3).** The matrix that represents the orthogonal projection onto V is $P = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$. **(4).** A basis of the kernel of P is formed by the vectors $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$.

Problem 10. **(1).** $B = \begin{pmatrix} -12/5 & 3/5 \\ 1/5 & 1/5 \end{pmatrix}$. **(2).** $S_{\mathcal{B}\to\mathcal{E}} = \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}$. $A = S_{\mathcal{B}\to\mathcal{E}} B S_{\mathcal{B}\to\mathcal{E}}^{-1} = \begin{pmatrix} -6/5 & 9/5 \\ 1 & -1 \end{pmatrix}$.

MAT 211: INTRODUCTION TO LINEAR ALGEBRA

Problems on subspaces of $\mathbf{R}^n$

Answer True or False to the following questions or statements. Moreover, if the False case occurs, give a concrete example of why the question is false, or give a motivation using your own words.

1) Is a basis of $U = \text{span}\left\{ \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\4\\6\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$ formed by $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\4\\6\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$? If not give a basis of U.

2) Is a basis of $U = \text{span}\left\{ \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$ formed by $v_1 = \begin{pmatrix}1\\2\\3\end{pmatrix}$ and $v_2 = \begin{pmatrix}0\\0\\1\end{pmatrix}$? Do these vectors also form a basis of $\mathbf{R}^3$? If not complete these two vectors to a basis of $\mathbf{R}^3$, namely a find vector v_3 in $\mathbf{R}^3$ such that v_1, v_2, v_3 form a basis of $\mathbf{R}^3$. Make a picture.

3) **True Fact (this is not an exercise):** In general there is a theorem that says that any basis of a subspace V of $\mathbf{R}^n$ can be completed to a basis of $\mathbf{R}^n$.

4) Do any four vectors in $\mathbf{R}^3$ span $\mathbf{R}^3$?

5) Are any three vectors in $\mathbf{R}^4$ linearly independent?

6) Do two not parallel vectors lying in a plane P of $\mathbf{R}^4$ span P? Do they span $\mathbf{R}^4$?

6') Do two parallel vectors lying in a plane P of $\mathbf{R}^4$ span P? Do they span $\mathbf{R}^4$? If not, what do they span?

7) Do seven linearly independent vectors in $\mathbf{R}^7$ span $\mathbf{R}^7$?

8) If in $U = \text{span}\{v_1, v_2, \ldots, v_p\}$ the vectors $v_1, v_2, \ldots, v_p$ are linearly independent (or equivalently not redundant), do they form a basis of U?

9) If four vectors in $\mathbf{R}^5$ span a subspace V of $\mathbf{R}^5$ of dimension four, are they linearly independent?

10) In general, if $v_1, \ldots v_p$ span a subspace V of $\mathbf{R}^n$ of dimension p, are they linearly independent?

11) Let U and V be two subspaces of $\mathbf{R}^n$ such that U is contained in V (we usually write $U \subseteq V$). Is it true that $\dim U \leq \dim V$?

11') Let U and V be two subspaces of $\mathbf{R}^n$ such that U is contained in V. If $\dim U = \dim V$, is it true that $U = V$?

12) Consider subspaces $U = \text{span}\left\{ \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\}$, $V = \text{span}\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\2\\2\end{pmatrix} \right\}$, and $W = \text{span}\left\{ \begin{pmatrix}3\\4\\7\end{pmatrix}, \begin{pmatrix}2\\0\\2\end{pmatrix} \right\}$. Is $U = V$? Is $U = W$? Is $V = W$?

MAT 211: INTRODUCTION TO LINEAR ALGEBRA

SOLUTIONS: Problems on subspaces of $\mathbf{R}^n$

1) Is a basis of $U = \text{span}\left\{ \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\4\\6\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$ formed by $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\4\\6\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$? If not give a basis of U.

The three vectors above do not form a basis of U because the second vector is redundant. We can omit it from the span keeping U unchanged. Therefore we have

$$U = \text{span}\left\{ \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}.$$

Notice that $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\begin{pmatrix}0\\0\\1\end{pmatrix}$ span U (this follows by the fact that U is defined as the span of these two vectors) and are linearly independent. Therefore they form a basis of U.

2) Is a basis of $U = \text{span}\left\{ \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$ formed by $v_1 = \begin{pmatrix}1\\2\\3\end{pmatrix}$ and $v_2 = \begin{pmatrix}0\\0\\1\end{pmatrix}$? Do these vectors also form a basis of $\mathbf{R}^3$? If not complete these two vectors to a basis of $\mathbf{R}^3$, namely find a vector v_3 in $\mathbf{R}^3$ such that v_1, v_2, v_3 form a basis of $\mathbf{R}^3$. Make a picture.

The vectors v_1 and v_2 form a basis of U because they span U and are linearly independent. The dimension of U is 2. However v_1 and v_2 do not form a basis of $\mathbf{R}^3$ because they do not span $\mathbf{R}^3$. In fact $\mathbf{R}^3$ has dimension 3 which means that a basis of $\mathbf{R}^3$ must contain 3 vectors. If we chose $v_3 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, then v_1, v_2, v_3 form a basis of $\mathbf{R}^3$. You can check this by using the criterion to be a basis of $\mathbf{R}^n$ (check that the rank of the matrix having the vectors as columns is 3). Equivalently v_1, v_2, v_3 form a basis because they are linearly independent, and we have seen in class that three independent vectors in $\mathbf{R}^3$ must span $\mathbf{R}^3$. Hence they form a basis. Notice that the choice of v_3 is not unique. In fact any vector v_3 not in U will do the job.

3) **True Fact (this is not an exercise):** In general there is a theorem that says that any basis of a subspace V of $\mathbf{R}^n$ can be completed to a basis of $\mathbf{R}^n$.

4) Do any four vectors in $\mathbf{R}^3$ span $\mathbf{R}^3$?

False. Take for instance $\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\2\\2\end{pmatrix}, \begin{pmatrix}3\\3\\3\end{pmatrix}, \begin{pmatrix}4\\4\\4\end{pmatrix}$. Their span is equal to the span of $\begin{pmatrix}1\\1\\1\end{pmatrix}$. But the span of one non-zero vector is a line through the origin spanned by that vector (which is obviously different from $\mathbf{R}^3$).

5) Are any three vectors in $\mathbf{R}^4$ linearly independent?

False. Take for instance $\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\2\\2\\2\end{pmatrix}, \begin{pmatrix}3\\3\\3\\3\end{pmatrix}$.

6) Do two not parallel vectors lying in a plane P of $\mathbf{R}^4$ span P? Do they span $\mathbf{R}^4$?

True. Two not parallel vectors are independent and span a subspace of dimension two, which is the plane P. However they do not span $\mathbf{R}^4$. In fact we need 4 linearly independent vectors to span $\mathbf{R}^4$.

6') Do two parallel vectors lying in a plane P of $\mathbf{R}^4$ span P? Do they span $\mathbf{R}^4$? If not, what do they span?

False. Two parallel vectors are dependent. Therefore they span a line.

7) Do seven linearly independent vectors in $\mathbf{R}^7$ span $\mathbf{R}^7$?

Yes, they do. Recall that in class we proved that if V is a subspace of $\mathbf{R}^n$ with $\dim(V) = p$ and $v_1, \ldots, v_p$ are linearly independent vectors in V, then they also span V.

8) If in $U = \operatorname{span}\{v_1, v_2, \ldots, v_p\}$ the vectors $v_1, v_2, \ldots, v_p$ are linearly independent (or equivalently not redundant), do they form a basis of U?

Yes, they do. In fact clearly $v_1, \ldots, v_p$ span U, and moreover they are linearly independent by hypothesis.

9) If four vectors in $\mathbf{R}^5$ span a subspace V of $\mathbf{R}^5$ of dimension four, are they linearly independent?

Yes, they are (see the next answer).

10) In general, if $v_1, \ldots v_p$ span a subspace V of $\mathbf{R}^n$ of dimension p, are they linearly independent?

Yes, they are. For the proof we proceed by contradiction. If they would be dependent, then one of them would be redundant which can be disregarded. Therefore we have at most $p-1$ vectors spanning a subspace of dimension p. This is impossible.

11) Let U and V be two subspaces of $\mathbf{R}^n$ such that U is contained in V (we usually write $U \subseteq V$). Is it true that $\dim U \leq \dim V$?

True. We complete a basis of U to a basis of V. Therefore the dimension of U is smaller or equal than the dimension of V.

11') Let U and V be two subspaces of $\mathbf{R}^n$ such that U is contained in V. If $\dim U = \dim V$, is it true that $U = V$?

True.

12) Consider subspaces $U = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$, $V = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \right\}$, and $W = \operatorname{span}\left\{ \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \right\}$. Is $U = V$? Is $U = W$? Is $V = W$?

Why are diagonal matrices important?

Introduction. Two frequent questions I get asked are: *Why do we need to perform a change of basis in order to get a diagonal matrix? Why are diagonal matrices "simpler" than other matrices?* I will try to convince you that diagonal matrices are important by describing a problem in Calculus. The content of these pages goes a little beyond the scope of our course. However if you feel somewhat comfortable with first-order differential equations, you may find the following example interesting.

A simple reason. First of all, a diagonal matrix is a matrix whose entries are zero outside the main diagonal (of course zero entries along the main diagonal are allowed). For instance, the following matrices are diagonal:

$$\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & \pi \end{pmatrix}.$$

One feature of diagonal matrices is that it is very simple to calculate their products with vectors. For instance

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & \pi \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2x \\ 4y \\ \pi z \end{pmatrix}.$$

A more serious reason. In general working with diagonal matrices simplifies calculations a lot. However this is not the only reason. In some situations we are able to solve problems *only if* we can deal with diagonal matrices. We now consider a system of first-order linear equations to illustrate this concept.

Consider two one-variable differentiable functions $x = x(t)$ and $y = y(t)$. These functions are functions of the variable t. You may think that $x(t)$ and $y(t)$ give the coordinates in the plane of a moving particle at time t. We denote by x' and y' the first derivatives of x and y with respect to t. Consider the following system of differential equations:

(1)
$$\begin{cases} x' = x + 2y \\ y' = 4x + 3y \end{cases}.$$

To solve this system it means that we are looking for all pairs of differentiable functions x and y such that themselves together with their derivatives x' and y' satisfy the above system. Solving the system (1) is a very difficult task, unless one has already taken a course in Ordinary Differential Equations. The main difficulty is that x' is expressed as a function of both x and y, and so is y'. We call this type of systems "mixed systems." In order to "unmix" the system we use the theory of change of basis and linear algebra.

First of all we rewrite the previous system in matrix notation:

(2)
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

where A is the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}.$$

The crucial observation is that A is similar to the following diagonal matrix

$$B = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}.$$

The matrix of change of basis is

$$S = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}.$$

Convince yourself that A and B are similar matrices via S by showing that $SB = SA$. *Question: How did you manage to see that A is similar to the diagonal matrix B? Answer: There is a general algebraic procedure to determine a diagonal matrix B and a matrix of change of basis S given a matrix A. This process is called diagonalization and we will learn this techniques towards the end of he course. Warning: There exist matrices that are not similar to any diagonal matrix! However this pathology does not occur in our situation as our matrix A is similar to the diagonal matrix B.*

As A and B are similar, they satisfy the relation $SB = AS$. Solving this equation for A we find $A = SBS^{-1}$. We plug this expression of A into equation (2) to find:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = SBS^{-1} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Multiply (on the left) the both sides of the previous equation by S^{-1} in order to find:

$$S^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix} = BS^{-1} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3}$$

Note that $S^{-1} \begin{pmatrix} x \\ y \end{pmatrix}$ is a vector as S^{-1} is a 2×2-matrix. We introduce new variables u and v (note that these are still functions of t) defined as:

$$\begin{pmatrix} u \\ v \end{pmatrix} = S^{-1} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{4}$$

Taking the derivatives of each side of the previous equation gives

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = S^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix}. \tag{5}$$

At this point by using (4) and (5) we can rewrite equation (3) as

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = B \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

We rewrite the previous equation as a linear system:

$$\begin{cases} u' &= 5u \\ & v' = -v \end{cases}. \tag{6}$$

Note that this system is "unmixed," namely each single equation depends on only one variable. Therefore we can solve each equation separately. The good news is that we are able to solve this type of equations! (I guess these equations are studied either in Calculus 2 or Calculus B-C). Via integration we can show that the solution of the first equation is $u(t) = C_1 e^{5t}$ where C_1 is an arbitrary constant (check that this is indeed the solution). Whereas the solution of the second equation is $v(t) = C_2 e^{-t}$ where C_2 is another arbitrary constant. To conclude we can write the solutions of the system (6) in vector form as:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} C_1 e^{5t} \\ C_2 e^{-t} \end{pmatrix}.$$

These are not yet the solutions of (1), but we are getting there. In view of equation (4) we have

$$S^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} C_1 e^{5t} \\ C_2 e^{-t} \end{pmatrix},$$

moreover by multiplying (on the left) both sides of the previous equation by S we obtain:

$$\begin{pmatrix} x \\ y \end{pmatrix} = S \begin{pmatrix} C_1 e^{5t} \\ C_2 e^{-t} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} C_1 e^{5t} \\ C_2 e^{-t} \end{pmatrix}.$$

Therefore the solutions x and y of (1) are

$$\begin{cases} x(t) &= C_1 e^{5t} + C_2 e^{-t} \\ & y(t) = 2C_1 e^{5t} - C_2 e^{-t} \end{cases}$$

where C_1 and C_2 are arbitrary real constants.

Exercise. Solve the following system of differential equations

$$\begin{cases} x' &= 5x - 4y \\ & y' = 3x - y \end{cases}.$$

Hint: The matrix $A = \begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$ is similar to the diagonal matrix $B = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$. The matrix of change of basis is $S = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Therefore the following relation holds $A = SBS^{-1}$.

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Notes about isomorphisms

Isomorphisms

Definition of isomorphism. A *real* vector space V is a vector space in which the scalars are real numbers. In our course and in the rest of these notes we only consider real vector spaces.

A function $T\colon V \to W$ between two vector spaces V and W is called ***linear*** if it preserves the sum and the scalar multiplication. In formulas this amounts to say that

$$T(x+y) = T(x) + T(y) \quad \text{for all } x, y \text{ in } V$$

and

$$T(cx) = cT(x) \quad \text{for all } x \text{ in } V \text{ and all scalars } c \text{ in } \mathbf{R}.$$

We say that a function $T\colon V \to W$ is an ***isomorphism*** if T is linear and T is invertible. We recall that T is invertible if there exists a linear function $T^{-1}\colon W \to V$ such that

$$T^{-1}\big(T(x)\big) = x \quad \text{for all } x \text{ in } V$$
$$T\big(T^{-1}(y)\big) = y \quad \text{for all } y \text{ in } W.$$

In many cases it is a difficult task to find the inverse T^{-1} of a linear function T. However in order to check that T is an isomorphism, we will usually calculate its kernel and image, as stated in the following theorem.

Theorem 1. *Suppose that $T\colon V \to W$ is a linear function between two vector spaces. Then T is an isomorphism if and only if $\mathrm{Im}(T) = W$ and $\mathrm{Ker}(T) = \{0\}$.*

The above theorem is very general, and it works even if either V or W has infinite dimension.

Can you think about a vector space of infinite dimension? If not, here is an example. The set of all infinite sequences of real numbers

$$(x_0, x_1, x_2, \ldots) \text{ such that } x_i \text{ are in } \mathbf{R} \text{ for all } i = 0, 1, 2, \ldots$$

is a vector space of infinite dimension. We denote it by $\mathbf{R}^\infty$.

Another example of a vector space of infinite dimension is the set $F(\mathbf{R}, \mathbf{R})$ of all functions $f\colon \mathbf{R} \to \mathbf{R}$ from $\mathbf{R}$ to $\mathbf{R}$. The vector spaces P_n of polynomials of degree at most n are finite-dimensional subspaces of $F(\mathbf{R}, \mathbf{R})$. Try to prove that e^x and e^{2x} are linearly independent elements of $F(\mathbf{R}, \mathbf{R})$, but that $\cos^2(x), \sin^2(x), 1$ are linearly dependent in $F(\mathbf{R}, \mathbf{R})$.

Criteria for finite dimensional vector spaces. We have easier criteria when the vector spaces V and W have both finite dimensions. (In our course we will only work with finite-dimensional vector spaces.) The big advantage of working with finite-dimensional spaces is that we have the Rank-Nullity Theorem, which states:

$$\dim \mathrm{Ker}(T) + \dim \mathrm{Im}(T) = \dim V$$

for any linear function $T\colon V \to W$.

In this way we obtain two criteria for a linear function between finite-dimensional vector spaces to be an isomorphism.

Theorem 2 (Criterion 1)**.** *Suppose that $T\colon V \to W$ is a linear function between two vector spaces of finite dimension. Then T is an isomorphism if and only if* $\dim \mathrm{Ker}(T) = 0$ *and* $\dim V = \dim W$*.*

Theorem 3 (Criterion 2)**.** *Suppose that $T\colon V \to W$ is a linear function between two vector spaces of finite dimension. Then T is an isomorphism if and only if* $\dim \mathrm{Im}(T) = \dim W$ *and* $\dim V = \dim W$*.*

Exercise. Prove the above criteria.

In particular we note that if $T\colon V \to W$ is a linear function between two finite-dimensional vector spaces having different dimensions, then T is never an isomorphism. An important theorem in linear algebra states that if V is a real vector spaces of finite dimension n, then we can always find an isomorphism between V and $\mathbf{R}^n$. Therefore V is isomorphic to $\mathbf{R}^n$, and any two real vector spaces of dimension n are isomorphic (because they are both isomorphic to $\mathbf{R}^n$).

Problem. I am going to solve Problem 3 part (3) of the Practice Midterm 2. The problem is the following. Let $T\colon P_2 \to P_2$ be the linear function defined as $T\big(f(x)\big) = f''(x) - 2f(x)$. Find bases of the image and kernel and their dimensions. Say whether T is an isomorphism.

Solution. A polynomial $f(x)$ in P_2 looks like $f(x) = a + bx + cx^2$. The derivatives are $f'(x) = b + 2cx$ and $f''(x) = 2c$. Therefore T can be described as

$$T\big(f(x)\big) = 2c - 2(a + bx + cx^2) = a(-2) + b(-2x) + c(2 - 2x^2).$$

Writing $T(f(x))$ in this way it gives immediately a basis of the image of T. In fact a basis of the image is

$$\mathcal{B} = \Big(-2, -2x, 2 - 2x^2\Big).$$

The dimension of the image is 3. Thanks to Theorem 3 we deduce that T is an isomorphism because $\dim \mathrm{Im}(T) = \dim P_2 = 3$. It follows that the kernel has dimension 0 by the Rank-Nullity Theorem. The only subspace that has dimension 0 is the zero subspace. Hence $\mathrm{Ker}(T) = \{0\}$ and the zero polynomial is its basis.

Problem. Consider the linear function $T\colon P_2 \to P_2$ defined as $T(f(x)) = f'(x)$. Say whether T is an isomorphism.

Solution. A polynomial $f(x)$ in P_2 looks like $f(x) = a + bx + cx^2$. Its derivative is $f'(x) = b + 2cx$. Therefore we can write $T(f(x))$ as

$$T(f(x)) = b + 2cx = b(1) + c(2x).$$

Hence a basis of the image is $\mathcal{B} = \big(1, 2x\big)$. Its dimension is 2. Thanks to Theorem 3 we already see that T is not an isomorphism because $\dim \mathrm{Im}(T) \neq \dim P_2$. The dimension of the kernel is 1, by the Rank-Nullity Theorem. To find a basis of the kernel we solve $T(f(x)) = 0$. Hence we want

$$b + 2cx = 0.$$

As a polynomial is equal to the zero polynomial if and only if all of its coefficients are zero, we obtain

$$b = 0, \quad 2c = 0.$$

Thus a polynomial $f(x) = a + bx + cx^2$ is in the kernel of T if and only if $b = c = 0$. It follows that all polynomials $f(x) = a = a(1)$ are in the kernel, and therefore $\mathcal{B} = \big(1\big)$ is a basis of $\mathrm{Ker}(T)$. There is no surprise here, because we already know from calculus that the first derivative of any constant is zero.

Further reading: Complex Numbers

I am going to show that the set of complex numbers has a vector space structure (over $\mathbf{R}$). What I am going to say here is not relevant to the course, but if you are interested you should continue reading.

Consider the set $\mathbf{C}$ of complex numbers defined as:

$$\mathbf{C} \;=\; \{x + yi \textit{ such that } x \textit{ and } y \textit{ are in } \mathbf{R} \textit{ and } i^2 = -1\}.$$

Note that i is not a real number! In fact the square root of a negative number does not exist. You should think at i as a symbol living outside $\mathbf{R}$, but satisfying some relations with real numbers. Note that any real number x is in particular a complex number because $x = x + 0i$.

The reason why one works with $\mathbf{C}$ rather than $\mathbf{R}$ is that the polynomial $x^2 + 1$ has no real roots, but the complex number i is a root of $x^2 + 1$ (the other root is $-i$). A deep result in algebra is that every non-constant one-single variable polynomial with real coefficients admits one complex root (this theorem is called Fundamental Theorem of Algebra, its proof is very difficult!).

Since we can write any complex number $x + yi$ as

$$x + yi \;=\; x \cdot 1 + y \cdot i,$$

where the elements 1 and i are in $\mathbf{C}$, and x and y are scalars in $\mathbf{R}$, we deduce that $\mathbf{C}$ is a real vector space whose basis is $\mathcal{B} = \big(1, i\big)$. The dimension of $\mathbf{C}$ is 2 and $\mathbf{C}$ is isomorphic to $\mathbf{R}^2$ as a vector space.

Exercise. Find an isomorphism between $\mathbf{C}$ and $\mathbf{R}^2$.

Note that $\mathbf{C}$ has a richer structure than $\mathbf{R}^2$. In fact we can multiply two complex numbers to obtain a third complex number, but we cannot multiply two vectors in $\mathbf{R}^2$ to obtain a third vector of $\mathbf{R}^2$. Certainly we can perform the dot product of two vectors of $\mathbf{R}^2$, but the result will be no longer a vector, rather a scalar.

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FORMULAS ABOUT BASE CHANGE AND MATRIX REPRESENTATION OF A LINEAR FUNCTION

Base change is considered to be one of the most difficult topics in linear algebra. In order to help you to understand this concept and solve the exercises, I prepared these notes in which I summarize the most important formulas about base change. You should compare or supplement these notes with what we covered in the class of Nov. 15th.

1. Base change

Denote by V a vector space of dimension n and fix a basis $\mathcal{B} = (v_1, \ldots v_n)$ of V. Then every element v of V can be uniquely written as a linear combination of the elements forming the basis $\mathcal{B}$:

$$v \;=\; c_1 v_1 + \ldots + c_n v_n \qquad \text{for a unique choice of scalars} \qquad c_1, \ldots, c_n.$$

The scalars $c_1, \ldots, c_n$ are the $\mathcal{B}$-coordinates of v and we write

$$[v]_{\mathcal{B}} \;=\; \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

In this way we can represent any element v of V as a column vector with n entries.

Now suppose that we have another basis $\mathcal{C} = (w_1, \ldots, w_n)$ of V. (You should think that $\mathcal{B}$ is the old basis, while $\mathcal{C}$ is a new basis.) Then every element v of V has two different sets of coordinates, the $\mathcal{B}$-coordinates and the $\mathcal{C}$-coordinates. We can pass from one type of coordinates to another thanks to the *matrix of change of basis* $S_{\mathcal{C}\to\mathcal{B}}$. This matrix transforms the $\mathcal{C}$-coordinates of every element v of V, into the $\mathcal{B}$-coordinates of v. Hence for any element v in V we have

$$\boxed{[v]_{\mathcal{B}} \;=\; S_{\mathcal{C}\to\mathcal{B}}\,[v]_{\mathcal{C}}.} \tag{1}$$

The matrix of change of basis $S_{\mathcal{C}\to\mathcal{B}}$ is defined as:

$$\boxed{S_{\mathcal{C}\to\mathcal{B}} \;=\; \begin{pmatrix} | & | & \cdots & | \\ [w_1]_{\mathcal{B}} & [w_2]_{\mathcal{B}} & \cdots & [w_n]_{\mathcal{B}} \\ | & | & \cdots & | \end{pmatrix}.}$$

In other words the columns of $S_{\mathcal{C}\to\mathcal{B}}$ are the $\mathcal{B}$-coordinates of the elements forming the basis $\mathcal{C}$.

Proof of equation (1). Say that $[v]_{\mathcal{C}} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ are the $\mathcal{C}$-coordinates of v with respect to the basis $\mathcal{C}$. This means that

$$v = c_1 w_1 + \ldots + c_n w_n.$$

Therefore starting from the right hand side of (1) we find that:

$$S_{\mathcal{C}\to\mathcal{B}}\,[v]_{\mathcal{C}} \;=\; \begin{pmatrix} | & | & \dots & | \\ [w_1]_{\mathcal{B}} & [w_2]_{\mathcal{B}} & \dots & [w_n]_{\mathcal{B}} \\ | & | & \dots & | \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \;=\; c_1[w_1]_{\mathcal{B}} + \ldots + c_n[w_n]_{\mathcal{B}} \;=$$

$$= \; [c_1w_1 + \ldots + c_nw_n]_{\mathcal{B}} \;=\; [v]_{\mathcal{B}}.$$

□

We can also pass from the $\mathcal{B}$-coordinates of an element v, into the $\mathcal{C}$-coordinates of v. This is achieved by the inverse $S^{-1}_{\mathcal{C}\to\mathcal{B}}$ of $S_{\mathcal{C}\to\mathcal{B}}$. Hence for any element v in V we have

$$\boxed{[v]_{\mathcal{C}} \;=\; S^{-1}_{\mathcal{C}\to\mathcal{B}}\,[v]_{\mathcal{B}}.}$$

2. The matrix associated to a linear function (after having fixed a basis)

After having chosen a basis $\mathcal{B}$ of V, we can represent a linear function $T\colon V \to V$ by a matrix, So let's say that V is a vector space of dimension n, that $T\colon V \to V$ is a linear function, and let's fix a basis $\mathcal{B} = (v_1, \ldots, v_n)$ of V. Then there exists a matrix $C_{\mathcal{B}}$ (depending on the basis $\mathcal{B}$) which transforms the $\mathcal{B}$-coordinates of an element v of V, into the $\mathcal{B}$-coordinates of its image $T(v)$ under T. Hence for any v in V the following relation holds:

$$\boxed{\Big[T(v)\Big]_{\mathcal{B}} \;=\; C_{\mathcal{B}}[v]_{\mathcal{B}}.} \tag{2}$$

The matrix $C_{\mathcal{B}}$ is defined as:

$$\boxed{C_{\mathcal{B}} \;=\; \begin{pmatrix} | & | & \dots & | \\ \Big[T(v_1)\Big]_{\mathcal{B}} & \Big[T(v_2)\Big]_{\mathcal{B}} & \dots & \Big[T(v_n)\Big]_{\mathcal{B}} \\ | & | & \dots & | \end{pmatrix}.}$$

Proof of equation (2). Say that $[v]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ are the $\mathcal{B}$-coordinates of v. This means that $v = c_1v_1 + \ldots + c_nv_n$. Starting from the right hand side of (2) we find that:

$$C_{\mathcal{B}}[v]_{\mathcal{B}} \;=\; \begin{pmatrix} | & | & \dots & | \\ \Big[T(v_1)\Big]_{\mathcal{B}} & \Big[T(v_2)\Big]_{\mathcal{B}} & \dots & \Big[T(v_n)\Big]_{\mathcal{B}} \\ | & | & \dots & | \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \;=\; c_1\Big[T(v_1)\Big]_{\mathcal{B}} + \ldots + c_n\Big[T(v_n)\Big]_{\mathcal{B}} \;=$$

$$= \; \Big[c_1T(v_1) + \ldots + c_nT(v_n)\Big]_{\mathcal{B}} \;=\; \Big[T\big(c_1v_1 + \ldots + c_nv_n\big)\Big]_{\mathcal{B}} \;=\; [T(v)]_{\mathcal{B}}.$$

□

Once we worked out the matrix $C_{\mathcal{B}}$ of a linear function T, it becomes very simple to check whether T is an isomorphism. In fact T is an isomorphism if and only if $\det(C_{\mathcal{B}}) \neq 0$. In addition we can find bases of the kernel and the image of T simply by computing the bases of the kernel and the image of $C_{\mathcal{B}}$. One issue in this context (that we will solve in the next section) is the following: Since we may have many different matrix-representations of a linear function (in fact for every choice of a basis $\mathcal{B}$ we obtain a different matrix $C_{\mathcal{B}}$), why are the information that we get from one representation equal to the information we obtain from a different representation? The point is that even though the matrices that represent a linear function with respect to different basis are not equal, they are however similar matrices. As such they share several important properties. For instance, the dimensions of the kernels and images of two similar matrices are equal, and they have the same determinant. From this we deduce that in order to check if T is an isomorphism, it does not matter which matrix-representation we work with.

3. From one representation of a linear function to one other

Suppose that we have two matrices $C_{\mathcal{B}}$ and $C_{\mathcal{C}}$ that represent a linear function $T\colon V \to V$ with respect to two bases $\mathcal{B}$ and $\mathcal{C}$ of V, respectively. What is the relation between $C_{\mathcal{B}}$ and $C_{\mathcal{C}}$? How much do they differ? Because we have two bases $\mathcal{B}$ (the old one) and $\mathcal{C}$ (the new one) of V, then we also have the matrix of change of basis $S_{\mathcal{C}\to\mathcal{B}}$ from the basis $\mathcal{C}$ to the basis $\mathcal{B}$. The relation between $C_{\mathcal{B}}$ and $C_{\mathcal{C}}$ is the following:

$$\boxed{C_{\mathcal{C}} \;=\; S^{-1}_{\mathcal{C}\to\mathcal{B}}\, C_{\mathcal{B}}\, S_{\mathcal{C}\to\mathcal{B}}.} \tag{3}$$

In other words $C_{\mathcal{B}}$ and $C_{\mathcal{C}}$ are similar matrices, because the matrix $S_{\mathcal{C}\to\mathcal{B}}$ is invertible and the previous equation can be rewritten as

$$S_{\mathcal{C}\to\mathcal{B}}\, C_{\mathcal{C}} \;=\; C_{\mathcal{B}}\, S_{\mathcal{C}\to\mathcal{B}}.$$

The equation (3) can also be rewritten as:

$$\boxed{C_{\mathcal{B}} \;=\; S_{\mathcal{C}\to\mathcal{B}}\, C_{\mathcal{C}}\, S^{-1}_{\mathcal{C}\to\mathcal{B}}.}$$

In conclusion we can calculate a matrix-representation of a linear function starting from any other matrix-representation. Moreover we conclude that:

the matrices that represent a linear function with respect to two different bases are similar.

What properties have similar matrices in common? Here is a list of them.

Theorem 3.1. *Suppose that A and B are similar matrices, so that there exists an invertible matrix S such that $SB = AS$. The the following statements hold.*

- *(i).* $\dim \mathrm{Ker}(A) = \dim \mathrm{Ker}(B)$ *and* $\dim \mathrm{Im}(A) = \dim \mathrm{Im}(B)$.
- *(ii).* $\det(A) = \det(B)$ *and* $\mathrm{tr}(A) = \mathrm{tr}(B)$.
- *(iii). A and B have the same eigenvalues with the same algebraic and geometric multiplicities.*
- *(iv). A and B have the same characteristic polynomial.*

Proof. We prove first the first point. Because A and B are similar matrices we have that $SB = AS$. We define a linear function $T_S\colon \mathrm{Ker}(B) \to \mathrm{Ker}(A)$ as $T(\vec{x}) = S\vec{x}$. This function is well defined because if if $\vec{x}$ belongs to $\mathrm{Ker}(B)$, then $S\vec{x}$ belongs to $\mathrm{Ker}(A)$. In fact

$$\vec{0} = S\vec{0} = S(B\vec{x}) = (SB)\vec{x} = (AS)\vec{x} = A(S\vec{x}).$$

The function T_S is an isomorphism because it admits an inverse, which is defined as $T_S^{-1}(\vec{y}) = S^{-1}\vec{y}$ (you should check that this is the inverse of T_S). Hence $\mathrm{Ker}(A)$ and $\mathrm{Ker}(B)$ are isomorphism vector spaces and in particular they have the same dimension. By the Rank-Nullity Theorem we also deduce that $\dim \mathrm{Im}(A) = \dim \mathrm{Im}(B)$.

Now we prove the fourth point. We have that $B = S^{-1}AS$ as A and B are similar. Recall that Binet's theorem says that $\det(EF) = \det(E)\det(F)$ for any two square matrices E and F of the same size. Hence

$$\begin{aligned} p_B(\lambda) &= \det(B-\lambda I) = \det(S^{-1}AS-\lambda I) = \det(S^{-1}AS - \lambda S^{-1}S) = \\ &= \det\left(S^{-1}(A-\lambda I)S\right) = \det(S^{-1})\det(A-\lambda I)\det(S) = \\ &= \det(S)^{-1}\det(A-\lambda I)\det(S) = \det(A-\lambda I) = p_A(\lambda). \end{aligned}$$

Points two and three follow directly from the last point because the eigenvalues, the determinant, and the trace of a matrix can be recovered from its characteristic polynomial.

□

4. Examples

Example 1. Suppose that $V = \mathbf{R}^{2\times 2}$ and fix the basis

$$\mathcal{B} = \left(\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right).$$

What are the $\mathcal{B}$-coordinates of $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$? In order to answer to this question, we need to write A as a linear combination of the matrices forming the basis $\mathcal{B}$:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = a\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -a & a+c \\ b+d & -b+c \end{pmatrix}.$$

Therefore $a = -1, b = -1, c = 3, d = 4$ and the $\mathcal{B}$-cooridnates are

$$\left[\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \right]_{\mathcal{B}} = \begin{pmatrix} -1 \\ -1 \\ 3 \\ 4 \end{pmatrix}.$$

On the other hand, if a matrix B has $\mathcal{B}$-coordinates $\begin{pmatrix} -2 \\ 1 \\ 2 \\ 3 \end{pmatrix}$, then

$$B = -2\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + 3\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix}.$$

Exercises with solutions about diagonalization

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Exercise 1. Diagonalize, if possible, the following 2×2 matrix:

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}.$$

Solution. First of all we look for the eigenvalues of A by finding the roots of the characteristic polynomial

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 3 \\ 2 & 6-\lambda \end{pmatrix} = (1-\lambda)(6-\lambda) - 6 = \lambda^2 - 7\lambda = \lambda(\lambda - 7) = 0.$$

Therefore the roots are $\lambda_1 = 0$ and $\lambda_2 = 7$, both with algebraic multiplicity one.

Now we compute bases for the eigenspaces associated to the two eigenvalues. The eigenspace E_0 is defined as

$$E_0 = \ker(A - 0I) = \ker \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}.$$

In order to find a basis of the kernel of the above matrix we find the solutions of the linear system

$$\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The associated linear system is

$$\begin{cases} x_1 + 3x_2 = 0 \\ 2x_1 + 6x_2 = 0 \end{cases}$$

from which we can see that the second equation is redundant (being twice the first). Therefore $x_1 = -3x_2$ and $x_2 = t$ is a free variable. The solutions are given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3t \\ t \end{pmatrix} = t \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

which implies that

$$E_1 = \operatorname{span}\left\{ \begin{pmatrix} -3 \\ 1 \end{pmatrix} \right\}.$$

The vector $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ forms a basis of E_0 and is an eigenvector of A.

Now we work out a basis for the eigenspace E_7 associated to the eigenvalue 7. This space is defined as:

$$E_7 = \ker(A - 7I) = \ker \begin{pmatrix} -6 & 3 \\ 2 & -1 \end{pmatrix}.$$

We solve the linear system

$$\begin{pmatrix} -6 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

in order to find a basis of E_7. The associated linear system is

$$\begin{cases} -6x_1 + 3x_2 = 0 \\ 2x_1 - x_2 = 0. \end{cases}$$

The first equation is -3 times the first, this means that it is redundant and hence we can just consider the second equation. Solving for x_1 we get $x_1 = \frac{1}{2}x_2$ and that $x_2 = t$ is the free variable. The solutions are given by:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}t \\ t \end{pmatrix} = t \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$$

and

$$E_7 = \operatorname{span}\left\{ \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \right\}.$$

The vector $\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$ forms a basis of E_7 and it is an eigenvector of A, linearly independent to the other eigenvector we previously found.

As we were able to find two linearly independent eigenvector, the matrix A is diagonalizable. Moreover the matrix that diagonalize A is:

$$S = \begin{pmatrix} -3 & 1/2 \\ 1 & 1 \end{pmatrix},$$

and the matrix

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 7 \end{pmatrix}$$

is the matrix that is similar to A. This means that the equation $S^{-1}AS = B$ holds (there is no need to work out the product, this is a byproduct of the diagonalization process). The idea behind the diagonalization procedure is the following. The matrix A represents a linear transformation T from $\mathbf{R}^2$ to $\mathbf{R}^2$ which is rather complicated. By choosing the "smarter" basis $\mathcal{B} = \left\{ \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \right\}$ for $\mathbf{R}^2$ we have that T is represented by the simpler matrix B, which is diagonal.

Exercise 2. Diagonalize, if possible, the following 3×3 matrix:

$$A = \begin{pmatrix} 3 & 0 & -2 \\ -7 & 0 & 4 \\ 4 & 0 & -3 \end{pmatrix}.$$

This exercise has lots of computations, but it is a good practice.

Solution. The characteristic equation of A is

$$\det(A - \lambda I) = (3-\lambda)(-\lambda)(-3-\lambda) - 8\lambda = -\lambda[(3-\lambda)(-3-\lambda)+8] = -\lambda[\lambda^2 - 1] = -\lambda(\lambda-1)(\lambda+1) = 0.$$

The solutions are $\lambda_1 = 0$, $\lambda_2 = 1$ and $\lambda_3 = -1$. All of them have algebraic multiplicity 1.

The first eigenspace is $E_0 = \ker(A - 0I)$. To find a basis of E_0 we solve the system

$$\begin{pmatrix} 3 & 0 & -2 \\ -7 & 0 & 4 \\ 4 & 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

In order to simplify things, we run Gauss–Jordan on this system which turns out to be equivalent to the following:

$$\begin{pmatrix} 1 & 0 & -2/3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We deduce that $x_3 = 0$ and that $x_1 = \frac{2}{3}x_3 = 0$. Therefore $x_2 = t$ is a free variable and the solutions are given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

The second eigenspace is $E_1 = \ker(A - I)$. To find a basis of E_1 we solve the system

$$\begin{pmatrix} 2 & 0 & -2 \\ -7 & -1 & 4 \\ 4 & 0 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

To simplify things, once again we run Gauss–Jordan to the system, so that it becomes equivalent to

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore $x_2 = -3x_3$ and $x_1 = x_3$, and $x_3 = t$ is a free variable. The solutions are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}.$$

The third eigenspace is $E_{-1} = \ker(A + I)$. To find a basis of E_{-1} we solve the system

$$\begin{pmatrix} 4 & 0 & -2 \\ -7 & 1 & 4 \\ 4 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We run Gauss–Jordan once last time so that it is enough to solve

$$\begin{pmatrix} -1 & 0 & -\frac{1}{2} \\ 0 & -1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We deduce that $x_2 = -\frac{1}{2}x_3$, $x_1 = \frac{1}{2}x_3$, and that $x_3 = t$ is a free variable. The solutions of the system are:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix}.$$

As there are three linearly independent eigenvectors, the matrix A is diagonalizable. The matrix S that diagonalize A is

$$S = \begin{pmatrix} 0 & 1 & 1/2 \\ 1 & -3 & -1/2 \\ 0 & 1 & 1 \end{pmatrix},$$

while the diagonal matrix B that is similar to A is

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

It results that $S^{-1}AS = B$. Notice that the matrix S is not unique, if one finds different basis for the eigenspaces, then S will be different. But in any event the product $S^{-1}AS$ is a diagonal matrix with entries the eigenvaues of A.

Exercise 3. Diagonalize, if possible, the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Solution. The characteristic equation is

$$\det \begin{pmatrix} 1-\lambda & 0 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^3-(1-\lambda) = (1-\lambda)[(1-\lambda)^2-1] = (1-\lambda)(\lambda)(\lambda-2) = 0.$$

The solutions are $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2$. All these solutions have algebraic multiplicity equal to one.

A basis of E_0 is given by the vector $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. A basis of E_1 is given by the vector $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. A basis of E_2 is given by the vector $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

As we found three linearly independent eigenvectors, the matrix A is diagonalizable. The matrices S and B for which $S^{-1}AS = B$ are given by

$$S = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Exercise 4. Diagonalize, if possible, the following 3×3 matrix:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution. The characteristic equation is

$$\det \begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^3 = 0.$$

The solution is $\lambda_1 = 1$ with algebraic multiplicity three.

A basis of E_1 is given by the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. As we found only one eigenvector, the matrix A is not diagonalizable.

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Solution Problem 21 in Section 3.4

Point (a). We first solve point (a) and find the matrix B via the formula $B = S^{-1}AS$. The matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ is the matrix that represents a linear transformation T with respect to the standard basis. In order words $T(\vec{x}) = A\vec{x}$. The matrix B is the matrix that represents T with respect to the new basis $\mathcal{B} = (\vec{v}_1, \vec{v}_2)$. In other words $[T(\vec{x})]_{\mathcal{B}} = B[\vec{x}]_{\mathcal{B}}$. In this problem $\vec{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. The matrix S is the matrix of change of basis from the basis $\mathcal{B}$ to the standard basis, its columns are the vectors forming the basis $\mathcal{B}$. Hence

$$S = \begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}.$$

The inverse of S is

$$S^{-1} = \frac{1}{7}\begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix}.$$

Now we can find B according to the formula

$$B = S^{-1}AS = \frac{1}{7}\begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}.$$

Solve now the triple product of matrices. As matrix multiplication is associative, it is the same thing if you first multiply the first matrix with the second matrix, and then with the third matrix, or if you first multiply the second matrix with the third matrix, and then with the first matrix. You should find $B = \begin{pmatrix} 7 & 0 \\ 0 & 0 \end{pmatrix}$.

Point (c). Now we solve point (c). We want to construct B column by column. In order to do this we recall how B is constructed: the columns of B are the $\mathcal{B}$-coordinates of the images (via A) of the vectors forming the basis $\mathcal{B}$. In other words

$$B = \begin{pmatrix} | & | \\ [T\begin{pmatrix} 1 \\ 3 \end{pmatrix}]_{\mathcal{B}} & [T\begin{pmatrix} -2 \\ 1 \end{pmatrix}]_{\mathcal{B}} \\ | & | \end{pmatrix}.$$

Let's compute the first column of B. The vector $T\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is computed as

$$T\begin{pmatrix} 1 \\ 3 \end{pmatrix} = A\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 21 \end{pmatrix}.$$

Now we look for the coordinates of $\begin{pmatrix} 7 \\ 21 \end{pmatrix}$ with respect to the basis $\mathcal{B}$. In order to do this we want to write $\begin{pmatrix} 7 \\ 21 \end{pmatrix}$ as linear combination of $\vec{v}_1, \vec{v}_2$:

$$\begin{pmatrix} 7 \\ 21 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2\begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

One minute of thinking (or by solving the system for c_1 and c_2) gives that $c_1 = 7$ and $c_2 = 0$. Therefore

$$[\begin{pmatrix} 7 \\ 21 \end{pmatrix}]_{\mathcal{B}} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}.$$

Now we compute the second column of B. We first compute

$$T\begin{pmatrix} 1 \\ 3 \end{pmatrix} = A\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Afterwards we write $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as linear combination of the elements of the basis $\mathcal{B}$, but this is very easy:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 0 \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Therefore

$$[\begin{pmatrix} 0 \\ 0 \end{pmatrix}]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and the matrix B is

$$B = \begin{pmatrix} 7 & 0 \\ 0 & 0 \end{pmatrix}.$$

Notice that B looks much simpler than A, so that working in a opportune basis different from the standard one may simplify the matrix that represents a transformation, and hence calculations.

Hints for Homework 8

Problem. Find a basis $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ of $\mathbf{R}^3$ such that the matrix B representing the reflection in $\mathbf{R}^3$ about the plane P defined by $x + y + z = 0$ is diagonal. Then find the $\mathcal{B}$-coordinates of the image under the reflection about P of the vector $\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$.

Solution. We are looking for a smart choice of a basis of $\mathbf{R}^3$, keeping in mind that we are dealing with a plane. I claim that a suitable basis would be that consisting of two vectors $\vec{v}_1$ and $\vec{v}_2$ that span the plane P, and a third vector $\vec{v}_3$ that is perpendicular to the plane. In the last part of this note we will check that the matrix B is diagonal.

We now find $\vec{v}_1$ and $\vec{v}_2$. These vectors are two vectors that span the plane. To find them, we write the solutions of $x+y+z = 0$ in parametric form. As $x = -y-z$, we have that $y = t$ and $z = s$ are free variables. The solutions are

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore we take

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Now we look for a vector $\vec{v}_3$ perpendicular to the plane. Recall that two vectors are perpendicular if their dot product is zero. Moreover, a vector is perpendicular to the plane if it is perpendicular to the vectors spanning the plane. Hence we need to find a vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 0.$$

Solving the dot products we end up with two equations:

$$-x + y = 0 \quad \text{and} \quad -x + z = 0.$$

One solution of this system is $\vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Now we check that the matrix B that represents the reflection T about the plane P with respect to the basis $\mathcal{B} = (\vec{v}_1, \vec{v}_2 \vec{v}_3)$ is diagonal. Denote by T the reflection about the plane $x + y + z = 0$. The matrix B is given by the formula:

$$B = \begin{pmatrix} | & | & | \\ [T(\vec{v}_1)]_{\mathcal{B}} & [T(\vec{v}_2)]_{\mathcal{B}} & [T(\vec{v}_3)]_{\mathcal{B}} \\ | & | & | \end{pmatrix}.$$

Now compute the first column of B. As $\vec{v}_1$ is a vector lying in the plane P, when we reflect it about the same plane we will end up with the same vector. Therefore $T(\vec{v}_1) = \vec{v}_1$. The $\mathcal{B}$-coordinates of $\vec{v}_1$ are

$$[\vec{v}_1]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

because $\vec{v}_1 = 1\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3$. Similarly $T(\vec{v}_2) = \vec{v}_2$ and

$$[\vec{v}_2]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Finally, as $\vec{v}_3$ is perpendicular to the plane, its reflection is equal to $-\vec{v}_3$, namely

$$T(\vec{v}_3) = -\vec{v}_3.$$

The $\mathcal{B}$-coordinates of $-\vec{v}_3$ are

$$[\vec{v}_3]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

Finally the matrix B is

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Now we solve the last question. Recall that the meaning of B is that it transforms the $\mathcal{B}$-coordinates of a vector, into the $\mathcal{B}$-coordinates of the image under T of that vector. In other words, we have the relation $[T(\vec{x})]_{\mathcal{B}} = B[\vec{x}]_{\mathcal{B}}$ for all vectors $\vec{x}$ in $\mathbf{R}^3$. Therefore the question simply asks to find $[T\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}]_{\mathcal{B}}$, which is then equal to $B[\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}]_{\mathcal{B}}$. The $\mathcal{B}$-coordinates of $\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$ are $[\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}]_{\mathcal{B}} = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$ because $\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} = 2\vec{v}_1 - 3\vec{v}_2 + 2\vec{v}_3$. We finally find that

$$[T\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}]_{\mathcal{B}} = B[\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix}.$$

Problem. Consider a basis $\mathcal{B}$ of $\mathbf{R}^3$ consisting of vectors $\mathcal{B} = \left(\vec{u}, \vec{v}, \vec{w}\right)$. Say whether the following vectors are linearly independent or not:

$$\vec{u} - \vec{v} + \vec{w}, \quad \vec{v} - \vec{u} - 3\vec{w}, \quad \vec{u} - \vec{v} - \vec{w}.$$

Then say what is the dimension of their span, and describe it geometrically.

Solution. We first find the $\mathcal{B}$-coordinates of the vectors $\vec{u} - \vec{v} + \vec{w}$, $\vec{v} - \vec{u} - 3\vec{w}$, and $\vec{u} - \vec{v} - \vec{w}$. As

$$\vec{u} - \vec{v} + \vec{w} \ = \ 1\vec{u} - 1\vec{v} + 1\vec{w},$$

we have that

$$[\vec{u} - \vec{v} + \vec{w}]_{\mathcal{B}} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Now we move to the second vector. We want to write it as a linear combination of the elements of the basis $\mathcal{B}$. As

$$\vec{v} - \vec{u} - 3\vec{w} \ = \ -1\vec{u} + 1\vec{v} - 3\vec{w},$$

we have that

$$[\vec{v} - \vec{u} - 3\vec{w}]_{\mathcal{B}} = \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix}.$$

Finally we study the third vector. Proceeding as before we find

$$[\vec{u} - \vec{v} - \vec{w}]_{\mathcal{B}} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

Now we put the three $\mathcal{B}$-coordinates vectors in a matrix

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -3 & -1 \end{pmatrix}.$$

The rref of this matrix is

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore only the first two vectors $\vec{u} - \vec{v} + \vec{w}, \quad \vec{v} - \vec{u} - 3\vec{w}$ are linearly independent. They span a subspace of dimension two in $\mathbf{R}^3$. This subspace looks like a plane of $\mathbf{R}^3$ passing through the origin.

Problem. Show that the subset W consisting of all 2×2 matrices A such that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A \;=\; A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is a subspace of $\mathbf{R}^{2\times 2}$. Find a basis and the dimension of W.

Solution. First of all we write W as a subset:

$$W \;=\; \{A \text{ in } \mathbf{R}^{2\times 2} \text{ such that } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A \;=\; A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\}.$$

In order to show that W is a subspace of $\mathbf{R}^{2\times 2}$, we need to show that: 1) W contains the zero matrix, 2) W is closed under addition, and 3) W is closed under scalar multiplication.

We start with 1). The zero matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ belongs to W because the following equation

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \;=\; \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is a true statement, as both sides are equal to the zero matrix.

No we prove that W is closed under addition. In other words we need to prove that if A_1 and A_2 are two elements of W, then also $A_1 + A_2$ is an element of W. To say that A_1 belongs to W it means that:

(1) $$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A_1 \;=\; A_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Similarly, to say that A_2 is an element of W it means that

(2) $$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A_2 \;=\; A_2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We would like to prove that $A_1 + A_2$ is also an element of W. In other words we need to prove the equality

(3) $$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (A_1 + A_2) \;=\; (A_1 + A_2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In order to do so, we start with the left hand side of equation (3), and we try to rewrite it until we get the right hand side of equation (3):

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (A_1 + A_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A_1 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A_2 = A_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + A_2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = (A_1 + A_2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that we have used the distributive property of matrix multiplication, and both equations (1) and (2).

Now we prove that W is closed under scalar multiplication. In other words we want to prove that if A is in W, then also cA is in W for any scalars c. To say that A is in W it means that

(4) $$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A \;=\; A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Recall that we want to prove that cA is also in W, this simply means that we need to show that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(cA) \;=\; (cA)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

We start with left hand side of equation (5), and through a series of equalities, we will get to its right hand side:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(cA) \;=\; c\Big(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}A\Big) = c\Big(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}A\Big) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}(cA).$$

Note that we have used the associativity property of matrix multiplication, and equation (4).

Now we look for a basis of W. First of all we need to understand how we can write a typical element of W in terms of some arbitrary constants. In our case, an element of W is a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We can rewrite this equation by performing the two matrix multiplications:

$$\begin{pmatrix} c & d \\ a & b \end{pmatrix} \;=\; \begin{pmatrix} a & -b \\ c & -d \end{pmatrix}.$$

We conclude that $c = a$ and $d = -b$. Plugging these relations into $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have that a typical element of W is of the form

$$A = \begin{pmatrix} a & b \\ a & -b \end{pmatrix}$$

which can be rewritten in terms of the constants a and b as

$$A = \begin{pmatrix} a & b \\ a & -b \end{pmatrix} \;=\; a\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \;+\; b\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}.$$

Therefore the matrices $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$ span W, and since they are also linearly independent, they actually form a basis of W. The dimension of W is 2.

Problem. Show that the subset W consisting of polynomials $p(x)$ of degree at most 3 such that $2p'(1) - 3p(1) = 0$ is a subspace of P_3. Find then a basis and the dimension of W.

Solution. We can write W as

$$W \;=\; \{\; p(x) = a + bx + cx^2 + dx^3 \text{ such that } a, b, c, d \text{ are scalars and } 2p'(1) - 3p(1) = 0 \;\}.$$

Obviously the zero polynomial belongs to W because the derivative of 0 is 0. To prove that W is closed under addition, we take two polynomials $p(x)$ and $q(x)$ in W and prove that $p(x) + q(x)$ still belongs to W. This amounts to prove that $2\big(p+q\big)'(1) - 3\big(p+q\big)(1) = 0$, but this is true because

$$\begin{aligned} 2\big(p+q\big)'(1) - 3\big(p+q\big)(1) \;&=\; 2\big(p'(1) + q'(1)\big) - 3\big(p(1) + q(1)\big) = \\ &=\; \big(2p'(1) - 3p(1)\big) \;+\; \big(2q'(1) - 3q(1)\big) = 0 + 0 = 0. \end{aligned}$$

Notice that the quantities $2p'(1) - 3p(1)$ and $2q'(1) - 3q(1)$ are both equal to zero because p and q are two polynomials in W.

Finally, in order to prove that W is closed under scalar multiplication, we take a polynomial $p(x)$ in W and prove that $cp(x)$ is in W for any scalar c. As

$$2(cp)'(1) - 3(cp)(1) \;=\; 2cp'(1) - 3cp(1) = c\big(2p'(1) - 3p(1)\big) = c \cdot 0 = 0,$$

we have that $cp(x)$ is in W.

Now we look for a basis of W. To this end we write a typical element of W in terms of arbitrary constants. A typical element of W is a polynomial $p(x) = a + bx + cx^2 + dx^3$ such that $2p'(1) - 3p(1) = 0$. We write this condition explicitly. The derivative of $p(x)$ is $p'(x) = b + 2cx + 3dx^2$. Moreover we note that $p(1) = a + b + c + d$ and $p'(1) = b + 2c + 3d$. To say that $2p'(1) - 3p(1) = 0$ it means that

$$2\big(b + 2c + 3d\big) - 3\big(a + b + c + d\big) = 0.$$

But this is equivalent to

$$-3a - b + c + 3d = 0.$$

We can solve for c, for instance, so that $c = 3a + b - 3d$. Then a typical element of W looks like

$$p(x) = a + bx + (3a + b - 3d)x^2 + dx^3 = a(1 + 3x^2) \ + \ b(x + x^2) \ + \ d(-3x^2 + x^3).$$

Therefore a basis of W is formed by the polynomials $\big(1 + 3x^2, x + x^2, -3x^2 + x^3\big)$, and the dimension of W is 3. In practice what this problem is saying is that the the polynomials of degree at most three satisfying $2p'(1) - 3p(1) = 0$ must be linear combinations of $1 + 3x^2, x + x^2$ and $-3x^2 + x^3$.

Problem. Consider the function $T\colon \mathbf{R}^{2\times 2} \to \mathbf{R}^{2\times 2}$ defined by $T(A) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} A$. Show that T is linear. Then find bases of the kernel and image of T. Say whether T is an isomorphism.

Solution. In order to prove that T is linear, we need to prove that it preserves the sum and the scalar multiplication. In other words we need to prove that

$$T(A + B) \ = \ T(A) + T(B) \quad \text{ for all } A \text{ and } B \text{ in } \mathbf{R}^{2\times 2}$$

and

$$T(cA) \ = \ cT(A) \quad \text{ for all } A \text{ in } \mathbf{R}^{2\times 2} \text{ and scalars } c.$$

We start by proving the first condition:

$$T(A + B) \ = \ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}(A + B) \ = \ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} A \ + \ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} B \ = \ T(A) \ + \ T(B).$$

Notice that we have only used the distributive property of matrix multiplication. Now we prove the other condition:

$$T(cA) \ = \ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}(cA) \ = \ c\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} A \ = \ cT(A).$$

In order to find the kernel of T, we solve the equation $T(A) = 0$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix in $\mathbf{R}^{2\times 2}$. Then

$$T(A) \ = \ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} A \ = \ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \ = \ \begin{pmatrix} a + 2c & b + 2d \\ 3a + 4c & 3b + 4d \end{pmatrix} \ = \ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

yields equations:

$$a + 2c = 0, \qquad b + 2d = 0, \qquad 3a + 4c = 0, \qquad 3b + 4d = 0.$$

But these equations admit the only solution $a = 0$, $b = 0$, $c = 0$, $d = 0$. Therefore $Ker(T) = \{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\}$. The basis of $Ker(T)$ is formed by the zero matrix, and its dimension is zero.

Now we find a basis of the image of T. For this we want solve $B = T(A)$. This is equivalent to

$$B \ = \ T(A) \ = \ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} A \ = \ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \ = \ \begin{pmatrix} a + 2c & b + 2d \\ 3a + 4c & 3b + 4d \end{pmatrix}.$$

Now we want to write the previous matrix in terms of the free variables a, b, c, d:

$$\begin{pmatrix} a + 2c & b + 2d \\ 3a + 4c & 3b + 4d \end{pmatrix} \ = \ a\begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} \ + \ b\begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} \ + \ c\begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix} \ + \ d\begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix}.$$

Therefore a basis of the image is formed by the matrices $\begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix}$ and its dimension is 4. Since also $\mathbf{R}^{2\times 2}$ is of dimension 4, we have that $Im(T) = \mathbf{R}^{2\times 2}$.

We can say that T is an isomorphism because $Ker(T) = \{0\}$ and $Im(T) = \mathbf{R}^{2\times 2}$.