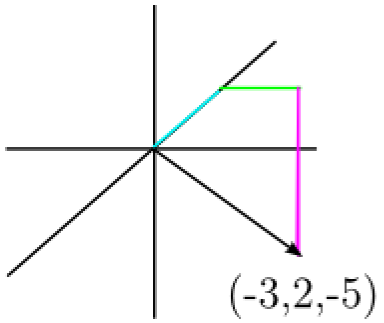
MAT 211 - Fall 2012

Introduction to Linear Algebra



$$Ax = b$$

$$Ax = \lambda x$$

Fall 2012 Office Hours	
M	By Appointment
T	3:00 - 5:00
W	10:00 - 11:00
H	By Appointment
F	By Appointment

Office hours are held in Math Tower 4-117.

Location for Final Exam: Old Engineering 145

Date and Time: December 13, 5:30 - 8:00pm

Handouts

- **Course Syllabus**
- **Style Guide for Mathematical Writing**
- Solutions to Midterm 1 and 2 are on **BlackBoard** (Registered Students Only)

Lecture Notes

I will occasionally post lecture notes on the webpage, especially for material not found in the textbook.

- **Modified Gram-Schmidt Algorithm** - October 26, 2012
- **Fundamental Theorem of Linear Algebra** - October 19, 2012
- **Four Fundamental Subspaces** - October 10, 2012
- **Solving Ax=b (Updated: Version 2)** - September 25, 2012
- **Solving Ax=0** - September 21, 2012
- **LU Factorization** - September 12, 2012
- **Gaussian Elimination** - August 29, 2012
- **Geometry of Linear Equations** - August 27, 2012

Important Dates

- Midterm 1: Monday, October 1, in class.
- Midterm 2: Monday, November 12, in class.
- **Final Exam: Thursday, December 13, 5:30 - 8:00 PM, in Old Engineering 145.**

Course Calendar

Upcoming lecture topics (subject to change). Click for description.

Homework

Homework 12
Due December 5

Homework 11
Due November 28

Homework 10 (Version 3)
Due November 14

Homework 9
Due November 7

Homework 8
Due October 24

Homework 7 (Version 3)
Due October 17

Homework 6
Due October 10

Homework 5
Due October 3

Homework 4
Due September 26

Homework 3
Due September 19

Homework 2
Due September 12

Homework 1
Due September 5

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STAPLE PAGE 1 OF ASSIGNMENT SHEET AS A COVER SHEET FOR YOUR HOMEWORK

MAT 211, Fall 2012, Homework 12 Due: December 5, 2012

Instructions.

(1) Complete the reading assignment.
(2) Do each of the following problems. Solutions must be NEAT and LEGIBLE.
(3) Put solutions IN THE SAME ORDER as the problems on this assignment sheet.
(4) Staple page 1 of assignment sheet as a cover sheet to front of your homework.
(5) Homework is due in class at the start of lecture.

Reading. Skim (quickly read through) the following sections of the textbook:

- Section 7.4

Problem 1. Exercise 7.4.2

Problem 2. Exercise 7.4.4.

Problem 3. Find a matrix A with eigenpairs (eigenvalue and corresponding eigenvector):

$$\lambda = 2 \longleftrightarrow x = \begin{bmatrix} 2 \\ 7 \end{bmatrix},$$

$$\lambda = 3 \longleftrightarrow x = \begin{bmatrix} 3 \\ 10 \end{bmatrix}.$$

Problem 4. Exercise 7.4.12.

Problem 5. Exercise 7.4.22.

Problem 6. Exercise 7.4.32.

Problem 7. Compute the matrix exponential e^A,

$$e^A = I + \frac{1}{1!}A + \frac{1}{2!}A^2 + \cdots + \frac{1}{n!}A^n + \cdots ,$$

where A is the matrix from Problem 6.

Problem 8. Find a formula for the numbers C_n, $n = 0, 1, 2 \ldots$ which satisfy the difference equation $C_{n+2} = 3C_{n+1} - C_n$ and initial conditions $C_0 = 0$, $C_1 = 1$.

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MAT 211, Fall 2012, Homework 11 Due: November 28, 2012

Instructions.

(1) Complete the reading assignment.
(2) Do each of the following problems. Solutions must be NEAT and LEGIBLE.
(3) Put solutions IN THE SAME ORDER as the problems on this assignment sheet.
(4) Staple page 1 of assignment sheet as a cover sheet to front of your homework.
(5) Homework is due in class at the start of lecture.

Reading. Skim (quickly read through) the following sections of the textbook:

- Minors and Laplace Expansion, pp. 268–272
- Eigenvalues and Eigenvectors, pp. 299–302
- Section 7.2, pp. 308–316
- Section 7.3, pp. 319–327
- Section 7.5, pp. 343–353

Problem 1. Let A be the matrix from Exercise 7.3.2.

(a) Find the determinant of A. Is the matrix invertible?
(b) Find the characteristic polynomial of A.
(c) Find the eigenvalues of A.
(d) Find the algebraic multiplicity of each eigenvalue of A.
(e) Find the geometric multiplicity of each eigenvalue of A.

Problem 2. Repeat Problem 1 with the matrix from Exercise 7.3.6.

Problem 3. Repeat Problem 1 with the matrix from Exercise 7.3.10.

Problem 4. Repeat Problem 1 with the matrix from Exercise 7.3.14.

Problem 5. Exercise 7.3.22.

Problem 6. Exercise 7.3.24.

Problem 7. Exercise 7.5.1.

Problem 8. Exercise 7.5.2.

Problem 9. Exercise 7.5.20. Also find an eigenvector for each complex eigenvalue.

Problem 10. Exercise 7.5.24. Also find an eigenvector for each complex eigenvalue.

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MAT 211, Fall 2012, Homework 10 (Version 3) Due: November 14, 2012

Instructions.

(1) Complete the reading assignment.
(2) Do each of the following problems. Solutions must be NEAT and LEGIBLE.
(3) Put solutions IN THE SAME ORDER as the problems on this assignment sheet.
(4) Staple page 1 of assignment sheet as a cover sheet to front of your homework.
(5) Homework is due in class at the start of lecture.

Let A be a $n \times n$ matrix, let λ be a scalar, and let x be a vector. If $Ax = \lambda x$ *and* $x \neq 0$, then we call λ an *eigenvalue* of A and we call x an *eigenvector* for A *corresponding to* λ.

Problem 1. Show that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 7 & -5 \\ 4 & -2 \end{bmatrix}$ corresponding to eigenvalue 2.

Problem 2. Find an eigenvector for $\begin{bmatrix} 7 & -5 \\ 4 & -2 \end{bmatrix}$ corresponding to eigenvalue 3.

Problem 3. Find the eigenvalues of $\begin{bmatrix} 7 & 8 \\ -3 & -4 \end{bmatrix}$. (There are two eigenvalues.)

Problem 4. Find an eigenvector for $\begin{bmatrix} 7 & 8 \\ -3 & -4 \end{bmatrix}$ for each eigenvalue of the matrix.

Problem 5. By definition the *trace* of a $n \times n$ matrix A is the sum of its diagonal entries:

$$\operatorname{tr} A = a_{11} + a_{22} + \cdots + a_{nn}.$$

(a) Find the trace of the matrix in Problem 3.
(b) Find the sum of the eigenvalues of the matrix in Problem 3.
(c) What do you notice about your answers to (a) and (b)?

Problem 6. Suppose x and y are eigenvectors for A corresponding to λ and $x \neq -y$. Show that $x + y$ is also an eigenvector for A corresponding to λ.

Problem 7. Suppose x is an eigenvector for A corresponding to λ and $c \neq 0$ is a scalar. Show that cx is also an eigenvector for A corresponding to λ.

Problem 8. Let A be a matrix with eigenvalue λ. Is the set of vectors $\mathcal{E}_\lambda = \{x : Ax = \lambda x\}$ a vector space? Explain your answer.

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STAPLE PAGE 1 OF ASSIGNMENT SHEET AS A COVER SHEET FOR YOUR HOMEWORK

MAT 211, Fall 2012, Homework 9 Due: October 31, 2012

Instructions.

(1) Complete the reading assignment.
(2) Do each of the following problems. Solutions must be NEAT and LEGIBLE.
(3) Put solutions IN THE SAME ORDER as the problems on this assignment sheet.
(4) Staple page 1 of assignment sheet as a cover sheet to front of your homework.
(5) Homework is due in class at the start of lecture.

Reading. *Skim (quickly read through) the following material.*

§§5.1 – 5.4

Problem 1. Compute $\operatorname{proj}_a b$ (the orthogonal projection of b in the direction of a) and compute $\operatorname{proj}_b a$ (the orthogonal projection of a in the direction of b) where

$$a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}.$$

For Problems 2 – 4 recall: Given a subspace W of $\mathbb{R}^m$ and a basis $w_1, \dots, w_n$ for W, set $A = [w_1| \dots |w_n]$. The *(orthogonal) projection matrix* P which maps $\mathbb{R}^m$ onto W is given by the formula

$$P = A(A^TA)^{-1}A^T.$$

In other words:

- Px belongs to W for all x in $\mathbb{R}^m$,
- $Pw = w$ for every w in $\mathbb{R}^m$; and,
- $x - Px \perp Px$ for all x in $\mathbb{R}^m$.

Problem 2. Suppose that W is a subspace of $\mathbb{R}^{10}$ and W has dimension 7, and let P be the corresponding projection matrix onto W. How many rows and columns does P have? What is the rank of P?

Problem 3. Let W be the plane in $\mathbb{R}^3$ defined by the equation $4x_1 + 5x_2 + 6x_3 = 0$. Find a basis for W and use your answer to compute the projection matrix P that maps $\mathbb{R}^3$ onto W.

Problem 4. Suppose that P is a symmetric $n \times n$ matrix such that $P^2 = P$ and let $W = \mathcal{R}(P)$ be the column space of P.

(a) Show that for every vector w in W, we have $Pw = w$.
(b) Show that for every vector x in $\mathbb{R}^n$, we have $Px - x \perp Px$.
(Hint: Use the rule $\langle Ay, z \rangle = \langle y, A^Tz \rangle$ for any matrix A and vectors y, z.)

Problem 5. Exercise 5.1.26

Problem 6. Exercise 5.4.22

Problem 7. Exercise 5.4.24

Problem 8. Exercise 5.4.30

For Problems 9–10 you may use the Gram-Schmidt Process (from the book) or the Modified Gram-Schmidt Process (from lecture and the class website)

Problem 9. Exercise 5.2.2

Problem 10. Exercise 5.2.4

Problem 11. Exercise 5.2.12

Problem 12. Exercise 5.2.14

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MAT 211, Fall 2012, Homework 8 Due: October 24, 2012

Instructions.

(1) Complete the reading assignment.
(2) Do each of the following problems. Solutions must be NEAT and LEGIBLE.
(3) Put solutions IN THE SAME ORDER as the problems on this assignment sheet.
(4) Staple page 1 of assignment sheet as a cover sheet to front of your homework.
(5) Homework is due in class at the start of lecture.

Reading. *Skim (quickly read through) the following material.*

§5.1 and Definition 5.5.1 on Page 234. Notes on "FTLA" on course website.

Problem 1. True or false.

(a) If the vectors $x_1, \ldots, x_m$ span a subspace S, then $\dim S = m$.
(b) The intersection of two subspaces of a vector space cannot be empty.
(c) If $Ax = Ay$, then $x = y$.
(d) The row space of A has a unique basis that can be computed by reducing A to echelon form.
(e) If a square matrix A has independent columns, so does A^2.

Problem 2. Find a counterexample to the statement: If v_1, v_2, v_3, v_4 is a basis for the vector space $\mathbb{R}^4$, and if W is a subspace, then some subset of the v's is a basis for W.

Problem 3. Let A be a 64×17 matrix of rank 11.

(a) How many independent vectors satisfy $Ax = 0$?
(b) How many independent vectors satisfy $y^T A = 0$?

Problem 4. Exercise 5.1.8

Problem 5. Exercise 5.1.10

Problem 6. Let x and y be arbitrary vectors in $\mathbb{R}^n$.

(a) Prove the parallelogram law: $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$.
(b) Draw a picture in $\mathbb{R}^2$ that explains the name "parallelogram law" for part (a).

Problem 7. Find a vector x orthogonal to the row space, and a vector y orthogonal to the column space, of

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix}.$$

Problem 8. In $\mathbb{R}^3$ find all vectors that are orthogonal to $[1, 1, 1]^T$ and $[1, -1, 0]^T$.
[Hint: $\mathcal{R}(A^T)^\perp = \mathcal{N}(A)$ by FTLA, Part II.]

Problem 9. Exercise 5.1.15 [Hint: Same Idea as Problem 8.]

Problem 10. Exercise 5.1.17 [Hint: Same Idea as Problem 8.]

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MAT 211, Fall 2012, Homework 7 (Version 3) Due: October 17, 2012

Instructions.

(1) Complete the reading assignment.
(2) Do each of the following problems. Solutions must be NEAT and LEGIBLE.
(3) Put solutions IN THE SAME ORDER as the problems on this assignment sheet.
(4) Staple page 1 of assignment sheet as a cover sheet to front of your homework.
(5) Homework is due in class at the start of lecture.

Reading. *Skim (quickly read through) the following material.*

§3.2 and 3.3 in the textbook,
"Four Fundamental Subspaces" on website
(You can also start reading §5.1 to get a head start on next week.)

Problem 1. Find a basis for the column space and a basis for the nullspace of the matrix in Exercise 3.3.6. (Use any method that you wish.)

Problem 2. Find a basis for the column space and a basis for the nullspace of the matrix in Exercise 3.3.8. (Use any method that you wish.)

Problem 3. Exercise 3.3.28

Problem 4. Exercise 3.3.30

Problem 5. Exercise 3.3.36 ($\mathrm{im}(A)$ = column space of A, $\ker(A)$ = nullspace of A.)

For the next set of problems, recall that if A is an $m \times n$ matrix, then

- $\mathcal{R}(A)$ is the column space of A,
- $\mathcal{N}(A)$ is the nullspace of A,
- $\mathcal{R}(A^T)$ is the row space of A
- $\mathcal{N}(A^T)$ is the left nullspace of A.

Each vector in $\mathcal{R}(A)$ or $\mathcal{N}(A^T)$ is a column vector with m entries.
Each vector in $\mathcal{N}(A)$ or $\mathcal{R}(A^T)$ is a column vector with n entries.

Problem 6. You are told that A is a 3×11 matrix with rank 2. What are the dimensions of the four fundamental subspaces of A: $\mathcal{R}(A)$, $\mathcal{N}(A)$, $\mathcal{R}(A^T)$, $\mathcal{N}(A^T)$?

Problem 7. You are told $\dim \mathcal{R}(B) = 4$, $\dim \mathcal{N}(B) = 2$, $\dim \mathcal{R}(B^T) = 4$, $\dim(\mathcal{N}(B^T) = 3$.
How many rows and columns does the matrix B have? What is the rank of B?
Give an example of a matrix B with all of these properties (try to make a simple example).

Problem 8. You are told $\dim \mathcal{R}(C) = 2$, $\dim \mathcal{N}(C) = 4$, $\dim \mathcal{R}(C^T) = 4$, $\dim(\mathcal{N}(C^T) = 3$.
Is there a matrix C whose fundamental subspaces have these dimensions? Explain.

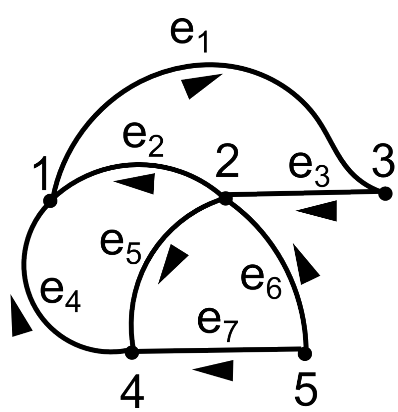
Problem 9. Find a basis for the four fundamental subspaces of the matrix

$$D = \begin{bmatrix} 2 & 7 \\ 4 & 14 \end{bmatrix}$$

Problem 10. Find a basis for the four fundamental subspaces of the matrix

$$E = \begin{bmatrix} 8 & 8 & 10 \\ 4 & 4 & 5 \\ 8 & 8 & 10 \\ 12 & 12 & 15 \end{bmatrix}$$

Let $G = (V, E)$ be a graph with n vertices (nodes) and m edges with arrows. Label the vertices $v_1, \ldots, v_n$ and label the edges $e_1, \ldots, e_m$. The incidence matrix I_G of G is an $m \times n$ matrix. If edge e_i has an arrow pointing from vertex v_j to vertex v_k, then row i of I_G looks as follows: there is a -1 in the j-th column, $+1$ in the k-th column, and there are 0s in all other columns.



Problem 11. Find the incidence matrix of the graph pictured above.

Problem 12. A *loop* in a graph is a path along the edges in the graph which starts and ends at the same vertex (loops do not need to respect the arrows). Two loops starting and ending at the same vertex can be joined to form a third loop. For example, the loops $v_1 \to v_3 \to v_2 \to v_1$ and $v_1 \to v_4 \to v_2 \to v_1$ which both start and end at v_1 can be joined (by concatenation) to form a third loop $v_1 \to v_3 \to v_2 \to v_1 \to v_4 \to v_2 \to v_1$. The *interior* a loop in the plane is the set of points inside the loop. The vertices and points along an edge are not in the interior of the loop. A loop L is *decomposable* if (1) it cotains some edge twice, *or* (2) there are two other loops M and N such that $\text{interior}\, L = \text{interior}\, M \cup \text{interior}\, N$ (interior of L is the union of interiors of M and N). A loop that is *indecomposible* if it is not decomposable. For example, the first two loops above are indecomposable. The third loop above is decomposable, because it contains a repeated edge. Finally, the loop $v_2 \to v_1 \to v_3 \to v_2 \to v_4 \to v_5 \to v_2$ is decomposable, because the interior of this loop is the union of the interiors of the loops $v_2 \to v_1 \to v_3$ and $v_2 \to v_4 \to v_5$.

How many indecomposable loops are in the graph above? How does this number relate to the four fundamental subspaces of the incidence matrix for the graph?

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MAT 211, Fall 2012, Homework 6 — Due: October 10, 2012

Instructions.

(1) Complete the reading assignment.
(2) Do each of the following problems. Solutions must be NEAT and LEGIBLE.
(3) Put solutions IN THE SAME ORDER as the problems on this assignment sheet.
(4) Staple page 1 of assignment sheet as a cover sheet to front of your homework.
(5) Homework is due in class at the start of lecture.

Reading. *Skim (quickly read through) the following material.*

§3.2 in the textbook (You can also start reading §3.3 to get a head start on next week.)

Problem 1. Exercises 3.2.10 and 3.2.12

Problem 2. Exercises 3.2.14 and 3.2.16

Problem 3. Exercise 3.2.34. (Remember 'kernel' is a synonym for 'nullspace'.)

Problem 4. Decide whether or not the following column vectors are linearly independent, by solving $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$:

$$v_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \quad v_1 = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \quad v_1 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}, \quad v_1 = \begin{bmatrix}0\\1\\0\\1\end{bmatrix}$$

Decide also if they span $\mathbb{R}^4$, by trying to solve $c_1v_1 + \cdots + c_4v_4 = [0,0,0,1]^T$.

Problem 5. Decide the dependence or independence of

(a) $[1,1,2]$, $[1,2,1]$, $[3,1,1]$
(b) $v_1 - v_2$, $v_2 - v_3$, $v_3 - v_4$, $v_4 - v_5$ for any vectors v_1, v_2, v_3, v_4
(c) $[1,1,0]$, $[1,0,0]$, $[0,1,1]$, $[x,y,z]$

(Please note that while Problem 4 used column vectors, Problem 5 uses row vectors.)

Problem 6. Prove that if any diagonal element of

$$T = \begin{bmatrix}a & b & c\\0 & d & e\\0 & 0 & f\end{bmatrix}$$

is zero, the rows of are linearly dependent.

Problem 7. Describe geometrically the subspace of $\mathbb{R}^3$ spanned by

(a) $[0,0,0]^T$, $[0,1,0]^T$, $[0,2,0]^T$
(b) $[0,0,1]^T$, $[0,1,1]^T$, $[0,2,1]^T$
(c) all six of these vectors. Which two form a basis?
(d) all vectors with positive components.

Problem 8. Describe, in words or in a sketch of the $x - y$ plane, the column space of $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ and of A^2. Give a basis for the column space.

Problem 9. Exercise 3.2.45.

Problem 10. Exercise 3.2.48. (Remember 'image' is a synonym for 'column space'.)

Problem 11. Exercise 3.2.49.

Problem 12. Find two different bases for the subspace of all vectors in $\mathbb{R}^3$ whose first two components are equal.

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Math 211, Fall 2012, Homework 5 — Due: October 3, 2012

Instructions.

(1) Complete the reading assignment.
(2) Do each of the following problems. Solutions must be NEAT and LEGIBLE.
(3) Put solutions IN THE SAME ORDER as the problems on this assignment sheet.
(4) Staple page 1 of assignment sheet as a cover sheet to front of your homework.
(5) Homework is due in class at the start of lecture.

Reading. *Skim (quickly read through) the following material.*

§§1.2 and 1.3 from textbook; "Solving $Ax = 0$" and "Solving $Ax = b$" on website.

Problem 1. Find all solutions to $Ax = b$ where

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 6 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 0 \\ -10 \end{bmatrix}$$

Problem 2. Find all solutions to $Bx = c$ where

$$B = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & -1 & 2 & 1 \\ 5 & 5 & 2 & -1 \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 4 \\ -4 \end{bmatrix}$$

Problem 3. Find all solutions to $Cx = d$ where

$$C = \begin{bmatrix} 2 & 1 & -5 \\ 4 & 6 & -6 \\ 2 & 5 & -1 \end{bmatrix}, \quad d = \begin{bmatrix} -7 \\ 6 \\ 13 \end{bmatrix}$$

Problem 4. Find all solutions to $Dx = e$ where

$$D = \begin{bmatrix} 4 & 1 & 1 \\ 8 & 2 & 2 \\ 16 & 4 & 4 \end{bmatrix}, \quad e = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$$

Problem 5. What are the ranks of the matrices A, B, C and D from Problems 1–4?

Problem 6. Give 5 examples of 4×4 matrices with ranks 0, 1, 2, 3 and 4, respectively.

Problem 7. Construct the smallest linear system you can with more unknowns than equations, but has no solution.

Problem 8. Find all possible reduced row echelon forms of a 2×3 matrix.

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Math 211, Fall 2012, Homework 4 — Due: September 26, 2012

Instructions.

(1) Complete the reading assignment.
(2) Do each of the following problems. Solutions must be NEAT and LEGIBLE.
(3) Put solutions IN THE SAME ORDER as the problems on this assignment sheet.
(4) Staple page 1 of assignment sheet as a cover sheet to front of your homework.
(5) Homework is due in class at the start of lecture.

Reading. *Skim (quickly read through) the following material.*

Read §3.1 and §3.2 of the textbook. This reading will include some extra material that, while we will not do it in lecture this week, we will cover it a few weeks from now. A "linear transformation" $T_A(x) = Ax$ is a function on columns vectors defined by left multiplying x by a matrix A. The "column space" of a matrix A is the "image" of T_A. The "nullspace" of a matrix A is the "kernel" of T_A.

Read §1.2 and §1.3. In lecture this week and next week, we will finally discuss how to find the "general solution" to a system of m equations in n variables.

Problem 1. An $n \times n$ matrix P is called a *permutation matrix* if the rows of P are obtained by rearranging the rows of the identity matrix I. For example,

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

is a 3×3 permutation: it is obtained from I by $R1 \leftrightarrow R2$, then $R2 \leftrightarrow R3$.

Let P be an $n \times n$ permutation matrix. Show that $P^{-1} = P^T$ (i.e. check $PP^T = I$). *(Hint: Write down the definition of the (i,j) entry of PP^T; then explain why $[PP^T]_{ij} = 1$ if $i = j$, and $[PP^T]_{ij} = 0$ if $i \neq j$.) If stuck, start with the specific 3×3 example above.*

Problem 2. Textbook 3.2.1.

Problem 3. Textbook 3.2.2.

Problem 4. Textbook 3.2.3.

Problem 5. Textbook 3.2.6. (The intersection $V \cap W$ means all vectors x which belong to V *and* W. The union $V \cup W$ means all vectors x which belong to V *or* W.)

Problem 6. Describe the column space (range) and the nullspace (kernel) of the matrices

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Problem 7. A square $n \times n$ matrix A is called *non-singular* if the only solution to the equation $Ax = 0$ is the zero vector $x = 0$. Show that if A has an inverse matrix A^{-1}, then A is non-singular.

Problem 8. A square $n \times n$ matrix A is called *singular* if there is a non-zero vector $x \neq 0$ such that $Ax = 0$. Give an example of a 2×2 singular matrix A with non-zero entries.

Problem 9. Find all solutions x to the equation $Ax = 0$ where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & -1 \\ 4 & 3 & 0 \end{bmatrix}$$

Problem 10. Find all solutions x to the equation $Bx = 0$ where

$$B = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 2 & 2 & 4 & 5 \end{bmatrix}$$

Math 211, Fall 2012, Homework 3 Due: September 19, 2012

STAPLE THIS COVER SHEET TO FRONT OF YOUR HOMEWORK

Name: ______________________________ (You are not required to record your student ID.)

Instructions.

(1) Do each of the following problems. Solutions must be NEAT and LEGIBLE.
(2) Put solutions IN THE SAME ORDER as the problems on this assignment sheet.
(3) Homework is due in class at the start of lecture.

Reading. *Skim (quickly read through) the following material.*

Website: Read notes on LU factorization (posted after lecture on September 12).

Textbook: pp. 214–216 (skip Theorem 5.3.7 and Theorem 5.3.9(c)).

Problem 1. Textbook 2.4.29.

Problem 2. Textbook 2.4.30.

Problem 3. From $AB = C$ find a formula for A^{-1}.

Problem 4. Suppose that A is an $n \times n$ matrix which satisfies the equation

$$A^3 - 2A^2 + 3A - I = 0.$$

Show that the $n \times n$ matrix B defined by the equation $B = A^2 - 2A + 3I$ is invertible and find its inverse.

Problem 5. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 5 & 7 & 8 \\ 0 & 2 & 3 \\ 0 & 0 & 6 \end{bmatrix}$.

What multiple of row 2 of A is subtracted from row 3 during Gaussian elimination? What are the pivots?

Problem 6. Apply forward elimination to produce the factors L and U of $A = LU$ for

(a) $A = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}$, (b) $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$, (c) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{bmatrix}$.

Problem 7. If $A = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, compute A^TB, B^TA, AB^T, and BA^T.

Problem 8. A square matrix S is called *symmetric* if $S^T = S$. Show that for any square matrix A, the matrix $S = A + A^T$ is symmetric.

Problem 9. A square matrix W is called *skew-symmetric* if $W^T = -W$. Show that for any square matrix A, the matrix $W = A - A^T$ is skew-symmetric.

Problem 10. Show that any square matrix A can be written as sum $A = S + W$ where S is symmetric and W is skew-symmetric. (Hint: Use the previous two problems.)

Math 211 (Badger, Fall 2012) Homework 2

Due: September 12, 2012

Instructions.

(1) Do each of the following problems.
(2) If hand-writing your homework, solutions must be NEAT and LEGIBLE.
(3) Put solutions IN THE SAME ORDER as the problems on this assignment sheet.
(4) STAPLE multiple pages together.
(5) Homework is due in class at the start of lecture.

Reading. *Skim (quickly read through) the following material.*

Textbook: pp. 28–33, 72–75 and 80–85.

Problem 1. Evaluate the following expressions:

(a)

$$\begin{bmatrix} 1 & 2 & -3 \\ 3 & 7 & 1 \end{bmatrix} + \begin{bmatrix} 7 & 7 & 8 \\ -1 & 0 & 1 \end{bmatrix}$$

(b)

$$4\begin{bmatrix} 2 & 2 \\ 3 & 6 \\ 1 & -1 \end{bmatrix}$$

Problem 2.

(a) Calculate

$$\begin{bmatrix} 1 & -4 & 0 \\ 2 & 3 & 9 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

by writing the product as a combination of the columns of the 3×3 matrix.

(b) Calculate

$$\begin{bmatrix} 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} 12 & 6 & 3 \\ -5 & -1 & 2 \\ 0 & 0 & 10 \end{bmatrix}$$

by writing the product as a combination of the rows of the 3×3 matrix.

Problem 3. Consider the 2×2 matrices

$$A = \begin{bmatrix} 2 & -7 \\ 11 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

(a) Calculate AB by writing the product as a combination of the columns of A.
(b) Calculate AB again by writing the product as a combination of the rows of B.

Problem 1. Give a short explanation why each of the following equations are true.

$$(a) \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$(b) \qquad \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Problem 5. Find the entries c_{14}, c_{23}, c_{42} and c_{55} of the matrix $C = AB$ where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ -2 & 7 & 0 & 3 & 2 \\ -1 & -1 & 11 & 12 & 1 \\ 8 & 1 & 2 & 3 & 9 \\ 9 & 1 & 14 & 11 & 12 \end{bmatrix}, \qquad B = \begin{bmatrix} 7 & -5 & 4 & 3 & 8 \\ 20 & 17 & -2 & 1 & 1 \\ -8 & 2 & -10 & -12 & 6 \\ 4 & -3 & 2 & 1 & 0 \\ 14 & -17 & 21 & 30 & 19 \end{bmatrix}.$$

Problem 6. Textbook 2.3.14.

Problem 7. Textbook 2.3.18 and 2.3.19.

Problem 8. Textbook 2.3.32.

Problem 9.

(a) If A is invertible and $AB = AC$, prove quickly that $B = C$.

(b) If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ find an example with $AB = AC$ but $B \neq C$.

Problem 10. Show that $\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$ has no inverse by trying to solve

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Problem 11. Textbook 2.4.1 and 2.4.2.

Problem 12. Textbook 2.4.4 and 2.4.5.

Math 211 (Badger, Fall 2012) — Homework 1

Due: September 5, 2012

Instructions.

(1) Do each of the following problems.
(2) If hand-writing your homework, solutions must be NEAT and LEGIBLE.
(3) Put solutions IN THE SAME ORDER as the problems on this assignment sheet.
(4) STAPLE multiple pages together.
(5) Homework is due in class at the start of lecture.

Problem 1. Consider the system of equations

$$\begin{aligned} -x + 3y &= 0 \\ 2x - 2y &= 4 \end{aligned}$$

(a) Write the system of equations in the form $Ax = b$.
(b) Draw the row picture.
(c) Draw the column picture. (Show how b is a combination of the columns of A.)

Problem 2. Consider the system of equations

$$\begin{aligned} -x + 4y \phantom{{}+z} &= -1 \\ 2x - 6y + z &= 3 \\ 4x \phantom{{}-6y} + 6z &= 10 \end{aligned}$$

(a) Write the system of equations in the form $Ax = b$.
(b) Sketch the column picture. (Show how b is a combination of the columns of A.)
(c) What is a solution to the equations?

Problems 3–6

(a) Convert the following linear system to an augmented matrix:

Problem 3

$$\begin{aligned} x - 4y &= 10 \\ x - y &= 12 \end{aligned}$$

Problem 4

$$\begin{aligned} 2x + y &= 2 \\ -8x - 5y &= 1 \end{aligned}$$

Problem 5

$$\begin{aligned} x + 2y \phantom{{}+z} &= 2 \\ -2x - 5y + z &= 0 \\ -2x - 3y + 3z &= 3 \end{aligned}$$

Problem 6

$$\begin{aligned} 10x + 7y + 6z &= -2 \\ 2x + y + z &= 2 \\ -2x + 5y + 4z &= -4 \end{aligned}$$

(b) Do Gaussian (forward) elimination and stop when the matrix is upper triangular. (A row exchange may be helpful for #6.) Write down all your steps.
(c) What are the pivots of the matrix?
(d) Perform back-substitution and write down the solution to the equations.

Problem 7. Solve $Ax = b$ for x where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 8 & 10 \\ 1 & 0 & 0 & -1 \\ -3 & -6 & -5 & -15 \end{bmatrix}, \qquad b = \begin{bmatrix} 3 \\ 1 \\ 8 \\ 6 \end{bmatrix}.$$

Show your work.

Problem 8. Textbook 1.1.29 (i.e. Section 1.1, Exercise 29). Show your work.

Problem 9. Textbook 1.1.31. Show your work.

Problem 10. Textbook 1.1.32. Show your work.

MAT 211, Introduction to Linear Algebra, Fall 2012 **Syllabus**

Simons Center 103 Monday, Wednesday, Friday 12:00 – 12:53

This syllabus contains the policies and expectations that the instructor has established for this course. Please read the entire syllabus carefully before continuing in this course. These policies and expectations are intended to create a productive learning atmosphere for all students. Unless you are prepared to abide by these policies and expectations, you risk losing the opportunity to participate further in the course.

Instructor: Dr. Matthew Badger (badger@math.sunysb.edu)
Office: Math Tower 4-117
Office Hours: Tuesdays 3:00 – 5:00, Wednesdays 10:00 – 11:00, and By Appointment

Course Description

This course introduces linear algebra, a well-developed theory which has many theorems, tools, and applications. In a semester, we will only scratch the surface. The prerequisite for this class is first year calculus, but the material in MAT 211 is of a different character altogether. Derivatives, integrals and limits will be set aside, while we turn back to following type of problem:

(P1) Find numbers x and y so that $3x + 2y = 0$ and $6x - 2y = 3$.

Of course, using high school algebra we quickly find that a solution to (P1) is $x = 1/3$, $y = -1/2$. A related but slightly harder problem is:

(P2) Is $x = 1/3$ and $y = -1/2$ the only solution to $3x + 2y = 0$ and $6x - 2y = 3$?

A third problem is:

(P3) If the equations $ax + by = c$ and $px + qy = r$ have a solution, then how many solutions can they have in total?

We start the course with a concrete, efficient method which solves problems like (P1). Then, taking small steps, we will study different approaches to state and solve (P2), (P3).

Topics to be covered (additional topics presented as time permits):

(1) geometry of linear equations (2 pictures), Gaussian elimination, matrix operations, LU factorization, permutations, matrix inverses, polynomial interpolation;
(2) vector spaces and subspaces in $\mathbb{R}^n$, the four fundamental subspaces of a matrix, linear independence, bases, dimension, rank, graphs and incidence matrices;
(3) dot product, orthogonal vectors, Gram-Schmidt algorithm, orthogonal subspaces, linear transformations, projection onto subspaces, method of least squares;
(4) eigenvectors, eigenvalues, trace and determinant, computing determinants (3 ways), characteristic polynomial, matrix diagonalization, difference equations.

Important Dates

Observed Holidays (Fall 2012)

- September 3–4: No classes (Labor day).
- November 21–25: No classes (Thanksgiving break).

Exam Dates

- October 1: Midterm #1 in class.
- November 5: Midterm #2 in class.
- December 13: Final Exam, 5:30pm – 8:00pm.

Registration Deadlines

- September 2: Last day to drop the class without tuition liability.
- September 11: Last day to drop the class without a "W" on student record.
- October 26: Last day to withdraw from the course; "W" recorded on student record.

Required Resources

- **Class Website:** `www.math.sunysb.edu/~badger/` → Link to MAT 211
- **Lecture Notes:** Attend lectures regularly and take your own notes.
- **Textbook:** Otto Bretcher, *Linear Algebra with Applications*, Fourth Edition, Pearson, Upper Saddle River, New Jersey, 2009. (Reading and homework assignments will be assigned out of this textbook. Make sure you can access a copy of the textbook.)

About Attendance

Attendance is highly encouraged. **Lectures may include material not in the textbook!** The lectures may also present material in a different order than it in the textbook.

Graded Components

- **Homework** – 20% of course average
  There will be a homework assignment due in class on most Wednesdays, starting on Wednesday, September 5. Homework assignments will be posted on the course webpage. The two lowest homework assignment scores for each student will be dropped.
- **Quizzes** – 10% of course average
  Pop quizzes (usually about 10–15 minutes long) will be given throughout the course. Quizzes will not be announced ahead of time. The two lowest quiz scores will be dropped.
- **Midterm Exams** – 40% of course average, 20% each exam
  There will be two closed book, closed notes midterm exams in class. The exams are hearby scheduled for Monday, October 1st and Monday, November 5th.
- **Final Exam** – 30% of course average
  There will be one closed book, closed notes final exam as scheduled by the university on Thursday, December 13th from 5:30 pm – 8:30 pm.

Your *course average* will be determined by a weighted average of the graded components above. Your *final grade* for the class will be based on your course average *and* on your participation. Grades will not be curved. Certain averages will initially guarantee the following grades:

95% guarantees an 'A',

80% guarantees a 'B',

65% guarantees a 'C'.

The average required to obtain a certain grade may be lowered before final grades are assigned, but will not be raised. Students that attend lectures regularly and actively participate in class may receive a higher final grade than is guaranteed by his or her course average alone.

Late Homework Policy

No late homework will be accepted. A student's homework assignment shall be considered late whenever it is not turned into the instructor by the end of lecture on the due date.

Missed Exam Policy

No make-up exams will be given. If a student misses a midterm exam, then the student's final exam grade will be substituted for the missed midterm. A student must sit the final exam at the scheduled time in order to receive a passing grade in the class.

Classroom Policies

Students are expected to arrive to lecture on time and remain until the lecture is concluded. (Leaving early creates distraction and is disrespectful to the instructor and your fellow students.) Cell phones should be silenced and music players disengaged for the duration of the lecture. Tablet and laptop computers should not be used during lecture, except for taking notes.

Disability Support Services

If you have a physical, psychological, medical, or learning disability that may impact your course work, please contact Disability Support Services (631) 632-6748 or

`studentaffairs.stonybrook.edu/dss/`

They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential. Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to the following website:

`www.sunysb.edu/facilities/ehs/fire/disabilities`

Academic Integrity

Each student must pursue his or her academic goals honestly and be personally accountable for all submitted work. Representing another person's work as your own is always wrong. Faculty are required to report any suspected instance of academic dishonesty to the Academic Judiciary. For more comprehensive information on academic integrity, including categories of academic dishonesty, please refer to the academic judiciary website at

`www.stonybrook.edu/uaa/academicjudiciary/`

Critical Incident Management

Stony Brook University expects students to respect the rights, privileges, and property of other people. Faculty are required to report to the Office of Judicial Affairs any disruptive behavior that interrupts their ability to teach, compromises the safety of the learning environment, and/or inhibits students' ability to learn.

Syllabus Revision

The standards and requirements set forth in this syllabus may be modified at any time by the course instructor. Notice of such changes will be by announcement in class and changes to this syllabus will be posted on the course website.

This syllabus was last revised on August 26, 2012.

Abstract: We show how to find an orthonormal set of vectors.

Recall that a set of vectors $\{q_1, \dots, q_k\} \subset \mathbb{R}^n$ is said to be **orthonormal** if

(1) *orthogonal*: $\langle q_i, q_j \rangle = q_i \cdot q_j = 0$ for each pair $i \neq j$
(2) *normalized*: $\|q_i\| = 1$ for all i

The basic problem we want to solve is this:

> Suppose $\{x_1, \dots, x_k\} \subset \mathbb{R}^n$ is a linearly independent set of vectors. How do we find an orthonormal set of vectors $\{q_1, \dots, q_k\}$ with the property that $\operatorname{span}\{q_1, \dots, q_k\} = \operatorname{span}\{x_1, \dots, x_k\}$?

The solution to this problem is called the *Gram-Schmidt Algorithm*. Actually we will present the **modified Gram-Schmidt Algorithm** which has the advantage that it does not introduce radicals until the end of the computation.

As usual we start with an example. Let's begin with vectors $\{x_1, x_2, x_3\} \subset \mathbb{R}^4$ where

$$x_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}.$$

The idea is to take one vector at a time and subtract off the projections in the direction of previous vectors.

Step 1: First Vector. The first step is easiest. Just relabel the first vector:

$$z_1 := x_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Step 2: Second Vector. Modify the second vector by subtracting off the projection of x_2 in the direction of z_1:

$$z_2 := x_2 - \operatorname{proj}_{z_1}(x_2) = x_2 - \frac{\langle x_2, z_1 \rangle}{\langle z_1, z_1 \rangle} z_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Step 3: Third Vector. Modify the third vector by subtracting off the projections of x_3 in the directions of z_1 and z_2:

$$\begin{aligned} z_3 : &= x_3 - \mathrm{proj}_{z_1}(x_3) - \mathrm{proj}_{z_2}(x_3) \\ &= x_3 - \frac{\langle x_3, z_1\rangle}{\langle z_1, z_1\rangle} z_1 - \frac{\langle x_3, z_2\rangle}{\langle z_2, z_2\rangle} z_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} - \frac{4}{4}\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/3 \\ 4/3 \\ -2/3 \end{bmatrix}. \end{aligned}$$

Step 4: Normalize. The vectors $\{z_1, z_2, z_3\}$ are an orthogonal set of vectors whose span is the same as $\{x_1, x_2, x_3\}$. To get an orthonormal set of vectors $\{q_1, q_2, q_3\}$, it remains to normalize the z's:

$$q_1 := \frac{z_1}{\|z_1\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad q_2 := \frac{z_2}{\|z_2\|} = \begin{bmatrix} 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \quad q_3 := \frac{z_3}{\|z_3\|} = \begin{bmatrix} 0 \\ -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1\sqrt{6} \end{bmatrix}.$$

Modified Gram-Schmidt Algorithm

Input: Suppose we have linearly independent vectors $x_1, x_2, \ldots, x_k$.
Output: An orthonormal set of vectors $q_1, \ldots, q_k$ with the property that $\mathrm{span}\{q_1, \ldots, q_j\} = \mathrm{span}\{x_1, \ldots, x_j\}$ for all $j \leq k$.
Algorithm:

Step 1 Compute orthogonal vectors $\{z_1, \ldots, z_k\}$:

$$z_1 = x_1$$

$$z_2 = x_2 - \mathrm{proj}_{z_1}(x_2) = x_2 - \frac{\langle x_2, z_1\rangle}{\langle z_1, z_1\rangle} z_1$$

$$\vdots$$

$$z_k = x_k - \mathrm{proj}_{z_1}(x_k) - \cdots - \mathrm{proj}_{z_{k-1}}(x_k) = x_k - \frac{\langle x_k, z_1\rangle}{\langle z_1, z_1\rangle} z_1 - \cdots - \frac{\langle x_k, z_{k-1}\rangle}{\langle z_{k-1}, z_{k-1}\rangle} z_{k-1}$$

Step 2 Normalize vectors:

$$q_1 = \frac{z_1}{\|z_1\|}, \quad q_2 = \frac{z_2}{\|z_2\|}, \quad \ldots, \quad q_k = \frac{z_k}{\|z_k\|}$$

The inner product (or dot product) of two vectors $x, y \in \mathbb{R}^n$ is defined by

$$\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n = y^Tx.$$

The important properties of the inner product are:

i) (Positivity) $\langle x, x\rangle \geq 0$ for all $x \in \mathbb{R}^n$, and $\langle x, x\rangle = 0 \Leftrightarrow x$ is the zero vector.

ii) (Symmetry) $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in \mathbb{R}^n$.

iii) (Linearity in Each Variable)
- $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$ for all $a, b \in \mathbb{R}$ and $x, y, z \in \mathbb{R}^n$
- $\langle z, ax + by\rangle = a\langle z, x\rangle + b\langle z, y\rangle$ for all $a, b \in \mathbb{R}$ and $x, y, z \in \mathbb{R}^n$

The vectors x and y are said to be *orthogonal* (written $x \perp y$) if $\langle x, y\rangle = 0$.

Suppose that X and Y are subspaces of $\mathbb{R}^n$. We say X and Y are *orthogonal* ($X \perp Y$) if $x \perp y$ for every vector $x \in X$ and for every vector $y \in Y$. For example, suppose that

$$X = \{x \in \mathbb{R}^n : x_2 = x_3 = 0\}, \quad Y = \{y \in \mathbb{R}^n : y_1 = y_3 = 0\}.$$

That is, let X be the x-axis in $\mathbb{R}^3$ and let Y be the y-axis in $\mathbb{R}^3$. Then these are orthogonal subspaces, because if $x = (x_1, 0, 0)$ is a vector on the x-axis and $y = (0, y_2, 0)$ is a vector on the y-axis, then $\langle x, y\rangle = (x_1)(0) + (0)(y_2) + (0)(0) = 0$.

Suppose that W is a subspace of $\mathbb{R}^n$. The *orthogonal complement* of W is another subspace of $\mathbb{R}^n$ (denoted $W^\perp$) defined by

$$W^\perp = \{x \in \mathbb{R}^n : \langle x, w\rangle \text{ for all } w \in W\}.$$

That is, a vector x belongs to the orthogonal complement of W precisely when the vector x is orthogonal to every vector in W. In other words, $W^\perp$ is the largest subspace that is orthogonal to W. It is always true that $(W^\perp)^\perp = W$. For example, if $W = X$ is the x-axis, then the orthogonal complement $X^\perp = \{b \in \mathbb{R}^3 : b_1 = 0\}$ is the yz-plane. Conversely, the orthogonal complement of the yz-plane is the x-axis.

An amazing fact is that if W is one of the four fundamental subspaces of a matrix A, then its orthogonal complement $W^\perp$ is another of the four fundamental subspaces of A. Remember our notation that $\mathcal{R}(A)$ is the column space of A, $\mathcal{N}(A)$ is the nullspace of A, $\mathcal{R}(A^T)$ is the row space of A and $\mathcal{N}(A^T)$ is the left nullspace of A.

Fundamental Theorem of Linear Algebra.

Let A be an $m \times n$ matrix, with rank r. Then:

(Part I - Dimensions)
- $\dim \mathcal{R}(A^T) = \dim \mathcal{R}(A) = r$
- $\dim \mathcal{R}(A^T) + \dim \mathcal{N}(A) = n$
- $\dim \mathcal{R}(A) + \dim \mathcal{N}(A^T) = m$

(Part II - Orthogonal Complements)
- $\mathcal{R}(A^T) \perp \mathcal{N}(A)$, $\quad \mathcal{R}(A^T)^\perp = \mathcal{N}(A)$, $\quad \mathcal{N}(A)^\perp = \mathcal{R}(A^T)$.
- $\mathcal{R}(A) \perp \mathcal{N}(A^T)$, $\quad \mathcal{R}(A)^\perp = \mathcal{N}(A^T)$, $\quad \mathcal{N}(A^T)^\perp = \mathcal{R}(A)$

For example, you can show that the row space $\mathcal{R}(A^T)$ and the nullspace $\mathcal{N}(A)$ are orthogonal by an easy computation. Suppose r is a vector in $\mathcal{R}(A^T)$ and n is a vector in $\mathcal{N}(A)$. Since r is in the row space, $r = A^T x$ for some vector x. Since n is in the nullspace, $An = 0$. To show $r \perp n$, we just compute:

$$\langle n, r\rangle = r^T n = (A^T x)^T n = (x^T A)n = x^T(An) = x^T 0 = 0.$$

(Here we used the facts that for any matrices M, N, $(M^T)^T = M$ and $(MN)^T = N^T M^T$.) We just checked that for any vector r in the row space and for any vector n in the nullspace, $\langle n, r\rangle = 0$. In other words, any vector r in the row space is orthogonal to any vector n in the nullspace. This is what we wanted: $\mathcal{R}(A^T) \perp \mathcal{N}(A)$.

In fact, the fundamental theorem says more. Let us now show that the orthogonal complement $\mathcal{R}(A^T)^\perp$ of the row space $\mathcal{R}(A^T)$ is the nullspace $\mathcal{N}(A)$. By the previous paragraph, we already know that $\mathcal{N}(A) \subset \mathcal{R}(A^T)^\perp$ (the nullspace is contained in the orthogonal complement of the row space). To show we actually have equality, we need to also need to show every vector in the orthogonal complement belongs to the nullspace. To that end, suppose $v \in \mathcal{R}(A^T)^\perp$. This means that $\langle v, r\rangle = 0$ for every vector $v \in \mathcal{R}(A^T)$. If $r = A^T x$, then we have

$$0 = \langle v, r\rangle = r^T v = (A^T x)^T v = (x^T A)v = x^T(Av).$$

There was nothing special about x (it could have been any vector), so we have $x^T(Av) = 0$ for every vector x. This means that $Av = 0$! (Take $x^T = (0, \ldots, 1, \ldots, 0)$, the vector with 1 in j-th position, 0 elsewhere; then the information that $x^T(Av) = 0$ implies that the j-th entry of $Av = 0$, for every j!) We have just shown that every vector v in the orthogonal complement of the row space belongs to the nullspace. This means that $\mathcal{R}(A^T)^\perp = \mathcal{N}(A)$.

The computations for the column space and the left nullspace are similar.

Four Fundamental Subspaces

Selected Notes — October 10, 2012

Abstract: We show how to find a basis for each of the four fundamental subspaces of a matrix. Then we discuss the Fundamental Theorem of Linear Algebra, Part I (the rank-nullity law).

If A is a $m \times n$ matrix, then the *four fundamental subspaces* of A are the following:

Subspaces of $\mathbb{R}^n$

1. The *nullspace* of A is $\mathcal{N}(A) = \{x \in \mathbb{R}^n : Ax = 0\}$.
2. The *row space* of A is $\mathcal{R}(A^T) = \{x \in \mathbb{R}^n : x = A^T y \text{ for some } y\}$.
   (This is all combinations of the rows of A, writing the rows as column vectors.)

Subspaces of $\mathbb{R}^m$

3. The *left nullspace* of A is $\mathcal{N}(A^T) = \{y \in \mathbb{R}^m : A^T y = 0\}$.
   (Equivalently this is all vectors y such that $y^T A = 0$.)
4. The *column space* of A is $\mathcal{R}(A) = \{y \in \mathbb{R}^m : y = Ax \text{ for some } x\}$.
   (This is all combinations of the columns of A.)

In this set of notes, we want to answer two questions. First how do we find a basis for each of the four subspaces? Second what are the dimensions of each of the four subspaces?

1. How to Find a Basis for the Four Fundamental Subspaces

As usual let us work with an example. We will find a basis for each fundamental subspace of the matrix A,

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 3 & 6 & 3 & 7 \end{bmatrix}.$$

Step 0) Use Gaussian elimination to transform $[A|b]$ into echelon form $[U|c]$

$$[A|b] = \begin{bmatrix} 1 & 2 & 1 & 2 & b_1 \\ 1 & 2 & 1 & 3 & b_2 \\ 3 & 6 & 3 & 7 & b_3 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 2 & 1 & 2 & b_1 \\ 0 & 0 & 0 & 1 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 - 2b_1 - b_2 \end{bmatrix} = [U|c]$$

Step 1) Find the pivot variables $x_{i_1}, x_{i_2}, \dots, x_{i_r}$ where $r = \operatorname{rank} A$ is the total number of pivots. A basis for the **column space** of A is the set of columns $i_1, \dots, i_r$ in the *original* matrix A.

Since the pivot variables of A are x_1 and x_4, a basis for the column space is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} \right\}.$$

We had to choose columns from the original matrix A, because Gaussian elimination changes the column picture at each step. To see this in our example, just note that the columns of U all have a zero in the third entry: if these spanned the column space of A then every column of A would also have to have zero in the third entry—which is false!

Step 2) Find the free variables $x_{j_1}, x_{j_2}, \ldots, x_{j_{n-r}}$ where $n-r$ is the number of columns of A without a pivot. There is a basis for the **nullspace** of A, with one vector associated to each free variable. Taking each free variable one at a time, set that free variable $= 1$ and the other free variables $= 0$. Then solve $Ux = 0$ for the vector x with this choice and put x in the basis. Repeat for each free variable.

In our example, the free variables are x_2 and x_3. Take turns setting $x_2 = 1$, $x_3 = 0$ and $x_2 = 0$, $x_3 = 1$ and solve $Ux = 0$ for each choice. A basis for the nullspace of is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Step 3) A basis for the **row space** of A is the set of non-zero rows of U (rewritten as column vectors).

In our example, a basis for the row space is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Notice the number of elements in the basis, which is the number of non-zero rows in U, is just the number of pivots. This means the column space and the row space of a matrix always have the same dimension.

Step 4) A basis for the **left nullspace** of A has $m-r$ vectors, which is the number of zero rows in U. We find a basis as follows. For each zero row, put a vector in the basis whose entries are the coefficients of the vector b in the entry of c corresponding to the zero row.

Returning to our example, there is one zero row of U. As an equation the row represents $0 = -2b_1 - b_2 + b_3$ (note we listed the b_i's in order). Thus a basis for the left nullspace is

$$\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

In general, the number of elements in this basis will equal the number of zero rows.

2. What are the dimensions of the fundamental subspaces?

Let A be $m \times n$ with rank r. Using the bases above, we observe the following:

- $\dim \mathcal{R}(A) = \dim \mathcal{R}(A^T) = r$ (the number of pivots).
- $\dim \mathcal{N}(A) = n - r$ (the number of free columns).
- $\dim \mathcal{N}(A^T) = m - r$ (the number of zero rows).

Notice that the column space and the row space of a matrix have the same dimension (even though the vectors in each subspace live in a different ambient space, $\mathbb{R}^m$ and $\mathbb{R}^n$.)

The nullspace and row space live in $\mathbb{R}^n$; the left nullspace and column space live in $\mathbb{R}^m$. There is an important relationship between the dimensions of the subspaces in each of these pairs of subspaces. It even has an important name:

Fundamental Theorem of Linear Algebra (Part I)**.**
Let A be an $m \times n$ matrix. Then:

- $\dim \mathcal{R}(A^T) + \dim \mathcal{N}(A) = n$
- $\dim \mathcal{R}(A) + \dim \mathcal{N}(A^T) = m$

The first equation is also called the *rank-nullity law*
(because $\operatorname{rank} A = \dim \mathcal{R}(A^T)$, $\operatorname{nullity} A = \dim \mathcal{N}(A)$).

A final word. To remember what are the dimensions of the four fundamental subspaces, it is best just to think about where the bases for each subspace comes from:

- The bases for the column space and row space come from the pivots, so the dimension of each of these subspaces is the rank of the matrix.

- The basis for the nullspace comes from the free columns, so the dimension of the nullspace is the number of free columns.

- The basis for the left nullspace is obtained from the zero rows, so the dimension of the left nullspace is the number of zero rows.

MAT 211 (Badger, Fall 2012) Solving $Ax = b$ (Updated: Version 2)

Selected Notes — September 25, 2012

Abstract: After stating a necessary and sufficient condition for $Ax = b$ to have a solution x, we discuss a procedure to find all possible solutions to $Ax = b$ when at least one exists.

Column Vector Interpretation of $Ax = b$

Let $A = [A_1|A_2|\cdots|A_n]$ be an $m \times n$ matrix, with columns A_1, A_2, …, A_n. Then a solution

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

to the matrix equation $Ax = b$ satisfies

$$x_1A_1 + x_2A_2 + \cdots + x_nA_n = b.$$

The equation $Ax = b$ has a solution exactly when b is a linear combination of the columns of A.

Example. Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$. Then $Ax = b$ has a solution exactly when

$$b = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = (x_1 + x_2) \begin{bmatrix} 1 \\ 2 \end{bmatrix} = (\text{some number}) \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

This means that $Ax = b$ has a solution exactly when the second entry of b is double the first entry of b. So $Ax = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$ has a solution, but $Ax = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ does not.

A few classes ago we defined the *column space* (or *range*) $\mathcal{R}(A)$ of a matrix A to by

$$\mathcal{R}(A) = \text{span(columns of } A),$$

where recall the span of a list of vectors is all possible linear combinations of those vectors. Thus, we can restate the condition for existence of a solution from above, as follows:

Existence Rule. *For any matrix A and right hand side vector b, the equation $Ax = b$ has a solution $x \iff b$ is in the column space $\mathcal{R}(A)$ of A.*

This rule is useful for many theoretical purposes, but is not as useful computationally.

Example: Finding a Particular Solution to $Ax = b$

Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 2 \\ 4 & -2 & 2 & -2 \end{bmatrix}, \qquad b = \begin{bmatrix} 2 \\ -10 \\ 18 \end{bmatrix}.$$

Suppose that we want to solve $Ax = b$. As usual, the first step is Gaussian elimination.

$$[A|b] = \left[\begin{array}{cccc|c} \boxed{1} & 0 & 1 & 0 & 2 \\ 0 & 2 & 2 & 2 & -10 \\ 4 & -2 & 2 & -2 & 18 \end{array}\right] \xrightarrow{R3-4R1} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 1 & 0 & 2 \\ 0 & \boxed{2} & 2 & 2 & -10 \\ 0 & -2 & -2 & -2 & 10 \end{array}\right] \xrightarrow{R3+R2} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 1 & 0 & 2 \\ 0 & \boxed{2} & 2 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = [U|c].$$

We see that A has 2 pivots total, which appear in columns 1 and 2 of U. Therefore, x_1 and x_2 are the pivot variables, and x_3 and x_4 are the free variables. Remember that we get to specify the values of the free variables. So if we want to find just one solution to $Ax = b$ (or $Ux = c$), we should make the simplest possible choice for the free variables.

Set $\boxed{x_3 = 0}$ and $\boxed{x_4 = 0}$. Then the equations $Ux = c$ simplify to

$$\begin{aligned} x_1 &= 2 \\ 2x_2 &= -10 \end{aligned}$$

Solving, we get $\boxed{x_1 = 2}$ and $\boxed{x_2 = -5}$. Thus one particular solution to $Ax = b$ is

$$x = \begin{bmatrix} 2 \\ -5 \\ 0 \\ 0 \end{bmatrix}.$$

FINDING ALL SOLUTIONS TO $Ax = b$.

Suppose that there are two vectors y and z that solve $Ax = b$, i.e. $Ay = b$ and $Az = b$. Subtracting the first equation from the second equation gives:

$$A(z - y) = Az - Ay = b - b = 0.$$

Hence, if y and z both solve $Ax = b$, then $z - y \in \mathcal{N}(A)$ (the nullspace of A). In other words, if y and z both solve $Ax = b$, then $z - y = n$ for some vector $n \in \mathcal{N}(A)$. Moving y to the right hand side, we get the following rule: *If y and z are any two solutions to $Ax = b$, then $z = y + n$ for some vector $n \in \mathcal{N}(A)$.*

On the other hand, suppose that x_p solves $Ax = b$ (so that $Ax_p = b$) and x_n is any vector in $\mathcal{N}(A)$ (so that $Ax_n = 0$). Then $A(x_p + x_n) = Ax_p + Ax_n = b + 0 = b$. That is, *If x_p is a solution to $Ax = b$ and x_n belongs to $\mathcal{N}(A)$, then $x_p + x_n$ is another solution to $Ax = b$.*

Combining the previous two paragraphs, we get the following rule:

> **General Solution.** Let x_p be a *particular solution* to $Ax = b$. Then the *general solution* to $Ax = b$ is given by $x_p + x_n$ where x_n is a vector in the nullspace of A.
>
> (In other words, each vector of the form $x = x_p + x_n$ solves $Ax = b$, and every vector x which solves $Ax = b$ can be written in this form.)

EXAMPLE CONTINUED: FINDING THE GENERAL SOLUTION TO $Ax = b$.

We have found that $x_p = \begin{bmatrix} 2 \\ -5 \\ 0 \\ 0 \end{bmatrix}$ is a particular solution to $Ax = b$ where

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 2 \\ 4 & -2 & 2 & -2 \end{bmatrix}, \qquad b = \begin{bmatrix} 2 \\ -10 \\ 18 \end{bmatrix}.$$

Last class we found that all solutions to $Ax = 0$,

$$\mathcal{N}(A) = \operatorname{span}\left(\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right).$$

Thus the general solution to $Ax = b$ is

$$x = x_p + x_n = \begin{bmatrix} 2 \\ -5 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

for any numbers c_1 and c_2.

Same Matrix, Different Right Hand Side

This time let

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 2 \\ 4 & -2 & 2 & -2 \end{bmatrix}, \qquad b = \begin{bmatrix} 2 \\ -2 \\ 18 \end{bmatrix}.$$

Suppose that we want to solve $Ax = b$. We start with Gaussian elimination.

$$[A|b] = \left[\begin{array}{cccc|c} \boxed{1} & 0 & 1 & 0 & 2 \\ 0 & 2 & 2 & 2 & -2 \\ 4 & -2 & 2 & -2 & 18 \end{array}\right] \xrightarrow{R3-4R1} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 1 & 0 & 2 \\ 0 & \boxed{2} & 2 & 2 & -2 \\ 0 & -2 & -2 & -2 & 10 \end{array}\right] \xrightarrow{R3+R2} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 1 & 0 & 2 \\ 0 & \boxed{2} & 2 & 2 & -2 \\ 0 & 0 & 0 & 0 & 8 \end{array}\right] = [U|c].$$

Look at equation represented by the last row: $0x_1 + 0x_2 + 0x_3 + 0x_4 = 8$, i.e. $0 = 8$.

This can never happen! So for the right hand side $b = \begin{bmatrix} 2 \\ -2 \\ 18 \end{bmatrix}$, $Ax = b$ has no solutions.

How to Determine if $Ax = b$ has a Solution

Applying the same analysis to a general right hand side b, we get the following way to determine if $Ax = b$ has a solution.

> **Solvability Criterion**. Perform Gaussian elimination to transform $[A|b]$ to $[U|c]$. The system of equations $Ax = b$ has a solution if and only if for every zero row of U the corresponding entry of c is zero.

Let's use this with our running example. Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 2 \\ 4 & -2 & 2 & -2 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Do Gaussian elimination.

$$[A|b] = \left[\begin{array}{cccc|c} \boxed{1} & 0 & 1 & 0 & b_1 \\ 0 & 2 & 2 & 2 & b_2 \\ 4 & -2 & 2 & -2 & b_3 \end{array}\right] \xrightarrow{R3-4R1} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 1 & 0 & b_1 \\ 0 & \boxed{2} & 2 & 2 & b_2 \\ 0 & -2 & -2 & -2 & b_3 - 4b_1 \end{array}\right] \xrightarrow{R3+R2} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 1 & 0 & b_1 \\ 0 & \boxed{2} & 2 & 2 & b_2 \\ 0 & 0 & 0 & 0 & b_3 - 4b_1 + b_2 \end{array}\right] =$$

Using the solvability criterion, we see that $Ax = b$ has a solution if and only if $b_3 - 4b_1 + b_2 = 0$.

For example, when $b = \begin{bmatrix} 2 \\ -10 \\ 18 \end{bmatrix}$, we get $(18) - 4(2) + (-10) = 0$.

On the other hand, when $b = \begin{bmatrix} 2 \\ -2 \\ 18 \end{bmatrix}$, we get $(18) - 4(2) + (-2) = 8 \neq 0$.

Rank of a Matrix

Definition. The **rank** of a matrix A is the number of pivots of A. The rank of a matrix A is always less than or equal to the number of rows and number of columns of A: if A is $m \times n$, then $\operatorname{rank} A \leq m$ *and* $\operatorname{rank} A \leq n$.

In the running example, the rank of A was 2.

Matrices with Full Row Rank

If A is $m \times n$ and $\operatorname{rank} A = m$, then after Gaussian elimination U can have no zero rows. Together with the solvability criterion, this observation shows:

Matrices with Full Row Rank. If A has the same number of pivots as rows, then $Ax = b$ has a solution for every b.

Matrices with Full Column Rank

If A is $m \times n$ and $\operatorname{rank} A = n$, then after Gaussian elimination U can have no free columns. This means that $\mathcal{N}(A) = \{\text{zero vector}\}$. Hence:

Matrices with Full Column Rank. If A has the same number of pivots as columns, then for each b the equation $Ax = b$ either has no solutions or has exactly one (a unique) solution.

MAT 211 (Badger, Fall 2012) **Solving $Ax = 0$**

Selected Notes — September 21, 2012

Abstract: We demonstrate an organized procedure to find all possible solutions to $Ax = 0$ for a system of m linear equations in n unknowns.

Nullspace of a Matrix

The nullspace $\mathcal{N}(A)$ of a $m \times n$ matrix A is a vector subspace of $\mathbb{R}^n$ defined by:

$$\mathcal{N}(A) = \{x \in \mathbb{R}^n : Ax = 0\}.$$

If A is a square matrix ($m = n$) and A^{-1} exists, then the null space $\mathcal{N}(A) = \{0\}$ (HOMEWORK). For general matrices A (including some square matrices), it is possible to have non-zero solutions to $Ax = 0$, i.e. there can exist vectors x which are not the zero vector but which still satisfy the equation $Ax = 0$. For example,

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}} \underset{x}{\begin{bmatrix} -2 \\ 1 \end{bmatrix}} = \underset{0}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}.$$

For this matrix A, the vector $x = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is not the only non-zero vector such that $Ax = 0$. Another example is

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}} \underset{x}{\begin{bmatrix} -4 \\ 2 \end{bmatrix}} = \underset{0}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}.$$

We include this example to reiterate, to find the nullspace $\mathcal{N}(A)$ of a matrix A, one needs to find *all solutions* to $Ax = 0$.

How to Solve $Ax = 0$: First Example

Let us start with an example. Suppose we want to find all solutions to $Ax = 0$ for the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 11 \end{bmatrix}.$$

A natural instinct is to convert $Ax = 0$ to augmented matrix form and do Gaussian elimination to simplify the system of equations. Let us first take a look at the augmented matrix for $Ax = 0$ in this example.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 3 & 6 & 11 & 0 \end{array}\right]$$

Note that the last column of the augmented matrix has a zero in each entry of the column. This means that no matter what row operations we do to the augmented matrix during Gaussian elimination, the last column will always just have zeros in each entry! (To convince yourself, perform the operation Row $2 - 3 \times$ Row 1 to the augmented matrix.) The upside here is that it is not necessary to keep track of the right hand side during Gaussian elimination; that is, we may as well do Gaussian elimination directly to the matrix A rather than to the augmented matrix $[A|0]$.

Let's proceed with Gaussian elimination to transform A into an "upper triangular" U:

$$A = \begin{bmatrix} \boxed{1} & 2 & 3 \\ 3 & 6 & 11 \end{bmatrix} \xrightarrow{R2-3R1} \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & 0 & \boxed{2} \end{bmatrix} = U.$$

A few comments are in order. The matrix A has two pivots — $\boxed{1}$ in the $(1,1)$ entry of U and $\boxed{2}$ in the $(2,3)$ entry of U. Note that there is not a pivot in the $(2,2)$ entry of U because 0 is never a pivot. To emphasize again, there is a pivot in columns 1 and 3 of U, but there is no pivot in column 2 of U.

> **Definition.** Suppose you have done Gaussian elimination to transform a $m \times n$ matrix A into an $m \times n$ upper triangular matrix U (for rectangular matrices "upper triangular" means that the (i,j) entry of U is zero whenever $i > j$.)
>
> We call column j of A a **pivot column** if there is a pivot in column j of U.
>
> We call column j of A a **free column** if there is not a pivot in column j of U.

Continuing with our example, columns 1 and 3 are pivots columns of A (and U), but column 2 is a free column of A (and U). This is important information for finding all solutions to $Ax = 0$.

So far we have done Gaussian elimination to transform the equations $Ax = 0$ into simpler equations $Ux = 0$. Writing the system $Ux = 0$ long hand, we have

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_3 &= 0 \end{aligned}$$

We now have two equations with three variables; since the number of equations is less than the number of variables, this system of equations will have more than one solution.

> **Definition.** Suppose A is an $m \times n$ matrix and you are trying to solve $Ax = 0$.
>
> We call x_j a **pivot variable** if column j of A is a pivot column.
>
> We call x_j a **free variable** if column j of A is a free column.

So in this example x_1 and x_3 are pivot variables, while x_2 is a free variable.

> **Rule.** When solving $Ax = 0$ (or equivalently $Ux = 0$), you are allowed to choose any values you want for the free variables.

Set the free variable $\boxed{x_2 = 1}$ in $Ux = 0$. With this choice, $Ux = 0$ looks like

$$\begin{aligned} x_1 + 3x_3 &= -2 \\ 2x_3 &= 0 \end{aligned}$$

Now this is an upper triangular system of two equations in two variables — we know how to solve this type of system — use back substitution!

$$\begin{aligned} 2x_3 = 0 &\Longrightarrow \boxed{x_3 = 0} \\ x_1 + 3x_3 = -2 &\Longrightarrow x_1 + 3(0) = -2 \Longrightarrow \boxed{x_1 = -2} \end{aligned}$$

Finally we have found out that one solution to $Ax = 0$ is $\boxed{x_1 = -2,\ x_2 = 1 \text{ and } x_3 = 0}$. But this is not the only solution, because we could have picked a different value for x_2.

Suppose that in the previous paragraph we picked $x_2 = t$ instead of $x_2 = 1$ (where t represents any real number). Substituting this choice into $Ux = 0$, the system of equations becomes

$$\begin{aligned} x_1 + 3x_3 &= -2t \\ 2x_3 &= 0. \end{aligned}$$

Using back substitution we can solve for x_1 and x_3.

$$2x_3 = 0 \Longrightarrow \boxed{x_3 = 0}$$

$$x_1 + 3x_3 = -2t \Longrightarrow x_1 + 3(0) = -2t \Longrightarrow \boxed{x_1 = -2t}$$

Thus another solution to $Ax = 0$ is $\boxed{x_1 = -2t,\ x_2 = t,\ x_3 = 0 \text{ where } t \text{ is any real number.}}$

If we let t range over all real numbers, then we have found *all solutions* to $Ax = 0$. Hence

$$\mathcal{N}(A) = \left\{ \begin{bmatrix} -2t \\ t \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

Finally note that the nullspace of A is simply the span of the first solution that we found:

$$\mathcal{N}(A) = \operatorname{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right)$$

How to Solve $Ax = 0$: Second Example

Let's find all solutions to $Ax = 0$ for the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 2 \\ 4 & -2 & 2 & -2 \end{bmatrix}.$$

The first step is to use Gaussian elimination to transform A into an "upper triangular" U:

$$A = \begin{bmatrix} \boxed{1} & 0 & 1 & 0 \\ 0 & 2 & 2 & 2 \\ 4 & -2 & 2 & -2 \end{bmatrix} \xrightarrow{R3-4R1} \begin{bmatrix} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{2} & 2 & 2 \\ 0 & -2 & -2 & -2 \end{bmatrix} \xrightarrow{R3+R2} \begin{bmatrix} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{2} & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U.$$

Therefore columns 1 and 2 of A are the pivot columns; columns 3 and 4 of A are the free columns. In particular, to solve $Ux = 0$, we are free to choose the values of the free variables x_3 and x_4.

> **Definition.** Suppose A has f free columns total. Suppose columns $j_1, j_2, \ldots, j_f$ of A are its free columns and $x_{j_1}, x_{j_2}, \ldots, x_{j_f}$ are the associated free variables. A **special solution** of $Ax = 0$ is the unique solution x to $Ax = 0$ that is found after setting one of the free variables $x_{j_k} = 1$ and all other free variables $x_{j_l} = 0$. A matrix A with f free columns total has exactly f special solutions to $Ax = 0$.

The next step is to find all of the special solutions to $Ax = 0$. Because A has 2 free columns, A has 2 special solutions. (In the previous example, there was only 1 special solution.)

To find the first special solution, set $\boxed{x_3 = 1}$ and $\boxed{x_4 = 0}$. Then $Ux = 0$ is simply

$$\begin{aligned} x_1 &= -1 \\ 2x_2 &= -2 \end{aligned}$$

Hence the first special solution is $\boxed{x_1 = -1,\ x_2 = -1,\ x_3 = 1,\ x_4 = 0}$.

To find the second special solution, set $\boxed{x_3 = 0}$ and $\boxed{x_4 = 1}$. Then $Ux = 0$ becomes

$$\begin{aligned} x_1 &= 0 \\ 2x_2 &= -2 \end{aligned}$$

Hence the second special solution is $\boxed{x_1 = 0,\ x_2 = -1,\ x_3 = 0,\ x_4 = 1}$.

Rule. Every solution to $Ax = 0$ is a linear combination of the special solutions of $Ax = 0$. In other words, the nullspace of A can be described as

$$\mathcal{N}(A) = \text{span}(\text{special solutions of } Ax = 0)$$

Thus for this second example, we have

$$\mathcal{N}(A) = \text{span}\left(\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right)$$

How to Solve $Ax = 0$: General Procedure

Here are the steps to find all solutions of $Ax = 0$.

(1) Do Gaussian elimination to transform $Ax = 0$ to $Ux = 0$ where U is upper triangular.
(2) Note which columns are pivots columns of A and which columns are free columns of A.
(3) Find the special solutions of $Ax = 0$. If A has f free columns, A has f special solutions.
  (a) Pick a free variable x_{j_k}. Set $x_{j_k} = 1$ and set all other free variables $x_{j_l} = 0$.
  (b) Substitute these values for the free variables into $Ux = 0$ and solve for the special solution using back substitution.
  (c) Repeat steps (a) and (b) once for each free variable.
(4) A generic solution to $Ax = 0$ is a linear combination of the special solutions of $Ax = 0$. This means that the nullspace of A is given by

$$\mathcal{N}(A) = \text{span}(f \text{ special solutions of } Ax = 0)$$

If A happens to have no free columns ($f = 0$), the only solution to $Ax = 0$ is $x = 0$.

MAT 211 (Badger, Spring 2012) LU Factorization

Selected Notes — September 12, 2012

Abstract: We describe the beautiful LU factorization of a square matrix (or how to write Gaussian elimination in terms of matrix multiplication).

WHAT IS AN ELEMENTARY MATRIX?

Our goal in these notes is to rewrite Gaussian elimination using matrix multiplication. The starting point for this project is a special kind of matrix:

> An **elementary matrix** E is a square matrix with 1s on the main diagonal and at most one non-zero entry off of the main diagonal.

For example,

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a 3×3 elementary matrix. We call this matrix E_{21} to signify that the non-zero entry in a non-diagonal position appears in the $(2, 1)$ entry.

Let's find the inverse of E_{21}. The general procedure states that we should form the matrix $[E_{21}|I]$ and apply elimination $\rightsquigarrow [I|E_{21}^{-1}]$ until the identity matrix is on the left:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R2-3R1} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

We are done after one step (there was only one non-zero entry in a non-diagonal entry, so we only needed to clear one entry to obtain the identity); the inverse of E_{21} is given by

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This pattern is easily shown to hold in general.

> Let E_{ij} $(i \neq j)$ be an elementary matrix with non-zero (i, j) entry. Then inverse E_{ij}^{-1} is the elementary matrix formed by negating the (i, j) of E_{ij}.

Why do we care about elementary matrices? We care because multiplication by an elementary matrix on the left is equivalent to an performing an elementary row operation. For example, if A is any 3×3 matrix and E_{21} is the matrix from above, then multiplying A by E_{21} on the left is the equivalent to adding $3 \times$ ROW 1 of A to ROW 2 of A:

$$E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \end{bmatrix} = \begin{bmatrix} \text{Row 1} \\ 3 \times \text{Row 1} + \text{Row 2} \\ \text{Row 3} \end{bmatrix}.$$

This is the type of operation we do in Gaussian elimination!

Rewriting Gaussian Elimination in Terms of Multiplication

As usual we will illustrate the general process with an example. Assign A be the matrix

$$A = \begin{bmatrix} 5 & -1 & 2 \\ 10 & 3 & 7 \\ 15 & 17 & 19 \end{bmatrix}.$$

Forward elimination will convert A to an upper triangular matrix U. After each step of elimination, we will record the same step using multiplication by an elementary matrix.

Step 1: We need to zero out the (2,1) entry. In elimination notation,

$$\begin{bmatrix} \boxed{5} & -1 & 2 \\ 10 & 3 & 7 \\ 15 & 17 & 19 \end{bmatrix} \xrightarrow{R2-2R1} \begin{bmatrix} 5 & -1 & 2 \\ 0 & 5 & 3 \\ 15 & 17 & 19 \end{bmatrix}.$$

In terms of multiplication,

$$\underset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 5 & -1 & 2 \\ 10 & 3 & 7 \\ 15 & 17 & 19 \end{bmatrix}} = \begin{bmatrix} 5 & -1 & 2 \\ 0 & 5 & 3 \\ 15 & 17 & 19 \end{bmatrix}.$$

Notice the (2,1) entry of E_{21} came from the multiple of row 1 added to row 2.

Step 2: We need to zero out the (3,1) entry. In elimination notation,

$$\begin{bmatrix} \boxed{5} & -1 & 2 \\ 0 & 5 & 3 \\ 15 & 17 & 19 \end{bmatrix} \xrightarrow{R3-3R1} \begin{bmatrix} 5 & -1 & 2 \\ 0 & 5 & 3 \\ 0 & 20 & 13 \end{bmatrix}.$$

In terms of multiplication,

$$\underset{E_{31}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}} \underset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 5 & -1 & 2 \\ 10 & 3 & 7 \\ 15 & 17 & 19 \end{bmatrix}} = \begin{bmatrix} 5 & -1 & 2 \\ 0 & 5 & 3 \\ 0 & 20 & 13 \end{bmatrix}.$$

Notice the (3,1) entry of E_{31} came from the multiple of row 1 added to row 3.

Step 3: We need to zero out the (3,2) entry. In elimination notation,

$$\begin{bmatrix} 5 & -1 & 2 \\ 0 & \boxed{5} & 3 \\ 0 & 20 & 13 \end{bmatrix} \xrightarrow{R3-4R2} \begin{bmatrix} 5 & -1 & 2 \\ 0 & 5 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

In terms of multiplication,

$$\underset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}} \underset{E_{31}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}} \underset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 5 & -1 & 2 \\ 10 & 3 & 7 \\ 15 & 17 & 19 \end{bmatrix}} \underset{=}{=} \underset{U}{\begin{bmatrix} 5 & -1 & 2 \\ 0 & 5 & 3 \\ 0 & 0 & 1 \end{bmatrix}}.$$

Notice the (3,2) entry of E_{21} came from the multiple of row 2 added to row 3.

What have we done so far? We have used the process of Gaussian elimination to find elementary matrices E_{21}, E_{31}, E_{32} so that $E_{32}E_{31}E_{21}A = U$ is upper triangular. This is what we wanted to do (write elimination using multiplication), but let's probe further.

What happens if we move the matrices E_{21}, E_{31}, E_{32} to the other side of the equation $E_{32}E_{31}E_{21}A = U$? To do this, we just multiply the equation by on the left $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$:

$$A = (E_{21}^{-1}E_{31}^{-1}E_{32}^{-1})U.$$

Fortunately, it is easy to compute the inverse of an elementary matrix (remember the example on page 1—just negate the (i,j) entry of E_{ij}). Let's calculate $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$:

$$E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix} \underbrace{\begin{bmatrix}1&0&0\\0&1&0\\3&0&1\end{bmatrix} \begin{bmatrix}1&0&0\\0&1&0\\0&4&1\end{bmatrix}}$$

$$= \begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix} \times \begin{bmatrix}1&0&0\\0&1&0\\3&4&1\end{bmatrix} = \begin{bmatrix}1&0&0\\2&1&0\\3&4&1\end{bmatrix} = L.$$

The matrix $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$ is just a **lower triangular** matrix L! Thus we have shown that

$$A = \begin{bmatrix}5&-1&2\\10&3&7\\15&17&19\end{bmatrix} = \begin{bmatrix}1&0&0\\2&1&0\\3&4&1\end{bmatrix} \begin{bmatrix}5&-1&2\\0&5&3\\0&0&1\end{bmatrix} = LU,$$

$A = LU$ is a lower triangular matrix times an upper triangular matrix!!!

LU Factorization

Theorem. Suppose that A is $n \times n$ matrix that can be reduced to an upper triangular matrix U by Gaussian elimination *without using row exchanges*. Then

$$A = LU$$

where L is lower triangular and U is upper triangular.

- Each entry on the diagonal of L is 1.
- For each $i > j$, the (i,j) entry of L is the multiple m of row j that was *subtracted* from row i during the elimination process.
- The diagonal entries of U are the pivots from the elimination process.

Example 1: Reading Off Information from an LU Factorizatoin

Suppose we are given a matrix A already written as $A = LU$:

$$A = \begin{bmatrix}1&0&0\\5&1&0\\2&-3&1\end{bmatrix} \begin{bmatrix}3&2&1\\0&-4&1\\0&0&2\end{bmatrix}.$$

Question: What multiple of row 1 was subtracted from row 3 during elimination?
Answer: By looking at the (3,1) entry of L, we know that 2 times row 2 was subtracted from row 3 during elimination.

Question: What were the pivots?
Answer: The pivots were 3, -4 and 2 (the diagonal entries of U).

Example 2: Finding an LU Factorization

Suppose we are asked to find the the factors L and U for the matrix

$$\begin{bmatrix} 6 & 0 & 2 \\ 24 & 1 & 8 \\ -12 & 1 & -3 \end{bmatrix}.$$

To do this, we perform Gaussian elimination and then read off the information we need from the elimination steps.

$$\begin{bmatrix} \boxed{6} & 0 & 2 \\ 24 & 1 & 8 \\ -12 & 1 & -3 \end{bmatrix} \xrightarrow{R2-4R1} \begin{bmatrix} \boxed{6} & 0 & 2 \\ 0 & 1 & 0 \\ -12 & 1 & -3 \end{bmatrix} \xrightarrow{R3+2R1} \begin{bmatrix} 6 & 0 & 2 \\ 0 & \boxed{1} & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R3-R2} \begin{bmatrix} 6 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}.$$

The matrix U is just the upper triangular matrix we end with doing elimination,

$$U = \begin{bmatrix} 6 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

To record L, we just start with a lower triangular matrix with 1s on the diagonal and then for each $i > j$ put the multiple m of row j which was *subtracted* from row i in the (i, j) entry of L:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix}.$$

(Note: Since we added $2 \times$ Row 1 to Row 3, we *subtracted* $-2 \times$ Row 1 from Row 3.)

Example 3: Solving a System of Equations Given the LU Factorization

Once one knows the LU factorization of a matrix A, solving $Ax = b$ is very quick. Let A, L and U be matrices from Example 2. Suppose we are asked to solve $Ax = b$ where

$$b = \begin{bmatrix} 4 \\ 19 \\ -6 \end{bmatrix}.$$

Since we already know the LU factorization $A = LU$, we can solve $Ax = b$ as follows: first solve the system of equations $Lc = b$; second solve the system of equations $Ux = c$. (This works because if $Lc = b$ and $Ux = c$, then $Ax = LUx = L(Ux) = Lc = b$.)
The system of equations $Lc = b$ looks like

$$\begin{bmatrix} 1 & & \\ 4 & 1 & \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 19 \\ -6 \end{bmatrix}.$$

Since L is lower triangular, we can find c_1, c_2 and c_3 using forward substitution:

$$\begin{aligned} c_1 &= 4 \\ 4c_1 + c_2 &= 19 \Longrightarrow 4(4) + c_2 = 19 \Longrightarrow c_2 = 3 \\ -2c_1 + c_2 + c_3 &= -6 \Longrightarrow -2(4) + (3) + c_3 = -6 \Longrightarrow c_3 = -1. \end{aligned}$$

The system of equations $Ux = c$ looks like

$$\begin{bmatrix} 6 & 0 & 2 \\ & 1 & 0 \\ & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}.$$

Since U is upper triangular, we can find x_1, x_2 and x_3 using back substitution:

$$\begin{aligned} x_3 &= -1 \\ x_2 &= 3 \\ 6x_1 + 2x_3 = 4 \Longrightarrow 6x_1 + 2(-1) &= 4 \Longrightarrow x_1 = 1 \end{aligned}$$

Therefore, the solution to $Ax = b$ is $\boxed{x_1 = 1,\ x_2 = 3 \text{ and } x_3 = -1}$. The point of this example is that if you already know $A = LU$, then to solve $Ax = b$ no elimination steps are required! Just use substitution (forward on $Lc = b$, then backward on $Ux = c$)!

Math 211 (Badger, Fall 2012)

Gaussian Elimination

Selected Notes — August 29, 2012

Abstract: We outline the method of forward elimination and back-substitution to find solutions of n linear equations in n unknowns.

As usual let's jump in with an example. Suppose we want to solve

$$\begin{aligned} x + 2y + z &= 5 \\ 3x + 10y + 6z &= 17 \\ 8y + 14z &= 20 \end{aligned}$$

The matrix form for this system is $Ax = b$ where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 10 & 6 \\ 0 & 8 & 14 \end{bmatrix}, \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 17 \\ 20 \end{bmatrix}.$$

The immediate idea is to leave the first equation alone and remove the variable x from the other equations. In terms of the coefficient matrix, this means we want the entries in the first column below the (1,1) entry to be 0. We can do this by subtracting a multiple of the row 1 (the first equation) from row 2 (the second equation).

$$\begin{bmatrix} \boxed{1} & 2 & 1 \\ 3 & 10 & 6 \\ 0 & 8 & 14 \end{bmatrix} \xrightarrow{\text{Row 2 } - \text{ } 3\times\text{Row 1}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 8 & 14 \end{bmatrix}$$

The boxed value in the (1,1) entry of the coefficient matrix above is called the **first pivot**. We use the pivot to determine what multiple of row 1 to subtract from row 2. To clear the (2,1) entry, we need to ask what number t times the pivot p equals the (2,1) entry; then we subtract t times row 1 from row 2. In our example, since the pivot is 1 and the (2,1) entry is 3, we need to subtract 3 times row 1 from row 2, as illustrated above. Since the resulting matrix has zero entries in the first column below the pivot, we have successfully eliminated x from the second and third equations.

Next we leave the first and second equations alone and remove the variable y from the third equation. This means we want to the entries in the second column below the (2,2) entry to be 0. The boxed number in the next equation is the **second pivot**. We use it to determine which multiple of row 2 to subtract from row 3.

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & \boxed{4} & 3 \\ 0 & 8 & 14 \end{bmatrix} \xrightarrow{\text{Row 3 } - \text{ } 2\times\text{Row 2}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & 8 \end{bmatrix}$$

We have eliminated y from the third equation. Notice that the matrix on the right hand side is **upper triangular**, because all the non-zero entries lie on or above the diagonal entries (1,1), (2,2), (3,3).

Although we do not need it for an elimination step, the **third pivot** is the boxed entry in the upper triangular matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & \boxed{8} \end{bmatrix}.$$

Whenever one manipulates an equation, one must change both sides of the equation at the same time. Since we manipulated the rows of the coefficient matrix (the left hand side of the original equation), we need to do the same thing to the rows of b (the right hand side of the original equation). There is a handy device for doing this computation. The **augmented matrix** of the linear system is the coefficient matrix with the column b tacked on to the end. In our example the adjoined matrix is

$$\begin{bmatrix} 1 & 2 & 1 & 5 \\ 3 & 10 & 6 & 17 \\ 0 & 8 & 14 & 20 \end{bmatrix}.$$

If we repeat the elimination steps (row operations) using the adjoined matrix instead of the coefficient matrix, then we can keep track of how both the left and right hand side of the original equations change while removing variables. Let's do it.

$$\begin{bmatrix} \boxed{1} & 2 & 1 & 5 \\ 3 & 10 & 6 & 17 \\ 0 & 8 & 14 & 20 \end{bmatrix} \xrightarrow{\text{Row 2} - 3\times\text{Row 1}} \begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & \boxed{4} & 3 & 2 \\ 0 & 8 & 14 & 20 \end{bmatrix} \xrightarrow{\text{Row 3} - 2\times\text{Row 1}} \begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 4 & 3 & 2 \\ 0 & 0 & \boxed{8} & 16 \end{bmatrix}$$

We have used **forward elimination** to convert the original system of equations $[A|b]$ to a new system of equations $[U|c]$ where the new coefficient matrix U is upper triangular. The pivots were 1, 4 and 8. The system of equations now look like

$$\begin{aligned} x + 2y + z &= 5 \\ 4y + 3z &= 2 \\ 8z &= 16 \end{aligned}$$

One of the advantages of using matrices to record the manipulations we did above was that we avoided writing the variables x, y and z over and over again in each line.

To solve a system of equations in upper triangular form, we use **back substitution**. This simply means that we start by solving the last equation and work backwards.

$$\begin{aligned} 8z = 16 &\implies \boxed{z = 2} \\ 4y + 3z = 2 &\implies 4y + 6 = 2 \implies \boxed{y = -1} \\ x + 2y + z = 5 &\implies x - 2 + 2 = 5 \implies \boxed{x = 5} \end{aligned}$$

The solution to our working example is $x = 5$, $y = -1$ and $z = 2$.

Let us check our solution in terms of the columns of A. We want to verify that b is a combination of the columns of A. Indeed

$$5\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 10 \\ 8 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 6 \\ 14 \end{bmatrix} = \begin{bmatrix} 5 \\ 17 \\ 20 \end{bmatrix}.$$

When combined as above the method of forward elimination and back substitution is called the method of **Gaussian elimination**.

Example of 4 Equations in 4 Unknowns

This time let's work with the following system of equations:

$$
\begin{aligned}
2x_1 + 4x_2 + 4x_3 + 2x_4 &= 16 \\
4x_1 + 8x_2 + 6x_3 + 8x_4 &= 32 \\
14x_1 + 29x_2 + 32x_3 + 16x_4 &= 112 \\
10x_1 + 17x_2 + 10x_3 + 2x_4 &= 28
\end{aligned}
$$

To solve the system we will (1) perform forward elimination to put the equations and (2) perform back substitution. Remember we start with the first pivot and do row operations to zero out the column below the pivot position:

$$
\begin{bmatrix} \boxed{2} & 4 & 4 & 2 & 16 \\ 4 & 8 & 6 & 8 & 32 \\ 14 & 29 & 32 & 16 & 112 \\ 10 & 17 & 10 & 2 & 28 \end{bmatrix} \xrightarrow{R2-2R1,\ R3-7R1,\ R4-5R1} \begin{bmatrix} 2 & 4 & 4 & 2 & 16 \\ 0 & 0 & -2 & 4 & 0 \\ 0 & 1 & 4 & 2 & 0 \\ 0 & -3 & -10 & -8 & -52 \end{bmatrix}.
$$

Next we want to examine the second pivot and zero out the entries in the column below. But if we look at the (2,2) pivot position, in our example,

$$
\begin{bmatrix} 2 & 4 & 4 & 2 & 16 \\ 0 & 0 & -2 & 4 & 0 \\ 0 & 1 & 4 & 2 & 0 \\ 0 & -3 & -10 & -8 & -52 \end{bmatrix}
$$

there is a zero in the pivot position.

> **A pivot cannot be zero!** We cannot use a zero in the pivot position, because subtracting multiples of the pivot row from the rows below will not change any of the entries below the pivot position. To fix this problem we need to reorder the equations, or in matrix terms, perform a **row exchange**.

Let's switch rows 2 and rows 3:

$$
\begin{bmatrix} 2 & 4 & 4 & 2 & 16 \\ 0 & 0 & -2 & 4 & 0 \\ 0 & 1 & 4 & 2 & 0 \\ 0 & -3 & -10 & -8 & -52 \end{bmatrix} \xrightarrow{R2 \leftrightarrow R3} \begin{bmatrix} 2 & 4 & 4 & 2 & 16 \\ 0 & 1 & 4 & 2 & 0 \\ 0 & 0 & -2 & 4 & 0 \\ 0 & -3 & -10 & -8 & -52 \end{bmatrix}
$$

There is now a non-zero term in the (2,2) pivot position. We can proceed with elimination.

$$
\begin{bmatrix} 2 & 4 & 4 & 2 & 16 \\ 0 & \boxed{1} & 4 & 2 & 0 \\ 0 & 0 & -2 & 4 & 0 \\ 0 & -3 & -10 & -8 & -52 \end{bmatrix} \xrightarrow{R4+3R2} \begin{bmatrix} 2 & 4 & 4 & 2 & 16 \\ 0 & 1 & 4 & 2 & 0 \\ 0 & 0 & -2 & 4 & 0 \\ 0 & 0 & 2 & -2 & -52 \end{bmatrix}
$$

Now we repeat this process with the third pivot.

$$\begin{bmatrix} 2 & 4 & 4 & 2 & 16 \\ 0 & 1 & 4 & 2 & 0 \\ 0 & 0 & \boxed{-2} & 4 & 0 \\ 0 & 0 & 2 & -2 & -52 \end{bmatrix} \xrightarrow{R4+R3} \begin{bmatrix} 2 & 4 & 4 & 2 & 16 \\ 0 & 1 & 4 & 2 & 0 \\ 0 & 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & \boxed{2} & -52 \end{bmatrix}$$

We have successfully used forward elimination to convert the original system of equations to an upper triangular system of equations. The pivots were 2, 1, -2 and 2; there was one row exchange.

Next we solve the upper triangular set of equations using back substitution.

$$\begin{aligned} 2x_4 = -52 &\Rightarrow \boxed{x_4 = -26} \\ -2x_3 + 4(-26) = 0 &\Rightarrow \boxed{x_3 = -52} \\ x_2 + 4(-52) + 2(-26) = 0 &\Rightarrow \boxed{x_2 = 520} \\ 2x_1 + 4(520) + 4(-52) + 2(-26) = 16 &\Rightarrow \boxed{x_1 = -902}. \end{aligned}$$

As an exercise you may want to check the solution in terms of the columns of the matrix.

Math 211 (Badger, Fall 2012)

Geometry of Linear Equations

Selected Notes — August 27, 2012

Abstract: We describe the "row picture" and "column picture" of a system of linear equations.

We start with an example in two dimensions. Consider the system of equations:

$$2x - y = 0$$
$$-x + 2y = 3$$

Using high school algebra, it is not hard to find the solution. Instead our goal is to understand two different geometric interpretations of these equations and their solution. We call the interpretations the "row picture" and the "column picture"—the name comes from rewriting the system of equations in *matrix form* $Ax = b$:

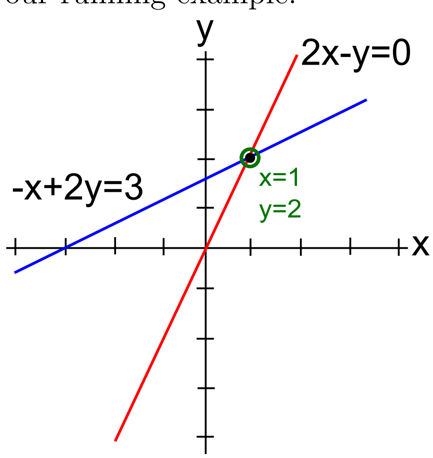
$$\underset{A}{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}} \underset{x}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{b}{\begin{bmatrix} 0 \\ 3 \end{bmatrix}}$$

There are standard names for A and x; we call A the *coefficient matrix* of the system of equations and call x the *vector of unknowns*. (As shown here we often use x to denote both a real-valued variable and a column vector.)

Geometric Interpretation #1: The Row Picture

Each row of the matrix represents a single equation. In the row picture, we draw a line (or a plane) for each equation. The solution of the system of equations (if it exists) is the unique intersection of all the lines (planes).

Here is the row picture in our running example:



A couple of remarks. Note that the first equation ($2x - y = 0$) passes through the origin because the right hand side is 0; the second equation ($-x + 2y = 3$) does not touch the origin because the right hand side is not 0. The solution ($x = 1$, $y = 2$) is the point of intersection between the two lines.

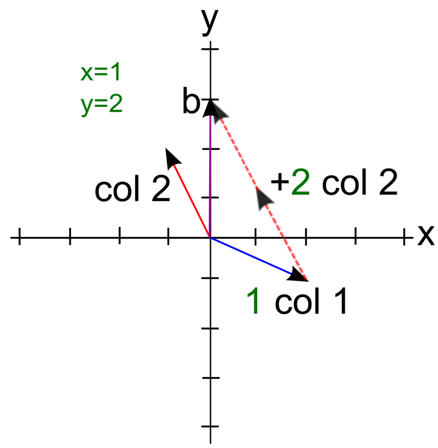
Geometric Interpretation #2: The Column Picture

Each column of the matrix corresponds to an unknown variable:

- *the 1st column of $A \longleftrightarrow$ the 1st variable of x,*
- *the 2nd column of $A \longleftrightarrow$ the 2nd variable of x, etc.*

In the column picture, we draw a vector (arrow) for each column of A and also draw the vector b. The solution of the system of equations (if it exists) is the unique combination of the matrix columns which equal b.

Here is the column picture in our running example:



So what does this picture mean? The solution to the system of equations ($x = 1$, $y = 2$) is the unique values of x and y such that

$$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

This is called a *linear combination* of the columns. The column picture means that to find a solution of a system of linear equations (put in matrix form) we may find the correct combination of the columns of the matrix A which equals the column vector b.

Now let's do an example in three dimensions. Consider the system of equations

$$\begin{aligned} 2x - y \quad\quad &= -1 \\ -x + 2y - z &= 1 \\ -3y + 4z &= 1 \end{aligned}$$

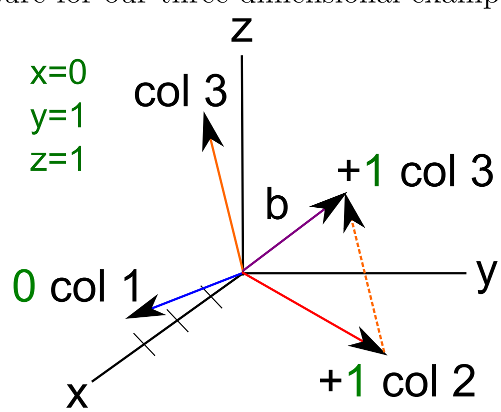
First we should put the equation in matrix form:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

I'll leave it to the reader to draw the row picture in this example. You need to plot each equation (row) separately. Since there are three variables, each equation represents a plane in xyz-space. In terms of the row picture, the solution will be the unique point of intersection of three planes—but this is very hard to identify using a picture.

Although the row picture becomes more complicated by adding a third variable, the column picture remains just a simple with the new variable. Switching from two to three variables in the row picture, we change from drawing lines (1d objects) to drawing planes (2d objects). But in the column picture, we always just draw vectors (1d objects)!

Here is the column picture for our three dimensional example:



The solution is $x = 0$, $y = 1$, $z = 1$. To write the column b on right hand side of the equation $Ax = b$ as a linear combination of the columns of the matrix, we just take zero of column 1, one of column 2, and one of column 3:

$$0\begin{bmatrix}2\\-1\\0\end{bmatrix}+1\begin{bmatrix}-1\\2\\-3\end{bmatrix}+1\begin{bmatrix}0\\-1\\4\end{bmatrix}=\begin{bmatrix}-1\\1\\1\end{bmatrix} \tag{1}$$

You will be asked to draw the row and column pictures of systems of linear equations of two and three variables on the first homework.