Introduction to Linear Algebra

MAT 211, Section 3, Spring 2008

Contact: Phone: (631) 632-8255; Fax 632-7631 Email: myonghi@math.sunysb.edu	Instructor: Myong-Hi (Nina) Kim Office: Math 3-109	Office Hours: Tuesday 2:45-4:15, Wednesday 9:30-10:30

Recitation: Problem Solving on every Thursday 3pm-4pm Life Science Library Room L-3	**Help** by all **MAT 211 staffs** Available to MAT211 students	Math Learning Center in room SL240A Math Tower, Free tutor available in all mathematics subjects

Please check your blackboard frequently for announcement, hint for homeworks etc.

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Homework stuff	Syllabus, Schedule	**Exam Related**	
#HWAssignment	Syllabus	Midterm 1: Review 1, Review1Solution Review2: OpenEndVersionWebwork	Midterm II Material To Know See BBL if the link does not work. Solution

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Recitations

A recitation class is now on! Day and time: Thursday 3:00-4:00pm. Place: Life Science Library, room L-3

Day of	Instructor	Topics
Th 2/21	Myong-hi Kim	Linear systems
Th 2/28	Tanvir Prince	Linear maps
Th 3/6	Julia Viro	Kernel and image, bases, dimension

Office Hours and Locations.

Instructor	Day and time	Office in Math Tower
Tanvir Prince (lecturer, section 1)	W 10:00-11:00am W 11:00am-1:00pm	2-121 MLC
Julia Viro (lecturer, section 2)	TuTh 4:00-5:15pm Th 10:00-11:00am	4-102 MLC
Myong-Hi Kim (lecturer, section 3)	Tu 2:45-4:15 pm W 9:30-10:30 am	3-109
Zhiyu Tian (grader)	MW 5:15-6:15pm Th 5:00-6:00pm	S-240C MLC

Math Leaning Center

Math Learning Center	Day and Evening	SL240A

Monday	Tuesday	Wednesday	Thursday
	10:00-11:00 (Viro, MLC)	9:30-10:30 (Kim, 3-109) 10:00-11:00 (Prince, 2-121)	
		11:00-1:00 (Prince, MLC)	
	2:45-4:15(Kim, 3-109)		
5:15-6:16(Tian S-240C)	4:00-5:15 (Viro, 4-102)	5:15-6:16(Tian S-240C)	4:00-5:15 (Viro, 4-102)

MAT 211 (Introduction to Linear Algebra) Section 3

Spring 2008

Schedule (tentative): The following is the basic syllabus. Please read the relevant parts of the book **before** class.

Day of	Homework due	Sections Covered
January 29		1.1 Introduction to Linear Systems
January 31	Half Homework 1	1.2 (Matrices, Vectors, and Gauss-Jordan Elimination)
February 5	Half Homework 1	1.3 (On the Solutions of Linear Systems; Matrix Algebra)
February 7		2.1 (Introduction to Linear Transformations And Their Inverses)
February 12	Homework 2	2.2 (Linear Transformations in Geometry)
February 19HWAssignment		2.3 (The Inverse of a Linear Transformation) 2.4 (Matrix Products)
February 21	Homework 3	2.4 (Matrix Products)
February 26		3.1 (Image and Kernel of a Linear Transformation)
February 28	Homework 4	3.2 (Subspaces of R^n; Bases and Linear Independence)
March 4		Review and 3.3 (The Dimension of a Subspace of R^n)
March 6	~~Homework 5~~	**Exam I (from 1.1 up to and including 3.2 and dimension from Section3.3)**
March 11		3.3 and 3.4 (Coordinates);
March 13	Homework 5	4.1 (Introduction to Linear Spaces)
March 18		Springr Recess
March 20		Springr Recess
March 25-April 1	Homework 6	4.2 (Linear Transformations and Isomorphisms) 4.3 (The Matrix of a Linear Transformation)
April 3		Review Section 4.1-4.3 6.1 (Introduction to Determinants)
April 8	Homework 7	6.2 (Properties of the Determinant)
April 10		6.3 (Geometrical Interpretations of the Determinant; Cramer's Rule)
		Review
April 15		**Exam II (from 3.4 up to and including 6.3)**
April 17	Homework 8	Ch 7.1: Dynamical systems and eigenvectors
April 22		7.2 (Finding the Eigenvalues of a Matrix)
April 22/24		7.3 (Finding the Eigenvectors of a Matrix)
April 24		7.4 (Diagonalization)
April 29	Homework 9	5.1 (Orthogonal Projections and Orthonormal Bases)
May 1		5.2 (Gram-Schmidt Process and QR Factorization)

May 6		5.3 (Orthogonal Transformations and Orthogonal Matrices)
May 8	Homework 10	Review
May 13		Review
May 15		**Final Exam (Cumulative)**

Homework Assignment:

#	Problems	Due Date
Homework 1	**§1.1**: 1,3,7,11,12,13,20,38,40 **§1.2**:1, 5, 6, 11, 20, 22,34,35,	Half 1/31/08
	§1.3:1, 5 , 6,10,20,21,29,30, 36, 38, 55, 57, **Chapter Exercise:** 1 through 10	Half 2/5/08
	Solutions to Selected Homework 1	
Homework 2	**§2.1: 7** **§2.2: 29,42**	2/26/08
Homework 3	**§2.3: 40,48** **§2.4:19,20,36,76**	2/26/08
	Solutions to Selected Homework 2 and 3	
Homework 4	**§3.1:** 30, 34, 51 **§ 3.2**: 2,6,8, 34, 36, 46	3/4
	Midterm I	
	Solutions to Selected Homework 5 and 6	Midterm I
Homework 5	**§3.3:**22, 28 30, 45, 46	3/13
Homework 6	**§3.4**:12, 16, 26, 28, 44, 47, 56, 62	4/1
Homework 7	**Show all your work.** **§ 4.1**: 4,6, 25, 30 **§ 4.2**:6,52 **§ 4.3:1**	4/8

	Solutions to Selected Homework 7	
Homework 8	**§ 6.1:** 10, 18,30,44 **§ 6.2:**35, 47,.59 **§ 6.3: 2,13, 30**	4/17
	Solutions to Selected Homework 8	
Homework 9	**§ 7.1: 8, 42** **§ 7.2: 4, 32, 44**	4/29
Homework 10	**§ 5.1**: 16, 17 **§ 5.2:** 14, 34	5/8

MAT 211 (Introduction to Linear Algebra) Section 3

Spring 2008

Lecture: Tuesday and Thursdays 12:50-2:10 in Earth&Space 079

Final Exam: Thur. May 15, 11:00AM- 1:30 PM

Prerequisites/Corequisites: at least one semester of calculus.

Academic Calendar

Instructor: Myong-hi Nina Kim

Office: Math Building 3-109:

Office Hours: Tuesday 2:45-4:15, Wednesday 9:30-10:30

Contact : Best way to contact me is via email.

Email: myonghi@math.sunysb.edu

Phone: 632-8255:

You are always welcome to contact me by email (myonghi@math.sunysb.edu)

Grader: Zhiyu Tian

Grader's Office Hours : Monday 1:30-3:30 at Math Tower S240 C. It is located next to the Math learning Center.

Textbook:

Bretscher, *Linear Algebra with Applications*, *3rd Ed.*, Pearson/Prentice-Hall

(two copies are available on reserve in the Math/Physics/Astronomy Library)

Link to an on line tutorial on Linear Algebra by Avi Goldman. It is worth looking at this from time to time since it contains useful outside links and notes on the topics of the course.

Click here for a link to the CURRENT HOMEWORK. This page also contains links to solutions.

Prerequisites/Corequisites: at least one semester of calculus

The Nature Of The Course:

This course is an introduction to the theory which has developed around the solution of systems of linear equations. The importance of this theory as a tool in the social, natural, and mathematical sciences cannot be overestimated. (To get some idea of why this is the case, click here, check out the links of interest below, or take a look through your textbook.) You should keep this in mind throughout the semester, especially if the course material ever seems "too weird" or "too abstract" to be useful.

Course Format:

You will get most out of the classes if you prepare beforehand by reading the relevant section in the textbook before class. I am always glad to answer questions during class. Since this class has no recitations, I will aim to set aside some class time each week for doing examples and discussing the homework. If you have more questions, please talk to me after class or come to my office hours (or go to the Math Learning Center in Math Building S-240A.) There will be a review session before each exam, to be scheduled later.

Some links of interest

A nice expository paper on the use of linear algebra in search engines.

A useful online linear algebra text with many worked examples and exercises with solutions.

Examinations:

If you miss an exam for an acceptable reason and provide me with an acceptable written excuse, the relevant exam will be dropped in computing your course grade. A letter stating that you were seen by a doctor or other medical personnel is **not** an acceptable document. An acceptable document should state that it was **reasonable/proper** for you to seek medical attention and medically **necessary** for you to miss the exam (for privacy reasons the note/letter need not state anything beyond this point).

Grading:

Your raw grade will be based on your examination performance and homework, weighted as follows:

Exam I	20%
Exam II	20%
Final Exam	40%
Homework	20%

DSS advisory:

If you have a physical, psychological, medical, or learning disability that may affect your course work, please contact Disability Support Services (DSS) office: ECC

(Educational Communications Center) Building, room 128, telephone (631) 632-6748/TDD. DSS will determine with you what accommodations are necessary and appropriate. Arrangements should be made early in the semester (before the first exam) so that your needs can be accommodated. All information and documentation of disability is confidential. Students requiring emergency evacuation are encouraged to discuss their needs with their professors and DSS. For procedures and information, go to the following web site http://www.ehs.sunysb.edu/ and search Fire safety and Evacuation and Disabilities.

Schedule (tentative): The following is the basic syllabus. Please read the relevant parts of the book **before** class.

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February 7		2.1 (Introduction to Linear Transformations And Their Inverses)
February 12	Homework 2	2.2 (Linear Transformations in Geometry)
February 14		2.3 (The Inverse of a Linear Transformation)
February 19	Homework 3	2.4 (Matrix Products)
February 21		3.1 (Image and Kernel of a Linear Transformation)
February 26	Homework 4	3.2 (Subspaces of R^n; Bases and Linear Independence)
February 28		3.3 (The Dimension of a Subspace of R^n)
March 4	Homework 5	3.4 (Coordinates); Review
March 6		**Exam I (on everything from 1.1 up to and including 3.3)**
March 11	Half Homework 6	4.1 (Introduction to Linear Spaces)
March 13		4.2 (Linear Transformations and Isomorphisms)
March 18		Spring Recess
March 20		Spring Recess
March 25	Homework 7	4.3 (The Matrix of a Linear Transformation)
March 27		5.1 (Orthogonal Projections and Orthonormal Bases)
April 1	Homework 8	5.2 (Gram-Schmidt Process and QR Factorization)
April 3		5.3 (Orthogonal Transformations and Orthogonal Matrices)
April 8	Homework 9	5.5 (Inner Product Spaces)
April 10		**Exam II (on everything from 3.4 up to and including 5.3)**
April 15	Half Homework 10	6.1 (Introduction to Determinants)
April 17		6.2 (Properties of the Determinant)
April 22	Homework 11	6.3 (Geometrical Interpretations of the Determinant; Cramer's Rule)
April 24		Ch 7.1: Dynamical systems and eigenvectors
April 29	Homework 12	7.2 (Finding the Eigenvalues of a Matrix)
May 1		7.3 (Finding the Eigenvectors of a Matrix)
May 6	Homework 13	7.4 (Diagonalization)
May 8		Review
May 13	Homework 14	
May 15		**Final Exam (Cumulative)**

MAT211 Fall 2007 Midterm 1 Review Sheet

The topics tested on Midterm 1 will be among the following.

(i) Finding the coefficient and augmented matrix of a system of linear equations.

(ii) Using Gauss-Jordan elimination or other elementary row operations to find the reduced row echelon form of a matrix.

(iii) Determining when a system of linear equations has no solution, a unique solution, or infinitely many solutions.

(iv) Finding the general solution of a system of linear equations.

(v) Finding the matrix representative of a linear transformation, i.e., finding A given T.

(vi) Adding and scaling matrices, determining when two matrices can be multiplied, and computing the matrix product when it exists.

(vii) Determining whether a matrix is or is not invertible, and computing the matrix inverse when it exists.

(viii) Given an ordered sequence $(\mathbf{v}_1, \ldots, \mathbf{v}_m)$ of elements in $\mathbb{R}^n$, determining which are redundant and irredundant, and expressing each redundant element as a linear combination of irredundant elements.

(ix) Finding an ordered basis for the image of a linear transformation.

(x) Finding an ordered basis for the kernel of a linear transformation (which is roughly the same as Item (iv) above).

You should know the meaning of all of the following words and phrases: linear equation, system of linear equations, consistent, inconsistent, zero matrix, identity matrix, coefficient matrix, augmented matrix, entries, column vector, vector, row vector, elementary row operation, reduced row echelon form, rank, leading 1, leading column, leading variable, free column, free variable, matrix product, linear combination, linear transformation, domain, codomain = target, inverse transformation, invertible, rotation, reflection, scaling, projection, shear, associativity, image, kernel, span, linear relation, trivial linear relation, linear subspace, linearly independent, linearly dependent, redundant term, irredundant term, ordered basis.

Problem 1 For the following linear system, find the augmented matrix and the reduced row echelon form of the augmented matrix. Say whether the solution is consistent or inconsistent. If the system is consistent, write down the general form of the solution.

$$\left\{\begin{array}{rcrcrcrcrcrcr} 4x_1 & + & 2x_3 & + & 22x_4 & + & x_5 & + & -14x_6 & = & 14 \\ -x_1 & - & x_3 & - & 8x_4 & + & & & 4x_6 & = & -4 \\ & & 3x_3 & + & 15x_4 & - & 7x_5 & + & 8x_6 & = & 3 \\ 3x_1 & + & & & 9x_4 & + & 2x_5 & - & 10x_6 & = & 9 \end{array}\right.$$

Problem 2 For the following augmented matrix $\widetilde{A} = [A|\vec{b}]$, find the reduced row echelon form of $\widetilde{A}$ and say whether or not the system $A\vec{x} = \vec{b}$ is consistent. If the system is consistent, write down the general form of the solution.

$$\left(\begin{array}{rrr|r} 3 & -2 & 7 & 0 \\ 5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 3 & 2 & 0 \\ 1 & 4 & 3 & 0 \end{array}\right).$$

Problem 3 Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation defined by

$$T\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}\right) = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

for the unique 4-tuple of real numbers b_1, b_2, b_3, b_4 such that

$$y'' + 2y = b_1 e^{-t}\cos(t) + b_2 e^{-t}\sin(t) + b_3 e^t \cos(t) + b_4 e^t \sin(t)$$

where

$$y(t) = a_1 e^{-t}\cos(t) + a_2 e^{-t}\sin(t) + a_3 e^{t}\cos(t) + a_4 e^{t}\sin(t).$$

Find the unique matrix A such that for every 4-vector $\vec{x}$

$$T(\vec{x}) = A\vec{x}.$$

Problem 4 Let $S : \mathbb{R}^3 \to \mathbb{R}^3$ be the function defined by

$$S(\vec{x}) = (\vec{e}_2 \cdot \vec{x})\vec{e}_1 - (\vec{e}_1 \cdot \vec{x})\vec{e}_2 + \vec{e}_3 \times \vec{x}$$

where

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}.$$

Is the function S a linear transformation? If so, write down the matrix of S.

Problem 5 Reflection in $\mathbb{R}^3$ about a plane containing the z-axis is a linear transformation with matrix

$$X = \begin{pmatrix} a & b & 0 \\ b & -a & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for some real numbers a and b satisfying $a^2 + b^2 = 1$.

Reflection in $\mathbb{R}^3$ about a plane containing the y-axis is a linear transformation with matrix

$$Y = \begin{pmatrix} c & 0 & d \\ 0 & 1 & 0 \\ d & 0 & -c \end{pmatrix}$$

for some real numbers c and d satisfying $c^2 + d^2 = 1$.

Compute the product matrices $Z = XY$ and $W = YX$. Does XY equal YX? Also compute ZW and WZ. Explain your answer.

Following are some practice problems. More practice problems are in the textbook as well as on the practice midterm.

Problem 6 For which values of k is the following system consistent? For which values are there infinitely many solutions? Whenever the solution is unique, compute its value.

$$\left\{\begin{array}{ccccccc} 2x_1 & + & k^2x_2 & + & (4k-4)x_3 & = & k+10 \\ 3x_2 & + & k^2x_2 & & & = & k+9 \\ -2x_2 & + & k^2x_2 & + & (5k-5)x_3 & = & k-1 \end{array}\right.$$

Problem 7 For which values of t is the following matrix invertible? Whenever it is invertible, write down the inverse. Whenever it is not invertible, write down an element in the kernel.

$$\begin{pmatrix} t & 1 \\ 1 & t \end{pmatrix}.$$

Problem 8 Find the inverse of the following invertible 3×3 matrix.

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Problem 9 Find the inverse of the following invertible 3×3 matrix.

$$\begin{pmatrix} 2 & 3 & 2 \\ 0 & 5 & 1 \\ -3 & 1 & 0 \end{pmatrix}$$

Problem 10 For the following matrix A, determine which column vectors are irredundant and which are redundant. Then write down an ordered basis for the image of T_A and an ordered basis for the kernel of T_A.

$$\begin{pmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ -2 & 1 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 2 & 3 \\ 0 & 0 & 0 & -1 & 1 & 5 \end{pmatrix}.$$

Problem 11 From the following ordered sequence of vectors, determine which terms are redundant and which are irredundant. Write every redundant term as a linear combination of the irredundant ones.

$$(2\vec{e}_1 + 2\vec{e}_2, \vec{0}, \vec{e}_2 + \vec{e}_3, -2\vec{e}_1 + \vec{e}_2 + 3\vec{e}_3, \vec{e}_1 + \vec{e}_4, -\vec{e}_1 - \vec{e}_2 + \vec{e}_3 + \vec{e}_4)$$

Problem 12 Let A be a 4×4 matrix such that $\text{Image}(T_A)$ equals $\text{Ker}(T_A)$. What can you say about A^2? Write down an example of such a matrix.

MAT211 Fall 2007 Midterm 1 Review Sheet

The topics tested on Midterm 1 will be among the following.

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(iv) Finding the general solution of a system of linear equations.

(v) Finding the matrix representative of a linear transformation, i.e., finding A given T.

(vi) Adding and scaling matrices, determining when two matrices can be multiplied, and computing the matrix product when it exists.

(vii) Determining whether a matrix is or is not invertible, and computing the matrix inverse when it exists.

(viii) Given an ordered sequence $(\mathbf{v}_1, \dots, \mathbf{v}_m)$ of elements in $\mathbb{R}^n$, determining which are redundant and irredundant, and expressing each redundant element as a linear combination of irredundant elements.

(ix) Finding an ordered basis for the image of a linear transformation.

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Problem 1 For the following linear system, find the augmented matrix and the reduced row echelon form of the augmented matrix. Say whether the solution is consistent or inconsistent. If the system is consistent, write down the general form of the solution.

$$\left\{\begin{array}{rcrcrcrcrcrcr} 4x_1 & + & 2x_3 & + & 22x_4 & + & x_5 & + & -14x_6 & = & 14 \\ -x_1 & - & x_3 & - & 8x_4 & + & & & 4x_6 & = & -4 \\ & & 3x_3 & + & 15x_4 & - & 7x_5 & + & 8x_6 & = & 3 \\ 3x_1 & + & & & 9x_4 & + & 2x_5 & - & 10x_6 & = & 9 \end{array}\right.$$

Solution to Problem 1 The augmented matrix is

$$\widetilde{A} = \boxed{\left(\begin{array}{rrrrrr|r} 4 & 0 & 2 & 22 & 1 & -14 & 14 \\ -1 & 0 & -1 & -8 & 0 & 4 & -4 \\ 0 & 0 & 3 & 15 & -7 & 8 & 3 \\ 3 & 0 & 0 & 9 & 2 & -10 & 9 \end{array}\right)}$$

By Gauss-Jordan elimination or some other sequences of elementary row operations, $\mathrm{rref}(\widetilde{A})$ equals

$$\mathrm{rref}(\widetilde{A}) = \boxed{\left(\begin{array}{rrrrrr|r} 1 & 0 & 0 & 3 & 0 & -2 & 3 \\ 0 & 0 & 1 & 5 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)}.$$

From the reduced row echelon form, the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -3 & 2 \\ 1 & 0 & 0 \\ 0 & -5 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \boxed{\begin{bmatrix} 3 - 3c_2 + 2c_3 \\ c_1 \\ 1 - 5c_2 + 2c_3 \\ c_2 \\ 2c_3 \\ c_3 \end{bmatrix}}$$

for arbitrary real numbers c_1, c_2 and c_3.

Problem 2 For the following augmented matrix $\widetilde{A} = [A|\vec{b}]$, find the reduced row echelon form of $\widetilde{A}$ and say whether or not the system $A\vec{x} = \vec{b}$ is consistent. If the system is consistent, write down the general form of the solution.

$$\left(\begin{array}{ccc|c} 3 & -2 & 7 & 0 \\ 5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 3 & 2 & 0 \\ 1 & 4 & 3 & 0 \end{array}\right).$$

Solution to Problem 2 The reduced row echelon form is

$$\text{rref}(\widetilde{A}) = \boxed{\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)}.$$

Since there is a leading 1 in the "constant column", i.e., the column for $\vec{b}$, the system is $\boxed{\text{inconsistent}}$.

Problem 3 Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation defined by

$$T\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}\right) = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

for the unique 4-tuple of real numbers b_1, b_2, b_3, b_4 such that

$$y'' + 2y = b_1 e^{-t}\cos(t) + b_2 e^{-t}\sin(t) + b_3 e^{t}\cos(t) + b_4 e^{t}\sin(t)$$

where

$$y(t) = a_1 e^{-t}\cos(t) + a_2 e^{-t}\sin(t) + a_3 e^{t}\cos(t) + a_4 e^{t}\sin(t).$$

Find the unique matrix A such that for every 4-vector $\vec{x}$

$$T(\vec{x}) = A\vec{x}.$$

Solution to Problem 3 The matrix is

$$A = \boxed{\left(\begin{array}{rr|rr} 2 & -2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ \hline 0 & 0 & 2 & 2 \\ 0 & 0 & -2 & 2 \end{array}\right)}$$

(the vertical and horizontal lines are just for effect).

Problem 4 Let $S : \mathbb{R}^3 \to \mathbb{R}^3$ be the function defined by

$$S(\vec{x}) = (\vec{e}_2 \cdot \vec{x})\vec{e}_1 - (\vec{e}_1 \cdot \vec{x})\vec{e}_2 + \vec{e}_3 \times \vec{x}$$

where

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}.$$

Is the function S a linear transformation? If so, write down the matrix of S.

Solution to Problem 4 Notice that for the vector

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

the following identities hold

$$\vec{e}_1 \cdot \vec{x} = x_1, \ \vec{e}_2 \cdot \vec{x} = x_2,$$

and

$$\vec{e}_3 \times \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix}.$$

Substituting this in gives

$$S(\vec{x}) = x_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-x_1) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix} =$$

$$\begin{bmatrix} x_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

In other words, for every $\vec{x}$ in $\mathbb{R}^3$,

$$S(\vec{x}) = \vec{0}.$$

This is indeed a linear transformation with matrix

$$A_S = \boxed{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}.$$

Problem 5 Reflection in $\mathbb{R}^3$ about a plane containing the z-axis is a linear transformation with matrix

$$X = \begin{pmatrix} a & b & 0 \\ b & -a & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for some real numbers a and b satisfying $a^2 + b^2 = 1$.

Reflection in $\mathbb{R}^3$ about a plane containing the y-axis is a linear transformation with matrix

$$Y = \begin{pmatrix} c & 0 & d \\ 0 & 1 & 0 \\ d & 0 & -c \end{pmatrix}$$

for some real numbers c and d satisfying $c^2 + d^2 = 1$.

Compute the product matrices $Z = XY$ and $W = YX$. Does XY equal YX? Also compute ZW and WZ. Explain your answer.

Solution to Problem 5 The matrix products are

$$Z = \begin{pmatrix} a & b & 0 \\ b & -a & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} c & 0 & d \\ 0 & 1 & 0 \\ d & 0 & -c \end{pmatrix} = \boxed{\begin{pmatrix} ac & b & ad \\ bc & -a & bd \\ d & 0 & -c \end{pmatrix}}$$

and

$$W = \begin{pmatrix} c & 0 & d \\ 0 & 1 & 0 \\ d & 0 & -c \end{pmatrix} \cdot \begin{pmatrix} a & b & 0 \\ b & -a & 0 \\ 0 & 0 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} ac & bc & d \\ b & -a & 0 \\ ad & bd & -c \end{pmatrix}}.$$

The matrices Z and W are $\boxed{\text{not equal,}}$ except in a small number of exceptional cases:

(i) if $(a, b) = (1, 0)$ and (c, d) is arbitrary,

(ii) if (a, b) is arbitrary and $(c, d) = (1, 0)$,

(iii) and if $(a, b) = ((-1)^m, 0)$ and $(c, d) = ((-1)^n, 0)$ for integers m and n, i.e., a is ± 1 and c is ± 1 independent of each other.

Because ZW equals $(XY)(YX)$, by associativity this is $X(YY)X$. Since Y is a reflection, YY equals I_3. Thus ZW equals $X(I_3)X$, i.e., XX. Since X is a reflection, XX equals I_3. Thus ZW equals I_3. By a similar argument, $WZ = (YX)(XY) = Y(XX)Y = Y(I_3)Y = YY$ also equals I_3. Thus

$$\boxed{ZW = WZ = I_3}$$

Problem 6 For which values of k is the following system consistent? For which values are there infinitely many solutions? Whenever the solution is unique, compute its value.

$$\left\{ \begin{array}{rcrcrcr} 2x_1 & + & k^2x_2 & + & (4k-4)x_3 & = & k+10 \\ 3x_2 & + & k^2x_2 & & & = & k+9 \\ -2x_2 & + & k^2x_2 & + & (5k-5)x_3 & = & k-1 \end{array} \right.$$

Solution to Problem 6 The augmented matrix of this linear system is

$$\widetilde{A} = \left(\begin{array}{rrr|r} 2 & k^2 & 4k-4 & k+10 \\ 3 & k^2 & 0 & k+9 \\ -2 & k^2 & 5k-5 & k-1 \end{array} \right).$$

Elementary row operations transform this to the row equivalent matrix

$$\widetilde{B} = \left(\begin{array}{rrr|l} 1 & 0 & 0 & 3 \\ 0 & k^2 & 0 & k \\ 0 & 0 & k-1 & 1 \end{array} \right).$$

There are 3 cases depending on whether $k = 0$, $k = 1$ or neither. First of all, if $k = 0$ then

$$\widetilde{B}|_{k=0} = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{array}\right)$$

whose reduced row echelon form is

$$\operatorname{rref}(\widetilde{B}|_{k=0}) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

So when $k = 0$ the system has infinitely many solutions,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} 3 \\ c \\ -1 \end{bmatrix}}$$

where c is an arbitrary real number.

Problem 7 For which values of t is the following matrix invertible? Whenever it is invertible, write down the inverse. Whenever it is not invertible, write down an element in the kernel.

$$\begin{pmatrix} t & 1 \\ 1 & t \end{pmatrix}.$$

The next case is when $k = 1$. In this case,

$$\widetilde{B}|_{k=1} = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

whose reduced row echelon form is

$$\operatorname{rref}(\widetilde{B}|_{k=1}) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

So when $k = 1$ the system is inconsistent.

Finally, when k is neither 0 nor 1, the reduced row echelon form is

$$\widetilde{B} = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1/k \\ 0 & 0 & 1 & 1/(k-1) \end{array} \right).$$

So $\boxed{\text{when } k \text{ is neither 0 nor 1}}$ the system $\boxed{\text{has a unique solution}}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boxed{\begin{bmatrix} 3 \\ 1/k \\ 1/(k-1) \end{bmatrix}.}$$

Problem 8 Find the inverse of the following invertible 3×3 matrix.

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Solution to Problem 8 The inverse of the matrix is

$$\boxed{\frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}.}$$

Problem 9 Find the inverse of the following invertible 3×3 matrix.

$$\begin{pmatrix} 2 & 3 & 2 \\ 0 & 5 & 1 \\ -3 & 1 & 0 \end{pmatrix}$$

Solution to Problem 9 The inverse of the matrix is

$$\boxed{\frac{1}{19} \begin{pmatrix} -1 & 2 & -7 \\ -3 & 6 & -2 \\ 15 & -11 & 10 \end{pmatrix}.}$$

Problem 10 For the following matrix A, determine which column vectors are irredundant and which are redundant. Then write down an ordered basis for the image of T_A and an ordered basis for the kernel of T_A.

$$\begin{pmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ -2 & 1 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 2 & 3 \\ 0 & 0 & 0 & -1 & 1 & 5 \end{pmatrix}.$$

Solution to Problem 10 The reduced row echelon form of A is

$$\text{rref}A = \begin{pmatrix} 1 & 0 & -7/5 & 0 & 0 & 0 \\ 0 & 1 & 11/5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The leading columns of the reduced row echelon matrix are the first, second, fourth and sixth columns. Thus the $\boxed{\text{irredundant}}$ columns of A are the $\boxed{1^{\text{st}}, 2^{\text{nd}}, 4^{\text{th}}, \text{and } 6^{\text{th}} \text{ columns.}}$ And the $\boxed{\text{redundant}}$ columns of A are the $\boxed{3^{\text{rd}} \text{ and } 5^{\text{th}} \text{ columns.}}$ Therefore an ordered basis for the image of T_A is

$$\boxed{\left(\begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 5 \end{bmatrix} \right).}$$

Finally, an ordered basis for the kernel of T_A is

$$\boxed{\left(\begin{bmatrix} 7/5 \\ -11/5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right).}$$

Problem 11 From the following ordered sequence of vectors, determine which terms are redundant and which are irredundant. Write every redundant term as a linear combination of the irredundant ones.

$$(2\vec{e}_1 + 2\vec{e}_2, \vec{0}, \vec{e}_2 + \vec{e}_3, -2\vec{e}_1 + \vec{e}_2 + 3\vec{e}_3, \vec{e}_1 + \vec{e}_4, -\vec{e}_1 - \vec{e}_2 + \vec{e}_3 + \vec{e}_4)$$

Solution to Problem 11 The ordered sequence of vectors $(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5, \vec{v}_6)$ is the same as the ordered sequence of column vectors of the following matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & -2 & 1 & -1 \\ 2 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

The reduced row echelon form is

$$\operatorname{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus the $\boxed{\text{irredundant}}$ terms are $\boxed{\vec{v}_1, \vec{v}_3, \vec{v}_5}$ and the $\boxed{\text{redundant}}$ terms are $\boxed{\vec{v}_2, \vec{v}_4, \vec{v}_6}$. Moreover, the redundant terms are linear combinations of the irredundant terms as follows

$$\boxed{\begin{bmatrix} \vec{v}_2 & \vec{v}_4 & \vec{v}_6 \end{bmatrix} = \begin{bmatrix} \vec{v}_1 & \vec{v}_3 & \vec{v}_5 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}}$$

Problem 12 Let A be a 4×4 matrix such that $\operatorname{Image}(T_A)$ equals $\operatorname{Ker}(T_A)$. What can you say about A^2? Write down an example of such a matrix.

Solution to Problem 12 If $\operatorname{Image}(T_A)$ equals $\operatorname{Ker}(T_A)$, or even if $\operatorname{Image}(T_A)$ is just contained in $\operatorname{Kern}(T_A)$, then $\boxed{A^2 \text{ equals the zero matrix.}}$ A typical example is

$$\boxed{A = \left(\begin{array}{r|r} -XR & -XRX \\ \hline R & RX \end{array}\right)}$$

where X is an arbitrary 2×2 matrix and R is an invertible 2×2 matrix, i.e.,

$$\boxed{A = \begin{pmatrix} -x & -y \\ -z & -w \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} r & s \\ t & u \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & x & y \\ 0 & 1 & z & w \end{pmatrix}}$$

for arbitrary real numbers x, y, z, w and for real numbers r, s, t, u for which $ru - st$ is nonzero. Just to be very specific, one such example is when $X = 0$ and $R = I_2$, i.e.,

$$\boxed{A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}}.$$

Spring2008-MAT211 1

The following is the open end version of Midterm1Preptest in the webwork. If you can handle the following you will score at least 80/

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1. $$\begin{aligned} -6\,x &+ 15\,y &= 6 \\ -14\,x &+ 35\,y &= k \end{aligned}$$

Find the value of k , for the above system of equations to be consistent.

---

2. If $A = \begin{bmatrix} 3 & -1 & -1 \\ -3 & 3 & 2 \\ -2 & -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 & -4 \\ 1 & 3 & -4 \\ 4 & 2 & 3 \end{bmatrix}$

Find $4A - 3B$
and $2A^T$

---

3. Solve the system

$$\begin{cases} x_1 + x_2 + 4x_3 = -9 \\ 4x_1 + 3x_2 + 3x_3 = 5 \end{cases}$$

---

4. The dot product of two vectors $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ in $\mathbb{R}^n$ is defined by $x \cdot y = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$.

The vectors x and y are called perpendicular if $x \cdot y = 0$.

Then a vector in $\mathbb{R}^3$ perpendicular to $\begin{bmatrix} -9 \\ 8 \\ 3 \end{bmatrix}$ can be written as a linear combination of two vectors. Express those vectors in $\mathbb{R}^3$ perpendicular to $\begin{bmatrix} -9 \\ 8 \\ 3 \end{bmatrix}$ as a linear combination of two vectors.

---

5. The reduced row-echelon forms of the augmented matrices of four systems are given below. How many solutions does each system have?

1. $\left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & 16 \\ 0 & 0 & 0 \end{array}\right]$
   - A. Infinitely many solutions
   - B. Unique solution
   - C. No solutions
   - D. None of the above
2. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & -13 \end{array}\right]$
   - A. Unique solution
   - B. No solutions
   - C. Infinitely many solutions
   - D. None of the above
3. $\left[\begin{array}{ccc|c} 1 & 0 & 15 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$
   - A. No solutions
   - B. Infinitely many solutions
   - C. Unique solution
   - D. None of the above
4. $\left[\begin{array}{ccc|c} 1 & 0 & -7 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$
   - A. Infinitely many solutions
   - B. No solutions
   - C. Unique solution
   - D. None of the above

---

6. Determine the value of k for which the system

$$\begin{cases} x + y + 4z = 3 \\ x + 2y - 2z = -1 \\ 3x + 9y + kz = -14 \end{cases}$$

has no solutions.

Remark: The question can be posted as a matrix form also.

7.

To see if $b = \begin{bmatrix} -7 \\ 1 \\ -3 \end{bmatrix}$ is a linear combination of the vectors $a_1 = \begin{bmatrix} -3 \\ -2 \\ -5 \end{bmatrix}$ and $a_2 = \begin{bmatrix} 6 \\ 10 \\ -8 \end{bmatrix}$ one can solve the matrix equation $Ax = c$. Find A and c

---

8. Write the system

$$\begin{cases} -6y + 3z = -3 \\ 4x - 7y = -4 \\ -2x + 9y - 5z = 2 \end{cases}$$

in matrix form.

---

9. Consider a linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^2$ for which

$T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$, $T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$,

and $T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$.

Find the matrix A of T.

10. Find the matrix A of the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ given by

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -5 \\ 7 \\ 6 \end{bmatrix} x_1 + \begin{bmatrix} -3 \\ 5 \\ -1 \end{bmatrix} x_2.$$

11. Consider the orthogonal projection onto the line L in $\mathbb{R}^2$ that consists of all scalar multiples of the vector $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$. I.e. for a given input $\begin{bmatrix} x \\ y \end{bmatrix}$, find the orthogonal projection onto the line in the direction of $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$. Note that it is a linear transformation from $\mathrm{R}^2 to R^2$.

12.) Let $A = \begin{bmatrix} -4 & 9 \\ 2 & 9 \\ 2 & -6 \end{bmatrix}$.

Define the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ **as** $T(x) = Ax$.

Find the images of $u = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ **and** $v = \begin{bmatrix} a \\ b \end{bmatrix}$ **under** T **.**

13. Determine which of the following transformations are linear transformations. If it is not a linear transformation, give a reason why it is not.

- **A. The transformation** T **defined by** $T(x_1, x_2, x_3) = (x_1, 0, x_3)$
- **B. The transformation** T **defined by** $T(x_1, x_2) = (4x_1 - 2x_2, 3|x_2|)$.
- **C. The transformation** T **defined by** $T(x_1, x_2, x_3) = (x_1, x_2, -x_3)$
- **D. The transformation** T **defined by** $T(x_1, x_2, x_3) = (1, x_2, x_3)$
- **E. The transformation** T **defined by** $T(x_1, x_2) = (2x_1 - 3x_2, x_1 + 4, 5x_2)$.

14. Find a linearly independent set of vectors that spans the same subspace of $\mathbb{R}^3$ **as that spanned by the vectors**

$$\begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix}.$$

Find the linearly independent set of vectors.
Find the basis of the subspace spanned by the above three vectors.

15.Let $A = \begin{bmatrix} 12 & 16 & -4 \\ 6 & 8 & -2 \end{bmatrix}$.

Find bases of the kernel and image of A **(or the linear transformation** $T(x) = Ax$**).**

16. Solve the equation $-8x_1 + 3x_2 + 8x_3 - 9x_4 = 0$ **and Find a basis of the subspace of** $\mathbb{R}^4$ **defined by the equation** $-8x_1 + 3x_2 + 8x_3 - 9x_4 = 0$.

17. Let W_1 **be the set:** $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Determine if W_1 **is a basis for** $\mathbb{R}^3$ **and check the correct answer(s) below.**

- **A.** W_1 **is a basis.**
- **B.** W_1 **is not a basis because it does not span** $\mathbb{R}^3$.
- **C.** W_1 **is not a basis because it is linearly dependent.**

Let W_2 **be the set:** $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Determine if W_2 **is a basis for** $\mathbb{R}^3$ **and check the correct answer(s) below.**

- **A.** W_2 **is not a basis because it is linearly dependent.**
- **B.** W_2 **is a basis.**
- **C.** W_2 **is not a basis because it does not span** $\mathbb{R}^3$.

18. Let

Let $A = \begin{bmatrix} 1 & 4 & 5 & -2 \\ 0 & 2 & -2 & -2 \\ -3 & -16 & -11 & 10 \end{bmatrix}$ **and** $b = \begin{bmatrix} 1 \\ 3 \\ -9 \end{bmatrix}$.

$\boxed{?}$**1. Is** b **in the image of** A?**. Support your answer.**

19. (1 pt) Library/TCNJ/TCNJ_NullColumnSpaces-/problem9.pg

A **is an** $m \times n$ **matrix.**
Check the true statements below and give a brief reason supporting your answer:

- **A. The kernel of a linear transformation is a vector space.**
- **B. The null space of an** $m \times n$ **matrix is in** $\mathbb{R}^m$.
- **C.** $ColA$ **is the set of all vectors that can be written as** Ax **for some** x.
- **D. If the equation** $Ax = b$ **is consistent, then** $ColA$ **is** $\mathbb{R}^m$.
- **E. The null space of** A **is the solution set of the equation** $Ax = 0$.
- **F. The column space of** A **is the range of the mapping** $x \to Ax$.

20. Find the projection of $v = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$ **onto the line** l **of** $\mathbb{R}^3$ **given by the parametric equation** $l = tu$**, where** $u = \begin{bmatrix} -1 \\ -5 \\ 4 \end{bmatrix}$

1

Midterm II
MAT 211 Section 3
Spring 2008

Show all your work

Your name: Student ID:

$Number$	$Score$	$Weight$
1		15
2		15
3		20
4 $ChooseOne$		15
5		15
6		15
$7TakeHome$ $DueThursday$		10
$Total$		105

Number 7 is a 10 pts Take home -webwork.

The following inverses might be useful for you.

$$\begin{bmatrix} 0 & 1 & 1/3 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1/3 & -1/3 \\ 0 & 1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 1/3 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1/3 & -1/3 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

(1) (15 pts) Consider a basis $\mathcal{B}$ of R^3.

$$\mathcal{B} = \left(\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ 1 \\ 0 \end{bmatrix} \right)$$

(a) Find $\vec{v}$, where $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$.

(b) Find the $\mathcal{B}-$coordinate of $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(2) (15 pts) Consider a transformation $T : R^3 \to R^3$ defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -x+y+z \\ 2y+z \\ z \end{bmatrix}$$

(a) Find the matrix of the transformation with respect to the standard basis $\mathcal{S}$.

(b) What is (Nullity of T + Rank of T)?

(c) Find the matrix of transformation with respect to $\mathcal{B}-$ basis, where let $\mathcal{B}$ as in (1),

$$\mathcal{B} = \left(\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ 1 \\ 0 \end{bmatrix}\right)$$

(3) (20pts) Let P_2 be the set of all polynomials with degree less than or equal to 2. (degree ≤ 2)and let $T : P_2 \to P_2$ be defined by $T(f) = 3f'$.

(a) Find the matrix of transformation with respect to the standard basis $\mathcal{S} = \{1, x, x^2\}$.

(b) Find the basis of kernel of T

(c) Show that T is linear transformation

(d) Is T an isomorphism? Justify your answer.

(4) (15 pts) Do one of the two

(a) Prove or disprove that

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 0 & 1 \end{bmatrix}$$

are linearly independent.

(b) Prove or disprove that $\{A \in R^{2x2} | det(A) = 1\}$ is a subspace of R^{2x2}

(5) (15 pts) Find the determinant of the following matrices.

(a) $$Det\left(\begin{bmatrix} 0 & 2 & 1 & 0 \\ -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 \end{bmatrix}\right)$$

(b) $$Det\left(\begin{bmatrix} 2 & 2 & -780 & 99 \\ 0 & 1 & 101 & -1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)$$

(c) Find $$det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix}\right)$$

(6) (15 pts) Let $T : R^4 \to R$ defined by

$$T(\vec{x}) = det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ x_1 & x_2 & x_3 & x_4 \end{bmatrix}\right), \text{ where } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

Suppose that

$$T(\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}\right) = det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 1 & 2 & 3 & 4 \end{bmatrix}\right) = 3, \quad T(\left(\begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix}\right) = 4$$

(a) Find $T(\begin{bmatrix} 11 \\ 12 \\ 13 \\ 14 \end{bmatrix})$

(b) Find $T\left(\begin{bmatrix} 20 \\ 20 \\ 20 \\ 20 \end{bmatrix}\right)$

(7) (10 pts) Take Home. Due Date Thursday April 17 noon.
Let

$$A = \begin{bmatrix} 0 & 1 & 1/3 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(a) Solve $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ using Cramer's Rule.

(b) Use Adjoint as in Fact 6.3.10 to find A^{-1}

Material to know

1. Coordinate with respect to a basis
2. Matrix of change of a basis
3. matrix of a linear transformation with respect to the given basis
4. Two matrices are similar
5. dimension of a liner space, subspace
6. Linear transformation:
    1. what is linear transformation ?
    2. Nullity
    3. Rank
    4. Kernel
    5. Image
    6. Rank-Nullity Theorem
    7. isomorphism
7. Comfortable with linear spaces
    1. spaces of polynomials, P_1, P_2, P_3 and its subspaces
    2. Spaces of matrices such as R^{2x2} and Subspaces of R^{2x2} such as
        1. the set of all upper triangular matrices,
        2. diagonal matrices,
        3. the set of matrices whose determinant is 0,
    3. basis, coordinate of a vector with respect to the given basis
    4. linear transformation and its matrix
8. Properties of determinant

1. Linearity in its columns
2. Determinant of the identity matrix I_n is 1.
3. Change in determinant with regards to elementary row operations, and column operations
4. Cramer's Rule for a solution for Ax=b
5. Volume of a parallelepiped
6. For nxn matrix A, the following are equivalent
    1. Determinant of A is non-zero,
    2. RREF is the identity,
    3. non-singular matrix,
    4. solution to Ax=0 has a unique solution(trivial), Ker(A)={0}
    5. Columns of A are linearly independent
    6. Rows of A are linearly independent

Sample Questions

1. Textbook Section 3.4: #28, 44, 16, 56
2. Section 4.1:4, 6, 16,18
    1. For #18 Find at least two basis for P_3
3. Section 4.2: #6,52
4. Section 4.3: 1 , 6, 20
5. Section 6.1: 39,
6. Section 6.2: #47 If the answer is yes, is T determinant map?

What condition does it violate to be a determinant?

#59. Hint: Consider the case when det =0.

7. Section 6.3: 224,
8. True False Chapter Tests
9. Web6, Web7, Web8
10. Consider P_2 the space of polynomials with degree <=2. Consider the linear map T:P_2 ----> P_2 defined by T(f(t))=f(t+1). Find the matrix of T with respect to the basis of B=(1,1+t,1+t+t^2)
11. Solution will be out soon.