Sylvain BONNOT Section I	MAT 211 Introduction to linear algebra	

We will meet on MWF : 9:35 am to 10:30 am in Harriman Hll 108.

First day of class: Wedn. Sept 6, 2006.
Final exam : it will take place on Wed. Dec 20, 8 to 10:30 am in Harriman Hll 108 (the usual room).

Office hours:
every Wedn. from 2:00 pm to 5:00 pm in my office, 5D-148 in the Math Tower.
My office is in the I.M.S (Institute for Math. Sciences), located on floor 5 and a half (true!) I offer you a campus map, in case you don't know where is the Math Tower...

How to contact me?
the best way is to email me there: `bonnot at math dot sunysb dot edu`

Our textbook:
Otto Bretscher: *Linear Algebra with applications, 3rd Ed., Pearson/Prentice-Hall*
We will cover the first seven chapters.

Link to Current Homework: Please have a look at the syllabus to know when it is due. See below if you want to know the grading policy. Click here to go to the homework page.

Course notes and announcements:

- The Final Exam has been graded, I posted your final grades on the Solar system, so you should be able to access them very soon, probably tonight or tomorrow... You did a good job, I think, and in many cases your final grade was much better than the beginning of the semester so for those cases, I was glad to apply my grading scheme...I have graded from 20/100 to 100/100, the average was around 72/100. By the way, I wrote some comments about the final, so read them, if you are bored during the break... Also I scanned a detailed correction of the final exam: the numbering of the questions is a bit different, but the content is what you want. final1, final2, final3, final4, final5. Anyway, it was nice having you as my students, I wish you good luck for your studies, and an excellent break!!! Au revoir!

- FINAL EXAM: Wed. Dec 20, 8:00am to 10:30am (morning), in Harriman 108.
This is the usual room. Please arrive 5 minutes earlier (I know it will be really early...) so that we can start on time. For this final, you are not allowed to use calculators. Good luck, and see you on Wednesday!
If you want to find me, send an email or stop by my office on monday afternoon starting at 2, and also on tuesday afternoon.

- I made a mistake in the correction of #3 second practice exam: the rank is 3 and not 2 as I said...

- Here comes the correction of the second practice exam!
scan1, scan2, scan3, scan4, scan5, scan6, scan7, scan8.

- Correction of the last homework is available.

- Try this at home: second practice exam !
 You will have a correction of this available on Thursday.

- Correction for Homework 11 is available on HW page

- Correction for the practice final: part1, part2, part3, part4, part5, part6.

- Brand new: a Practice Final exam for you
. The correction will be available in few days, together with another practice exam...

- The new HW 12 is on the homework page, due on Monday 12/11.
 There might be one last HW, that I will give you on friday, due on Friday december 15th(last day of class).

- Special office hours: Tuesday 21st from 2pm to 5pm
. Depending on the number of people who will go to these, I might have to find a larger room...I will put a note at the door of 5D-148...

- Also the correction of HW9 is on the homework page.

- Correction for Midterm II has arrived:
scan1, scan2, scan3, scan4, scan5, scan6, scan7.

- Midterm II is graded:
 You will have it tomorrow. Lowest grade: 12/100 ----- Highest grade: 98/100. Average grade is 73/100. On Friday, you will have a correction available on the web as usual.

- Solutions of the new practice problems!
 You can now check the solutions1, solutions2.

- Some more problems for you to practice!
 I knew you wanted some more, so here is some new stuff. On monday I will post the solutions for that...

- Midterm is next week on Wednesday 15th, in Harriman Hll 108, 9:35am to 10:30am
 (Usual place and time) Content: everything from 1.1 to 5.1 included. The focus is on chapter 4, but you need to remember the previous chapters, that's why it is cumulative... The class on Monday will be a review session, so prepare your questions for that day! The office hours next week will be on Tuesday afternoon (from 2pm to ??...whatever you will need!However it must end before 9:35am on wednesday for obvious reasons)

- The correction for the practice exam has arrived:
 read it just to make sure that you would have obtained 100pts... Page1, Page2, Page3,Page4.

- Correction for HW8 is on HW page

- a Practice exam is ready
 Please try both of these: scan1, scan2. Also new HW is on HW page.

- HW8 is now on HW page

- Hum Hum
 Ok, now you have the correction of the correction available on the page! (Thanks to Joe Pastore for telling me I was wrong...)

- I still have some midterm papers with me

For the people who weren't here last friday: I still have your midterm, I will bring it back on wednesday, so don't forget to claim your paper at the end of the class! Otherwise, you can also find me in my office.

- Correction for HW5 is on HW page

About the lecture notes: I treated most of the content this morning, so I am not sure to write them actually, I will see...

- < New HW is on HW page

Coming soon: (soon or never...) some lectures notes about 3.2, 3.3. (You will have to read them in detail, all of them, and yes you will need them for the next midterm...)

- Correction of the midterm is here:

Please have a look at these scans: correc1, correc2,correc3, correc4,correc5, correc6.

- **MIDTERM I**

Date: Wednesday Oct. 11th, from 9:35 am to 10:30 am in Harriman Hll 108 (usual time and hour). Please arrive 5 minutes in advance so that we can start on time!

- (Posted on Monday 10/09) A remark about the correction of the practice exam

This morning No Eul told me that my correction for question (g) was wrong: so the answer is that question (g) is False, but the matrix I provided doesn't work...Instead Min Sung proposed to take a 2 by 2 matrix with first column made of 1, second column made of 0, and this works...Thanks to both of you! Joe Pastore also told me there was a mistake in (5): for the case b=0, a=4/3, the solutions are actually given by (x1, x2, x3)=s(0,1,0)+t(-3,0,1)...thanks for the remark!

- (Posted on Monday 10/09) Solutions of HW4 !

scan1, scan2.

- (Posted on Friday 10/06) < Solutions of the PRACTICE EXAM !

Try it first by yourself and then have a look at these scan1, scan2,scan3,scan4.

- (Posted on Thursday 10/05) NEW!!! PRACTICE EXAM !

Here is a brand new practice exam.Please try it...you will have a correction for this...

- (Posted on Wednesday 10/04) Correction for HW3

Here are the scans of my correction for HW3:
scan #3.1, scan #3.2, scan #3.3, scan #3.4.

- (Posted on friday 09/29) Please see the HW page for the next homework...

Lecture notes, please read!!!! Click on this link.

- (Posted on monday 09/25) Correction for HW2

Here are the scans of my correction for HW2:
scan #2.1, scan #2.2.

Some hints for #48: actually I gave you an answer this morning. You just need to understand the difference between these two phrases:"for any vector y the system A.x=y has a unique solution" (meaning that A is invertible), and "there exists a vector y such that the system A.x=y"...

- (Posted Friday 09/22) HW3 is assigned on HW page

Since I already defined the product, you can already solve most of them. I'll give some hints on monday. Coming soon: scans of the correction for HW2, available on monday.

- (Posted on monday 09/18) Correction for HW1

Scans of my correction are now available:
scan #1.1, scan #1.2,scan #1.3, scan #1.4,scan #1.5.

- (Posted on Friday 09/15) HW 2 is on homework page I scanned my detailed correction for HW1, it will be available on Monday.

- (Posted on Monday 09/11) Some hints for HW 1 :
Please have a look at the following lecture notes #1 (.pdf file), it might help you for HW 1!! Here are the same notes #1(.ps) in Postscript (.ps).

- (Posted on Friday 09/08)Please read 1.1 and 1.2 for monday (1.1 is just an introduction). We will cover on monday 1.2 and therefore the first HW will be due on **FRIDAY Sep. 15** (and NOT Wedn. as I said earlier). HW1 is about the solution of linear systems. Don't worry about what we saw today, it will be covered again later (2.1). See below for the link to HW1: please notice that I will give indications about it on monday, so basically you don't need to start it right now... HW1 is posted now, see below for the link.

Quick intro: Linear algebra is all about solving systems of linear equations (a nice circular "definition"...). It's an old subject where all the main concepts have been clarified and polished over the years, that's why it's possible now to present them in a concise way. Even if the subject is pretty old, there are applications everywhere nowadays. I found some examples just for you:

- Face recognition by computers (related to biometry,etc...): have a quick look at that page, you'll see eigenvectors everywhere (we will see these in the class);
- Information retrieval (data mining);
- Compression of pictures (for the Web);
- Quantum mechanics (where "physical observables such as energy and momentum are no longer considered as functions on some phase space, but as eigenvalues of operators which act on such functions"): check this link;
- Cryptography (you can read this);
- Search engines (read this article if you want to know how Google works!)...

And the list goes on and on...

Prerequisites:
You must have had at least one semester of calculus. If you have not yet studied integration, you should be taking the relevant calculus course (e.g. MAT 126) concurrently with this one, as some important problems and examples in this course require a knowledge of integration.

Math Learning Center:
This is a very useful place for you: there you can ask questions about the class or the homework problems. It is located in the Math Tower S-240A (basement level). You should definitely check their webpage.

Link to Current Homework: Please have a look at the syllabus to know when it is due. See below if you want to know the grading policy. Click here to go to the homework page.

Syllabus (very tentative schedule):

Day of	Homework due	Sections Covered
September 6		1.1 (Introduction to Linear Systems)
September 8		Intro to linear transformations
September 11		1.2 (Matrices, Vectors, and Gauss-Jordan Elimination)
September 13		1.3 (On the Solutions of Linear Systems; Matrix Algebra)

September 15	Homework 1	2.1 (Introduction to Linear Transformations And Their Inverses)
September 18		2.2 (Linear Transformations in Geometry)
September 20		2.3 (The Inverse of a Linear Transformation)
September 22	Homework 2	2.4 (Matrix Products)
September 25		2.4 (Continued)
September 27		3.1 (Image and Kernel of a Linear Transformation)
September 29	Homework 3	3.2 (Subspaces of R^n; Bases and Linear Independence)
October 2:**NO CLASS**		
October 4		3.2 (Continued)
October 6	Homework 4	3.3 (The Dimension of a Subspace of R^n)
October 9		Review session
October 11: **Midterm I**		**Midterm I: Everything up to and including 3.3**
October 13	Homework 5	3.4 (Coordinates)
October 16		4.1 (Introduction to linear spaces)
October 18		4.2 (Linear Transformations and Isomorphisms)
October 20	Homework 6	4.3 (The matrix of a linear transformation)
October 23		4.3 (Continued)
October 25		5.1 (Orthogonal Projections and Orthonormal Bases)
October 27	Homework 7	5.2 (Gram-Schmidt Process and QR Factorization)
October 30		5.2 (Continued)
November 1		5.3 (Orthogonal transformations)
November 3	Homework 8	5.4 (Least squares and data fitting)
November 6		5.5 (Inner Product Spaces)
November 8		6.1 (Introduction to Determinants)
November 10	Homework 9	6.1 (Continued)
November 13		Review session
November 15:**Midterm II**		**Midterm II: Everything from 1.1 to 5.1 (included)**
November 17	Homework 10	6.2 (Properties of the Determinant)
November 20		6.2 (Continued)
November 22	Homework 11	6.3 (Geometrical Interpretations of the Determinant; Cramer's Rule)
November 24:**NO**		

CLASS		
November 27		Ch 7.1: Dynamical systems and eigenvectors
November 29		7.2 (Finding the Eigenvalues of a Matrix)
December 1	Homework 12	7.2 (Continued)
December 4		7.3 (Finding the Eigenvectors of a Matrix)
December 6		7.3 (Continued)
December 8	Homework 13	7.4 (Diagonalization)
December 11		7.4 (Continued)
December 13:**CORRECTION DAY for 10/02**	Homework 14	7.5 (Complex eigenvalues)
December 15: **LAST CLASS**		**Review session**
December 20:**FINAL EXAM**		**Final Exam (Cumulative: from 1.1 to 7.4 included)**

Exams: (tentative schedule)
You will get very soon the definitive schedule.

Midterm 1	October 11, Wednesday 9:35-10:30 a.m.	Usual room
Midterm 2	November 15, Wednesday 9:35-10:30 a.m.	Usual room
Final	December 20, Wednesday 8:00-10:30 a.m.	Usual room Harriman 108

Homework and grading policy: The grading will not be based on a curve. Here is how your final grade will be computed.First I'll take a weighted average of the following:

Exam I	25%
Exam II	25%
Final Exam	35%
Homework	15%

This gives me a first grade. A second grade is given by 90% of your final exam grade. The grade you will receive at the end of the class will be the maximum of these two grades. Late homework will not be accepted.

DSS advisory:

If you have a physical, psychological, medical, or learning disability that may affect your course work, please contact Disability Support Services (DSS) office: ECC (Educational Communications Center) Building, room 128, telephone (631) 632-6748/TDD. DSS will determine with you what accommodations are necessary and appropriate.

Arrangements should be made early in the semester (before the first exam) so that your needs can be accommodated. All information and documentation of disability is confidential. Students requiring emergency evacuation are encouraged to discuss their needs with their professors and DSS. For procedures and information, go to the following web site http://www.ehs.sunysb.edu and search Fire safety and Evacuation and Disabilities.

MAT 211 Section 1 Homework Assignments

Fall 2006

Link to main page for MAT 211 Section 1.
Mathematics department

#	Problems	Due Date
1	**Section 1.1**: 10 , 22, 26, and 33 **Section 1.2 :** 7, 11 and 48 See the Hints on main page for the class... Complete correction: scan #1.1, scan #1.2,scan #1.3, scan #1.4,scan #1.5.	**Friday** 9/15/06
2	**Section 1.3**: 19, 24, 28, 34, 44, 50 Complete correction: scan #2.1, scan #2.2.	**Friday** 9/22/06
3	**Section 2.2**: 2, 12, 29, 42 **Section 2.3**: 30, 40, 42, 44, 48 **Section 2.4**: 28, 30 Complete correction: scan #3.1, scan #3.2 scan #3.3,scan #3.4.	**Friday** 9/29/06
4	**Section 2.4**: 36, 76, 19, 20 **Section 3.1**: 30, 34, 48(a and b only), 51 **Section 3.2**: 2,6,8 Complete correction: scan #4.1, scan #4.2.	**Friday** 10/06/06
5	**Section 3.2**: 34, 36, 46 **Section 3.3**: 22, 28 Complete correction: scan #5.1, scan #5.2.	**MONDAY** 10/16/06
6	**Section 3.3**: 24, 30, 45, 46, 56 Complete correction: scan #6.1, scan #6.2.	**Friday** 10/20/06
7	**Section 3.4**:12, 16, 26, 28, 44, 47, 56, 62 Complete correction: scan #7.1, scan #7.2.	**Friday** : 10/27/06
8	**Section 4.1**: 6, 25, 30, 55 **Section 4.2**: 26, 28, 58 Complete correction: scan #8.1, scan #8.2.	**Friday** 11/3/06
9	**Section 4.2** : 72, 73, 74 **Section 4.3**: 14, 28, 38, 57, 64 Complete correction: scan #9.1, scan #9.2, scan #9.3.	**Friday** 11/10/06
10	**Section 4.3**: 33, 54, 60, 68 **Section 5.1**: 16, 17, 32. Complete correction : scan #10.1, scan #10.2	**MONDAY** 11/20/06
11	**Section 5.2:** 14, 34 **Section 6.1:** 10, 18, 30, 36, 44, 54 **Section 6.2:**35, 47, 48.	**Monday** 12/04/06

	Complete correction: scan #11.1, scan #11.2, scan #11.3 , scan #11.4, scan #11.5, scan #11.6.	
12	**Section 6.2: 26, 59** **Section 6.3: 2, 30, 36** **Section 7.1: 8, 42** **Section 7.2: 4, 32, 44** Complete correction: scan #12.1, scan #12.2, scan #12.3.	**Monday** 12/11/2006

Some comments about the Final

Ok, I know that you are still struggling with your last finals, but I'm asking one last mathematical effort from you! Just read these quick notes, and then, only then enjoy your well-deserved break!

I wish you a happy New Year!

1. **(Problem 1.)** You had to compute the rank of a matrix. That was easy, I only took points away when your answers were not consistent (e.g: if you say that the rank was 2, and just after that the vectors were linearly independent!)
2. **(Problem 2.)** Find the inverse of a matrix: no particular comments.
3. **(Problem 3.)** Finding a basis for a subspace V was OK, but then many people had forgotten the definition of the orthogonal complement of V, and lost points. Many of you thought that I had asked to find an orthonormal basis of V, but no, that wasn't the question...
4. **(Problem 4.)** Find a formula for A^n. This was treated in the practice exam, so it should have been ok, but I agree that this was probably the most difficult question.
5. **(Problem 5.)** Question with the polynomials. Unfortunately many of you thought that the map was:
$$T\colon f(t) \mapsto f(5t+1).f'(t)$$
when it was:
$$T\colon f(t) \mapsto (5t+1).f'(t)$$
so they got a wrong matrix...Well this was again the same exercise as one in the practise exam...
6. **(Problem 6.)** Computing a det. This was ok for most of you. Be careful, you cannot use Sarrus rule for a determinant that is not a 3 by 3!

7. **(Problem 7.)** Orthogonal projection onto a line. This was ok again, except for those who had forgotten that formula:

$$p(\vec{x}\,) = \frac{\vec{x}\,.\vec{u}}{\|\vec{u}\,\|^2}.\vec{u}$$

do not forget the exponent 2 in the denominator!

8. **(Problem 8.)** Diagonalize a 3 by 3 matrix. I gave generous partial credit for that one. However, I was happy to see that many of you managed to factor that difficult characteristic polynomial. However, that happiness didn't last too long (after all, it's not Christmas yet...): many of you concluded that the linear map wasn't diagonalizable because it had only 2 eigenvalues, but this is a FALSE argument!!! I told you that in class and I insisted: think about the identity (3x3) matrix, it has only one eigenvalue, but it is evidently diagonalizable (it's even already in diagonal form!).

   What is true is the following: if a (3x3) matrix has 3 distinct eigenvalues THEN it is diagonalizable (it's a theorem). But you have plenty of (3x3) matrices that are diagonalizable and that have strictly less than 3 eigenvalues (e.g the identity, or unfortunately the one in this problem)...

Correction of the Final exam:

(1) As usual we reduce

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ -4 & 3 & -1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2+4R_1 \\ R_3-2R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & -5 & -1 & 4 & 1 & 0 \\ 0 & 6 & 1 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \times(-\frac{1}{5}) \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{5} & -\frac{4}{5} & -\frac{1}{5} & 0 \\ 0 & 6 & 1 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} R_1+2R_2 \\ \\ R_3-6R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{5} & -\frac{3}{5} & -\frac{2}{5} & 0 \\ 0 & 1 & \frac{1}{5} & -\frac{4}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & -\frac{1}{5} & \frac{14}{5} & \frac{6}{5} & 1 \end{array}\right] \begin{array}{l} R_1+2R_3 \\ R_2+R_3 \\ (\times -5) \end{array}$$

(continued)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 2 & 2 \\ 0 & 1 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & -14 & -6 & -5 \end{array}\right]$$

Therefore $A^{-1} = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 1 & 1 \\ -14 & -6 & -5 \end{bmatrix}$

We check the result:

$$\begin{bmatrix} 1 & -2 & 0 \\ -4 & 3 & -1 \\ 2 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 & 2 & 2 \\ 2 & 1 & 1 \\ -14 & -6 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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(2) Let's reduce

$$\begin{bmatrix} 3 & 5 & 1 \\ 4 & 4 & 2 \\ 1 & -1 & 1 \\ 0 & -4 & 1 \end{bmatrix} \begin{array}{l} \\ \\ R_1 \leftrightarrow R_3 \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 4 & 4 & 2 \\ 3 & 5 & 1 \\ 0 & -4 & 1 \end{bmatrix} \begin{array}{l} \\ R_2-4R_1 \\ R_3-3R_1 \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 8 & -2 \\ 0 & 8 & -2 \\ 0 & -4 & 1 \end{bmatrix} \begin{array}{l} \\ R_2/8 \\ \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -\frac{1}{4} \\ 0 & 8 & -2 \\ 0 & -4 & 1 \end{bmatrix} \begin{array}{l} R_1+R_2 \\ \\ R_3-8R_2 \\ R_4+4R_2 \end{array}$$

(continued)

$$\begin{bmatrix} 1 & 0 & 3/4 \\ 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore rank $= 2$ and the columns are not linearly independent.

Actually we have $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \frac{3}{4} \cdot \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 5 \\ 4 \\ -1 \\ -4 \end{bmatrix}$.

3 a) 2/

V is the kernel of the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto [1\,0\,4] \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Thus we want to find a basis for ker $[1 0 4]$:

- $\vec{v}_2 = \vec{0}$ so $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is a first vector,
- $\vec{v}_3 = 4\vec{v}_1$ so $\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$ is a second vector : therefore $\underset{\vec{u}_1}{\underset{\shortparallel}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}}, \underset{\vec{u}_2}{\underset{\shortparallel}{\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}}}$ is a basis for V.

b) The orthogonal complement of V is the set of vectors that are orthogonal to both $\vec{u}_1, \vec{u}_2$,

so it is the kernel of $S: \mathbb{R}^3 \to \mathbb{R}^2$

$$\vec{x} \mapsto \begin{bmatrix} \vec{u}_1 \cdot \vec{x} \\ \vec{u}_2 \cdot \vec{x} \end{bmatrix}$$

So we need to find a basis for ker $\begin{bmatrix} 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$ $R_2 \leftrightarrow R_1$ and $\times -\frac{1}{4}$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & 0 \end{bmatrix}$$

We have $\vec{v}_3 = -\frac{1}{4}\vec{v}_1$ so $\begin{bmatrix} 1/4 \\ 0 \\ 1 \end{bmatrix}$ is a basis for $V^\perp$.

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4) Since the characteristic polynomial of A is $(2-\lambda)(1-\lambda)$, it has 2 distinct eigenvalues (1 and 2) and thus A is diagonalizable.

Let's find a basis of eigenvectors:

$E_1 = \ker(A - I) = \ker \begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix}$ $R_2 - 4R_1$

$\mathrm{rref}(A-I) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and we get $\vec{v}_2 = \vec{0}$ so $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a basis for E_1.

$E_2 = \ker(A - 2I) = \ker \begin{bmatrix} 0 & 0 \\ 4 & -1 \end{bmatrix}$

$\mathrm{rref}(A-2I) = \begin{bmatrix} 1 & -\frac{1}{4} \\ 0 & 0 \end{bmatrix}$ so $\vec{v}_2 = -\frac{1}{4}\vec{v}_1$ and $\begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix}$ is a basis for E_2.

In conclusion $\underset{\vec{e}_1}{\underset{\shortparallel}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}}, \underset{\vec{e}_2}{\underset{\shortparallel}{\begin{bmatrix} 1/4 \\ 1 \end{bmatrix}}}$ is an eigenbasis, written $\mathcal{B}$.

(4) Continued.

Since $A\vec{e}_1 = \vec{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}_B$ and $A\vec{e}_2 = 2\vec{e}_2 = \begin{bmatrix}0\\2\end{bmatrix}_B$, the matrix of the linear map in the new basis is the diagonal $D = \begin{bmatrix}1&0\\0&2\end{bmatrix}$.

The change of basis matrix is $P = P_{B\to \text{standard}} = \begin{bmatrix}0&1/4\\1&1\end{bmatrix}$ and $P^{-1} = \begin{bmatrix}-4&1\\4&0\end{bmatrix}$.

Now we know $D = P^{-1}AP$ so $A = PDP^{-1}$ and also $A^n = (PDP^{-1})^n = PD^nP^{-1}$

$$= P.\begin{pmatrix}1&0\\0&2^n\end{pmatrix}P^{-1}$$

$$= \begin{bmatrix}0&1/4\\1&1\end{bmatrix}\underbrace{\begin{bmatrix}1&0\\0&2^n\end{bmatrix}\cdot\begin{bmatrix}-4&1\\4&0\end{bmatrix}}$$

$$= \begin{bmatrix}0&1/4\\1&1\end{bmatrix}\cdot\begin{bmatrix}-4&1\\4.2^n&0\end{bmatrix}$$

so $$\boxed{A^n = \begin{bmatrix}2^n&0\\-4+4.2^n&1\end{bmatrix}}$$

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(5) (a) Eigenvalues: $M - \lambda I = \begin{bmatrix}-6-\lambda&4&4\\-4&4-\lambda&2\\-8&4&6-\lambda\end{bmatrix}$ so the char. polynomial is:

$$-(6+\lambda)\left((4-\lambda)(6-\lambda)-8\right) + 4.\left(4(6-\lambda)-16\right) - 8.\left(8-4(4-\lambda)\right)$$

$$= -(6+\lambda)\left(\lambda^2-10\lambda+16\right) + 4.(8-4\lambda) - 8.(4\lambda-8)$$

$$= -(6+\lambda)\left((\lambda-2)(\lambda-8)\right) + (\lambda-2)(-16-32)$$

$$= (\lambda-2)\left(-(\lambda-8)(6+\lambda)-48\right)$$

$$= -(\lambda-2)\left(\lambda^2-2\lambda\right)$$

$$= -\lambda.(\lambda-2)^2.$$ So the eigenvalues are 0 and 2.

(b) Eigenvectors:

$E_0 = \mathrm{Ker}\begin{bmatrix}-6&4&4\\-4&4&2\\-8&4&6\end{bmatrix}$ $R_2 \leftrightarrow R_1$ and $\times -\frac{1}{4}$

$\begin{bmatrix}1&-1&-\frac{1}{2}\\-6&4&4\\-8&4&6\end{bmatrix}$ $\begin{matrix}\\R_2+6R_1\\R_3+8R_1\end{matrix}$ $\to \begin{bmatrix}1&-1&-\frac{1}{2}\\0&-2&1\\0&-4&2\end{bmatrix}$ $\times -\frac{1}{2}$

$\begin{bmatrix}1&-1&-\frac{1}{2}\\0&1&-\frac{1}{2}\\0&-4&2\end{bmatrix}$ $\begin{matrix}R_1+R_2\\ \\R_3+4R_2\end{matrix}$ $\to \begin{bmatrix}1&0&-1\\0&1&-\frac{1}{2}\\0&0&0\end{bmatrix}$

$\vec{v}_3 = -\vec{v}_1 - \frac{1}{2}\vec{v}_2$

so $\begin{bmatrix}1\\1/2\\1\end{bmatrix}$ is a basis for Ke

(5) continued:

$$E_2 = \ker(M - 2I) = \ker \begin{bmatrix} -8 & 4 & 4 \\ -4 & 2 & 2 \\ -8 & 4 & 4 \end{bmatrix} \begin{array}{l} R_1/(-8) \\ R_2 - \frac{1}{2}R_1 \\ R_3 - R_1 \end{array}$$

$$\mathrm{Rref}(M-2I) = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ so $\vec{v}_2 = -\frac{1}{2}\vec{v}_1$ and we get $\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$ in $\ker(M-2I)$

and $\vec{v}_3 = -\frac{1}{2}\vec{v}_1$ so $\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$ is in $\ker(M-2I)$.

Therefore $\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}$ is a basis of $\ker(M-2I)$.

(c) In conclusion, we found $\begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}$ which is an eigenbasis : so M is diagonalizable

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(6) (a) We have $T(1) = (5t+1).0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; $T(t) = (5t+1).1 = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}$; $T(t^2) = (5t+1)2t = 10t^2 + 2t = \begin{bmatrix} 0 \\ 2 \\ 10 \end{bmatrix}$

So the matrix is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 5 & 2 \\ 0 & 0 & 10 \end{bmatrix}$. Its determinant is 0 so:

(b) T is not an isomorphism.

(7) Expand along the 3rd column: $\det M = (-1) \cdot \det \underbrace{\begin{bmatrix} 4 & 1 & 3 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}}_{4(1-2) - 2(1-6) + 3(1-3)}$

$$= (-1)(-4 + 10 - 6)$$
$$= 0$$

Therefore M is not invertible.

8 (a) We have $p_L\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \dfrac{\begin{bmatrix}1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\end{bmatrix}}{\left\|\begin{bmatrix}1\\-1\end{bmatrix}\right\|^2}\cdot\begin{bmatrix}1\\-1\end{bmatrix} = \dfrac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\end{bmatrix}$.

and $p_L\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \dfrac{\begin{bmatrix}0\\1\end{bmatrix}\cdot\begin{bmatrix}1\\-1\end{bmatrix}}{2}\cdot\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}-1/2\\1/2\end{bmatrix}$

So the matrix is $\begin{bmatrix}1/2 & -1/2\\-1/2 & 1/2\end{bmatrix}$.

(b) The characteristic polynomial is $\lambda^2 - \lambda + 0 = \lambda(\lambda-1)$ so the eigenvalues are $0, 1$

We know that $p(\vec{u}) = \vec{u}$ because $\vec{u} \in L$, therefore $\begin{bmatrix}1\\-1\end{bmatrix}$ is an eigenvector (for the eigenva

Correction : Second Practice Final exam: 1/

① ⓐ V is the kernel of $T: \mathbb{R}^3 \longrightarrow \mathbb{R}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto [3\ 1\ 0] \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Since $\mathrm{rref}([3\ 1\ 0]) = [1\ \tfrac{1}{3}\ 0]$, we can find a basis for the kernel:

$$\left.\begin{array}{l} \bullet\ \vec{v}_2 = \tfrac{1}{3}\vec{v}_1 \text{ so } \begin{bmatrix} -\tfrac{1}{3} \\ 1 \\ 0 \end{bmatrix} \text{ is a first vector in the kernel.} \\ \bullet\ \vec{v}_3 = \vec{0} \text{ so } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ is a second vector in kernel} \end{array}\right\} \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -\tfrac{1}{3} \\ 1 \\ 0 \end{bmatrix} \text{ is a basis for the kernel.}$$

ⓑ $\left\| \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\| = 1$ so we keep this vector,

$\begin{bmatrix} -\tfrac{1}{3} \\ 1 \\ 0 \end{bmatrix}$ is already orthogonal to $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, so we just need to normalize it: $\left\| \begin{bmatrix} -\tfrac{1}{3} \\ 1 \\ 0 \end{bmatrix} \right\| = \frac{\sqrt{10}}{3}$,

therefore $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \\ 0 \end{bmatrix}$ is an orthonormal basis of V.

---

② $A = \begin{bmatrix} 0 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & 3 & 4 \end{bmatrix}$

We write

$$\left[\begin{array}{ccc|ccc} 0 & 3 & 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] R_2 \leftrightarrow R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \times\tfrac{1}{3} \\ R_3 - R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1/3 & 1/3 & 0 & 0 \\ 0 & 3 & 2 & 0 & -1 & 1 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 3R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \begin{array}{l} R_1 - 2R_3 \\ R_2 - \tfrac{1}{3}R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 3 & -2 \\ 0 & 1 & 0 & 2/3 & 1/3 & -1/3 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

So the inverse is $\begin{bmatrix} 2 & 3 & -2 \\ 2/3 & 1/3 & -1/3 \\ -1 & -1 & 1 \end{bmatrix}$

(Don't forget to check the result!).

(3) a) We need to reduce A:

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & -1 & 5 \\ 4 & -1 & 19 \\ 1 & 0 & 3 \end{bmatrix} \begin{matrix} \\ R_2 - 4R_1 \\ R_3 - 4R_1 \\ R_4 - R_1 \end{matrix}$$

Error! Rank is 3 not 2!

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & -7 \\ 0 & -1 & 7 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \times(-1) \\ R_3 - R_2 \\ \\ \end{matrix}$$

$$\text{RRef } A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore a basis for im A is $\begin{bmatrix} 1 \\ 4 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \end{bmatrix}$. So dim im $A = 2$.

3 leading ones! so dim ker=0!!

(b) The dimension formula says: $\underset{3}{\text{dim domain}} = \text{dim ker } A + \underset{2}{\text{dim im } A}$, therefore dim ker $A = 1$.

---

(4) (a) We need to calculate: $T(1) = (2t-1) \cdot 0 = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_B$

$$T(t) = (2t-1) \cdot 1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}_B$$

$$T(t^2) = (2t-1) \cdot 2t = \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix}_B$$

So the matrix of T in the basis B is:

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

(b) Eigenvalues:

we compute $\det \begin{bmatrix} -\lambda & -1 & 0 \\ 0 & 2-\lambda & -2 \\ 0 & 0 & 4-\lambda \end{bmatrix} = (-\lambda) \cdot (2-\lambda) \cdot (4-\lambda)$ so the eigenvalues are $0, 2, 4$.

Eigenspaces :

- $E_0 = \ker(A - 0.I) = \ker \begin{bmatrix} 0 & -1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 4 \end{bmatrix}$

Since we want to find a basis of $\ker A$, we need to reduce

$$\begin{array}{cl} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 4 \end{bmatrix} & \begin{array}{l} \times(-1) \\ R_2 + 2R_1 \\ R_3 \times \frac{1}{4} \end{array} \end{array}$$

$$\begin{array}{cl} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} & \begin{array}{l} \\ \times \frac{-1}{2} \\ R_3 + \frac{1}{2}R_2 \end{array} \end{array}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\vec{v}_1 = \vec{0}$ so $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a basis of E_0. ($\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the polynomial $f(t) = 1$).

- $E_2 = \ker(A - 2I) = \ker \begin{bmatrix} -2 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 2 \end{bmatrix} \begin{array}{l} \times \frac{-1}{2} \\ \times -\frac{1}{2} \\ R_3 + R_2 \end{array}$

Again we reduce this matrix:

$$\mathrm{rref}(A - 2I) = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since $\vec{v}_2 = \frac{1}{2}\vec{v}_1$ we get that $\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$ is a basis of $E_2 = \ker(A - 2I)$. ($\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$ is also $t - \frac{1}{2}$).

- $E_4 = \ker(A - 4I) = \ker \begin{bmatrix} -4 & -1 & 0 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \times -\frac{1}{4} \\ \times -\frac{1}{2} \\ \end{array}$

$$\begin{array}{cl} \begin{bmatrix} 1 & \frac{1}{4} & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} & \begin{array}{l} R_1 - \frac{1}{4}R_2 \\ \\ \\ \end{array} \end{array}$$

$\mathrm{rref}(A - 4I) = \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, and $\vec{v}_3 = -\frac{1}{4}\vec{v}_1 + \vec{v}_2$ gives $\begin{bmatrix} \frac{1}{4} \\ -1 \\ 1 \end{bmatrix}$ as a basis for E_4.

($\begin{bmatrix} 1/4 \\ -1 \\ 1 \end{bmatrix}$ is also $t^2 - t + \frac{1}{4}$).

(c) Since $\dim E_0 + \dim E_2 + \dim E_4 = 3$, A is diagonalizable, and an eigenbasis is given by: $1, \ t - \frac{1}{2}, \ t^2 - t + \frac{1}{4}$.

5 First, let's find the eigenvalues of M: 4/

The characteristic polynomial of M is $\det(M-\lambda I) = \det \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 2 \\ 1 & 2 & -\lambda \end{bmatrix}$

Let's expand it along the first column: $(-\lambda)\left(\lambda^2-4\right) - 1.\left(-\lambda-2\right) + 1.\left(2+\lambda\right)$

and factor by $\lambda+2$: $= (\lambda+2)\left[(-\lambda)(\lambda-2)+1+1\right]$

$= (\lambda+2)\left(-\lambda^2+2\lambda+2\right)$

$= -(\lambda+2)\left(\lambda-(1+\sqrt{3})\right)\left(\lambda-(1-\sqrt{3})\right)$. So the eigenvalues are: $-2, 1+\sqrt{3}, 1-\sqrt{3}$.

Since we have 3 distinct eigenvalues, we already know that M is diagonalizable.

Eigenspaces:

- E_{-2} is $\ker\left(A-(-2)I\right) = \ker \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$ $R_2 \leftrightarrow R_1$

$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ $\begin{matrix} R_2 - 2R_1 \\ R_3 - R_1 \end{matrix}$

$\begin{pmatrix} 1 & 2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{pmatrix}$ $R_2 \times \left(\frac{-1}{3}\right)$

$\begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $R_1 - 2R_2$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ Since $\vec{v}_3 = \vec{v}_2$ we obtain that $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ is a basis for E_{-2}.

- $E_{1+\sqrt{3}}$ is $\ker\left(A-(1+\sqrt{3})I\right) = \begin{pmatrix} -(1+\sqrt{3}) & 1 & 1 \\ 1 & -(1+\sqrt{3}) & 2 \\ 1 & 2 & -(1+\sqrt{3}) \end{pmatrix}$ $R_2 \leftrightarrow R_1$

$\begin{pmatrix} 1 & -(1+\sqrt{3}) & 2 \\ -(1+\sqrt{3}) & 1 & 1 \\ 1 & 2 & -(1+\sqrt{3}) \end{pmatrix}$ $\begin{matrix} R_2 + (1+\sqrt{3})R_1 \\ R_3 - R_1 \end{matrix}$

$$\begin{pmatrix} 1 & -(1+\sqrt{3}) & 2 \\ 0 & 1-(4+2\sqrt{3}) & 1+2(1+\sqrt{3}) \\ 0 & 3+\sqrt{3} & -3-\sqrt{3} \end{pmatrix} \text{ which is } \begin{pmatrix} 1 & -1-\sqrt{3} & 2 \\ 0 & -3-2\sqrt{3} & 3+2\sqrt{3} \\ 0 & 3+\sqrt{3} & -3-\sqrt{3} \end{pmatrix} \begin{matrix} \\ \times\frac{-1}{3+2\sqrt{3}} \\ \times\frac{1}{3+\sqrt{3}} \end{matrix}$$

$$\begin{pmatrix} 1 & -1-\sqrt{3} & 2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \begin{matrix} R_1+(1+\sqrt{3})R_2 \\ \\ R_3-R_2 \end{matrix}$$

$$RRef(A-(1+\sqrt{3})I) = \begin{pmatrix} 1 & 0 & 1-\sqrt{3} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Since $\vec{v_3} = (1-\sqrt{3})\vec{v_1} - \vec{v_2}$, we get that $\begin{bmatrix} \sqrt{3}-1 \\ 1 \\ 1 \end{bmatrix}$ is ~~an eigenvector of~~ a basis of $E_{1+\sqrt{3}}$.

- $E_{1-\sqrt{3}}$ is Ker $\begin{pmatrix} -1+\sqrt{3} & 1 & 1 \\ 1 & -1+\sqrt{3} & 2 \\ 1 & 2 & -1+\sqrt{3} \end{pmatrix} R_2 \leftrightarrow R_1$

$$\begin{pmatrix} 1 & \sqrt{3}-1 & 2 \\ \sqrt{3}-1 & 1 & 1 \\ 1 & 2 & \sqrt{3}-1 \end{pmatrix} \begin{matrix} \\ R_2+(1-\sqrt{3})R_1 \\ R_3-R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & \sqrt{3}-1 & 2 \\ 0 & -3+2\sqrt{3} & 3-2\sqrt{3} \\ 0 & 3-\sqrt{3} & \sqrt{3}-3 \end{pmatrix} \begin{matrix} \\ \times\frac{-1}{3-2\sqrt{3}} \\ \times\frac{1}{3-\sqrt{3}} \end{matrix}$$

$$\begin{pmatrix} 1 & \sqrt{3}-1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \begin{matrix} R_1-(\sqrt{3}-1)R_2 \\ \\ R_3-R_2 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & \sqrt{3}+1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$ So $\vec{v_3} = (1+\sqrt{3})\vec{v_1} - \vec{v_2}$ and $\begin{pmatrix} -1-\sqrt{3} \\ 1 \\ 1 \end{pmatrix}$ is a basis of $E_{1-\sqrt{3}}$.

So an eigenbasis will be $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} \sqrt{3}-1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1-\sqrt{3} \\ 1 \\ 1 \end{pmatrix}$ corresponding to the eigenvalues $-2, 1+\sqrt{3}, 1-\sqrt{3}$ (in this order)

If we write $Q = \begin{pmatrix} 0 & \sqrt{3}-1 & -1-\sqrt{3} \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ then Q is invertible and is such that $\underbrace{\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1+\sqrt{3} & 0 \\ 0 & 0 & 1-\sqrt{3} \end{pmatrix}}_{= D} = Q^{-1} A Q$.

(6). $A = \begin{bmatrix} 4 & 1 & 0 & 2 \\ 1 & -1 & 1 & 1 \\ 3 & 0 & 0 & 1 \\ 2 & 2 & 0 & 1 \end{bmatrix}$ Expand it along the third column:

$$-1 \cdot \left(\det \begin{pmatrix} 4 & 1 & 2 \\ 3 & 0 & 1 \\ 2 & 2 & 1 \end{pmatrix} \right)$$

$$= -1 \cdot \left(4(-2) - 3(1-4) + 2 \cdot 1 \right) = -1(-8+9+2) = -3.$$

---

(7) The characteristic polynomial is: $\lambda^2 - \mathrm{tr}C \cdot \lambda + \det C = \lambda^2 - 6\lambda + 5 = (\lambda-1)(\lambda-5)$.

There are 2 distinct eigenvalues so C is diagonalizable: (RK: each eigenspace will have dim = 1 so we just need to find one nonzero vector in each eigenspace...)

Eigenspaces:

$E_1 := \mathrm{Ker}(A - I) = \mathrm{Ker}\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$. (Sorry I wrote A, but you should read C instead...)

It's easy to see that $\begin{pmatrix} 1 \\ -1 \end{pmatrix} \in \mathrm{Ker}(A-I)$. Since it has dim = 1, we know that $E_1 = \mathrm{Span}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$E_5 := \mathrm{Ker}(A - 5I) = \mathrm{Ker}\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}$. Similarly $\begin{bmatrix} 1 \\ 3 \end{bmatrix} \in \mathrm{Ker}\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}$, so $E_5 = \mathrm{Span}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Let's call $P = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$ then the new matrix is $P^{-1} A P = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} = D$ (a diagonal with the eigenvalues of A as entries).

Now $P^{-1} = \frac{1}{\det P} \cdot \begin{bmatrix} 3 & -1 \\ +1 & 1 \end{bmatrix}$ and since $D^n = P^{-1} A^n P$, and $D^n = \begin{pmatrix} 1 & 0 \\ 0 & 5^n \end{pmatrix}$

we get that $A^n = P \begin{bmatrix} 1 & 0 \\ 0 & 5^n \end{bmatrix} P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5^n \end{bmatrix} \begin{bmatrix} 3/4 & -1/4 \\ 1/4 & 1/4 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3/4 & -1/4 \\ 5^n/4 & 5^n/4 \end{bmatrix}$$

so $$A^n = \begin{bmatrix} 3/4 + 5^n/4 & -\frac{1}{4} + \frac{5^n}{4} \\ -3/4 + \frac{3 \cdot 5^n}{4} & \frac{1}{4} + \frac{3 \cdot 5^n}{4} \end{bmatrix}$$

(8) (a) Assume we have a linear relation:

$$\alpha + 2\beta t + \gamma \cos t = 0 \quad \text{for any } t \in \mathbb{R}.$$

Take the second derivative of it:

$$-\gamma \cos t = 0 \text{ for any } t, \text{ therefore } \gamma \text{ must be } 0.$$

Now $\alpha + 2\beta t = 0$ for any t, but a polynomial function is zero if and only if its coefficients are zero, so $\alpha = \beta = 0$.

Conclusion: these 3 functions are linearly independent.

(b) Let's reduce the matrix:

$$\begin{bmatrix} 1 & 5 & 1 \\ 5 & -30 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{matrix} \\ R_2 - 5R_1 \\ R_3 - 2R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 5 & 1 \\ 0 & -55 & -5 \\ 0 & -11 & -1 \end{bmatrix} \begin{matrix} \\ \times \frac{-1}{55} \\ R_3 - \frac{1}{5}R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 5 & 1 \\ 0 & 1 & \frac{1}{11} \\ 0 & 0 & 0 \end{bmatrix}$$

This has Rank $= 2$ not 3 so the 3 vectors are not linearly independent.

(9.) The orthogonal complement of V is the kernel of $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$\vec{x} \longmapsto \begin{bmatrix} 1 & 1 & 0 \\ 1 & -3 & 1 \end{bmatrix} \cdot \vec{x}$$

Let's reduce $\begin{bmatrix} 1 & 1 & 0 \\ 1 & -3 & 1 \end{bmatrix} R_2 - R_1$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} R_2/-4$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -\frac{1}{4} \end{bmatrix} R_1 - R_2$$

$$\begin{bmatrix} 1 & 0 & 1/4 \\ 0 & 1 & -1/4 \end{bmatrix}$$

$\to$ we have $\vec{v}_3 = \frac{1}{4}\vec{v}_1 - \frac{1}{4}\vec{v}_2$ so $\begin{bmatrix} -1/4 \\ 1/4 \\ 1 \end{bmatrix}$ is a basis for $V^{\perp}$.

10. (a) $\|\vec{u}_1\|^2 = \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 = 1$

$\|\vec{u}_2\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = 1$

and $\vec{u}_1 \cdot \vec{u}_2 = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{2}} + 0 - \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{2}} = 0$

Therefore $\vec{u}_1, \vec{u}_2$ is an orthonormal family (or collection) of vectors.

(b) We know that $p_W(\vec{x}) = (\vec{x} \cdot \vec{u}_1)\vec{u}_1 + (\vec{x} \cdot \vec{u}_2)\vec{u}_2$.

So $p_W\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \left(\begin{bmatrix}1\\0\\0\end{bmatrix} \cdot \begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}\right)\vec{u}_1 + \left(\begin{bmatrix}1\\0\\0\end{bmatrix} \cdot \begin{bmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{bmatrix}\right) \cdot \vec{u}_2$

$= \frac{1}{\sqrt{3}}\vec{u}_1 + \frac{1}{\sqrt{2}}\vec{u}_2$

$= \begin{bmatrix}1/3\\1/3\\1/3\end{bmatrix} + \begin{bmatrix}1/2\\0\\-1/2\end{bmatrix} = \begin{bmatrix}5/6\\1/3\\-1/6\end{bmatrix}$

$p_W\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \frac{1}{\sqrt{3}}\vec{u}_1 + 0\,\vec{u}_2 = \begin{bmatrix}1/3\\1/3\\1/3\end{bmatrix}$.

and finally

$p_W\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \frac{1}{\sqrt{3}}\vec{u}_1 - \frac{1}{\sqrt{2}}\vec{u}_2$

$= \begin{bmatrix}1/3\\1/3\\1/3\end{bmatrix} + \begin{bmatrix}-1/2\\0\\1/2\end{bmatrix}$

$= \begin{bmatrix}-1/6\\1/3\\5/6\end{bmatrix}$

So the matrix is $\begin{bmatrix}5/6 & 1/3 & -1/6\\1/3 & 1/3 & 1/3\\-1/6 & 1/3 & 5/6\end{bmatrix}$

(c) To find the eigenvalues one can compute the characteristic polynomial, etc...

I will use a geometric argument instead:

. The line orthogonal to $\mathrm{span}(\vec{u}_1, \vec{u}_2)$ projects to $\vec{0}$: therefore 0 is an eigenvalue.

. Now $p_W(\vec{u}_1) = \vec{u}_1$ (because $\vec{u}_1 \in W$, or check it by the formula); similarly $p_W(\vec{u}_2) = \vec{u}_2$.

Therefore $\vec{u}_1, \vec{u}_2$ are 2 (independent) vectors that are eigenvectors for the eigenvalue 1.

Since we got an eigenbasis, there are no other eigenvalues than 0 and 1.

MAT 211, Linear Algebra Fall 2006
Second Practice Final Exam

1. Find an **orthonormal** basis for the subspace:

$$V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \Big/ \; 3x + y = 0 \right\}.$$

2. Compute A^{-1} using row reduction, where

$$A = \begin{bmatrix} 0 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & 3 & 4 \end{bmatrix}.$$

3. Let T: $\mathbb{R}^3 \longrightarrow \mathbb{R}^4$ be the linear map defined by $T(\vec{x}) = A.\vec{x}$, where

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & -1 & 5 \\ 4 & -1 & 19 \\ 1 & 0 & 3 \end{bmatrix}.$$

   a) Find a basis for $\operatorname{im} A$. What is the dimension of $\operatorname{im} A$?

   b) What is the dimension of $\ker A$?

4. Let P_2 be the space of polynomials of degree less than or equal to 2. We consider the basis $\mathcal{B}$: 1, t, t^2. Let's define the following linear map:

$$\begin{array}{rccc} T: & P_2 & \longrightarrow & P_2 \\ & f(t) & \longmapsto & (2t-1).f'(t) \end{array}$$

   a) Find the matrix of T in the basis $\mathcal{B}$.

   b) Find all the real eigenvalues of T and the eigenvectors.

   c) If possible, give an eigenbasis.

5. Is the matrix $M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ diagonalizable?

   If yes, find an invertible matrix P, and a diagonal matrix D such that PA = DP.

6. Compute $\det(A)$ when $A = \begin{bmatrix} 4 & 1 & 0 & 2 \\ 1 & -1 & 1 & 1 \\ 3 & 0 & 0 & 1 \\ 2 & 2 & 0 & 1 \end{bmatrix}$.

7. Consider the matrix $C = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$. Find a formula for C^n.

   (Hint: write A as $P^{-1}.D.P$ where D is a diagonal matrix).

8. Linearly independent families:

   a) Let V be the space of all C^∞ functions from $\mathbb{R}$ to $\mathbb{R}$. Consider the following three functions in V: $f_1(t) = 1$ (the constant function); $f_2(t) = 2t$; $f_3(t) = \cos(t)$. Are these three functions linearly independent?

   b) In $\mathbb{R}^3$, are the following vectors linearly independent?

$$\begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ -30 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

9. In $\mathbb{R}^3$, let V be the plane spanned by $\vec{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

   Find a basis for the orthogonal complement of V in $\mathbb{R}^3$.

10. In $\mathbb{R}^3$, consider a plane W, together with an orthonormal basis of W given by $\vec{u}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$.

    a) Check that $\vec{u}_1, \vec{u}_2$ form an orthonormal family.

    b) Find the matrix (in the standard basis) of the orthogonal projection p_W onto the plane W.

    c) Find the eigenvalues of that matrix. Using a geometric argument (without any computation) find at least one eigenvector.

Correction of Practice final exam:

1 ~~2 0 1 1~~

The augmented matrix of the system is

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 \\ 3 & 0 & -1 & -1 & 0 \\ 4 & 1 & 2 & 1 & 9 \end{array}\right] \begin{array}{l} R_1/2 \\ \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1/2 & 1/2 & 5/2 \\ 0 & 1 & 0 & -1 & -1 \\ 3 & 0 & -1 & -1 & 0 \\ 4 & 1 & 2 & 1 & 9 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 3R_1 \\ R_4 - 4R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1/2 & 1/2 & 5/2 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -5/2 & -5/2 & -15/2 \\ 0 & 1 & 0 & -1 & -1 \end{array}\right] \begin{array}{l} \\ \\ R_3 \times (-\frac{2}{5}) \\ R_4 - R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1/2 & 1/2 & 5/2 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1 - \frac{1}{2} R_3 \\ \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We have one free variable $x_4 = s$ and the system is equivalent to

$$\begin{aligned} x_1 &= 1 \\ x_2 &= -1 + s \\ x_3 &= 3 - s \\ x_4 &= s \end{aligned}$$

So the solutions are given by $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$.

2

(2) (a) We have $p\begin{bmatrix}1\\0\end{bmatrix} = \dfrac{\begin{bmatrix}1\\0\end{bmatrix}\cdot\vec{u}}{\|\vec{u}\|^2}\cdot\vec{u} = \dfrac{3}{13}\begin{bmatrix}3\\-2\end{bmatrix} = \dfrac{1}{13}\begin{bmatrix}9\\-6\end{bmatrix}$

and $p\begin{bmatrix}0\\1\end{bmatrix} = \dfrac{\begin{bmatrix}0\\1\end{bmatrix}\cdot\vec{u}}{13}\cdot\vec{u} = \dfrac{-2}{13}\begin{bmatrix}3\\-2\end{bmatrix} = \dfrac{1}{13}\begin{bmatrix}-6\\4\end{bmatrix}$

So $M = \dfrac{1}{13}\begin{bmatrix}9 & -6\\-6 & 4\end{bmatrix}$.

(b) The characteristic polynomial is $\lambda^2 - \mathrm{tr} M\cdot\lambda + \det M = \lambda^2 - \lambda + \left(\dfrac{36-36}{13^2}\right) = \lambda(\lambda-1)$.

Therefore the eigenvalues are 0 and 1.

(c) We know that $p(\vec{u}) = \vec{u}$ so $\vec{u}$ is one eigenvector of M (for the eigenvalue $\lambda = 1$).

---

(3) (a) The orth. complement of span$(\vec{e})$ is the kernel of $T: \mathbb{R}^3 \to \mathbb{R}$

$\vec{x} \mapsto [4\ 1\ 2]\cdot[\vec{x}]$

So we need to find a basis for Ker A, where $A = [4\ 1\ 2]$.

$\mathrm{RRef}(A) = [1\ \ 1/4\ \ 1/2]$ and $\vec{v}_2 = \frac{1}{4}\vec{v}_1$ so $\begin{bmatrix}-1/4\\1\\0\end{bmatrix} = \vec{w}_1$ is a first vector of ~~the~~ the basis.

and $\vec{v}_3 = \frac{1}{2}\vec{v}_1$ so $\begin{bmatrix}-1/2\\0\\1\end{bmatrix} = \vec{w}_2$ is the second vector of a basis for Ker A.

(b) Now $\|\vec{w}_1\| = \sqrt{1^2 + (1/4)^2} = \dfrac{\sqrt{17}}{4}$.

. So we take $\vec{u}_1 = \begin{bmatrix}-1/\sqrt{17}\\4/\sqrt{17}\\0\end{bmatrix}$ as first vector for the orthonormal basis.

. Take $\vec{w}_2$ and subtract to it the proj. onto the line L spanned by $\vec{u}_1$:

$p_L(\vec{w}_2) = (\vec{w}_2\cdot\vec{u}_1)\vec{u}_1 = \dfrac{1}{2\sqrt{17}}\cdot\vec{u}_1 = \begin{bmatrix}-\frac{1}{34}\\2/17\\0\end{bmatrix}$

So we get $\vec{w}_2 - p_L(\vec{w}_2) = \begin{bmatrix}-1/2\\0\\1\end{bmatrix} - \begin{bmatrix}-1/34\\2/17\\0\end{bmatrix} = \begin{bmatrix}-16/34\\-2/17\\1\end{bmatrix} = \begin{bmatrix}-8/17\\-2/17\\1\end{bmatrix}$ of length $\sqrt{\frac{64}{17^2} + \frac{4}{17^2} + \frac{17^2}{17^2}} = \dfrac{\sqrt{357}}{17}$

So the second vector is $\vec{u}_2 = \dfrac{1}{\sqrt{357}}\begin{bmatrix}-8\\-2\\17\end{bmatrix}$.

④ No: take $\begin{pmatrix}1&1\\1&1\end{pmatrix}$ it has $\det = 0$ but an expansion along the diagonal would give $1\cdot\det 1 + 1\cdot\det 1 = 2$. ③

⑤ We have:
(a) $d_1 = 1$, $d_2 = 0$, $d_3 = \det\begin{bmatrix}1&1&1\\1&1&0\\1&0&1\end{bmatrix} = \det\begin{bmatrix}1&1&1\\0&0&-1\\0&-1&0\end{bmatrix} = 1\cdot(-1) = -1$

(b) expand along the ~~first~~ second column: $d_n = -1\cdot\det I_{n-1} - 1\cdot\det\begin{bmatrix}1 & \cdots & 1\\ | & & \\ 1 & & \end{bmatrix}$

(b) expand along the second column:

$$d_n = -1\cdot\det\begin{pmatrix}1 & & & 0\\ | & 0 & \ddots & \\ | & 1 & 0 & \\ 1 & 0 & \cdots & 0\ \ 1\end{pmatrix} + 1\cdot d_{n-1} = d_{n-1} - 1.$$

(c) Since $d_n = d_{n-1} - 1$ and $d_1 = 1$ we deduce that $d_n = 2 - n$

. Indeed the formula is true for $n=1, n=2$

. Assume $d_n = 2-n$ for $n \geq 2$, then $d_{n+1} = d_n - 1 = 2 - n - 1$
$= 2 - (n+1)$.

So the formula is true for any n.

---

⑥ $T(1) = 1 - 1 = 0$; $T(t+1) = (3t+1+1) - (t+2+1) = 2t - 1 = 2(t+1) - 3 = \begin{bmatrix}-3\\2\\0\end{bmatrix}_B$

and $T((t-1)^2) = (3t+1-1)^2 - (t+2-1)^2 = 9t^2 - (t+1)^2 = 8t^2 - 2t - 1$

$= 8(t-1)^2 + 16t - 8 - 2t - 1$

$= 8(t-1)^2 + 14t - 9$

$= 8(t-1)^2 + 14(t+1) - 14 - 9$

$= \begin{bmatrix}-23\\14\\8\end{bmatrix}_B$

So the matrix is $\begin{bmatrix}0 & -3 & -23\\0 & 2 & 14\\0 & 0 & 8\end{bmatrix}$

Its determinant is 0.

7

(a) $M-\lambda I = \begin{pmatrix}-\lambda & -1 & -1\\ -1 & -\lambda & -1\\ -1 & -1 & -\lambda\end{pmatrix}$ Now if we set $\lambda = \underline{1}$ we get a matrix with (-1) everywhere. such a matrix is not invertible so $\det(M-I)=0$ and 1 is an eigenvalue of M.

(b) $A - I_3 = \begin{pmatrix}-1 & -1 & -1\\ -1 & -1 & -1\\ -1 & -1 & -1\end{pmatrix}$ has rank $= 1$ so $\dim \operatorname{Ker}(A-I_3)$ is 2,

because $\dim \mathbb{R}^3 = \dim \ker(A-I_3) + \underbrace{\dim \operatorname{im}(A-I_3)}_{= 1}$

(c) see above.

---

8 $\begin{pmatrix}1 & 5 & 2 & | & 1 & 0 & 0\\ 1 & 1 & 7 & | & 0 & 1 & 0\\ 0 & -3 & 4 & | & 0 & 0 & 1\end{pmatrix} R_2 - R_1$

$\begin{pmatrix}1 & 5 & 2 & | & 1 & 0 & 0\\ 0 & -4 & 5 & | & -1 & 1 & 0\\ 0 & -3 & 4 & | & 0 & 0 & 1\end{pmatrix} \times\left(-\frac{1}{4}\right)$

$\begin{pmatrix}1 & 5 & 2 & | & 1 & 0 & 0\\ 0 & 1 & -\frac{5}{4} & | & \frac{1}{4} & -\frac{1}{4} & 0\\ 0 & -3 & 4 & | & 0 & 0 & 1\end{pmatrix} \begin{matrix}R_1 - 5R_2\\ \\ R_3 + 3R_2\end{matrix}$

~~1 5 2 1 0 0~~

$\begin{pmatrix}1 & 0 & \frac{33}{4} & | & -\frac{1}{4} & \frac{5}{4} & 0\\ 0 & 1 & -\frac{5}{4} & | & \frac{1}{4} & -\frac{1}{4} & 0\\ 0 & 0 & \frac{1}{4} & | & \frac{3}{4} & -\frac{3}{4} & 1\end{pmatrix} \begin{matrix}R_1 - 33R_3\\ R_2 + 5R_3\\ R_3 \times 4\end{matrix}$

$\begin{pmatrix}1 & 0 & 0 & | & -25 & 26 & -33\\ 0 & 1 & 0 & | & 4 & -4 & 5\\ 0 & 0 & 1 & | & 3 & -3 & 4\end{pmatrix}$

So the inverse is $\begin{bmatrix}-25 & 26 & -33\\ 4 & -4 & 5\\ 3 & -3 & 4\end{bmatrix}$ (Check it!).

---

9 The characteristic polynomial of A is $\det \begin{bmatrix}5-\lambda & 4\\ 0 & 1-\lambda\end{bmatrix} = (5-\lambda)(1-\lambda)$. So the eigenvalues are $\begin{cases}\lambda = 1\\ \lambda = 5\end{cases}$.

The 2×2 matrix has 2 distinct eigenvalues, therefore it is diagonalizable.

(a) E_1: we want to solve $\begin{pmatrix}5 & 4\\ 0 & 1\end{pmatrix}\cdot \vec{x} = 1.\vec{x}$ so the augmented matrix is $\begin{pmatrix}4 & 4 & | & 0\\ 0 & 0 & | & 0\end{pmatrix}$ whose solutions are $\begin{pmatrix}x_1\\ x_2\end{pmatrix} = s.\begin{pmatrix}-1\\ 1\end{pmatrix}$.

so $E_1 = \operatorname{span}\begin{pmatrix}-1\\ 1\end{pmatrix}$

(b) E_5: we want to solve $\begin{pmatrix}5 & 4\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}5x_1\\ 5x_2\end{pmatrix}$ so the augmented matrix is $\begin{pmatrix}0 & 4 & | & 0\\ 0 & -4 & | & 0\end{pmatrix}$ whose solutions are $\begin{pmatrix}x_1\\ x_2\end{pmatrix} = s.\begin{pmatrix}1\\ 0\end{pmatrix}$.

therefore $E_5 = \operatorname{span}\begin{pmatrix}1\\ 0\end{pmatrix}$.

(9) continued:

The change of basis matrix is $\underbrace{\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}}_{P}$

We know that the matrix of C in the eigenbasis $\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$.

---

(10) Eigenvalues:

$$T - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 1 \\ 2 & -\lambda & -2 \\ -1 & 2 & 3-\lambda \end{bmatrix}$$

Let's expand along the first column:

$$(1-\lambda)\Big((-\lambda)(3-\lambda)+4\Big) - 2\Big(2(3-\lambda)-2\Big) - 1\Big(-4+\lambda\Big)$$

$$= (1-\lambda)(\lambda^2 - 3\lambda + 4) - 12 + 4\lambda + 4 + 4 - \lambda$$

$$= \lambda^2 - 3\lambda + \cancel{4} - \lambda^3 + 3\lambda^2 - 4\lambda - \cancel{12} + 4\lambda + \cancel{8} - \lambda$$

$$= -\lambda^3 + 4\lambda^2 - 4\lambda$$

$$= -\lambda(\lambda - 2)^2$$

So the eigenvalues are $\lambda = 0$ and $\lambda = 2$.

Eigenspaces:

(a) E_0 (Kernel of T): we want to solve $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 0 & -2 & 0 \\ -1 & 2 & 3 & 0 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & 4 & 4 & 0 \end{array}\right] \begin{array}{l} \\ \times -\frac{1}{4} \\ R_3 + R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1 - 2R_2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So the solutions are $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = s \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

Therefore $E_0 = \text{span}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

(b) $E_2 = \ker(A - 2I) = \ker\begin{pmatrix} -1 & 2 & 1 \\ 2 & -2 & -2 \\ -1 & 2 & 1 \end{pmatrix}$

we notice already that rank $(A-2I)$ is 2, because im $(A-2I)$ is spanned by the 2 first columns,

therefore, by the dimension formula $\dim \ker(A-2I) = 3-2 = 1$.

But $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is in $\ker(A-2I)$ (the last column is (-1) times the first one),

so we know that $E_2 = \operatorname{span}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

MAT 211, Linear Algebra Fall 2006
Practice Final Exam

1. Solve the following system of linear equations:

$$\begin{array}{rrrrcr} 2x & & +z & +w & = & 5 \\ & y & & -w & = & -1 \\ 3x & & -z & -w & = & 0 \\ 4x & +y & +2z & +w & = & 9 \end{array}$$

2. Let $p\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the orthogonal projection onto the line L spanned by the vector $\vec{u} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

   a) Find the matrix M of p expressed in the standard basis of $\mathbb{R}^2$.

   b) Find the eigenvalues of M.

   c) Give one example of an eigenvector of M.

3. Let $\vec{e} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$.

   a) Find a basis of the orthogonal complement of span$(\vec{e}\,)$.

   b) Find an **orthonormal basis** of the orthogonal complement of span$(\vec{e}\,)$.

4. Can we compute in general the determinant of any matrix by expanding down the diagonal?

   - If you think this is true, then prove it for 3×3 matrices;
   - If you think this isn't true for all the matrices, then give me a counterexample.

5. For any integer $n \geqslant 1$, let's call M_n the $n \times n$ matrix that has 1's everywhere on the first row, the first column and the diagonal, and zeroes everywhere else. For example, $M_1 = [1]$, $M_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $M_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, etc...We also call $d_n = \det(M_n)$.

   a) Compute d_1, d_2, d_3.

b) For $n \geqslant 3$, find a relation between d_n and d_{n-1}.

c) Find a closed formula for d_n (this means an exact formula like $d_n = n^2 + 3$, or $4n$, or 11).

6. Let P_2 be the space of polynomials of degree less than or equal to 2. We consider the basis $\mathcal{B}$: 1, $(t+1)$, $(t-1)^2$. Let's define the following linear map:

$$\begin{array}{rccc} T: & P_2 & \longrightarrow & P_2 \\ & f(t) & \longmapsto & f(3t+1) - f(t+2) \end{array}$$

a) Find the matrix of T in the basis $\mathcal{B}$.

b) Find the determinant of the linear map T.

7. Let $M = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$.

a) Show that 1 is an eigenvalue of M.

b) What is the dimension of $\ker(A - I_3)$? (I_3 is the identity matrix).

c) Use the dimension formula to deduce the rank of $A - I_3$.

8. Find the inverse of $A = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 1 & 7 \\ 0 & -3 & 4 \end{bmatrix}$.

9. Diagonalize the matrix $C = \begin{bmatrix} 5 & 4 \\ 0 & 1 \end{bmatrix}$.

10. Find the eigenvectors and the eigenvalues of

$$T = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & -2 \\ -1 & 2 & 3 \end{bmatrix}.$$

Name:

Student Id:

MAT 211, Section I Fall 2006
Midterm II

(1) (**10 points**) *For each of the questions below, indicate if the statement is true* (**T**) *or false* (**F**).

a T	f F
b F	g T
c F	h T
d F	i F
e T	j T

T (a) *There exists an isomorphism between* P_3 *(the space of polynomials of degree less than or equal to 3) and* $\mathbb{R}^4$.

F (b) *If* $T : \mathbb{R}^2 \to \mathbb{R}^5$ *is a linear map then necessarily* $kerT = \{\vec{0}\}$.

F (c) *One can find a linear map* $T : S \to \mathbb{R}^3$ *such that* $kerT = \{\vec{0}\}$ *where* S *is the space of all sequences of real numbers* $(a_0, a_1, \ldots)$.

F (d) There exists an isomorphism between $\mathbb{R}^3$ and $\mathbb{R}^4$.

T (e) *The linear map* $T : P_2 \to P_2$ *given by* $T(f(t)) = f(5t - 7)$ *is invertible.*

F (f) *For a linear map* $T : \mathbb{R}^3 \to P_5$ *one has* $dimP_5 = dim(kerT) + dim(imT)$.

T (g) *The family of functions* (e^x, e^{3x}, e^{5x}) *is linearly independent* .

T (h) *The matrices* $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ *form a basis for the space of upper-triangular* 2×2 *matrices.*

F (i) *The vector* $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ *is in the orthogonal complement of the plane spanned by* $\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$ *and* $\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$.

T (j) *The image of* $T : P_2 \to \mathbb{R}$ *defined by* $T(f(t)) = f(3)$ *has dimension* 1 .

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TRUE / FALSE :

(a) TRUE : a basis for P_3 is $1, x, x^2, x^3$.

(b) False : example $T: \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ has a kernel of dim 1 (equal to span $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$).

(c) False :

Take 4 (or more) linearly independent vectors in S. Since $\ker T = \{\vec{0}\}$ we know that the images by T are linearly independent which is impossible (because $\dim \mathbb{R}^3 = 3$).

(d) False : they don't have the same dimension

(e) TRUE : the inverse is $T(f(t)) = f\left(\frac{t+7}{5}\right)$.

(f) False : $\dim \underline{\text{domain}} = \dim \ker T + \dim \operatorname{im} T$.

(g) TRUE : proved in class...

(h) TRUE : $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ is not in span $\left(\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}\right)$ and $\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$ is not a multiple of $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$.

(i) False : $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = 2 + 6 - 2 = 6 \neq 0$.

(j) TRUE : $\operatorname{im} T$ has dimension 0 or 1. Since T is not the zero map, $\dim \operatorname{im} T \neq 0$ so it is $= 1$.

(2) (**25 points**)Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map. Its matrix in the standard basis is given by:

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ 1 & 5 & 1 \end{bmatrix}$$

Find the matrix of T in the following new basis

$$\mathcal{B} : \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}.$$

Let's consider the change of basis matrix.

$$P_{\mathcal{B} \to std} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 4 \\ -1 & 1 & 2 \end{bmatrix}$$

Let's find its inverse.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 3 & 4 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2/3 \\ R_3 + R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4/3 & 0 & 1/3 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 - 2R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1/3 & 1 & -1/3 & 0 \\ 0 & 1 & 4/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1/3 & 1 & -2/3 & 1 \end{array}\right] \begin{array}{l} R_1 + R_3 \\ R_2 - 4R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 1 \\ 0 & 1 & 0 & -4 & 3 & -4 \\ 0 & 0 & 1 & 3 & -2 & 3 \end{array}\right]$$

Let's check the result.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 4 \\ -1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 1 \\ -4 & 3 & -4 \\ 3 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now the matrix of T in $\mathcal{B}$ is $B = P^{-1} A P = \underbrace{\begin{bmatrix} 2 & -1 & 1 \\ -4 & 3 & -4 \\ 3 & -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ 1 & 5 & 1 \end{bmatrix}}_{\shortparallel} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 4 \\ -1 & 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 7 & 1 & 2 \\ -16 & -10 & -5 \\ 12 & 8 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 4 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 12 & 15 \\ -11 & -51 & -66 \\ 8 & 40 & 52 \end{bmatrix}$$

(3) (**25 points**)Let $T : P_2 \to P_2$ be defined by $T(f(t)) = f(t^2+1)$.

- Find the matrix of T in the basis $4-t, t+1, t^2$.
- Is T an isomorphism ? If yes find its inverse, if no find a basis for $kerT$.

We start with:

$$T(4-t) = 4-(t^2+1) = 3-t^2 = \frac{3}{5}(4-t) + \frac{3}{5}(t+1) - t^2 = \begin{bmatrix} 3/5 \\ 3/5 \\ -1 \end{bmatrix}_B$$

then

$$T(t+1) = t^2+1+1 = t^2+2 = \frac{2}{5}(4-t) + \frac{2}{5}(t+1) + t^2 = \begin{bmatrix} 2/5 \\ 2/5 \\ 1 \end{bmatrix}_B$$

then $T(t^2) = (t^2+1)^2 = t^4 + 2t^2 + 1$ which doesn't belong to P_2,

therefore T cannot be an isomorphism. (and $\ker T$ isn't even defined).

Remark:

Even if you consider $T : P_2 \to P_4$, this isn't an isomorphism:

For example because for any polynomial $T(f(t))$ has only terms with an even degree, therefore all the polynomials that are sums of monomials with odd degree are not in $\operatorname{im} T$.

(4) (**20 points**) In $\mathbb{R}^4$, find a basis for the orthogonal complement of the plane spanned by

$$\vec{v}_1 = \begin{bmatrix}1\\0\\1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\\1\\1\end{bmatrix} = \vec{v}_2$$

The orthogonal complement of span $(\vec{v}_1, \vec{v}_2)$ is the kernel of the following linear map

$$T : \mathbb{R}^4 \to \mathbb{R}^2 \qquad \vec{x} \mapsto \begin{bmatrix}\vec{v}_1 \cdot \vec{x} \\ \vec{v}_2 \cdot \vec{x}\end{bmatrix}$$

and the matrix of T in the standard basis is:

$$A = \begin{bmatrix}1 & 0 & 1 & 2 \\ -1 & 1 & 1 & 1\end{bmatrix}. \quad R_2 + R_1$$

In order to find a basis for ker T we reduce A:

$$\text{RRef} A = \begin{bmatrix}1 & 0 & \boxed{1} & \boxed{2} \\ 0 & 1 & \boxed{2} & \boxed{3}\end{bmatrix}$$

$$\qquad\qquad\qquad\qquad \vec{w}_3 \quad \vec{w}_4$$

Rank $A = 2$ so dim ker $T = 4 - 2 = 2$.

ⓐ $\vec{w}_3 = \vec{w}_1 + 2\vec{w}_2$ therefore $-\vec{w}_1 - 2\vec{w}_2 + \vec{w}_3 + 0\vec{w}_4 = \vec{0}$

so $\begin{bmatrix}-1\\-2\\1\\0\end{bmatrix}$ is a first vector for the basis of ker T.

ⓑ $\vec{w}_4 = 2\vec{w}_1 + 3\vec{w}_2$ so $-2\vec{w}_1 - 3\vec{w}_2 + 0\vec{w}_3 + 1\vec{w}_4 = \vec{0}$

and $\begin{bmatrix}-2\\-3\\0\\1\end{bmatrix}$ is a 2^{d} vector of the basis.

In conclusion, a basis for the orthogonal complement of span $(\vec{v}_1, \vec{v}_2)$ is given by $\left(\begin{bmatrix}-1\\-2\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\-3\\0\\1\end{bmatrix}\right)$.

(5) (**20 points**) Let V be the set of all sequences of real numbers $(a_0, a_1, \ldots)$ such that for any non-negative integer $K \geq 0$ we have $a_{K+2} = a_{K+1} + a_K$.

- Show that V is a subspace of the space of all sequences of real numbers.
- Consider $T : V \to \mathbb{R}^2$ defined by $T((a_0, a_1, \ldots)) = \binom{a_0}{a_1}$. Find $kerT$, and imT. What can you conclude?

(6) (**Extra credit: 5 points**) Show that the family of functions $f_a : x \mapsto \cos(ax)$ (where a is an arbitrary real number) is linearly independent.

#5 a) 1) $(0,0,\ldots)$ is in V because $0 = 0+0$!

2) if $\left.\begin{array}{l}(a_0, a_1, \ldots) \in V \\ (b_0, b_1, \ldots) \in V\end{array}\right\}$ then $(a_{K+2} + b_{K+2}) = a_{K+1} + a_K + b_{K+1} + b_K = (a_{K+1} + b_{K+1}) + (a_K + b_K)$

So V is closed under addition.

3) if $(a_0, a_1, \ldots) \in V$ and α is any real number then $\alpha \cdot (a_0, a_1, \ldots)$ is in V

because $\alpha a_{K+2} = \alpha a_{K+1} + \alpha a_K$ for any $K \geq 0$.

So V is closed under scalar multiplication.

In conclusion, V is a subspace of the space of all sequences of real numbers.

b) Let $(a_0, a_1, \ldots)$ be in $\ker T$. therefore $\begin{cases} a_0 = 0 \\ a_1 = 0 \end{cases}$

Let's prove by induction that $a_i = 0$ for any $i \geq 0$:

1) $a_0 = 0$ by assumption.

2) Assume $a_i = 0$ for $i \in \{0, 1, \ldots, K+1\}$.

Then since $(a_0, a_1, \ldots) \in V$ we know that $a_{K+2} = a_{K+1} + a_K = 0 + 0 = 0$

Therefore $a_i = 0$ for $i \in \{0, 1, \ldots, K+2\}$.

So we proved that $\ker T = \{\vec{0}\}$.

Now we can prove by induction (the same way) that a vector in V is uniquely defined by a_0 and a_1. Therefore $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ are in $\operatorname{im} T$. But this implies that $\operatorname{im} T = \mathbb{R}^2$.

Conclusion: T is an isomorphism.

Extra credit:

Suppose that we have a linear relation between the f_a: then you can pick one with the minimal number of terms (say n):

so $\lambda_1 \cos(a_1 x) + \cdots + \lambda_n \cos(a_n x) = 0$ (1)

Derive this expression twice to get:

$-\lambda_1 \cdot a_1^2 \cos(a_1 x) + \cdots - \lambda_n a_n^2 \cos(a_n x) = 0$ (2)

Multiply (1) by a_n^2 and add (2): you get a linear relation between $(n-1)$ such functions, which is impossible (n was the minimal number of terms).

Some more problems: the correction:

① $T: P_2 \to \mathbb{R}^3$

$$p(t) \mapsto \begin{bmatrix} p(1) \\ p'(1) \\ p''(1) \end{bmatrix}$$

ⓐ We compute $T(1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $T(x) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $T(x^2) = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ so the matrix is $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$.

ⓑ Similarly: $T(1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $T(x-1) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $T((x-1)^2) = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ so the matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

---

② Since
$$\begin{cases} \cos(t + \frac{\pi}{3}) = \cos t \cdot \cos\frac{\pi}{3} - \sin t \cdot \sin\frac{\pi}{3} & \in \operatorname{span}(\cos t, \sin t) \\ \sin(t - \frac{\pi}{2}) = -\cos t & \in \operatorname{span}(\cos t, \sin t) \end{cases}$$

we know that $\operatorname{span}(\cos t, \sin t, \cos(t+\frac{\pi}{3}), \sin(t-\frac{\pi}{2})) = \operatorname{span}(\cos t, \sin t)$. But $\cos t$, and $\sin t$ are linearly independent therefore $(\cos t, \sin t)$ is a basis for $\operatorname{span}(\cos t, \sin t, \cos(t+\frac{\pi}{3}), \sin(t-\frac{\pi}{2}))$ which has then dim =

---

③ $T: P_2 \to P_2$

$$f(t) \mapsto f(3t-2)$$

We have $T(2+t) = 2 + 3t - 2 = 3t = -3(2+t) + 6(t+1) = \begin{bmatrix} -3 \\ 6 \\ 0 \end{bmatrix}_B$.

$T(t+1) = 3t - 2 + 1 = 3t - 1 = -4(t+2) + 7(t+1)$

$$= \begin{bmatrix} -4 \\ 7 \\ 0 \end{bmatrix}_B.$$

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$$T\left((t-1)^2\right) = (3t-2-1)^2 = 9(t-1)^2 = \begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix}_B$$

So the matrix is $\begin{bmatrix} -3 & -4 & 0 \\ 6 & 7 & 0 \\ 0 & 0 & 9 \end{bmatrix}$. T is an isomorphism, its inverse is $f(t) \mapsto f\left(\frac{t+2}{3}\right)$.

(4) 1) $(0,0,\cdots) \in V$

2) if $(a_0, a_1, \cdots)$, $(b_0, b_1, \cdots)$ are in V then $a_{k+2} + b_{k+2} = (a_{k+1} + b_{k+1}) + 2(a_k + b_k)$

so V is closed under addition.

3) Let α be any real number, and $(a_0, \cdots) \in V$ then $(\alpha a_0, \alpha a_1, \cdots) \in V$

because $\alpha a_{k+2} = \alpha a_{k+1} + 2\alpha a_k$.

Some more problems:

①

Consider the following map:

$$T : P_2 \longrightarrow \mathbb{R}^3$$
$$p(t) \mapsto \begin{bmatrix} p(1) \\ p'(1) \\ p''(1) \end{bmatrix}$$

ⓐ Find the matrix of T, for the following choice of basis:
$1, x, x^2$ for P_2, and standard basis for $\mathbb{R}^3$

ⓑ Find the matrix of T, for the following choice of basis:
$1, (x-1), (x-1)^2$ for P_2, and standard basis for $\mathbb{R}^3$.

Rk: please notice that you can't use the change of basis formula, because T goes from one space to another one (different one), but we only defined change of basis for $T : V \to V$ ...

② What is the dimension of the space: $\text{span}\left(\cos t, \sin t, \cos\left(t+\frac{\pi}{3}\right), \sin\left(t-\frac{\pi}{2}\right)\right)$

③ Let $T : P_2 \to P_2$ be defined by $T(f(t)) = f(3t-2)$.

- Find the matrix of T in the basis $2+t, t+1, (t-1)^2$.
- is T an isomorphism? If yes, find its inverse.

④ Let V be the set of all real sequences $(a_0, a_1, a_2, \ldots\ldots)$ such that for any ~~~~ integer $k \geq 0$ we have $a_{k+2} = a_{k+1} + 2a_k$.
Is V a subspace of the space of all real sequences?

Correction Practice Midterm II:

① True/False:

① F : {0} is not in it

② F : $T(3.P(x)) = 3P(x).3P(-x) =$ ~~...~~ $9T(P(x))$.

③ T : the change of basis matrix from $1, x, x^2$ is $\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ which is invertible.

④ T : im T has dim at most 1, and im T is not $\{\vec{0}\}$, so it is $\mathbb{R}$.

⑤ F : $\ker \Delta \neq \{\vec{0}\}$ (contains all the cst functions).

⑥ F : examples: take space (x, y, z), then (x, y, t), (x, z, t) etc...

⑦ T : $\ker T = \{\vec{0}\}$ and $\operatorname{im} T =$ target space.

⑧ F : example ~~...~~ $\vec{0}$ is in both.

⑨ T : there can be at most 4 leading 1's in the reduced matrix.

⑩ F : span $\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)$ doesn't contain any invertible matrix.

② Let's consider the change of basis matrix $P = P_{\mathcal{B} \to std} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 4 \\ 1 & 1 & 2 \end{bmatrix}$.

Let's find its inverse:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & 2 & -2 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \; R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 3 & 2 & -2 & 1 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 3R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 1 & -3 \end{array}\right] \begin{array}{l} R_1 + R_3 \\ R_2 + R_3 \\ R_3 \times (-1) \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & -3 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -1 & -1 & 3 \end{array}\right]$$

So $P^{-1} = \begin{bmatrix} 2 & 1 & -3 \\ 0 & 1 & -2 \\ -1 & -1 & 3 \end{bmatrix}$.

(We check this to be sure:

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 4 \\ 1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & -3 \\ 0 & 1 & -2 \\ -1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$).

(2) Continued:

Now the matrix B of T in the new basis is given by $B = P^{-1} A P$.

So: $B = \begin{bmatrix} 2 & 1 & -3 \\ 0 & 1 & -2 \\ -1 & -1 & 3 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} 3 & 0 & 5 \\ 4 & 2 & 6 \\ 1 & 7 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 4 \\ 1 & 1 & 2 \end{bmatrix}}$

$B = \begin{bmatrix} 2 & 1 & -3 \\ 0 & 1 & -2 \\ -1 & -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 8 & 5 & 13 \\ 14 & 12 & 24 \\ 16 & 22 & 31 \end{bmatrix}$

$B = \begin{bmatrix} -18 & -44 & -43 \\ -18 & -32 & -38 \\ 26 & 49 & 56 \end{bmatrix}$

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(3) Let's find the matrix of T in the basis $B: 1, x, x^2$ of P_2:

$T(1) = 1 - 1 + 0 = 0$

$T(x) = x+1 - x + 1 = 2$

$T(x^2) = (x+1)^2 - x^2 + 2x = 4x + 1$

So the matrix of T is $A = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

We have $\operatorname{RRef}(A) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Therefore $\begin{cases} \text{Rank } (= \dim \operatorname{im} T) = 2 \\ \dim \ker T = 1 \end{cases}$

• A basis for $\operatorname{im} T$ is given by the vectors $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}_B$ and $\begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}_B$, but since $\operatorname{span}\left(\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}_B, \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}_B\right) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_B, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_B\right)$,

we get that $\operatorname{im} T = \operatorname{span}(1, x) = P_1$, and $(1, x)$ is a basis for $\operatorname{im} T$.

• A basis for $\ker T$:

the first column $\vec{v}_1 = \vec{0}$ so we get the vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_B$ as a <u>basis for $\ker T$.</u>

In other words $\ker T = \operatorname{span}(1)$ = space of all constant polynomials.

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④ The orthogonal of W is the kernel of the orthogonal projection onto W.
We could find an orthonormal basis for W in order to get the expression of this orthogonal projection, but we will do something simpler:

let $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 6 \end{bmatrix}$, and let $T : \mathbb{R}^4 \to \mathbb{R}^2$
$$\vec{x} \mapsto \begin{bmatrix} \vec{x}\cdot\vec{u}_1 \\ \vec{x}\cdot\vec{u}_2 \end{bmatrix}$$

Now we are reduced to the problem of finding a basis for Ker T.

Let's compute the matrix A of T in the standard basis:

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 6 \end{bmatrix}.$$

So $A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 4 & 2 & 6 \end{bmatrix}$ and $\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 5/4 & 0 \end{bmatrix}$.

We have the linear relations $\begin{cases} \vec{v}_3 = -\vec{v}_1 + 5/4\,\vec{v}_2 \\ \text{and} \\ \vec{v}_4 = 2\vec{v}_1 \end{cases}$ so a basis for Ker T is $\begin{bmatrix} 1 \\ -5/4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Therefore $\begin{bmatrix} 1 \\ -5/4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is a basis for $W^\perp$.

---

⑤ ⓐ First we notice that tr is linear: ⓐ $\mathrm{tr}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right) = \mathrm{tr}\begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} = a+e+d+h = \mathrm{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \mathrm{tr}\begin{pmatrix} e & f \\ g & h \end{pmatrix}$.

ⓑ $\mathrm{tr}\left(k.\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \mathrm{tr}\begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix} = ka + kd = k(a+d) = k\,\mathrm{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

But now, $M \mapsto AM$ is linear (because $A(M+N) = AM + AN$, and $A(k.M) = k.A.M$)
So $M \mapsto \mathrm{tr}(AM)$ is linear (because $\mathrm{tr}(A(M+N)) = \mathrm{tr}(AM+AN) = \mathrm{tr}\,AM + \mathrm{tr}\,AN$, and $\mathrm{tr}(A.(k.M)) = \mathrm{tr}(k.AM) = k.\mathrm{tr}\,AM$).
So this is true in particular for $A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$.

(5) Continued:

an alternative proof of (1):

if $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $AM = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+2c & b+2d \\ -a+3c & -b+3d \end{pmatrix}$

and $tr(AM) = a+2c-b+3d$ which is linear.

Let's take the following basis for the 2x2 matrices $\vec{e}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\vec{e}_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\vec{e}_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Let A be the matrix of $T : \mathbb{R}^{2\times 2} \to \mathbb{R}$ in this basis:

$T(\vec{e}_1) = tr\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} = 1$, $T(\vec{e}_2) = tr\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} = -1$, $T(\vec{e}_3) = tr\begin{pmatrix} 0 & 2 \\ 0 & 3 \end{pmatrix} = 3$, $T(\vec{e}_4) = tr\begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix} = 2$

So $A = \begin{bmatrix} 1 & -1 & 3 & 2 \end{bmatrix}$ The image of T is spanned by 1, so it is $\mathbb{R}$ itself.

By the dimension formula: $\underbrace{\dim \mathbb{R}^{2\times 2}}_{4} = \dim \ker T + \underbrace{\dim \operatorname{im} T}_{1}$

we get that $\boxed{\dim \ker T = 3}$.

Et voilà...

Practice Midterm II

① True or False questions:

1/ The 1×1 invertible matrices form a vector space.

2/ $T: P_2 \to P_2$ defined by $T(P(x)) = P(x).P(-x)$ is linear

3/ x^2-x+1, x^2+1, x^2 is a basis of P_2.

4/ The image of $T: P[x] \to \mathbb{R}$ has dim 1, where T is given by $T(P(x)) = P(8)$.

5/ The derivation map $\Delta: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is an isomorphism.

6/ $\mathbb{R}^4$ has only one subspace of dim 3.

7/ If T is a linear transformation from P_5 to a space W, such that $\ker T = \{\vec{0}\}$ and $\dim W = 6$ then T is an isomorphism.

8/ If T is a linear transformation from P_2 to itself then there is no vector that is in both $\ker T$ and $\operatorname{im} T$.

9/ If $W = \operatorname{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ then $\dim W \leq 4$.

10/ Every subspace of dim ≥ 1 of $\mathbb{R}^{2\times2}$ ($\mathbb{R}^{2\times2}$ means the 2×2 matrices) contains one invertible matrix (at least one).

② Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be defined by its matrix A in the standard basis:

$$A = \begin{bmatrix} 3 & 0 & 5 \\ 4 & 2 & 6 \\ 1 & 7 & 1 \end{bmatrix}.$$

Let $\mathcal{B}$ be the basis $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$.

Find the matrix of T in the basis $\mathcal{B}$.

---

③ ⓐ Find the image, the rank, the kernel and dimension of the kernel of $T: P_2 \to P_2$

$$P(x) \longmapsto P(x+1) - P(x) + P'(x).$$

ⓑ Find a basis for $\ker(T)$.

---

④ Find a basis for the orthogonal of W, where $W = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 2 \\ 6 \end{bmatrix} \right)$

[You will be able to solve this <u>next week</u>].

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⑤ For a square 2×2 matrix M, we call "trace of M" (and we write $tr(M)$) the sum of its entries on the diagonal $\left(\text{ex: } tr\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d\right)$.

① Let $A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$. Show that $T: \mathbb{R}^{2\times 2} \to \mathbb{R}$ given by $T(M) = tr(AM)$ is linear.

② What is the image of T? What is $\dim \ker T$?

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MAT 211, Section I Fall 2006
Midterm I

(1) **(10 points)** *For each of the questions below, indicate if the statement is true* (**T**) *or false* (**F**).

a	T	f	F
b	T	g	T
c	T	h	F
d	F	i	T
e	T	j	F

(a) *A* 1×1 *matrix* $[x]$ *is invertible if and only if* x *is nonzero.* T (the inverse is $[1/x]$.)

(b) *There exist non invertible matrices for which* $\ker A = \{0\}$. T (ex: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$).

(c) *The rank of* $\begin{pmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \end{pmatrix}$ *is one.* T (RRef(A) is $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$).

(d) *The matrix* $\begin{pmatrix} 3 & 5 \\ 0 & 5 \end{pmatrix}$ *is invertible and its inverse is* $\begin{pmatrix} 1/3 & 1/5 \\ 0 & 1/5 \end{pmatrix}$. F (it is invertible, but the inverse is $\begin{pmatrix} 1/3 & -1/3 \\ 0 & 1/5 \end{pmatrix}$).

(e) *The vectors* $\binom{1}{1}, \binom{2}{2}, \binom{3}{0}$ *generate* $\mathbb{R}^2$. T ($\begin{bmatrix} x \\ y \end{bmatrix} = \frac{x-y}{3} \cdot \begin{bmatrix} 3 \\ 0 \end{bmatrix} + y \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$).

(f) *For A, B square matrices,* $\ker(A.B)$ *is included in* $\ker A$. F (Take $A = I$, $B = 0$ then $\ker AB = \mathbb{R}^n$, $\ker A = \{\vec{0}\}$)

(g) *The span of* $(\binom{3}{2}, \binom{4}{5}, \binom{1}{3})$ *is equal to the span of* $(\binom{3}{2}, \binom{1}{3})$ T (because $\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix}$).

(h) *The union of two subspaces of* $\mathbb{R}^2$ *is a subspace of* $\mathbb{R}^2$. F (The union of horiz. and vert. axis in $\mathbb{R}^2$ is not a subspace)

(i) *Ker A contains a non zero vector if and only if there exists a non trivial linear relation between the columns of A.* T (because $\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_m \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1 \vec{v}_1 + \cdots + x_m \vec{v}_m$).

(j) *If AB is invertible then A is invertible.*

F ($\begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = [2]$ invertible, but $[1 \ 1]$ is not.)

(2) (**20 points**)Show that the following matrix is invertible and compute its inverse:

$$A = \begin{bmatrix} 2 & 5 & 4 \\ 2 & 4 & 2 \\ -1 & 0 & 2 \end{bmatrix}$$

As usual we write:

$$\left[\begin{array}{ccc|ccc} 2 & 5 & 4 & 1 & 0 & 0 \\ 2 & 4 & 2 & 0 & 1 & 0 \\ -1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \times(-1) \text{ and swap } R_1 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & 0 & -1 \\ 2 & 5 & 4 & 1 & 0 & 0 \\ 2 & 4 & 2 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} R_2 - 2R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & 0 & -1 \\ 0 & 5 & 8 & 1 & 0 & 2 \\ 0 & 4 & 6 & 0 & 1 & 2 \end{array}\right] \times \frac{1}{5}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & 0 & -1 \\ 0 & 1 & 8/5 & 1/5 & 0 & 2/5 \\ 0 & 4 & 6 & 0 & 1 & 2 \end{array}\right] R_3 - 4R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & 0 & -1 \\ 0 & 1 & 8/5 & 1/5 & 0 & 2/5 \\ 0 & 0 & -2/5 & -4/5 & 1 & 2/5 \end{array}\right] \times\left(-\frac{5}{2}\right)$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & 0 & -1 \\ 0 & 1 & 8/5 & 1/5 & 0 & 2/5 \\ 0 & 0 & 1 & 2 & -\frac{5}{2} & -1 \end{array}\right] \begin{array}{l} R_1 + 2R_3 \\ R_2 - 8/5\, R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -5 & -3 \\ 0 & 1 & 0 & -3 & 4 & 2 \\ 0 & 0 & 1 & 2 & -\frac{5}{2} & -1 \end{array}\right]$$

The reduced form of A is the identity, so A is invertible and $A^{-1} = \begin{bmatrix} 4 & -5 & -3 \\ -3 & 4 & 2 \\ 2 & -\frac{5}{2} & -1 \end{bmatrix}$.

(Just to be sure let's compute $\begin{bmatrix} 2 & 5 & 4 \\ 2 & 4 & 2 \\ -1 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 4 & -5 & -3 \\ -3 & 4 & 2 \\ 2 & -\frac{5}{2} & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$).

(3) (**20 points**)Solve the following system

$$x + 3y + 5t = -1$$
$$3x + 9y + 2z = 2$$
$$6x + 19y + 32t = -7$$
$$7x + 4y + t = 10$$

We write the augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 5 & -1 \\ 3 & 9 & 2 & 0 & 2 \\ 6 & 19 & 0 & 32 & -7 \\ 7 & 4 & 0 & 1 & 10 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 6R_1 \\ R_7 - 7R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 5 & -1 \\ 0 & 0 & 2 & -15 & 5 \\ 0 & 1 & 0 & 2 & -1 \\ 0 & -17 & 0 & -34 & 17 \end{array}\right] \begin{array}{l} \\ \text{Swap } R_2 \leftrightarrow R_3 \\ \\ \times\left(\frac{-1}{17}\right) \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 5 & -1 \\ 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 2 & -15 & 5 \\ 0 & 1 & 0 & 2 & -1 \end{array}\right] \begin{array}{l} R_1 - 3R_2 \\ \\ R_3/2 \\ R_4 - R_2 \end{array}$$

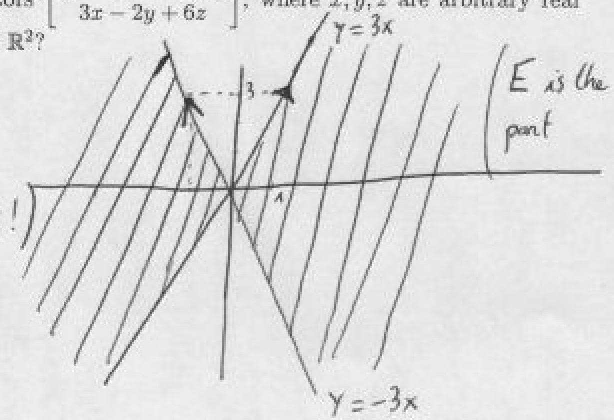
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 1 & -\frac{15}{2} & \frac{5}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So the rank is 3, there is one free variable $t = u$,

and the solutions of the system are given by $\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 5/2 \\ 0 \end{bmatrix} + u \cdot \begin{bmatrix} 1 \\ -2 \\ 15/2 \\ 1 \end{bmatrix}$. (It's a line).

(4) **(20 points)**

- Let E be the set of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ satisfying : $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ **or** ($x \neq 0$ and $-3 \leq \frac{y}{x} \leq 3$).
  Is E a subspace of $\mathbb{R}^2$?
- Let F be the set of all vectors $\begin{bmatrix} 4x - 17y + 1000z \\ 3x - 2y + 6z \end{bmatrix}$, where x, y, z are arbitrary real numbers. Is F a subspace of $\mathbb{R}^2$?



(E is the shaded part)

(a)

1) E contains $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

2/ Now $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is in E (because $-3 \leq \frac{3}{1} \leq 3$!)

and also $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ (because $-3 \leq \frac{3}{-1} \leq 3$)

but their sum $\begin{bmatrix} 0 \\ 6 \end{bmatrix}$ is not.

So E is <u>not</u> a subspace of $\mathbb{R}^2$.

(b) F is a subspace of $\mathbb{R}^2$ because it is im(A) where $A = \begin{bmatrix} 4 & -17 & 1000 \\ 3 & -2 & 6 \end{bmatrix}$.

(5) • **(15 points)** Let M be a square (3×3) matrix such that the sum of the entries of each row is zero. Show that M is not invertible. (Hint: what is the image of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$?)

• **(5 points)** Let N be a square matrix such that the ~~some~~ sum of the entries of each column is zero. Show that N is not invertible.

(a) $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b+c \\ d+e+f \\ g+h+i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Since $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a non zero vector in the Kernel of M, we know that M is not invertible.

(b) Multiply N to the left by the matrix $P = \underbrace{\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}}_{\text{square matrix}}$ (all the entries are 1).

Then we get $P.N = \begin{bmatrix} 0 & - & 0 \\ | & & | \\ 0 & - & 0 \end{bmatrix}$ (the zero matrix).

But now, if N were invertible, we could multiply by N^{-1} to the right on each side of the equation $P.N = [0]$ and we would get $P.N.N^{-1} = [0].N^{-1}$
$\Rightarrow P = [0]$
which is not true.

Therefore N is <u>not</u> invertible.

(6) Let us call E_{ij} the 3×3 matrix with a 1 in position (i, j) (meaning: the entry that is on row number i and column j is 1) and zeroes everywhere else.

(a) (**10 points**)Find all the (3×3) matrices commuting with each of the 9 matrices E_{ij} (we say that M and E_{ij} commute if and only if: $M.E_{ij} = E_{ij}.M$).

(b) (**extra credits: 10 points**)Find all the matrices N that commute with all the (3×3) matrix.

$\boxed{\text{First solution:}}$

ⓐ Call m_{ij} the entries of M. We also write $M = \left[\vec{c}_1 \,\vdots\, \vec{c}_2 \,\vdots\, \vec{c}_3\right]$ (column vectors)

and $M = \begin{bmatrix} \vec{R}_1 \\ \vec{R}_2 \\ \vec{R}_3 \end{bmatrix}$ (Row vectors).

Then: $M.E_{ij} = \begin{array}{c} \quad j \\ \begin{pmatrix} 0 & \vec{c}_i & 0 \end{pmatrix} \end{array}$ (means copy $\vec{c}_i$ from M and paste it in column j.)

whereas $E_{ij}.M = \begin{array}{cc} \begin{pmatrix} 0 \\ - \vec{R}_j - \\ 0 \end{pmatrix} & i \end{array}$ (i.e copy Row $\vec{R}_j$ from M and paste it in Row i).

By comparing these 2 matrices you get that $m_{ii} = m_{jj}$ and that $\begin{cases} m_{\ell i} = 0 \text{ if } \ell \neq i \\ \text{and} \\ m_{jk} = 0 \text{ if } k \neq j \end{cases}$

Since this must be true for any matrix E_{ij} $\left(\begin{array}{l}\text{where } i = 1,2,3 \\ \text{and } j = 1,2,3\end{array}\right)$,

we get that necessarily M is such that: $m_{11} = m_{22} = m_{33}$ and all other coefficients are 0.

This means that M must be a multiple of the identity matrix. Conversely, any multiple of the identity commutes with all the 3×3 matrices.

$\boxed{\text{Second solution}}$: write $M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ and compute the products with $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, etc...

Remark: the first solution gives the answer for all $n \times n$ matrices. The second one is more what I expected to see.

ⓑ If N commutes with all 3×3 matrices, then in particular it commutes with the E_{ij}, therefore it must be equal to $\alpha.I$. Conversely every matrix $\alpha.I$ commutes with every matrix.
So the matrices commuting with all the others are exactly the multiples of the identity matrix.

Correction of HW4:

2.4 : #36:

Let $X = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ then $AX = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a+2b & c+2d \\ 2a+4b & 2c+4d \end{bmatrix}$

Now $AX = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ if and only if $\begin{cases} a+2b = 0 \\ 2a+4b = 0 \\ c+2d = 0 \\ 2c+4d = 0 \end{cases}$. The solutions of this system

are given by $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ where s, t are arbitrary real numbers.

Therefore the matrices X we are looking for are all $\begin{bmatrix} -2s & -2t \\ s & t \end{bmatrix}$, s, t being arbitrary.

#76:

If B commutes with a 2×2 matrix then necessarily it must be a 2×2 matrix itself.

Let $B = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. Then $A.B = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2a+3b & 2c+3d \\ -3a+2b & -3c+2d \end{bmatrix}$

and $B.A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2a-3c & 3a+2c \\ 2b-3d & 3b+2d \end{bmatrix}$

These two products are equal if and only if $\begin{cases} 2a+3b = 2a-3c \\ -3a+2b = 2b-3d \\ 2c+3d = 3a+2c \\ -3c+2d = 3b+2d \end{cases}$

This is equivalent to $\begin{cases} b = -c \\ a = d \end{cases}$ so the matrices B are of the form $\begin{bmatrix} s & -t \\ t & s \end{bmatrix}$, where

s and t are arbitrary real numbers.

#19: False: the 1×1 matrices $[1]$ and $[-1]$ are invertible but their sum is $[0]$ (not invertible!)

#20: False: remember that $A.B \neq B.A$ in general! Example: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

but $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

But still these 2 matrices are invertible.

2/2

Section 3.1:

#30: Take $A = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ then im(A) is the set of all $A.\vec{x}$, that is the set of all $\begin{bmatrix} 1 \\ 5 \end{bmatrix}.[x]$.

But this is exactly the span of the vector $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$.
Another example, with 2×2 matrices: $\begin{bmatrix} 1 & 2 \\ 5 & 10 \end{bmatrix}$.

#34:
You want to find A such that the solutions of $A.\vec{x} = \vec{0}$ are given by $\vec{x} = s.\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$, $s \in \mathbb{R}$.

Therefore our system must have one free variable, say $x_2 = s$. But then our system becomes $\begin{cases} x_1 = -x_2 \\ x_3 = 2x_2 \end{cases}$ which is equivalent to $\begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

So the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ is one possible solution.

#48 ⓐ: $\vec{w} \in \text{im}(A)$ implies $\vec{w} = A\vec{x}$ for some $\vec{x} \in \mathbb{R}^2$. But then $A\vec{w} = A^2\vec{x} = A\vec{x} = \vec{w}$.
ⓑ If rank $A = 2$ then A is invertible, but then $A^2 = A$ can be simplified: $A^{-1}.A^2 = A^{-1}.A$ which says that $A = I$ (identity).
If rank $A = 0$ then A is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

#51: $\vec{x} \in \ker(AB) \Leftrightarrow AB\vec{x} = \vec{0} \Rightarrow B.\vec{x} \in \ker A \Rightarrow B.\vec{x} = \vec{0}$ (because $\ker A = \{\vec{0}\}$)
$\Leftrightarrow \vec{x} \in \ker B$ (by def.)
$\Rightarrow \vec{x} = \vec{0}$ (because $\ker B = \{\vec{0}\}$).

Therefore we proved: $\ker AB = \{\vec{0}\}$.

Section: 3.2: #2: $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in W$ but $(-1).\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \notin W$ therefore W is not a subspace.

#6: $V \cap W$ contains $\vec{0}$; if $\vec{x}, \vec{y}$ are in $V \cap W$ then they are in V, but $\vec{x} + \vec{y}$ also (V is a subspace), and similarly $\vec{x} + \vec{y} \in W$ but then $\vec{x} + \vec{y} \in V \cap W$. Same thing for a multiple of a vector.
$V \cup W$ is not necessarily a subspace: take V as the horizontal axis in $\mathbb{R}^2$, W the vertical one: then $\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin V \cup W$.

#8: $\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 2.\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is a non trivial relation between the 3 vectors.

Practice Midterm solutions:

(1) (a) False: Rank $\begin{pmatrix}1&0\\1&0\end{pmatrix}$ = Rank $\begin{pmatrix}0&1\\0&1\end{pmatrix} = 1$ but Rank $\begin{pmatrix}1&1\\1&1\end{pmatrix} = 1$.

(b) False: $\underbrace{f\begin{pmatrix}8\pi/2\\1\end{pmatrix}}_{0} \neq \underbrace{8\, f\begin{pmatrix}\pi/2\\1/8\end{pmatrix}}_{1}$ for example

(c) False: not a square, so it's not invertible.

(d) True

(e) True (Rank is 2).

(f) True: Actually $A^2=A$ so $A^3=A^2=A$

(g) False: Take $A=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ see webpage for correction!

(h) True: f is the zero function.

(i) False: Take a proj. p on a line L and take $3.p$.

(j) False: Take $A=\begin{pmatrix}0&1\\0&0\end{pmatrix}$. Then $A^2=\begin{pmatrix}0&0\\0&0\end{pmatrix}$.

(2) Let us show it's invertible:

$$\left[\begin{array}{ccc|ccc}4&3&2&1&0&0\\1&1&0&0&1&0\\7&6&3&0&0&1\end{array}\right] \; R_2 \leftrightarrow R_1$$

$$\left[\begin{array}{ccc|ccc}1&1&0&0&1&0\\4&3&2&1&0&0\\7&6&3&0&0&1\end{array}\right] \; \begin{array}{l}\\R_2-4R_1\\R_3-7R_1\end{array}$$

$$\left[\begin{array}{ccc|ccc}1&1&0&0&1&0\\0&-1&2&1&-4&0\\0&-1&3&0&-7&1\end{array}\right] \; \begin{array}{l}\\(-R_2)\\(-R_3)\end{array}$$

$$\left[\begin{array}{ccc|ccc}1&1&0&0&1&0\\0&1&-2&-1&4&0\\0&1&-3&0&7&-1\end{array}\right] \; \begin{array}{l}R_1-R_2\\\\R_3-R_2\end{array}$$

$$\left[\begin{array}{ccc|ccc}1&0&2&1&-3&0\\0&1&-2&-1&4&0\\0&0&-1&1&3&-1\end{array}\right] \; \begin{array}{l}\\\\-R_3\end{array}$$

(continued)

$$\left[\begin{array}{ccc|ccc}1&0&2&1&-3&0\\0&1&-2&-1&4&0\\0&0&1&-1&-3&1\end{array}\right] \; \begin{array}{l}R_1-2R_3\\R_2+2R_3\\\\\end{array}$$

$$\left[\begin{array}{ccc|ccc}1&0&0&3&3&-2\\0&1&0&-3&-2&2\\0&0&1&-1&-3&1\end{array}\right]$$

So A is invertible, and $A^{-1}=\begin{bmatrix}3&3&-2\\-3&-2&2\\-1&-3&1\end{bmatrix}$

(to be sure let's verify this by computing:

$$\begin{bmatrix}4&3&2\\1&1&0\\7&6&3\end{bmatrix}\cdot\begin{bmatrix}3&3&-2\\-3&-2&2\\-1&-3&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$).

(Actually this is how I realized I had made a mistake... shame on me)

(3) As usual, we write the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 5 \\ 7 & 8 & 1 & 37 \\ 2 & 3 & 16 & 12 \end{array}\right] \begin{array}{l} \\ R_2 - 7R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 5 \\ 0 & 1 & 22 & 2 \\ 0 & 1 & 22 & 2 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 - R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -25 & 3 \\ 0 & 1 & 22 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The rank is 2, x_3 is a free variable so we set $x_3 = s$, and the system is equivalent to

$$x_1 = 3 + 25s$$
$$x_2 = 2 - 22s$$
$$x_3 = s$$

So the solutions are given by $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + s.\begin{bmatrix} 25 \\ -22 \\ 1 \end{bmatrix}$, s arbitrary.

This is the line going through $\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$, directed by $\begin{bmatrix} 25 \\ -22 \\ 1 \end{bmatrix}$.

(4) E is not a subspace: for example $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is in E but $(-1).\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not.

(5) Let us determine Ker A: we want to solve $A\vec{x} = \vec{0}$ so we write the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ a & b & 4 & 0 \end{array}\right] R_2 - a.R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & b & 4-3a & 0 \end{array}\right]$$

There are several cases:

(a) $b = 0$:

See webpage for correction!

(1) $a = \frac{4}{3}$: then the system is equivalent to $x_1 + 3x_3 = 0$

There are 2 free variables $x_2 = s$

$x_3 = t$

The solutions are given by $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s.\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + t.\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

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(5) Continued:

Therefore the kernel is spanned by two linearly independent vectors.

② $a \neq \frac{4}{3}$: the rank is 2, there is only one free variable $x_2 = s$

so the system is equivalent to $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ (we divided by $4-3a \neq 0$).

equivalent to $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

The solutions are $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ so $\ker A$ is the line directed by $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(b) $b \neq 0$:

the system is equivalent to $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & \frac{4-3a}{b} & 0 \end{array}\right]$

The solutions are given by: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \cdot \begin{bmatrix} -3 \\ \frac{3a-4}{b} \\ 1 \end{bmatrix}$

Therefore the kernel is ~~a~~ the line spanned by $\begin{bmatrix} -3 \\ \frac{3a-4}{b} \\ 1 \end{bmatrix}$.

Conclusion:

The kernel of $\begin{pmatrix} 1 & 0 & 3 \\ a & b & 4 \end{pmatrix}$ is spanned by a single vector

if and only if: $\left(b = 0 \text{ and } a \neq \frac{4}{3}\right)$

or

$(b \neq 0)$.

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(6) (a)

We have $M = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 6 & 8 & 10 \\ 3 & 6 & 9 & 12 & 15 \\ 4 & 8 & 12 & 16 & 20 \\ 5 & 10 & 15 & 20 & 25 \end{bmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - 4R_1 \\ R_5 - 5R_1 \end{array}$

$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ so the rank is 1.

(b) $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is non zero, so let's call x_i the first non-zero entry in $\vec{x}$.

Then $M = \begin{array}{c} \\ i \\ \\ \end{array} \begin{bmatrix} \boxed{0} & 0 & \cdots & \cdots & 0 \\ & x_i^2 & x_i x_{i+1} & \cdots & x_i x_n \\ \boxed{0} & \vdots & & & \vdots \\ & x_n x_i & \cdots & \cdots & x_n x_n \end{bmatrix} \begin{array}{l} \\ R_i / x_i^2 \\ \\ \\ \end{array}$ (we can do this because $\underline{\underline{x_i \neq 0}}$).

now we keep the row R_i, and replace R_{i+1} by $R_{i+1} - \frac{x_{i+1}}{x_i} R_i$
$\vdots$
R_n by $R_n - \frac{x_n}{x_i} R_i$

we can do this because $x_i \neq 0$.

we get : $\begin{array}{c} \\ i \\ \\ \end{array} \begin{bmatrix} \boxed{0} & 0 & \cdots & 0 \\ & x_i^2 & \cdots & x_i x_n \\ \boxed{0} & 0 & \cdots & 0 \end{bmatrix}$ so the rank is 1 again (divide R_i by x_i^2).

The end

MAT 211, Linear Algebra   Fall 2006

Section I, Prof. S. Bonnot

Practice Midterm I

(1) *For each of the questions below, indicate if the statement is true* (**T**) *or false* (**F**).

a	f
b	g
c	h
d	i
e	j

(a) *For any matrices A, B, we have $rank(A+B) = rank(A) + rank(B)$.*

(b) *The map $f : \binom{x}{y} \mapsto \sin(x).y$ is a linear map from $\mathbb{R}^2$ to $\mathbb{R}$.*

(c) *The matrix $\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \end{pmatrix}$ is invertible because of rank 2.*

(d) *The matrix $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ has a rank equal to 1.*

(e) *The matrix $\begin{pmatrix} 7 & 2 \\ 0 & 13.2 \end{pmatrix}$ is invertible.*

(f) *If A is the matrix of an orthogonal projection onto a line, then $A^3 = A$.*

(g) *For a square matrix, $A^2 = A$ implies that $A = 0$ or A is the identity matrix .*

(h) *If the image of a linear map is reduced to zero, then the kernel is the whole domain.*

(i) *If two maps have same kernel and same image, then they are equal.*

(j) *If a matrix A is non zero, then A^n is never zero, for any integer n .*

(2) Let A be the following matrix

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 1 & 1 & 0 \\ 7 & 6 & 3 \end{bmatrix}$$

(a) *Is A invertible?.*

(b) *If yes, find its inverse,*

(c) *If A is not invertible, find a nonzero vector that is in $ker(A)$.*

(3) Solve the following system

$$\begin{aligned} x + y - 3z &= 5 \\ 7x + 8y + z &= 37 \\ 2x + 3y + 16z &= 12 \end{aligned}$$

and describe the set of solutions (is it a point, a line, the whole plane?)

(4) Let E be the set of all vectors in $\mathbb{R}^2$ satisfying both of the following conditions:
(a) $x \geq 0$,

(b) $x \leq 2y$.
Is E a subspace of $\mathbb{R}^2$?

(5) Find all values of a, b for which the kernel of $\left(\begin{smallmatrix} 1 & 0 & 3 \\ a & b & 4 \end{smallmatrix}\right)$ is spanned by a single vector (meaning that the kernel is a line going through the origin).

(6) (a) Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$. Consider now the (5×5) matrix M given by $M = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} . \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}$.
What is the rank of M?

(b) Let $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ be a nonzero vector in $\mathbb{R}^n$ (meaning that at least one of the x_i is non zero). Consider the $(n \times n)$ matrix M given by the product $M = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}$.

What is the rank of M?

1

Correction : HW 3:

Section 2.2 :

#2:

$$M = \begin{bmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

#12:

We proved that $T(\vec{x}) = \frac{\vec{x}\cdot\vec{u}}{\|\vec{u}\|^2}\cdot\vec{u} = \frac{x_1v_1 + x_2v_2}{v_1^2+v_2^2}\cdot\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

so $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \frac{v_1}{v_1^2+v_2^2}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \frac{v_2}{v_1^2+v_2^2}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

and the matrix is:

$$M = \frac{1}{v_1^2+v_2^2}\cdot\begin{bmatrix} v_1^2 & v_2v_1 \\ v_1v_2 & v_2^2 \end{bmatrix}$$

#29: We proved this in class : we want to show $L(\vec{x}+\vec{y}) = L(\vec{x})+L(\vec{y})$.

Since L is the inverse of T we have $\vec{x} = T(L(\vec{x}))$ and $\vec{y} = T(L(\vec{y}))$,

so $\vec{x}+\vec{y} = T(L(\vec{x})) + T(L(\vec{y})) = T(L(\vec{x})+L(\vec{y}))$ because T is linear.

Then $L(\vec{x}+\vec{y}) = L(T(L(\vec{x})+L(\vec{y}))) = L(\vec{x})+L(\vec{y})$ because L is the inverse of T.

Similarly $L(k.\vec{x}) = k.L(\vec{x})$.

#42:

For any vector $\vec{v}$ on the line L, $T(\vec{v}) = \vec{v}$ by definition of the projection.

Since for any vector $\vec{x}\in\mathbb{R}^2$ we have that $T(\vec{x})$ is a vector on the line L,

we deduce immediately that $T(T(\vec{x})) = T(\vec{x})$.

Section 2.3:

\#30: ~~To~~ Let's reduce $\begin{pmatrix} 0 & 1 & b \\ -1 & 0 & c \\ -b & -c & 0 \end{pmatrix}$ $R_2 \leftrightarrow R_1$

$$\begin{pmatrix} 1 & 0 & -c \\ 0 & 1 & b \\ -b & -c & 0 \end{pmatrix} \; R_3 + bR_1$$

$$\begin{pmatrix} 1 & 0 & -c \\ 0 & 1 & b \\ 0 & -c & -bc \end{pmatrix} \; R_3 + cR_2$$

$$\begin{pmatrix} 1 & 0 & -c \\ 0 & 1 & b \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore the rank is always 2, and there ~~is~~ are no values of b, c for which the matrix is invertible.

\#40:

Let's write $A = \left[\begin{array}{c|c|c|c} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ \vdots & \vdots & & \vdots \end{array}\right]$ and assume that the two columns $\vec{v}_i$ and $\vec{v}_j$ are equal.

Then we notice that $A \cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{v}_i - \vec{v}_j = \vec{0}$, where the vector $\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ has a 1 in row i, and a -1 in row j.

So we proved that the system $A \cdot \vec{x} = \vec{0}$ has a non-zero solution, therefore A is not invertible.

\#42:

Since there is exactly a 1 in each row and each column (the rest being 0's), just by swaping the rows we get that the reduced matrix is a diagonal of 1, therefore the matrix is invertible.

To get the inverse we write $\left[\begin{array}{c|c} \boxed{A} & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \end{array}\right]$ and reduce the whole matrix, just by swaping rows.

The inverse of A is obtained just by swaping the rows of the Identity, therefore it is also a permutation matrix.

#44:

Let's reduce $\begin{pmatrix} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{pmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - 4R_1 \end{matrix}$

$$\begin{pmatrix} 1 & 5 & 9 & 13 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \\ 0 & -12 & -24 & -36 \end{pmatrix} \begin{matrix} \\ R_2 \times \left(-\frac{1}{4}\right) \\ R_3 + (-2)R_2 \\ R_4 + (-3)R_2 \end{matrix}$$

$$\begin{pmatrix} 1 & 5 & 9 & 13 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ so the rank is 2.

This standard procedure works, but the following is easier: $\begin{pmatrix} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{pmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_2 \\ R_4 - R_3 \end{matrix}$

$$\begin{pmatrix} 1 & 5 & 9 & 13 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{matrix} R_2 - R_1 \\ R_3 - R_2 \\ R_4 - R_3 \end{matrix}$$

$$\begin{pmatrix} 1 & 5 & 9 & 13 \\ 0 & 4 & 8 & 12 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let us consider now $n \geq 2$:

In the general case, $M = \begin{bmatrix} 1 & n+1 & 2n+1 & \cdots & n(n-1)+1 \\ \vdots & \vdots & \vdots & & \vdots \\ n & 2n & 3n & & n^2 \end{bmatrix}$. By definition of M, the row $(i+1)$ is obtained from row (i) by adding 1 to each component.

Subtract to each Row R_i the row R_{i-1} (for $i=2$ to $i=n$), you get

$$\begin{bmatrix} 1 & (n+1) & \cdots & n(n-1)+1 \\ 1 & 1 & \cdots & 1 \\ \vdots & & & \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

all those rows are the same

Keep the first 2 rows and subtract R_2 to all the others, you get $\begin{bmatrix} 1 & (n+1) & \cdots & n(n-1)+1 \\ 1 & 1 & \cdots & 1 \\ 0 & & \text{———} & 0 \\ \vdots & & & \vdots \\ 0 & & \text{———} & 0 \end{bmatrix}$

44 (continued).

Now keep the first row and replace R_2 by $R_2 - R_1$ to get $\begin{pmatrix} 1 & (n+1) & \cdots & n.(n-1)+1 \\ 0 & n & 2n \cdots & n.(n-1) \\ 0 & — & — & 0 \\ | & & & | \\ 0 & — & — & 0 \end{pmatrix}$

Divide by n the second line to get $\begin{pmatrix} 1 & (n+1) & & n.(n-1)+1 \\ 0 & 1 & 2 \cdots & (n-1) \\ 0 & — & — & 0 \\ | & & & | \\ 0 & — & — & 0 \end{pmatrix}$

So Rank $(M_n) = 2$ as well.

Let's summarize: for $n=1$, $M = (1)$, Rank is 1, M is invertible.

for $n=2$, Rank $=2$ therefore M is invertible.

For $n>3$, Rank is 2, therefore it's not equal to n, thus M is <u>not</u> invertible.

48:

Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$, then notice that $A.\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \\ a \end{bmatrix}$. Therefore the system $A.\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

has a unique solution $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Notice that A is not square, so the fact is not contradicted! The matrix A is not invertible: you can see it by seeing that A is not square, or by noticing that there exists <u>no</u> vector $\vec{x}$ such that, for example, $A.\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Section 2.4: #28 Take $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. It's a non zero matrix such that $B^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O$ (the zero matrix).

Section 2.4: #30: A is square, non invertible, therefore by our invertibility criterion there exists a non-zero vector $\vec{v}$ such that $A.\vec{v} = \vec{0}$.

Now take the matrix B with all columns equal to $\vec{v}$. By definition of the product of matrices we have that

$$A.B = \underbrace{\begin{bmatrix} A.\vec{v} & \cdots & A.\vec{v} \end{bmatrix}}_{\text{all columns equal to } A.\vec{v} = \vec{0}} = \begin{bmatrix} \vec{0} & \cdots & \vec{0} \end{bmatrix} = O \text{ (the zero matrix)}.$$

Lecture Notes II

1 Section 2.3: Inverse transformations

Complements about the invertibility. Here are some important notions. Let $f : X \to Y$ be a map between two sets. (f is not necessarily a linear map).

1. We say that f is *injective* if for any x, x' in X, $f(x) = f(x')$ implies that $x = x'$;
2. we say that f is *surjective* if for any y in Y there exists (at least one) x in X such that $f(x) = y$;
3. we say that f is *bijective* or *invertible* if f is both injective and surjective.

Exercise 1.1. *Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a linear map. Show that f is injective if and only if $f(\vec{x}) = \vec{0}$ has $\vec{0}$ as a unique solution.*

Exercise 1.2. *Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a linear map. Show that (f is injective) implies that ($m \leq n$).*

Exercise 1.3. *Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a linear map. Show that (f is surjective) implies that ($m \geq n$).*

The following exercise shows that for linear maps from a space $\mathbb{R}^n$ to itself (notice that there are 3 conditions in here: linear, from a space $\mathbb{R}^n$, and f goes from $\mathbb{R}^n$ to itself) the situation is simplified a lot:

Exercise 1.4. *Let f be a linear map from $\mathbb{R}^n$ to itself. Show that f is injective if and only if f is surjective.*

Each hypothesis is important: the result is not necessarily true if you consider a non-linear map (think of $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ which is surjective but non injective. Think of $f : \mathbb{R} \to \mathbb{R}$ defined by the exponential, it is injective but not surjective). It's not true for example, for linear maps going from $\mathbb{R}$ to $\mathbb{R}^2$ (think about $x \mapsto \binom{x}{0}$, which is injective not surjective) and it's not true for linear maps going from $\mathbb{R}^2$ to $\mathbb{R}$ for example (the orthogonal projection onto a line is surjective but not injective)... Therefore you really need all these hypotheses.

2 About Section 2:4

I now realize that I didn't insist enough about the following fact which is useful:

$$\left[\begin{array}{c|c|c} \vec{v_1} & \dots & \vec{v_n} \end{array}\right] . \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1. \begin{bmatrix} \vec{v_1} \end{bmatrix} + \ldots + x_n. \begin{bmatrix} \vec{v_n} \end{bmatrix}$$

3 About invertible matrices

Let me summarize some points for you, because I didn't finish this discussion in class. First let us start with two equivalent definitions of invertible matrices

Definition 3.1. *An $n \times m$ matrix M is invertible if and only if : for any $\vec{y}$ in $\mathbb{R}^n$ there exists one and only one $\vec{x}$ in $\mathbb{R}^m$ such that $M.\vec{x} = \vec{y}$.*

Definition 3.2. *A matrix M is invertible if and only if there exists a matrix N such that $M.N = I$ (the identity matrix, with just a diagonal of ones) and $N.M = I$.*

Remarks. You see from the second definition that only square matrices can be invertible. But then for square matrices the following is true:

Theorem 3.3. *A square $n \times n$ matrix M is invertible if and only if there exists a square $n \times n$ matrix such that $M.N = I$.*

The following criterion can be useful:

Theorem 3.4. *A square matrix M is invertible if and only if the linear system $M.\vec{x} = \vec{0}$ has the vector $\vec{0}$ as unique solution.*

At this point, I can give you a proof of that using simply the resolution of linear systems, but we will see in few weeks a simpler argument, so let's wait a little bit... Another criterion is the following:

Theorem 3.5. *A matrix M is invertible if and only if it is a square matrix ($n \times n$ for some n) and rank$(M) = n$.*

This was the theory, here are some worked-out examples.

3.1 How to prove invertibility of matrices

Exercise 3.6. *Is the following matrix invertible?*

$$M = \begin{bmatrix} 1 & -1 & 5.1 \\ 3 & 2 & 1/3 \end{bmatrix}$$

Answer. Since M is not a square matrix, it is not invertible.
$\square$ That was easy. What about that:

Exercise 3.7. *Is the following matrix invertible?*

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$$

I gave you many different criteria to determine this. May be you don't know which one to choose, so I will do all of them, just to show you how they work, and that you get the same result...Needless to say, one method is enough, so pick your favorite.

Answer 1. I'll use the first definition of invertibility (perhaps it's not the most economic choice...) So for any $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ I want to see how many solutions the system $M.\vec{x} = \vec{y}$ has. Let us solve that. As usual we write the augmented matrix:

$$\begin{bmatrix} 1 & 1 & 1 & \vdots & y_1 \\ 1 & 2 & 3 & \vdots & y_2 \\ 1 & 3 & 6 & \vdots & y_3 \end{bmatrix} \quad \begin{array}{c} \\ R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & \vdots & y_1 \\ 0 & 1 & 2 & \vdots & y_2 - y_1 \\ 0 & 2 & 5 & \vdots & y_3 - y_1 \end{bmatrix} \quad \begin{array}{l} R_1 - R_2 \\ \\ R_3 - 2R_1 \end{array}$$

$$\begin{bmatrix} 1 & 0 & -1 & \vdots & 2y_1 - y_2 \\ 0 & 1 & 2 & \vdots & y_2 - y_1 \\ 0 & 0 & 1 & \vdots & y_3 - y_1 \end{bmatrix} \quad \begin{array}{l} R_1 + R_3 \\ R_2 - 2R_3 \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & -y_1 - y_2 + y_3 \\ 0 & 1 & 0 & \vdots & -3y_1 + y_2 - 2y_3 \\ 0 & 0 & 1 & \vdots & -y_1 + y_3 \end{bmatrix}$$

If we plug again the variables x_i we see that the linear system has a unique solution given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -y_1 - y_2 \\ -3y_1 + y_2 - 2y_3 \\ -y_1 + y_3 \end{bmatrix}.$$

Notice that on the right-hand side, we have a column vector (not a 3×3 matrix). At this point we proved, using the first definition, that the matrix is invertible. Now you can say more, by rewriting this:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -y_1 - y_2 \\ -3y_1 + y_2 - 2y_3 \\ -y_1 + y_3 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 0 \\ -3 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix} . \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

This means that not only we proved that the matrix was invertible, but we also have computed without knowing it the inverse...

Answer 2. Can we use the second definition? (namely the existence of a matrix N such that $M.N = I$ where I is the identity matrix, a diagonal of 1 with zeroes everywhere else). The answer is no, unless you can guess the form of N... Therefore this definition is used rather for theoretical problems where you don't have the explicit form of the matrix. Remember that I used it to show that if a square matrix M is such that $M^2 = 0$ then $I - M$ is invertible.

Answer 3. Let us use the criterion saying that a square matrix is invertible if and only if the linear system $A.\vec{x} = \vec{0}$ has the unique solution $\vec{x} = \vec{0}$. As usual, write the augmented matrix:

$$\begin{bmatrix} 1 & 1 & 1 & \vdots & 0 \\ 1 & 2 & 3 & \vdots & 0 \\ 1 & 3 & 6 & \vdots & 0 \end{bmatrix}$$

And then you reduce the whole thing. I don't really need to do it here because the operations on the rows are exactly the same as above. Notice that the last column stays the same. At the end you will get:

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \end{bmatrix}$$

Plug again the variables x_i, and then conclude by saying: the system has a unique solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore our criterion says that the matrix is invertible. Notice that the computation are a bit simpler, because the last column is made of zeroes. The inconvenient is that, if by any chance, like here, you prove the invertibility, then you don't have an expression for the inverse, so basically you would have to do it again if you want the inverse...

Answer 4. Use the criterion with the rank, saying that a **square** $n \times n$ matrix M is invertible if and only if its rank is n. So let us compute the rank: you write the matrix and you reduce it:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$$

and then you do all the operations on the rows (again the same combinations will appear as above). At the end, you get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This has three leading ones, so the rank is three, which is the size of our square matrix, hence it is invertible.

Answer 5. You might want to show the invertibility and compute the inverse (if it exists) at the same time. Then you just need to write a big matrix, with two square blocks. The first block is the matrix we study, the other one is a diagonal of ones. And then you reduce the whole thing. It goes like this:

$$\begin{bmatrix} 1 & 1 & 1 & \vdots 1 & 0 & 0 \\ 1 & 2 & 3 & \vdots 0 & 1 & 0 \\ 1 & 3 & 6 & \vdots 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{ccc} & & \\ R_2 & - & R_1 \\ R_3 & - & R_1 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & \vdots +1 & 0 & 0 \\ 0 & 1 & 2 & \vdots -1 & 1 & 0 \\ 0 & 2 & 5 & \vdots -1 & 0 & 1 \end{bmatrix} \quad \begin{array}{ccc} R_1 & - & R_2 \\ & & \\ R_3 & - & 2R_2 \end{array}$$

You continue the reduction like this and at the end you will get:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & -3 & 1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

So the conclusion is: the matrix is invertible (because the block on the left is the identity matrix) and its inverse is given by the block on the right.

Conclusion. Now that you have seen all of them at the same time, I guess that you see that all these methods are equivalent. The same basic principles (elementary operations on the rows to get a reduced form) appear under different disguises. If you need to compute the inverse of a matrix, there is no doubt that the last one is the simplest...

Correction of HW 2:

#19: $\begin{bmatrix} 1 & 1 & -1 \\ -5 & 1 & 1 \\ 1 & -5 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

#24:

Since $A\vec{x} = \vec{b}$ has a unique solution, we know that the reduced form

of $\left[\begin{array}{c|c} \boxed{A} & \begin{matrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{matrix} \end{array}\right]$ looks like this: $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & b_1' \\ 0 & 1 & 0 & 0 & b_2' \\ 0 & 0 & 1 & 0 & b_3' \\ 0 & 0 & 0 & 1 & b_4' \end{array}\right]$.

Therefore the reduced form of the other system $A\vec{x} = \vec{c}$ will be: $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & c_1' \\ 0 & 1 & 0 & 0 & c_2' \\ 0 & 0 & 1 & 0 & c_3' \\ 0 & 0 & 0 & 1 & c_4' \end{array}\right]$.

Consequently, $A\vec{x} = \vec{c}$ also has a unique solution.

#28:

Since $\text{rank}(A) = 3$ we know that there are 3 leading 1's in $\text{rref}(A)$, one in each column.

So this means that necessarily $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ & \boxed{0} & \end{bmatrix}$.

#34:

ⓐ $A\vec{e}_1 = \begin{bmatrix} a \\ d \\ g \end{bmatrix}$; $A\vec{e}_2 = \begin{bmatrix} b \\ e \\ h \end{bmatrix}$ and $A\vec{e}_3 = \begin{bmatrix} c \\ f \\ k \end{bmatrix}$.

#34 (continued):

(b)

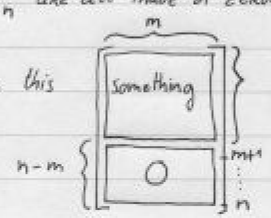
Now $B = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ a_n & b_n & c_n \end{bmatrix}$ and we call $\vec{v}_1 = \begin{bmatrix} a_1 \\ | \\ a_n \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} b_1 \\ | \\ b_n \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} c_1 \\ | \\ c_n \end{bmatrix}$.

So $B.\vec{e}_1 = B.\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 \\ | \\ a_n \end{bmatrix} = \vec{v}_1$. Similarly $B.\vec{e}_2 = \vec{v}_2$ and $B.\vec{e}_3 = \vec{v}_3$.

#44:

A is an $n \times m$ matrix, with $n > m$. Therefore rref(A) has at most m leading 1's. Because of the particular form of rref(A) ("staircase") we also know that the rows $R_{m+1}, R_{m+2}, \dots, R_n$ are all made of zeroes. These extra rows exist because $n > m$.

So rref(A) looks like this



. Now take $\vec{b} = \begin{bmatrix} 0 \\ | \\ 0 \\ 1 \\ 0 \\ | \\ 0 \end{bmatrix}$ (0 everywhere, and 1 in position m+1).

Immediately you see that the row (m+1) of your augmented matrix is $[0 \text{——} 0 \,\vdots\, 1]$.
So for this particular ~~system~~ $\vec{b}$ the system $A\vec{x} = \vec{b}$ is inconsistent.

#50:

The augmented matrix $[A \,\vdots\, \vec{b}]$ is a 4×4 matrix. Its rank is 4 so this implies that the reduced form of the augmented matrix is $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$.

Since the last row is $[0\ 0\ 0 \,\vdots\, 1]$, the system is inconsistent.
Therefore $A\vec{x} = \vec{b}$ has <u>no solution</u>.

HW #1:

1

Section 1.1:

#10: The augmented matrix is $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 7 & 2 \\ 3 & 7 & 11 & 8 \end{array}\right]$

Let's find its RREF form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 5 \end{array}\right] R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} R_1 - 2R_2 \\ R_2 - 2R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -9 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \end{array}\right] R_1 + R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

So the system has only one solution, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -9 \\ 5 \\ 0 \end{bmatrix}$.

#22:

Assume that $x(t) = a\sin(t) + b\cos(t)$. Then $x'(t) = a\cos t - b\sin t$

$x''(t) = -a\sin t - b\cos t$

If $x(t)$ is a solution of the equation then necessarily:

$$(-a\sin t - b\cos t) - (a\cos t - b\sin t) - (a\sin t + b\cos t) = \cos t$$

therefore $(-2a + b)\sin t + (-a - 2b - 1)\cos t = 0$ for all $t \in \mathbb{R}$.

In particular for $t=0$ and $t = \frac{\pi}{2}$, we have that necessarily $\begin{pmatrix} -2a + b = 0 \\ -a - 2b = 1 \end{pmatrix}$.

This system has a unique solution $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1/5 \\ -2/5 \end{bmatrix}$

Then $x(t) = -\frac{1}{5}\left(\sin t + 2\cos t\right) = \left(-\frac{1}{5}\right)\left(\sqrt{5}\right)\cdot\left(\frac{1}{\sqrt{5}}\sin t + \frac{2}{\sqrt{5}}\cos t\right)$.

Since $\left(\frac{1}{\sqrt{5}}\right)^2 + \left(\frac{2}{\sqrt{5}}\right)^2 = 1$ there exists $\theta \in \mathbb{R}$ such that $\cos\theta = \frac{1}{\sqrt{5}}$ and $\sin\theta = \frac{2}{\sqrt{5}}$.

So $x(t) = -\frac{\sqrt{5}}{5}\cdot\sin(t+\theta)$ and the graph is obtained from the graph of sin by transl., dilation

#26:

The augmented matrix is
$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & k^2-5 & k \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & k^2-4 & k-2 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & k^2-4 & k-2 \end{array}\right]$$

- 1st case: $k = -2$:

  Last row is $[0 \;\; 0 \;\; 0 \;|\; -4]$ so the system is inconsistent.

- 2d case: $k = 2$:

  The system has infinitely many solutions given by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$
  where s is an arbitrary real number.

- 3rd case: $k \neq 2$ and $k \neq -2$:

  The system is equivalent to $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 + 3/k+2 \\ 0 & 1 & 0 & 1 - 2/k+2 \\ 0 & 0 & 1 & \frac{1}{k+2} \end{array}\right]$

It has a unique solution $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 + \frac{3}{k+2} \\ 1 - \frac{2}{k+2} \\ \frac{1}{k+2} \end{bmatrix}$

#33:

Assume $f(t) = a + bt + ct^2$ then $f'(t) = b + 2ct$,

and $\begin{cases} f(1) = 1 \\ f(3) = 3 \\ f'(2) = 1 \end{cases}$ is equivalent to $\begin{bmatrix} a+b+c = 1 \\ a+3b+9c = 3 \\ b+4c = 1 \end{bmatrix}$. The matrix is $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 3 \\ 0 & 1 & 4 & 1 \end{array}\right]$

It becomes $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & 8 & 2 \\ 0 & 1 & 4 & 1 \end{array}\right]$, then $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 4 & 1 \\ 0 & 1 & 4 & 1 \end{array}\right]$ and finally $\left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

It has infinitely many solutions given by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$ where $s \in \mathbb{R}$ is arbitrary.

Section 1.2:

#7: The augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} R_1 - 2R_3 \\ R_2 - 3R_3 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 9 & 0 \\ 0 & 0 & 1 & 0 & 11 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} R_1 - 9R_4 \\ R_2 - 11R_4 \\ R_3 + 3R_4 \\ \\ \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

It has one free variable $(x_2 = s)$. So the set S of solutions is $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} ; s \in \mathbb{R} \right\}$

#11: The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 3 & 4 & -6 & 8 & 0 \\ 0 & -1 & 3 & 4 & -12 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 3R_1 \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 4 & -12 & -4 & 24 \\ 0 & -1 & 3 & 4 & -12 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 4R_2 \\ R_4 + R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & -6 \end{array}\right] \begin{array}{l} \\ \\ \\ \left(\times \frac{1}{3}\right) \text{ and } R_4 \leftrightarrow R_3 \end{array}$$

#11 (continued): $\left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ which becomes $\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

The solutions are given by $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \\ -2 \end{bmatrix} + s \cdot \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$, where s is an arbitrary real number.

#48: For $n=3$ the system is $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \end{array}\right]$

For $n=4$ it is $\left[\begin{array}{cccc|c} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \end{array}\right]$ with reduced form $\left[\begin{array}{cccc|c} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -2 & 1 & 0 \end{array}\right]$

For $n=5$ it is $\left[\begin{array}{ccccc|c} 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right]$ reducing to $\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -4 & 3 & 0 \\ 0 & 1 & 0 & -3 & 2 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right]$.

From that we guess that the reduced form for n is $\left[\begin{array}{cccc|c} 1 & & 0 & -(n-1) & n-2 & 0 \\ & \ddots & & \vdots & \vdots & \vdots \\ & & & -3 & 2 & \\ 0 & & 1 & -2 & 1 & 0 \end{array}\right]$

Let's prove this by induction:

1. The property is true for $n=3$.
2. Assume that the property is true for k.

Then the matrix for $k+1$ is $\left[\begin{array}{ccccccc|c} 1 & -2 & 1 & & & & & 0 \\ & \ddots & \ddots & \ddots & & & & \vdots \\ & & 1 & -2 & 1 & 0 & & \\ & & & 1 & -2 & 1 & & 0 \end{array}\right]$.

You can notice that it contains a rectangular block that is exactly the matrix for $n=k$. Since we assumed that the property is true for $n=k$ we know that our matrix can be reduced to: $\left[\begin{array}{ccccccc|c} 1 & & 0 & -(k-1) & k-2 & 0 & & 0 \\ & \ddots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & & 1 & -2 & 1 & 0 & & \\ & & & 1 & -2 & 1 & & 0 \end{array}\right]$. But this can be reduced further by: $\begin{cases} R_1 + (k-1) R_{k-1} \\ \vdots \\ R_{k-1} + 2R_{k-1} \end{cases}$

#48 continued

By doing this we get a block $\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ of size $(k-1)\times(k-1)$

followed by a block $\begin{bmatrix} (k-2)+(k-1)(-2) & k-1 \\ (k-3)+(k-2)(-2) & k-2 \\ \vdots & \vdots \\ -2 & 1 \end{bmatrix}$ equal to $\begin{bmatrix} -k & k-1 \\ -(k-1) & k-2 \\ \vdots & \vdots \\ -2 & 1 \end{bmatrix}$

Therefore we proved that the property is true for $n=k+1$.
So we proved by induction that it is true for any n.

Now the system has $(n-2)$ leading variables and two free variables $x_{n-1}=s$, $x_n=t$

So the solutions of the system are given by

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix} = s.\begin{bmatrix} n-1 \\ \vdots \\ 2 \\ 1 \\ 0 \end{bmatrix} - t\begin{bmatrix} n-2 \\ \vdots \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

Lecture Notes I

1 Section 1.1

A remark : Since this is the first homework and you didn't have time to use the techniques I showed to you on Monday, that's ok if you solve the linear systems in your "own" way, which still must be mathematically correct! However, I really want you to know how to solve those linear systems by the systematic way of the textbook!

About 22 : About 22 : Do not try to solve the equations, just take the derivatives of the proposed form $x(t) = a.\sin(t) + b.\cos(t)$ and plug them into the equation!

Then you will need to use the following lemma (we call a lemma any small auxiliary theorem that you need in order to prove something else)

Lemma 1.1 *Let C and D be two real numbers. The following propositions are equivalent :*

1. *$C = D = 0$;*
2. *for any real number t, $C.\sin(t) + D.\cos(t) = 0$.*

Proof. Well, one implication is immediate. For the other, try to plug some clever values (two is enough) for t and see what it implies for C and D!

About 33 : Again do not try to solve the equations! Just do what they propose, namely to plug in particular values. This will give you a linear system in the variables a, b, c if you choose to write your polynomial function as $f(t) = a + bt + ct^2$.

2 Section 1.2

About 48 : They want you to prove something for an arbitrary $n \leq 3$. I advise you to choose $n = 3$ and to reduce the matrix as we did today. Then take $n = 4$ and do the same, then do it for $n = 5$. Now by looking to the reduced matrix you get, you should start to see a certain common pattern (if not do the case $n = 6$!). Since you want to prove the thing for all n you need to write down *a proof by induction* :

Proofs by induction, a quick summary . The idea is simple : in order to prove that a property is true for any non-negative integer, it is enough to show that it is true for $n = 0$, and that each time it is true for an integer k then it is true for $k + 1$. Here is an example to show you how it works, and how to write down such a proof :

Exercise 2.1 *Show that for any non negative integer n we have $2^n \geq n + 1$.*

Proof. Let $P(n)$ be the proposition $(2^n \geq n + 1)$.

- $P(0)$ is true : indeed $1 = 2^0 \geq 0 + 1$;
- $P(k) implies P(k + 1)$: we assume that $P(k)$ is true and we want to prove that $P(k + 1)$ is true. We have

$$2^{k+1} = 2.2^k \geq 2.(k + 1)$$

  because we assumed that $P(k)$ was true. But now $2.(k + 1) = k + 1 + k + 1 \geq k + 2$. Therefore we proved that $P(k+1)$ is true and this concludes the proof of the theorem.

Now you will have to prove by induction that the reduced form for the matrix that you guessed is the right one...

Correction HW 5:

Section 3.2:

#34: $A.\begin{bmatrix}1\\2\\3\\4\end{bmatrix} = \vec{0}$ so $\vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3 + 4\vec{v}_4 = \vec{0}$ and $\vec{v}_4 = -\frac{1}{4}\vec{v}_1 - \frac{1}{2}\vec{v}_2 - \frac{3}{4}\vec{v}_3$

#36:

$\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ are linearly dependent so there exists a non-trivial linear relation between them: $a_1\vec{v}_1 + \dots + a_m\vec{v}_m = \vec{0}$, where one a_i at least is non zero.

Then $T(a_1\vec{v}_1 + \dots + a_m\vec{v}_m) = T(\vec{0}) = \vec{0}$ (T is linear)

and $a_1 T(\vec{v}_1) + \dots + a_m T(\vec{v}_m) = \vec{0}$, where one a_i at least is non zero.

But this means exactly that the $T(\vec{v}_i)$ are linearly dependent.

#46:

$A = \begin{bmatrix}1 & 2 & 0 & 3 & 5\\ 0 & 0 & 1 & 4 & 6\end{bmatrix}$ can be reduced to $\mathrm{RRef}(A) = \begin{bmatrix}1 & 2 & 0 & 3 & 5\\ 0 & 0 & 1 & 4 & 6\end{bmatrix}$ $(= A\,!)$

Now: $\vec{v}_2 = 2\vec{v}_1$ therefore $\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix} \in \ker(A)$

$\vec{v}_4 = 3\vec{v}_1 + 4\vec{v}_3$ therefore $\begin{bmatrix}-3\\0\\-4\\1\\0\end{bmatrix} \in \ker(A)$

and $\vec{v}_5 = 5\vec{v}_1 + 6\vec{v}_3$ and this implies that $\begin{bmatrix}-5\\0\\-6\\0\\1\end{bmatrix} \in \ker(A)$.

So a basis for the kernel is given by $\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\-4\\1\\0\end{bmatrix}, \begin{bmatrix}-5\\0\\-6\\0\\1\end{bmatrix}$.

Section 3.3:

~~#22~~ $\cancel{A = \begin{bmatrix}2 & 4 & 8\\ 4 & 5 & 1\\ 7 & 9 & 3\end{bmatrix}}$ ~~and~~ $\mathrm{RRef}(A) = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ NO!!! ~~so a basis for im(A) is~~ $\cancel{\begin{bmatrix}2\\4\\7\end{bmatrix}, \begin{bmatrix}4\\5\\9\end{bmatrix}, \begin{bmatrix}8\\1\\3\end{bmatrix}}$.

~~and Ker(A) is~~ $\cancel{\{\vec{0}\}}$. ~~(has a basis made of 0 vectors...)~~

See next page !!!

#28: Let's reduce $\begin{bmatrix}1 & 0 & 0 & 2\\ 0 & 1 & 0 & 3\\ 0 & 0 & 1 & 4\\ 2 & 3 & 4 & k\end{bmatrix} = A$, we get $\begin{bmatrix}1 & 0 & 0 & 2\\ 0 & 1 & 0 & 3\\ 0 & 0 & 1 & 4\\ 0 & 0 & 0 & k-29\end{bmatrix}$

- If $k \neq 29$ then Rank is 4 and the four vectors in A form a basis.
- If $k = 29$ then a basis for $\mathrm{im}(A)$ is formed by the first 3 column vectors in A.

Conclusion: the four column vectors of A form a basis of $\mathbb{R}^4$ if and only if $k \neq 29$.

Section 3.3:

#22:

$$A = \begin{bmatrix} 2 & 4 & 8 \\ 4 & 5 & 1 \\ 7 & 9 & 3 \end{bmatrix} \times \tfrac{1}{2}$$

Let's reduce it:

$$\begin{bmatrix} 1 & 2 & 4 \\ 4 & 5 & 1 \\ 7 & 9 & 3 \end{bmatrix} \begin{matrix} \\ R_2 - 4R_1 \\ R_3 - 7R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & -15 \\ 1 & -5 & -25 \end{bmatrix} \begin{matrix} \\ \times(-\tfrac{1}{3}) \\ \times(-\tfrac{1}{5}) \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & 1 & 5 \end{bmatrix}$$

Therefore $\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

* So a basis for $\operatorname{im} A$ is given by $\begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix}$.

* Since $\vec{v}_3 = -6\vec{v}_1 + 5\vec{v}_2$, we get that $\begin{bmatrix} 6 \\ -5 \\ 1 \end{bmatrix}$ is a basis for $\ker A$.

HW6: Correction:

Section 3.3:

#24:

We get $\mathrm{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. Therefore a basis for $\mathrm{im}(A)$ is given by $\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 10 \\ 0 \end{bmatrix}$.

We notice that $\vec{v}_2 = 2\vec{v}_1$ (equivalently $-2\vec{v}_1 + \vec{v}_2 + 0\vec{v}_3 + 0\vec{v}_4 + 0\vec{v}_5 = \vec{0}$),

therefore $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is a basis for $\mathrm{Ker}(A)$.

See next page for #30

#45:

Let $A = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_p & \vec{w}_1 & \cdots & \vec{w}_q \end{bmatrix}$. Since the $\vec{v}_i$ are linearly independent, there will be a leading 1 in each one of the first p columns of $\mathrm{rref}(A)$.

If there are some more leading 1's among the next columns, we add the corresponding $\vec{w}_i$ in order to get a basis for $\mathrm{im}\, A$.

If there ~~are~~ is no other leading 1, then the $\vec{v}_i$ form a basis for $\mathrm{im}\, A$.

In any case, we have: $\underbrace{V = \mathrm{span}(\vec{w}_1, \ldots, \vec{w}_q)}_{\text{by hypothesis}} \subset \underbrace{\mathrm{im}(A) \subset V}_{\text{because all the columns are vectors of } V.}$

Thus $\mathrm{im}(A) = V$, and we get a basis of V made of all the $\vec{v}_i$ and (possibly) some of the $\vec{w}_j$.

#46:

As in 45, let's reduce $\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 4 & 0 & 1 & 0 & 0 \\ 3 & 6 & 0 & 0 & 1 & 0 \\ 4 & 8 & 0 & 0 & 0 & 1 \end{bmatrix} = A$. One finds $\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & -1/4 \\ 0 & 1 & -1 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 & -1/2 \\ 0 & 0 & 0 & 0 & 1 & -3/4 \end{bmatrix}$

So a basis for $\mathbb{R}^4$ is given by $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 6 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

#56:

Assume we have a linear relation $c_0\vec{v} + c_1 A\vec{v} + \cdots + c_{m-1}A^{m-1}\vec{v} = \vec{0}$. Multiply each side by A^{m-1} to the left to get $c_0 A^{m-1}\vec{v} + \underbrace{c_1 A^m \vec{v}}_{=\vec{0}} + \cdots + \underbrace{c_{m-1} A^{2m-2}}_{=\vec{0}}\vec{v} = \vec{0}$ which implies $c_0 = 0$.

Now we have $c_1 A\vec{v} + \cdots + c_{m-1}A^{m-1}\vec{v} = \vec{0}$. Multiply to the left by A^{m-2}: you get $c_1 A^{m-1}\vec{v} = \vec{0}$, therefore $c_1 = 0$.

And continue the same way.

Section 3.3:

#30: The subspace considered is $\ker A$, where $A = \begin{bmatrix} 2 & -1 & 2 & 4 \end{bmatrix}$.

The kernel has dimension 3.

Let's find a basis for it:

1) since $\vec{v}_2 = -\frac{1}{2}\vec{v}_1$ we get that $\begin{bmatrix} 1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \in \ker A$.

2) since $\vec{v}_3 = \vec{v}_1$, $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \in \ker A$.

3) since $\vec{v}_4 = 2\vec{v}_1$, $\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \in \ker A$.

So a basis for our subspace is given by: $\begin{bmatrix} 1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

HW7 Correction:

Section 3.4:

/10 #12:

Let's reduce $\begin{bmatrix} 8 & 5 & 1 \\ 4 & 2 & -2 \\ -1 & -1 & -2 \end{bmatrix} = A$. We get $\text{RRef}(A) = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

Therefore $\begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} = -3 . \begin{bmatrix} 8 \\ 4 \\ -1 \end{bmatrix} + 5 . \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$ (so $\vec{x}$ is in span $(\vec{v}_1, \vec{v}_2)$)

The coordinates in the basis B, of the vector $\vec{x}$ are $[\vec{x}]_B = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$.

/10 #16:

Let's reduce $\begin{bmatrix} 1 & 1 & 1 & 7 \\ 1 & 2 & 3 & 1 \\ 1 & 3 & 6 & 3 \end{bmatrix} = A$. We obtain $\text{RRef}(A) = \begin{bmatrix} 1 & 0 & 0 & 21 \\ 0 & 1 & 0 & -22 \\ 0 & 0 & 1 & 8 \end{bmatrix}$

Therefore $\vec{x}$ belongs to span $\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} \right)$ and the coordinates are $[\vec{x}]_B = \begin{bmatrix} 21 \\ -22 \\ 8 \end{bmatrix}$.

/10 #26:

The change of basis matrix is $P_{B \to std} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$. We have $P_{B \to std}^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$.

So the matrix of the linear map T in the new basis $\mathcal{B}$ is $B = P^{-1} . A . P$

$$= \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} . \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} . \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 4 \\ -4 & -3 \end{bmatrix}$$

/20 #28:

We have $P_{B \to std} = \begin{bmatrix} 2 & 1 & 0 \\ 2 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix}$. We compute $P_{B \to std}^{-1} = \begin{bmatrix} 2/9 & 2/9 & 1/9 \\ 5/9 & -4/9 & -2/9 \\ 1/9 & 1/9 & -4/9 \end{bmatrix}$

And then the new matrix is $B = P^{-1} . A . P$.

after computation,

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}.$$

#44:

/10

By definition: $\left[\vec{x}\right]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ means that $\vec{x} = 2\vec{v}_1 - \vec{v}_2$

$$= 2 \cdot \begin{bmatrix} 8 \\ 4 \\ -1 \end{bmatrix} - \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 11 \\ 6 \\ -1 \end{bmatrix}.$$

#47:

/10

The New basis is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We have $B = P^{-1}AP$ where $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (change of basis matrix).

Therefore $A = PBP^{-1}$. Here (it's a coincidence) $P^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

So $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} d & c \\ b & a \end{bmatrix}.$

#56:

/10

We are looking for $\vec{v}_1, \vec{v}_2$ such that: $\begin{cases} \vec{v}_1 + 2\vec{v}_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \\ 3\vec{v}_1 + 4\vec{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \end{cases}$ $R_2 - 3R_1$

$$\begin{cases} \vec{v}_1 + 2\vec{v}_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \\ -2\vec{v}_2 = \begin{bmatrix} -7 \\ -12 \end{bmatrix} \end{cases}$$

So we find $\vec{v}_2 = \begin{bmatrix} 7/2 \\ 6 \end{bmatrix}$ and $\vec{v}_1 = \begin{bmatrix} -4 \\ -7 \end{bmatrix}$.

#62:

/20

The new matrix is $B = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$: the first column means that $T(\vec{v}_1) = 5.\vec{v}_1 + 0.\vec{v}_2$

Therefore if we write $\vec{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$ (in standard basis), we get $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = 5 \cdot \begin{bmatrix} a \\ b \end{bmatrix}$ ①

Similarly the second column of B tells us that $T(\vec{v}_2) = 0.\vec{v}_1 - 1.\vec{v}_2$.

If we write $\vec{v}_2 = \begin{bmatrix} c \\ d \end{bmatrix}$ this translates into: $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = -1 \cdot \begin{bmatrix} c \\ d \end{bmatrix}$. ②

Now ① is equivalent to $\begin{cases} a + 2b = 5a \\ 4a + 3b = 5b \end{cases}$ which is $\begin{cases} -4a + 2b = 0 \\ 4a - 2b = 0 \end{cases}$ equivalent to $2a - b = 0$

So $\begin{cases} a = 1 \\ b = 2 \end{cases}$ is a solution of ①. Similarly ② is equivalent to $c + d = 0$, and $\begin{cases} c = 1 \\ d = -1 \end{cases}$ is a solution.

So a possible choice for the new basis is $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

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Correction for HW 8:

Section 4.1:

/10 # 6: The invertible 3×3 matrices do not form a subspace of $\mathbb{R}^{3\times 3}$, for example because $\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0\end{pmatrix}$ is not in it.

Or, if you prefer: $+I$ and $-I$ are in the subset but not their sum.

/10 #25:
The space of all polynomials $f(t)$ in P_2 s.t. $f(1)=0$ is the kernel of $T: \begin{cases} P_2 \to \mathbb{R} \\ f(t) \mapsto f(1)\end{cases}$

Since $T(\underbrace{c}_{\text{constant polynomial} = c}) = c \in \mathbb{R}$ we have: $\operatorname{im} T = \mathbb{R}$.

The matrix of T is $[1\ 1\ 1]$ so $\dim \operatorname{Ker} T = 2$, and $X-1$ and X^2-1 form a basis for $\operatorname{Ker} T$.

/20 #30:
Let $A = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$. Then $\begin{bmatrix} 1 & 2 \\ 3 & 6\end{bmatrix} . A = \begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix}$ is equivalent to $\begin{cases} a+2c=0 \\ 3a+6c=0 \\ b+2d=0 \\ 3b+6d=0\end{cases}$

The system is equivalent to $\begin{cases} a+2c=0 \\ b+2d=0 \end{cases}$. Let $\begin{cases} c=s \\ d=t\end{cases}$

The solutions are $\begin{bmatrix} a\\b\\c\\d\end{bmatrix} = s.\begin{bmatrix} -2\\0\\1\\0\end{bmatrix} + t.\begin{bmatrix} 0\\-2\\0\\1\end{bmatrix}$. Therefore A has the form $\begin{bmatrix} -2s & -2t \\ s & t\end{bmatrix}$, where s and t are arbitrary real numbers.

/10 # 55:
In the class we proved that the (infinite) family of functions $(e^{ax}, a \in \mathbb{R})$ is linearly independent. Therefore $F(\mathbb{R},\mathbb{R})$ has infinite dimension.

Section 4.2:

/10 #26: If $T(f(t)) = f(-t)$ then $T(T(f(t))) = f(t)$ therefore T is an isomorphism, with inverse T itself.

HW0081.jpeg %d×%d pixels

/20 #28 : $T(f(t)) = f(2t) - f(t)$.

Let 1 be the constant polynomial equal to 1. Then $T(1) = 1 - 1 = 0$.
Therefore $\ker T$ is not equal to $\{\vec{0}\}$ and T is not an isomorphism.

/20 #58: $T(x_0, x_1, \ldots) = (0, x_0, x_1, \ldots)$

$T(x_0, x_1 \ldots) = (0, 0, 0, \ldots)$ if and only if $(x_0, x_1, \ldots) = (0, 0, \ldots)$
so $\ker T = \{\vec{0}\}$.

Image of $T = \left\{ (x_0, x_1, \ldots) \text{ such that } x_0 = 0 \right\}$.

(1)

Correction HW 9:

Section 4.2:

#72. Z_n is a subspace of P_n because it is the kernel of $L: \begin{cases} P_n \to \mathbb{R} \\ f(t) \mapsto f(0) \end{cases}$.

/10

A basis for Z_n is $x, x^2, \dots, x^n$ so $\dim Z_n = n$.

#73: ~~$T(a_1 t + a_2 t^2 + \dots + a$~~ $T(a_0 + a_1 t + \dots + a_{n-1} t^{n-1}) = a_0 t + \frac{a_1}{2} t^2 + \dots + \frac{a_{n-1}}{n} t^n \in Z_n$

not graded. T is an isomorphism with inverse the derivation.

/10

#74: The derivation is such an isomorphism.

Section 4.3: /20

#14.

$$T\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}_B$$

$$T\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}_B$$

$$T\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 6 & 0 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}_B$$

$$T\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 0 & 6 \end{bmatrix} = 3\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}_B$$

So the matrix is $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$. T is not an isomorphism. A basis for Ker T is $\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$,

And a basis for im T is $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$

(2)

#28: /20

$T(f(t)) = f(2t-1)$ so $T(1) = 1$, $T(t-1) = 2t-1-1 = 2(t-1) = \begin{bmatrix}0\\2\\0\end{bmatrix}_B$

~~$f(t)=$~~

and $T((t-1)^2) = (2t-1-1)^2 = 4(t-1)^2 = \begin{bmatrix}0\\0\\4\end{bmatrix}_B$

so the matrix of T in B is $\begin{bmatrix}1&0&0\\0&2&0\\0&0&4\end{bmatrix}$. T is an isomorphism (because rank(matrix of T) = 3).

#38: /10

$$T\left(\begin{bmatrix}1&0\\1&0\end{bmatrix}\right) = \begin{bmatrix}0&1\\1&0\end{bmatrix}\cdot\begin{bmatrix}1&0\\1&0\end{bmatrix} - \begin{bmatrix}1&0\\1&0\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix} = \begin{bmatrix}1&0\\1&0\end{bmatrix} - \begin{bmatrix}1&0\\1&0\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}.$$

$$T\left(\begin{bmatrix}1&0\\-1&0\end{bmatrix}\right) = \begin{bmatrix}0&1\\1&0\end{bmatrix}\cdot\begin{bmatrix}1&0\\-1&0\end{bmatrix} - \begin{bmatrix}1&0\\-1&0\end{bmatrix}\cdot\begin{bmatrix}1&0\\0&-1\end{bmatrix} = \begin{bmatrix}-1&0\\1&0\end{bmatrix} - \begin{bmatrix}1&0\\-1&0\end{bmatrix} = -2\begin{bmatrix}1&0\\-1&0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0&1\\0&1\end{bmatrix}\right) = \begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&1\\0&1\end{bmatrix} - \begin{bmatrix}0&1\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix} = \begin{bmatrix}0&1\\0&1\end{bmatrix} - \begin{bmatrix}0&-1\\0&-1\end{bmatrix} = \begin{bmatrix}0&2\\0&2\end{bmatrix} = 2\begin{bmatrix}0&1\\0&1\end{bmatrix}.$$

$$T\left(\begin{bmatrix}0&1\\0&-1\end{bmatrix}\right) = \begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&1\\0&-1\end{bmatrix} - \begin{bmatrix}0&1\\0&-1\end{bmatrix}\cdot\begin{bmatrix}1&0\\0&-1\end{bmatrix} = \begin{bmatrix}0&-1\\0&1\end{bmatrix} - \begin{bmatrix}0&-1\\0&1\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$$

So the matrix of T in the basis B is: $\begin{bmatrix}0&0&0&0\\0&-2&0&0\\0&0&2&0\\0&0&0&0\end{bmatrix}$. T is not an isomorphism.

A basis for $\ker T$ is $\left(\begin{bmatrix}1&0\\1&0\end{bmatrix}, \begin{bmatrix}0&1\\0&-1\end{bmatrix}\right)$, a basis for $\operatorname{im} T$ is $\left(\begin{bmatrix}1&0\\-1&0\end{bmatrix}, \begin{bmatrix}0&1\\0&1\end{bmatrix}\right)$.

#57: /10

$$T\left(\begin{bmatrix}1\\1\\-1\end{bmatrix}\right) = \begin{bmatrix}-2&-3&1\\1&0&-2\\0&1&1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\-1\end{bmatrix} = \begin{bmatrix}-6\\3\\0\end{bmatrix} = -\begin{bmatrix}1\\1\\-1\end{bmatrix} - \begin{bmatrix}5\\-4\\1\end{bmatrix} = \begin{bmatrix}-1\\-1\end{bmatrix}_B.$$

$$T\begin{bmatrix}5\\-4\\1\end{bmatrix} = \begin{bmatrix}-2&-3&1\\1&0&-2\\0&1&1\end{bmatrix}\cdot\begin{bmatrix}5\\-4\\1\end{bmatrix} = \begin{bmatrix}3\\3\\-3\end{bmatrix} = 3\begin{bmatrix}1\\1\\-1\end{bmatrix} = \begin{bmatrix}3\\0\end{bmatrix}_B$$

Therefore the matrix of the restriction of T to V in the chosen basis is $\begin{bmatrix}1&3\\-1&0\end{bmatrix}$.

\# 64: /20

$$T\left(\begin{bmatrix}1&0\\0&0\end{bmatrix}\right) = 1.\,I_2 + 0\,P + 0\,P^2 = \begin{bmatrix}1&0\\0&0\end{bmatrix} + \begin{bmatrix}0&0\\0&1\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix}_B$$

$$T\left(\begin{bmatrix}0&1\\0&0\end{bmatrix}\right) = 1.\,P = \begin{bmatrix}1&2\\0&3\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}_B.$$

$$T\left(\begin{bmatrix}0&0\\0&1\end{bmatrix}\right) = 1.\,P^2 = \begin{bmatrix}1&2\\0&3\end{bmatrix}\cdot\begin{bmatrix}1&2\\0&3\end{bmatrix} = \begin{bmatrix}1&8\\0&9\end{bmatrix} = \begin{bmatrix}1\\8\\9\end{bmatrix}_B.$$

So the matrix is $\begin{bmatrix}1&1&1\\0&2&8\\1&3&9\end{bmatrix}$. Its reduced form is $\begin{bmatrix}1&0&-3\\0&1&4\\0&0&0\end{bmatrix}$. So the rank is 2,

a basis for the image is $\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&2\\0&3\end{bmatrix}$. A basis for the kernel is $\begin{bmatrix}3\\-4\\1\end{bmatrix}$., obtained by writing

$$\vec{v}_3 = -3\vec{v}_1 + 4\vec{v}_2 \Rightarrow +3\vec{v}_1 - 4\vec{v}_2 + \vec{v}_3 = \vec{0}.$$

Correction HW 10:

Section 4.3:

#33: $T(1) = 1 = \begin{bmatrix}1\\0\\0\end{bmatrix}_B$; $T(t-1) = 0 + 1.(t-1) = \begin{bmatrix}0\\1\\0\end{bmatrix}_B$; $T((t-1)^2) = 0 + 0.$

So the matrix is $\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$. $\begin{cases}\text{Basis for the image: } 1, t-1.\\ \text{Basis for the kernel: } (t-1)^2.\end{cases}$

#54: Call $\vec{v}_1 = \begin{bmatrix}1\\1\\-1\end{bmatrix}$ and p the orthogonal projection on the line L directed by $\vec{v}_1$.

Since $\vec{v}_1 \in L$ we know that $p(\vec{v}_1) = \vec{v}_1 = \begin{bmatrix}1\\0\end{bmatrix}_B$.

Now $p\begin{pmatrix}5\\-4\\1\end{pmatrix} = \dfrac{\begin{pmatrix}5\\-4\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix}}{1^2+1^2+(-1)^2} \cdot \begin{bmatrix}1\\1\\-1\end{bmatrix} = \vec{0}$ because $\begin{bmatrix}5\\-4\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\-1\end{bmatrix} = 0.$

So the matrix is $\begin{bmatrix}1&0\\0&0\end{bmatrix}$.

#60:

a) $\boxed{P_{\mathcal{B}\to\mathcal{U}}}$: we write $\vec{v}_1 = 1.\vec{u}_1 + 0.\vec{u}_2 = \begin{bmatrix}1\\0\end{bmatrix}_{\mathcal{U}}$ and $\vec{v}_2 = 1.\vec{u}_1 + 1.\vec{u}_2 = \begin{bmatrix}1\\1\end{bmatrix}_{\mathcal{U}}$.

So $P_{\mathcal{B}\to\mathcal{U}} = \begin{bmatrix}1&1\\0&1\end{bmatrix}$.

b) $\boxed{P_{\mathcal{U}\to\mathcal{B}}}$: we write $\vec{u}_1 = 1.\vec{v}_1 + 0.\vec{v}_2 = \begin{bmatrix}1\\0\end{bmatrix}_{\mathcal{B}}$ and $\vec{u}_2 = 1\vec{v}_2 - 1.\vec{v}_1 = \begin{bmatrix}1\\-1\end{bmatrix}_{\mathcal{B}}$.

(Or we could say that $P_{\mathcal{U}\to\mathcal{B}} = (P_{\mathcal{B}\to\mathcal{U}})^{-1}$). So $P_{\mathcal{U}\to\mathcal{B}} = \begin{bmatrix}1&1\\0&-1\end{bmatrix}$.

c) Just notice that: $\begin{cases}\vec{v}_1 = 1.\vec{u}_1 + 0\,\vec{u}_2\\ \vec{v}_2 = 1.\vec{u}_1 + 1.\vec{u}_2\end{cases}$

Therefore: $\begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix} = \begin{bmatrix}\vec{u}_1 & \vec{u}_2\end{bmatrix}\cdot\begin{bmatrix}1&1\\0&1\end{bmatrix}$.

68:

(a) see correction of midterm II.

(b) Consider $T: W \to \mathbb{R}^2$, $(x_0, x_1, \ldots) \mapsto \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$ and see that T is an isomorphism. (see correction of Mid 2 again

(c) If $(1, c, c^2, \ldots) \in W$ then necessarily $c^2 = c + 6 \Rightarrow \begin{cases} c = -2 \\ \text{OR} \\ c = 3 \end{cases}$.

Now if $c = -2$ or $c = 3$ we have $c^{n+2} = c^n \cdot c^2 = c^n \cdot (c+6) = c^{n+1} + 6 \cdot c^n$,

therefore $(1, c, c^2, \ldots) \in W$.

In conclusion $(1, -2, (-2)^2, \ldots)$ and $(1, 3, 3^2, \ldots)$ are the only geometric sequences in W.

(d) $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ form a basis of $\mathbb{R}^2$ therefore (by question (b)) $(1, -2, \ldots)$ and $(1, 3, \ldots)$

form a basis of W.

(e) $(0, 1, 1, 7, 13, \ldots)$ We have $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{5}\left[\begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right]$, so $(x_0, x_1, \ldots) = \frac{1}{5}\left((1, 3, 3^2, \ldots) - (1, -2, \ldots\right.$

implying that $\boxed{x_n = \frac{1}{5}\left[3^n - (-2)^n\right]}$.

Section 5.1:

16: The vector $\begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$ works. Since the orthogonal complement of $\operatorname{span}(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is a line. There are only 2 such vectors (of length 1) on that line.

17: $W^\perp$ is the kernel of $T: \mathbb{R}^4 \to \mathbb{R}^2$, $\vec{x} \mapsto \begin{bmatrix} \vec{v}_1 \cdot \vec{x} \\ \vec{v}_2 \cdot \vec{x} \end{bmatrix}$ which has the following matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} = A$.

Now $\operatorname{rref} A = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$ and $\vec{w}_3 = -\vec{w}_1 + 2\vec{w}_2$ gives the vector $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$,

and $\vec{w}_4 = -2\vec{w}_1 + 3\vec{w}_2$ gives $\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$.

32:
- If $\vec{v}_1 = \vec{0}$ then G is not invertible.
- If $\vec{v}_1 \neq \vec{0}$ then we have $\vec{v}_2 \cdot \vec{v}_1 = \frac{\vec{v}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \cdot (\vec{v}_1 \cdot \vec{v}_1)$ and $\operatorname{rref} G = \begin{bmatrix} 1 & \frac{\vec{v}_1 \cdot \vec{v}_2}{\vec{v}_1 \cdot \vec{v}_1} \\ 0 & \vec{v}_2 \cdot \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \cdot (\vec{v}_1 \cdot \vec{v}_2) \end{bmatrix}$.

  and G is invertible only if $\vec{v}_2 \cdot \vec{v}_2 \neq \frac{(\vec{v}_1 \cdot \vec{v}_2)^2}{\vec{v}_1 \cdot \vec{v}_1}$.

Remark now that $p(\vec{v}_2) = \frac{\vec{v}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$ is the orthogonal proj. of $\vec{v}_2$ on $\operatorname{span}(\vec{v}_1)$. So G isn't invertible iff $\vec{v}_2 \cdot \vec{v}_2 = p(\vec{v}_2) \cdot \vec{v}_2$,

but this is equivalent to $\vec{v}_2 \cdot (\vec{v}_2 - p(\vec{v}_2)) = 0$ which is possible iff $\vec{v}_2$ is a multiple of $\vec{v}_1$.

[illegible]

Correction of HW 11

5.2:

14:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 7 \\ 2 \\ 7 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 8 \\ 1 \\ 6 \end{bmatrix}$$

(a) We start with $\vec{v}_1$: First we compute $\|\vec{v}_1\| = \sqrt{1^2+7^2+1^2+7^2} = 10$

So our first vector is $\vec{u}_1 = \dfrac{\vec{v}_1}{\|\vec{u}_1\|} = \begin{bmatrix} 1/10 \\ 7/10 \\ 1/10 \\ 7/10 \end{bmatrix}$.

(b) Second vector:

Take $\vec{v}_2$ and subtract to it the projection of $\vec{v}_2$ onto the line L generated by $\vec{u}_1$:

$$P_L(\vec{v}_2) = \frac{(\vec{v}_2 \cdot \vec{u}_1)}{\|\vec{u}_1\|^2} \cdot \vec{u}_1 = (\vec{v}_2 \cdot \vec{u}_1) \cdot \vec{u}_1 \quad (\text{because } \|\vec{u}_1\| = 1).$$

$$= \left(\frac{49}{10} + \frac{2}{10} + \frac{49}{10}\right)\vec{u}_1$$

$$= 10\,\vec{u}_1$$

So we get $\vec{v}_2 - P_L(\vec{v}_2) = \vec{v}_2 - 10\vec{u}_1 = \begin{bmatrix} 0 \\ 7 \\ 2 \\ 7 \end{bmatrix} - \begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

The length of this vector is: $\sqrt{(-1)^2 + 1^2} = \sqrt{2}$.

So our second (normalized) vector is $\begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$ or if you prefer $\vec{u}_2 = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$.

Rk: notice that $\vec{u}_2 \perp \vec{u}_1$, and that $\|\vec{u}_1\| = \|\vec{u}_2\| = 1$

(c) Third vector:

. Take $\vec{v}_3$ and subtract to it the ~~p~~ orthogonal projection of $\vec{v}_3$ onto the plane spanned by $\vec{u}_1, \vec{u}_2$.

. $P_3(\vec{v}_3) = (\vec{v}_3 \cdot \vec{u}_1)\,\vec{u}_1 + (\vec{v}_3 \cdot \vec{u}_2)\,\vec{u}_2$ (this is the formula for the orthog. proj. onto a plane spanned by an <u>orthonormal</u> basis $(\vec{u}_1, \vec{u}_2)$).

$$= \left(\frac{1}{10} + \frac{56}{10} + \frac{1}{10} + \frac{42}{10}\right)\vec{u}_1 + \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)\vec{u}_2$$

$$= 10\,\vec{u}_1$$

~~So ..~~

So we get:

$$\vec{v}_3 - P_{\mathcal{P}}(\vec{v}_3) = \begin{bmatrix}1\\8\\1\\6\end{bmatrix} - \begin{bmatrix}1\\7\\1\\7\end{bmatrix} = \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \text{ of length } \sqrt{1^2+1^2} = \sqrt{2}$$

We still need to normalize it:

$$\vec{u}_3 = \begin{bmatrix}0\\1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix}.$$

So the Gram-Schmidt procedure gives us this orthonormal basis: $\vec{u}_1 = \begin{bmatrix}1/10\\7/10\\1/10\\7/10\end{bmatrix}, \vec{u}_2 = \begin{bmatrix}-\sqrt{2}/2\\0\\\sqrt{2}/2\\0\end{bmatrix}, \vec{u}_3 = \begin{bmatrix}0\\\sqrt{2}/2\\0\\\sqrt{2}/2\end{bmatrix}$.

Rk 1: You can start also from $\vec{v}_2$ or $\vec{v}_3$ and continue the procedure: you will end up with a different orth. basis, for the same space.

Rk 2: The vectors have been chosen so that computations will be easy. Don't be surprised if in general you get messy computations ...

#34:

First we find a basis for $\ker A$:

We need to reduce $\begin{bmatrix}1&1&1&1\\1&2&3&4\end{bmatrix} R_2 - R_1$

$\begin{bmatrix}1&1&1&1\\0&1&2&3\end{bmatrix} R_1 - R_2$

$\begin{bmatrix}1&0&-1&-2\\0&1&2&3\end{bmatrix}$

- $\vec{v}_3 = -\vec{v}_1 + 2\vec{v}_2 \Rightarrow \vec{v}_1 - 2\vec{v}_2 + \vec{v}_3 = \vec{0}$ so we get $\begin{bmatrix}1\\-2\\1\\0\end{bmatrix}$ in the kernel.
- $\vec{v}_4 = -2\vec{v}_1 + 3\vec{v}_2 \Rightarrow 2\vec{v}_1 - 3\vec{v}_2 + \vec{v}_4 = \vec{0}$ so we get $\begin{bmatrix}2\\-3\\0\\1\end{bmatrix}$ in the kernel.

Now we need to apply Gram-Schmidt:

First vector: $\left\|\begin{bmatrix}1\\-2\\1\\0\end{bmatrix}\right\| = \sqrt{1^2+2^2+1^2} = \sqrt{6}$

so we get $\vec{u}_1 = \begin{bmatrix}1/\sqrt{6}\\-2/\sqrt{6}\\1/\sqrt{6}\\0\end{bmatrix}$.

Second vector:

Take $\vec{w}_2 = \begin{bmatrix}2\\-3\\0\\1\end{bmatrix}$ and subtract to it the projection of $\vec{w}_2$ onto the line L spanned by $\vec{u}_1$:

$$P_L(\vec{w}_2) = \left(\begin{bmatrix}2\\-3\\0\\1\end{bmatrix}\cdot\begin{bmatrix}1/\sqrt{6}\\-2/\sqrt{6}\\1/\sqrt{6}\\0\end{bmatrix}\right)\cdot\vec{u}_1 = \left(\frac{2}{\sqrt{6}} + \frac{6}{\sqrt{6}} + 0\right)\vec{u}_1$$

$$= \frac{8\sqrt{6}}{6}\vec{u}_1$$

$$= \begin{bmatrix}4/3\\-8/3\\4/3\\0\end{bmatrix}$$

So we get $\vec{w}_2 - P_L(\vec{w}_2) = \begin{bmatrix}2\\-3\\0\\1\end{bmatrix} - \begin{bmatrix}4/3\\-8/3\\4/3\\0\end{bmatrix} = \begin{bmatrix}2/3\\-1/3\\-4/3\\1\end{bmatrix}$ with length $\sqrt{\left(\frac{2}{3}\right)^2+\left(\frac{-1}{3}\right)^2+\left(\frac{-4}{3}\right)^2+1^2}$

$= \sqrt{\frac{10}{3}}$.

Therefore $\vec{u}_2 = \begin{bmatrix}\frac{2}{3}\frac{\sqrt{3}}{\sqrt{10}}\\-\frac{1}{3}\frac{\sqrt{3}}{\sqrt{10}}\\-\frac{4}{3}\frac{\sqrt{3}}{\sqrt{10}}\\\frac{\sqrt{3}}{\sqrt{10}}\end{bmatrix} = \begin{bmatrix}\frac{2}{\sqrt{30}}\\-\frac{1}{\sqrt{30}}\\-\frac{4}{\sqrt{30}}\\\frac{\sqrt{3}}{\sqrt{10}}\end{bmatrix}$

Section: 6.1:

#10:

$$A = \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array} \end{array}$$

Let's compute the det by using row operations.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$ whose det is $5 - 2.2 = 1$ so A is invertible.

#18:

$$\begin{bmatrix} 0 & 1 & k \\ 3 & 2k & 5 \\ 9 & 7 & 5 \end{bmatrix}$$

Expand the det along the first column:

$$-3\left(5 - 7k\right) + 9.\left(5 - k.2k\right) = -15 + 21k + 45 - 18k^2$$
$$= -18k^2 + 21k + 30$$
$$= (-3)\left(6k^2 - 7k - 10\right)$$
$$= (-3).6\left(k - 2\right)\left(k + \tfrac{5}{6}\right)$$

So the matrix is invertible if and only if $\left(k \neq 2 \text{ and } k \neq -\frac{5}{6}\right)$.

#30:

$$\begin{bmatrix} 4-\lambda & 2 & 0 \\ 4 & 6-\lambda & 0 \\ 5 & 2 & 3-\lambda \end{bmatrix}$$

Expand along the last column:

$$\det(A - \lambda I_n) = (3-\lambda)\left((4-\lambda)(6-\lambda) - 4.2\right)$$
$$= (3-\lambda)\left(\lambda^2 - 10\lambda - 8 + 24\right)$$
$$= (3-\lambda)(\lambda-8)(\lambda-2).$$

So $(A - \lambda I_n)$ is <u>not invertible</u> if and only if $\left(\lambda = 3 \text{ or } \lambda = 8 \text{ or } \lambda = 2\right)$.

#36:

$$A = \begin{bmatrix} 2 & 0 & 2 & 2 \\ 1 & 0 & 2 & 2 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 \end{bmatrix}$$

Expand along the second column:

$$\det A = -2.\det \begin{bmatrix} 2 & 2 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix} \begin{array}{l} R_1 - 2R_3 \\ R_2 - R_3 \\ \end{array}$$

$$= -2.\det \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= -2.2$$
$$= -4$$

#44:

By linearity on the columns: $\det(kA) = k^n \cdot \det A$.

54:

(a) Expand along 1^{st} column: $d_n = 5 \cdot d_{n-1} - 1 \cdot 6 \cdot d_{n-2}$

$$d_n = 5d_{n-1} - 6d_{n-2}.$$

(b) $d_1 = 2$, $d_2 = \det\begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix} = 4$, $d_3 = \begin{bmatrix} 5 & 6 & 0 \\ 1 & 5 & 6 \\ 0 & 1 & 2 \end{bmatrix} = 5 \cdot (10-6) - 1 \cdot (6 \cdot 2) = 20 - 12 = 8$.

(c) Let's prove that $d_n = 2^n$.

1) $d_1 = 2^1$, $d_2 = 2^2$.

2/ Assume that for any $1 \leq k \leq n-1$ we have $d^k = 2^k$.

Now $d^n = 5 \cdot 2^{n-1} - 6 \, 2^{n-2} = (5 \cdot 2 - 6) \cdot 2^{n-2} = 2^{2+n-2} = 2^n$.

6.2:

#35: $\det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & a_1 & b_1 \\ x_2 & a_2 & b_2 \end{bmatrix} = 1(a_1b_2 - a_2b_1) - x_1(b_2 - a_2) + x_2(b_1 - a_1)$.

This is the eq. of a line in the plane (it's of the form $\alpha x + \beta y + \gamma = 0$).

If we replace $\begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix}$ by $\begin{bmatrix} 1 \\ a_1 \\ a_2 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ b_1 \\ b_2 \end{bmatrix}$ the det is 0 (two columns are identical).

47:

Linearity in the first column: $T\begin{bmatrix} a+a' & b \\ c+c' & d \end{bmatrix} = (a+a')d + b(c+c') = T\begin{bmatrix} a & b \\ c & d \end{bmatrix} + T\begin{bmatrix} a' & b \\ c' & d \end{bmatrix}$

and $T\begin{bmatrix} ka & b \\ kc & d \end{bmatrix} = kad + kcb = k \cdot T\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Now the formula $ad+bc$ is unchanged if you swap $\begin{bmatrix} a \\ c \end{bmatrix}$ and $\begin{bmatrix} b \\ d \end{bmatrix}$, therefore T is linear in the second column, and also if you replace $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ by its transpose, therefore T is linear in the rows.

HW11_0006.jpeg %d×%d pixels

#48:

$T: \mathbb{R}^n \to \mathbb{R}$

. Pick $\vec{x} \notin \text{span}(\vec{v}_2, \dots, \vec{v}_n)$: then $T(\vec{x}) \neq \vec{0}$ so $\dim \operatorname{im} T \neq 0$ so $\operatorname{im} T = \mathbb{R}$.

. $\vec{x} \in \ker T$ if and only if $\vec{x} \in \underbrace{\text{span}(\vec{v}_2, \dots, \vec{v}_n)}_{\text{has dim } (n-1)}$.

HW 12: correction:

#26:

A basis for V is B: $\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}$.

$$T\begin{bmatrix}1&0\\0&0\end{bmatrix}=\begin{bmatrix}1&2\\2&3\end{bmatrix}\cdot\begin{bmatrix}1&0\\0&0\end{bmatrix}+\begin{bmatrix}1&0\\0&0\end{bmatrix}\cdot\begin{bmatrix}1&2\\2&3\end{bmatrix}=\begin{bmatrix}1&0\\2&0\end{bmatrix}+\begin{bmatrix}1&2\\0&0\end{bmatrix}=\begin{bmatrix}2&2\\2&0\end{bmatrix}=\begin{bmatrix}2\\0\\2\end{bmatrix}_B$$

$$T\begin{bmatrix}0&0\\0&1\end{bmatrix}=\begin{bmatrix}1&2\\2&3\end{bmatrix}\cdot\begin{bmatrix}0&0\\0&1\end{bmatrix}+\begin{bmatrix}0&0\\0&1\end{bmatrix}\cdot\begin{bmatrix}1&2\\2&3\end{bmatrix}=\begin{bmatrix}0&2\\0&3\end{bmatrix}+\begin{bmatrix}0&0\\2&3\end{bmatrix}=\begin{bmatrix}0&2\\2&6\end{bmatrix}=\begin{bmatrix}0\\6\\2\end{bmatrix}_B$$

$$T\begin{bmatrix}0&1\\1&0\end{bmatrix}=\begin{bmatrix}1&2\\2&3\end{bmatrix}\cdot\begin{bmatrix}0&1\\1&0\end{bmatrix}+\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}1&2\\2&3\end{bmatrix}=\begin{bmatrix}2&1\\3&2\end{bmatrix}+\begin{bmatrix}2&3\\1&2\end{bmatrix}=\begin{pmatrix}4&4\\4&4\end{pmatrix}=\begin{bmatrix}4\\4\\4\end{bmatrix}_B$$

So the matrix is: $B=\begin{bmatrix}2&0&4\\0&6&4\\2&2&4\end{bmatrix}$ and $\det B = 2\left(6\cdot4-2\cdot4\right)+2\cdot\left(0-24\right)=-16$.

#59:

False: $\det\begin{pmatrix}1&0\\0&1\end{pmatrix}=\det\underbrace{\begin{pmatrix}1&1\\0&1\end{pmatrix}}_{B}=1$ but $\begin{pmatrix}1&1\\0&1\end{pmatrix}=B$ can't be similar to I_2: $P^{-1}(I_2)P$ is always equal to I_2!

6.3: #2:

area $=\frac{1}{2}$ area of parallelogram $=\frac{1}{2}\left|\det\begin{bmatrix}8&3\\2&7\end{bmatrix}\right|=25$.

\# 30:

$$A=\begin{bmatrix}1&0&0\\2&3&0\\4&5&6\end{bmatrix}$$

$A_{11}=18$, $A_{12}=12$, $A_{13}=10-12=-2$

$A_{21}=0$, $A_{22}=6$, $A_{23}=5$

$A_{31}=0$, $A_{32}=0$, $A_{33}=3$

So $$\operatorname{adj}(A)=\begin{pmatrix}+18&-0&+0\\-12&+6&-0\\+(-2)&-5&+3\end{pmatrix}=\begin{pmatrix}18&0&0\\-12&6&0\\-2&-5&3\end{pmatrix}.$$

\#36: A is invertible:

$\operatorname{adj}(A)=(\det A\cdot A^{-1})$ so $\operatorname{adj}(\operatorname{adj}A)=\det(\operatorname{adj}A)\cdot(\operatorname{adj}A)^{-1}=\det\left(\det A\cdot A^{-1}\right)\cdot\left(\det A\cdot A^{-1}\right)^{-1}$

$$=(\det A)^n\cdot\frac{1}{\det A}\cdot\frac{1}{\det A}\cdot A$$

$$=(\det A)^{n-2}\cdot A$$

7.1:

#42:

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ if and only if:

$$\underbrace{\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\begin{bmatrix} a \\ d \\ g \end{bmatrix}} = \lambda \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

this implies $d = g = 0$.

Similarly $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector if $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ f \\ i \end{bmatrix} = \mu \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow f = c = 0$.

So A is necessarily of the form $\begin{bmatrix} a & b & 0 \\ 0 & e & 0 \\ 0 & h & i \end{bmatrix}$. Reciprocally, any such matrix has both $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ as eigenvectors.

Now a basis for this space V is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$: so $\dim = 5$.

#8:

We are looking for $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 5 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, but this is equivalent to $\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$.

So this is the set of all matrices of the form $\begin{bmatrix} 5 & b \\ 0 & d \end{bmatrix}$. (This is a set, not a subspace!).

7.2:

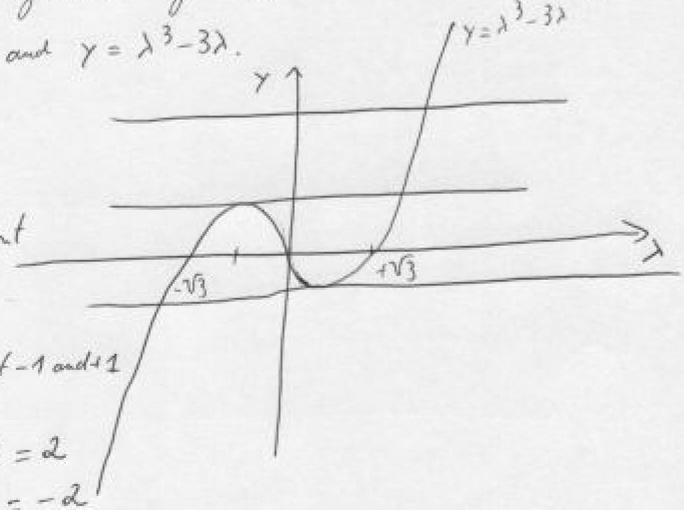
#4. $A = \begin{bmatrix} 0 & 4 \\ -1 & 4 \end{bmatrix}$ so $\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 4 \\ -1 & 4-\lambda \end{pmatrix} = \lambda^2 - 4\lambda + 4$
$= (\lambda - 2)^2$

So we have only one eigenvalue $\lambda = 2$.

#32:

$A - \lambda I = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ k & 3 & -\lambda \end{bmatrix}$ and $\det(A - \lambda I) = (-\lambda)(\lambda^2 - 3) - 1(-k)$
$= -\lambda^3 + 3\lambda + k.$

So the real eigenvalues are given by the intersections of the graph of $y = k$ and $y = \lambda^3 - 3\lambda$.



The two graphs have exactly 3 distinct intersections if and only if k is taken between the two values y_1, y_2 where the tangent to the graph of $y = \lambda^3 - 3\lambda$ is horizontal.

Now $\frac{d}{d\lambda}(\lambda^3 - 3\lambda) = 3\lambda^2 - 3$ which is zero at -1 and $+1$

The value of $\lambda^3 - 3\lambda$ at -1 is $-1 + 3 = 2$
$+1$ is $1 - 3 = -2$

. So A has exactly 3 distinct eigenvalues if and only if $k \in (-2, 2)$.

. For $k = -2$ or $k = 2$, A has 2 real eigenvalues.

. For $k > 2$ or $k < -2$, A has only 1 real eigenvalue.

#44: Assume we have A, B invertible, such that: $AB = BA + A$ then necessarily $ABA^{-1} = B + I$
and $\underbrace{\mathrm{tr}(ABA^{-1})}_{\mathrm{tr}\,B} = \mathrm{tr}\,B + \mathrm{tr}\,I$
$\cancel{\mathrm{tr}\,B} = \cancel{\mathrm{tr}\,B} + n$

Which is impossible.