MAT 211 (Introduction to Linear Algebra) Section 1

Spring 2006

Department of Mathematics, Stony Brook University

The course meets Mondays and Wednesdays in Lgt Engr Lab 152 from 2:20 pm to 3:40 pm.

Instructor: Dusa McDuff

Instructor's Office Hours: Math Building 3-111: *Monday 1-2:10, Tuesday 12:15 - 1:30 and by appt.* You are always welcome to contact me by email (dusa at math.sunysb.edu) either to ask a short question or to set up an appointment to see me.

Grader: Mohamed Hassan

Grader's Office Hours: *Monday 5--7pm* in Physics C117 and by appt.

Textbook:

Bretscher, *Linear Algebra with Applications, 3rd Ed.,* Pearson/Prentice-Hall

(two copies are available on reserve in the Math/Physics/Astronomy Library)

Link to an on line tutorial on Linear Algebra by Avi Goldman. It is worth looking at this from time to time since it contains useful outside links and notes on the topics of the course.

Click here for a link to the CURRENT HOMEWORK. This page also contains links to solutions.

Course Notes:

- *(posted May 12)* I have now posted the grades. In general you did pretty well on the final, and some of you did extremely well; one student even got 120/120! The average was 84.6/120. Distribution: <50: 7; 60-69: 3; 70-79: 1; 80-89: 5; 90-99: 8; 100-109: 7; 110-120: 3. For most people the final grade was based on the total number of points they had earned, but a few people did better on the final than over the semester and so their grade came from the final. Have a good summer!

- *(posted May 8)* Mohamed asks me to tell you that there is a typo on p 7 of the solutions to the final review sheet. E0 should be (1,0,0) not (0,0,1) and E1 should be (1,1,0) and not (0,1,1). E2 is correct. If you see any other inconsistences please email Mohamed at: mrhassan@ic.sunysb.edu. (You can also email me so that I can post a correction.)

- *(posted May 5)* Here are the SOLUTIONS to the problems on the review sheet.

- *(posted May 1)* The FINAL EXAM is on Wednesday May 10 at 2:00-4:30 pm in our usual classroom. q

- *(posted May 1)* I read over the final review sheet and found a few typos -- in Q 5,7, 10 and 13. Nothing too serious, but you might want to look at the new version now posted.

- *(posted May 1)* My OFFICE HOURS for next week: Monday May 8: 1-2 and Tuesday May 9: 2-4 both times in my office.

- *(posted April 25)* I have just posted the last homework. This is a Half Homework due on MONDAY May 1. It should be graded by Wed May 3. This is the last day of classes. I cannot be there, but Mohamed will take a review class based on a review sheet that is posted HERE. I plan to hold some review sessions just before the final; the times will arranged in class soon, and I will post them here.

- *(posted April 25)* Here are the solutions to the Extra Homework. And here is the second midterm.

- *(posted April 17)* Someone pointed out in class today that there is a typo in question 1 on the Extra Homework (make up exam 2): it should be R^3 and not R^4. I can't change the pdf file at the moment, since its source is on my home computer which is out of commission right now.

- *(posted April 9)* Here are the solutions to Midterm 2. Also I posted Homework 12 today.

- *(posted April 4)* Midterm 2 was harder than I meant it to be, and so I have decided to give those of you who did poorly on it another chance. I have written a simplified version which I am assigning as an extra homework. Everyone is welcome to do it (you can use it as extra practice). However, it will affect the grades only of those who got less than 50 on the midterm. For those people I will add 1/3 of your score on this make up to your midterm grade, to a maximum of 50. i.e. your new Midterm grade will be the MINIMUM of 50 and (Old Midterm grade + 1/3 of make up grade.) This make up is due on April 19, the Wednesday after the break. Click here for the assignment. As always, I expect the solutions you hand in to be your own work, though you can get any help you want, including studying the solutions to Midterm 2 (which I will post very soon.)

- *(posted April 3)* I have slightly rearranged the schedule for the last few weeks of semester. I will give 3 classes on Ch 6 and 4 on Chapter 7. The last class will be review.

- *(posted April 3)* I slightly changed the Homework for this week; one of the problems from Sec 5.5 has been deleted. You certainly don't need to read the whole section to do the remaining problem; look at defs 5.5.1 and 5.5.2 and examples 2 and 4.

- *(posted April 3)* The scores for this Midterm were significantly lower. Average grade: 52.5 I will post solutions very soon. Please come to office hours if you have questions. (I am willing to make special appointments with students who cannot manage the regular times.)

Distribution: < 20: 5; 20-29: 3; 30-39: 4; 40-47: 0; 48-59: 5; 60-69: 4; 70-79: 2; 80-89: 2; 90-99: 4.

Grade cutoffs: < 30: F; 30-47: D; 48-59: C, C+; 60-79: B-,B,B+; 80-89: A-, 90-99: A.

- *(posted March 31)* Are any of you interested in being an undergraduate math TA next semester? Applications forms are in the Math Undergraduate Office; the interviews will be soon. For the first semester you work for 3 credits (taking MAT 475), but in subsequent semesters you get paid. If you are interested, it is much the best to start in the Fall; there are usually no openings in the Spring.

- *(posted March 30)* I decided to make the next HW a half HW. It will be posted today. I haven't graded the exam yet, but it's obvious you found it harder than the first one. Hang in there! The grade cut offs will be adjusted to make the exam fair. And those of you who know you did badly, please COME to CLASS and DO the Homework -- it really makes a difference. It's never too late to start. We will begin the determinant on Monday. This is completely new material, not related to Ch 4 or Ch 5.

- *(posted March 28)* ANOTHER TYPO on Review sheet. In 5(iv) I should ask for the matrix of T, not its coordinates. Here are the Solutions to Review sheet.

- *(posted March 22)* Here is the REVIEW SHEET for the second midterm. We will discuss it in class on MONDAY. Note that it has a typo: Definition 5.1.3 should be Def 5.3.1. Note also that there is a half Homework due on Monday (5 short questions on Sec 5.3).

- *(posted Mar 6)* IMPORTANT NOTICE ABOUT FINAL EXAM The final is on Wed May 10 from 2-4:30 in the usual classroom. The time posted on the departmental syllabus was incorrect.

- *(posted Mar 6)* The class did Midterm 1 very well. Average 70, distribution: 0-39: 6; 40-49: 1; 50-59: 3; 60-69: 3; 70-79: 9; 80-89: 7; 90-100: 10. Congratulations! Very rough grade equivalencies: < 50 = F; 50-59 = D,D+; 60-69 = C; 70-89 divided equally among C+ to A-; 90-100: A.

- *(posted Feb 27)* Mohamed pointed out that my solution to sec 3.1 number 22 is wrong (page 2 of sols to HW5). The matrix A has rank 2: the second row in the reduced form should be 0 5/2 -5/2 which is a multiple of the last row. In fact the second row is 5/4 times the last row. Hence there is a solution iff (b_2-3/2 b_1) = 5(b_3 - 3 b_1)/4; ie iff 9 b_1 + 4 b_2 - 5 b_3 = 0. I hope this is correct now.

- *(posted Feb 2)* Here are some notes on vectors that should help with the homework this week. I will probably add to these notes over the weekend.

- *(posted Jan 25)* The most important thing for you to do right now is to review/learn the geometric properties of vectors in 3-space. You need to know the parametric equations for a line, a little about the dot product (i.e. two vectors are perpendicular if their dot product vanishes), and how to determine a plane in 3-space by a point on it and the normal vector. (Don't worry at this stage about how to use the dot product to calculate other angles. That'll come up later.) This is all explained in the on-line text; Ch 1, section 2. I recommend you do all exercises in Sec 2.1 and Ex 2:10, 2:11, 2:13, 2:14 from sec 2.2.) You might also do ex 34 and 35 from p 22 (sec 1.2) of *Bretscher*. Almost any first year calculus text that does more than differentiation and integration (so is not a "Brief calculus") has a chapter on vectors. Multivariable calculus texts also usually have a chapter on vectors. Wikipedia also has useful information: look up the words: *vector, plane*.

- (*posted Jan 28*) The homework due 2/1 is now posted. You can find find more notes about vectors, lines and planes on the website of the other section.

- Fred Girao, the TA for section 2, has office hours in MLC (Math Learning Center) on Tuesdays 1-2pm. The MLC itself is open M-W 10-9; Th 10-6; F 10-2.

Prerequisites/Corequisites:

You are assumed to have had at least one semester of calculus. If you have not yet studied integration, you should be taking the relevant calculus course (e.g. MAT 126) concurrently with this one, as some important problems and examples in this course require a knowledge of integration.

The Nature Of The Course:

This course is an introduction to the theory which has developed around the solution of systems of linear equations. The importance of this theory as a tool in the social, natural, and mathematical sciences cannot be overestimated. (To get some idea of why this is the case, click here, check out the links of interest below, or take a look through your textbook.) You should keep this in mind throughout the semester, especially if the course material ever seems "too weird" or "too abstract" to be useful.

You should also keep in mind that this course is quite distinct in nature from others you have taken. *Regardless of your performance in previous math courses, do not be discouraged if you find yourself wrestling with a problem or a concept for hours.* Doing many computations is essential to understanding the material, but mindlessly applying memorized techniques while ignoring their theoretical framework will not get you very far. It helps to be proactive in analyzing and even creating examples (*not necessarily complicated ones!*) that illuminate the theory.

Course Format:

You will get most out of the classes if you prepare beforehand by reading the relevant section in the textbook before class. I am always glad to answer questions during class. Since this class has no recitations, I will aim to set aside some class time each week for doing examples and discussing the homework. If you have more questions, please talk to me after class or come to my office hours (or go to the Math Learning Center in Math Building S-240A.) There will be a review session before each exam, to be scheduled later.

Some links of interest

A nice expository paper on the use of linear algebra in search engines.

A useful online linear algebra text with many worked examples and exercises with solutions.

Homework:

As always in a mathematics class, doing the homework is an essential part of learning the course material. It is often good to study together with other students. However the work you hand in must be written in your own words, not copied from someone else. The homework assignments are listed here. Due dates for homework assignments are listed in the schedule below. They are due at the beginning of class on their due date, typically Wednesday unless it just before a test. (The first one (a half homework) is due on Wednesday January 25!) Submissions consisting of multiple pages must be stapled together. If you cannot get to class, hand them in to my office (Math 3-111) before class. **Late homework will be penalized severely (by at least 20%), and will not be accepted if it is too much overdue.** The instructor will hand the homeworks back at the beginning of class. **The grader has the final say on all homework grades.**

If all goes as planned, 11 full homeworks and 3 half homeworks will be assigned during the semester. Your total homework grade will be the sum of the grades of your best 10 homeworks. (One of these might consist of two halves.) Each full homework will be graded on a 20-point system as follows. Five problems (not necessarily the five most difficult ones) will be graded completely, and they each account for 3 points. In order to receive full credit for your solutions to these, each step must be stated clearly and in the correct order, and each statement in English must be a complete and correct sentence. The remaining 5 points are accounted for by a substantial attempt at solving **all** the remaining problems; a penalty of at least 1 point will be incurred for each one which has not been adequately attempted. A sequence of relevant calculations and/or a list of relevant ideas counts as an adequate attempt, whereas leaving a blank or writing "I don't know" does not.

Examinations:

Please note that examinations are not graded on a curve, nor are the final grades curved. Thus you are not competing against your fellow students for a limited number of top grades. There will be two methods of computing the final grade, and you will get the better of the two. One is a weighted average of your work throughout the semester; the other is based entirely on the final, but your grade here will be somewhat discounted. Thus in order to get a C on the basis of the final alone you would have to get at least a C+ on the exam. If at any point in the semester you are seriously concerned with your standing in the class, you are invited to discuss your concerns in detail with me.

Any use of cellphones, calculators, books, or notes while an exam is underway will be considered cheating. If you miss an exam for an acceptable reason and provide me with an acceptable written excuse, the relevant exam will be dropped in computing your course grade. A letter stating that you were seen by a doctor or other medical personnel is **not** an acceptable document. An acceptable document should state that it was **reasonable/proper** for you to seek medical attention and medically **necessary** for you to miss the exam (for privacy reasons the note/letter need not state anything beyond this point). Incomplete grades will be granted only if **documented circumstances beyond your control** prevent you from completing 50% or more of all class assignments.

Grading:

Your raw grade will be based on your examination performance and homework, weighted as follows:

Exam I	25%
Exam II	25%
Final Exam	30%
Homework	20%

The grade you receive in the course will be the maximum of your raw grade and 90% of your final exam grade.

DSS advisory:

If you have a physical, psychological, medical, or learning disability that may affect your course work, please contact Disability Support Services (DSS) office: ECC (Educational Communications Center) Building, room 128, telephone (631) 632-6748/TDD. DSS will determine with you what accommodations are necessary and appropriate. Arrangements should be made early in the semester (before the first exam) so that your needs can be accommodated. All information and documentation of disability is confidential. Students requiring emergency evacuation are encouraged to discuss their needs with their professors and DSS. For procedures and information, go to the following web site http://www.ehs.sunysb.edu and search Fire safety and Evacuation and Disabilities.

Schedule (tentative):

The following is the basic syllabus. Please read the relevant parts of the book **before** class.

Day of	**Homework due**	**Sections Covered**
January 23		1.1 (Introduction to Linear Systems)
January 25	Half Homework 1	1.2 (Matrices, Vectors, and Gauss-Jordan Elimination)
January 30		1.3 (On the Solutions of Linear Systems; Matrix Algebra)

February 1	Homework 2	2.1 (Introduction to Linear Transformations And Their Inverses)
February 6		2.2 (Linear Transformations in Geometry)
February 8	Homework 3	2.3 (The Inverse of a Linear Transformation)
February 13		2.4 (Matrix Products)
February 15	Homework 4	3.1 (Image and Kernel of a Linear Transformation)
February 20		3.2 (Subspaces of R^n; Bases and Linear Independence)
February 22	Homework 5	3.3 (The Dimension of a Subspace of R^n)
February 27	Half Homework 6	3.4 (Coordinates); Review
March 1		**Exam I (on everything from 1.1 up to and including 3.3)**
March 6		4.1 (Introduction to Linear Spaces)
March 8	Homework 7	4.2 (Linear Transformations and Isomorphisms)
March 13		4.3 (The Matrix of a Linear Transformation)
March15	Homework 8	5.1 (Orthogonal Projections and Orthonormal Bases)
March 20		5.2 (Gram-Schmidt Process and QR Factorization)
March 22	Homework 9	5.3 (Orthogonal Transformations and Orthogonal Matrices)
March 27	Half Homework 10	5.5 (Inner Product Spaces)
March 29		**Exam II (on everything from 3.4 up to and including 5.3)**
April 3		6.1 (Introduction to Determinants)
April 5	Half Homework 11	6.2 (Properties of the Determinant)
April 17		6.3 (Geometrical Interpretations of the Determinant; Cramer's Rule)
April 19	Homework 12	Ch 7.1: Dynamical systems and eigenvectors
April 24		7.2 (Finding the Eigenvalues of a Matrix)
April 26	Homework 13	7.3 (Finding the Eigenvectors of a Matrix)
May 1		7.4 (Diagonalization)

May 3	Homework 14	Review
May 10		**Final Exam (Cumulative)**

MAT 211.01 Homework Assignments

Spring 2006

#	Problems	Due Date
Homework 1	Mostly the HW will be from Bretscher, but I wrote it out this week since you may not all have the book yet. Download it by clicking on the blue prompt to the left.	1/25/06
	Solutions to Homework 1	
Homework 2	sec 1.1: 20; sec 1.2: 6, 10, 18, 22, 26, 34, 36, 46. Also: (a) Find parametric equations for the line of intersection of the planes 2x+3y+4z=1 and x+y+z=1. (b) Find the equation of the plane through the point (1,1,1) and perpendicular to the line (x,y,z) = (1+2t,2-3t,t).	2/1/06
	Solutions to Homework 2	
Homework 3	sec 1.3: 6, 14, 20, 30, 36, 49. T/F questions on p 38: 20; sec 2.1 6, 10, 14. (This should be easier homework.)	2/8/06
	Solutions to Homework 3	
Homework 4	sec 2.1: 42; sec 2.2: 4 (draw a diagram), 26(a)-(d),30,44 (draw a diagram); sec 2.3: 20,30; sec 2.4: 4,14,28.	2/15/06
	Solutions to HW 4 Page 1 Page 2 Page 3	
Homework 5	sec 2.4: 40, 50, 52, 86; sec 3.1: 22, 24, 44; sec 3.2: 2, 16, 32.	2/22/06
	Solutions to HW 5 Page 1 Page 2 Page 3	
Half Homework 6	Sec 3.2: 22, 46; Sec 3.3: 16, 26, 30.	2/27/06
	Solutions to HW 6 Page 1	
Homework 7	Sec 3.4: 8, 10, 14 (read sec 3.4 through Def 3.4.1); Ch 3 T/F questions: 4, 14, 16; Sec 4.1: 4, 6, 20, 30. Solutions to HW 7	3/8/06
	Solutions to Midterm I Page 1 Page 2 Page 3 Page 4	
Homework 8	Sec 4.1: 36, 38; Sec 4.2: 6, 16, 20, 30, 56; Sec 4.3: 4, 6, 20. Solutions	3/15/06
Homework 9	Sec 4.3: 23, 24, 46, 60; Sec 5.1: 2, 6, 26. Sec 5.2: 6, 32. Solutions	3/22/06
Half Homework 10	Sec 5.3: 4, 8, 10, 22, 36 Solutions	3/27/06
Half Homework 11	Sec 5.5: 10; Sec 6.1: 8, 30, 34. Solutions	4/5/06
Homework 12	Sec 6.2: 2, 10, 12, 14, 30, 38; Sec 6.3: 2, 7, 24, 30.	

	Solutions	4/19/06
Homework 13	Sec 7.1: 10, 16, 18, 22, 34, 50; Sec 7.2: 10, 22(for n=2,3 only), 24, 26. Solutions	4/26/06
Half Homework 14	Sec 7.3: 6, 20, 36; Sec 7.4: 8, 16 Solutions	5/1/06

Your answers will be graded both on the correctness of the mathematics and on the way in which you have written it down. Always explain what you are doing.

IMPORTANT: Answers to "True or False" and "Show that..." problems MUST include a full justification.

For example... If you are given the statement "all odd integers are prime," first determine whether it is true or false. Then make the resulting claim, and give its justification. The following is acceptable:

CLAIM: Not all odd integers are prime. PROOF: Even though 9 is odd, it is not prime, since 9 = (3)(3).
(This is not the only possible answer -- you could use, say, 15 or 25 instead of 9. The point is that this is the level of precision expected in your answer.)

If you are given the statement "the product of two odd integers is odd," first determine whether it is true or false. Then make the resulting claim, and give its justification. The following is acceptable:

CLAIM: The product of any two odd integers is odd. PROOF: If m and n are odd integers, then m=2k+1 and n=2l+1 for some integers k and l, so mn = (2k+1)(2l+1) = 4kl+2k+2l+1 = 2(2kl+k+l)+1. Therefore mn is odd.

However, the following is NOT acceptable:
CLAIM: The product of any two odd integers is odd. PROOF: 5 and 7 are odd. (5)(7)=35, and 35 is odd.
(Since the claim concerns ANY pair of odd integers, not just 5 and 7, checking the statement for just one pair of odd integers does not constitute a valid answer.)

The only difference between a "True or False" problem and a "Show that..." problem is that with the latter, you are given a statement which is KNOWN to be true and then asked to justify it.

Review Sheet

Prob 1

same det same trace

let's find a matrix that proves the similarity (S)

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a+3c & b+3d \\ c & d \end{bmatrix} = \begin{bmatrix} a & 2a+b \\ c & 2c+d \end{bmatrix}$$

$$\therefore c=0, \quad d=\frac{2a}{3} \Rightarrow S = \begin{bmatrix} a & b \\ 0 & \frac{2a}{3} \end{bmatrix}$$

confirm,

$$AS = \begin{bmatrix} a & b+2a \\ 0 & \frac{2a}{3} \end{bmatrix}, \quad SB = \begin{bmatrix} a & b+2a \\ 0 & \frac{3a}{2} \end{bmatrix} \Rightarrow \text{ok.}$$

Prob2 $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow (\lambda-1)(1-\lambda^2)=0 \Rightarrow \lambda = 1, \pm 1$

$E_{-1} \longrightarrow$ one dim $\quad E_{-1} = \text{Ker}\begin{bmatrix} +1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & +1 \end{bmatrix} = \text{Span}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$E_{+1} = \text{Ker}\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\} \to 2\text{ dim}$

$\because$ A is 3×3 matrix and E_{-1}, E_{+1} add up to 3 dim

$\therefore$ A is diagonalizable

$$S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad \text{Det } S = 2$$

$$S^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\therefore D = S^{-1}AS = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Prob 3

i) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 2 & 0 & 2 \end{bmatrix} \longrightarrow (3-\lambda)\lambda(\lambda-3) = 0$

$\Rightarrow \lambda = 0, 3, 3$

ii) $\lambda = 0 \longrightarrow$ no free term in the characteristic eqn.

$\longrightarrow \det = 0 \longrightarrow A$ is non-invertible

iii) $\lambda = 3$ has algebric multiplicity of 2

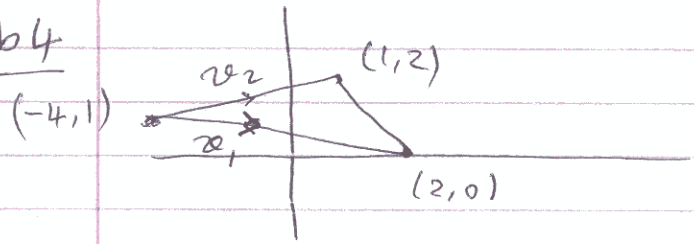
$\longrightarrow \lambda = 3$ " geometric " of 2 or 1

$\longrightarrow$ the matrix might not be diagonalizable

iv) if we have distinct λ's $\longrightarrow$ matrix is diagonalizable

Here, this is not the case $\longrightarrow$ more work is needed.

Prob 4



$v_1 = (2,0) - (-4,1) = (6,-1)$

$v_2 = (1,2) - (-4,1) = (5,1)$

$\therefore \text{Area} = \frac{1}{2} \operatorname{Det} \begin{vmatrix} 6 & 5 \\ -1 & 1 \end{vmatrix} = \frac{1}{2}(11) = 5.5$

Prob 5

i) we have $SA = BS$

$* A^{-1} \longrightarrow SAA^{-1} = BSA^{-1} \longrightarrow S = BSA^{-1}$

$* B^{-1} \longrightarrow B^{-1}S = B^{-1}BSA^{-1} = SA^{-1}$

$$B^{-1}S = SA^{-1}$$

$\therefore B^{-1}, A^{-1}$ are similar True

ii) False,

Consider $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

let $S = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$

$$AS = \begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}, \quad SB = \begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}$$

$AS = SB \Longrightarrow A, B$ are similar

iii) True Fact 7.3.6

iv) True Def. 7.4.2

v) True,

Consider, $A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$, $S = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$

$SA = \begin{pmatrix} 2 & 2 \\ -4 & 4 \end{pmatrix}$ $BS = \begin{pmatrix} 2 & 2 \\ -4 & 4 \end{pmatrix} \Longrightarrow A, B$ are similar

Prob 6

$$A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \longrightarrow (1-\lambda)^2 - 4 = 0 \implies \lambda = 3, -1$$

$$E_3 = \text{Ker}\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} = \text{span}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$E_{-1} = \text{Ker}\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} = \text{span}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\therefore S = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} \longrightarrow S^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}$$

$$\therefore D = S^{-1}AS = \frac{1}{4}\begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A^{10} = S D^{10} S^{-1} = \cancel{\frac{1}{4}\begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3^{10} & 0 \\ 0 & (-1)^{10} \end{pmatrix}\begin{pmatrix} & \\ & \end{pmatrix}}$$

$$= \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3^{10} & 0 \\ 0 & (-1)^{10} \end{pmatrix}\left(\frac{1}{4}\right)\begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 2 \cdot 3^{10} + 2 & 4 \cdot 3^{10} - 4 \\ 3^{10} - 1 & 2 \cdot 3^{10} + 2 \end{pmatrix}$$

Prob 7

let $f(t) = at^2 + bt + c$

$Tf = (t+1) f'(t) = (t+1)(2at+b) = 2at^2 + (2a+b)t + b$

if the transformation result is $\lambda f \Rightarrow f$ is eigenvector

i.e., $Tf = \lambda f$

$2at^2 + (2a+b)t + b = \lambda (at^2 + bt + c)$

$= \lambda a t^2 + \lambda b t + \lambda c$

Let's study the possible solutions of this equality
Compare λ coefficients on both sides,

(*)

$2a = \lambda a \longrightarrow \lambda = 2$

$2a + b = \lambda b \longrightarrow 2a + b = 2b \longrightarrow b = 2a$

$b = \lambda c \longrightarrow c = \frac{b}{2} = a$

$\Rightarrow f(t) = at^2 + 2at + a \quad , \lambda = 2$

(*)(*) $2a = \lambda a \longrightarrow a = 0$

take $\lambda = 1 \Rightarrow b = c$

$\therefore f(t) = 0t^2 + bt + b \quad , \lambda = 1$

(*)(*)(*) $2a = \lambda a \longrightarrow a = 0$

take $\lambda = 0 \longrightarrow f(t) = 0t^2 + 0t + c \quad , \lambda = 0$

ii) Three distinct eigenvalues $\longrightarrow$ there is an eigenbasis

iii) Let's apply the matrix formalism and see if we can obtain the same results in (i),

$$T(at^2+bt+c) = 2at^2+(2a+b)t+b$$

or

$$T\begin{pmatrix} c \\ b \\ a \end{pmatrix} = \begin{pmatrix} b \\ 2a+b \\ 2a \end{pmatrix}$$

$\therefore$ T can be $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} \longrightarrow \lambda = 0, 1, 2$

$E_0 = \text{span}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\qquad E_1 = \text{span}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$E_2 = \text{span}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

Same resullts like those in (i)

Prob 8

Method 1,

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ y \\ -2x-3y \\ x+4y \end{pmatrix} \longrightarrow \underbrace{x\begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ -3 \\ 4 \end{pmatrix}}_{2\ \text{dim}}$$

Method 2, Dusa's

$$\begin{bmatrix} 2 & 3 & 1 & 0 \\ 1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = 0$$

Get the Rref form,

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 5 & -1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 4/5 & 3/5 \\ 0 & 1 & -1/5 & -2/5 \end{bmatrix}$$

take z, t as parameters,

$$\therefore y = \tfrac{1}{5}z + \tfrac{2}{5}t \quad , \quad x = -\tfrac{4}{5}z - \tfrac{3}{5}t$$

$$\therefore \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \longrightarrow \begin{pmatrix} -4/5\, z - 3/5\, t \\ 1/5\, z + 2/5\, t \\ z \\ t \end{pmatrix} \longrightarrow \underbrace{z\begin{pmatrix} -4/5 \\ 1/5 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3/5 \\ 2/5 \\ 0 \\ 1 \end{pmatrix}}_{2\ \text{dim}}$$

You may easily check that the space in both methods is the same

Prob 9

i) let $T \equiv \begin{bmatrix} \underline{v}_1 & \underline{v}_2 & \underline{v}_3 \end{bmatrix}$ $\qquad \underline{v}_3 = \underline{v}_1 + \underline{v}_2$

$\therefore$ Img basis $=$ Span $\{\underline{v}_1, \underline{v}_2\}$ $\longrightarrow$ 2 dim

ii) $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$

This eqn describes a $(n-1)$ dimensional surface in n dimensional space.

Here, we have 4-dim space, but our surface is only 2 dim (since it has only two basis vectors)

$\Longrightarrow$ It's not possible to write $Im(T)$ in this format.

Prob 10

i) $A^{-1} = \begin{bmatrix} 3 & 1/3 & -1 \\ 2 & 2/3 & -1 \\ -2 & -1/3 & 1 \end{bmatrix}$

ii) remember, $(A^T)^{-1} = (A^{-1})^T$

Prob 11

if $A = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$$A^TA = \begin{pmatrix} v_1 v_1 & v_1 v_2 & v_1 v_3 \\ v_2 v_1 & v_2 v_2 & v_2 v_3 \\ v_3 v_1 & v_3 v_2 & v_3 v_3 \end{pmatrix}$$

in this problem, $\underline{v}_1, \underline{v}_2, \underline{v}_3$ are mutually orthogonal

$\Rightarrow$ off diagonal elements vanish

$$\therefore A^TA = \begin{pmatrix} v_1^2 & 0 & 0 \\ 0 & v_2^2 & 0 \\ 0 & 0 & v_3^2 \end{pmatrix}$$

Prob 12

i) $\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right)$

$\Rightarrow X_3 = -1 \quad , \quad X_1 + X_2 = 3$

let $x_1 = t$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ 3-t \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

ii) Direction vector is $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

iii) $$\underline{n}_1 \wedge \underline{n}_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

iv) $\therefore$ $\underline{n}_1 \wedge \underline{n}_2$ is parallel to the direction vector

This is logical since the intersection ~~point~~ line is normal to both planes

$\therefore$ Direction vector $\parallel (\underline{n}_1 \wedge \underline{n}_2)$

Prob 13

i) 2-fold dilation in X-component + a reflection in Y component

ii) SDS^{-1} has the same geometric effect but with respect to a different basis (basis described by the change of basis matrix S)

Prob 14

i) $\operatorname{Det}\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & -1 \\ 2 & 1 & 3 \end{vmatrix} = \text{zero} \Rightarrow$ The vectors are linearly dependent

ii) 2-dim

iii) Any two vectors form a basis.

Prob (15)

i) v_1 is parallel to the line. v_2 is perpendicular

ii) in page (216) $QQ^T = [u_1\ u_2]$

$$\therefore QQ^T = \begin{pmatrix}1\\0\end{pmatrix}(1\ \ 0) = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$$

iii) $u_1 = \frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\end{bmatrix} \Rightarrow QQ^T = \frac{1}{5}\begin{pmatrix}2\\1\end{pmatrix}(2\ \ 1) = \frac{1}{5}\begin{pmatrix}4 & 2\\2 & 1\end{pmatrix}$

Prob (16)

i) $\begin{pmatrix}x\\y\\z\end{pmatrix} \longrightarrow \begin{pmatrix}x\\y\\-2x\end{pmatrix} \longrightarrow x\underbrace{\begin{pmatrix}1\\0\\-2\end{pmatrix}}_{\text{basis (but it's not normalized)}} + y\begin{pmatrix}0\\1\\0\end{pmatrix}$

$$\therefore \underline{u}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\0\\-2\end{pmatrix}, \quad \underline{u}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

ii) one vector

iii) $\underline{v}_3 = \underline{u}_1 \wedge \underline{u}_2 = \frac{1}{\sqrt{5}}\begin{vmatrix} i & j & k\\ 1 & 0 & -2\\ 0 & 1 & 0\end{vmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\0\\1\end{pmatrix} \quad \therefore \underline{u}_3 = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\0\\1\end{pmatrix}$

(*) Another way is to use Gram-Schmidt procedure,

$$\underline{v}_3^{\perp} = \underline{v}_3 - (\underline{u}_1 \cdot \underline{v}_3)\,\underline{u}_1 - (\underline{u}_2 \cdot \underline{v}_3)\,\underline{u}_2$$

let $\underline{v}_3 = \begin{pmatrix}1\\1\\1\end{pmatrix} \Rightarrow \underline{v}_3^{\perp} = \frac{3}{5}\begin{pmatrix}2\\0\\1\end{pmatrix}$ normalize this, $\underline{u}_3 = \begin{pmatrix}2/\sqrt{5}\\0\\1/\sqrt{5}\end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\0\\1\end{pmatrix}$

iv) notice that $(2,0,1)$ is normal to the plane,

$\Rightarrow$ it has only one coordinate in the perpendicular direction (i.e. $\underline{u}_3$). And the coordinate value is the vector length $(\sqrt{5})$

Math 211.01 Review for Final

The final will be cumulative, with a slight extra emphasis on the last two chapters. You are expected to know (almost – see below) all the topics that I listed on the other two review sheets. In addition:

• Chapter 6.1 through Fact 6.1.6; Ch 6.2 through Fact 6.2.8; much of Chapter 6.3 but not Fact 6.3.4, 6.3.6, 6.3.7, 6.3.8.

• Chapter 7.1 (except for p 300, 301); all of Ch 7.2, 7.3 and 7.4 (except for the infinite dimensional examples).

• We omitted Ch 3.4, which on retrospect was a mistake since we have used the ideas a lot. Some good exercises from this section: 25, 27, 37, 41, 43, 59, 60, 63. There will not be an explicit question from this section on the exam.

• Some topics the final exam will NOT contain: Sec 5.5; the QR factorization; row operations expressed as multiplication by elementary matrices (as in ex 50-52 in Ch 2.4); the change of basis formula in Fact 4.3.5.

• You should be able to calculate dot products, cross products, area of triangles, determinants, products of matrices. You should also know what orthogonal matrices are and about similar matrices.

Review problems:

Note: The actual exam will mostly contain simple questions like 4, 6 and 8. Only a few will ask for explanations as in 3 and 9. But I think these are good review problems.

1. Show that the matrices $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ are similar.

2 Is $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ diagonalizable? If so, find matrices S, D, where D is diagonal, such that $AS = SD$.

3. (i) Find the characteristic polynomial of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 2 & 0 & 2 \end{bmatrix}$.

(ii) Notice that 0 is a root. What does this tell you about A?

(iii) What do the other roots tell you about A?

(iv) Can you tell from this polynomial that A is diagonalizable, or would you have to do more work to find this out?

4. (i) Use a determinant to find the area of the triangle in $\mathbb{R}^2$ with vertices at $A = (2, 0), B = (1, 2)$ and $C = (-4, 1)$. Draw a diagram.

(ii) Calculate $\det(A)$ where $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 \\ 3 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix}$. Is A invertible?

5. (This question is a bit too hard for the exam, but is a good review question, I think.) Suppose that A and B are similar, invertible $n \times n$ matrices. Are the following statements true or false? Give a reason for your answer or a counterexample. (If you are unsure of an answer, try some 2×2 examples.)

(i) A^{-1} is similar to B^{-1}.

(ii) It is impossible for A to be upper triangular (with some nonzero entries above the diagonal) while B is diagonal.

(iii) A and B have the same trace and the same determinant.

(iv) If A is diagonalizable so is B.

(v) It is possible for A to have zeros on the diagonal while B is diagonal.

6. Let $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$. Find A^{10}. (**Hint**: diagonalize A.)

7. Let $V = P_2$ the space of polynomials of degree ≤ 2. Define $T : V \to V$ by $(Tf)(t) = (t+1)f'(t)$.

(i) Find all eigenvalues and eigenvectors of T by finding all solutions to the equation $Tf = \lambda f$, for $f \in P_2$.

(ii) Does T have an eigenbasis?

(iii) Find the matrix of T with respect to the standard basis $\mathcal{B} = (1, t, t^2)$, and use it to check your answers to (i) and (ii) above.

8. (i) Find a basis for the subspace

$$W = \{(x, y, z, t) \in \mathbb{R}^4 : 2x + 3y + z = 0, x - y + z + t = 0.\}$$

(**Hint**: Express this as the kernel of a certain matrix A, and then find this kernel.)

(ii) What is $\dim W$?

9. Define $T : \mathbb{R}^3 \to \mathbb{R}^4$ by $T\vec{x} = A\vec{x}$ where $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 3 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix}$.

(i) Find a basis for the image of T. What is its dimension?

(ii) Is it possible to write $\mathrm{im}(T)$ as the solution set of a single linear equation $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$? Explain your answer.

10. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 3 \\ 2 & 1 & 4 \end{bmatrix}$.

(i) Calculate A^{-1} using row reduction.

(ii) Use your answer to (i) to find the inverse of A^T. Check your solution.

11. Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & -1 \end{bmatrix}$. Calculate $A^T A$. Interpret your result in terms of the columns of A.

12. (i) Find the line of intersection of the planes $x_1+x_2+2x_3 = 1$ and $x_1+x_2+x_3 = 2$.
(ii) What is its direction vector?

(iii) Find the cross product of the normals to these planes.

(iv) Compare your answers to (ii) and (iii) and explain what you find.

13. Let $D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ and let $T_D : \mathbb{R}^2 \to \mathbb{R}^2, \vec{x} \mapsto D(\vec{x})$ be the corresponding transformation.
(i) Describe T_D geometrically.
(ii) Let $S = \begin{bmatrix} 1 & 4 \\ -4 & 1 \end{bmatrix}$. Describe the transformation $T_{SDS^{-1}}$ geometrically.

14. (i) Are the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

linearly dependent or independent?

(ii) What is the dimension of the space they span.?

(iii) Find a basis for the space they span.

15. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the orthogonal projection onto the line

$$L = \{(x, y) \in \mathbb{R}^2 : x - 2y = 0\}.$$

(i) Let

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

How are these vectors related to L? Draw a diagram.
(ii) Find the matrix P of T with respect to the basis $\mathcal{B} := (\vec{v}_1, \vec{v}_2)$.

(iii) Find the matrix A of T with respect to the standard basis of $\mathbb{R}^2$.

16. (i) Find an orthonormal basis for the subspace

$$V = \{(x, y, z) \in \mathbb{R}^3 : 2x + z = 0\}.$$

(ii) How many vectors do you need to add to get an orthonormal basis for $\mathbb{R}^3$?

(iii) Add vectors to the basis found in (i) to get an orthonormal basis $\mathcal{B}$ for $\mathbb{R}^3$.

(iv) You should find that the coordinates of the vector $\vec{x} = (2, 0, 1)$ with respect to $\mathcal{B}$ are:

$$[\vec{x}]_{\mathcal{B}} == \begin{bmatrix} 0 \\ 0 \\ \sqrt{5} \end{bmatrix}.$$

Why?

Extra HW Solutions

Prob. 1:

i) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2y - 3z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$. So the basis is , $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$. The choice of the basis is not unique but it will be always 2 dimensional

ii) $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = -1\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - 1\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ Hence, the vector is expressed as $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$

Prob. 2:

Testing the linearity,

i)

$$T(aM_1 + bM_2) = aM_1 + bM_2 - I$$

$$aT(M_1) + bT(M_2) = aM_1 - aI + bM_2 - bI$$

They are not equal → not linear

ii) $T(aM_1 + bM_2) = aM_1J + bM_2J = aT(M_1) + bT(M_2)$ → linear transformation

If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $T(M) = \begin{bmatrix} 0 & a+2b \\ 0 & c+2d \end{bmatrix} = \begin{bmatrix} 0 & x_1 \\ 0 & x_2 \end{bmatrix}$. This is the image

The rank is 2 since we have two basis matrices, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Prob. 3:

i) $T(t^2) = (1-t)2t = 2t - 2t^2$

ii) $T(1) = (1-t)(0) = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad T(t) = (1-t) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\quad T(t^2) = 2t - 2t^2 = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$

$$\Rightarrow TransformationMatrix = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

Prob. 4:

i) Check the book.

ii) The normalization condition ; $c^2 + k^2 = 1$

The orthogonality condition ; $-\frac{c}{\sqrt{5}} + \frac{2k}{\sqrt{5}} = 0 \Rightarrow k = \frac{c}{2}$

$c^2 + \frac{c^2}{4} = 1 \Rightarrow c = \pm\frac{2}{\sqrt{5}} and , k = \pm\frac{1}{\sqrt{5}}$

Therefore, the matrix A could be either , $\begin{bmatrix} \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} or \begin{bmatrix} \frac{-1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix}$

Prob. 5:

a) $v_1 . v_2 = 2 + 1 - 3 = 0 \Rightarrow orthogonal$

b) The vectors are orthogonal already → enough to normalize them ,

$$u_1 = \frac{(2,1,3)}{\sqrt{14}} and , u_2 = \frac{(1,1,-1)}{\sqrt{3}}$$

c)

$$Orthogonal projection = (u_1.x)u_1 + (u_2.x)u_2$$

$$= \left(\frac{(2,1,3)}{\sqrt{14}}.\begin{bmatrix}0\\1\\1\end{bmatrix}\right)\frac{1}{\sqrt{14}}\begin{bmatrix}2\\1\\3\end{bmatrix} + \left(\frac{(1,1,-1)}{\sqrt{3}}.\begin{bmatrix}0\\1\\1\end{bmatrix}\right)\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\end{bmatrix}$$

$$= \frac{4}{14}\begin{bmatrix}2\\1\\3\end{bmatrix} + 0 = \frac{1}{7}\begin{bmatrix}4\\2\\6\end{bmatrix}$$

Notice that the vector (0,1,1) is orthogonal to one of the basis vectors (v2), therefore, the projection has no component in this direction

$$\|x\|^2 = 2$$

$$\|y\|^2 = \frac{56}{49}$$

d) $x - y = (0,1,1) - \frac{1}{7}(4,3,6) = \frac{1}{7}(-4,5,1)$

$$\|x - y\|^2 = \frac{16 + 25 + 1}{49} = \frac{42}{49}$$

$$\|y\|^2 + \|x - y\|^2 = \frac{56}{49} + \frac{42}{49} = \frac{98}{49} = 2 = \|x\|^2$$

Math 211.01: Second Midterm

March 29, 2006

Name: **School ID:**

Answer all the following questions, justifying all your statements. Write neatly so that we can read and follow your answers. You are not allowed to use calculators, and please turn off cell phones. Use the back of the exam for scrap. There are five questions. Good luck!

Problem 1. (20 points) Let $V = \mathbb{R}^{2\times 2}$. the 2×2 matrices. Which of the following transformations $T : V \to V$ are linear? Explain your answer. If T is linear, describe its image and rank.

(i) $T(M) = M^2$.

(ii) $T(M) = JM - MJ$, where $J := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

1	20pt	
2	20pt	
3	20pt	
4	20pt	
5	20pt	
Total	100pt	

Problem 2. (20 points) (i) Find a basis for the subspace $V := \{(x, y, z, t) \in \mathbb{R}^4 : x - y + 2z + t = 0\}$.

(ii) Find the coordinates of the vector $\vec{x} = (1, 1, -1, 2)$ with respect to this basis.

Problem 3. (20 points) Let $V = P_2$ the space of polynomials of degree ≤ 2. Define $T : V \to V$ by $T(f) = (1+t)f'(t)$. Consider the basis $\mathcal{B} := (1+t, t, 1+t^2)$.

(i) Find the coordinates $[f]_{\mathcal{B}}$ of $f(t) = a + bt + ct^2$ with respect to this basis.

(ii) Check that the coordinates of $f = 1+t$ with respect to this basis are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Why is this so?

(iii) Find the matrix that represents T with respect to this basis.

Problem 4. (20 points) (i) What is an orthogonal matrix?

(ii) Are there constants c, k such that the following matrix is orthogonal? Justify your answer (and calculate c, k if they exist.)

$$A = \begin{bmatrix} 1/\sqrt{3} & 1\sqrt{3} & -1\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & c & k \end{bmatrix}$$

Problem 5. (20 points) Let V be the subspace of $\mathbb{R}^4$ spanned by the vectors $\vec{v}_1 = (1,0,1,2)$ and $\vec{v}_2 = (1,1,1,-1)$.

(i) Check that $\vec{v}_1, \vec{v}_2$ are orthogonal.

(ii) Find an orthonormal basis for V.

(ii) Find the orthogonal projection $\vec{y}$ of $\vec{x} := (0,0,1,1)$ onto V.

(iii) Check that $\|\vec{x}\|^2 = \|\vec{y}\|^2 + \|\vec{x} - \vec{y}\|^2$.

Math 211.01: Second Midterm

March 29, 2006

Name: **School ID:**

Answer all the following questions, justifying all your statements. Write neatly so that we can read and follow your answers. You are not allowed to use calculators, and please turn off cell phones. Use the back of the exam for scrap. There are five questions. Good luck!

Problem 1. (20 points) (i) Find a basis for the subspace $V := \{(x, y, z, t) \in \mathbb{R}^4 : x - 2y + z + t = 0\}$.

$v_1 = [1\ 0\ 0\ -1]$

$v_2 = [0\ 1\ 0\ 2]$

$v_3 = [0\ 0\ 1\ -1]$

$v_1, v_2, v_3 \in V$

they are indep as

$av_1 + bv_2 + cv_3 = (a, b, c, -a+2b-c)$

$\neq 0$ unless $a, b, c = 0$

10 for answer

5 for explanation

they span as a,b,c can be anything & t is det by a,b,c

(ii) Find the coordinates of the vector $\vec{x} = (1, 1, -1, 2)$ with respect to this basis.

5

~~the~~ coords are $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

check $-1 + 2 + (-1) = 2$ = last coord.

1	20pt	
2	20pt	
3	20pt	
4	20pt	
5	20pt	
Total	100pt	

Problem 2. (20 points) Let $V = \mathbb{R}^{2\times 2}$, the 2×2 matrices. Which of the following transformations $T : V \to V$ are linear? Explain your answer. If T is linear, describe its image and rank.

(i) $T(M) = M^2$.

(ii) $T(M) = MJ - JM$, where $J := \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

1) T is not linear. eg $T\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

8

$T\left(3\cdot\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right) = \begin{pmatrix} 36 & 0 \\ 0 & 36 \end{pmatrix} \neq 3\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

2). T is linear. Since $(A+B)J - J(A+B) = AJ - JA + BJ - JB$

$(kA)J - J(kA) = k(AJ - JA)$

12

$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} -a & -b \\ c & d \end{bmatrix} - \begin{bmatrix} -a & b \\ -c & d \end{bmatrix} = \begin{bmatrix} 0 & -2b \\ 2c & 0 \end{bmatrix}$

$\therefore$ Im T = {all matrices that = 0 on diagonal}

Im T has dim 2 — with basis $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$\therefore$ rank T = 2

Problem 3. (20 points) Let $V = P_2$ the space of polynomials of degree ≤ 2. Define $T : V \to V$ by $T(f) = (1-t)f'(t)$. Consider the basis $\mathcal{B} := (1-t, t, 1-t^2)$.

(i) Find the coordinates $[f]_{\mathcal{B}}$ of $f(t) = a + bt + ct^2$ with respect to this basis.

~~8~~ 10

Want x, y, z s.t $\quad a + bt + ct^2 = x(1-t) + yt + z(1-t^2)$

$\therefore z = -c \qquad x + z = a \qquad$ so $\quad x = a + c$

$-x + y = b \quad \therefore \quad y = b + a + c$

$$\therefore \text{coords} = \begin{bmatrix} a+c \\ a+b+c \\ -c \end{bmatrix}$$

(ii) Check that the coordinates of $f = 1 - t$ with respect to this basis are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Why is this so?

2

if $a = 1, b = -1, c = 0$ we get $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

ie $\quad 1 - t = (1-t) + 0 \cdot t + 0 \cdot (1-t^2)$

because $1-t$ is the first basis element.

(iii) Find the matrix that represents T with respect to this basis.

~~10~~ 8

$$B = \begin{bmatrix} [Tf_1]_{\mathcal{B}} & [Tf_2]_{\mathcal{B}} & [Tf_3]_{\mathcal{B}} \end{bmatrix}$$

$Tf_1 = -1 + t \qquad Tf_2 = 1 - t$

$Tf_3 = (1-t)(-2t) = -2t + 2t^2 \quad$ has coords $\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$

$$\therefore B = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

5

Problem 4. (20 points) (i) What is an orthogonal matrix?

- A is orthogonal if it is $n \times n$ and $A^T A = I$

or A is $n \times n$ and $\|Ax\| = \|x\|$ for all x

or the cols of A form orthonormal basis

(ii) Are there constants c, k such that the following matrix is orthogonal? Justify your answer (and calculate c, k if they exist.)

15

$$A = \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{6} & c & k \end{bmatrix}$$

Let v_1, v_2, v_3 be cols of A $\quad \|v_1\|^2 = \frac{1}{3} + \frac{1}{2} + \frac{1}{6} = 1$

$v_1 \cdot v_3 = 0 \iff -\frac{1}{3} + \frac{k}{\sqrt{6}} = 0 \quad \therefore k = \frac{\sqrt{6}}{3} = \sqrt{\frac{2}{3}}$

Then $\|v_3\|^2 = \frac{1}{3} + \frac{2}{3} = 1.$

$v_2 \cdot v_3 = 0 \iff \frac{1}{3} + c \cdot \sqrt{\frac{2}{3}} = 0 \quad \therefore$ need $c = -\sqrt{\frac{3}{2}} \cdot \frac{1}{3}$

$= -\frac{1}{\sqrt{6}}$

then $\|v_2\|^2 = \frac{1}{3} + \frac{1}{2} + \frac{1}{6} = 1$

also need $v_1 \cdot v_2 = 0$ ie $-\frac{1}{3} + \frac{1}{2} + \frac{c}{\sqrt{6}} = \frac{1}{6} - \frac{1}{6} = 0$ ✓

$\therefore k = \sqrt{\frac{2}{3}}$, $c = -\frac{1}{\sqrt{6}}$ give orthog matrix.

Problem 5. (20 points) Let V be the subspace of $\mathbb{R}^4$ spanned by the vectors $\vec{v}_1 = (1,0,2,1)$ and $\vec{v}_2 = (1,1,-1,1)$.

(i) Check that $\vec{v}_1, \vec{v}_2$ are orthogonal.

(ii) Find an orthonormal basis for V.

(iii) Find the orthogonal projection $\vec{y}$ of $\vec{x} := (0,0,1,1)$ onto V.

(iv) Check that $\|\vec{x}\|^2 = \|\vec{y}\|^2 + \|\vec{x} - \vec{y}\|^2$.

(i) 2 ✓5

(ii) 4 ✓5

(iii) 9 ✓5

(iv) 5 ✓5.

i) $\underline{v}_1 \cdot \underline{v}_2 = 1 - 2 + 1 = 0$

ii) They are already orthogonal $\Rightarrow$ They only need to be normalized,

$$\underline{u}_1 = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\0\\2\\1\end{pmatrix} \qquad \underline{u}_2 = \frac{1}{2}\begin{pmatrix}1\\1\\-1\\1\end{pmatrix}$$

iii) $\underline{y} = \mathrm{Proj}_V(\underline{x}) = (\underline{u}_1 \cdot \underline{x})\underline{u}_1 + (\underline{u}_2 \cdot \underline{x})\underline{u}_2$

$$= \begin{pmatrix}1/2\\0\\1\\1/2\end{pmatrix} \Rightarrow \|\underline{y}\|^2 = \frac{6}{4}$$

iv) $$\underline{x} - \underline{y} = \begin{pmatrix}0\\0\\1\\1\end{pmatrix} - \begin{pmatrix}1/2\\0\\1\\1/2\end{pmatrix} = \begin{pmatrix}-1/2\\0\\0\\1/2\end{pmatrix} \Rightarrow \|\underline{x} - \underline{y}\|^2 = \frac{2}{4}$$

$\frac{8}{4} = 2$

$$\underline{x} = \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \Rightarrow \|\underline{x}\|^2 = 2$$

#

Math 211.01: Extra homework

due April 19, 2006

Name: **School ID:**

Answer all the following questions, justifying all your statements. This is an easier version of Midterm 2.

Problem 1. (20 points) (i) Find a basis for the subspace $V := \{(x, y, z) \in \mathbb{R}^3 : x - 2y + 3z = 0\}$.
(ii) Find the coordinates of the vector $\vec{x} = (1, -1, -1)$ with respect to this basis.

Problem 2. (20 points) Let $V = \mathbb{R}^{2\times 2}$. the 2×2 matrices. Which of the following transformations $T : V \to V$ are linear? Explain your answer. If T is linear, describe its image and rank.

(i) $T(M) = M - I$ where $I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(ii) $T(M) = MJ$ where $J := \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$.

Problem 3. (20 points) Let $V = P_2$ the space of polynomials of degree ≤ 2. Define $T : V \to V$ by $T(f) = (1 - t)f'(t)$. Consider the standard basis $\mathcal{B} := (1, t, t^2)$.
(i) What is $T(t^2)$?
(ii) Find the matrix that represents T with respect to the basis $\mathcal{B}$.

Problem 4. (20 points) (i) What is an orthogonal matrix?

(ii) Find constants c, k so that the following matrix is orthogonal.

$$A = \begin{bmatrix} -1/\sqrt{5} & c \\ 2/\sqrt{5} & k \end{bmatrix}$$

Problem 5. (20 points) Let V be the subspace of $\mathbb{R}^3$ spanned by the vectors $\vec{v}_1 = (2, 1, 3)$ and $\vec{v}_2 = (1, 1, -1)$.

(i) Check that $\vec{v}_1, \vec{v}_2$ are orthogonal.

(ii) Find an orthonormal basis for V.

(ii) Find the orthogonal projection $\vec{y}$ of $\vec{x} := (0, 1, 1)$ onto V.

(iii) Check that $\|\vec{x}\|^2 = \|\vec{y}\|^2 + \|\vec{x} - \vec{y}\|^2$.

Prob 1

i) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a+d=1 \Rightarrow \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$

Can we find neutral element? NO, a and $(1-a)$ cannot both be zero $\Rightarrow$ Not a subspace

ii) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a+d=0 \Rightarrow \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$

There is a "zero" element, and closed under addition and multiplication $\longrightarrow$ a subspace

Prob 2

i) That means,

$$C = xA_1 + yA_2 + zA_3 = \begin{bmatrix} x & x-y \\ 0 & y+z \end{bmatrix}$$

ii) let,

$$\begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} x & x-y \\ 0 & y+z \end{bmatrix} \Rightarrow x=2,\ y=-1,\ z=5$$

$$= C = (2,-1,5)_B$$

Prob 3

i) $A^T = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix}$

ii) let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \Rightarrow \|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$

$$TX = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_4 \\ x_1 \\ x_3 \end{bmatrix}$$

$\|TX\| = \sqrt{x_2^2 + x_4^2 + x_1^2 + x_3^2} = \|x\| \longrightarrow T$ is orthogonal

since T is orthogonal, $T^{-1} = T^T$

$$\therefore B^{-1} = T^{-1} = T^T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

check!

$$B^{-1} \cdot B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Prob 4

i) let $(1,1,3)$ and $(0,1,2)$ be two linearly independent vectors on the plane,

$$\underline{u}_1 = \frac{1}{\sqrt{11}} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \qquad \underline{v}_2^{\perp} = \underline{v}_2 - (\underline{u}_1 \cdot \underline{v}_2)\underline{u}_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \frac{7}{11} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \frac{1}{11} \begin{pmatrix} -7 \\ 4 \\ 1 \end{pmatrix}$$

$$\hat{u}_2 = \frac{v_2^\perp}{\|v_2^\perp\|} = \frac{1}{\sqrt{66}}\begin{pmatrix}-7\\4\\1\end{pmatrix}$$

$\therefore \frac{1}{\sqrt{66}}\begin{pmatrix}-7\\4\\1\end{pmatrix}$ and $\frac{1}{\sqrt{11}}\begin{pmatrix}1\\1\\3\end{pmatrix}$ are the orthonormal basis of the plane $x+2y-z=0$. obviously, this choice is not unique.

ii) $x = \begin{pmatrix}1\\1\\2\end{pmatrix}$ $\qquad Y = Proj(X) = (\underline{u}_1 \cdot \underline{x})\underline{u}_1 + (\underline{u}_2 \cdot \underline{x})\underline{u}_2$

$$Y = \frac{8}{11}\begin{pmatrix}1\\1\\3\end{pmatrix} - \frac{1}{66}\begin{pmatrix}-7\\4\\1\end{pmatrix} = \begin{pmatrix}55/66\\44/66\\143/66\end{pmatrix}$$

iii) $X - Y = \begin{pmatrix}11/66\\22/66\\-11/66\end{pmatrix} = \frac{1}{6}\begin{pmatrix}1\\2\\-1\end{pmatrix}$

From the plane eqn., the normal to the plane is, $(1,2,-1) \Rightarrow (X-Y)$ is parallel to the normal to the plane $\Rightarrow (X-Y)$ is orthogonal to the plane.

Prob 5

i) $T(f_1+f_2) = f_1' + f_2' - f_1(3) - f_2(3)$

$= f_1' - f_1(3) + f_2' - f_2(3) = T(f_1) + T(f_2)$

$T(kf) = (kf)' - kf(3) = k\left(f' - f(3)\right) = kT(f)$

$\longrightarrow T$ is linear

ii) let $f(t) = at^2 + bt + c$

$T(f) = f' - f(3) = 2at + b - 9a - 3b - c$

$= 2a(t) + (-9a - 2b - c)$

$$\Rightarrow T\begin{pmatrix} c \\ b \\ a \end{pmatrix} = \begin{pmatrix} -9a-2b-c \\ 2a \\ 0 \end{pmatrix}_{1,t,t^2} \qquad (*)$$

$$\operatorname{Img} T = \underbrace{(-9a-2b-c)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2a\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\text{Img basis}} \longrightarrow \text{rank} = 2$$

$(*)$ leads to $T = \begin{pmatrix} -1 & -2 & -9 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}_{1,t,t^2}$

to find the Kernel of T, solve the system,

$$\begin{pmatrix} -1 & -2 & -9 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow z = 0,\ y = t,\ x = -2t$$

$\therefore$ Kernel $T = \begin{pmatrix} -2t \\ t \\ 0 \end{pmatrix} = t\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \longrightarrow \text{Dim} = 1$

rank + nullity $= 2+1 = 3 \longrightarrow$ same dimension of the space P_2

iv) $T = \begin{pmatrix} -1 & -2 & -9 \\ 0 & 0 & 2 \\ 0 & 0 & 6 \end{pmatrix}$

Prob 6

i) $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}_A$

ii) $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \rightarrow \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}_A$

iii) $a\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} + b\begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a-b \\ a+b \end{pmatrix}$

iv) $\begin{pmatrix} a \\ b \end{pmatrix}$

v) $S_{B\to A} = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$, check, $S_{B\to A}\begin{pmatrix} a \\ b \end{pmatrix}_B = \frac{1}{2}\begin{pmatrix} a-b \\ a+b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}_A$

#

Math 211.01 Review for Midterm 2

• The exam will be based on Ch 4 (everything except for Facts 4.3.4 and 4.3.5) and Ch 5, secs 1 through 3 (everything except for Facts 5.1.9, 5.1.10, 5.1.11, correlations (cf Def 5.1.13), the QR factorization, Facts 5.3.9 and 5.3.10.).

• You are expected to be able to work with the definitions of **linear space, subspace, basis and dimension of a linear space, coordinates; linear transformations (image, kernel, rank, nullity); isomorphisms; the $\mathcal{B}$-matrix of a linear transformation, and change of basis matrix. Orthogonality and length, orthonormal bases, orthogonal projection, angle, orthogonal transformation and matrix, transpose of matrix.**

• You are expected to memorize and be able to reproduce the easier definitions: **Def 4.1.3, 4.2.1, 4.2.2, Def 5.1.1, 5.1.2, 5.1.12, 5.3.1.**

• You are expected to know how to:

- decide whether a given set is a linear space;
- decide whether a given subset is a subspace;
- find a basis for a linear space;
- find the coordinates of an element $\vec{x} \in V$ in terms of a given basis $\mathcal{B}$ for V;
- decide whether a given function $T : V \to W$ is linear, and whether it is an isomorphism;
- find the matrix of a linear transformation in terms of a given basis $\mathcal{B}$;
- find the change of basis matrix;
- find an orthonormal basis for a given subspace of $\mathbb{R}^n$;
- find the orthogonal projection of an element $f \in \mathbb{R}^n$ onto a given subspace $V \subset \mathbb{R}^n$;
- find the angle between two given vectors;
- decide if a given linear transformation is orthogonal;
- find the transpose of a matrix A and the inverse of an orthogonal matrix.‘

Sample questions Note: you must EXPLAIN all your answers. The exam will have about 5 questions similar to those given below.
1: Let $V = \mathbb{R}^{2\times 2}$, the linear space consisting of all 2 by 2 matrices. Which of the following are subspaces of V?
(i) The set of matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a + d = 1$;
(ii) The set of matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a + d = 0$.

2: Let $V = U^{2\times 2}$ the space of upper triangular 2 by 2 matrices with basis

$$\mathcal{B} = \left\{ A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

(i) What does it mean to say that (x, y, z) are the coordinates of a matrix C with respect to $\mathcal{B}$?

(ii) Find the coordinates of the matrix $C = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$ with respect to $\mathcal{B}$.

3: (i) Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 3 \end{bmatrix}$. Find A^T.

(ii) Let B be the permutation matrix $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. Show that B is orthogonal. What is B^{-1}?

4 (i) Find an orthonormal basis for the subspace $x + 2y - z = 0$ of $\mathbb{R}^3$.

(ii) Find the orthogonal projection $\vec{y}$ of the vector $\vec{x} := (1, 1, 2)$ onto this subspace V.
(iii) Check that $\vec{x} - \vec{y}$ is perpendicular to V.

5 Let $V = P_2$ the space of polynomials of degree ≤ 2. Define $T : P_2 \to P_2$ by $T(f) = f' - f(3)$, where f' is the derivative of f.
(i) Show that T is linear.
(ii) What is the kernel of T? What is its image?
(iii) What are the rank and nullity of T? Why is their sum equal to 3?
(iv) Find the matrix of T with respect to the standard basis for P_2, i.e. $\mathcal{B} = (1, t, t^2)$.

6: Consider the linear space $V = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$ with the two bases:

$$\mathcal{B} = \left\{ f_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, f_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}, \quad \mathcal{A} = \left\{ v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\}.$$

(i) Find the $\mathcal{A}$-coordinates of f_1.

(ii) Find the $\mathcal{A}$-coordinates of f_2.

(iii) Find the $\mathcal{A}$-coordinates of $f = af_1 + bf_2$.

(iv) Find the $\mathcal{B}$-coordinates of $f = af_1 + bf_2$.

(v) Find the change of coordinates matrix $S_{\mathcal{B}\to\mathcal{A}}$.

Math 211.01 Notes on Vectors

Linear combinations of vectors (cf p 31 in Bretscher: This is useful background for sec 1.3 ex 6.)

A vector $\vec{b}$ is said to be a **linear combination** of the vectors $\vec{v_1}, \vec{v_2}$ if there are scalars x_1, x_2 such that $\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2$. For example,

$$\begin{bmatrix} -1 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

If we write $\vec{e}_1 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then any vector in $\mathbb{R}^2$ can be written as an linear combination of $\vec{e}_1, \vec{e}_2$ as follows:

$$\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

See Fig 1.

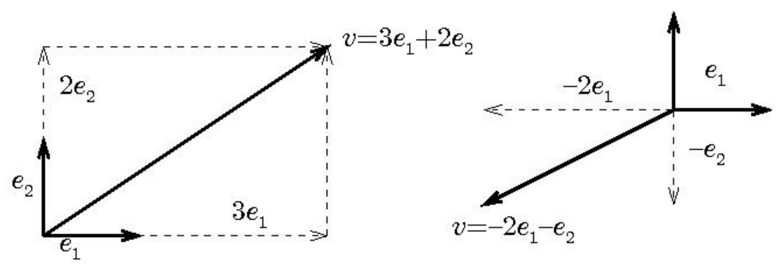


FIGURE 1. Writing $\vec{v}$ as a linear combination of $\vec{e}_1, \vec{e}_2$

IMPORTANT FACT Given *any* pair of noncollinear vectors $\vec{v}_1, \vec{v}_2$ in $\mathbb{R}^2$ one can write an *arbitrary* vector $\vec{b}$ in $\mathbb{R}^2$ as a linear combination of $\vec{v}_1, \vec{v}_2$. This is obvious geometrically: see Fig 2.

An explicit example is worked out on the lower half of page 33 in Bretscher. Exactly the same argument works in general. Here are the details.

Notice that to say that $\vec{v}_1, \vec{v}_2$ are noncollinear means that $\vec{v}_1 \neq k\vec{v}_2$ for any k and also that $\vec{v}_2 \neq k\vec{v}_1$ for any k, i.e. neither of the vectors is a multiple of the other. (You need to write both these inequalities in case one of the vectors is zero.) Hence if you row reduce the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with row vectors $\vec{v}_1 = [a, b], \vec{v}_2 = [c, d]$ you get a matrix with two nonzero rows, i.e. rank $(A) = 2$. We saw in class that this is equivalent to the condition $ad - bc \neq 0$.

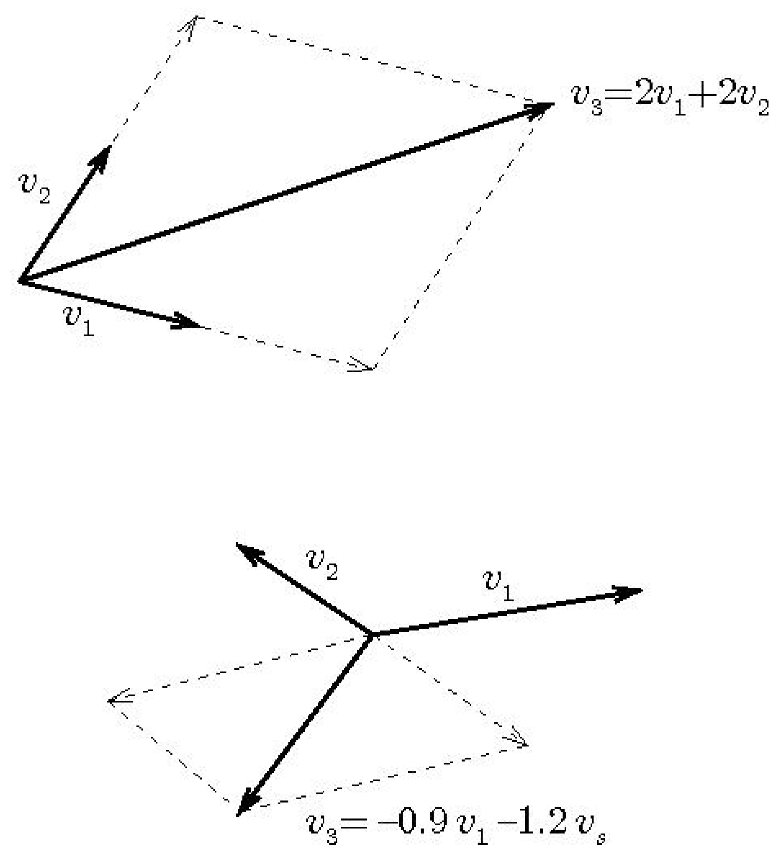


FIGURE 2. Writing $\vec{v}_3$ as a linear combination of $\vec{v}_1, \vec{v}_2$

You can also check this directly. For example, if $ad - bc = 0$ and $a \neq 0$ then $d = bc/a$. So

$$\vec{v}_2 = \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} \frac{c}{a}a \\ \frac{c}{a}b \end{bmatrix} = \frac{c}{a} \begin{bmatrix} a \\ b \end{bmatrix} = \frac{c}{a}\, \vec{v}_1.$$

Now notice that if $\vec{v}_1, \vec{v}_2$ are not collinear (and so satisfy the above condition), the equation $x\vec{v_1} + y\vec{v_2} = \vec{v_3}$ (with unknowns x, y) that expresses $\vec{v_3} = [p, q]$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$ can be written out as

$$x \begin{bmatrix} a \\ b \end{bmatrix} + y \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$

which is the same as the system

$$\begin{aligned} ax + cy &= p \\ bx + dy &= q. \end{aligned}$$

The matrix of coefficients $B = \begin{bmatrix} a & c \\ b & c \end{bmatrix}$ is called the **transpose** of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and has the same rank, i.e. 2. Hence these equations have a unique solution.

To be continued.

Math 211.01 Homework 1

Due Wednesday January 25, 2006, at the beginning of class

This homework is based on Sec 1.1 of Bretscher. Explain your work clearly and write neatly! (But pencil is fine.) Put your name on the paper and please staple the pages together.

Problem 1. Find all solutions to the linear system:

$$\left|\begin{array}{rcr} x+3y+2z &=& 8 \\ x+3y+3z &=& 10 \\ x+4y+2z &=& 9 \end{array}\right|$$

Problem 2. Find all solutions to the linear system below. Represent the solution graphically as the intersection of two (carefully drawn) lines in the x, y plane.

$$\left|\begin{array}{rcr} x+2y &=& 3 \\ x-3y &=& 1 \end{array}\right|$$

Problem 3. Consider the following linear system that depends on a real number k:

$$\left|\begin{array}{lcr} x+2y+3z &=& 1 \\ 3x+2y+z &=& 1 \\ 2x+2y+kz &=& 1 \end{array}\right|$$

(a) For which value(s) of k does this system have a unique solution? Give this solution in terms of k.

(b) For which value(s) of k does this system have infinitely many solutions? Find all these solutions.

(c) Are there any values of k for which there are no solutions?

Problem 4. Find the polynomial of degree 2 (so of the form $f(t) = at^2 + bt + c$) whose graph goes through the points $(1, 2)$, $(-1, 4)$ and $(2, 4)$.

Problem 5. You have 22 bills, a mixture of \$1s, \$5s and \$10s, in your pocket worth a total of \$100. How many do you have of each? There are two possible solutions. Find both.

Problem 1 Numbers in circles refer to eqns.

$$\left\{\begin{array}{l} x+3y+2z=8 \\ x+3y+3z=10 \\ x+4y+2z=9 \end{array}\right\} -② \longrightarrow \left\{\begin{array}{l} x+3y+2z=8 \\ x+3y+3z=10 \\ 0+y-z=-1 \end{array}\right\} -① \longrightarrow$$

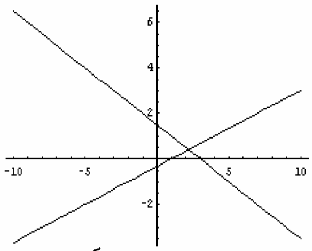
$$\left\{\begin{array}{l} x+3y+2z=8 \\ 0+0+z=2 \\ 0+y-z=-1 \end{array}\right\} \text{swap} \longrightarrow \left\{\begin{array}{l} x+3y+2z=8 \\ 0+y-z=-1 \\ 0+0+z=2 \end{array}\right\} +③ \longrightarrow$$

$$\left\{\begin{array}{l} x+3y+2z=8 \\ 0+y+0=1 \\ 0+0+z=2 \end{array}\right\} -3\times② \longrightarrow \left\{\begin{array}{l} x+0+2z=5 \\ 0+y+0=1 \\ 0+0+z=2 \end{array}\right\} -2\times③ \longrightarrow$$

$$\left\{\begin{array}{l} x=1 \\ y=1 \\ z=2 \end{array}\right\}$$

Problem 2

$$\left\{\begin{array}{l} x+2y=3 \\ x-3y=1 \end{array}\right\}\begin{array}{l} \\ -① \end{array} \longrightarrow \left\{\begin{array}{l} x+2y=3 \\ 0-5y=-2 \end{array}\right\}\begin{array}{l} +\frac{2}{5}\times② \\ \end{array} \longrightarrow \left\{\begin{array}{ll} x & =11/5 \\ & y=2/5 \end{array}\right\}$$



Problem 3

$$\left\{\begin{array}{l} x+2y+3z=1 \\ 3x+2y+z=1 \\ 2x+2y+Kz=1 \end{array}\right\}\begin{array}{l} \\ -3\times① \\ -2\times① \end{array} \longrightarrow \left\{\begin{array}{cccc} 1 & 2 & 3 & 1 \\ 0 & -4 & -8 & -2 \\ 0 & -2 & K-6 & -1 \end{array}\right\}\begin{array}{l} \\ \div(-2) \\ \\ \end{array}$$

$$\left\{\begin{array}{cccc} 1 & 2 & 3 & 1 \\ 0 & 2 & 4 & 1 \\ 0 & -2 & K-6 & -1 \end{array}\right\}\begin{array}{l} \\ \\ +② \end{array} \longrightarrow \left\{\begin{array}{cccc} 1 & 2 & 3 & 1 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & K-2 & 0 \end{array}\right\}$$

Now, the 3rd eqn is $\boxed{(K-2)z=0}$. If $z=0$, we may continue and solve 2nd eqn $\Rightarrow y=1/2$. Then solve 1st eqn $\Rightarrow x=0$

$\Rightarrow (0, 1/2, 0)$ is a solution [independent of K!]

$\Downarrow$

Part (c) in the problem has the answer $\boxed{NO}$

For any K, the 3 planes intersect in a point (at least)

★ If $K=2$, only the 1st and 2nd eqns survive $\Longrightarrow$ two eqns. in three variables

$\Downarrow$

infinite number of solutions

take $z = t \Rightarrow y = \frac{1-4t}{2}$, substitute in ① $\Rightarrow x + 2\left(\frac{1-4t}{2}\right) + 3t = 1$

$$\Rightarrow x = t$$

$\Rightarrow$ the general solution is $\left(t, \frac{1-4t}{2}, t\right)$ Note: the point $(0, \frac{1}{2}, 0)$ is on this line.

Problem 4

In a straight forward mannar, you can form the system,

$$\left\{\begin{array}{l} 2 = a + b + c \\ 4 = a - b + c \\ 4 = 4a + 2b + c \end{array}\right\} \begin{array}{l} \\ -① \\ -4 \times ① \end{array} \longrightarrow \left(\begin{array}{cccc} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & 2 \\ 0 & -2 & -3 & -4 \end{array}\right) \begin{array}{l} \\ \div(-2) \\ -② \end{array} \longrightarrow$$

$$\left(\begin{array}{cccc} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -3 & -6 \end{array}\right) \begin{array}{l} \\ \\ \div(-3) \end{array} \longrightarrow \left(\begin{array}{cccc} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right) \begin{array}{l} -② \\ \\ \\ \end{array} \longrightarrow$$

$$\left(\begin{array}{cccc} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right) \begin{array}{l} -③ \\ \\ \\ \end{array} \longrightarrow \left(\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$\therefore a = 1, \ b = -1, \ c = 2$

$\therefore f(t) = t^2 - t + 2 \longrightarrow$ The polynomial required.

Problem 5

You can directly write down these two eqns,

$$\left\{\begin{array}{l} x + 5y + 10z = 100 \\ x + y + z = 22 \end{array}\right\}$$

Two eqns in three variables $\longrightarrow$ Infinite number of solutions (if any)

$\therefore$ let $z = t$; a parameter. Then solve the system,

$$\left\{\begin{array}{l} x + 5y = 100 - 10t \\ x + y = 22 - t \end{array}\right\} - ② \longrightarrow \left\{\begin{array}{l} 0 + 4y = 78 - 9t \\ x + y = 22 - t \end{array}\right\} \div 4 \longrightarrow$$

$$\left\{\begin{array}{l} y = \frac{78-9t}{4} \\ x + y = 22 - t \end{array}\right\} - ① \longrightarrow \left\{\begin{array}{l} y = \frac{78-9t}{4} \\ x = 22 - t - \frac{78}{4} + \frac{9}{4}t \end{array}\right\}$$

$$\Rightarrow (x, y, z) = \left(\frac{10}{4} + \frac{5}{4}t,\ \frac{78-9t}{4},\ t\right)$$

This infinite number of solutions is filtered using the fact that x, y, z are all integers.

$\Rightarrow$ Try $t = 1 \longrightarrow x, y$ are fractions $\longrightarrow$ not acceptable

$\boxed{t = 2} \longrightarrow x = 5,\ y = 15 \longrightarrow (5, 15, 2)$ is acceptable solution

$t = 3 \longrightarrow x, y$ are fractions $\longrightarrow$ not acceptable

$t = 4$ " "

$t = 5$ " "

$\boxed{t = 6} \longrightarrow x = 10,\ y = 6 \longrightarrow (10, 6, 6)$ is the second accepted solution

5

There is a constraint on the number of solutions because x, y, z are positive numbers,

if $x + 5y + 10z = 100 \Rightarrow \underbrace{(x + y + z)}_{22 \text{ (from eqn. 2)}} + 4y + 9z = 100$

$\Rightarrow 4y + 9z = 78$

$z = 9$ would violate this eqn for positive $y \Rightarrow z \leq 8$

Graded Problems :-

Ex 1.1 — 20

Ex 1.2 — 10 , 22 , 46 ,

And the additional problem.

Ex 1.1

20

From the graph, you can write,

$$b = 0.2a + 780 \;,\; a = 0.1b + 1000 \Rightarrow \boxed{a = 1100 \;,\; b = 1000} \rightarrow \text{in millions of \$}$$

Ex. 1.2

6 let $x_5 = t \Rightarrow x_4 = 1 - t$

$x_3 = 2 + 2t$

let $x_2 = s \Rightarrow x_1 = 3 - t + 7s$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 - t + 7s \\ s \\ 2 + 2t \\ 1 - t \\ t \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 7 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

10

$$\begin{bmatrix} 4 & 3 & 2 & -1 & 4 \\ 5 & 4 & 3 & -1 & 4 \\ -2 & -2 & -1 & 2 & -3 \\ 11 & 6 & 4 & 1 & 11 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{3}{4} & \frac{1}{2} & \frac{-1}{4} & 1 \\ 5 & 4 & 3 & -1 & 4 \\ -2 & 2 & -1 & 2 & -3 \\ 11 & 6 & 4 & 1 & 11 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & \frac{3}{4} & \frac{1}{2} & \frac{-1}{4} & 1 \\ 0 & 1 & 2 & 1 & -4 \\ 0 & -\frac{1}{2} & 0 & \frac{3}{2} & -1 \\ 0 & \frac{-9}{4} & \frac{-3}{2} & \frac{15}{4} & 0 \end{bmatrix} \longleftarrow \begin{bmatrix} 1 & \frac{3}{4} & \frac{1}{2} & \frac{-1}{4} & 1 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & -1 \\ 0 & -\frac{1}{2} & 0 & \frac{3}{2} & -1 \\ 0 & \frac{-9}{4} & \frac{-3}{2} & \frac{15}{4} & 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & -1 & -1 & 4 \\ 0 & 1 & 2 & 1 & -4 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 3 & 6 & -9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & -1 & 4 \\ 0 & 1 & 2 & 1 & -4 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & 2 & -3 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

let $x_4 = t \Rightarrow x_3 = -3-2t$

$x_2 = 2+3t$

$x_1 = 1-t$

18

b, c, d are in the reduced row-echelon form

22

Seven!

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & K \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & K & \ell \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & K \\ 0 & 1 & \ell \end{bmatrix}, \begin{bmatrix} 1 & K & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

K, ℓ are arbitrary numbers

26 Yes

34

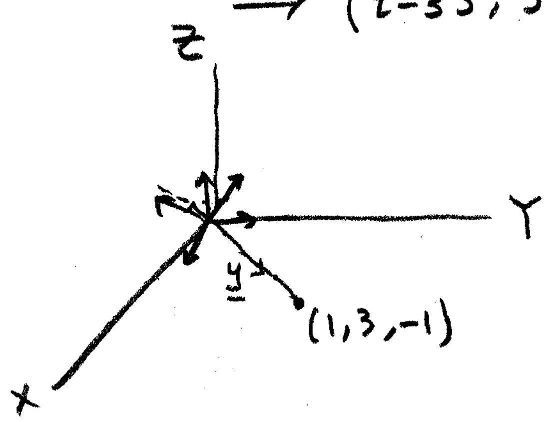
$\underline{x} \cdot \underline{y} = 0$ let $\underline{x} = (x, y, z)$, $\underline{y} = (1, 3, -1)$

$\Rightarrow \underline{x} \cdot \underline{y} = x + 3y - z = 0$ (if perpendicular)

solve the system,

let $z = t$, $y = s$

$\Rightarrow (t-3s, s, t)$ represent the set of all normal vectors.



36

Form the matrix,

$$\begin{array}{c}\begin{bmatrix} 1 & 2 & 4 & -8 \\ 4 & 5 & 6 & -1 \\ 7 & 8 & 9 & 2 \\ 5 & 3 & 1 & 15 \end{bmatrix}\end{array}\begin{array}{l} \\ -4\times 1 \\ -7\times 1 \\ -5\times 1 \end{array} \longrightarrow \begin{array}{c}\begin{bmatrix} 1 & 2 & 4 & -8 \\ 0 & -3 & -10 & 31 \\ 0 & -6 & -19 & 58 \\ 0 & -7 & -19 & 55 \end{bmatrix}\end{array}\begin{array}{l} \\ \\ -4 \\ \\ \end{array}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 2 & 4 & -8 \\ 0 & -3 & -10 & 31 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -19 & 76 \end{bmatrix} \longleftarrow \begin{bmatrix} 1 & 2 & 4 & -8 \\ 0 & -3 & -10 & 31 \\ 0 & 1 & 0 & 3 \\ 0 & -7 & -19 & 55 \end{bmatrix}\begin{array}{l} \\ \\ \\ +7\times 3 \end{array}$$

$$\Big\downarrow\ 4 \div -19$$

$$\begin{bmatrix} 1 & 2 & 4 & -8 \\ 0 & -3 & -10 & 31 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{bmatrix}\begin{array}{l} -4\times 4 \\ +10\times 4 \\ \\ \\ \end{array} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 8 \\ 0 & -3 & 0 & -9 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{bmatrix}\begin{array}{l} -2\times 3 \\ \div -3 \\ \\ \\ \end{array}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

Notice the redundant eqn. (since we have four eqns in three variables)

$\therefore\ (x_1, x_2, x_3) = (2, 3, -4)$

Prob 46 obtaining the row-echelon form,

$$\begin{bmatrix} 0 & 1 & 2K & 0 \\ 1 & 2 & 6 & 2 \\ K & 0 & 2 & 1 \end{bmatrix} \xrightarrow{\text{swap}} \begin{bmatrix} K & 0 & 2 & 1 \\ 1 & 2 & 6 & 2 \\ 0 & 1 & 2K & 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 2/K & 1/K \\ 0 & 2 & 6-\frac{2}{K} & 2-1/K \\ 0 & 1 & 2K & 0 \end{bmatrix} \longleftarrow \begin{bmatrix} 1 & 0 & 2/K & 1/K \\ 1 & 2 & 6 & 2 \\ 0 & 1 & 2K & 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 2/K & 1/K \\ 0 & 1 & 2K & 0 \\ 0 & 2 & 6-\frac{2}{K} & 2-\frac{1}{K} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2/K & 1/K \\ 0 & 1 & 2K & 0 \\ \boxed{0} & \boxed{0} & \boxed{6-\frac{2}{K}-4K} & \boxed{2-1/K} \end{bmatrix}$$

* Consider the third eqn,

1) if it is on the form 0 0 0 number → inconsistent system → no solution

2) " " " 0 0 number number → there is a unique solution

3) " " " 0 0 0 0 → z is a free parameter and there is infinite no. of solutions

4) " " " 0 0 number 0 → Unique solution with $z = 0$

So, let $6-\frac{2}{K}-4K=0$

$\Rightarrow 3K-1-2K^2=0$

$2K^2-3K+1=0 \Rightarrow (2K-1)(K-1)=0$

$\Rightarrow \boxed{K=+1}$ or $\boxed{K=1/2}$

⊙ if $K=1$ → case (1) → no solutions

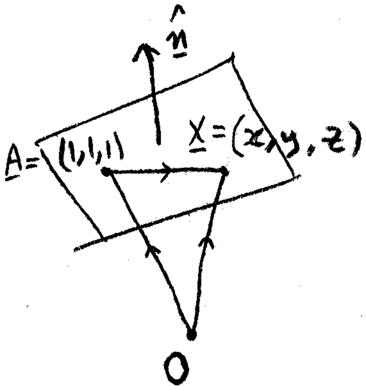
⊙ if $K=1/2$ → Case (3) → parametric solutions, let $z=t \Rightarrow y=-t$, $x=2-4t$

⊙ Unique solution elsewhere

The written problem,

ⓐ Assume $z=t$ and solve the system $\longrightarrow$ $z=t$
$y=-1-2t$
$x=2+t$

ⓑ



$\hat{\underline{n}} \equiv$ (the vector normal to the plane)

$= (2,-3,1) \longleftrightarrow$ from the line parametric eqn.

$$\hat{\underline{n}} \cdot \underbrace{(\underline{X}-\underline{A})}_{\text{vector in the plane}} = 0$$

$$\Rightarrow (2,-3,1)\cdot\big((x,y,z)-(1,1,1)\big)=0$$

$$\Rightarrow 2x-2-3y+3+z-1=0$$

$$\Rightarrow 2x-3y+z=0$$

HW3

Ex 1.3

Prob 6

There are no solutions for this system. Geometrically, we can't construct a combination of $\underline{v}_1, \underline{v}_2$ that gives $\underline{v}_3$

1

if $\underline{v}_1 \parallel \underline{v}_2 \Rightarrow \underline{v}_1 = n\underline{v}_2 \quad , n \in \mathbb{R}$

$$\Rightarrow x\underline{v}_1 + y\underline{v}_2 = nx\underline{v}_2 + y\underline{v}_2 = (nx+y)\underline{v}_2 \neq \underline{v}_3$$

Prob 14

1

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix} \quad \text{Def 1.3.6}$$

$$= \begin{bmatrix} [1 \ 2 \ 3] \cdot [-1 \ 2 \ 1] \\ [2 \ 3 \ 4] \cdot [-1 \ 2 \ 1] \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix} \quad \text{Fact 1.3.8}$$

Prob 20

1

ⓐ $$\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} 7 & 5 \\ 3 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 9 & 8 \\ 7 & 6 \\ 6 & 6 \end{bmatrix}$$

ⓑ $$9\begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 9 & -9 & 18 \\ 27 & 36 & 45 \end{bmatrix}$$

Prob 30

Assume $A \equiv \begin{pmatrix} a & b & c \\ ka & kb & kc \\ la & lb & lc \end{pmatrix}$ a matrix of rank 1. $\Rightarrow A\begin{pmatrix} 5 \\ 3 \\ -9 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow \boxed{K=0}$, $(5a+3b-9c) = (2)$, $l(5a+3b-9c) = 1$ $\rightarrow \boxed{l = 1/2}$ now, choose any a, b, c such that, $5a+3b-9c=2$

i.e.) $a=1 \quad b=1 \quad c=2/3$

Prob 36

Assume $A = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$ 3

$$A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore x_1 = 1, \quad y_1 = 2, \quad z_1 = 3$

Similarly, we can find all unknowns,

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Prob 49

3

(a) 4 variables 3 eqns $A = 3 \times 4$

if rank $A_{arg} = 2$ $\longrightarrow$ rank A is two or one

if one $\longrightarrow$ inconsistent system since we have $\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 1 \end{array}\right]$

if two $\longrightarrow$

# of free variables $= m - \text{rank } A$
$= 4 - 2 = 2 \longrightarrow$ infinite no. of solutions

(b) 3 variables, 4 equations $A = 4 \times 3$

rank $A = 3$

# of free variables $= m - \text{rank } A$
$= 3 - 3 = 0$

$\therefore$ the system has either unique or no solutions [Depending on rank A_{arg}]

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 \end{array}\right]$ unique or, $\left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{array}\right]$ no solutions

(C) 4 eqns 3 variables Rank $A_{arg} = 4$

$$A_{arg} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \text{inconsistent system with no solutions}$$

(D) 3 eqns 4 Unknowns Rank $A = 3$

$\#$ of free variables $= 4 - 3 = 1 \longrightarrow$ infinite no of solutions

True Or False (20) (1)

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ⟶ substitute in the system, and you'll get

$c+d=2$ and $c+d=\frac{1}{2}$

inconsistency!

Also,

$$A\begin{bmatrix}2\\2\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix} \Rightarrow 2A\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix} \Rightarrow A\begin{bmatrix}1\\1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}1\\ 1/2\end{bmatrix}$$

but $A\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$ ← inconsistent

∴ The statement is incorrect

Ex. 2.1

(6) To test the linearity, (3)

* let $\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}v_1+w_1\\v_2+w_2\end{bmatrix}$

$$\therefore T\begin{bmatrix}x_1\\x_2\end{bmatrix}=T\begin{bmatrix}v_1+w_1\\v_2+w_2\end{bmatrix}=(v_1+w_1)\begin{bmatrix}1\\2\\3\end{bmatrix}+(v_2+w_2)\begin{bmatrix}4\\5\\6\end{bmatrix}$$

$$=v_1\begin{bmatrix}1\\2\\3\end{bmatrix}+v_2\begin{bmatrix}4\\5\\6\end{bmatrix}+w_1\begin{bmatrix}1\\2\\3\end{bmatrix}+w_2\begin{bmatrix}4\\5\\6\end{bmatrix}$$

$$=T\begin{bmatrix}v_1\\v_2\end{bmatrix}+T\begin{bmatrix}w_1\\w_2\end{bmatrix}$$

* Also, let $\begin{bmatrix}x_1\\x_2\end{bmatrix}=k\begin{bmatrix}u_1\\u_2\end{bmatrix}=\begin{bmatrix}ku_1\\ku_2\end{bmatrix}$

$$\therefore Tk\begin{bmatrix}u_1\\u_2\end{bmatrix}=T\begin{bmatrix}ku_1\\ku_2\end{bmatrix}=ku_1\begin{bmatrix}1\\2\\3\end{bmatrix}+ku_2\begin{bmatrix}4\\5\\6\end{bmatrix}$$

$$=k\left[u_1\begin{bmatrix}1\\2\\3\end{bmatrix}+u_2\begin{bmatrix}4\\5\\6\end{bmatrix}\right]=kT\begin{bmatrix}u_1\\u_2\end{bmatrix}$$

⟶ T is linear

Finding the Transformation matrix,

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} x_1 + 4x_2 \\ 2x_1 + 5x_2 \\ 3x_1 + 6x_2 \end{bmatrix}$$

$3\times 2 \quad 2\times 1 \qquad\qquad 3\times 1$

$\therefore$ T is 3×2 matrix

let $T = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \Rightarrow T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \\ ex_1 + fx_2 \end{bmatrix}$

$$T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \longleftarrow \text{compare}$$

(1)

(10) let $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ then solve the system for x, y

$\Rightarrow$ $x = 9a - 2b$, $y = -4a + b$ $\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow A^{-1} \equiv \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}$

Another solution,

$$\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} \longrightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 4 & 9 & 0 & 1 \end{array}\right]_{-4\times(1)} \longrightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -4 & 1 \end{array}\right] \xrightarrow{-2\times(2)} \left[\begin{array}{cc|cc} 1 & 0 & 9 & -2 \\ 0 & 1 & -4 & 1 \end{array}\right]$$

$\underbrace{\begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}}_{A^{-1}}$

(14)

(a) $2K - 15 = 0 \Rightarrow K = 7.5$ Matrix is invertible if $K \in \mathbb{R} - \{7.5\}$

(3)

(b) $\begin{bmatrix} 2 & 3 \\ 5 & K \end{bmatrix}^{-1} = \dfrac{1}{2K-15}\begin{bmatrix} K & -3 \\ -5 & 2 \end{bmatrix}$

$2K-15$ is ± 1 or a fraction,

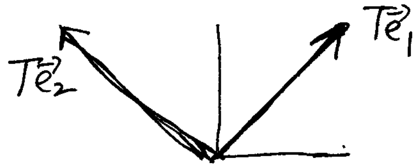
$2K - 15 = \pm 1 \Rightarrow 2K = 15 \pm 1 \Rightarrow K = 8$ or 7

The only valid fractions are, $\dfrac{K}{2K-15}$ = integer $= n \Rightarrow K = 2Kn - 15n \Rightarrow$

$K = \dfrac{15n}{2n-1} \Rightarrow n = \dfrac{K}{2K-15}$

Homework 4.

2.2 #4



Rotation through $\pi/4$ clockwise followed by scaling by factor $\sqrt{2}$.

2.2 #26 (a) $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

(b) This is a projection onto the line $y=0$

$\therefore$ ~~rotate~~ it is $T(x,y) = (x,0)$

Matrix $= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. $[(u_1,u_2) = (1,0)]$

c) A rotation matrix has the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ with $a^2+b^2=1$.

Since $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 5\begin{bmatrix} -b \\ a \end{bmatrix}$ we must have $a = 4/5$, $b = -3/5$

$\therefore$ Matrix is $\begin{bmatrix} 4/5 & 3/5 \\ -3/5 & 4/5 \end{bmatrix}$

d) This shear fixes the y coordinate

$\therefore$ has the form $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+3a \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

$\therefore$ matrix $= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

30. We need $A\vec{e}_1$, $A\vec{e}_2$ both to be parallel to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

These are the columns of A.

$\therefore$ A could be $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ or $\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$ or $\begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$ ....

2.2 #44 $A^{-1} = \frac{1}{a^2+b^2}\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

If A rotates clockwise through θ (with $\tan\theta = -b/a$) and scales by c
A^{-1} " ~~anti~~ clockwise through $-\theta$ " $\tan\theta = b/a$ and scales by $\frac{1}{c}$

Sec 2.3 #20

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 8 & 0 & 1 & 0 \\ 2 & 7 & 12 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & -1 & 1 & 0 \\ 0 & 1 & 6 & -2 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -12 & 4 & -3 & 0 \\ 0 & 1 & 5 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -8 & -15 & 12 \\ 0 & 1 & 0 & 4 & 6 & -5 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

" A^{-1}

<u>Check</u> $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 8 \\ 2 & 7 & 12 \end{bmatrix}\begin{bmatrix} -8 & -15 & 12 \\ 4 & 6 & 5 \\ -1 & -1 & 1 \end{bmatrix} = \begin{matrix} -8+12-3 & -15+18-3 & \\ -8+16-8 & & \\ & \text{etc} & \end{matrix}$

#30 $\begin{bmatrix} 0 & 1 & b \\ -1 & 0 & c \\ -b & -c & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -c \\ 0 & 1 & b \\ -b & -c & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -c \\ 0 & 1 & b \\ 0 & -c & -bc \end{bmatrix}$

we just need to find the rank
so row reduce

$\to \begin{bmatrix} 1 & 0 & -c \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix}$ $\therefore$ rank $= 2$
<u>never</u> invertible.

2.4 # 4 $\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3-1 & 2+0 \\ 0+2 & 0+0 \\ 6+1 & 4+0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 7 & 4 \end{bmatrix}$

#14. $A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

$$BC = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+4+9 & 0+2+6 & -1+0+3 \end{bmatrix} = \begin{bmatrix} 14 & 8 & 2 \end{bmatrix}$$

$$BD = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \end{bmatrix}$$

$$C^2 = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -2 & -2 \\ 4 & 1 & -2 \\ 10 & 4 & -2 \end{bmatrix}$$

$$CD = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$$

$$DE = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$$

$$DB = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

$$EB = \begin{bmatrix} 5 & 10 & 15 \end{bmatrix} \qquad E^2 = \begin{bmatrix} 25 \end{bmatrix}$$

#28 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ will do. or $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

HW5 Sec 2.4

40 $B = \frac{1}{5-6}\begin{bmatrix}5 & -2\\ -3 & 1\end{bmatrix} = \begin{bmatrix}-5 & 2\\ 3 & -1\end{bmatrix}$

$(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix}1 & 3\\ 2 & 5\end{bmatrix}$

$\therefore\ B(B^{-1}A^{-1}) = (BB^{-1})A^{-1} = A^{-1} = \begin{bmatrix}-5 & 2\\ 3 & -1\end{bmatrix}\begin{bmatrix}1 & 3\\ 2 & 5\end{bmatrix} = \begin{bmatrix}-1 & -5\\ 1 & 4\end{bmatrix}$

$\therefore\ A = \frac{1}{-4+5}\begin{bmatrix}4 & 5\\ -1 & -1\end{bmatrix} = \begin{bmatrix}4 & 5\\ -1 & -1\end{bmatrix}$

50 a) $EA = \begin{bmatrix}a & b & c\\ d-3a & e-3b & f-3c\\ g & h & k\end{bmatrix}$ gotten from A by a row operation: add (-3)row1 to row3

b) $EA = \begin{bmatrix}a & b & c\\ d/4 & e/4 & f/4\\ g & h & k\end{bmatrix}$ gotten from A by the row operation: multiply row 2 by $1/4$

c) Take $E = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$

matrices of type a) have 1's on diagonal and one off diagonal non zero entry - They add a multiple of one row to another
type b) has only diagonal entries, all 1's except for one
They multiply a row by a scalar.
type c): is diagonal except that two diagonal entries ii, jj are put in ij and ji position. They effect a row permutation.

52 a) Each row operation is given by premultiplying A by some elementary matrix so $A \rightsquigarrow E_pA \rightsquigarrow E_{p-1}E_pA \rightsquigarrow \cdots \rightsquigarrow E_1\cdots E_pA = \text{rref}A$

b) $\begin{bmatrix}0 & 2\\ 1 & 3\end{bmatrix} \rightsquigarrow \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 2\\ 1 & 3\end{bmatrix} = \begin{bmatrix}1 & 3\\ 0 & 2\end{bmatrix} \rightsquigarrow \begin{bmatrix}1 & 0\\ 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 2\\ 1 & 3\end{bmatrix} = \begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}$

$\rightsquigarrow \underbrace{\begin{bmatrix}1 & -3\\ & 1\end{bmatrix}}_{E_1}\underbrace{\begin{bmatrix}1 & 0\\ 0 & \frac{1}{2}\end{bmatrix}}_{E_2}\underbrace{\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}}_{E_3}\begin{bmatrix}0 & 2\\ 1 & 3\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

Sec 2.4 86. a) $P = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1 & -1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

c) The transformation is $\begin{bmatrix} R \\ G \\ B \end{bmatrix} \xrightarrow{T_A} \begin{bmatrix} R \\ G \\ 0 \end{bmatrix} \xrightarrow{T_P} \begin{bmatrix} I \\ L \\ S \end{bmatrix} = \begin{bmatrix} \frac{R+G}{3} \\ R-G \\ -\frac{R+G}{2} \end{bmatrix}$

$T_P \circ T_A$ has matrix $\begin{bmatrix} 1/3 & 1/3 & 0 \\ 1 & -1 & 0 \\ -1/2 & -1/2 & 0 \end{bmatrix} = PA$

Sec 3.1 (22) $\operatorname{Im} A = \{\vec{b} : A\vec{x} = \vec{b}\}$ has a solution

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & b_1 \\ 3 & 4 & 2 & b_2 \\ 6 & 5 & 7 & b_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1/2 & 3/2 & b_1/2 \\ 0 & 5/2 & 1/2 & b_2 - \frac{3}{2}b_1 \\ 0 & 2 & -2 & b_3 - 3b_1 \end{array}\right]$$

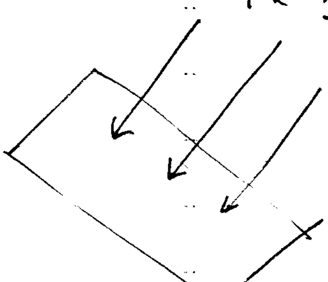
The matrix A has rank 3

$\therefore$ The image is the whole of $\mathbb{R}^3$.

(24) The image is the plane $x + 2y + 3z = 0$. in $\mathbb{R}^3$

The <u>Kernel</u> is the <u>line</u> along which you are projecting

ie the line through 0 in the direction of the normal vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.



orthog projection onto a plane

(44) a) $\ker A = \{\vec{x} : A\vec{x} = 0\}$ But $A\vec{x} = 0 \iff \operatorname{rref} A\vec{x} = 0$

$\therefore \ker A = \ker \operatorname{rref}(A)$.

b) $\operatorname{Im} A$ need not equal $\operatorname{Im} \operatorname{rref}(A)$ (They are equal if T_A is onto).

<u>ex</u> $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\operatorname{Im} A = \{c\begin{bmatrix} 1 \\ 1 \end{bmatrix} : c \in \mathbb{R}\}$ – the line in direction $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\operatorname{rref} A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ with image the line $\{c\begin{bmatrix} 1 \\ 0 \end{bmatrix} : c \in \mathbb{R}\}$

Sec 3.2

HW5 3

2 $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x \leq y \leq z \right\}$ is not a subspace as it is not closed under multiplication by -1.

eg $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in W$ but $\begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$ is <u>not</u>.

16 $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$ $\qquad \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\therefore$ The last vector is redundant

$\therefore$ <u>NOT</u> lin indep

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$$\begin{array}{c} \begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccccc} & \uparrow & & \uparrow & \uparrow & \\ & \vec{v}_1 & & \vec{v}_2 & \vec{v}_3 & \end{array} \end{array}$$

The three indicated col. vectors form a basis of the image since all the other columns are combinations of these ones (therefore they span) and they are obviously linearly independent.

HW #6 3.2: 22, 46 3.3: 16, 26, 30

22. $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ is redundant ; $\begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$\therefore \begin{bmatrix} 3 \\ -1 \end{bmatrix} \in \ker A$.

46. $A = \begin{bmatrix} 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 1 & 4 & 6 \end{bmatrix}$ There are 3 free variables

$x_2 = r, \quad x_4 = s, \quad x_5 = t$

$x_1 = -2r - 3s - 5t$

$x_3 = -4s - 6t$.

$\therefore$ Solution set $= \ker A = \left\{ r \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{v}_1} + s \underbrace{\begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_2} + t \underbrace{\begin{bmatrix} -5 \\ 0 \\ -6 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}_3} : r, s, t \in \mathbb{R} \right\}$

This exhibits $\ker A$ as the span of the 3 vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

They are lin indep because $\begin{pmatrix} -2r - 3s - 5t \\ r \\ -4s - 6t \\ s \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \iff r = s = t = 0$.

3.3 16. If the cols are $\vec{v}_1 \; \vec{v}_2 \; \vec{v}_3 \; \vec{v}_4$, $\vec{v}_3 = 3\vec{v}_1 + 2\vec{v}_2$ $\therefore \vec{v}_3$ is redundant

But $\vec{v}_4$ is not : any vector $c_1\vec{v}_1 + c_2\vec{v}_2$ has ~~1st entry~~.

the form $\begin{pmatrix} c_1 + c_2 \\ c_2 \\ c_2 \\ c_2 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$. $\therefore \ker A$ has dim 1, spanned by $\begin{pmatrix} 3 \\ 2 \\ -1 \\ 0 \end{pmatrix}$.

$\vec{v}_1, \vec{v}_2, \vec{v}_4$ are basis of $\operatorname{Im} A$.

26. $\ker C$ is spanned by $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. The only other matrix with this vector in its kernel is L. $\therefore$ answer to 1 is L.

$\operatorname{Im} C$ is spanned by $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. So its elts have form $\begin{bmatrix} a \\ b \\ a \end{bmatrix}$ i.e. $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ where $y_1 = y_3$.

Which other matrices have cols with 1st and 3rd entries the same?

X and H. Now check that these have the same image. $\therefore$ answer to 2 is X, H

answer to 3 is L (as $\operatorname{Im} Y = \operatorname{Im} T$)

30. There are 3 free variables $x_2 = r, \; x_3 = s, \; x_4 = t$.

Get basis (as in 46 above) by taking the vectors with $r = 1, \; s = t = 0$; $s = 1, \; r = t = 0$; $t = 1, \; r = s = 0$

$\therefore \begin{bmatrix} 1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

3.4

(8) No

(10) yea $3v_1 + v_2$

$$[X]_B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

(14) yea $3v_1 + 4v_2 + 6v_3$

$$[X]_B = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

CH3 T/F

(4) No

(14) yea

(16) No , for example consider the set of vectors that form a line passing through the origin. Any line will be a subspace of the plane $\mathbb{R}^2$ [Do not choose the X axis or Y axis as your line, as they contain $\underline{e}_1$, $\underline{e}_2$]

Also, you might consider any plane (2D) that contains the origin. Choose a tilted plane that contains none of $\underline{e}_1, \underline{e}_2, \underline{e}_3$.

Now, this plane is a legitimate subspace of the 3D space $\mathbb{R}^3$

Sec 4.1

(4) $\int_0^1 P(t)\,dt = 0$ It's subset of P_2 $\therefore P(t) = at^2 + bt + c$

It is a subspace since,

1) $\int_0^1 (0)\,dt = 0$

2) $\int_0^1 K\,P(t)\,dt = K\int_0^1 P(t)\,dt = 0$

3) $\int_0^1 (P_1 + P_2)\,dt = \int_0^1 P_1\,dt + \int_0^2 P_2\,dt = 0 + 0 = 0$

The basis of any element in P_2 is $(1, t, t^2)$. the vanishing integration

Puts a relation on the coefficients a, b, c.

$$\int_0^1 (at^2+bt+c)\,dt = \left(\frac{at^3}{3}+\frac{bt^2}{2}+ct\right)\Big|_0^1 = \frac{a}{3}+\frac{b}{2}+c-0 = 0$$

$$\Downarrow$$

$$\boxed{c = -\left(\frac{a}{3}+\frac{b}{2}\right)}$$

$\therefore$ The subspace contains elements on the form,

$t^2-\frac{1}{3}$

$t-\frac{1}{2}$

$$P(t) = at^2+bt-\left(\frac{a}{3}+\frac{b}{2}\right) \Rightarrow P(t) = \underbrace{a\begin{bmatrix}1\\0\\-\frac{1}{3}\end{bmatrix}+b\begin{bmatrix}0\\1\\-\frac{1}{2}\end{bmatrix}}_{\text{basis}}$$

(6) Not a subspace,

Consider the matrix $A=\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}$ that is invertible.

You can find another matrix on the form,

$$B=\begin{bmatrix}-a & -b & -c\\ j & k & l\\ m & n & o\end{bmatrix}$$ also invertible

Try adding both, $A+B=\begin{bmatrix}0 & 0 & 0\\ & & \\ & & \end{bmatrix}$ $\Rightarrow$ Non invertible

$\Rightarrow$ Not closed under addition $\Rightarrow$ Not a subspace

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$a+d=0 \Rightarrow a=-d$

$$\Rightarrow A=\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{\text{Basis}}$$

Dim = 3

30

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Longrightarrow \begin{bmatrix} a+2c & b+2d \\ 3a+6c & 3b+6d \end{bmatrix} = 0$

$\Rightarrow \boxed{a=-2c}$, $\boxed{b=-2d}$

$$\therefore A=\begin{bmatrix} -2c & -2d \\ c & d \end{bmatrix} = c\underbrace{\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix}}_{\text{basis}}$$

Dim = 2

Math 211.01: First Midterm

March 1, 2006

Name: **School ID:**

Answer all the following questions, justifying all your statements. Write neatly so that we can read and follow your answers. You are not allowed to use calculators, and please turn off cell phones. Use the back of the exam for scrap. There are five questions. Good luck!

Problem 1. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$.

(i) (8 points) Which of the products A^2, B^2, AB, BA are defined?

B^2, BA are defined. The others are not

eg $\underset{2\times 3}{A}\ \underset{2\times 2}{B}$ do not match.

(ii) (12 points) Calculate the products that are defined.

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 3\cdot 0 & 1\cdot 3 + 3\cdot 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 4 & 3 \\ 2 & 1 & 1 \end{bmatrix}$$

1	20pt	
2	15pt	
3	15pt	
4	30pt	
5	20pt	
Total	100pt	

Problem 2. (i) (10 points) Find all solutions to the equations $A\vec{x} = \vec{0}$ where $A := \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{array}{l} x_1 + x_2 = 0 \\ x_2 - x_3 = 0 \end{array}$$

(this is so easy; no need to row reduce.)

put $x_3 = s$ then $x_2 = s$, $x_1 = -s$

$\therefore$ Solns $\vec{x} = \{[s, s, -s] : s \in \mathbb{R}\}$

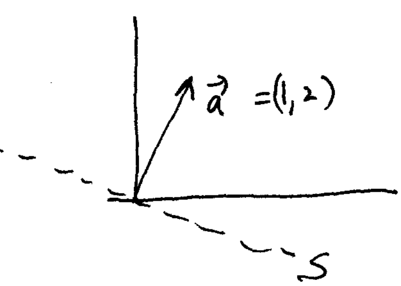
(ii) (5 points) What is the direction vector of this line? $[1, 1, -1]$

Problem 3. (15 points) Let $\vec{a}$ be the vector $(1, 2) \in \mathbb{R}^2$. Is the set

$$S = \{\vec{x} \in \mathbb{R}^2 \mid \vec{x} \cdot \vec{a} = 0\}$$

a subspace? Explain your answer, and draw a diagram of S.

$\vec{x} \cdot (1,2) = x_1 + 2x_2$ $\therefore$ this is the line $x_1 + 2x_2 = 0$ perp to $\vec{a} = (1,2)$



S is a subspace: if $\vec{x}, \vec{y} \in S$ then $(\vec{x} + \vec{y}) \cdot \vec{a} = \vec{x} \cdot \vec{a} + \vec{y} \cdot \vec{a} = \vec{0}$

and $(k\vec{x}) \cdot \vec{a} = k(\vec{x} \cdot \vec{a}) = 0$.

$\therefore$ it satisfies the two defining conditions for a subspace.

(you can also say $\vec{x} + \vec{y} \in S$ because $(\vec{x} + \vec{y}) \cdot (1,2) = (x_1 + y_1) + 2(x_2 + y_2) = 0$ when $\vec{x}, \vec{y} \in S$ }

Problem 4. (i) (5 points) Define the rank of an $n \times m$ matrix A.

rank A = number of leading 1s in rref(A).

(ii) (10 points) Find rank A if $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 0 & 1 \\ 2 & 2 & 6 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & 0 & 1 \\ 2 & 2 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ ② - 2① \\ ③ - 2① \end{matrix} \to \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 5/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\to \underbrace{\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 5/2 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{rref}(A)}$$ has two leading 1s

$\therefore$ rank $= 2$

(iii) (10 points) With A as above, find a vector $\vec{b}$ such that the equation $A\vec{x} = \vec{b}$ has no solution.

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & b_1 \\ 2 & 0 & 1 & b_2 \\ 2 & 2 & 6 & b_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 3 & b_1 \\ 0 & -2 & -5 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - 2b_1 \end{array}\right]$$

- This set of equations is consistent iff $b_3 = 2b_1$

$\therefore$ if $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $A\vec{x} = \vec{b}$ has no solution.

(You just need to find a vector that is <u>not</u> in $\operatorname{Im} A$, ie not a linear comb of the cols of A.)

(iv) (5 points) Is A invertible? Explain.

A is not invertible because T_A is not surjective.

Problem 5. Here is a matrix A together with its row reduced form rref(A):

$$A = \begin{bmatrix} 2 & 1 & -7 & 0 \\ -1 & -1 & 4 & 3 \\ 1 & 0 & -3 & 1 \\ 1 & 1 & -4 & 2 \end{bmatrix}, \quad \text{rref}(A) = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(i) (10 points) Find a basis for Ker(A). (You must briefly explain why your chosen vectors are linearly independent and why they span the given subspace.)

rref(A) has 3 leading 1's $\therefore$ one free variable

$\therefore$ ker A = ~~ker~~ ker rref(A) has dimension 1 (ie a basis with a single element.) To find it solve rref(A)$\vec{x} = 0$.

The eq^s are $x_4 = 0$ $\quad x_2 - x_3 = 0$ $\quad x_1 - 3x_3 = 0$

$\therefore x_3 = t$ (arbitrary) $\quad x_2 = t$ $\quad x_1 = 3t$ $\quad \therefore \vec{x} = t\begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

$\therefore$ a basis is $\begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

(ii) (10 points) Find a basis for Im(A), explaining your answer as above.

Im (A) = Span (col vectors of A). Since ker A has dim 1, there is one linear relation between these col vectors, namely $3\vec{v}_1 + \vec{v}_2 + \vec{v}_3 = 0$

$\therefore \vec{v}_3$ is redundant and $\vec{v}_1, \vec{v}_2, \vec{v}_4$ form the basis, where

$\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 0 \\ 3 \\ 1 \\ 2 \end{bmatrix}$. They are lin indep as explained above and span Im (A) because any vector $\sum_{i=1}^{4} a_i \vec{v}_i$ can be written as the combination $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3(-3\vec{v}_1 - \vec{v}_2) + a_4\vec{v}_4$ of $\vec{v}_1, \vec{v}_2, \vec{v}_4$.

Sec 4.1

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$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & e & f \\ 0 & g & h \end{bmatrix} \qquad \text{Dim} = 5$$

Basis,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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let $B = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$ and A is 4×4 matrix with 16 parameter

Using the condition $BA = AB$, many different cases appear,

* if $a = b = c = d \Rightarrow$ The 16 parameter in A survive

  $\Rightarrow D = 16$

* if $a = b = c \neq d \Rightarrow$ 6 parameters die

  $\Rightarrow D = 10$

* if $(a = b) \neq (c = d) \Rightarrow$ 8 parameters die

  $\Rightarrow D = 8$

* if $a=b\neq c\neq d \Rightarrow$ 10 parameters die

$\Rightarrow D=6$

* if $a\neq b\neq c\neq d \Rightarrow$ 12 parameters die

$\Rightarrow D=4$

Sec 4.2

(6) $T(M)=M\begin{bmatrix}1&2\\3&6\end{bmatrix}$ $\quad R^{2\times2}\longrightarrow R^{2\times2}$

$T(M_1+M_2)=(M_1+M_2)\begin{bmatrix}1&2\\3&6\end{bmatrix}=M_1\begin{bmatrix}1&2\\3&6\end{bmatrix}+M_2\begin{bmatrix}1&2\\3&6\end{bmatrix}$

$=T(M_1)+T(M_2)$

$T(KM)=KM\begin{bmatrix}1&2\\3&6\end{bmatrix}=KT(M)$, K is ascalar

$\Rightarrow T$ is a linear transformation

$T: R^{2\times2}\longrightarrow R^{2\times2} \Rightarrow$ if $Ker(T)=\{0\}$, then it is isomorphism

let $M=\begin{bmatrix}M_1&M_2\\M_3&M_4\end{bmatrix}$ $\quad\therefore T(M)=\begin{bmatrix}M_1+3M_2 & 2M_1+6M_2\\M_3+3M_4 & 2M_3+6M_4\end{bmatrix}$

$\Rightarrow$ conditions for the kernel,

$M_1=-3M_2$, $M_3=-3M_4$

$\Rightarrow Ker(T)=\begin{bmatrix}-3M_2&M_2\\-3M_4&M_4\end{bmatrix}=\begin{bmatrix}-3a&a\\-3b&b\end{bmatrix}\neq 0$

$\Rightarrow$ Not Isomorphism

(16) checking the linearity is easy.

let $M = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} \Rightarrow T(M) = \begin{bmatrix} -M_1 & 0 \\ -2M_3 & -M_4 \end{bmatrix}$

$\Rightarrow$ The Kernel contains $M_1 = M_3 = M_4 = 0$

But M_2 is arbitrary.

$\therefore \; \mathrm{Ker}(T) = \begin{bmatrix} 0 & M_2 \\ 0 & 0 \end{bmatrix} \neq \{0\}$

$\Rightarrow$ not isomorphism

(20)

if $x_1 + iy_1$, $x_2 + iy_2$ are two complex number

$\Rightarrow T(x_1 + iy_1 + x_2 + iy_2) = x_1 + x_2 - i(y_1 + y_2)$

$= x_1 - iy_1 + x_2 - iy_2$

$= T(x_1 + iy_1) + T(x_2 + iy_2)$

$T(k(x+iy)) = kx - iky = k(x - iy) = kT(x+iy)$

$\therefore$ It is linear.

obviously, $\mathrm{Ker}(T) = \{0\} \Rightarrow$ isomorphism

(30) $T(f(t)) = t\, f'(t)$ $\qquad P_2 \longrightarrow P_2$

Linear transformation.

study the Kernel $\Rightarrow t f'(t) = 0 \Rightarrow f(t) = \text{const} \neq 0$

$\Rightarrow$ not isomorphism

(56)

* we found that $Ker(T) = \text{const} \Rightarrow \text{basis} = \{1\}$

$Dim(Ker) = \text{nullity} = 1$ —— ①

* The image is in P_2 by definition, Also, it is multiplied by t all the time $\Rightarrow$ basis $= \{t, t^2\}$

systematicaly, let $f(t) = at^2 + bt + c$

$$\Rightarrow T(f(t)) = 2at^2 + bt$$

$Dim(img) = \text{rank} = 2$ —— ②

① + ② gives 3 as the space dimension (correct since we have P_2)

Sec 4.3

(4)

$f(t)$	$tf(t)$	$g(t)$
$t+1$	t^2+t	$(t+2)(t+K)$

Can we form the third basis out of the first two?

let's see,

$$(t+2)(t+K) = c_1(t+1) + c_2(t^2+t)$$

$$t^2 + (2+K)t + 2K = c_2 t^2 + (c_1+c_2)t + c_1 \quad (\star)$$

$\Rightarrow c_1 = 2K \quad , \quad c_2 = 1 \quad \Rightarrow \boxed{c_1 + c_2 = 2K+1}$

But by comparing t coefficients in the eqn $(\star)$,

$$c_1 + c_2 = 2 + K \Rightarrow 2K+1 = 2+K \Rightarrow \boxed{K=1}$$

$\therefore$ if $K=1$, there is a relation between the three functions $\longrightarrow$ They can't form a set of bases

(6) To construct the transformation matrix, we can use the basis and construct the matrix a column by a column,

$$T(e_1) = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_B$$

$$T(e_2) = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_B$$

$$T(e_3) = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}_B$$

$$\therefore T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \rightarrow \text{invertible} \rightarrow \text{isomorphism}$$

(20)

Also, transfer the basis as in problem (6),

$$T(1) = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ T(t) = 1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ T(t^2) = 2t = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \text{non invertible} \rightarrow \text{non isomorphism}$$

$Ker(T)$ = set of constant functions Dim = 1

$Img(T) = span[1, t]$ Dim = 2

space dimension $= 1 + 2 = 3$

Sec 4.3

23

$f(t) = at^2 + bt + c$

$f(3) = 9a + 3b + c$

$$T(f(t)) = 9a+3b+c = \begin{pmatrix} 9a+3b+c \\ 0 \\ 0 \end{pmatrix}_{(1,t,t^2)}$$

$$T\begin{pmatrix} c \\ b \\ a \end{pmatrix}_{1,t,t^2} = \begin{pmatrix} 9a+3b+c \\ 0 \\ 0 \end{pmatrix} \Rightarrow T = \begin{pmatrix} 1 & 3 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{1,t,t^2} \to \text{not iso}$$

$$\text{Img basis} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{Dim}(\text{Img}) = 1$$

$$\text{Ker basis} = \begin{pmatrix} 1 \\ 0 \\ -1/9 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1/3 \end{pmatrix}$$

$$\text{Or} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 9 \\ 0 \\ -1 \end{pmatrix}$$

$$\text{Dim}(\text{Ker}) = 2$$

(24) Transform the basis $1,\ t-3,\ (t-3)^2 = t^2-6t+9$

$$\begin{pmatrix} 1 & 3 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 3 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 9 \\ -6 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{1,\,t-3,\,(t-3)^2} \longrightarrow \text{not iso}$$

$$\text{Img basis} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{Ker basis} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(46) (a) Express the new basis interms of the old one, and construct the matrix column by column,

$$1 \longrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad t-3 \longrightarrow \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} \qquad (t-3)^2 \longrightarrow \begin{pmatrix} 9 \\ -6 \\ 1 \end{pmatrix}$$

$$\therefore S_{B\to A} = \begin{bmatrix} 1 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

(b)

$$SB = \begin{bmatrix} 1 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$AS = \begin{bmatrix} 1 & 3 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = SB$$

(c)

$$\underset{A\to B}{S} = \underset{B\to A}{S^{-1}} = \begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}$$

(60) (a) Try to construct each vector in B from the two basis vectors in A. write down the components (coordinates) you need.

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Longrightarrow \underset{B\to A}{S} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

(b) Reverse your procedure in (a) or simply find S^{-1}

$$\Longrightarrow \underset{A\to B}{S} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

(c)

$$[v] = [u][S_{B\to A}]$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 2 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 2 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 2 & 3 \end{bmatrix}$$

sec 5.1

(2) $\sqrt{29}$

(6)

$$\cos\theta = \frac{u \cdot v}{|u||v|} = \frac{-3}{\sqrt{10}\sqrt{54}} \Rightarrow \theta = 1.7\ \text{rad} = 97.4^\circ$$

θ in the 2nd quadrant $\longrightarrow$ Cosine θ is negative $\longrightarrow$ OK

(26) Is the basis orthogonal?

$$(2,3,6)\cdot(3,-6,2) = 0 \Longrightarrow u \perp v$$

Create unit vectors out of the basis,

$$|\hat{u}| = \frac{1}{\sqrt{49}}\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \qquad |\hat{v}| = \frac{1}{\sqrt{49}}\begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}$$

Use the formula for the orthogonal projection,

$$\text{Proj}(x) = (u \cdot x)u + (v \cdot x)v$$

$$\text{proj}\begin{pmatrix}49\\49\\49\end{pmatrix} = \frac{1}{7}\left\{\begin{pmatrix}2\\3\\6\end{pmatrix}\cdot\begin{pmatrix}49\\49\\49\end{pmatrix}\right\}\frac{1}{7}\begin{pmatrix}2\\3\\6\end{pmatrix} + \frac{1}{7}\left\{\begin{pmatrix}3\\-6\\2\end{pmatrix}\cdot\begin{pmatrix}49\\49\\49\end{pmatrix}\right\}\frac{1}{7}\begin{pmatrix}3\\-6\\2\end{pmatrix}$$

$$= (2+3+6)\begin{pmatrix}2\\3\\6\end{pmatrix} + (3-6+2)\begin{pmatrix}3\\-6\\2\end{pmatrix} = \begin{pmatrix}19\\39\\64\end{pmatrix}$$

Sec 5.2

6 The unit vector of $v_1 = \begin{pmatrix}2\\0\\0\end{pmatrix}$ is $\frac{1}{2}\begin{pmatrix}2\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$

$\therefore \underline{u}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$

$$\underline{v}_2^{\perp} = \underline{v}_2 - (\underline{u}_1 \cdot \underline{v}_2)\underline{u}_1 = \begin{pmatrix}3\\4\\0\end{pmatrix} - 3\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\4\\0\end{pmatrix}$$

$\underline{u}_2$ is the unit vector of $v_2^{\perp} = \frac{1}{4}\begin{pmatrix}0\\4\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$

$$\underline{v}_3^{\perp} = \underline{v}_3 - (\underline{u}_1 \cdot \underline{v}_3)\underline{u}_1 - (\underline{u}_2 \cdot \underline{v}_3)\underline{u}_2$$

$$= \begin{pmatrix}5\\6\\7\end{pmatrix} - 5\begin{pmatrix}1\\0\\0\end{pmatrix} - 6\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\7\end{pmatrix}$$

$\underline{u}_3$ is the unit vector of $\underline{v}_3^{\perp} = \frac{1}{7}\begin{pmatrix}0\\0\\7\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$\Rightarrow$ the basis is $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$

(32) Any two vectors $\underline{u}, \underline{v}$ will do if,

1) They both satisfy the plane equation

2) " " are normalized

3) $\underline{u} \cdot \underline{v} = 0$

for example, start with any two linearly independent vectors on the plane,

$$\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$$

$$\underline{u}_1 = \hat{\underline{u}} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \qquad \text{———} \ (1)$$

$$\underline{v}_2^{\perp} = \underline{v} - (\underline{u}_1 \cdot \underline{v})\,\underline{u}_1 = \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix} - \frac{1}{\sqrt{6}} \left(\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix} \right) \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix} - \frac{12}{6} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\therefore \underline{u}_2 = \hat{\underline{v}}_2^{\perp} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad \text{———} \ (2)$$

These two vectors satisfy the three conditions above

Homework 10 : Solutions

5.3 #4. This is not orthogonal − col 1 and col 3 are not orthogonal.

#8 If A, B are orthogonal, $A+B$ need not be.
eg $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = B$ is orthogonal, but $A+B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is not.

#10 If A, B are orthog, $B^{-1}AB$ is also orthog.
Proof B^{-1} is orthog by Fact 5.3.4 b) and $B^{-1}A$ and $(B^{-1}A)B$ are also orthog by Fact 5.3.4 a).

#22 $(BB^T)^T = (B^T)^T B^T$ ~~by Fact~~ since $(AB)^T = B^T A^T$
$= BB^T$ since $(B^T)^T = B$.
$\therefore BB^T$ is symmetric

#36. we need $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \quad \therefore a - b = 0 \quad \therefore a = b$

also $\begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \quad \therefore 2a + 2b + c = 0$
$\therefore 4a + c = 0 \quad \therefore c = -4a$
$\therefore$ vector is $\begin{bmatrix} a \\ a \\ -4a \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$

It must be a unit vector. Since $\left\| \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix} \right\| = \sqrt{18} = 3\sqrt{2}$, Take $a = \frac{1}{3\sqrt{2}}$

$\therefore$ vector is $\frac{1}{3\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$

5.5

Problem 10

This integral vanishes for any $(f(t)\cdot g(t)) =$ odd function

$\Rightarrow$ if $f(t) = t \Longrightarrow g(t)$ must contain $1, t^2$ components only

$\therefore$ in general, $\boxed{g(t) = a + bt^2}$

This is the space, but we need basis. Could it be a, bt^2?

let's check,

normalization,

$$\langle a, a\rangle = \frac{1}{2}\int_{-1}^{1} a\cdot a\,dt = 1 \Rightarrow a^2 = 1 \rightarrow a = \pm 1$$

$$\langle bt^2, bt^2\rangle = \frac{1}{2}\int_{-1}^{1} bt^2\cdot bt^2\,dt = 1 \Rightarrow \frac{1}{2}b^2\left.\frac{t^5}{5}\right|_{-1}^{1} = 1$$

$$\Rightarrow \frac{b^2}{5} = 1 \Rightarrow b = \pm\sqrt{5}$$

$\therefore$ It seems that $(\pm 1, \pm\sqrt{5}t^2)$ might work as a basis

However, they are not orthogonal to each other. Their product is even function and the integral will not vanish.

There is two ways to find orthonormal basis,

Method one, start with $1, t^2$. These are neither

orthogonal nor parallel. Use these as starting vectors in "Gram-Schmidt" process

$\underline{u}_1 = \underline{v}_1 = 1$

$$\underline{v}_2^{\perp} = \underline{v}_2 - (\underline{u}_1 \cdot \underline{v}_2)\underline{u}_1$$
$$= t^2 - \frac{1}{2}\int_{-1}^{1}(1 \cdot t^2)\,dt = t^2 - \frac{1}{3}$$

$$\|\underline{v}_2^{\perp}\| = \sqrt{\langle t^2 - \tfrac{1}{3}, t^2 - \tfrac{1}{3}\rangle} = \sqrt{\frac{1}{2}\int_{-1}^{1}\left(t^2 - \frac{1}{3}\right)^2 dt} = \sqrt{\frac{4}{45}}$$

$$\Rightarrow \underline{u}_2 = \frac{t^2 - 1/3}{\sqrt{4/45}}$$

$\therefore$ $\underline{u}_1, \underline{u}_2$ form the orthonormal basis

Method two,

The space is $a + bt^2$. Dimension $= 2$. Assume very general basis, $\left\{ \begin{matrix} a_1 + b_1 t^2 \\ a_2 + b_2 t^2 \end{matrix} \right\}$

we need to satisfy three conditions;

1) $\frac{1}{2}\int_{-1}^{1}(a_1 + b_1 t^2)(a_1 + b_1 t^2)\,dt = 1$

2) $\frac{1}{2}\int_{-1}^{1}(a_2 + b_2 t^2)(a_2 + b_2 t^2)\,dt = 1$

$\rightarrow$ normalization

3) $\frac{1}{2}\int_{-1}^{1}(a_1 + b_1 t^2)(a_2 + b_2 t^2)\,dt = 0$ $\longrightarrow$ orthogonality

$\Rightarrow$ Three equations in Four variables $\rightarrow$ Many solutions

And this makes sense because the basis choice is not unique.

You may choose $\boxed{a_1 = 1}$ and solve the three eqn system to find an orthonormal basis

Sec 6.1

8) $\text{Det} = 1\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} = -1 + 4 - 3 = 0$

$\Rightarrow$ Matrix is noninvertible

30) $$\begin{bmatrix} 4 & 2 & 0 \\ 4 & 6 & 0 \\ 5 & 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\Rightarrow (3-\lambda)\underbrace{\Big[(6-\lambda)(4-\lambda) - 8\Big]}_{(\lambda^2 - 10\lambda + 16) = (\lambda - 2)(\lambda - 8)} = 0$$

$\Rightarrow \lambda = 2, 3, 8$

34) $$\text{Det} = -4\underbrace{\begin{bmatrix} 4 & 5 & 0 \\ 3 & 6 & 0 \\ 1 & 8 & 2 \end{bmatrix}}_{2(24-15)} + 3\underbrace{\begin{bmatrix} 4 & 5 & 0 \\ 3 & 6 & 0 \\ 2 & 7 & 1 \end{bmatrix}}_{1(24-15)}$$

$= -8 \times 9 + 3 \times 9 = -45$

HW12 Solutions

Sec 6.2 :

Prob. 2: The matrix can be easily transformed into the following form using Gaussian elimination ,

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$ and its determinant equals 1.4.6=24 *No row swap or multiplication was used*

Prob. 10: After successive elimination steps, the matrix becomes,

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$ So, the determinant equals unity. *No row swap or multiplication was used*

Prob. 12:

$$\begin{bmatrix} V_4 \\ V_2 \\ V_3 \\ V_1 \end{bmatrix} \text{ one swap} \rightarrow \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}$$ So, del = 8 . -1 = -8

Prob. 14:

$$\begin{bmatrix} V_1 \\ V_2 + 9V_4 \\ V_3 \\ V_4 \end{bmatrix} \text{ -9* IV +II @ II} \rightarrow \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}$$ So, det 1st = det 2nd

Prob. 30:

a) F(t) = del $\begin{bmatrix} 1 & 1 & 1 \\ a & b & t \\ a^2 & b^2 & t^2 \end{bmatrix}$ To prove that the function is quadratic, enough to show that the coefficient of t^2 is not zero. If we used the third column to evaluate the matrix, we would find that the coefficient of t^2 is (b-a). So, it's quadratic unless (b-a) vanishes.

b) If t = a or t = b , we'll have two identical columns. What is the value of this determinant?

Let's look at the matrix transpose.

It will be something like, $\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & a & a^2 \end{bmatrix}$. 1st row is the same like 3rd . So, the system of equations is clearly inconsistent. The determinant is necessarily zero. Using fact 6.2.7, we prove that the f(a) = f(b) = 0

Another way to show that, is to think of the matrix that has two identical columns as if it expresses a volume in 3 dimensions. If two edges of that volume are expressed with the same vector, this means that it is a shape in two dimensions instead of three ; the volume = zero. → determinant is zero.

If f(t) = k(t-a)(t-b)→ k is the coefficient of t^2. From part (a), **k = (b-a)**

c) f(t) = k (t-a)(t-b) . Obviously, the matrix will be **noninvertible** when

Det = f(t) = 0 → t = a or t = b

Prob. 38:

$$Det\,(A^T A) = DetA^T .DetA = (DetA)^2 = 3^2 = 9$$

Sec 6.3 :

Prob. 2:

Form a matrix out of the two vectors as $\begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix}$. The determinant of this matrix is twice the area surrounded by the two vectors. Area = Det/2 = | (6-56) |/2 = 25

Prob. 7:

Form a matrix out of each two successive pair of vectors. Four matrices are produced. Their determinants give twice of the area shaded in the graph

$$\text{Area}= \frac{1}{2}\left\{ Det\begin{bmatrix} 5 & -7 \\ 5 & 7 \end{bmatrix} + Det\begin{bmatrix} -7 & -5 \\ 7 & -6 \end{bmatrix} + Det\begin{bmatrix} -5 & 3 \\ -6 & -4 \end{bmatrix} + Det\begin{bmatrix} 3 & 5 \\ -4 & 5 \end{bmatrix} \right\} =110$$

Prob. 24:

Using Cramer's rule,

$$X = \frac{\begin{bmatrix} 8 & 3 & 0 \\ 3 & 4 & 5 \\ -1 & 0 & 7 \end{bmatrix}}{\begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 5 \\ 6 & 0 & 7 \end{bmatrix}} = 1 \qquad Y = \frac{\begin{bmatrix} 2 & 8 & 0 \\ 0 & 3 & 5 \\ 6 & -1 & 7 \end{bmatrix}}{\begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 5 \\ 6 & 0 & 7 \end{bmatrix}} = 2 \qquad Z = \frac{\begin{bmatrix} 2 & 3 & 8 \\ 0 & 4 & 3 \\ 6 & 0 & -1 \end{bmatrix}}{\begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 5 \\ 6 & 0 & 7 \end{bmatrix}} = -1$$

Check by yourself that these values satisfy the equation system.

Prob. 24:

The classical adjoint should NOT be calculated from the inverse matrix. Actually, most of the time, we calculate the adjoint in order to obtain the inverse.

- ***How to calculate the classical adjoint :***

1) Calculate the *matrix of the minor determinants* of each element in the matrix

i.e.) from the matrix, $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$ we get, $\begin{bmatrix} 18 & 12 & -2 \\ 0 & 6 & 5 \\ 0 & 0 & 3 \end{bmatrix}$ ***take a look at "Minors" at page 250***

2) Multiply each element by the sign in its location from $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

i.e.) our matrix here becomes, $\begin{bmatrix} 18 & -12 & -2 \\ 0 & 6 & -5 \\ 0 & 0 & 3 \end{bmatrix}$

3) Take the Transpose,

i.e.) $\begin{bmatrix} 18 & 0 & 0 \\ -12 & 6 & 0 \\ -2 & -5 & 3 \end{bmatrix}$ and that is the final result ; the classical adjoint

Apply this method to any square matrix of any size.

To check that the answer is correct, we know that,

$$A^{inverse} = \frac{A^{adjoint}}{DetA} \Rightarrow A.A^{inverse} = I = \frac{A.A^{adjoint}}{DetA} \Rightarrow A.A^{adjoint} = DetA.I \qquad I \text{ ,is the identity matrix}$$

So, you can multiply the adjoint matrix with the one you started with. The result is expected to be a matrix with equal diagonal elements and zero’s elsewhere (the identity matrix multiplied by the determinant value)

HW13

Sec 7.1

Prob 10

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} a+2c \\ b+2d \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} \Rightarrow \begin{array}{l} a = 5-2c \\ b = 10-2d \end{array}$$

$\therefore$ the solution is $\begin{bmatrix} 5-2c & c \\ 10-2d & d \end{bmatrix}$

Prob 16

Any vector will stay as it is but pointing in opposite direction $\therefore T\underline{A} = -\underline{A}$ eigenvalue is -1

Prob 18

* vectors in the reflection plane stay the same
  $\rightarrow$ eigen value $= 1$, basis contain two vectors (plane)
* vectors normal to the reflection plane stay the same also but with opposite direction $\rightarrow$ eigenvalue $= -1$
* Any "tilted" vector is not an eigen vector of this transformation.

Prob 34 $\quad A\underline{v} = 4\underline{v} \Rightarrow A \cdot A\underline{v} = A\,4\underline{v} = 16\underline{v}$

$$\therefore (A^2 + 2A + 3I_n)\underline{v} = (16 + 2\cdot 4 + 3\cdot 1)\underline{v} = 27\underline{v}$$

It's an eigenvector with $\underline{27}$ as eigenvalue

Prob 50

~~h(t+1) = A(~~ $\qquad A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$

ⓐ $h(0) = f(0) = 100$

eigen vector

$$A\begin{pmatrix} 100 \\ 100 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 100 \\ 100 \end{pmatrix} = \begin{pmatrix} 200 \\ 200 \end{pmatrix} = 2\begin{pmatrix} 100 \\ 100 \end{pmatrix}$$

So, Applying A once to this initial condition $\begin{pmatrix} 100 \\ 100 \end{pmatrix}$ gives back the same vector with a factor of 2

$\Rightarrow$ Applying A t times gives back $\begin{pmatrix} 100 \\ 100 \end{pmatrix}$ with a factor of 2^t

$$\therefore \begin{pmatrix} h(t) \\ f(t) \end{pmatrix} = 2^t\begin{pmatrix} 100 \\ 100 \end{pmatrix}$$

ⓑ $h(0) = 200 \qquad f(0) = 100$

eigen vector

$$A\begin{pmatrix} 200 \\ 100 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 200 \\ 100 \end{pmatrix} = \begin{pmatrix} 600 \\ 300 \end{pmatrix} = 3\begin{pmatrix} 200 \\ 100 \end{pmatrix}$$

$\therefore$ similar to (a) $\quad \therefore \quad \begin{pmatrix} h(t) \\ f(t) \end{pmatrix} = 3^t \begin{pmatrix} 200 \\ 100 \end{pmatrix}$

(c)

$$A \begin{pmatrix} 600 \\ 500 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 600 \\ 500 \end{pmatrix} = \begin{pmatrix} 1400 \\ 1100 \end{pmatrix}$$, not a multiple of $\begin{pmatrix} 600 \\ 500 \end{pmatrix}$

but, we can factorize the initial condition in terms of the eigenvalues found in (a), (b)

$$A \begin{pmatrix} 600 \\ 500 \end{pmatrix} = A \left[\begin{pmatrix} 200 \\ 100 \end{pmatrix} + 4 \begin{pmatrix} 100 \\ 100 \end{pmatrix} \right] \xrightarrow[\text{(a), (b)}]{\text{using}} = 3 \begin{pmatrix} 200 \\ 100 \end{pmatrix} + 4 \cdot 2 \begin{pmatrix} 100 \\ 100 \end{pmatrix}$$

And after t time,

$$\begin{pmatrix} h(t) \\ f(t) \end{pmatrix} = \begin{pmatrix} 400\, 2^t + 200\, 3^t \\ 500\, 2^t + 100\, 3^t \end{pmatrix}$$

Sec 7.2

Prob 10

$$\operatorname{Det} \begin{bmatrix} -3-\lambda & 0 & 4 \\ 0 & -1-\lambda & 0 \\ -2 & 7 & 3-\lambda \end{bmatrix} = 0$$

$\Rightarrow (-3-\lambda)(-1-\lambda)(3-\lambda) + 4(2(-1-\lambda)) = 0$

$\Rightarrow (-1-\lambda)(-9+\lambda^2+8) = 0 \quad \Rightarrow \lambda = -1 \quad , \lambda = \pm 1$

$\therefore$ eigenvalue $+1$ Algebric multiplicity one

    " -1 "   "   two

Prob 22

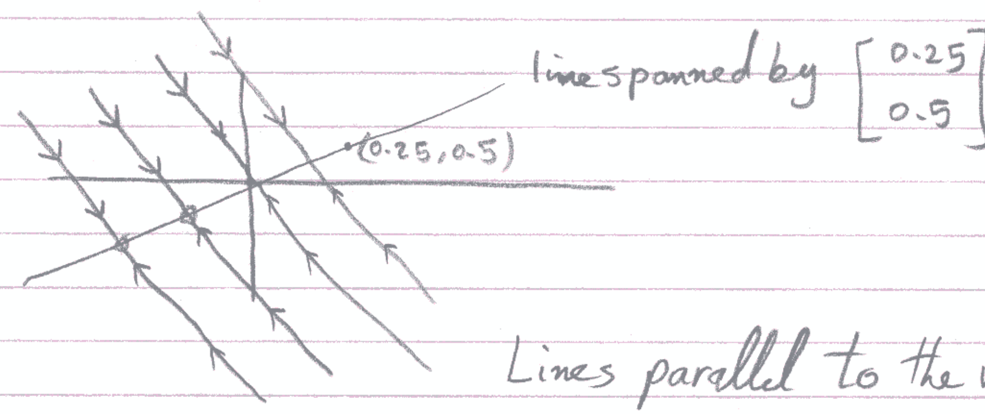
A, A^T have the same eigenvalues
Also " " " trace

$\Longrightarrow$ the characteristic polynomial is the same

Prob 24

$\lambda = 1 \;,\; 1/4$

Prob 26



Lines parallel to the vector $(1, -1)$

HW14

Sec 7.3

Prob 6

the characteristic eqn is $\lambda^2 - 7\lambda - 2 = 0$

roots are $\lambda_{1,2} = \dfrac{7 \mp \sqrt{57}}{2}$

eigenvector are obtained using,

$$E_{1,2} = \text{Ker}\left[A - \lambda_{1,2} I\right]$$

$$\therefore E_{\lambda = \frac{7-\sqrt{57}}{2}} = \begin{bmatrix} -3 \\ \frac{\sqrt{57}-3}{2} \end{bmatrix}$$

similarly, find $E_{\lambda = \frac{7+\sqrt{57}}{2}}$

Prob 20

Upper Triangular matrix $\longrightarrow \lambda = 1, 1, 2$

for $\lambda = 2$, algebric multiplicity $= 1 \longrightarrow$ geometric multiplicity $= 1$

for $\lambda = 1$, " " $= 2 \longrightarrow$ " " $=$ either 1 or 2

$$E_1 = \text{Ker}\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{row operations}} \text{Ker}\begin{bmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} ay \\ 0 \\ z \end{bmatrix}$$

if $a = 0 \Rightarrow E_1 = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \longrightarrow$ 2 dim $\longrightarrow$ geom mult. $= 2$

if $a \neq 0 \Rightarrow E_1 = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} \longrightarrow$ 1 dim $\longrightarrow$ " " $= 1$

* if we need an eigenbasis to exist for this 3x3 matrix,

the dimensions must add up to $\underline{\underline{3}}$

$\Rightarrow a=0$ gives an eigenbasis b/c here,

E_1 is 2-dim and E_2 is 1-dim

Prob 36

trace 1 $\neq$ trace 2 $\Rightarrow$ not similar

Sec 7.4

Prob 8

$$A=\begin{bmatrix}1&3\\3&1\end{bmatrix} \Rightarrow (1-\lambda)^2-9=0 \Rightarrow \lambda=4,-2$$

$$E_4=\mathrm{Ker}\begin{bmatrix}-3&3\\3&-3\end{bmatrix}=\mathrm{span}\begin{bmatrix}1\\1\end{bmatrix}$$ — ~~and~~ one-dim

$$E_{-2}=\mathrm{Ker}\begin{bmatrix}3&3\\3&3\end{bmatrix}=\mathrm{span}\begin{bmatrix}1\\-1\end{bmatrix}$$ — one-dim

They add up to 2 dimensions and A is 2×2 matrix

$\Rightarrow A$ is diagonalizable

$$S=\begin{pmatrix}1&1\\-1&1\end{pmatrix} \qquad S^{-1}=\frac{1}{2}\begin{pmatrix}1&-1\\1&1\end{pmatrix}$$

$$\therefore D=\frac{1}{2}\underbrace{\begin{pmatrix}1&-1\\1&1\end{pmatrix}}_{S^{-1}}\underbrace{\begin{pmatrix}1&3\\3&1\end{pmatrix}}_{A}\underbrace{\begin{pmatrix}1&1\\-1&1\end{pmatrix}}_{S}=\begin{pmatrix}-2&0\\0&4\end{pmatrix}$$

the eigenvalues on the diagonal

correct

Prob 16

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \longrightarrow (1-\lambda)(\lambda-2)(\lambda-3)=0$$

$$\Rightarrow \lambda = 1, 2, 3$$

Distinct eigenvalues $\longrightarrow$ A is diagonalizable

and we expect D to be $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Let's find S,

$$E_1 = \begin{bmatrix} 3 & 0 & -2 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \text{span} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 2 & 0 & -2 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} = \text{Span} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} = \text{span} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad S^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 2 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\therefore D = S^{-1} A S = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 2 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$