MAT 203: Calculus III with Applications
Fall 2013

Click here for the content and prerequisite for the course.

Textbook

Larson, Edwards, Multivariable Calculus, 10th edition.

Schedule and location

Instructors

	Name	Office	Office hour	MLC hour	Email
Lecture	Dr. Chi Li	3-120	T/Th 1-2:30pm		chi.li@stonybrook.edu
R01	Xiaojie Wang	P-136	Wed 4-5pm	Wed 5-7pm	wang@math.sunysb.edu
R02	Shaosai Huang	S-240G	Wed 9-10am		shaosai.huang@stonybrook.edu

Syllabus and homework

Homework

Problem sets will be assigned weekly; check the syllabus webpage for the assigments. If you have the 9th edition of the textbook, you can also find the typed or scanned version of homework on the page. The solution will be be given for each homework.

Each homework is due during your recitation class of the following week (unless otherwises stipulated). **No late homework.** The recitation instructor will collect the homework and grade three of the problems.

Write the problem up carefully in your own words even if you have consulted the book for the final answer: always show your work. It is OK to discuss homework problems with other students. However, each student must write up the homework individually in his/her own words, rather than merely copying someone else's.

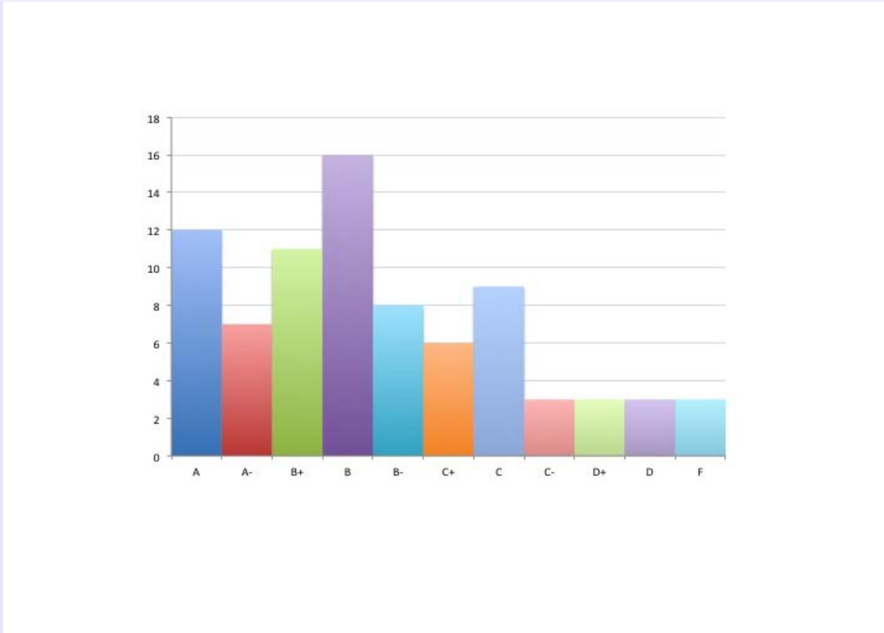
Quizzes

There are quizzes from the homework problems every **three** weeks in the recitation. No make up quiz. Check the syllabus webpage for the schedule.

Grading

15% Homework
10% Quizzes
20% Midterm 1
20% Midterm 2
35% Final Exam

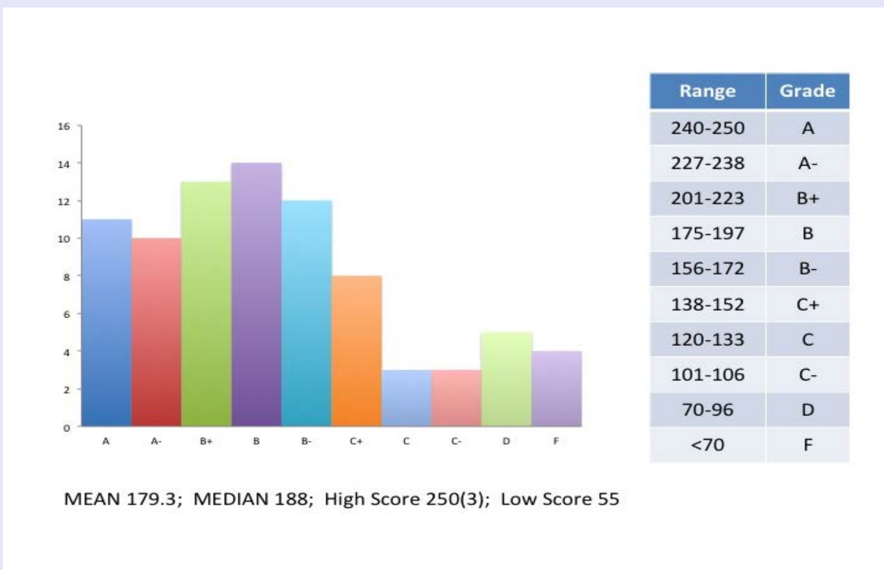
Overall Grades:



Midterm exam

There are two midterms which are are given in class.

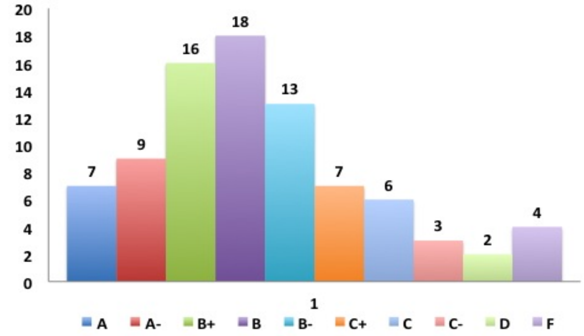
Midterm 2



Range	Grade
240-250	A
227-238	A-
201-223	B+
175-197	B
156-172	B-
138-152	C+
120-133	C
101-106	C-
70-96	D
<70	F

Midterm 1

The tentative curve for this midterm is shown in the picture. This is just to give you some idea of the distribution of the grades. The real curve will be made only after the final exam.

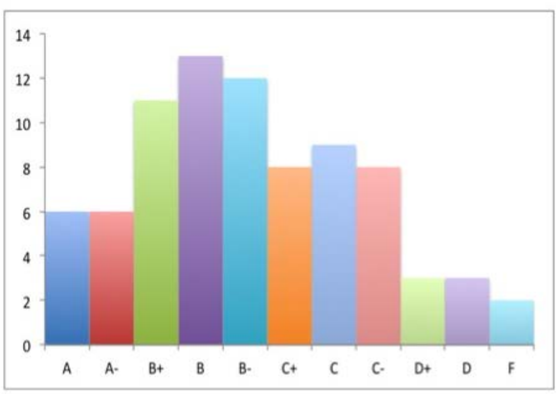


MEAN: 188.294; MEDIAN: 190; High Score: 250; Low Score: 83

Range	Grade
250	A
248-243	A-
240-212	B+
208-181	B
179-156	B-
155-140	C+
126-120	C
114-113	C-
106-105	D
94-83	F

Final exam

Wednesday, December 18, 8:00AM-10:45AM. More details to come.



MEAN 266; MEDIAN 280; High Score 445; Low Score 85

Range	Grade
390-445	A
375-385	A-
320-370	B+
295-315	B
248-285	B-
225-240	C+
180-220	C
147-175	C-
105-135	D+/D
<100	F

Read and Take Notes

Read the relevant materials on the textbook both before and after the lecture. If you really want to master the course, it is wise to attempt or solve as many problems as you can in the relevant section of the book.

Help

A very useful resource is the **Math Learning Center (MLC)** located in room S240-A of the mathematics building basement. The Math Learning Center is open every day and most evenings. Check the schedule on the door.

Another useful resource are your teachers, whose office hours are listed above.

Stony Brook University expects students to maintain standards of personal integrity that are in harmony with the educational goals of the institution; to observe national, state, and local laws as well as University regulations; and to respect the rights, privileges, and property of other people. Faculty must notify the Office of Judicial Affairs of any disruptive behavior that interferes with their ability to teach, compromises the safety of the learning environment, or inhibits students' ability to learn.

DSS advisory: If you have a physical, psychiatric, medical, or learning disability that may affect your ability to carry out the assigned course work, please contact the office of Disabled Student Services (DSS), Humanities Building, room 133, telephone 632-6748/TDD. DSS will review your concerns and determine what accommodations may be necessary and appropriate. All information regarding any disability will be treated as strictly confidential.

Students who might require special evacuation procedures in the event of an emergency are urged to discuss their needs with both the instructor and DSS. For important related information, click here.

Syllabus

Week	Sections	Homework	Notes
8/26-8/30	**11.1:** Vectors in the Plane **11.2:** Space Coordinate, Vectors in Space **11.3:** Dot Product	**11.1:** 8,28,36,50, 84 **11.2:** 6,26,44,58, 84 **11.3:** 4,10 Homework Week 1 (for 9th edition textbook) Solution for homework 1	Hint for HW11.2.84: The velocity of the plane is the sum of the wind-velocity and the air-velocity. The air-velocity is the scalar multiplication of the airspeed with the compass direction) 9/1 Last day to drop a class or withdraw without tuition liability.
9/2-9/6	**11.4:** Cross Product	**11.3:** 14, 42, 57, 62 **11.4:** 1-6, 10, 16, 28, 38 Homework Week 2 (for 9th edition textbook) Solution for homework 2	In problem 11.3.57, the edge is assumed to be adjacent to the diagonal. 9/2 Labor Day: No class
9/9-9/13	**11.5:** Lines and Planes **11.6:** Surfaces in Space	**11.5:** 10, 28, 29, 40, 42, 54, 82 **11.6:** 1-6, 10, 16, 32, 44 Homework Week 3 (for 9th edition textbook) Solution for homework 3	Notes on quadric surface 9/10: Last day to drop or withdraw without a "W"
9/16-9/20	**12.1:** Vector-Valued Function **12.2:** Differentiation, Integration of Vector-Valued Functions **12.3:** Velocity and Acceleration	**12.1:** 2, 19-22, 24, 52, 68, 72 **12.2:** 4, 26, 42, 46, 52, 58 **12.3:** 18, 22 Homework Week 4 (for 9th edition textbook) Solution for homework 4	Quiz 1 in recitation
9/23-9/27	**12.4:** Tangent vectors, principal normal vectors **12.5:** Arc Length **13.1:** Functions of Several Variables	**12.3:** 26, 36 **12.4:** 10, 19, 22, 29-32 **12.5:** 10, 12, 17 **13.1:** 16, 24 Homework Week 5 (for 9th edition textbook) Solution for Homework 5	Example: principal normal vector for a plane curve
9/30-10/4	**13.2:** Limits and Continuity Review for Midterm I **Midterm I**	Practice Midterm 1 Solution to Practice Midterm 1 Solution to midterm: Solution 1 , Solution 2 **Homework 6: 13.1:** 27, 38, 45-48, 52. (for both 10/9th edition textbook) Solutions to Homework 6	10/4: Midterm I in class covering up to 12.5. 10/4: Last day to drop-down or move-up.
10/7-10/11	**13.3:** Partial Derivatives **13.4:** Differentials **13.5:** Chain Rules	**13.2:** 24, 74, 76 **13.3:** 18, 52, 60, 66, 78, 110 **13.4:** 14, 26, 30 **13.5:** 12, 16, 20, 21 Homework Week 7 (for 9th edition textbook) Solution to Homework 7	
10/14-10/18	**13.6:** Directional Derivatives, Gradients **13.7:** Tangent Planes, Normal Lines **13.8:** Extrema of Functions of Two Variables	**The numbers in the round bracket is the numbering for 9th edition textbook** **13.6:** 2,24,36,52,61 (9th: 14,28,40,56,66) **13.7:** 10,18,24,42 (9th: 18,28,34,52) **13.8:** 18,27-30,42,46 (9th: 26,31-34,46,50) Solution to Homework 8	
10/21-10/25	**13.9:** Applications of Extrema **13.10:** Lagrange Multipliers	**13.8:** 6,12 (9th:6, 10) **13.9:** 6, 10, 14, 18 (9th: 6,10,14,16) **13.10:** 6, 24, 34, 46 (9th: 8,26,36,50) Solution to Homework 9	Optimizing problem discussed in class Quiz 2 in recitation 10/25: Last day to drop a course with "W".
	14.1: Iterated Integrals, Area in the plane	**14.1:** 28,32,42,56,60 (9th:28,32,43,58,62)	

10/28-11/1	**14.2:** Double Integrals, Volume **14.4:** Center of Mass	**14.2:** 8,12,18,24,34,48,54 (9th:10,14,20,28,42,56,62) **14.4:** 2,6 (9th:2,6) Solution to Homework 10	
11/4-11/8	**14.3:** Double integrals in Polar Coordinates Review for Midterm II **Midterm II**	Practice Midterm 2 Solution to Practice Midterm 2 Solution to midterm 2: Solution 1 , Solution 2 **Homework 11:** **14.3:** 6,18,22,26,30,40,44 (9th:6,18,22,26,30,40,46) **14.4:** 20,22 (9th:20,22) Solution to Homework 11	11/8: Midterm II in class covering up to 14.2.
11/11-11/15	**14.6:** Triple Integrals **11.7:** Cylindrical and Spherical Coordinates **14.7:** Triple Integrals in Cylindrical and Spherical Coordinates	**14.6:** 6,14,21,26,36,38 (9th:6,16,23,28,38,40) **11.7:** 4,8,26,32,38,44 (9th:4,8,26,32,38,44) **14.7:** 10,12,18,36 (9th:10,12,18,36) Solution to Homework 12	
11/18-11/22	**15.1:** Vector Fields **15.2:** Line Integrals	**14.7:** 26,34 (9th:26,24) **15.1:** 1-4, 8, 34, 40, 44, 60 (9th:1-6, 9, 40, 46, 50, 66) **15.2:** 6,10,16,20,22 (9th:6,10,16,20,22) Solution to Homework 13	Hand in homework 13 and 14 together in the week after thanksgiving
11/25-11/29	**15.3:** Conservative Vector Fields, Independence of Path	(for both 9/10th edition) **15.2:** 30,38,41-44,54,72,78 **15.3:** 2,16,26,36 Solution to Homework 14	11/29: Thanksgiving no class
12/2-12/6	**15.4:** Green's Theorem Review for Final	**15.3:** 30,34 (9th:30,34) **15.4:** 2,4,10,22,38,44,47,48 (9th: 2,4,10,22,38,44^{10},44,45,46) Solution to Homework 15	Quiz 3 in recitation covering 14.3-14.7 12/6: Last Day of Class
12/9-12/13		Practice Final Solution to Practice Final	No class: Reading Period
12/18	**Final exam**	December 18, 8:00-10:45 am.	

1 1.1

8. Find the vectors whose initial and terminal points are given. Show that $\vec{u}$ and $\vec{v}$ are equivalent.

$$\vec{u} : (-4, -1), (11, -4); \quad \vec{v} : (10, 13), (25, 10)$$

28. The vector $\vec{v}$ and its initial point are given. Find the terminal point.

$$\vec{v} = \langle 4, -9 \rangle;\ \text{Initial point:}(5, 3).$$

36. Find the unit vector in the direction of $\vec{v}$ and verify that it has length 1.

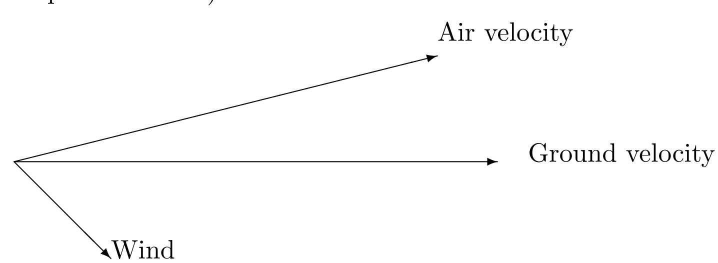
$$\vec{v} = \langle -5, 15 \rangle$$

50. Find the component form of $\vec{v}$ given its magnitude and the angle it makes with the positive x-axis.

$$\|\vec{v}\| = 5, \quad \theta = 120^\circ.$$

84. A plane flies at a constant groundspeed of 400 miles per hour due east and encounters a 50-miles-per-hour wind from the northwest. Find the airspeed and compass direction that will allow the plane to maintain its groundspeed and eastward direction.

(Hint: The velocity of the plane is the sum of the wind-velocity and the air-velocity. The air-velocity is the scalar multiplication of the airspeed with the compass direction)



2 1.2

6. Find the coordinates of the point. The point is located seven units in front of the yz-plane, two units to the left of the xz-plane, and one unit below the xy-plane.

26. Find the distance between the points.

$$(2, 2, 3), \quad (4, -5, 6)$$

44. Complete the square to write the equation of the sphere in standard form. Find the center and radius.

$$4x^2 + 4y^2 + 4z^2 - 24x - 4y + 8z - 23 = 0.$$

58. Given $\vec{u} = \langle 1, 2, 3 \rangle$, $\vec{v} = \langle 2, 2, -1 \rangle$, and $\vec{w} = \langle 4, 0, -4 \rangle$. Find

$$\vec{z} = 5\vec{u} - 3\vec{v} - \frac{1}{2}\vec{w}.$$

84. Find the vector $\vec{v}$ with the given magnitude and the same direction as $\vec{u}$

$$\text{Magnitude: } \|\vec{v}\| = 3; \quad \text{Direction: } \vec{u} = \langle 1, 1, 1 \rangle.$$

3 1.3

4. Find **(a)** $\vec{u} \cdot \vec{v}$, **(b)** $\vec{u} \cdot \vec{u}$, **(c)** $\|\vec{u}\|^2$, **(d)** $(\vec{u} \cdot \vec{v})\vec{v}$, and **(e)** $\vec{u} \cdot (2\vec{v})$.

$$\vec{u} = \langle -4, 8 \rangle, \quad \vec{v} = \langle 7, 5 \rangle.$$

10. Find the θ between the vectors **(a)** in radians and **(b)** in degrees.

$$\vec{u} = \langle 3, 1 \rangle, \quad \vec{v} = \langle 2, -1 \rangle$$

1 1.1

8. Find the vectors whose initial and terminal points are given. Show that $\vec{u}$ and $\vec{v}$ are equivalent.

$$\vec{u} : (-4, -1), (11, -4); \quad \vec{v} : (10, 13), (25, 10)$$

Solution: $\vec{u} = \langle 15, -3 \rangle$, $\vec{v} = \langle 15, -3 \rangle$. So $\vec{u} = \vec{v}$.

28. The vector $\vec{v}$ and its initial point are given. Find the terminal point.

$$\vec{v} = \langle 4, -9 \rangle; \text{ Initial point:}(5, 3).$$

Solution: Terminal point is $(9, -6)$.

36. Find the unit vector in the direction of $\vec{v}$ and verify that it has length 1.

$$\vec{v} = \langle -5, 15 \rangle$$

Solution: Unit vector in the direction of $\vec{v}$ is

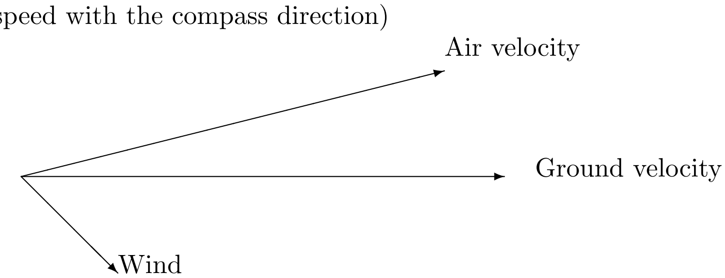
$$\frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle -5, 15 \rangle}{\|\langle -5, 15 \rangle\|} = \left\langle -\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle.$$

50. Find the component form of $\vec{v}$ given its magnitude and the angle it makes with the positive x-axis.

$$\|\vec{v}\| = 5, \quad \theta = 120°.$$

Solution: $\vec{v} = \|\vec{v}\| \langle \cos\theta, \sin\theta \rangle = \left\langle -\frac{5}{2}, \frac{5\sqrt{3}}{2} \right\rangle$.

84. A plane flies at a constant groundspeed of 400 miles per hour due east and encounters a 50-miles-per-hour wind from the northwest. Find the airspeed and compass direction that will allow the plane to maintain its groundspeed and eastward direction. (Hint: The velocity of the plane is the sum of the wind-velocity and the air-velocity. The air-velocity is the scalar multiplication of the airspeed with the compass direction)



Solution: Ground velocity=$\langle 400, 0 \rangle$. Wind velocity=$\langle 25\sqrt{2}, -25\sqrt{2} \rangle$. So air velocity is

$$\vec{v} = \langle 400 - 25\sqrt{2}, 25\sqrt{2} \rangle \approx \langle 364.6, 35.4 \rangle.$$

The airspeed is $\|\vec{v}\| = 366.4$ miles per hour. The compass direction is

$$\frac{\vec{v}}{\|\vec{v}\|} \approx \langle 0.995, 0.097 \rangle.$$

Or equivalently, the angle θ is about $\arctan(25\sqrt{2}/(400 - 25\sqrt{2})) \approx 5.5°$ north of due east.

2 1.2

6. Find the coordinates of the point. The point is located seven units in front of the yz-plane, two units to the left of the xz-plane, and one unit below the xy-plane.

Solution: $(7, 2, -1)$.

26. Find the distance between the points.

$$(2, 2, 3), \quad (4, -5, 6)$$

Solution: distance$=\sqrt{(4-2)^2+(-5-2)^2+(6-3)^2} = \sqrt{62}$.

44. Complete the square to write the equation of the sphere in standard form. Find the center and radius.

$$4x^2 + 4y^2 + 4z^2 - 24x - 4y + 8z - 23 = 0.$$

Solution: Standard form:

$$(x-3)^2 + (y - \frac{1}{2})^2 + (z+1)^2 = 16.$$

The center is $(3, 1/2, -1)$. Radius is 4.

58. Given $\vec{u} = \langle 1, 2, 3\rangle$, $\vec{v} = \langle 2, 2, -1\rangle$, and $\vec{w} = \langle 4, 0, -4\rangle$. Find

$$\vec{z} = 5\vec{u} - 3\vec{v} - \frac{1}{2}\vec{w}.$$

Solution: $\vec{z} = \langle -3, 4, 20\rangle$.

84. Find the vector $\vec{v}$ with the given magnitude and the same direction as $\vec{u}$

$$\text{Magnitude: } \|\vec{v}\| = 3; \quad \text{Direction: } \vec{u} = \langle 1, 1, 1\rangle.$$

Solution: $\vec{v} = \|\vec{v}\|\frac{\vec{u}}{\|\vec{u}\|} = \langle \sqrt{3}, \sqrt{3}, \sqrt{3}\rangle$.

3 1.3

4. Find **(a)** $\vec{u}\cdot\vec{v}$, **(b)** $\vec{u}\cdot\vec{u}$, **(c)** $\|\vec{u}\|^2$, **(d)** $(\vec{u}\cdot\vec{v})\vec{v}$, and **(e)** $\vec{u}\cdot(2\vec{v})$.

$$\vec{u} = \langle -4, 8\rangle, \quad \vec{v} = \langle 7, 5\rangle.$$

Solution: **(a)** 12. **(b)** 80. **(c)** 80. **(d)** $\langle 84, 60\rangle$. **(e)** 24.

10. Find the θ between the vectors **(a)** in radians and **(b)** in degrees.

$$\vec{u} = \langle 3, 1\rangle, \quad \vec{v} = \langle 2, -1\rangle$$

Solution:

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{\sqrt{2}}{2}.$$

So $\theta = \frac{\pi}{4} = 45°$.

1 1.3

14: Find the angle θ between the vectors **(a)** in radians and **(b)** in degrees

$$\vec{u} = 3\vec{i} + 2\vec{j} + \vec{k}, \quad \vec{v} = 2\vec{i} - 3\vec{j}.$$

42: **(a)** Find the projection of $\vec{u}$ onto $\vec{v}$, and **(b)** Find the vector component of $\vec{u}$ orthogonal to $\vec{v}$.

$$\vec{u} = \vec{i} + 4\vec{k}, \quad \vec{v} = 3\vec{i} + 2\vec{k}.$$

57: Find the angle between the diagonal of a cube and one of its adjacent edges. Show your work.

62: A toy wagon is pulled by exerting a force of 25 pounds on a handle that makes a $20°$ angle with the horizontal. Find the work done in pulling the wagon 50 feet.

2 1.4

1-6: Find the cross product of unit vectors and sketch your result in pictures:
1: $\vec{j} \times \vec{i}$ **2:** $\vec{i} \times \vec{j}$ **3:** $\vec{j} \times \vec{k}$ **4:** $\vec{k} \times \vec{j}$ **5:** $\vec{i} \times \vec{k}$ **6:** $\vec{k} \times \vec{i}$.

10: Find **(a)** $\vec{u} \times \vec{v}$, **(b)** $\vec{v} \times \vec{u}$ and **(c)** $\vec{v} \times \vec{v}$.

$$\vec{u} = \langle 3, -2, -2 \rangle, \quad \vec{v} = \langle 1, 5, 1 \rangle.$$

16: Find $\vec{u} \times \vec{v}$ and show that it is orthogonal to both $\vec{u}$ and $\vec{v}$.

$$\vec{u} = \vec{i} + 6\vec{j}, \quad \vec{v} = -2\vec{i} + \vec{j} + \vec{k}.$$

28: Find the area of the triangle with given vertices.

$$\mathbf{A}(2, -3, 4), \quad \mathbf{B}(0, 1, 2), \quad \mathbf{C}(-1, 2, 0).$$

(Hint: $\frac{1}{2}\|\vec{u} \times \vec{v}\|$ is the area of the triangle having $\vec{u}$ and $\vec{v}$ as adjacent sides.)

38: Use the triple scalar product to find the volume of the parallelpiped having adjacent edges $\vec{u}$, $\vec{v}$ and $\vec{w}$.

$$\vec{u} = \langle 1, 3, 1 \rangle, \quad \vec{v} = \langle 0, 6, 6 \rangle, \quad \langle -4, 0, -4 \rangle.$$

1 1.3

14: Find the angle θ between the vectors **(a)** in radians and **(b)** in degrees

$$\vec{u} = 3\vec{i} + 2\vec{j} + \vec{k}, \quad \vec{v} = 2\vec{i} - 3\vec{j}.$$

Solution: $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = 0$. So $\theta = \frac{\pi}{2} = 90°$.

42: **(a)** Find the projection of $\vec{u}$ onto $\vec{v}$, and **(b)** Find the vector component of $\vec{u}$ orthogonal to $\vec{v}$.

$$\vec{u} = \vec{i} + 4\vec{k}, \quad \vec{v} = 3\vec{i} + 2\vec{k}.$$

Solution:

$$(\mathbf{a}): \quad \mathrm{Proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{11}{13}(3\vec{i} + 2\vec{k}).$$

(b): Vector component orthogonal to $\vec{v}$:

$$\vec{u} - \mathrm{Proj}_{\vec{v}}\vec{u} = \frac{-20\vec{i} + 30\vec{j}}{13}.$$

57: Find the angle between the diagonal of a cube and one of its adjacent edges. Show your work.

Solutions: Vector in the direction of diagonal $\vec{u} = \langle 1, 1, 1\rangle$. Vector along adjacent side $\vec{v} = \langle 1, 0, 0\rangle$. So $\cos\theta = \frac{1}{\sqrt{3}}$ and $\theta = Arccos(1/\sqrt{3}) \approx 54.7°$.

62: A toy wagon is pulled by exerting a force of 25 pounds on a handle that makes a 20° angle with the horizontal. Find the work done in pulling the wagon 50 feet.

Solutions: Work=$25 \times \cos(20°) \times 50 = 1174.62$ ft-lbs.

2 1.4

1-6: Find the cross product of unit vectors and sketch your result in pictures:
1: $\vec{j} \times \vec{i}$ **2:** $\vec{i} \times \vec{j}$ **3:** $\vec{j} \times \vec{k}$ **4:** $\vec{k} \times \vec{j}$ **5:** $\vec{i} \times \vec{k}$ **6:** $\vec{k} \times \vec{i}$.
Solution: 1: $-\vec{k}$ **2:** $\vec{k}$ **3:** $\vec{i}$ **4:** $-\vec{i}$ **5:** $-\vec{j}$ **6:** $\vec{j}$.

10: Find **(a)** $\vec{u} \times \vec{v}$, **(b)** $\vec{v} \times \vec{u}$ and **(c)** $\vec{v} \times \vec{v}$.

$$\vec{u} = \langle 3, -2, -2\rangle, \quad \vec{v} = \langle 1, 5, 1\rangle.$$

Solution: (a): $\langle 8, -5, 17\rangle$, **(b):** $\langle -8, 5, -17\rangle$, **(c):** 0.

16: Find $\vec{u} \times \vec{v}$ and show that it is orthogonal to both $\vec{u}$ and $\vec{v}$.

$$\vec{u} = \vec{i} + 6\vec{j}, \quad \vec{v} = -2\vec{i} + \vec{j} + \vec{k}.$$

Solution: $\vec{u} \times \vec{v} = \langle 5, -3, 13\rangle$.

28: Find the area of the triangle with given vertices.

$$\mathbf{A}(2,-3,4), \quad \mathbf{B}(0,1,2), \quad \mathbf{C}(-1,2,0).$$

(Hint: $\frac{1}{2}\|\vec{u}\times\vec{v}\|$ is the area of the triangle having $\vec{u}$ and $\vec{v}$ as adjacent sides.

Solution: $\overrightarrow{\mathbf{AB}} = \langle -2,4,-2\rangle$, $\overrightarrow{\mathbf{AC}} = \langle -3,5,-4\rangle$.

$$\overrightarrow{\mathbf{AB}} \times \overrightarrow{\mathbf{AC}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 4 & -2 \\ -3 & 5 & -4 \end{vmatrix} = \langle -6,-2,2\rangle.$$

$$\text{Area}(\triangle_{\mathbf{ABC}}) = \frac{1}{2}\left|\overrightarrow{\mathbf{AB}} \times \overrightarrow{\mathbf{AC}}\right| = \sqrt{11}.$$

38: Use the triple scalar product to find the volume of the parallelpiped having adjacent edges $\vec{u}$, $\vec{v}$ and $\vec{w}$.

$$\vec{u} = \langle 1,3,1\rangle, \quad \vec{v} = \langle 0,6,6\rangle, \quad \vec{w} = \langle -4,0,-4\rangle.$$

Solution:

$$\vec{u}\times\vec{v}\cdot\vec{w} = \begin{vmatrix} 1 & 3 & 1 \\ 0 & 6 & 6 \\ -4 & 0 & -4 \end{vmatrix} = -72.$$

The volume of the parallelepiped=72.

1 1.5

10: Find sets of **(a)** parametric equations and **(b)** symmetric equations of the line through the two points. (if possible). (For each line, write the direction numbers as integers.)

$$(0, 4, 3), \quad (-1, 2, 5).$$

28: Determine whether any of the lines are parallel or identical.

$$L_1 : \frac{x-3}{2} = \frac{y-2}{1} = \frac{z+2}{2}.$$

$$L_2 : \frac{x-1}{4} = \frac{y-1}{2} = \frac{z+3}{4}.$$

$$L_3 : \frac{x+2}{1} = \frac{y-1}{0.5} = \frac{z-3}{1}.$$

$$L_4 : \frac{x-3}{2} = \frac{y+1}{4} = \frac{z-2}{-1}.$$

29: Determine whether the lines intersect, and if so, find the point of intersection and the cosine of the angle of intersection.

$$x = 4t + 2, \quad y = 3, \quad z = -t + 1$$

$$x = 2s + 2, \quad y = 2s + 3, \quad z = s + 1.$$

40: Find an equation of the plane passing through the point perpendicular to the given line.

$$(3, 2, 2); \quad \frac{x-1}{4} = y + 2 = \frac{z+3}{-3}.$$

42: Find an equation of the plane which passes through $(3, -1, 2)$, $(2, 1, 5)$, and $(1, -2, -2)$.

54: Find an equation of the plane that contains all the points that are equidistant from the given points.

$$(1, 0, 2), \quad (2, 0, 1).$$

82: Find the distance between the point and the plane

$$(0, 0, 0), \quad 5x + y - z = 9.$$

2 1.6

1-6 in text book.

10: Describe and sketch the surface $y^2 + z = 6$.

16: Classify and sketch the quadric surface.

$$-8x^2 + 18y^2 + 18z^2 = 2.$$

32: Find the equation for the surface of revolution formed by revolving the curve in the indicated coordinate plane about the given axis Equation of curve: $z = 3y$. Coordinate plane: yz-plane. Axis of revolution: y-axis

44: Find an equation of the surface satisfying the conditions and identify the surface: the set of all points equidistant from the point $(0, 0, 4)$ and the xy-plane.

1 1.5

10: Find sets of **(a)** parametric equations and **(b)** symmetric equations of the line through the two points. (if possible). (For each line, write the direction numbers as integers.)

$$(0,4,3), \quad (-1,2,5).$$

Solution: **(a)** Parametric equation:

$$x = -t, y = 4 - 2t, z = 3 + 2t.$$

(b) Symmetric equations:

$$\frac{x}{-1} = \frac{y-4}{-2} = \frac{z-3}{2}.$$

28: Determine whether any of the lines are parallel or identical.

$$L_1 : \frac{x-3}{2} = \frac{y-2}{1} = \frac{z+2}{2}.$$

$$L_2 : \frac{x-1}{4} = \frac{y-1}{2} = \frac{z+3}{4}.$$

$$L_3 : \frac{x+2}{1} = \frac{y-1}{0.5} = \frac{z-3}{1}.$$

$$L_4 : \frac{x-3}{2} = \frac{y+1}{4} = \frac{z-2}{-1}.$$

Solution: L_1, L_2 and L_3 have the parallel direction vector $\langle 2,1,2\rangle$. So they are parallel to each other. Pick points $P_1 = (3,2,-2) \in L_1$, $P_2 = (1,1,-3) \in L_2$, $P_3 = (-2,1,3) \in L_3$. By calculation, P_i does not lie on the L_j if $i \neq j$. So they are parallel but different lines.

29: Determine whether the lines intersect, and if so, find the point of intersection and the cosine of the angle of intersection.

$$x = 4t + 2, \quad y = 3, \quad z = -t + 1$$

$$x = 2s + 2, \quad y = 2s + 3, \quad z = s + 1.$$

Solution Solve these equations simultaneously, one gets solution $t = 0$, $s = 0$. So the two lines intersect at a point $(2,3,1)$.

40: Find an equation of the plane passing through the point perpendicular to the given line.

$$(3,2,2); \quad \frac{x-1}{4} = y + 2 = \frac{z+3}{-3}.$$

Solution: The plane has normal vector $\langle 4,1,-3\rangle$.

$$4(x-3) + (y-2) - 3(z-2) = 0.$$

In simpler form: $4x + y - 3z = 8$.

42: Find an equation of the plane which passes through $(3, -1, 2)$, $(2, 1, 5)$, and $(1, -2, -2)$.

Solution: $\overrightarrow{AB} = \langle -1, 2, 3\rangle$. $\overrightarrow{AC} = \langle -2, -1, -4\rangle$. A normal vector for the plane is $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \langle -5, -10, 5\rangle = 5\langle -1, -2, 1\rangle$. So the equation for the plane is:

$$-1(x-3) - 2(y+1) + 1(z-2) = 0 \Longleftrightarrow x + 2y - z = -1.$$

It's better to check the points are indeed on the plane.

54: Find an equation of the plane that contains all the points that are equidistant from the given points.

$$(1, 0, 2), \quad (2, 0, 1).$$

Solution: By geometry, the required plane passes through the middle point C of $\overline{AB}$ and is perpendicular to the vector $\overrightarrow{AB}$. The middle point C of $\overline{AB}$ satisfies $\overrightarrow{OC} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB}) = \frac{1}{2}\langle 3, 0, 3\rangle$. O is the origin. So $C = \frac{1}{2}(3, 0, 3)$. The vector $\overrightarrow{AB} = \langle 1, 0, -1\rangle$. So the plane has equation:

$$(x - 3/2) + 0(y - 0) - (z - 3/2) = 0 \Longleftrightarrow x - z = 0.$$

82: Find the distance between the point and the plane

$$(0, 0, 0), \quad 5x + y - z = 9.$$

Solution: Choose a point, e.g. $Q = (2, -1, 0)$ on the plane. Let $P = (0, 0, 0)$ be the given point. The normal vector for the plane is $\vec{n} = \langle 5, 1, -1\rangle$. The distance is equal to

$$\left\|\mathrm{Proj}_{\vec{n}}\overrightarrow{QP}\right\| = \frac{\|\overrightarrow{QP} \cdot \vec{n}\|}{\|\vec{n}\|} = \frac{9}{\sqrt{27}} = \sqrt{3}.$$

2 1.6

1-6 in text book.

Solution: (a)$\leftrightarrow$ 6: hyperbolic paraboloid; (b)$\leftrightarrow$ 4: elliptical cone; (c)$\leftrightarrow$ 1: ellipsoid; (d)$\leftrightarrow$ 5: elliptical paraboloid; (e)$\leftrightarrow$ (2): hyperboloid of two sheets; (f)$\leftrightarrow$ (3): hyperboloid of one sheet.

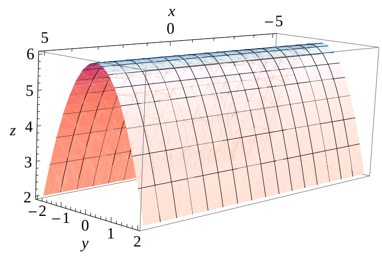


Figure 1: $y^2 + z = 6$

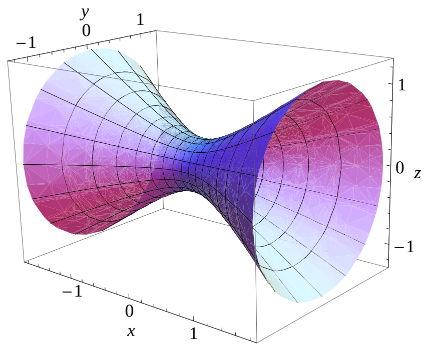


Figure 2: $-8x^2 + 18y^2 + 18z^2 = 2$

10: Describe and sketch the surface $y^2 + z = 6$. **Solution:** This is a cylindrical surface, parallel to the x-axis.

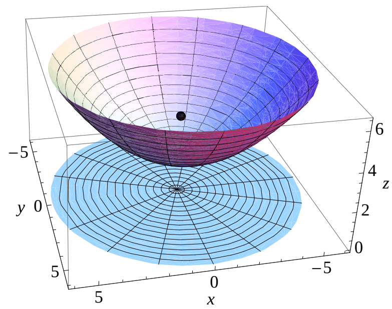
16: Classify and sketch the quadric surface.

$$-8x^2 + 18y^2 + 18z^2 = 2.$$

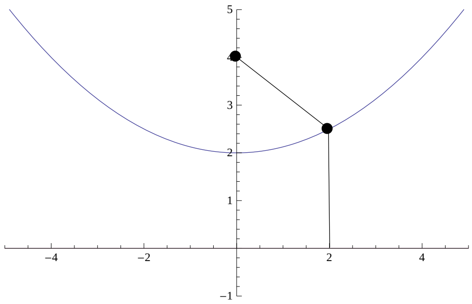
Solution: This is a hyperboloid of one sheet. The axis variable is x.

32: Find the equation for the surface of revolution formed by revolving the curve in the indicated coordinate plane about the given axis Equation of curve: $z = 3y$. Coordinate plane: yz-plane. Axis of revolution: y-axis

Solution: $x^2 + z^2 = 9y^2$. This is an elliptical cone.



(a) $x^2 + y^2 - 8z + 16 = 0$



(b) $x^2 - 8z + 16 = 0$

Figure 3: $x^2 + y^2 - 8z + 16 = 0$

44: Find an equation of the surface satisfying the conditions and identify the surface: the set of all points equidistant from the point $(0, 0, 4)$ and the xy-plane.

Solution:

$$\sqrt{x^2 + y^2 + (z-4)^2} = |z| \iff x^2 + y^2 - 8z + 16 = 0.$$

This is an elliptical paraboloid with vertex $(0, 0, 2)$.

Step 0: If three variables have the same sign **and** all have order 2, then it's an ellipsoid. Otherwise do the following.

Step 1: Find the axis variable for the surface. The axis variable is the variable which has different sign or different degree.

Step 2: Figure out if there are gaps or restrictions on the axis variable.

Step 3: Change the axis variable from 0 to find slices as one changes the axis variable to cut out the surface.

Step 4: Combine and connect the slices to visualize the surface.

Example:

1: $z^2 - x^2 - \frac{y^2}{4} = 1$.
Step 1: This axis variable is z. $z^2 = x^2 + \frac{y^2}{4} + 1$.
Step 2: There is the gap: $|z| \geq 1$, i.e. $z \in (-\infty, -1] \cup [1, \infty)$.
Step 3: The slice when $|z| = 1$ is a point. The slice for $|z| > 1$ is ellipse.
Step 4: It's the hyperboloid of 2 sheets

2: $x^2 - 2y^2 - 2z^2 = 0$.
Step 1: The axis variable is x. $x^2 = 2y^2 + 2z^2$.
Step 2: Gap is zero $|x| \geq 0$, i.e. $x \in (-\infty, +\infty)$.
Step 3: The slice when $x = 0$ is a point. The slices are ellipses when $|x| > 0$.
Step 4: This is elliptic cone.

3: $x^2 + 4y^2 - z = 0$.
Step 1: The axis variable is z. $x^2 + 4y^2 = z$.
Step 2: There is restriction $z \geq 0$.
Step 3: The slice when $z = 0$ is a point. The slices when $z > 0$ are ellipses.
Step 4: This is elliptic paraboloid.

4: $3x + y^2 - z^2 = 0$.
Step 1: The axis variable is x. $3x = -y^2 + z^2$.
Step 2: There is no restriction on x, i.e. $x \in (-\infty, +\infty)$.
Step 3: The slices are hyperbola.
Step 4: This is hyperbolic paraboloid.

5: $16x^2 - y^2 + 16z^2 = 4$.
Step 1: The axis variable is y. $y^2 + 4 = 16x^2 + 16z^2$.
Step 2: There is no gap for y, i.e. $y \in (-\infty, +\infty)$.
Step 3: The slice when $y = 0$ is a small ellipse $16x^2 + 16z^2 = 4$. The slices when $|y| > 0$ are growing ellipses as one increases $|y|$.
Step 4: This is hyperboloid of one sheet.

6: $1 - x^2 - y^2/4 - 16z^2 = 0$.
Step 0: All variables have the same sign and same order. It's an ellipsoid.

(The numbering for most problems is according to 9-th edition. However, 26^{10} means the problem 26 in edition 10.)

1 12.1

Homework: **2, 21-24, 28, 56, 74, 78**

2 12.2

Homework: **6**
Homework: $\mathbf{26}^{10}$: Find **(a)** $\vec{r}\,'(t)$, **(b)** $\vec{r}\,''(t)$, **(c)** $\vec{r}\,'(t)\cdot\vec{r}\,''(t)$, **(d)** $\vec{r}\,'(t)\times\vec{r}\,''(t)$.

$$\vec{r}(t) = t^3\vec{i} + (2t^2+3)\vec{j} + (3t-5)\vec{k}.$$

Homework: **46, 56, 62, 68**

3 12.3

Homework: **20, 26**

(2^{10} **means problem 2 in 10-th edition textbook,** 6^9 **means problem 6 in 9-th edition textbook**)

1 12.1

$\mathbf{2^{10}} = \mathbf{2^9}$: Domain$=\{t; |t| \le 2\} = [-2, 2]$.

$\mathbf{19^{10}} - \mathbf{22^{10}} = \mathbf{21^9} - \mathbf{24^9}$: One can use the projection of the curve onto xy-plane to match the curve.

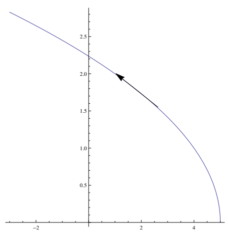
(a)$\leftrightarrow$ 22. The projection of the curve to the xy-plane is the graph of function $y = \ln x$.

(b)$\leftrightarrow$ 19. The projection to the xy-plane is the line $y = 2x$.

(c)$\leftrightarrow$ 20. The projection to the xy-plane is the circle $x^2 + y^2 = 1$.

(d)$\leftrightarrow$ 21. The projection to the xy-plane is the parabola: $y = x^2$.

$\mathbf{24^{10}} = \mathbf{28^9}$: Cancelling the parameter t, we get half of the parabola: $\{(x, y); x = 5 - y^2, y \ge 0\}$. When t increases, x decreases. The curve with the orientation are shown below



$\mathbf{52^{10}} = \mathbf{56^9}$: $\vec{r}(t) = 3\cos t\ \vec{i} + 4\sin t\ \vec{j}$.

$\mathbf{68^{10}} = \mathbf{74^9}$:

$$\text{The limit} = \lim_{t\to\infty} e^{-t}\vec{i} + \lim_{t\to\infty} \frac{1}{t}\vec{j} + \lim_{t\to\infty} \frac{t}{t^2+1}\vec{k} = \vec{0}.$$

$\mathbf{72^{10}} = \mathbf{78^9}$: $\vec{r}(t)$ is continuous in its domain $\{t \in \mathbb{R}; t > 1\}$.

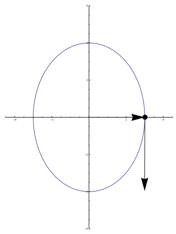
2 12.2

$\mathbf{4^{10}} = \mathbf{6^9}$: $\vec{r}\,'(t) = 3\cos t\vec{i} - 4\sin t\vec{j}$. The curve is an ellipse: $\frac{x^2}{9} + \frac{y^2}{16} = 1$. $\vec{r}(\pi/2) = 3\vec{i}$. $\vec{r}(\pi/2) = -4\vec{j}$.

$\mathbf{26^{10}}$: **(a):** $\vec{r}\,'(t) = 3t^2\vec{i} + 4t\vec{j} + 3\vec{k}$. **(b):** $\vec{r}\,''(t) = 6t\vec{i} + 4\vec{j}$. **(c):** $\vec{r}\,'(t)\cdot\vec{r}\,''(t) = 18t^3 + 16t$. **(d):** $\vec{r}\,'(t)\times\vec{r}\,''(t) = -12\vec{i} + 18t\vec{j} - 12t^2\vec{k}$.

$\mathbf{42^{10}} = \mathbf{46^9}$: **(a):**

1. $\vec{r}(t)\cdot\vec{u}(t) = \sin t + t^2$. $\frac{d}{dt}(\vec{r}(t)\cdot\vec{u}(t)) = \cos t + 2t$.



2. $\frac{d}{dt}\vec{r}(t) = -\sin t\vec{i} + \cos t\vec{j} + \vec{k}$. $\frac{d}{dt}\vec{u}(t) = \vec{k}$. If we use the product rule:

$$\frac{d}{dt}\vec{r}(t) \cdot \vec{u}(t) + \vec{r}(t) \cdot \frac{d}{dt}\vec{u}(t) = (\cos t + t) + t = \cos t + 2t.$$

So the two ways give the same result. **(b):**

1. The cross product:

$$\vec{r}(t) \times \vec{u}(t) = (t\sin t - t)\vec{i} - t\cos t\vec{j} + \cos t\vec{k}.$$

$$\frac{d}{dt}(\vec{r}(t) \times \vec{u}(t)) = (\sin t + t\cos t - 1)\vec{i} + (-\cos t + t\sin t)\vec{j} - \sin t\vec{k}.$$

2. If we use the product rule

$$\begin{aligned} & \frac{d}{dt}\vec{r}(t) \times \vec{u}(t) + \vec{r}(t) \times \frac{d}{dt}\vec{u}(t) = \left[(t\cos t - 1)\vec{i} + t\sin t\vec{j} - \sin t\vec{k}\right] + \left[\sin t\vec{i} - \cos t\vec{j}\right] \\ = & \quad (t\cos t + \sin t - 1)\vec{i} + (t\sin t - \cos t)\vec{j} - \sin t\vec{k}. \end{aligned}$$

The two ways give the same result.

$\mathbf{46^{10} = 56^{9}}$: For the $\vec{i}$ component, use integration by parts

$$\int \ln t dt = t\ln t - \int t d\ln t = t\ln t - t + C_1.$$

So

$$\int \left(\ln t\vec{i} + \frac{1}{t}\vec{j} + \vec{k}\right) dt = (t\ln t - t)\vec{i} + \ln|t|\vec{j} + t\vec{k} + \vec{C}.$$

$\mathbf{52^{10} = 62^{9}}$:

$$\int_{-1}^{1} (t\vec{i} + t^3\vec{j} + t^{1/3}\vec{k})dt = \left.\frac{1}{2}t^2\vec{i} + \frac{1}{4}t^4\vec{j} + \frac{3}{4}t^{4/3}\right]_{-1}^{1} = 0.$$

$\mathbf{58^{10} = 68^{9}}$:

$$\begin{aligned} \vec{r}(t) &= \int_0^t \vec{r}\,'(s)ds + \vec{r}(0) = \int_0^t (3t^2\vec{j} + 6\sqrt{t}\vec{k})dt + (\vec{i} + 2\vec{j}) \\ &= (t^3\vec{j} + 4t^{3/2}\vec{k}) + (\vec{i} + 2\vec{j}) = \vec{i} + (t^3 + 2)\vec{j} + 4t^{3/2}\vec{k}. \end{aligned}$$

3 12.3

$18^{10} = 20^{9}$: (a): Velocity: $\vec{v}(t) = \vec{r}\,'(t) = \langle 1/t, -1/t^2, 4t^3 \rangle$. Speed: $v = \|\vec{v}\| = \sqrt{16t^{10} + t^2 + 1}/t^2$.

Acceleration: $\vec{a}(t) = \vec{r}\,''(t) = \langle -t^{-2}, 2t^{-3}, 12t^2 \rangle$.

(b): When $t = 2$, the velocity $\vec{v}(2) = \langle 1/2, -1/4, 32 \rangle$, the acceleration $\vec{a}(2) = \langle -1/4, 1/4, 48 \rangle$.

$22^{10} = 26^{9}$:

$$\vec{v}(t) = \int_0^t \vec{a}(s)ds + \vec{v}(0) = 3\vec{i} - 2\vec{j} + (-32t + 1)\vec{k}.$$

$$\vec{r}(t) = \int_0^t \vec{v}(s)ds + \vec{r}(0) = 3t\vec{i} + (-2t + 5)\vec{j} + (-16t^2 + t + 2)\vec{k}.$$

So $\vec{r}(2) = 6\vec{i} + \vec{j} - 60\vec{k}$.

(If you have 10th edition textbook, don't look at this. Use this only when you have 9-th edition textbook. The numbering is according to 9-th edition.)

1 12.3

Homework: **30, 42**

2 12.4

Homework: **14, 29, 36, 45-48**

3 12.5

Homework: **10, 12, 19**

4 13.1

Homework: **16, 24**

(26^{10} **means problem 26 in 10-th edition textbook,** 30^9 **means problem 30 in 9-th edition textbook**)

1 12.3

$\mathbf{26^{10}} = \mathbf{30^{9}}$: Note the acceleration is $\vec{a} = -32\vec{j}$. The path of the projectile is described by the vector-valued function:

$$\vec{r}(t) = \langle 450\sqrt{2}\ t, 3 + 450\sqrt{2}\ t - 16t^2 \rangle.$$

1. Maximal height: $450\sqrt{2} - 32t_1 = 0 \Rightarrow t_1 = \frac{450}{32}\sqrt{2}$. So the maximal height is:
$$h_{\max} = y(t_1) = 3 + 450\sqrt{2}\frac{450}{32}\sqrt{2} - 16\frac{450^2}{32^2}2 \approx 6331.13ft.$$
2. Range: $3 + 450\sqrt{2}t_2 - 16t_2^2 = 0 \Rightarrow t_2 = 39.78$. The range is $450\sqrt{2}t_2 = 25315.5$ ft.

$\mathbf{36^{10}} = \mathbf{42^{9}}$: Note that $12^\circ = \frac{12}{180}\pi = \frac{\pi}{15}$. The corresponding vector-valued function is

$$\vec{r}(t) = \langle v_0 \left(\cos\frac{\pi}{15}\right) t, v_0 \left(\sin\frac{\pi}{15}\right) t - 16t^2 \rangle.$$

The time when the projectile hits the ground is obtained by

$$v_0(\sin(\pi/15))t_2 - 16t_2^2 = 0 \Longrightarrow t_2 = \frac{v_0}{16}\sin(\pi/15).$$

The range for the projectile is

$$R = v_0 t_2 = \frac{v_0^2}{16}(\sin\frac{\pi}{15})(\cos\frac{\pi}{15}) = \frac{v_0^2}{32}\sin\frac{2\pi}{15}.$$

So

$$R = 200 = \frac{v_0^2}{32}\sin\frac{2\pi}{15} \Longrightarrow v_0 = \sqrt{\frac{200 \cdot 32}{\sin\frac{2\pi}{15}}} \approx 125.44ft/sec.$$

2 12.4

$\mathbf{10^{10}} = \mathbf{14^{9}}$: $\vec{r}\,'(t) = \langle 1, 1, -t(4-t^2)^{-1/2} \rangle$. $\vec{r}(t_1) = \langle 1, 1, \sqrt{3} \rangle \Rightarrow t_1 = 1$. So the tangent vector at P is $\langle 1, 1, -1/\sqrt{3} \rangle$. The unit tangent vector is

$$\vec{T}(1) = \frac{\vec{r}\,'(1)}{\|\vec{r}\,'(1)\|} = \frac{\langle 1, 1, -1/\sqrt{3} \rangle}{\sqrt{7/3}} = \langle \sqrt{3/7}, \sqrt{3/7}, -1/\sqrt{7} \rangle.$$

The tangent line at point $P = (1, 1, \sqrt{3})$ is given by

$$x = 1 + \sqrt{\frac{3}{7}}\ t, y = 1 + \sqrt{\frac{3}{7}}\ t, z = \sqrt{3} - \frac{1}{\sqrt{7}}t.$$

$\mathbf{19^{10}} = \mathbf{29^9}$: $\vec{r}\,'(t) = -6\sin t\vec{i} + 6\cos t\vec{j}$. $\vec{T} = -\sin t\vec{i} + \cos t\vec{j}$. $\vec{T}\,'(t) = -\cos t\vec{i} - \sin t\vec{j}$. The principal normal vector is

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|} = -\cos t\;\vec{i} - \sin t\;\vec{j}, \quad \vec{N}\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}\vec{i} - \frac{\sqrt{2}}{2}\vec{j}.$$

$\mathbf{22^{10}} = \mathbf{36^9}$: $\vec{r}\,'(t) = 2t\vec{i} + 2\vec{j}$. The unit tangent vector is

$$\vec{T}(t) = \frac{\vec{T}(t)}{\|\vec{T}(t)\|} = \frac{t\vec{i} + \vec{j}}{\sqrt{t^2+1}}.$$

$$\begin{aligned}\vec{T}\,'(t) &= \frac{\vec{i}}{\sqrt{t^2+1}} + (t\vec{i}+\vec{j})(-1/2)(t^2+1)^{-3/2}(2t) = \frac{\vec{i}(t^2+1) - (t\vec{i}+\vec{j})t}{(t^2+1)^{3/2}} \\ &= \frac{\vec{i} - t\vec{j}}{(t^2+1)^{3/2}}.\end{aligned}$$

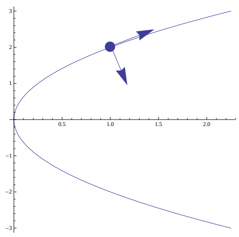
So the principal normal vector is

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|} = \frac{\vec{i} - t\vec{j}}{\sqrt{1+t^2}}.$$

Note that another way to get principal normal vector for a plane curve is to use the following trick.

$$\vec{r}\,'(t) = \langle 2t, 2\rangle \Rightarrow n_1 = \langle -2, 2t\rangle, n_2 = \langle 2, -2t\rangle.$$

n_1, n_2 are normal vectors in opposite directions. From the geometry of the



curve, at $t = 1$, the turning direction is downward-right, so we need to choose

$$n_2 = \langle 2, -2t\rangle \Longrightarrow \vec{N}(t) = \frac{n_2(t)}{\|n_2(t)\|} = \frac{\vec{i} - t\vec{j}}{\sqrt{1+t^2}}.$$

The acceleration vector is:

$$\vec{a}(t) = \vec{r}\,''(t) \equiv 2\vec{i}.$$

$$a_T = \vec{a}\cdot\vec{T} = 2\vec{i}\cdot\frac{t\vec{i}+\vec{j}}{\sqrt{1+t^2}} = \frac{2t}{\sqrt{1+t^2}}.$$

$$a_N = \sqrt{\|a\|^2 - a_T^2} = \sqrt{4 - \frac{4t^2}{1+t^2}} = \frac{2}{\sqrt{1+t^2}}.$$

When $t = 1$, we get

$$\vec{T}(1) = \frac{\vec{i} + \vec{j}}{\sqrt{2}}, \quad \vec{N}(1) = \frac{\vec{i} - \vec{j}}{2}, \quad a_T = \sqrt{2}, \quad a_N = \sqrt{2}.$$

$\mathbf{29^{10} - 32^{10} = 45^9 - 48^9}$:

29: $\vec{r}\,'(t) = -a\omega \sin(\omega t)\vec{i} + a\omega \cos(\omega t)\vec{j}$. Unit tangent vector:

$$\vec{T}(t) = -\sin(\omega t)\vec{i} + \cos(\omega t)\vec{j}.$$

$\vec{T}\,'(t) = -\omega \cos(\omega t)\vec{i} - \omega \sin(\omega t)\vec{j}$. The principal normal vector

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|} = -\cos(\omega t)\vec{i} - \sin(\omega t)\vec{j}.$$

$$\vec{a}(t) = \vec{r}\,'' = -a\omega^2 \cos(\omega t)\vec{i} - a\omega^2 \sin(\omega t)\vec{j}.$$

$$a_T = \vec{a}(t) \cdot \vec{T} = 0. \quad a_N = \sqrt{\|\vec{a}\|^2 - a_T^2} = a\omega^2.$$

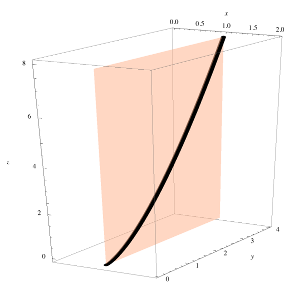
30: $\vec{T}(t)$ is orthogonal to the position vector $\vec{r}(t)$. $\vec{N}(t) = -\omega^2\vec{r}(t)$ is in the opposite direction of $\vec{r}(t)$.

31: The speed of the object at any time t is a constant: $\|\vec{v}(t)\| = a\omega$. So $a_T = \frac{d}{dt}\|\vec{v}\| = 0$ as seen above.

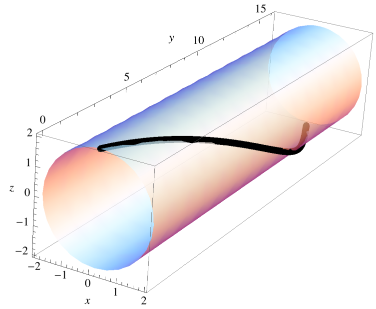
32: If ω is halved, i.e. $\omega \rightsquigarrow \omega/2$, then $a_N \rightsquigarrow a_N/4$.

3 12.5

$\mathbf{10^{10} = 10^9}$: $\vec{r}\,'(t) = 2t\vec{j} + 3t^2\vec{k}$. $\|\vec{r}\,'(t)\| = \sqrt{4t^2 + 9t^4} = |t|\sqrt{4 + 9t^2}$.



$$\begin{aligned} L &= \int_0^2 \|\vec{r}\,'(t)\| dt = \int_0^2 t\sqrt{4 + 9t^2} dt = \frac{1}{18}\int_0^2 \sqrt{4 + 9t^2} d(4 + 9t^2) \\ &= \frac{1}{18}\int_4^{40} \sqrt{u} du = \frac{1}{18}\frac{2}{3}\, u^{3/2}\Big]_4^{40} = \frac{8}{27}(10\sqrt{10} - 1). \end{aligned}$$



$\mathbf{12^{10}} = \mathbf{12^9}$: $\vec{r}\,'(t) = \langle 2\cos t, 5, -2\sin t\rangle$. $\|\vec{r}\,'(t)\| = \sqrt{29}$.

$$L = \int_0^{\pi} \|\vec{r}\,'(t)\| dt = \sqrt{29}\pi.$$

$\mathbf{17^{10}} = \mathbf{19^9}$: **(a):** $\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 1\rangle$. $\|\vec{r}\,'(t)\| = \sqrt{5}$. So

$$s(t) = \int_0^{t} \|\vec{r}\,'(u)\| du = \sqrt{5}t.$$

(b): $t = s/\sqrt{5}$. So the arc length parametrization is:

$$\vec{r}(s) = \left\langle 2\cos\frac{s}{\sqrt{5}}, 2\sin\frac{s}{\sqrt{5}}, \frac{s}{\sqrt{5}}\right\rangle.$$

(c): $\vec{r}(\sqrt{5}) = \langle 2\cos 1, 2\sin 1, 1\rangle$. The coordinate of the point=$(2\cos 1, 2\sin 1, 1)$ for the arc length $s = \sqrt{5}$. Similarly, the coordinate of the point for the arc length $s = 4$ is $(2\cos(4/\sqrt{5}), 2\sin(4/\sqrt{5}), 4/\sqrt{5})$.
(d):

$$\vec{r}\,'(s) = \left\langle -\frac{2}{\sqrt{5}}\sin\frac{s}{\sqrt{5}}, \frac{2}{\sqrt{5}}\cos\frac{s}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right\rangle, \quad \|\vec{r}\,'(s)\| = 1.$$

4 13.1

$\mathbf{16^{10}} = \mathbf{16^9}$:

$$g(x,y) = \ln|t|\big|_x^y = \ln\left|\frac{y}{x}\right|.$$

So **(a):** $g(4,1) = \ln(1/4) = -\ln 4$. **(b):** $g(6,3) = -\ln 2$. **(c):** $g(2,5) = \ln(5/2)$.
(d): $g(1/2,7) = \ln 14$.

$\mathbf{24^{10}} = \mathbf{24^9}$: Domain=$\{(x,y) \in \mathbb{R}^2; x \neq y\}$. Range=$\mathbb{R} = (-\infty, +\infty)$

Consider the plane curve given by:

$$\vec{r}(t) = t\vec{i} + t^3\vec{j}.$$

The tangent vector is

$$\vec{r}\,'(t) = \vec{i} + 3t^2\vec{j} \quad \Longrightarrow \quad \vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{\vec{i} + 3t^2\vec{j}}{\sqrt{1+9t^4}}.$$

1 Principal normal vector

To find the principal normal vector, we can use two different methods:

1 **Standard calculation:** Derivative of **unit tangent vectors**:

$$\begin{aligned}
\vec{T}'(t) &= \frac{6t\vec{j}}{\sqrt{1+9t^4}} - (\vec{i} + 3t^2\vec{j})\frac{1}{2}(1+9t^4)^{-3/2} \cdot 36t^3 \\
&= \frac{6t\vec{j}(1+9t^4) - (\vec{i}+3t^2\vec{j})18t^3}{(1+9t^4)^{3/2}} = \frac{-18t^3\vec{i} + 6t\vec{j}}{(1+9t^4)^{3/2}} \\
&= \frac{6t}{(1+9t^4)^{3/2}}(-3t^2\vec{i} + \vec{j}).
\end{aligned}$$

It's norm is

$$\|\vec{T}'(t)\| = \frac{6|t|}{(1+9t^4)^{3/2}}(9t^4+1)^{1/2}.$$

When $t \neq 0$, $\vec{T}'(t) \neq 0$. So when $t \neq 0$, the principal normal vector is

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} = \frac{t}{|t|}\frac{-3t^2\vec{i} + \vec{j}}{\sqrt{9t^4+1}} = \begin{cases} \frac{-3t^2\vec{i}+\vec{j}}{\sqrt{9t^4+1}}, & t > 0; \\ \\ \frac{3t^2\vec{i}-\vec{j}}{\sqrt{9t^4+1}}, & t < 0. \end{cases}$$

2 **Use geometry of plane curves:** Since $\vec{r}\,'(t) = \langle 1, 3t^2 \rangle$, we get two normal vectors in opposite direction:

$$\vec{n}_1(t) = \langle -3t^2, 1 \rangle, \quad \vec{n}_2(t) = \langle 3t^2, -1 \rangle.$$

When $t > 0$, by **turning direction**, choose $\vec{n}_1(t) = \langle -3t^2, 1 \rangle$.

$$\vec{N}(t) = \frac{\vec{n}_1(t)}{\|\vec{n}_1\|} = \frac{\langle -3t^2, 1 \rangle}{\sqrt{9t^4+1}}, \text{ when } t > 0.$$

When $t < 0$, by **turning direction**, choose $\vec{n}_2(t) = \langle 3t^2, -1 \rangle$.

$$\vec{N}(t) = \frac{\vec{n}_2(t)}{\|\vec{n}_2\|} = \frac{\langle 3t^2, -1 \rangle}{\sqrt{9t^4+1}}, \text{ when } t < 0.$$

2 Acceleration in the $\vec{T}$ and $\vec{N}$ direction

Acceleration vector:

$$\vec{a}(t) = \vec{r}\,''(t) = 6t\vec{j}.$$

$$a_T = \vec{a} \cdot \vec{T} = \frac{18t^3}{\sqrt{1+9t^4}}.$$

$$a_N = \sqrt{\|\vec{a}\|^2 - a_T^2} = \frac{6|t|}{\sqrt{1+9t^4}} \geq 0.$$

MAT 203 FALL 2013 Practice MIDTERM I

NAME : ID :

RECITATION NUMBER:

THERE ARE SIX (6) PROBLEMS. THEY HAVE THE INDICATED VALUE.

SHOW YOUR WORK

DO NOT TEAR-OFF ANY PAGE

NO CALCULATORS NO CELLS ETC.

ON YOUR DESK: ONLY test, pen, pencil, eraser.

1		40pts
2		50pts
3		45pts
4		40pts
5		45pts
6		40pts
Total		260pts

!!! WRITE YOUR NAME, STUDENT ID AND LECTURE N. BELOW !!!

NAME : ID :

LECTURE N.

1. (40pts)

(a): Find the area of parallelogram on the plane with the following vertices

$$A(0,0), B(7,3), C(9,8), D(2,5).$$

(b): Calculate the cosines of the angles of the parallelogram.

2. (50pts) (a): Classify each of the following surfaces. Sketch the surface if possible.

(1) $x^2 + y^2 - 2x + 4y = 0$.

(2) $y^2 + z^2 = 4$.

(3) $x^2 - z^2 - y^2 = 1$.

(b): A quadric surface is a revolution surface obtained by rotating the curve $x = -y^2$ around the x-axis. Write down the equation for this surface and classify it.

3. (45pts)

Consider a point and a plane given by

$$P = (1, 0, -1); \quad H : -4x + y + z = 4.$$

(a): Find the equation of the line passing through P and perpendicular to the plane H.

(b): Find the intersection point of L with H.

(c): Find the distance between the point and the plane.

4. (40pts)

Assume we have a vector valued function satisfying

$$\vec{r}\,''(t) = -32\vec{j}, \quad r(0) = 8\vec{j} + 8\vec{k}, \quad \vec{r}\,'(0) = 8\vec{i} + 8\vec{j}.$$

(a): Find the expression for $\vec{r}(t)$.

(b): Assume the curve $\mathcal{C}$ is described by the vector-valued function $\vec{r}(t)$. Find the intersection points of $\mathcal{C}$ with the xz-plane.

5. (45pts)

Consider a motion is described by the smooth plane curve

$$\vec{r}(t) = (2\cos t)\vec{i} + (\sin t)\vec{j}.$$

(a): Sketch this curve. Calulate $\vec{v}(t)$ and $\vec{a}(t)$ for any t.

(b): Find the unit tangent vector for any t. Calculate the component of acceleration in the direction of $\vec{T}$: $a_T = \vec{a} \cdot \vec{T}$.

(c): Find the principal normal vector $\vec{N}(2\pi/3)$ when $t = \frac{2\pi}{3}$.

6. (40pts)

Calculate the length of the curve within the given interval

$$\vec{r}(t) = (\cos^3 t)\vec{j} + (\sin^3 t)\vec{k}, \quad 0 \le t \le \pi/2.$$

!!! WRITE YOUR NAME, STUDENT ID AND LECTURE N. BELOW !!!

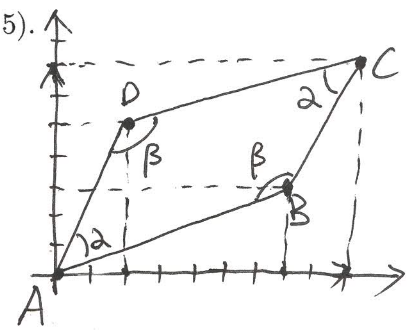
NAME : ID :

LECTURE N.

1. (40pts)

(a): Find the area of parallelogram on the plane with the following vertices

$$A(0,0), B(7,3), C(9,8), D(2,5).$$



$\vec{AB} = \langle 7,3\rangle$. $\vec{AD} = \langle 2,5\rangle$

$$\vec{AB} \times \vec{AD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 7 & 3 & 0 \\ 2 & 5 & 0 \end{vmatrix} = 29\vec{k}.$$

so $\text{Area}\left(\overset{D\quad C}{\underset{A\quad B}{\square}}\right) = 29.$

(b): Calculate the cosines of the angles of the parallelogram.

$$\cos\alpha = \frac{\vec{AB}\cdot\vec{AD}}{\|\vec{AB}\|\cdot\|\vec{AD}\|} = \frac{29}{\sqrt{58}\cdot\sqrt{29}} = \frac{1}{\sqrt{2}} \Rightarrow \alpha = \frac{\pi}{4} = 45^\circ.$$

$$\Rightarrow \underset{\pi-\alpha}{\overset{\|}{\beta}} = \frac{3\pi}{4} = 135^\circ.$$

For β, we can also use ~~inner~~ dot product:

$\vec{BA} = \langle -7,-3\rangle$, $\vec{BC} = \langle 2,5\rangle$

$$\cos\beta = \frac{\vec{BA}\cdot\vec{BC}}{\|\vec{BA}\|\|\vec{BC}\|} = -\frac{1}{\sqrt{2}} \Rightarrow \beta = \frac{3\pi}{4}.$$

2. (50pts) (a): Classify each of the following surfaces. Sketch the surface if possible.

(1) $x^2 + y^2 - 2x + 4y = 0$.
(2) $y^2 + z^2 = 4$.
(3) $x^2 - z^2 - y^2 = 1$.

(1). Complete the squares: $(x-1)^2+(y+2)^2=5$ On the xy-plane, this is a circle with center $(1,-2)$ and radius $\sqrt{5}$. In 3-dim. space, this equation defines a circular cylindrical surface parallel to the z-axis.

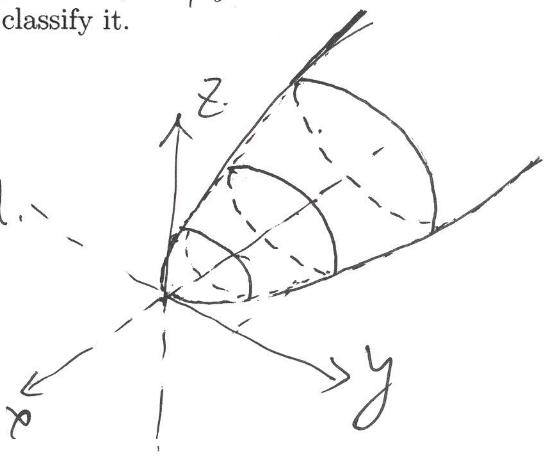
(I was trying to give $x^2+y^2-2x+4y+z^2=0$ which defines a sphere).

(2) This is a cylindrical surface parallel to x-axis with projection to yz-plane being a circle.

(b): A quadric surface is a revolution surface obtained by rotating the curve $x = -y^2$ around the x-axis. Write down the equation for this surface and classify it.

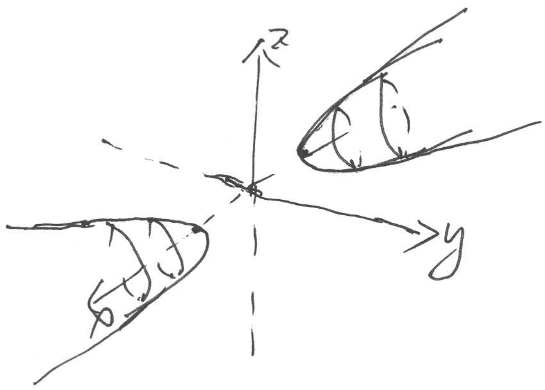
$$x=-y^2 \xrightarrow[\text{x-axis}]{\text{around}} x=-(y^2+z^2)$$

This is an elliptical paraboloid.



(a).3: $x^2=y^2+z^2+1$.

- x is the axis variable.
- there is restriction on x: $|x|\geq 1$.
- The slices are ellipses $y^2+z^2=x^2-1$ (circles) when $|x|\geq 1$.
- This is a hyperboloid of 2 sheets.



3. (45pts)

Consider a point and a plane given by

$$P = (1, 0, -1); \quad H : -4x + y + z = 4.$$

(a): Find the equation of the line passing through P and perpendicular to the plane H.

Parametric equations $x = 1 - 4t,\ y = t,\ z = -1 + t.$ (1)

or Symmetric equation: $\dfrac{x-1}{-4} = \dfrac{y}{1} = \dfrac{z+1}{1}.$

(b): Find the intersection point of L with H.

Substitute (1) into equation for H:

$-4 \cdot (1-4t) + t + (-1+t) = 4 \Rightarrow 18t = 9 \Rightarrow t = \frac{1}{2}.$

Intersection pt: $Q = (-1, \frac{1}{2}, -\frac{1}{2}).$

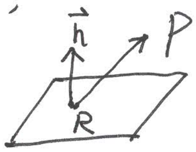
(c): Find the distance between the point and the plane.

Method 1: $\text{dist}(P, H) = \text{dist}(P, Q) = \| \underbrace{\langle 2, -\tfrac{1}{2}, -\tfrac{1}{2} \rangle}_{\overrightarrow{QP}} \| = \frac{3\sqrt{2}}{2}.$

Or: Method 2: Choose a point, say $R = (-1, 0, 0)$ on the plane H.

Use projection to find distance:

$\text{dist}(P, H) = \| \text{Proj}_{\vec{n}} \overrightarrow{RP} \| = \dfrac{|\overrightarrow{RP} \cdot \vec{n}|}{\|\vec{n}\|} = \dfrac{9}{\sqrt{18}} = \dfrac{3\sqrt{2}}{2}.$



$\overrightarrow{RP} = \langle 2, 0, -1 \rangle$

$\vec{n} = \langle -4, 1, 1 \rangle$

4. (40pts)

Assume we have a vector valued function satisfying

$$\vec{r}\,''(t) = -32\vec{j}, \quad r(0) = 8\vec{j} + 8\vec{k}, \quad \vec{r}\,'(0) = 8\vec{i} + 8\vec{j}.$$

(a): Find the expression for $\vec{r}(t)$.

$$\vec{r}\,'(t) = \int_0^t \vec{r}\,''(s)\,ds + \vec{r}\,'(0) = -32t\vec{j} + 8\vec{i} + 8\vec{j}$$

$$= 8\vec{i} + (-32t+8)\vec{j}$$

$$\vec{r}(t) = \int_0^t \vec{r}\,'(s)\,ds + \vec{r}(0) = 8t\vec{i} + (-16t^2 + 8t)\vec{j} + (8\vec{j} + 8\vec{k})$$

$$= 8t\vec{i} + (-16t^2 + 8t + 8)\vec{j} + (\cancel{8t+}8)\vec{k}.$$

(b): Assume the curve C is described by the vector-valued function $\vec{r}(t)$. Find the intersection points of C with the xz-plane.

The intersection points $C \cap \{xz\text{-plane}\}$ have y-coordinate, i.e. $\vec{j}$-component being 0.

So we solve:

$$-16t^2 + 8t + 8 = 0 \Rightarrow t = -\tfrac{1}{2},\ 1.$$

$$\| $$

$$-8(2t^2 - t - 1) = -8(2t+1)(t-1)$$

2 intersection points

So we get 2 intersection points:

$t = -\frac{1}{2}$: $\vec{r}(-\frac{1}{2}) = \langle -4, 0, 8 \rangle$. $\boxed{(-4, 0, \cancel{4}\ 8)}$

$t = 1$: $\vec{r}(1) = \langle 8, 0, \cancel{16}\ 8 \rangle$ $\boxed{(8, 0, \cancel{16}\ 8)}$

Note: The problem on the actual exam is much easier for calculations.

For the last part, don't panic.

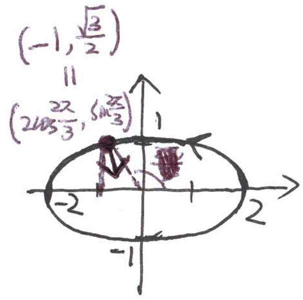
5. (45pts) and there will be more spaces for writing.

Consider a motion is described by the smooth plane curve

$$\vec{r}(t) = (2\cos t)\vec{i} + (\sin t)\vec{j}.$$

(a): Sketch this curve. Calulate $\vec{v}(t)$ and $\vec{a}(t)$ for any t.

This curve is an ellipse: $\frac{x^2}{4}+y^2=1$.



$\vec{v}(t) = \vec{r}\,'(t) = -2\sin t\,\vec{i} + \cos t\,\vec{j}$

$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t) = -2\cos t\,\vec{i} - \sin t\,\vec{j}$

(b): Find the unit tangent vector for any t. Calculate the component of acceleration in the direction of $\vec{T}$: $a_T = \vec{a}\cdot\vec{T}$.

Method 2:

$\vec{r}\,'(\frac{2\pi}{3}) = -\sqrt{3}\,\vec{i} - \frac{1}{2}\vec{j}$

unit tangent vector: $\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{-2\sin t\,\vec{i} + \cos t\,\vec{j}}{\sqrt{4\sin^2 t + \cos^2 t}}$

$= \boxed{\frac{-2\sin t\,\vec{i} + \cos t\,\vec{j}}{\sqrt{1+3\sin^2 t}}} = \vec{T}(t)$

$\Rightarrow$ normal vectors $\vec{n}_1 = \langle \frac{1}{2}, -\sqrt{3}\rangle$, $\vec{n}_2 = \langle -\frac{1}{2}, \sqrt{3}\rangle$

by turning direction, choose $\vec{n}_1 = \langle \frac{1}{2}, -\sqrt{3}\rangle$

Principal normal: $\vec{N}(\frac{2\pi}{3}) = \frac{\vec{n}_1}{\|\vec{n}_1\|} = \boxed{\sqrt{\frac{4}{13}}\langle \frac{1}{2}, -\sqrt{3}\rangle}$.

$a_T = \vec{a}(t)\cdot\vec{T} = \frac{4\cos t\cdot\sin t - \sin t\cdot\cos t}{\sqrt{1+3\sin^2 t}} = \frac{3\sin t\cos t}{\sqrt{1+3\sin^2 t}} = \boxed{\frac{\frac{3}{2}\sin(2t)}{\sqrt{1+3\sin^2 t}}} = a_T(t)$.

Or

Method 1:

(c): Find the principal normal vector $\vec{N}(2\pi/3)$ when $t = \frac{2\pi}{3}$.

$$\frac{d}{dt}\bigg|_{t=\frac{2\pi}{3}} \frac{(-2)\sin t}{\sqrt{1+3\sin^2 t}} = (-2)\left[\frac{\cos t}{\sqrt{1+3\sin^2 t}} + \sin t\cdot\left(-\frac{1}{2}\right)(1+3\sin^2 t)^{-\frac{3}{2}}\cdot 6\sin t\cdot\cos t\right]\bigg|_{t=\frac{2\pi}{3}} = \sqrt{\frac{4}{13}}\cdot\left(\frac{-2}{13}\right)\cdot(-2)$$

$$\frac{d}{dt}\bigg|_{t=\frac{2\pi}{3}} \frac{\cos t}{\sqrt{1+3\sin^2 t}} = \left[\frac{-\sin t}{\sqrt{1+3\sin^2 t}} + \cos t\cdot\left(-\frac{1}{2}\right)\cdot(1+3\sin^2 t)^{-\frac{3}{2}}\, 6\sin t\cdot\cos t\right]\bigg|_{t=\frac{2\pi}{3}} = \sqrt{\frac{4}{13}}\left(-\frac{8\sqrt{3}}{13}\right).$$

$$\Rightarrow \vec{T}\,'(\tfrac{2\pi}{3}) = \left\langle \sqrt{\frac{4}{13}}\cdot\frac{4}{13}, \sqrt{\frac{4}{13}}\left(-\frac{8\sqrt{3}}{13}\right)\right\rangle = \sqrt{\frac{4}{13}}\,\frac{4}{13}\langle 1, -2\sqrt{3}\rangle$$

$$\vec{N}(\tfrac{2\pi}{3}) = \frac{\vec{T}\,'(\frac{2\pi}{3})}{\|\vec{T}\,'(\frac{2\pi}{3})\|} = \frac{\langle 1, -2\sqrt{3}\rangle}{\sqrt{13}} = \boxed{\left\langle \frac{1}{\sqrt{13}}, -\frac{2\sqrt{3}}{\sqrt{13}}\right\rangle}$$

6. (40pts)

Calculate the length of the curve within the given interval

$$\vec{r}(t) = (\cos^3 t)\vec{j} + (\sin^3 t)\vec{k}, \quad 0 \le t \le \pi/2.$$

$$\vec{r}\,'(t) = -3\cos^2 t \cdot \sin t \, \vec{j} + 3 \sin^2 t \cdot \cos t \, \vec{k}.$$

$$\|\vec{r}\,'(t)\|^2 = 9 \cdot \cos^2 t \sin^2 t \cdot (\cos^2 t + \sin^2 t)$$

$$= \frac{9}{4} \cdot (2 \sin t \cdot \cos t)^2 = \frac{9}{4} \sin^2(2t)$$

$\sin(2t) \geq 0$ for $t \in [0, \frac{\pi}{2}]$

$$L = \int_0^{\frac{\pi}{2}} \frac{3}{2} |\sin(2t)|\, dt = \frac{3}{2} \cdot \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin(2t) \cdot d(2t)$$

$$= \frac{3}{4} \int_0^{\pi} \sin(s)\, ds = \frac{3}{4} \cdot (-\cos s) \Big]_0^{\pi} = \frac{3}{2}.$$

MAT 203 FALL 2013 MIDTERM I

NAME : ID :

RECITATION DAY (WED OR FRI):

THERE ARE SIX (6) PROBLEMS. THEY HAVE THE INDICATED VALUE.

SHOW YOUR WORK

DO NOT TEAR-OFF ANY PAGE

NO CALCULATORS NO CELLS ETC.

ON YOUR DESK: ONLY test, pen, pencil, eraser.

1		40pts
2		40pts
3		40pts
4		40pts
5		50pts
6		40pts
Total		250pts

!!! WRITE YOUR NAME, STUDENT ID BELOW !!!

NAME : ID :

1. (40pts)

$$\vec{u} = 2\vec{i} - 2\vec{j} + \vec{k}, \quad \vec{v} = \vec{i} + \vec{j}.$$

(a)(20pts): Find the angle between $\vec{u}$ and $\vec{v}$.

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \frac{2 \cdot 1 + (-2) \cdot 1}{\sqrt{9} \cdot \sqrt{2}} = 0$$

$$\Rightarrow \theta = \frac{\pi}{2} = 90°.$$

(b)(20pts): Find the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$.

Since $\theta = \frac{\pi}{2}$, the parallelogram is a rectangle.

the area is

$$\|\vec{u}\| \cdot \|\vec{v}\| \cdot \sin\theta = \sqrt{9} \cdot \sqrt{2} \sin\frac{\pi}{2} = \boxed{3\sqrt{2}.}$$

Or, use the cross product:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\vec{i} + \vec{j} + 4\vec{k}.$$

Area of the parallelogram $= \|\vec{u} \times \vec{v}\| = \sqrt{18} = \boxed{3\sqrt{2}.}$

2. (40pts)

(a)(20pts): Write down the equation for the surface: sphere with center $(0, 0, 1)$ and radius 1.

$$x^2+y^2+(z-1)^2=1$$

$$\Updownarrow$$

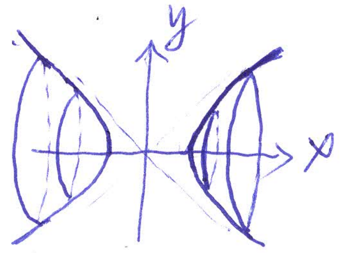
$$x^2+y^2+z^2-2z=0.$$

(b)(20pts): Write down the equation and classify the surface: the revolution surface obtained by rotating the curve $x^2 - y^2 = 1$ on the xy-plane around the x-axis.

Fix x, change y to $\sqrt{y^2+z^2}$, we get equation

$x^2-(y^2+z^2)=1$ for the revolution surface.

$x^2=y^2+z^2+1 \Rightarrow |x|\geq 1$. So there is a gap for x variable, which is the axis variable so the surface is a hyperboloid of 2 sheets.



3. (40pts)

Consider the point P and the line L given by

$$P = (0, 1, 0); \quad L : x = 2 - t, y = 2 + t, z = t.$$

(a)(20pts): Find the plane H which contains P and is perpendicular to L.

The normal vector for H is given by direction vector of L: $\vec{n} = \langle -1, 1, 1 \rangle$.

H is given by $\langle -1, 1, 1 \rangle \cdot \langle x, y-1, z \rangle = 0$

$$\Updownarrow$$

$$-x + (y-1) + z = 0$$

$$\Updownarrow$$

$$x - y - z = -1.$$

(b)(20pts): Find the intersection point Q of H with L.

Substitute paramametric equation for L into H:

$$\underbrace{(2-t)-(2+t)-t}_{=\,-3t} = -1 \Rightarrow t = \frac{1}{3}.$$

So the intersection point

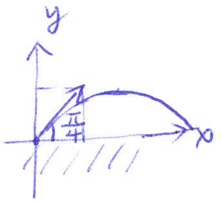
$$Q = \left(\frac{5}{3}, \frac{7}{3}, \frac{1}{3}\right).$$

4. (40pts)

A projectile is fired from the origin over the horizontal ground at an initial speed of $64ft/sec$ and a launch angle of 45° above the horizontal.

(a)(20pts): Find the vector-valued function describing the path of the projectile. Use the value $g = 32ft/s^2$ so the acceleration is $\vec{a}(t) = -32\vec{j}$.

$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} = \sin\frac{\pi}{4}$.



$\vec{a}(t) \equiv -32\vec{j}$. $\vec{v}(0) = V_0\cos\theta\,\vec{i} + V_0\sin\theta\,\vec{j} = 32\sqrt{2}(\vec{i}+\vec{j})$

Integrate: $\vec{r}(0) = \vec{0}$.

$\vec{v}(t) = \int_0^t \vec{a}(s)ds + \vec{v}(0) = \langle V_0\cos\theta, V_0\sin\theta - gt\rangle = \langle 32\sqrt{2}, 32\sqrt{2} - 32t\rangle$

$\vec{r}(t) = \int_0^t \vec{v}(s) + \vec{r}(0) = \langle V_0\cos\theta\, t, V_0\sin\theta\, t - \frac{1}{2}gt^2\rangle$

$= \langle 32\sqrt{2}\cdot t, 32\sqrt{2}t - 16t^2\rangle$.

(b)(20pts): What's the range of the projectile?

Solve t when it hits the ground:

$32\sqrt{2}t - 16t^2 = 0 \Rightarrow \underline{t_1 = 2\sqrt{2}}$, $t_2 = 0$.

So the horizontal range of the projectile is

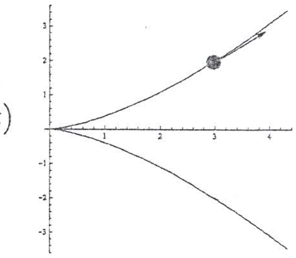
$32\sqrt{2}\cdot t_1 = 32\sqrt{2}\cdot 2\sqrt{2} = 128\ ft$.

5. (50pts)

A motion is described by the vector-valued function

$$\vec{r}(t) = 3t^2\vec{\imath} + 2t^3\vec{\jmath}.$$

The curve of motion is shown in the figure.



(a)(20pts): Find the velocity $\vec{v}(t)$ and the acceleration $\vec{a}(t)$ for any t.

$$\vec{v}(t) = \vec{r}\,'(t) = 6t\,\vec{i} + 6t^2\vec{j}$$

$$\vec{a}(t) = \vec{v}\,'(t) = 6\vec{i} + 12t\,\vec{j}.$$

(b)(20pts): When $t = 1$, find the unit tangent vector $\vec{T}(1)$. Calculate the component of acceleration in the direction of $\vec{T}$ given by $a_T = \vec{a} \cdot \vec{T}$ when $t = 1$.

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{6t\vec{i} + 6t^2\vec{j}}{\sqrt{(6t)^2 + (6t^2)^2}} = \frac{\vec{i} + t\vec{j}}{\sqrt{1+t^2}}. \quad \text{for } t > 0.$$

$$a_T = \vec{a}(t)\cdot\vec{T} = \frac{6 + 12t^2}{\sqrt{1+t^2}}.$$

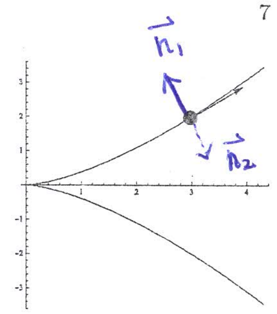
$$\text{So} \quad \vec{T}(1) = \frac{\vec{i} + \vec{j}}{\sqrt{2}}, \quad a_T(1) = \frac{18}{\sqrt{2}} = 9\sqrt{2}.$$

Problem 5 continued: A motion is described by the vector-valued function

$$\vec{r}(t) = 3t^2\vec{i} + 2t^3\vec{j}.$$

The curve of motion is shown in the figure.

(c)(10pts): Calculate the principal normal vector $\vec{N}(1)$ when $t = 1$. (Note that the principal normal vector points to the turning direction)



Method 1: $\vec{T}(t) = \dfrac{\vec{i} + t\vec{j}}{\sqrt{1+t^2}}$ for $t > 0$.

$$\vec{T}'(t) = \frac{\vec{j}}{\sqrt{1+t^2}} + (\vec{i} + t\vec{j})\left(-\frac{1}{2}\right)(1+t^2)^{-\frac{3}{2}}(2t)$$

$$= \frac{-t\vec{i} + \vec{j}}{(1+t^2)^{3/2}} \quad \text{for } t > 0.$$

Principal normal vector:

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} = \frac{-t\vec{i} + \vec{j}}{\sqrt{1+t^2}} \Rightarrow \vec{N}(1) = \frac{-\vec{i} + \vec{j}}{\sqrt{2}}.$$

Method 2: $\vec{T}(1) = \dfrac{\vec{i} + \vec{j}}{\sqrt{2}}$.

$\Rightarrow$ 2 normal vectors $\vec{n}_1 = \dfrac{-\vec{i} + \vec{j}}{\sqrt{2}}$, $\vec{n}_2 = \dfrac{\vec{i} - \vec{j}}{\sqrt{2}}$.

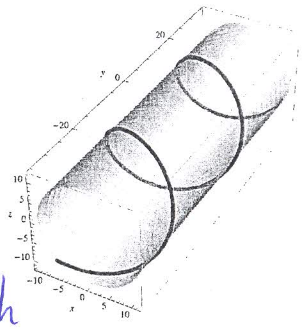
by turning direction, choose $\vec{n}_1 = \dfrac{-\vec{i} + \vec{j}}{\sqrt{2}}$.

so $\vec{N}(1) = \dfrac{\vec{n}_1}{\|\vec{n}_1\|} = \dfrac{-\vec{i} + \vec{j}}{\sqrt{2}}.$

6. (40pts)

Consider a helix curve

$$\vec{r}(t) = (12\sin t)\vec{i} + 5t\vec{j} - (12\cos t)\vec{k}.$$



(a)(20pts): Write down the equation of the circular cylindrical surface which contains the helix curve and is parallel to the y-axis.

$x^2 + z^2 = 144.$

This is a circular cylindrical surface with rulings parallel to y-axis, because y variable is missing and the trace on the xz-plane is a circle.

(b)(20pts): Calculate the arc length for part of the curve from $\vec{r}(-\pi)$ to $\vec{r}(\pi)$.

$$\vec{r}\,'(t) = 12\cos t\,\vec{i} + 5\vec{j} + 12\sin t\,\vec{k}.$$

$$\|\vec{r}\,'(t)\| = \sqrt{12^2(\cos^2 t + \sin^2 t) + 25} = \sqrt{169} = 13.$$

So $$L = \int_{-\pi}^{\pi} \|\vec{r}\,'(t)\|\,dt = 13 \cdot 2\pi = 26\pi.$$

!!! WRITE YOUR NAME, STUDENT ID BELOW !!!

NAME : ID :

1. (40pts)

$$\vec{u} = \vec{i} - \vec{j} + \vec{k}, \quad \vec{v} = \vec{i} + \vec{j}.$$

(a)(20pts): Find the angle between $\vec{u}$ and $\vec{v}$.

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|} = \frac{1\cdot 1 + (-1)\cdot 1}{\sqrt{3}\cdot\sqrt{2}} = 0$$

$$\Rightarrow \theta = \frac{\pi}{2} = 90^\circ.$$

(b)(20pts): Find the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$.

Since $\theta = \frac{\pi}{2}$, the parallelogram is a rectangle.

The area is

$$\|\vec{u}\|\cdot\|\vec{v}\|\cdot\sin\theta = \sqrt{3}\cdot\sqrt{2}\cdot\sin\frac{\pi}{2} = \sqrt{6}.$$

Or, use the cross product:

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\vec{i} + \vec{j} + 2\vec{k}.$$

Area of the parallelogram $= \|\vec{u}\times\vec{v}\| = \sqrt{6}$.

2. (40pts)

(a)(20pts): Write down the equation for the surface: sphere with center $(-1, 0, 0)$ and radius 1.

$$(x+1)^2+y^2+z^2=1$$

$$\Updownarrow$$

$$x^2+y^2+z^2+2x=0$$

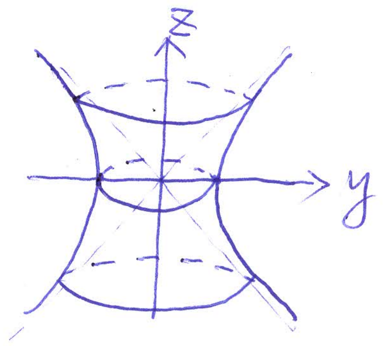
(b)(20pts): Write down the equation and classify the surface: the revolution surface obtained by rotating the curve $y^2 - z^2 = 1$ on the yz-plane around the z-axis.

Around z-axis $\longrightarrow$ Fix z, change y to $\sqrt{x^2+y^2}$. we get equation

$$x^2+y^2-z^2=1. \iff x^2+y^2=z^2+1.$$

z can be any real number. $z \in (-\infty, \infty)$. No gap for z.

So it's a hyperboloid of 1 sheet.



3. (40pts)

Consider the point P and the line L given by

$$P = (0,0,1); \quad L: x = 2+t, y = 2-t, z = t.$$

(a)(20pts): Find the equation of the plane H which contains P and is perpendicular to L.

The normal vector for H is given by the direction vector of L:

$\vec{n} = \langle 1, -1, 1 \rangle$.

H is given by $1\cdot(x-0) - 1\cdot(y-0) + 1\cdot(z-1) = 0$

$$\Updownarrow$$

$$x - y + z = 1.$$

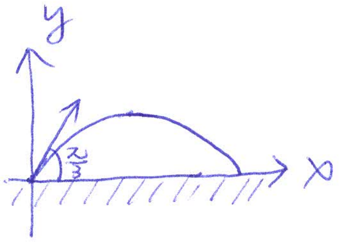
(b)(20pts): Find the intersection point Q of H with L.

Substitute parametric equation for L into H:

$$\underbrace{(2+t)-(2-t)+t}_{=3t} = 1 \Rightarrow t = \frac{1}{3}.$$

So the intersection point is

$$Q = \left(\frac{7}{3}, \frac{5}{3}, \frac{1}{3}\right).$$



4. (40pts)

A projectile is fired from the origin over the horizontal ground at an initial speed of $32ft/sec$ and a launch angle of $60°$ above the horizontal.

(a)(20pts): Find the vector-valued function describing the path of the projectile. Use the value $g = 32ft/s^2$ so the acceleration is $\vec{a}(t) = -32\vec{j}$.

$\cos\frac{\pi}{3} = \frac{1}{2}$, $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

$\vec{a}(t) = -32\vec{j}$. $\vec{v}(0) = V_0\cos\theta\,\vec{i} + V_0\sin\theta\,\vec{j} = 16\cdot(\vec{i} + \sqrt{3}\vec{j})$. $\vec{r}(0) = \vec{0}$.

Integrate: $\vec{v}(t) = \int_0^t \vec{a}(s)\,ds + \vec{v}(0) = \langle V_0\cos\theta, V_0\sin\theta - gt\rangle = \langle 16, 16\sqrt{3} - 32t\rangle$.

$$\vec{r}(t) = \int_0^t \vec{v}(s)\,ds + \vec{r}(0) = \langle V_0\cos\theta\cdot t, V_0\sin\theta\cdot t - \tfrac{1}{2}gt^2\rangle$$
$$= \langle 16t, 16\sqrt{3}t - 16t^2\rangle.$$

(b)(20pts): What's the range of the projectile?

Solve t when it hits the ground:

$16\sqrt{3}t - 16t^2 = 0 \Rightarrow t_1 = \sqrt{3}$, $t_2 = 0$.

so the (horizontal) range of the projectile is:

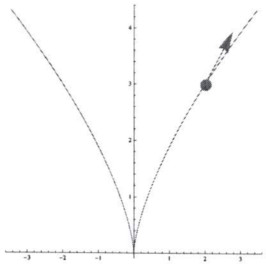
$16\cdot t_1 = 16\cdot\sqrt{3}$ ft.

5. **(50pts)**

A motion is described by the vector-valued function

$$\vec{r}(t) = 2t^3\vec{\imath} + 3t^2\vec{\jmath}.$$

The curve of motion is shown in the figure.



(a)(20pts): Find the velocity $\vec{v}(t)$ and the acceleration $\vec{a}(t)$ for any t.

$$\vec{v}(t) = \vec{r}\,'(t) = 6t^2\vec{\imath} + 6t\vec{\jmath}.$$

$$\vec{a}(t) = \vec{v}\,'(t) = 12t\vec{\imath} + 6\vec{\jmath}.$$

(b)(20pts): When $t = 1$, find the unit tangent vector $\vec{T}(1)$. Calculate the component of acceleration in the direction of $\vec{T}$ given by $a_T = \vec{a} \cdot \vec{T}$ when $t = 1$.

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{6t^2\vec{\imath} + 6t\vec{\jmath}}{\sqrt{(6t^2)^2 + (6t)^2}} = \frac{t\vec{\imath} + \vec{\jmath}}{\sqrt{1+t^2}} \text{ for } t>0.$$

$$a_T = \vec{a}(t) \cdot \vec{T}(t) = \frac{12t^2 + 6}{\sqrt{1+t^2}}.$$

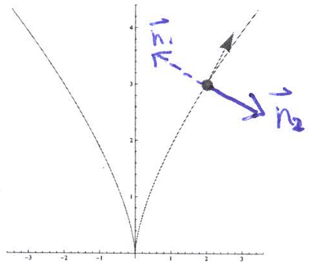
So $\vec{T}(1) = \dfrac{\vec{\imath} + \vec{\jmath}}{\sqrt{2}}$, $a_T(1) = \dfrac{18}{\sqrt{2}} = 9\sqrt{2}$.

Problem 5 continued: A motion is described by the vector-valued function

$$\vec{r}(t) = 2t^3\vec{i} + 3t^2\vec{j}.$$

The curve of motion is shown in the figure.

(c)(10pts): Calculate the principal normal vector $\vec{N}(1)$ when $t = 1$. (Note that the principal normal vector points to the turning direction)



Method 1: $\vec{T}(t) = \dfrac{t\vec{i}+\vec{j}}{\sqrt{1+t^2}}$ for $t>0$.

$$\vec{T}'(t) = \frac{\vec{i}}{\sqrt{1+t^2}} + (t\vec{i}+\vec{j})\cdot\left(-\tfrac{1}{2}\right)(1+t^2)^{-\frac{3}{2}}\cdot 2t$$

$$= \frac{\vec{i}-t\vec{j}}{(1+t^2)^{3/2}} = \frac{\vec{i}-t\vec{j}}{(1+t^2)^{3/2}} \text{ for } t>0$$

Principal normal vector:

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} = \frac{\vec{i}-t\vec{j}}{\sqrt{1+t^2}} \Rightarrow \vec{N}(1) = \frac{\vec{i}-\vec{j}}{\sqrt{2}} = \boxed{\frac{\vec{i}-\vec{j}}{\sqrt{2}}}$$

Method 2: $\vec{T}(1) = \dfrac{\vec{i}+\vec{j}}{\sqrt{2}}$.

$\Rightarrow$ 2 normal vectors $\vec{n_1} = \dfrac{-\vec{i}+\vec{j}}{\sqrt{2}}$, $\vec{n_2} = \dfrac{\vec{i}-\vec{j}}{\sqrt{2}}$.

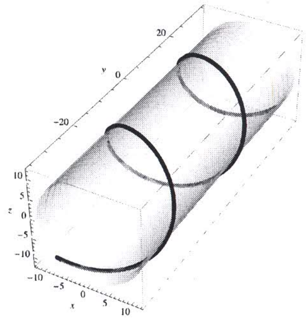
by turning direction, Choose $\vec{n_2} = \dfrac{\vec{i}-\vec{j}}{\sqrt{2}}$.

so $\vec{N}(1) = \dfrac{\vec{n_2}}{\|\vec{n_2}\|} = \boxed{\dfrac{\vec{i}-\vec{j}}{\sqrt{2}}}$

6. **(40pts)**

Consider a helix curve

$$\vec{r}(t) = (10\sin t)\vec{\imath} + 5t\vec{\jmath} - (10\cos t)\vec{k}.$$



(a)(20pts): Write down the equation of the circular cylindrical surface which contains the helix curve and is parallel to the y-axis.

$x^2 + z^2 = 100.$

This is a circular cylindrical surface with rulings parallel to y-axis because y-variable is missing, and on the xz-plane, the trace is a circle.

(b)(20pts): Calculate the arc length for part of the curve from $\vec{r}(-\pi)$ to $\vec{r}(\pi)$.

$\vec{r}\,'(t) = 10\cos t\,\vec{i} + 5\vec{j} + 10\sin t\,\vec{k}$

$\|\vec{r}\,'(t)\| = \sqrt{10^2(\cos^2 t + \sin^2 t) + 25} = \sqrt{125} = 5\sqrt{5}.$

So $L = \int_{-\pi}^{\pi} \|\vec{r}\,'(t)\|\,dt = 5\sqrt{5}\cdot 2\pi = 10\sqrt{5}\cdot\pi.$

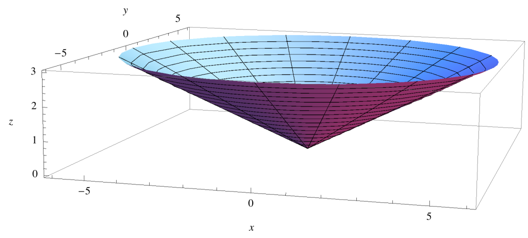


Figure 1: $z = \frac{1}{2}\sqrt{x^2 + y^2}$

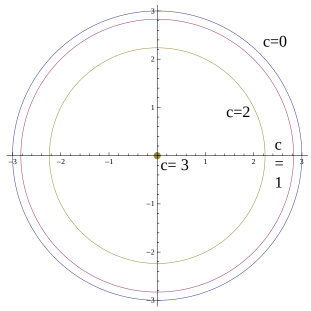


Figure 2: $x^2 + y^2 = 9 - c^2$

1 13.1

27: $f(x, y) = \arccos(x + y)$. Domain=$\{(x, y) \in \mathbb{R}^2; |x + y| \leq 1\}$. Range=$[0, \pi]$.

38: $z = \frac{1}{2}\sqrt{x^2 + y^2}$. The graph of this function is the upper half of a elliptical cone:

$$z^2 = \frac{1}{4}(x^2 + y^2), \quad z \geq 0.$$

See figure 1.

45-48: **(a)**$\longleftrightarrow$ (48); **(b)**$\longleftrightarrow$ (47); **(c)**$\longleftrightarrow$ (45); **(d)**$\longleftrightarrow$ (46).

52: The level curves for $f(x, y) = \sqrt{9 - x^2 - y^2}$ are: $\sqrt{9 - x^2 - y^2} = c$. The level curves form a family of circles: $x^2 + y^2 = 9 - c^2$ with common center $(0, 0)$. Note that c needs to satisfy $|c| \leq 3$. See figure 2.

(**The following problems are numbered according to 9-th edition textbook, except that** 30^{10} **is a question from 10-th edition. If you have 10-th edition textbook, don't use this.**)

1 13.2

Homework: **24, 84, 86**.

2 13.3

Homework: **20, 54, 66, 82, 76, 118**.

3 13.4

Homework: **14, 26**.

Homework: $\mathbf{30^{10}}$: The possible error involved in measuring each dimension of a right circular cylinder is ± 0.05 centimeter. The radius is 3 centimeters and the height is 10 centimeters. Approximate the propagated error and the relative error in the calculated volume of the cylinder.

4 13.5

Homework: **12, 18, 26, 27**.

(24^{10} **means problem 24 in 10-th edition textbook,** 24^9 **means problem 24 in 9-th edition textbook**)

1 13.2

$\mathbf{24^{10}} = \mathbf{24^9}$:

$$\lim_{(x,y)\to(1,-1)} \frac{x^2y}{1+xy^2} = \frac{1^2(-1)}{1+1(-1)^2} = -\frac{1}{2}.$$

$\mathbf{74^{10}} = \mathbf{84^9}$: **(a):** Along any line $y = ax$, the limit is:

$$\lim_{x\to 0} \frac{x^2ax}{x^4+(ax)^2} = \lim_{x\to 0} \frac{ax}{x^2+a^2} = 0.$$

(b): Along the parabola $y = x^2$, the limit is:

$$\lim_{x\to 0} \frac{x^2x^2}{x^4+x^4} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}.$$

(c): The limit does not exist, because the limits in **(a)** and **(b)** do not coincide. If the limit exist, then the limit should be the same along any path approaching $(0,0)$.

$\mathbf{76^{10}} = \mathbf{86^9}$:

$$\begin{aligned} \lim_{(x,y,z)\to(0,0,0)} \tan^{-1}\left[\frac{1}{x^2+y^2+z^2}\right] &= \tan^{-1}\left[\lim_{(x,y,z)\to(0,0,0)} \frac{1}{x^2+y^2+z^2}\right] \\ &= \tan^{-1}(+\infty) = \frac{\pi}{2}. \end{aligned}$$

2 13.3

$\mathbf{18^{10}} = \mathbf{20^9}$: $z = ye^{y/x}$.

$$\frac{\partial z}{\partial x} = ye^{y/x}\left(-\frac{y}{x^2}\right) = -\frac{y^2}{x^2}e^{y/x}, \quad \frac{\partial z}{\partial y} = e^{y/x} + ye^{y/x}\frac{1}{x}$$

$\mathbf{52^{10}} = \mathbf{54^9}$: $h_x = 2x$. So the slope in the x-direction is $h_x(-2,1) = -4$. $h_y = -2y$. So the slope in the y-direction is $h_y(-2,1) = -2$.

$\mathbf{60^{10}} = \mathbf{66^9}$: $f(x,y,z) = x^2y^3 + 2xyz - 3yz$. $f_x = 2xy^3 + 2yz$, $f_y = 3x^2y^2 + 2xz - 3z$, $f_z = 2xy - 3y$. So

$$f_x(-2,1,2) = 0, \quad f_y(-2,1,2) = -2, \quad f_z(-2,1,2) = -7.$$

$\mathbf{66^{10}} = \mathbf{82^9}$: $f(x,y) = x^2 - xy + y^2 - 5x + y$. $f_x = 2x - y - 5$, $f_y = -x + 2y + 1$.

$$\left\{\begin{array}{r} 2x - y - 5 = 0 \\ -x + 2y + 1 = 0 \end{array}\right. \Longrightarrow x = 3, y = 1.$$

78^{10} $=$ **76^{9}**: $z = \ln(x-y)$. $\frac{\partial z}{\partial x} = \frac{1}{x-y}$, $\frac{\partial z}{\partial y} = -\frac{1}{x-y}$.

$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{(x-y)^2}, \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{1}{(x-y)^2};$$

$$\frac{\partial^2 z}{\partial y^2} = -\frac{1}{(y-x)^2}, \quad \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{1}{(x-y)^2}.$$

110^{10} $=$ **118^{9}**: $C(x,y) = 32\sqrt{xy} + 175x + 205y + 1050$.
(a): $\partial C/\partial x = 16x^{-1/2}y^{1/2} + 175$. $\partial C/\partial y = 16x^{1/2}y^{-1/2} + 205$.

$$\frac{\partial C}{\partial x}(80,20) = 183, \quad \frac{\partial C}{\partial y}(80,20) = 237.$$

(b): $(\partial C/\partial y)(80,20) > (\partial C/\partial x)(80,20)$, so increasing the production of fireplace-insert stove (corresponding to variable y) results in the cost increasing at a higher rate.

3 13.4

14^{10} $=$ **14^{9}**: $f(x,y) = y/x$.
(a): $f(2,1) = 0.5$, $f(2.1, 1.05) = 1.05/2.1 = 0.5$, $\Delta z = f(2.1, 1.05) - f(2,1) = 0$.
(b): $df = f_x dx + f_y dy = -\frac{y}{x^2}dx + \frac{1}{x}dy$.

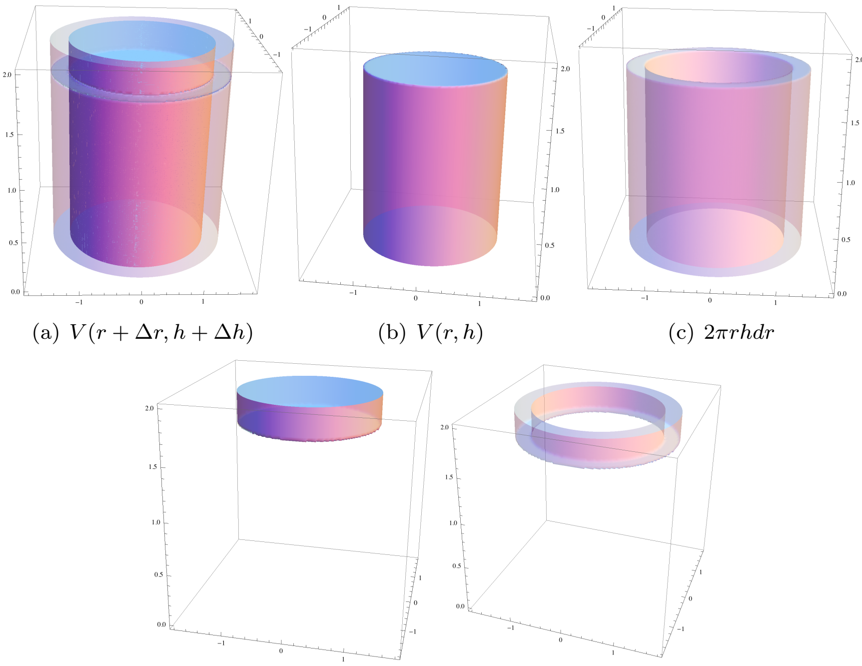
$$df = -\frac{1}{2^2} \cdot 0.1 + \frac{1}{2} \cdot (0.05) = -0.025 + 0.025 = 0.$$

26^{10} $=$ **26^{9}**: $V(r,h) = \pi r^2 h$. $dV = 2\pi rh dr + \pi r^2 dh$. See figure 3.
30^{10}: $V(r,h) = \pi r^2 h$. $V(3,10) = \pi 3^2 \cdot 10 = 90\pi \approx 282.74\text{cm}^3$. $dV = 2\pi rh dr + \pi r^2 dh$.

$$|dV| \leq 2\pi 3 \cdot 10 \cdot 0.05 + \pi 3^2 \cdot 0.05 \approx 10.84.$$

The possible error is about 10.84cm^3. The relative error is about

$$\frac{dV}{V} = \frac{10.84}{282.74} \approx 0.038.$$



(a) $V(r + \Delta r, h + \Delta h)$ (b) $V(r, h)$ (c) $2\pi rhdr$

(d) $\pi r^2 dh$ (e) $\Delta V - dV$

Figure 1: Problem 26

4 13.5

12^10 = 12^9: $d(t)^2 = (x_1(t) - x_2(t)^2 + (y_1(t) - y_2(t))^2$. Take the derivative on both sides with respect to t and use chain rule, we get

$$2dd' = 2(x_1(t) - x_2(t))(x_1'(t) - x_2'(t)) + 2(y_1(t) - y_2(t))(y_1'(t) - y_2'(t)).$$

So we get

$$d'(t) = \frac{(x_1(t) - x_2(t))(x_1' - x_2') + (y_1(t) - y_2(t))(y_1'(t) - y_2'(t)}{d(t)}. \tag{1}$$

When $t = 1$,

$$x_1(1) = 48\sqrt{2}, x_1'(1) = 48\sqrt{2}; \quad y_1(1) = 48\sqrt{2} - 16, y_1'(1) = 48\sqrt{2} - 32.$$

$$x_2(1) = 48\sqrt{3}, x_2'(1) = 48\sqrt{3}; \quad y_2(1) = 32, y_2'(1) = 16.$$

$$d(1) = \sqrt{(x_1(1) - x_2(1))^2 + (y_1(1) - y_2(1))^2} = 48\sqrt{(\sqrt{2} - \sqrt{3})^2 + (\sqrt{2} - 1)^2}$$

Substitute the data into (1), we get

$$\begin{aligned} d'(1) &= \frac{48^2(\sqrt{2} - \sqrt{3})^2 + 48^2(\sqrt{2} - 1)^2}{48\sqrt{(\sqrt{2} - \sqrt{3})^2 + (\sqrt{2} - 1)^2}} = 48\sqrt{(\sqrt{2} - \sqrt{3})^2 + (\sqrt{2} - 1)^2} \\ &= 48\sqrt{8 - 2\sqrt{6} - 2\sqrt{2}}. \end{aligned}$$

$16^{10} = 18^{9}$: $x(3,\pi/4) = 3\cos(\pi/4) = 3/\sqrt{2}$, $y(3,\pi/4) = 3\sin(\pi/4) = 3/\sqrt{2}$.

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} = 2x \cdot \cos t - 2y \cdot \sin t \\ &= 2 \cdot 3/\sqrt{2} \cdot 1/\sqrt{2} - 2 \cdot 3/\sqrt{2} \cdot 1/\sqrt{2} = 0.\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} = 2x \cdot (-s\sin t) - 2y \cdot (s\cos t) \\ &= -2 \cdot 3/\sqrt{2} \cdot 3/\sqrt{2} - 2 \cdot 3/\sqrt{2} \cdot 3/\sqrt{2} = -18.\end{aligned}$$

$20^{10} = 26^{9}$: $w = x\cos(yz)$, $x = s^2$, $y = t^2$, $z = s - 2t$.

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\ &= \cos(yz) \cdot 2s - xz\sin(yz) \cdot 0 - xy\sin(yz) \cdot 1 \\ &= 2s\cos(t^2(s-2t)) - s^2t^2\sin(t^2(s-2t)).\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} \\ &= \cos(yz) \cdot 0 - xz\sin(yz) \cdot 2t - xy\sin(yz) \cdot (-2) \\ &= -2s^2(s-2t)t\sin(t^2(s-2t)) + 2s^2t^2\sin(t^2(s-2t)).\end{aligned}$$

$21^{10} = 27^{9}$: $x^2 - xy + y^2 - x + y = 0$. This identity defines $y = y(x)$ implicitly. Differentiate both sides with respect to x:

$$2x - y - x\frac{dy}{dx} + 2y\frac{dy}{dx} - 1 + \frac{dy}{dx} = 0 \Longleftrightarrow (2x - y - 1) - (x - 2y - 1)\frac{dy}{dx} = 0.$$

So

$$\frac{dy}{dx} = \frac{2x - y - 1}{x - 2y - 1}.$$

(2^{10} **means problem 24 in 10-th edition textbook,** 14^9 **means problem 24 in 9-th edition textbook**)

1 13.6

$\mathbf{2^{10}} = \mathbf{14^9}$: $f(x,y) = y/(x+y)$, $P(3,0)$, $\theta = -\pi/6$.

$$f_x(3,0) = -\left.\frac{y}{(x+y)^2}\right|_{(3,0)} = 0, \quad f_y = \left.\frac{x}{(x+y)^2}\right|_{(3,0)} = \frac{1}{3}.$$

So

$$D_{\vec{u}}f(3,0) = f_x \cos\left(-\frac{\pi}{6}\right) + f_y \sin\left(-\frac{\pi}{6}\right) = \frac{1}{3}\cdot(-\frac{1}{2}) = -\frac{1}{6}.$$

$\mathbf{24^{10}} = \mathbf{28^9}$: $f(x,y) = 3x^2 - y^2 + 4$. $f_x = 6x$, $f_y = -2y$. $\nabla f(-1,4) = \langle -6, -8\rangle$. $\overrightarrow{PQ} = \langle 4, 2\rangle$. The unit vector in the direction of $\overrightarrow{PQ}$ is

$$\vec{u} = \frac{\overrightarrow{PQ}}{\|\overrightarrow{PQ}\|} = \frac{\langle 4, 2\rangle}{\sqrt{20}} = \frac{\langle 2, 1\rangle}{\sqrt{5}}.$$

The directional derivative is:

$$(D_{\vec{u}}f)(-1,4) = \nabla f(-1,4)\cdot\vec{u} = -\frac{20}{\sqrt{5}} = -4\sqrt{5}.$$

$\mathbf{36^{10}} = \mathbf{40^9}$: $f(x,y,z) = xe^{yz}$. $P(2,0,-4)$.

$$(\nabla f)(2,0,-4) = \langle e^{yz}, xze^{yz}, xye^{yz}\rangle(2,0,-4) = \langle 1, -8, 0\rangle.$$

The maximum value of the directional derivative is:

$$(D_{\frac{\nabla f}{\|\nabla f\|}} f)(2,0,4) = \|\nabla f\|(2,0,-4) = \|\langle 1,-8,0\rangle\| = \sqrt{65}.$$

$\mathbf{52^{10}} = \mathbf{56^9}$: $f(x,y) = x - y^2$. $c = 4$. $P(4,-1)$. **(a):** $(\nabla f)(4,-1) = \langle 1, -2y\rangle(4,-1) = \langle 1, 2\rangle$. **(b):** Unit normal vector $\vec{n} = (\nabla f)/\|\nabla f\| = \langle 1, 2\rangle/\sqrt{5}$. **(c):** Tangent line equation:

$$\langle 1, 2\rangle \cdot \langle x-4, y+1\rangle = 0 \quad\Longleftrightarrow\quad x + 2y = 2.$$

(d): See Figure 1.

$\mathbf{61^{10}} = \mathbf{66^9}$: $h(x,y) = 5000 - 0.001x^2 - 0.004y^2$. $\nabla h = \langle -0.002x, -0.008y\rangle$. At point $(500, 300, 4390)$ the gradient is

$$(\nabla h)(500, 300) = \langle -1, -2.4\rangle.$$

The direction to get greatest rate of ascendence is the direction of gradient vector:

$$\vec{u} = \frac{\nabla h}{\|\nabla h\|} = \frac{\langle -1, -2.4\rangle}{\sqrt{6.76}} \approx \langle -0.38, -0.92\rangle.$$

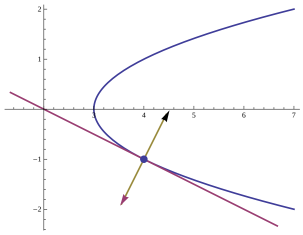


Figure 1: $52^{10} = 56^9$

2 13.7

10^{10} = 18^9: $f(x, y) = y/x$. $(1, 2, 2)$ The graph is also a level surface $g(x, y, z) = \frac{y}{x} - z = 0$. So the normal vector is given by the gradient of function

$$(\nabla g)(1, 2, 2) = \left\langle -\frac{y}{x^2}, \frac{1}{x}, -1 \right\rangle (1, 2, 2) = \langle -2, 1, -1 \rangle.$$

So the tangent plane at $(1, 2, 2)$ is:

$$\langle -2, 1, -1 \rangle \cdot \langle x - 1, y - 2, z - 2 \rangle = 0 \quad \Longleftrightarrow \quad 2x - y + z = 2.$$

18^{10} = 28^9: $x^2 + 2z^2 - y^2 = 0$. The gradient gives the normal vector $\langle 2x, -2y, 4z \rangle (1, 3, -2) = \langle 2, -6, -8 \rangle$. So the tangent plane is:

$$2(x - 1) - 6(y - 3) - 8(z + 2) = 0 \Longleftrightarrow x - 3y - 4z = 0$$

24^{10} = 34^9: $z = 16 - x^2 - y^2$. $(2, 2, 8)$. Write this as the level set: $x^2 + y^2 + z - 16 = 0$. The normal vector is given by the gradient:

$$\langle 2x, 2y, 1 \rangle (2, 2, 8) = \langle 4, 4, 1 \rangle.$$

The tangent plane is:

$$4(x - 2) + 4(y - 2) + (z - 8) = 0 \Longleftrightarrow 4x + 4y + z = 24.$$

The set of symmetric equations for the normal lines at $(2, 2, 8)$ is:

$$\frac{x - 2}{4} = \frac{y - 2}{4} = \frac{z - 8}{1}.$$

42^{10} = 52^9: $z = 3x^2 + 2y^2 - 3x + 4y - 5$. The tangent plane is horizontal if and only if the partial derivatives both vanish.

$$\frac{\partial z}{\partial x} = 6x - 3 = 0, \frac{\partial z}{\partial y} = 4y + 4 = 0 \Longrightarrow (x, y) = \left(\frac{1}{2}, -1 \right).$$

3 13.8

$18^{10} = 26^9$: $f(x,y) = 2xy - \frac{1}{2}(x^4 + y^4) + 1$. We set the partial derivatives to be 0. $f_x = 2y - 2x^3 = 0$, $f_y = 2x - 2y^3 = 0$. From the 1st equation we get $y = x^3$. Substitute this into the 2nd equation, we get

$$0 = 2x - 2x^9 = 2x(1-x^8) = 2x(1-x^4)(1+x^4) = 2x(1-x)(1+x)(1+x^2)(1+x^4).$$

So $x = 0, 1, -1$ and we get 3 critical points: $P_1(0,0)$, $P_2(1,1)$, $P_3(-1,-1)$. To classify the critical points, we calculate 2nd partial derivatives: $f_{xx} = -6x^2, f_{xy} = 2, f_{yy} = -6y^2$. $d = f_{xx}f_{yy} - f_{xy}^2 = 36x^2y^2 - 4$. Using the 2nd partial test:

1. $P_1(0,0)$: $d = -4 < 0$. This is a saddle point.
2. $P_2(1,1)$: $d = 32 > 0$, $f_{xx} = -6 < 0$. This is a relative maximum.
3. $P_3(-1,-1)$: $d = 32 > 0$, $f_{xx} = -6 < 0$. This is a relative maximum.

$27 - 30^{10} = 31 - 34^9$:

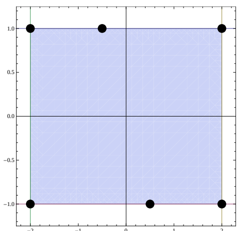
27: $d = f_{xx}f_{yy} - f_{xy}^2 = 9 \times 4 - 6^2 = 0$. Inconclusive.

28: $d = 20 > 0$, $f_{xx} = -3 < 0$. This is a relative maximum.

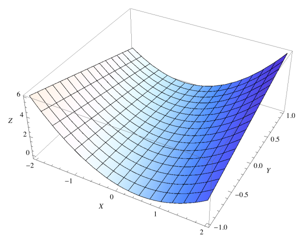
29: $d = -154 < 0$. Saddle point.

30: $d = 100 > 0$. $f_{xx} = 25 > 0$. Relative minimum.

$42^{10} = 46^9$: $f(x,y) = x^2 + xy$. $R = \{(x,y) : |x| \le 2, |y| \le 1\}$.



(a) Regions and candidate points



(b) Graph of the function

1. Find critical point in the interior:

$$f_x = 2x + y = 0, \quad f_y = x = 0 \Longrightarrow (x,y) = (0,0).$$

   The 2nd partial test: $f_{xx}f_{yy} - f_{xy}^2 = 2 \cdot 0 - 1^2 = -1 < 0.$. $(0,0)$ is a saddle point. So it does not obtain either maximum or minimum.

2. Restrict $f(x,y)$ to the boundary. There are four parts of the boundary:

- $C_1 = \{x = 2, |y| \le 1\}$. $g_1(y) = f(2, y) = 4 + 2y$ on the interval $[-1, 1]$. $\partial g_1/\partial y = 2 \neq 0$. There is no critical point in the interior of interval. We need to consider the end points: $(2, -1)$, $(2, 1)$. The values at these points are:

$$f(2, -1) = g_1(-1) = 2, \quad f(2, 1) = g_1(1) = 6.$$

- $C_2 = \{y = 1, |x| \le 2\}$. $g_2(x) = f(x, 1) = x^2 + x$ on the interval $[-2, 2]$. $\partial g_2/\partial x = 2x + 1$. So there is a critical point for g_2 at $x = -1/2$. So we get two more candidate points: $(-1/2, 1)$ and $(-2, 1)$(end point). The values at these points are:

$$f(-1/2, 1) = g_2(-1/2) = -\frac{1}{4}, \quad f(-2, 1) = g_2(-2) = 2.$$

- $C_3 = \{x = -2, |y| \le 1\}$. $g_3(y) = f(-2, y) = 4 - 2y$ on the interval $[-1, 1]$. $\partial g_3/\partial y = -2 \neq 0$. There is no critical point in the interior of interval. We need to consider the new end point: $(-2, -1)$. The value at this point is:

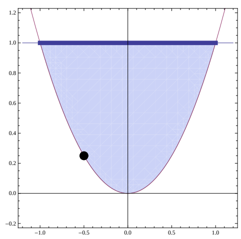
$$f(-2, -1) = g_3(-1) = 6.$$

- $C_4 = \{y = -1, |x| \le 2\}$. $g_4(x) = f(x, -1) = x^2 - x$ on the interval $[-2, 2]$. $\partial g_4/\partial x = 2x - 1$. So there is a critical point for g_4 at $x = 1/2$. So we get the last candidate point: $(1/2, -1)$. (The end points are already counted in) The value at this point is:

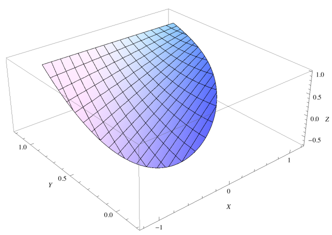
$$f(1/2, -1) = g_4(1/2) = -\frac{1}{4}.$$

3. By comparison, we get the 2 absolute maximum points: $f(2, 1) = f(-2, -1) = 6$, and 2 absolute minimum points: $f(-1/2, 1) = f(1/2, -1) = -1/4$.

$\mathbf{46^{10}} = \mathbf{50^{9}}$: $f(x, y) = 2x - 2xy + y^2$. $R = \{(x, y) : x^2 \le y \le 1\}$.



(c) Regions and candidate points



(d) Graph of the function

1. Find critical point in the interior:

$$f_x = 2 - 2y = 0, f_y = -2x + 2y = 0 \Longrightarrow (x, y) = (1, 1).$$

This point is on the boundary.

2. Restrict $f(x,y)$ to the boundary. There are 2 parts of the boundary:

   - $C_1 = \{y = x^2, |x| \leq 1\}$. $g_1(x) = f(x,x^2) = 2x - 2x^3 + x^4$ on the interval $[-1,1]$. $\partial g_1/\partial x = 2 - 6x^2 + 4x^3 = 2(1 - 3x^2 + 2x^3)$. We factorize to find the critical point:

     $$\begin{aligned} 1 - 3x^2 + 2x^3 &= 1 - x^2 - 2x^2 + 2x^3 = (1-x)(1+x) - 2x^2(1-x) \\ &= (x-1)(2x^2 - x - 1) = (x-1)(x-1)(2x+1). \end{aligned}$$

     So we get 2 critical potions for $g_1(x)$: $x = 1, -1/2$. We get 3 candidate points: $(-1/2, 1/4)$, $(1,1)$, $(-1,1)$. The values at these points are:

     $$f(-1/2, 1/4) = g_1(-1/2) = -\frac{11}{16}, f(1,1) = g_1(1) = 1, f(-1,1) = 1.$$

   - $C_2 = \{y = 1, |x| \leq 1\}$. $g_2(x) = f(x,1) = 1$ on the interval $[-1,1]$. So f is a constant along C_2. All the points on this line segment are candidate points.

3. By comparison, we get the absolute minimum: $f(-1/2, 1/4) = -\frac{11}{16}$, and f obtains absolute maximum value 1 at every point on the line segment $C_2 = \{y = 1, |x| \leq 1\}$.

(6^{10} **means problem 6 in 10-th edition textbook,** 6^9 **means problem 6 in 9-th edition textbook**)

1 13.8

$\mathbf{6^{10}} = \mathbf{6^9}$: $f(x,y) = -x^2 - y^2 + 10x + 12y - 64 = -(x-5)^2 - (y-6)^2 - 3$. So $f(5,6) = -3$ is a **absolute maximum**. Use partial derivative to verify:

$$f_x = -2x + 10 = 0, f_y = -2y + 12 = 0 \Longrightarrow x = 5, y = 6.$$

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \Longrightarrow d = f_{xx}f_{yy} - f_{xy} = 4 > 0, f_{xx} = -2 < 0.$$

By 2nd partial test the critical point $(5,6)$ is a relative maximum.

$\mathbf{12^{10}} = \mathbf{10^9}$: $f(x,y) = 2x^2 + 2xy + y^2 + 2x - 3$.

$$f_x = 4x + 2y + 2 = 0, f_y = 2x + 2y = 0 \Longrightarrow (x,y) = (-1,1).$$

2nd partial test:

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix} \Longrightarrow d = f_{xx}f_{yy} - f_{xy} = 4 > 0, f_{xx} = 2 > 0.$$

So the critical point $(-1,1)$ is a relative minimum.

2 13.9

$\mathbf{6^{10}} = \mathbf{6^9}$: $x + y + z = 32$. Maximize $P = xy^2z$. We can solve $z = 32 - x - y$. So $P(x,y) = xy^2(32 - x - y)$. We find critical point by setting:

$$P_x = 32y^2 - 2xy^2 - y^3 = y^2(32 - 2x - y) = 0, P_y = 64xy - 2x^2y - 3xy^2 = xy(64 - 2x - 3y) = 0.$$

We get the following critical points:

$$(x,0) \text{ for any } x; \quad (0,32); \quad (8,16).$$

The maximum is obtained when $(x,y) = (8,16)$ for which $z = 32 - x - y = 8$. $P_{max} = 8 \times 16^2 \times 8 = 16384$.

$\mathbf{10^{10}} = \mathbf{10^9}$: If the length, width and height of the box are denoted by x, y, z. Then the cost for constructing the box is: $1.5xy + 2(xz + yz)$. To find the maximum volume we can assume $1.5xy + 2(xz+yz) = C$. Under this constraint, we want to maximize $V = xyz$. We can solve $z = (C - 1.5xy)/(2(x+y))$. Then we get

$$V(x,y) = xy \cdot \frac{C - 1.5xy}{2(x+y)}.$$

We find critical point (maximum) by solving:

$$V_x = \frac{(Cy - 3xy^2)(x+y) - (Cxy - 1.5x^2y^2)}{2(x+y)^2} = 0,$$

and
$$V_y = \frac{(Cx - 3x^2y)(x+y) - (Cxy - 1.5x^2y^2)}{2(x+y)^2} = 0.$$
We can assume $x > 0$ and $y > 0$, then, after some calculations, these equations simply to
$$1.5x^2 + 3xy = C, 1.5y^2 + 3xy = C.$$
We can solve these two equations to get $x = y = \sqrt{C/4.5} = \sqrt{2C}/3$. The height is
$$z = \frac{C - 1.5xy}{x+y} = \frac{\sqrt{C}}{2\sqrt{2}}.$$

$\mathbf{14^{10}} = \mathbf{14^9}$: The profit
$$P(x_1, x_2) = 15(x_1 + x_2) - (0.02x_1^2 + 4x_1 + 500) - (0.05x_2^2 + 4x_2 + 275).$$
We find maximum using vanishing of partial derivatives:
$$P_{x_1} = 15 - 0.04x_1 - 4 = 0, P_{x_2} = 15 - 0.1x_2 - 4 = 0 \Longrightarrow (x_1, x_2) = (275, 110).$$

$\mathbf{18^{10}} = \mathbf{16^9}$: The area equals
$$A(x, \theta) = (30 - 2x + x\cos\theta) \cdot x\sin\theta = 30x\sin\theta - 2x^2\sin\theta + x^2\sin\theta\cos\theta.$$
We find the maximum by setting:
$$A_x = (30 - 4x + 2x\cos\theta)\sin\theta = 0, A_\theta = x(30\cos\theta - 2x\cos\theta + x(\cos^2\theta - \sin^2\theta)) = 0.$$
From the first equation, we get $\cos\theta = (2x-15)/x$ since we can assume $\sin\theta \neq 0$. Substituting $\cos\theta$ into the 2nd equation and using the relation $\cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$, we get the equation for x:
$$30\frac{2x-15}{x} - 2x\frac{2x-15}{x} + x\left(2\frac{(2x-15)^2}{x^2} - 1\right) = 0.$$
Simplying this equation, we get: $3x(x - 10) = 0$. So $x = 10$ and $\cos\theta = 1/2$. So $\theta = \pi/3 = 60°$.

3 13.10

$\mathbf{6^{10}} = \mathbf{8^9}$: $f(x, y) = 3x + y + 10$. Constraint $g(x, y) = x^2y - 6 = 0$. By Lagrange Multiplier method, we get three equations:
$$\begin{cases} \nabla f = \lambda\nabla g \\ x^2y - 6 = 0. \end{cases} \Longleftrightarrow \begin{cases} 3 = \lambda 2xy \\ 1 = \lambda x^2 \\ x^2y = 6. \end{cases}$$
Canceling λ from the first 2 equations, we get $y = 3x/2$. Substitute this into the last equation we get $3x^3/2 = 6$. So we get $x = 4^{1/3}$. From this we get $y = 1.5 \times 4^{1/3}$ and $\lambda = 4^{-2/3}$. So the maximum of $f(x, y)$ under the constraint is equal to
$$f(4^{1/3}, 1.5 \times 4^{1/3}) = 4.5 \times 4^{1/3} + 10.$$

$\mathbf{24^{10} = 26^9}$: Minimize the squared-distance function $f(x, y) = x^2 + (y - 10)^2$ under the constraint $g(x, y) = (x - 4)^2 + y^2 - 4 = 0$. Using Lagrange Multiplier method, we get three equations:

$$\begin{cases} 2x & = \lambda 2(x-4) \\ 2(y-10) & = \lambda 2y \\ (x-4)^2 + y^2 & = 4. \end{cases}$$

From the 1st equation, we get $x = \frac{4\lambda}{\lambda - 1}$. From then 2nd equation, we get $y = -\frac{10}{\lambda - 1}$. Substitute these into the last equation, we get an equation for λ:

$$\left(\frac{4\lambda}{\lambda - 1} - 4\right)^2 + \frac{100}{(\lambda - 1)^2} = 4 \Longleftrightarrow (\lambda - 1)^2 = 29.$$

So we get two solutions: $\lambda = 1 \pm \sqrt{29}$.

- If $\lambda = 1 + \sqrt{29}$, $x = 4(1 + \sqrt{29})/\sqrt{29}$ and $y = -10/\sqrt{29}$. The squared-distance is

$$f\left(\frac{4(\sqrt{29}+1)}{\sqrt{29}}, \frac{-10}{\sqrt{29}}\right) = \frac{16(\sqrt{29}+1)^2 + 100(\sqrt{29}+1)^2}{29} = 4(\sqrt{29}+1)^2.$$

- If $\lambda = 1 - \sqrt{29}$, $x = 4(\sqrt{29} - 1)/\sqrt{29}$ and $y = 10/\sqrt{29}$. The squared-distance is:

$$f\left(\frac{4(\sqrt{29}-1)}{\sqrt{29}}, \frac{10}{\sqrt{29}}\right) = \frac{16(\sqrt{29}-1)^2 + 100(\sqrt{29}-1)^2}{29} = 4(\sqrt{29}-1)^2.$$

The 2nd value is smaller, so the minimal distance is equal to $2(\sqrt{29} - 1)$.

Remark 3.1 *Note that one can solve the problem using geometry. The closest point on the circle to a point (outside of the circle) is the intersection point of the circle and line segment connecting the point and the center.*

$\mathbf{34^{10} = 36^9}$: We want to maximize $P(x, y, z) = xy^2z$ under the constraint $g(x, y, z) = x + y + z - 32 = 0$. Using Lagrange Multiplier method, we get equations:

$$\begin{cases} \nabla P = \lambda \nabla g \\ g(x,y,z) = 0. \end{cases} \Longleftrightarrow \begin{cases} y^2 z & = \lambda \cdot 1 \\ 2xyz & = \lambda \cdot 1 \\ xy^2 & = \lambda \cdot 1 \\ x + y + z & = 32. \end{cases}$$

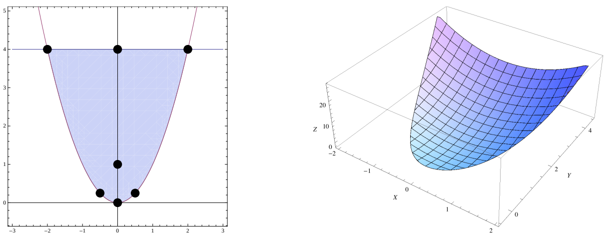
To find the maximum value of P, we can assume x, y, z are nonzero. So $\lambda \neq 0$ for the solution. Divide the first 2 equations, we get $y = 2x$. Divide the 2nd and 3rd equation, we get $y = 2z$. So $x = z = y/2$. Substitute this into the last equation, we get $2y = 32$. So $y = 16$ and $x = z = 8$. The maximum value of P is $P(8, 16, 8) = 16384$.

$\mathbf{46^{10} = 50^9}$: We assume the area is fixed, so $lh + \pi(l/2)^2/2 = A$ is fixed. In other words, we have the constraint $g(l, h) = l \cdot h + \pi l^2/8 - A = 0$. We want to minimize the diameter $f(l,h) = \pi \cdot l/2 + l + 2h$. Using the Lagrange Multiplier, we get the equation:

$$\begin{cases} \pi/2 + 1 & = & \lambda(h + \pi l/4) \\ 2 & = & \lambda l \\ lh + \pi l^2/8 & = & A. \end{cases}$$

From the 2nd equation, we get $l = 2/\lambda$. Substitute this into first equation, we solve $h = 1/\lambda$. So $l = 2h$, i.e. the length of the rectangle is twice its height.

Problem 1: Find the absolute extrema of function $f(x,y) = 3x^2 + 2y^2 - 4y$ on the region bounded by the curves $C_1 = \{y = x^2\}$ and $C_2 = \{y = 4\}$.



(a) Regions and candidate points

(b) Graph of the function

Solution:

1. Find the critical points in the interior:

$$\begin{cases} f_x = 6x = 0 \\ f_y = 4y - 4 = 0 \end{cases} \Longrightarrow x = 0, y = 1.$$

   There is a unique critical point $(0,1)$ in the interior. Note that $f(x,y) = 3x^2 + 2(y-1)^2 - 2 \geq -2$ and $f(0,1) = -2$. So we immediately get $(0,1)$ is **an** absolute minimum point. We can compute the value:

$$f(0,1) = -2.$$

2. Restrict $f(x,y)$ to the boundary:

   - Restrict $f(x,y)$ to the curve C_1. $g_1(x) = f(x,x^2) = 2x^4 - x^2$ on the interval $[-2,2]$.

$$g_1'(x) = 8x^3 - 2x = 2x(4x^2 - 1) = 0 \Longrightarrow x = 0, \pm 1/2.$$

     We get the following candidate points corresponding the critical points in the interior of the interval: $(0,0), (1/2,1/4), (-1/2,1/4)$. There are extra two candidates corresponding to the ends points of the interval $[-2,2]$: $(-2,4), (2,4)$. We compute the values at candidate points:

$$f(0,0) = g_1(0) = 0; \quad f(\pm 1/2, 1/4) = g_1(\pm 1/2) = -\frac{1}{8};$$

$$f(\pm 2, 4) = g_1(\pm 2) = 28.$$

   - Restrict $f(x,y)$ to the curve C_2. We define $g_2(x) = f(x,4) = 3x^2 + 16$. $g_2'(x) = 6x = 0 \to x = 0$. So we get the following candidate points: $(0,4)$, $(-2,4)$ and $(2,4)$. The latter two coming from end points of the interval $[-2,2]$ and were already counted.

$$f(0,4) = g_2(0) = 16.$$

3. Calculate the value of $f(x, y)$ at candidate points and compare to find absolute extrema. From above, we get altogether 7 candidate points:

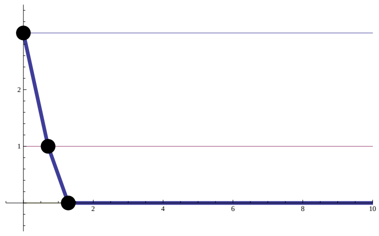
$$(0,1),(0,0),(-1/2,1/4),(1/2,1/4),(0,4),(-2,4),(2,4).$$

By the above calculation of the values of f at these candidate points, we get that the absolute maxima is $f(\pm 2, 4) = 28$. The absolute minimum is $f(0, 1) = -2$.

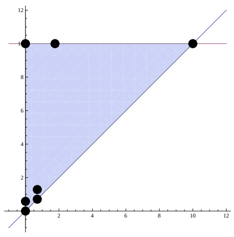
Problem 2 ($\mathbf{17^{10} = 19^{10}}$): The problem is translated to the following optimization problem: Minimize the function

$$f(x,y) = 3\sqrt{x^2+4} + 2\sqrt{1+(y-x)^2} + 10 - y$$

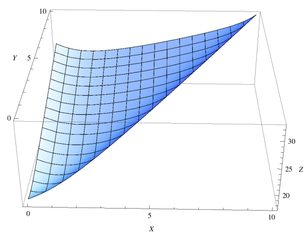
on the region $\{0 \le x \le y \le 10\}$.



(c) Cost optimizing configuration



(d) Regions and candidate points



(e) Graph of the function

1. Find the critical point in the interior:

$$\begin{cases} f_x &= \frac{3x}{\sqrt{4+x^2}} + \frac{2(x-y)}{\sqrt{1+(x-y)^2}} = 0 \\ f_y &= \frac{2(y-x)}{\sqrt{1+(x-y)^2}} - 1 = 0 \end{cases} \Rightarrow \begin{cases} \frac{3x}{\sqrt{4+x^2}} = 1 \\ 3(y-x)^2 = 1 \end{cases} \Rightarrow x = \frac{1}{\sqrt{2}}, y = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}.$$

We calculate the value at this critical point:

$$f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}) = 10 + 4\sqrt{2} + \sqrt{3} \approx 17.39.$$

2. Restrict $f(x, y)$ to the boundary.

- Restrict $f(x,y)$ to $C_1 = \{x = 0\}$. $g_1(y) = f(0,y) = 2\sqrt{1+y^2} - y + 16$ on $[0,10]$.
$$g_1'(y) = \frac{2y}{\sqrt{1+y^2}} - 1 = 0 \Longrightarrow y = \frac{1}{\sqrt{3}}.$$
So we get candidate points: $(0, 1/\sqrt{3})$, and $(0,0)$, $(0,10)$ (two end points). Calculate the values:
$$f(0, 1/\sqrt{3}) = g_1(1/\sqrt{3}) = \sqrt{3} + 16 \approx 17.73;$$
$$f(0,0) = g_1(0) = 18; \quad f(0,10) = g_1(10) = 2\sqrt{101} + 6 \approx 26.10.$$

- Restrict $f(x,y)$ to $C_2 = \{y = 10\}$. $g_2(x) = f(x,10) = 3\sqrt{x^2+4} + 2\sqrt{1+(10-x)^2}$ on $[0,10]$. To find critical point on this interval, we can differentiate:
$$g_2'(x) = \frac{3x}{\sqrt{x^2+4}} + \frac{2(x-10)}{\sqrt{1+(10-x)^2}} = 0.$$
It's not easy to find the solution to this equation. One could use computer to get approximate minimal point:
$$f(1.76557) \approx 24.59.$$
Also for the new corner point $(10,10)$, we calculate
$$f(10,10) = g_2(10) = 3\sqrt{104} + 2 \approx 32.59.$$

- Restrict $f(x,y)$ to $C_3 = \{y = x\}$. $g_3(x) = f(x,x) = 3\sqrt{x^2+4} - x + 12$ on $[0,10]$.
$$g_3'(x) = \frac{3x}{\sqrt{x^2+4}} - 1 = 0 \Longrightarrow x = \frac{1}{\sqrt{2}}.$$
So we get one new candidate point: $(1/\sqrt{2}, 1/\sqrt{2})$. Calculate
$$f(1/\sqrt{2}, 1/\sqrt{2}) = g_3(1/\sqrt{2}) = 12 + 4\sqrt{2} \approx 17.66.$$

3. By comparison of the above values, we find the absolute minimum:
$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}\right) = 10 + 4\sqrt{2} + \sqrt{3} \approx 17.39.$$

(28^{10} **means problem 24 in 10-th edition textbook,** 28^{9} **means problem 24 in 9-th edition textbook**)

1 14.1

$\mathbf{28^{10}} = \mathbf{28^{9}}$:

$$\begin{aligned}
\int_0^{\pi/4}\int_{\sqrt{3}}^{\sqrt{3}\cos\theta} r dr d\theta &= \int_0^{\pi/4} \frac{1}{2}r^2\Big]_{\sqrt{3}}^{\sqrt{3}\cos\theta} d\theta = \frac{3}{2}\int_0^{\pi/4}(\cos^2\theta - 1)d\theta = -\frac{3}{2}\int_0^{\pi/4}\sin^2\theta d\theta \\
&= -\frac{3}{2}\int_0^{\pi/4}\frac{1-\cos(2\theta)}{2}d\theta = -\frac{3}{4}\left(\theta - \frac{\sin(2\theta)}{2}\right)\Big]_0^{\pi/4} \\
&= -\frac{3}{4}(\frac{\pi}{4} - \frac{1}{2}) = \frac{3}{8} - \frac{3\pi}{16}.
\end{aligned}$$

$\mathbf{32^{10}} = \mathbf{32^{9}}$:

$$\begin{aligned}
\int_0^3\int_0^\infty \frac{x^2}{1+y^2} dy dx &= \int_0^3 x^2\left[\tan^{-1} y\right]_0^\infty dx = \int_0^3 x^2\frac{\pi}{2}dx \\
&= \frac{\pi}{2}\left[\frac{x^3}{3}\right]_0^3 = \frac{9\pi}{2}.
\end{aligned}$$

$\mathbf{42^{10}} = \mathbf{43^{9}}$:

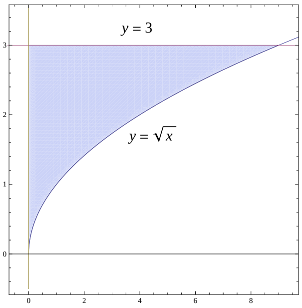
$$\begin{aligned}
\iint_R dA &= \int_{-a}^{a}\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} dy dx = \int_{-a}^{a} 2b\sqrt{1-\frac{x^2}{a^2}}dx \\
&= \int_{-\pi/2}^{\pi/2} 2b|\cos\theta| a\cos\theta d\theta = 2ab\int_{-\pi/2}^{\pi/2}\cos^2\theta d\theta \\
&= 2ab\int_{-\pi/2}^{\pi/2}\frac{1-\cos(2\theta)}{2}d\theta = ab\left[\theta - \frac{\sin(2\theta)}{2}\right]_{-\pi/2}^{\pi/2} \\
&= \pi ab.
\end{aligned}$$

$\mathbf{56^{10}} = \mathbf{58^{9}}$:

$$\begin{aligned}
\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy dx &= \int_{-2}^{2} 2\sqrt{4-x^2}dx = \int_{-\pi/2}^{\pi/2} 8\cos^2\theta d\theta \\
&= \int_{-\pi/2}^{\pi/2} 4(1+\cos(2\theta))d\theta = 4\left[1 + \frac{\sin(2\theta)}{2}\right]_{-\pi/2}^{\pi/2} \\
&= 4\pi.
\end{aligned}$$

Integrate x first:

$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dx dy = 4\pi.$$



$60^{10} = 62^{9}$: Region: $0 \le x \le 9$, $\sqrt{x} \le y \le 3$.

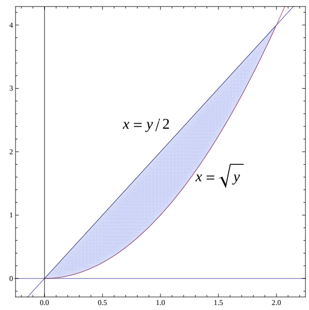
$$\int_0^9 \int_{\sqrt{x}}^3 dydx = \int_0^9 (3-\sqrt{x})dx = \left[3x - \frac{2}{3}x^{3/2}\right]_0^9 = 27 - 18 = 9.$$

Integrate x first:

$$\int_0^3 \int_0^{y^2} dxdy \quad = \quad \int_0^3 y^2 dy = \left[\frac{y^3}{3}\right]_0^3 = 9.$$

2 14.2

$8^{10} = 10^{9}$: Region: $0 \le y \le 4$, $y/2 \le x \le \sqrt{y}$.



$$\begin{aligned}\int_0^4 \int_{y/2}^{\sqrt{y}} x^2y^2 dxdy \quad &= \quad \int_0^4 y^2 \left[x^3/3\right]_{y/2}^{\sqrt{y}} dy = \frac{1}{3}\int_0^4 (y^{7/2} - \frac{y^5}{8})dy \\ &= \quad \frac{1}{3}\left[\frac{2}{9}y^{9/2} - \frac{1}{48}y^6\right]_0^4 = \frac{256}{27}.\end{aligned}$$

$12^{10} = 14^{9}$:

$$\begin{aligned}\iint_R \sin x \sin y \; dA \quad &= \quad \int_{-\pi}^{\pi}\int_0^{\pi/2} \sin x \sin y \; dydx = \int_{-\pi}^{\pi} \sin x[-\cos y]_0^{\pi/2} dx \\ &= \quad \int_{-\pi}^{\pi} \sin x dx = -[\cos x]_{-\pi}^{\pi} = 2.\end{aligned}$$

$$\begin{aligned}\iint_R \sin x \sin y \; dA &= \int_0^{\pi/2}\int_{-\pi}^{\pi} \sin x \sin y \; dxdy = \int_0^{\pi/2} \sin y[-\cos x]_{-\pi}^{\pi} dy \\ &= \int_0^{\pi/2} 2\sin y \; dy = -2[\cos x]_0^{\pi/2} = 2.\end{aligned}$$

18^{10} = 20^{9}: (It's much easier to use polar coordinate for this problem.)

$$\begin{aligned}&\iint_R (x^2+y^2)dA = \int_{-2}^{2}\int_0^{\sqrt{4-x^2}} (x^2+y^2)\; dydx = \int_{-2}^{2} \left[x^2 y + \frac{1}{3}y^3\right]_0^{\sqrt{4-x^2}} dx \\ =\; & \int_{-2}^{2}\left(x^2\sqrt{4-x^2} + \frac{1}{3}(4-x^2)\sqrt{4-x^2}\right) dx = \int_{-\pi/2}^{\pi/2} \frac{4}{3}\sqrt{4-x^2} + \frac{2}{3}x^2\sqrt{4-x^2}dx \\ =\; & \frac{1}{3}\int_{-\pi/2}^{\pi/2} (4\cdot 2\cos\theta + 2\cdot 4\sin^2\theta\cdot 2\cos\theta)2\cos\theta d\theta = \frac{8}{3}\int_{-\pi/2}^{\pi/2}(2\cos^2\theta + 4\sin^2\theta\cos^2\theta)d\theta \\ =\; & \frac{8}{3}\int_{-\pi/2}^{\pi/2}(1+\cos(2\theta)+\sin^2(2\theta))d\theta = \frac{8}{3}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_{-\pi/2}^{\pi/2} + \frac{8}{3}\int_{-\pi/2}^{\pi/2}\frac{1-\cos(4\theta)}{2}d\theta \\ =\; & \frac{8}{3}\pi + \frac{4}{3}\pi = 4\pi.\end{aligned}$$

Polar coordinate:

$$\iint_R (x^2+y^2)dA = \int_0^{\pi}\int_0^2 r^2 \; rdrd\theta = \int_0^{\pi} \left.\frac{r^4}{4}\right]_0^2 d\theta = 4\pi.$$

24^{10} = 28^{9}: The volume is calculated by

$$\begin{aligned}\int_0^2\int_x^2 (4-y^2)\; dydx &= \int_0^2 \left[4y - \frac{1}{3}y^3\right]_x^2 dx = \int_0^2 \left(\frac{16}{3} - 4x + \frac{x^3}{3}\right) dx \\ &= \left[\frac{16}{3}x - 2x^2 + \frac{x^4}{12}\right]_0^2 = 4.\end{aligned}$$

If we first integrate x variable, then we get:

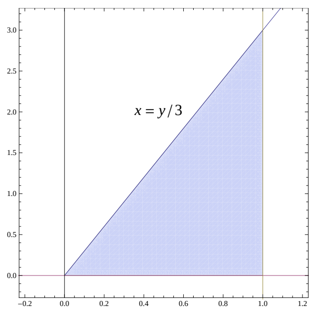
$$\int_0^2\int_0^y (4-y^2)\; dxdy \;=\; \int_0^2 (4-y^2)ydy = \left[2y^2 - \frac{y^4}{4}\right]_0^2 = 4.$$

34^{10} = 42^{9}: First find the set of points of intersection: $x^2+y^2 = 2x$. This defines a circle: $(x-1)^2+y^2 = 1$. We can solve $y = \pm\sqrt{2x-x^2}$. The volume of the solid region is over the disk $R = \{(x-1)^2 + y^2 \le 1\}$.

$$\begin{aligned}
\iint_R (2x - x^2 - y^2) dA &= \int_0^2 \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} (2x - x^2 - y^2)\ dydx \\
&= \int_0^2 \left[(2x-x^2)y - \frac{1}{3}y^3\right]_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} dx = \frac{4}{3}\int_0^2 (1-(x-1)^2)\sqrt{1-(x-1)^2}dx \\
&= \frac{4}{3}\int_{-1}^{1} (1-z^2)\sqrt{1-z^2}dz \\
&= \frac{4}{3}\int_{-\pi/2}^{\pi/2} \cos^2\theta\cos\theta\cos\theta d\theta = \frac{4}{3}\int_{-\pi/2}^{\pi/2} \cos^4\theta\ d\theta \\
&= \frac{\pi}{2}.
\end{aligned}$$

Easier to compute in polar coordinate (**good exercise**).

48$^{\mathbf{10}}$ $=$ **56**$^{\mathbf{9}}$: Region: $0 \le y \le 3$, $y/3 \le x \le 1$.



$$\begin{aligned}
\int_0^3 \int_{y/3}^{1} \frac{1}{1+x^4}\ dxdy &= \int_0^1 \int_0^{3x} \frac{1}{1+x^4}\ dydx = \int_0^1 \frac{3x}{1+x^4}dx \\
&= \frac{3}{2}\int_0^1 \frac{d(x^2)}{1+(x^2)^2} = \frac{3}{2}\int_0^1 \frac{du}{1+u^2} \\
&= \frac{3}{2}\left[\tan^{-1} u\right]_0^1 \\
&= \frac{3}{2}\frac{\pi}{4} = \frac{3}{8}\pi.
\end{aligned}$$

54$^{\mathbf{10}}$ $=$ **62**$^{\mathbf{9}}$: $Area(R) = \frac{1}{2}$. Integral:

$$\begin{aligned}
\iint_R f(x,y)\ dA &= \int_0^1 \int_0^x \frac{1}{x+y}\ dydx = \int_0^1 \left[\ln|x+y|\right]_0^x dx \\
&= \int_0^1 (\ln(2x) - \ln(x))dx = \int_0^1 \ln 2dx = \ln 2.
\end{aligned}$$

So the average value is

$$\frac{1}{Area(R)} \iint_R f(x,y)\ dA = 2\ln 2.$$

3 14.4

$2^{10} = 2^9$: The mass is

$$\begin{aligned}
\int_0^3 \int_0^{9-x^2} xy \; dydx &= \int_0^3 \left[\frac{x}{2}y^2\right]_0^{9-x^2} dx = \frac{1}{2}\int_0^3 x(9-x^2)^2 dx \\
&= \frac{1}{2}\int_0^3 (81x - 18x^3 + x^5)dx = \frac{1}{2}\left[\frac{81}{2}x^2 - \frac{18}{4}x^4 + \frac{1}{6}x^6\right]_0^3 \\
&= \frac{243}{4}.
\end{aligned}$$

$6^{10} = 6^9$: **(a):** The mass is

$$\text{mass} = \iint_R \rho(x,y) \; dA = \int_0^a \int_0^b kxy \; dydx = \int_0^a \frac{k}{2}b^2 x dx = \frac{k}{4}a^2b^2.$$

The integral of x coordinate:

$$\int_0^a \int_0^b x(kxy) \; dydx = \int_0^a \frac{k}{2}b^2x^2 dx = \frac{k}{6}a^3b^2.$$

The integral of y coordinate:

$$\int_0^a \int_0^b y(kxy) \; dydx = \int_0^a \frac{k}{3}b^3 x dx = \frac{k}{6}a^2b^3.$$

So the center of mass is

$$\frac{1}{\text{mass}}\left(\iint_R x\rho(x,y) \; dA, \iint_R y\rho(x,y) \; dA\right) = \left(\frac{2}{3}a, \frac{2}{3}b\right).$$

(b): The mass is

$$\begin{aligned}
\iint_R \rho(x,y) \; dA &= \int_0^a \int_0^b k(x^2+y^2) \; dydx = \int_0^a k\left(x^2b + \frac{b^3}{3}\right) dx = k\frac{ba^3 + ab^3}{3} \\
&= \frac{k}{3}ab(a^2+b^2).
\end{aligned}$$

The integral of x coordinate:

$$\begin{aligned}
\int_0^a \int_0^b xk(x^2+y^2)) \; dydx &= \int_0^a k\left(bx^3 + \frac{xb^3}{3}\right) dx = k\left(\frac{ba^4}{4} + \frac{a^2b^3}{6}\right) \\
&= \frac{k}{12}a^2b(3a^2+2b^2).
\end{aligned}$$

The integral of y coordinate:

$$\begin{aligned}
\int_0^a \int_0^b yk(x^2+y^2) \; dydx &= \int_0^a k\left(\frac{x^2b^2}{2} + \frac{b^4}{4}\right) dx = k\left(\frac{a^3b^2}{6} + \frac{ab^4}{4}\right) \\
&= \frac{k}{12}ab^2(2a^2+3b^2).
\end{aligned}$$

So the center of mass is

$$\begin{aligned}
\frac{1}{mass}\left(\iint_R x\rho(x,y)\ dA, \iint_R y\rho(x,y)\ dA\right) &= \frac{1}{ab(a^2+b^2)k/3}\left(\frac{k}{12}a^2b(3a^2+2b^2), \frac{k}{12}ab^2(2a^2+3b^2)\right) \\
&= \left(\frac{a(3a^2+2b^2)}{4(a^2+b^2)}, \frac{b(2a^2+3b^2))}{4(a^2+b^2)}\right).
\end{aligned}$$

MAT 203 FALL 2013 Practice MIDTERM II

NAME : ID :

RECITATION NUMBER:

THERE ARE SIX (6) PROBLEMS. THEY HAVE THE INDICATED VALUE.

SHOW YOUR WORK

DO NOT TEAR-OFF ANY PAGE

NO CALCULATORS NO CELLS ETC.

ON YOUR DESK: ONLY test, pen, pencil, eraser.

1		40pts
2		50pts
3		45pts
4		40pts
5		45pts
6		40pts
Total		260pts

!!! WRITE YOUR NAME, STUDENT ID AND LECTURE N. BELOW !!!

NAME : ID :

LECTURE N.

1. (40pts)

Find the domain and range of the given function and identify its level curves. Sketch a typical level curve.

$$f(x,y) = \frac{1}{x^2+y^2+1}.$$

2. (50pts)
(a): Let $g(x,y) = e^x + y\sin(xy)$. Find the directional derivative of $g(x,y)$ at point $(1,\pi)$ in the direction of $\vec{v} = \vec{i} + \vec{j}$.

(b): For a right circular cylinder, if there are both errors of measurements of r and h of 1%, what's the relative (percentage) error of measurement of the volume $V = \pi r^2 h$?

3. (45pts)

(a): Find $\partial w/\partial u$ and $\partial w/\partial v$ when $u = v = 0$ if $w = \ln\sqrt{1+x^2} - \tan^{-1} x$ and $x = 2e^u \cos v$.

(b): Find the value of dy/dx at point P.

$$2xy + e^{x+y} - 2 = 0, \quad P(0, \ln 2).$$

4. (40pts) Find an equation for the tangent plane and normal line of the graph of following function at the given point:

$$z = \frac{1}{x^2 + y^2}, (1, 1, 1/2).$$

5. (45pts)

(a): Find relative maxima, relative minima and saddle points over the xy-plane.

$$f(x, y) = x^2 + y^2 - 3x - xy.$$

(b): Find extreme values of $f(x, y) = x^2 + y^2 - 3x - xy$ on the disk $x^2 + y^2 \leq 9$.

6. (40pts)

(a): Sketch the region of integration and evaluate the integral.

$$\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}}\, dydx.$$

(b): Find the volume of the solid that is bounded above by the cylinder $z = x^2$ and below by the region enclosed by the parabola $y = 2 - x^2$ and the line $y = x$ in the xy-plane.

!!! WRITE YOUR NAME, STUDENT ID AND LECTURE N. BELOW !!!

NAME : ID :

LECTURE N.

1. (40pts)

Find the domain and range of the given function and identify its level curves. Sketch a typical level curve.

$$f(x,y) = \frac{1}{x^2+y^2+1}.$$

Domain : $\{(x,y) \in \mathbb{R}\times\mathbb{R} ; -\infty < x < \infty, -\infty < y < \infty\}$

= the whole xy-plane.

Range : $(0, 1]$

This is because $1 \le x^2+y^2+1 < \infty$

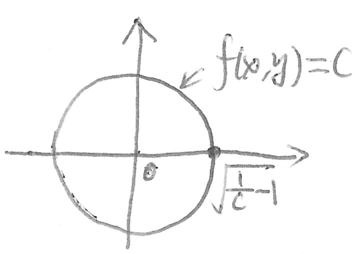
$$\Downarrow$$

$$0 < \frac{1}{x^2+y^2+1} \le 1.$$

Level curves : for $c \in (0,1]$,

$$\frac{1}{x^2+y^2+1} = c \iff x^2+y^2 = \frac{1}{c} - 1.$$

This is a circle of radius $\sqrt{\frac{1}{c}-1}$, centered at $(0,0)$



2. (50pts)
(a): Let $g(x, y) = e^x + y\sin(xy)$. Find the directional derivative of $g(x, y)$ at point $(1, \pi)$ in the direction of $\vec{v} = \vec{i} + \vec{j}$.

The gradient of g:

$$\nabla g = \langle g_x, g_y \rangle = \langle e^x + y^2\cos(xy),\ \sin(xy) + xy\cos(xy) \rangle$$

at $(1,\pi)$, $(\nabla g)(1,\pi) = \langle e^1 + \pi^2\cos\pi,\ \sin\pi + \pi\cos\pi \rangle$

$$= \langle e - \pi^2, -\pi \rangle$$

The direction of $\vec{v} = \vec{i} + \vec{j}$ is represented by the unit vector

$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{2}}(\vec{i} + \vec{j})$. so the directional derivative

is $(D_{\vec{u}}g)(1,\pi) = \nabla g \cdot \vec{u} = \frac{1}{\sqrt{2}}(e - \pi^2 - \pi)$.

(b): For a right circular cylinder, if there are both errors of measurements of r and h of 1%, what's the relative (percentage) error of measurement of the volume $V = \pi r^2 h$?

By assumption, $\frac{dr}{r} = 1\% = \frac{dh}{h}$.

The differential of $V = \pi r^2 h$ is

$$dV = 2\pi r h \cdot dr + \pi r^2 dh.$$

This gives the estimated error. The relative error is

$$\frac{dV}{V} = \frac{2\pi r h\, dr + \pi r^2 dh}{\pi r^2 h} = \frac{2dr}{r} + \frac{dh}{h} = 3\%.$$

3. (45pts)

$\frac{1}{2}\ln(1+x^2)$
$\parallel$

(a): Find $\partial w/\partial u$ and $\partial w/\partial v$ when $u = v = 0$ if $w = \ln\sqrt{1+x^2} - \tan^{-1}x$ and $x = 2e^u \cos v$.

$$\frac{dw}{dx} = \frac{1}{2}\frac{2x}{1+x^2} - \frac{1}{1+x^2} = \frac{x-1}{1+x^2}$$

$$\begin{array}{ccc} & w & \\ & | & \\ & x & \\ / & & \backslash \\ u & & v \end{array}$$

$$\frac{\partial x}{\partial u} = 2e^u \cos v, \quad \frac{\partial x}{\partial v} = -2e^u \sin v$$

when $u=v=0$. $x = 2e^0 \cos 0 = 2$.

$$\left.\frac{dw}{dx}\right|_{x=2} = \frac{1}{5} \qquad \left.\frac{\partial x}{\partial u}\right|_{(0,0)} = 2, \quad \left.\frac{\partial x}{\partial v}\right|_{(0,0)} = 0.$$

By chain rule,

$$\frac{\partial w}{\partial u} = \frac{dw}{dx}\cdot\frac{\partial x}{\partial u} = \frac{1}{5}\cdot 2 = \frac{2}{5}$$

$$\frac{\partial w}{\partial v} = \frac{dw}{dx}\cdot\frac{\partial x}{\partial v} = \frac{1}{5}\cdot 0 = 0.$$

(b): Find the value of dy/dx at point P.

$$2xy + e^{x+y} - 2 = 0, \quad P(0, \ln 2).$$

$y = y(x)$ is defined implicitly by

$$2x\,y(x) + e^{x+y(x)} - 2 = 0.$$

Taking derivative with respect to x, we get

$$2y + 2xy' + e^{x+y}(1+y') = 0$$
$$\parallel$$
$$(2x + e^{x+y})y' + (2y + e^{x+y})$$

$$\Rightarrow y' = -\frac{2y + e^{x+y}}{2x + e^{x+y}}$$

at $(x,y) = (0, \ln 2)$,

$$\frac{dy}{dx} = y' = -\frac{2\ln 2 + e^{\ln 2}}{2\cdot 0 + e^{\ln 2}} = -\frac{2\ln 2 + 2}{2} = -(\ln 2 + 1).$$

4. (40pts) Find an equation for the tangent plane and normal line of the graph of following function at the given point:

$$z = \frac{1}{x^2+y^2}, (1, 1, 1/2).$$

The graph is a level surface of the function:

$g(x,y,z) = (x^2+y^2)z - 1$ (or use: $z - \frac{1}{x^2+y^2}$)

$\nabla g(x,y,z) = \langle 2xz, 2yz, x^2+y^2 \rangle$ ($\nabla g = \langle \frac{2x}{(x^2+y^2)^2}, \frac{2y}{(x^2+y^2)^2}, 1 \rangle$)

at $(x,y,z) = (1,1,\frac{1}{2})$, $\nabla g = \langle 1,1,2 \rangle$ ($\nabla g = \langle \frac{1}{2}, \frac{1}{2}, 1 \rangle$)

So the tangent plane is:

$1(x-1) + 1\cdot(y-1) + 2(z-\frac{1}{2}) = 0 \Leftrightarrow x+y+2z = 3.$

The normal line is:

$$\frac{x-1}{1} = \frac{y-1}{1} = \frac{z-\frac{1}{2}}{2} \text{ (symmetric equations)}$$

or by parametric equation:

$$\begin{cases} x = 1+t \\ y = 1+t \\ z = \frac{1}{2}+2t \end{cases} \quad -\infty < t < \infty$$

5. (45pts)

(a): Find relative maxima, relative minima and saddle points over the xy-plane.

$$f(x,y) = x^2 + y^2 - 3x - xy.$$

$f_x = 2x-3-y=0$
$f_y = 2y - x = 0$ $\Rightarrow$ $x=2$, $y=1$

$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ $\det = f_{xx}f_{yy} - f_{xy}^2 = 3 > 0$

$f_{xx} = 2 > 0$.

By 2nd partial test,

the (unique) critical point $(2,1)$ is a relative minimum.
$f(2,1) = -3$.

(b): Find extreme values of $f(x,y) = x^2 + y^2 - 3x - xy$ on the disk $x^2 + y^2 \le 9$.

By (a), $(2,1)$ is a candidate point for absolute extremal point. (since $2^2+1^2<9$)

We need to restrict $f(x,y)$ to the boundary to find other candidate points. This is a constraint problem with constraint:

$g(x,y) = x^2+y^2-9=0$.

$f(x,y)$ becomes $f_1(x,y) = 9-3x-xy$ (since $x^2+y^2=9$).

Using Lagrange Multiplier method, we get equations:

$$\begin{cases} -3-y = \lambda \cdot 2x & ① \\ -x = \lambda \cdot 2y & ② \\ x^2+y^2 = 9 & ③ \end{cases}$$

Substitute ② into ①, we get

$(4\lambda^2-1)y = 3 \Rightarrow y = \frac{3}{4\lambda^2-1} \overset{②}{\Rightarrow} x = \frac{-6\lambda}{4\lambda^2-1}$

substitute these into ③ we get: $\frac{9}{(4\lambda^2-1)^2} + \frac{36\lambda^2}{(4\lambda^2-1)^2} = 9 \Leftrightarrow 1+4\lambda^2 = (4\lambda^2-1)^2 = 16\lambda^4 - 8\lambda^2 + 1$

so we get $0 = 16\lambda^4 - 12\lambda^2 = 4\lambda^2(4\lambda^2-3) \Rightarrow \lambda = 0$, or $\pm\frac{\sqrt{3}}{2}$.

i) $\lambda = 0 \overset{①②}{\Rightarrow} x=0, y=-3$. back page

$f(x,y)=x^2+y^2-3x-xy$

ii) $\lambda=\frac{\sqrt{3}}{2} \Rightarrow y=\frac{3}{4\lambda^2-1}=\frac{3}{2},\ x=-2\lambda y=-\frac{3\sqrt{3}}{2}$

iii) $\lambda=-\frac{\sqrt{3}}{2} \Rightarrow y=\frac{3}{4\lambda^2-1}=\frac{3}{2},\ x=-2\lambda y=\frac{3\sqrt{3}}{2}$

So we get all the candidate points:

$(2,1)$; $(0,-3)$; $\left(\frac{-3\sqrt{3}}{2},\frac{3}{2}\right)$; $\left(\frac{3\sqrt{3}}{2},\frac{3}{2}\right)$

$f(2,1)=-3$, $f(0,-3)=9$, $f\left(-\frac{3\sqrt{3}}{2},\frac{3}{2}\right)=9+\frac{27\sqrt{3}}{4}$; $f\left(\frac{3\sqrt{3}}{2},\frac{3}{2}\right)=9-\frac{27\sqrt{3}}{4}$.

So the absolute maximum is: $f\left(-\frac{3\sqrt{3}}{2},\frac{3}{2}\right)=9+\frac{27\sqrt{3}}{4}$.

absolute minimum is $f(2,1)=-3$.

(Note that $-3<9-\frac{27\sqrt{3}}{4}$: $\Leftarrow -3-\left(9-\frac{27\sqrt{3}}{4}\right)=\frac{27\sqrt{3}}{4}-12<0$

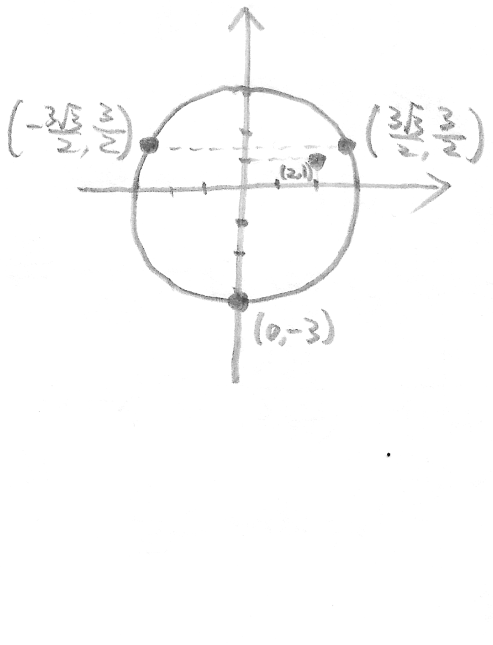
$\Uparrow$

$27\sqrt{3}-48=3(9\sqrt{3}-16)<0$

$\Uparrow$

$(9\sqrt{3})^2-(16)^2=81\times3-256$

$=243-256<0$)



$x^2+y^2-3x-xy$
$\shortparallel$

Note: when we want to find the extrema of f restricted to the boundary, we can also parametrize the boundary curve $x^2+y^2=9$ by $x=3\cos\theta$, $y=3\sin\theta$. $0\le\theta<2\pi$.

we can define

$$g(\theta)=f(3\cos\theta, 3\sin\theta)=9-9\cos\theta-9\sin\theta\cdot\cos\theta. \text{ on } [0,2\pi)$$

we can now find extrema of $g(\theta)$ by differentiation:

$$g'(\theta)=9(\sin\theta-\cos^2\theta+\sin^2\theta)=9(\sin\theta-(1-\sin^2\theta)+\sin^2\theta)$$

$$=9\cdot(2\sin^2\theta+\sin\theta-1)=9\cdot(2\sin\theta-1)\cdot(\sin\theta+1).$$

$g'(\theta)=0 \Rightarrow \sin\theta=\frac{1}{2}$ or $\sin\theta=-1$.

i) $\sin\theta=\frac{1}{2} \Rightarrow \cos\theta=\pm\frac{\sqrt{3}}{2}$. so we get 2 candidate points

$\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$, $\left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$

ii). $\sin\theta=-1 \Rightarrow \cos\theta=0$. so we get 1. candidate point

$(0,-3)$.

Now we can proceed as before to the comparison of values.

6. (40pts)

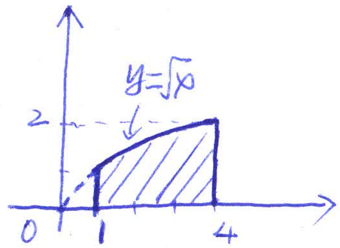
(a): Sketch the region of integration and evaluate the integral.

$$\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}}\, dydx.$$

$$\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{\frac{y}{\sqrt{x}}}\, dy\, dx = \int_1^4 \frac{3}{2}\sqrt{x}\, e^{\frac{y}{\sqrt{x}}}\Big]_0^{\sqrt{x}}\, dx$$

$$= \int_1^4 \frac{3}{2}\sqrt{x}\cdot(e-1) = (e-1)\, x^{\frac{3}{2}}\Big]_1^4$$

$$= (e-1)(8-1) = 7(e-1).$$

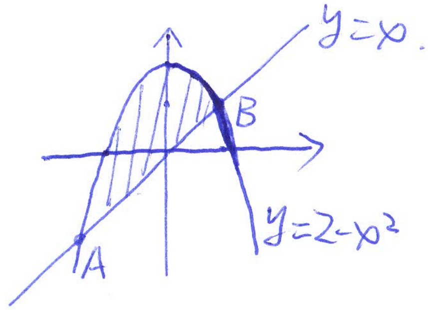


Region $1 \le x \le 4,\ 0 \le y \le \sqrt{x}$

(b): Find the volume of the solid that is bounded above by the cylinder $z = x^2$ and below by the region enclosed by the parabola $y = 2 - x^2$ and the line $y = x$ in the xy-plane.

Solve A, B first:

$$\begin{cases} y = 2 - x^2 \\ y = x \end{cases} \Rightarrow \underset{\underset{(x+2)(x-1)}{\parallel}}{x^2 + x - 2} = 0 \Rightarrow x = -2 \text{ or } 1.$$



So $A = (-2,-2),\ B = (1,1)$

$$\text{Volume} = \int_{-2}^{1} \int_{x}^{2-x^2} x^2\, dy\, dx = \int_{-2}^{1} x^2(2 - x^2 - x)\, dx$$

$$= \left[\frac{2}{3}x^3 - \frac{1}{5}x^5 - \frac{1}{4}x^4\right]_{-2}^{1} = \left(\frac{2}{3} - \frac{1}{5} - \frac{1}{4}\right) - \left(-\frac{16}{3} + \frac{32}{5} - \frac{16}{4}\right)$$

$$= 6 - \frac{33}{5} + \frac{15}{4} = \frac{63}{20} = 3.15.$$

!!! WRITE YOUR NAME, STUDENT ID AND LECTURE N. BELOW !!!

NAME : ID :

1. **(40pts)** Let $f(x, y) = \ln(y - x^2)$.

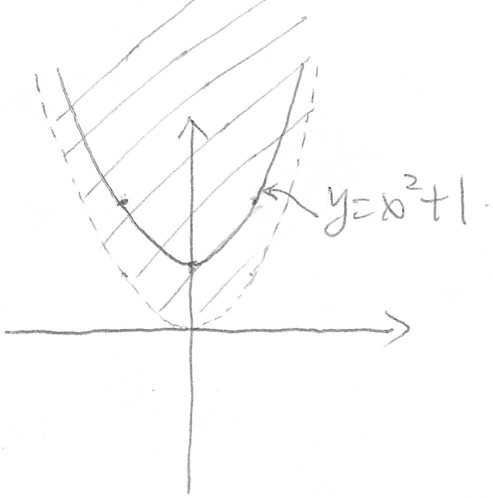
(a): Find the domain and range of the $f(x, y)$.

Domain: $\{(x, y) \in \mathbb{R}\times\mathbb{R},\ y > x^2\}$.

Range: $\mathbb{R} = (-\infty, +\infty)$.

(b): Sketch the domain of $f(x, y)$, and the level curve $C = \{(x, y) : \ln(y - x^2) = 0\}$.

$\ln(y - x^2) = 0 \Leftrightarrow y - x^2 = 1 \Leftrightarrow y = x^2 + 1$.



2. (40pts)

$$f(x,y) = e^{xy}\sin x, \quad P = \left(\frac{\pi}{2}, 0\right).$$

Find the directions in which the function increases and decreases most rapidly at P. Then find the directional derivatives of the function in these directions.

$f_x = ye^{xy}\sin x + e^{xy}\cos x$, $f_x(\frac{\pi}{2}, 0) = 0$

$f_y = xe^{xy}\sin x$, $f_y(\frac{\pi}{2}, 0) = \frac{\pi}{2}$.

So the gradient $\nabla f\big|_{(\frac{\pi}{2},0)} = \langle 0, \frac{\pi}{2}\rangle$.

The direction of most rapid increase is

$$\vec{u}_1 = \frac{\nabla f}{\|\nabla f\|}\Big|_{(\frac{\pi}{2},0)} = \langle 0, 1\rangle.$$

$$D_{\vec{u}_1} f\Big|_{(\frac{\pi}{2},0)} = \|\nabla f\| = \frac{\pi}{2}.$$

The direction of most rapid decrease is

$$\vec{u}_2 = -\vec{u}_1 = \langle 0, -1\rangle$$

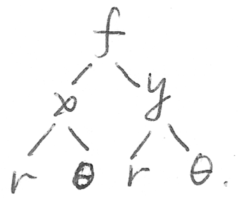
$$D_{\vec{u}_2} f\Big|_{(\frac{\pi}{2},0)} = -\|\nabla f\| = -\frac{\pi}{2}.$$

3. (40pts)

(a): Let $f(x,y) = \ln(y^2 - x^2)$ and $x = r\cos\theta$, $y = r\sin\theta$. Find $\partial f/\partial r$ and $\partial f/\partial\theta$ when $r = 2$ and $\theta = \pi/3$.

Method 1:

Use chain rule:



$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$\frac{\partial f}{\partial x} = \frac{-2x}{y^2-x^2}, \quad \frac{\partial f}{\partial y} = \frac{2y}{y^2-x^2} \qquad \left.\frac{\partial f}{\partial x}\right|_{(1,\sqrt{3})} = -1, \quad \left.\frac{\partial f}{\partial y}\right|_{(1,\sqrt{3})} = \sqrt{3}$$

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \left.\frac{\partial x}{\partial r}\right|_{(2,\frac{\pi}{3})} = \frac{1}{2}, \quad \left.\frac{\partial x}{\partial \theta}\right|_{(2,\frac{\pi}{3})} = -\sqrt{3}$$

$$\frac{\partial y}{\partial r} = \sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta. \qquad \left.\frac{\partial y}{\partial r}\right|_{(2,\frac{\pi}{3})} = \frac{\sqrt{3}}{2}, \quad \left.\frac{\partial y}{\partial \theta}\right|_{(2,\frac{\pi}{3})} = 1.$$

when $r=2$, $\theta = \frac{\pi}{3}$, $x = 2\cdot\cos\frac{\pi}{3} = 1$, $y = 2\cdot\sin\frac{\pi}{3} = \sqrt{3}$

so

$$\left.\frac{\partial f}{\partial r}\right|_{(r,\theta)=(2,\frac{\pi}{3})} = -1\cdot\frac{1}{2} + \sqrt{3}\cdot\frac{\sqrt{3}}{2} = 1$$

$$\left.\frac{\partial f}{\partial \theta}\right|_{(r,\theta)=(2,\frac{\pi}{3})} = -1\cdot(-\sqrt{3}) + \sqrt{3}\cdot 1 = 2\sqrt{3}.$$

Method 2: $f(r,\theta) = f(r\cos\theta, r\sin\theta) = \ln\left(r^2(\sin^2\theta - \cos^2\theta)\right) = 2\ln r + \ln(-\cos(2\theta$

so

$$\left.\frac{\partial f}{\partial r}\right|_{(2,\frac{\pi}{3})} = \left.\frac{2}{r}\right|_{r=2} = 1.$$

$$\left.\frac{\partial f}{\partial \theta}\right|_{(2,\frac{\pi}{3})} = \left.\frac{-2\sin(2\theta)}{\cos(2\theta)}\right|_{\theta=\frac{\pi}{3}} = \frac{-2\sin(\frac{2\pi}{3})}{\cos(\frac{2\pi}{3})} = 2\sqrt{3}.$$

(b): $y = y(x)$ is implicitly defined by the following identity. Find the value of dy/dx at point P.

$$xy + y^2 + 3x - 3 = 0, \quad P(1, -1).$$

$y = y(x)$ satisfies:

$$x \cdot y(x) + y(x)^2 + 3x - 3 = 0.$$

differentiate with respect to x to get:

$$y + xy' + 2y \cdot y' + 3 = 0$$

$$\| $$

$$(x+2y)\,y' + (y+3)$$

so $y' = -\dfrac{y+3}{x+2y}$

so $\left.\dfrac{dy}{dx}\right|_{(x,y)=(1,-1)} = -\dfrac{-1+3}{1+2\cdot(-1)} = 2.$

4. (40pts) Find an equation for the **tangent plane**, and equations for the **normal line** of the graph of the following function at the given point:

$$z = \cos(x - y), \quad \left(0, \frac{\pi}{2}, 0\right).$$

The graph is also a level surface:

$g(x,y,z) = \cos(x-y) - z = 0.$

The normal vectors are given by gradient vectors:

$\nabla g = \langle -\sin(x-y),\ \sin(x-y), -1 \rangle$

$\nabla g|_{(0,\frac{\pi}{2},0)} = \langle -\sin(-\frac{\pi}{2}),\ \sin(-\frac{\pi}{2}), -1 \rangle = \langle 1, -1, -1 \rangle.$

So the tangent plane is:

$1\cdot(x-0) - 1\cdot(y-\frac{\pi}{2}) - (z-0) = 0 \iff -x + y + z = \frac{\pi}{2}.$

The normal line is:

$$\frac{x-0}{1} = \frac{y-\frac{\pi}{2}}{-1} = \frac{z-0}{-1}.$$

or

$$\begin{cases} x = t \\ y = \frac{\pi}{2} - t \\ z = -t \end{cases} \qquad -\infty < t < \infty.$$

5. (50pts)

(a): Find relative maxima, relative minima and saddle points over the xy-plane.

$$f(x,y) = x^3 - y^3 + 3xy + 15.$$

Find critical points:

$f_x = 3x^2 + 3y = 0$ ① $\Rightarrow y = -x^2 \overset{②}{\Rightarrow} x^4 - x = 0$

$f_y = -3y^2 + 3x = 0$ ②

$x(x^3-1) (= x(x-1)(x^2+x+1))$

So $x = 0$ or $x = 1^{\frac{1}{3}} = 1$.

So there are 2 critical points $(0,0)$ and $(1,-1)$.

Classify them by 2nd partial test:

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} 6x & 3 \\ 3 & -6y \end{pmatrix}. \quad \det = f_{xx} f_{yy} - f_{xy}^2 = -36xy - 9.$$

i) $(0,0)$: $\det = -9 < 0$, saddle point. $f(0,0) = 15$.

ii) $(1,-1)$: $\det = 27 > 0$, $f_{xx} = 6 > 0$. relative minimum.

$f(1,-1) = 14$.

(b): Use the Lagrange Multiplier method to find the extrema of $f(x, y, z) = x - y + z$ under the constraint $2x^2 + 2y^2 + z^2 = 8$.

Let $g(x,y,z) = 2x^2+2y^2+z^2-8=0$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y,z)=0 \end{cases} \Leftrightarrow \begin{cases} 1 = 4\lambda x & ① \Rightarrow x = \frac{1}{4\lambda} \\ -1 = 4\lambda y & ② \Rightarrow y = -\frac{1}{4\lambda} \\ 1 = 2\lambda z & ③ \Rightarrow z = \frac{1}{2\lambda} \\ 2x^2+2y^2+z^2=8 & ④ \end{cases}$$

Substitute into ④: $2\cdot\left(\frac{1}{4\lambda}\right)^2+2\left(-\frac{1}{4\lambda}\right)^2+\left(\frac{1}{2\lambda}\right)^2=8$

$$\frac{1}{8\lambda^2}+\frac{1}{8\lambda^2}+\frac{1}{4\lambda^2}=\frac{1}{2\lambda^2}$$

So $\lambda^2=\frac{1}{16} \Rightarrow \lambda=\pm\frac{1}{4}$, we get 2 candidate points:

i) $\lambda=\frac{1}{4} \Rightarrow (x,y,z)=(1,-1,2)$

$f(\underset{1}{x},\underset{-1}{y},\underset{2}{z})=4$. absolute maximum.

ii) $\lambda=-\frac{1}{4} \Rightarrow (x,y,z)=(-1,1,-2)$

$f(-1,1,-2)=-4$. absolute minimum

6. (40pts)

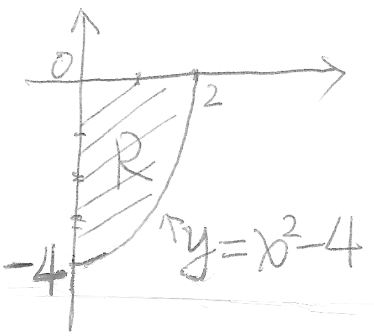
Consider the iterated integral:

$$\int_0^2 \int_{x^2-4}^0 \frac{x\, e^{+2y}}{4+y}\, dydx.$$

(a): Sketch the region of integration.

From the limits, we get inequalities:

$0 \le x \le 2,\ x^2-4 \le y \le 0.$



(b): Switch the order of integration and evaluate the integral.

$$\iint_R \frac{xe^{+2y}}{4+y}\, dy\, dx = \int_{-4}^{0} \int_0^{\sqrt{4+y}} \frac{xe^{+2y}}{4+y}\, dx\, dy$$

$$= \int_{-4}^{0} \frac{e^{+2y}}{4+y} \cdot \left[\frac{x^2}{2}\right]_0^{\sqrt{4+y}} dy$$

$$= \frac{1}{2}\int_{-4}^{0} e^{+2y}\, dy = \frac{1}{4} e^{2y}\Big]_{-4}^{0}$$

$$= \frac{1}{4}(1-e^{-8})$$

!!! WRITE YOUR NAME, STUDENT ID AND LECTURE N. BELOW !!!

NAME : ID :

1. (40pts) Let $f(x, y) = \ln(x^2 - y)$.

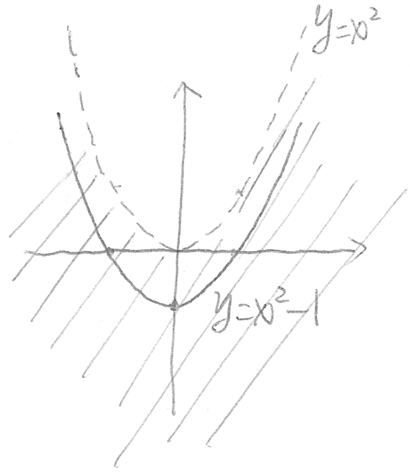
(a): Find the domain and range of the $f(x, y)$.

Domain: $\{(x,y) \in \mathbb{R}\times\mathbb{R},\ x^2 > y\}$

Range: $\mathbb{R} = (-\infty, +\infty)$

(b): Sketch the domain of $f(x, y)$, and the level curve $C = \{(x, y) : \ln(x^2 - y) = 0\}$.

$\ln(x^2-y)=0 \Leftrightarrow x^2-y=1 \Leftrightarrow y=x^2-1$.



2. (40pts)

$$f(x,y) = e^{xy}\sin y, \quad P = \left(0, \frac{\pi}{2}\right).$$

Find the directions in which the function increases and decreases most rapidly at P. Then find the directional derivatives of the function in these directions.

$f_x = y e^{xy} \sin y, \qquad f_x(0, \frac{\pi}{2}) = \frac{\pi}{2}$

$f_y = x e^{xy} \sin y + e^{xy} \cos y. \quad f_y(0, \frac{\pi}{2}) = 0.$

The gradient $\nabla f|(0, \frac{\pi}{2}) = \langle \frac{\pi}{2}, 0 \rangle.$

The direction of most rapid increase is

$$\vec{u}_1 = \frac{\nabla f}{\|\nabla f\|}\Big|_{(0,\frac{\pi}{2})} = \langle 1, 0 \rangle.$$

$$D_{\vec{u}_1} f\Big|_{(0,\frac{\pi}{2})} = \|\nabla f\| = \frac{\pi}{2}.$$

The direction of most rapid decrease is

$$\vec{u}_2 = -\vec{u}_1 = \langle -1, 0 \rangle$$

$$D_{\vec{u}_2} f\Big|_{(0,\frac{\pi}{2})} = -\|\nabla f\| = \frac{\pi}{2}.$$

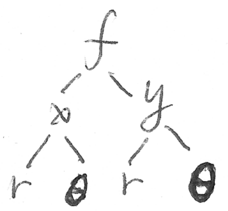
3. (40pts)

(a): Let $f(x,y) = \ln(x^2 - y^2)$ and $x = r\cos\theta$, $y = r\sin\theta$. Find $\partial f/\partial r$ and $\partial f/\partial\theta$ when $r = 2$ and $\theta = \pi/6$.

Method 1:

Use chain rule: $\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}$.

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$



$\frac{\partial f}{\partial x} = \frac{2x}{x^2-y^2}$, $\frac{\partial f}{\partial y} = -\frac{2y}{x^2-y^2}$, $\left.\frac{\partial f}{\partial x}\right|_{(\sqrt{3},1)} = \sqrt{3}$, $\left.\frac{\partial f}{\partial y}\right|_{(\sqrt{3},1)} = -1$.

$\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial x}{\partial \theta} = -r\cdot\sin\theta$ $\left.\frac{\partial x}{\partial r}\right|_{(2,\frac{\pi}{6})} = \frac{\sqrt{3}}{2}$, $\left.\frac{\partial x}{\partial \theta}\right|_{(2,\frac{\pi}{6})} = -1$

$\frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$. $\left.\frac{\partial y}{\partial r}\right|_{(2,\frac{\pi}{6})} = \frac{1}{2}$. $\left.\frac{\partial y}{\partial \theta}\right|_{(2,\frac{\pi}{6})} = \sqrt{3}$

when $r=2, \theta = \frac{\pi}{6}$, $x = r\cos\theta = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$. $y = r\sin\theta = 2\cdot\frac{1}{2} = 1$.

so $\left.\frac{\partial f}{\partial r}\right|_{(r,\theta)=(2,\pi/6)} = \sqrt{3}\cdot\frac{\sqrt{3}}{2} - 1\cdot\frac{1}{2} = 1$.

$$\left.\frac{\partial f}{\partial \theta}\right|_{(r,\theta)=(2,\frac{\pi}{6})} = \sqrt{3}\cdot(-1) - 1\cdot\sqrt{3} = -2\sqrt{3}.$$

Method 2: $f(r,\theta) = f(r\cos\theta, r\sin\theta) = \ln\left(r^2(\cos^2\theta - \sin^2\theta)\right) = 2\ln r + \ln(\cos 2\theta)$.

so $\left.\frac{\partial f}{\partial r}\right|_{(2,\frac{\pi}{6})} = \left.\frac{2}{r}\right|_{r=2} = 1$.

$$\left.\frac{\partial f}{\partial \theta}\right|_{(2,\frac{\pi}{6})} = \left.\frac{-2\sin(2\theta)}{\cos(2\theta)}\right|_{\theta=\frac{\pi}{6}} = \left.\frac{-2\sin(\frac{\pi}{3})}{\cos(\frac{\pi}{3})}\right|_{\theta=\frac{\pi}{6}} = -2\sqrt{3}.$$

(b): $y = y(x)$ is defined implicitly by the following equation. Find the value of dy/dx at point P.

$$xy + y^2 - 3x - 3 = 0, \quad P(-1, 1).$$

$$x \cdot y(x) + y(x)^2 - 3x - 3 = 0.$$

differentiate with respect to x to get:

$$y + x y' + 2y \cdot y' - 3 = 0$$

$$\parallel$$

$$(x+2y)\,y' + (y-3).$$

so $y' = -\dfrac{y-3}{x+2y}$.

so $\left.\dfrac{dy}{dx}\right|_{(x,y)=(-1,1)} = -\dfrac{1-3}{-1+2\cdot 1} = 2.$

4. **(40pts)** Find an equation for the **tangent plane**, and a ~~parametric~~ equation for the **normal line** of the graph of the following function at the given point:

$$z = \sin(x - y), \quad (0, \pi, 0).$$

The graph is also a level surface: ~~of~~

$g(x, y, z) = \sin(x-y) - z = 0.$

The normal vector is given by the gradient:

$\nabla g = \langle \cos(x-y), -\cos(x-y), -1 \rangle.$

$\nabla g|_{(0,\pi,0)} = \langle \cos(-\pi), -\cos(-\pi), -1 \rangle = \langle -1, +1, -1 \rangle$

so the tangent plane is:

$-1\cdot(x-0) + 1\cdot(y-\pi) - 1\cdot(z-0) = 0 \iff -x + y - z = \pi.$

The normal line is:

$$\frac{x-0}{-1} = \frac{y-\pi}{+1} = \frac{z-0}{-1}$$

or

$$\begin{cases} x = -t \\ y = \pi + t \\ z = -t. \end{cases} \quad -\infty < t < \infty$$

5. (50pts)

(a): Find relative maxima, relative minima and saddle points over the xy-plane.

$$f(x, y) = x^3 + y^3 - 3xy + 15.$$

Find critical points:

$f_x = 3x^2 - 3y = 0$ ① $\Rightarrow y = x^2 \overset{②}{\Rightarrow} x^4 - x = 0$

$f_y = 3y^2 - 3x = 0$. ②

$x(x^3-1) (= x(x-1)(x^2+x+1))$

So $x = 0$, or $x = 1^{\frac{1}{3}} = 1$

So there are 2 critical points $(0,0)$, and $(1,1)$.

Classify them by 2nd partial test:

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}. \quad \det = f_{xx}f_{yy} - f_{xy}^2 = 36xy - 9.$$

i) $(0,0)$: $\det = -9 < 0$, Saddle point. $f(0,0) = 15$

ii) $(1,1)$ $\det = 27 > 0$, $f_{xx} = 6 > 0$, relative minimum.

$f(1,1) = 14$.

(b): Use the Lagrange Multiplier method to find the extrema of $f(x, y, z) = x+y-z$ under the constraint $2x^2 + y^2 + 2z^2 = 8$.

Let $g(x, y, z) = 2x^2 + y^2 + 2z^2 - 8 = 0.$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y,z) = 0 \end{cases} \iff \begin{cases} 1 = \lambda \cdot 4x & ① \Rightarrow x = \frac{1}{4\lambda} \\ 1 = \lambda \cdot 2y & ② \Rightarrow y = \frac{1}{2\lambda} \\ -1 = \lambda \cdot 4z & ③ \Rightarrow z = -\frac{1}{4\lambda} \\ 2x^2 + y^2 + 2z^2 = 8 & ④ \end{cases}$$

Substitute into ④: $2\cdot\left(\frac{1}{4\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 + 2\cdot\left(-\frac{1}{4\lambda}\right)^2 = 8$

$$\frac{1}{8\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{8\lambda^2} = \frac{1}{2\lambda^2}$$

So $\lambda^2 = \frac{1}{16} \Rightarrow \lambda = \pm\frac{1}{4}.$ We get 2 candidate points:

i) $\lambda = \frac{1}{4} \Rightarrow (x, y, z) = (1, 2, -1).$

$f(1, 2, -1) = 4.$ absolute maximum.

ii) $\lambda = -\frac{1}{4} \Rightarrow (x, y, z) = (-1, -2, 1).$

$f(-1, -2, 1) = -4.$ absolute minimum

6. (40pts)

Consider the iterated integral:

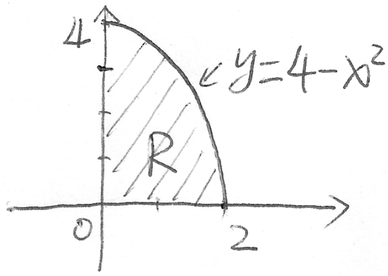
Find the region and.

$$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\, dydx.$$

(a): Sketch the region of integration.

From the limits we get inequalities:

$0 \le x \le 2, \quad 0 \le y \le 4 - x^2.$



(b): Switch the order of integration and evaluate the integral.

$$\iint_R \frac{xe^{2y}}{4-y}\, dy\, dx = \int_0^4 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y}\, dx\, dy$$

$$= \int_0^4 \frac{e^{2y}}{4-y} \left[\frac{x^2}{2}\right]_0^{\sqrt{4-y}} dy$$

$$= \frac{1}{2}\int_0^4 e^{2y}\, dy = \frac{1}{4} e^{2y} \Big]_0^4$$

$$= \frac{1}{4}(e^8 - 1)$$

(6^{10} **means problem 24 in 10-th edition textbook,** 6^9 **means problem 24 in 9-th edition textbook**)

1 14.3

$\mathbf{6^{10}} = \mathbf{6^9}$: In rectangular coordinate, $x^2 + (y-2)^2 \le 4 \Longleftrightarrow x^2 + y^2 - 2y \le 0$. Substituting $x = r\cos\theta$, $y = r\sin\theta$, we get

$$r^2 - 2r\sin\theta \le 0, \quad 0 \le \theta \le \pi \quad \Longleftrightarrow \quad 0 \le r \le 2\sin\theta, \quad 0 \le \theta \le \pi.$$

$\mathbf{18^{10}} = \mathbf{18^9}$:

$$\begin{aligned}\int_0^a \int_0^{\sqrt{a^2-x^2}} x \; dydx &= \int_0^{\pi/2} \int_0^a r\cos\theta \; rdrd\theta = [\sin\theta]_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^a \\ &= \frac{a^3}{3}.\end{aligned}$$

$\mathbf{22^{10}} = \mathbf{22^9}$: Region: $0 \le y \le 2$, $y \le x \le \sqrt{8-y^2}$.

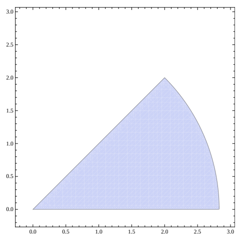


Figure 1: Problem 22

$$\begin{aligned}\int_0^2 \int_y^{\sqrt{8-y^2}} \sqrt{x^2+y^2} \; dxdy &= \int_0^{\pi/4} \int_0^{2\sqrt{2}} r \; r\mathrm{d}rd\theta \\ &= \frac{\pi}{4}\left[\frac{r^3}{3}\right]_0^{2\sqrt{2}} = \frac{\pi}{4}\frac{2^{9/2}}{3} \\ &= \frac{4\sqrt{2}\pi}{3}.\end{aligned}$$

$\mathbf{26^{10}} = \mathbf{26^9}$:

$$\begin{aligned}\int_0^2 \int_0^{\sqrt{4-x^2}} \sin\sqrt{x^2+y^2} \; dydx &= \int_0^{\pi/2} \int_0^2 \sin r \; rdrd\theta \\ &= \frac{\pi}{2}\left(-\int_0^2 r \; \mathrm{d}(\cos r)\right) = -\frac{\pi}{2}\left([r\cos r]_0^2 - \int_0^2 \cos r \; \mathrm{d}r\right) \\ &= -\frac{\pi}{2}(2\cos 2) + \frac{\pi}{2}[\sin r]_0^2 = \frac{\pi}{2}(\sin 2 - 2\cos 2).\end{aligned}$$

$\mathbf{30^{10}} = \mathbf{30^9}$: The region is the right half disk.

$$\begin{aligned} \iint_R e^{-(x^2+y^2)/2}\ \mathrm{d}A &= \int_{-\pi/2}^{\pi/2}\int_0^5 e^{-r^2/2}\ r\mathrm{d}r d\theta \\ &= \pi\int_0^5 e^{-r^2/2}\ \mathrm{d}\left(\frac{r^2}{2}\right) = -\pi e^{-r^2/2}]_0^5 \\ &= \pi(1-e^{-25/2}). \end{aligned}$$

$\mathbf{40^{10}} = \mathbf{40^9}$:

$$\begin{aligned} Vol &= \int_0^{2\pi}\int_0^a 2\sqrt{a^2-r^2}\ r\mathrm{d}r d\theta = -2\pi\int_0^a (a^2-r^2)^{1/2}\ \mathrm{d}(a^2-r^2) \\ &= -2\pi\frac{2}{3}\left[(a^2-r^2)^{3/2}\right]_0^a \\ &= \frac{4}{3}\pi a^3. \end{aligned}$$

$\mathbf{44^{10}} = \mathbf{46^9}$:

$$\begin{aligned} Area &= \int_0^{2\pi}\int_0^{2+\sin\theta} r\mathrm{d}r d\theta = \int_0^{2\pi}\frac{1}{2}(2+\sin\theta)^2\ \mathrm{d}\theta \\ &= \int_0^{2\pi}\left(2+2\sin\theta+\frac{\sin^2\theta}{2}\right)\ \mathrm{d}\theta \\ &= [2\theta-2\cos\theta]_0^{2\pi}+\frac{1}{2}\int_0^{2\pi}\frac{1-\cos(2\theta)}{2}d\theta \\ &= 4\pi+\frac{1}{4}\left[\theta-\frac{\sin(2\theta)}{2}\right]_0^{2\pi} \\ &= 4\pi+\frac{\pi}{2} = \frac{9\pi}{2}. \end{aligned}$$

2 14.4

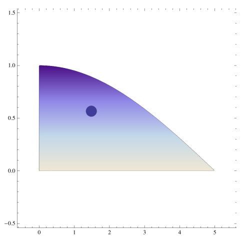


Figure 2: Problem 20

$20^{10} = 20^9$:

$$\begin{aligned}
\text{mass} &= \iint_R \rho(x,y)\ dA = \int_0^{L/2} \int_0^{\cos(\pi x/L)} ky \ \ \mathrm{d}y\ \mathrm{d}x \\
&= \int_0^{L/2} \frac{k}{2}\cos^2(\pi x/L) dx = \int_0^{L/2} \frac{k}{2} \frac{1+\cos(2\pi x/L)}{2}\ \mathrm{d}x \\
&= \frac{k}{4}\left[x + \frac{L}{2\pi}\sin(2\pi x/L)\right]_0^{L/2} \\
&= \frac{kL}{8}.
\end{aligned}$$

$$\begin{aligned}
\iint_R x\rho(x,y)\ dA &= \int_0^{L/2} \int_0^{\cos(\pi x/L)} kxy\ dydx \\
&= \int_0^{L/2} \frac{k}{2} x\cos^2(\pi x/L) dx = \frac{k}{2}\int_0^{L/2} x\frac{1+\cos(2\pi x/L)}{2}\ \mathrm{d}x \\
&= \frac{k}{4}\left[\frac{x^2}{2}\right]_0^{L/2} + \frac{k}{4}\int_0^{\pi} \frac{L}{2\pi} u(\cos u)\frac{L}{2\pi}\ \mathrm{d}u \\
&= \frac{kL^2}{32} + \frac{kL^2}{16\pi^2}\int_0^{\pi} u\ \mathrm{d}(\sin u) \\
&= \frac{kL^2}{32} + \frac{kL^2}{16\pi^2}\left([u\sin u]_0^{\pi} - \int_0^{\pi}\sin u\ \mathrm{d}u\right) \\
&= \frac{kL^2}{32} + \frac{kL^2}{16\pi^2}\left(0 + [\cos u]_0^{\pi}\right) \\
&= \frac{kL^2}{32} - \frac{kL^2}{16\pi^2}\cdot 2 \\
&= \frac{kL^2(\pi^2-4)}{32\pi^2}.
\end{aligned}$$

$$\begin{aligned}
\iint_R y\rho(x,y)\ dA &= \int_0^{L/2} \int_0^{\cos(\pi x/L)} ky^2\ dydx \\
&= \int_0^{L/2} \frac{k}{3}\cos^3(\pi x/L) dx \\
&= \frac{k}{3}\int_0^{\pi/2} \cos^3(u)\frac{L}{\pi} du \\
&= \frac{kL}{3\pi}\int_0^{\pi/2} (1-\sin^2 u) d(\sin(u)) \\
&= \frac{kL}{3\pi}\left[\sin u - \frac{\sin^3 u}{3}\right]_0^{\pi/2} \\
&= \frac{kL}{3\pi}\frac{2}{3} = \frac{2kL}{9\pi}.
\end{aligned}$$

So

$$(\bar{x},\bar{y}) = \frac{1}{kL/8}\left(\frac{kL^2(\pi^2-4)}{32\pi^2}, \frac{2kL}{9\pi}\right) = \left(\frac{L(\pi^2-4)}{4\pi^2}, \frac{16}{9\pi}\right).$$

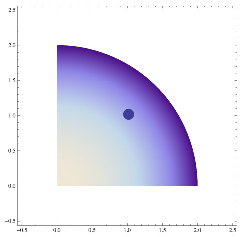


Figure 3: Problem 22

$22^{10} = 22^{9}$:

$$mass = \int_0^{\pi/2}\int_0^a kr^2 \ r\mathrm{d}r\mathrm{d}\theta = \frac{\pi}{2}k\frac{a^4}{4} = \frac{k\pi a^4}{8}.$$

$$\begin{aligned}\iint_R x\rho(x,y)\ dA &= \int_0^{\pi/2}\int_0^a r\cos\theta kr^2 \ r\mathrm{d}r\mathrm{d}\theta = k[\sin\theta]_0^{\pi/2}\left[\frac{r^5}{5}\right]_0^a \\ &= k\frac{a^5}{5}.\end{aligned}$$

So
$$\bar{x} = \frac{\iint_R x\rho(x,y)dA}{mass} = \frac{8a}{5\pi}.$$
By symmetry, $\bar{y} = \bar{x}$. So the center of mass has coordinate $(\frac{8a}{5\pi}, \frac{8a}{5\pi})$.

(6^{10} **means problem 6 in 10-th edition textbook,** 6^{9} **means problem 6 in 9-th edition textbook**)

1 14.6

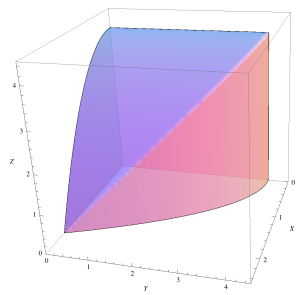
$\mathbf{6^{10}} = \mathbf{6^{9}}$:

$$\begin{aligned}\int_1^4\int_1^{e^2}\int_0^{1/(xz)} \ln z\; dydzdx &= \int_1^4\int_1^{e^2}\frac{1}{xz}\ln z dzdx = \int_1^4\int_1^{e^2}\ln z d(\ln z)\frac{1}{x}dx\\ &= \int_1^4\left[\frac{(\ln z)^2}{2}\right]_1^{e^2}\frac{1}{x}dx = 2\int_1^4\frac{1}{x}dx\\ &= 2\ln 4.\end{aligned}$$

$\mathbf{14^{10}} = \mathbf{16^{9}}$:

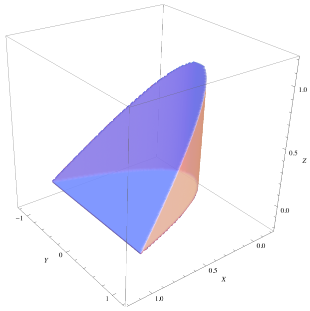
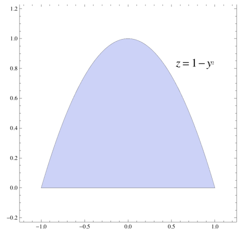
$$\begin{aligned}\iiint_Q dV &= \iint_{\{x^2+y^2\le 16\}}\int_0^{\sqrt{16-x^2-y^2}} dzdA = \iint_{x^2+y^2\le 16}\sqrt{16-x^2-y^2}dA\\ &= \int_0^{2\pi}\int_0^4\sqrt{16-r^2}\; rdrd\theta = \pi\int_0^4\sqrt{16-r^2}\;\left(-d\left(16-r^2\right)\right)\\ &= \left[-\pi\frac{2}{3}(16-r^2)^{3/2}\right]_0^4 = \frac{128\pi}{3}.\end{aligned}$$

$\mathbf{21^{10}} = \mathbf{23^{9}}$:



$$\begin{aligned}\iiint_Q dV &= \iint_{R_{xy}}\int_0^{4-x^2} dzdA = \iint_{R_{xy}}(4-x^2)dA\\ &= \int_0^2\int_0^{4-x^2}(4-x^2)dydx = \int_0^2(4-x^2)^2dx\\ &= \int_0^2(16-8x^2+x^4)dx = \left[16x-\frac{8}{3}x^3+\frac{x^5}{5}\right]_0^2\\ &= \frac{256}{15}.\end{aligned}$$

$\mathbf{26^{10}} = \mathbf{28^{9}}$: The solid is defined by

$$-1 \le y \le 1, \quad y^2 \le x \le 1, \quad 0 \le z \le 1-x.$$

The projection to the yz-plane is the region:

$$0 \le z \le 1-y^2, \quad -1 \le y \le 1.$$

$$\int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{1-x} dzdxdy = \int_{-1}^{1}\int_{0}^{1-y^2}\int_{y^2}^{1-z} dxdzdy = \frac{8}{15}.$$

$\mathbf{36^{10}}$ = $\mathbf{38^{9}}$:

1. The projection to the xy-plane is the region:

   $$0 \le x \le 3, \quad 0 \le y \le x.$$

   So
   $$\int_{0}^{3}\int_{0}^{x}\int_{0}^{9-x^2} dzdydx = \int_{0}^{3}\int_{y}^{3}\int_{0}^{9-x^2} dzdxdy.$$

2. The projection to the yz-plane is the region:

   $$0 \le y \le 3, \quad 0 \le z \le 9-y^2.$$

   So the corresponding two iterated integrals are

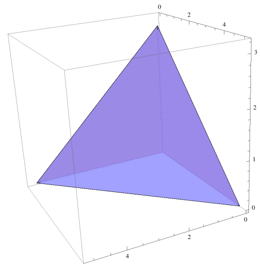
   $$\int_{0}^{3}\int_{0}^{9-y^2}\int_{y}^{\sqrt{9-z}} dxdzdy = \int_{0}^{9}\int_{0}^{\sqrt{9-z}}\int_{y}^{\sqrt{9-z}} dxdydz.$$

3. The projection to the xz-plane is the region:

   $$0 \le x \le 3, \quad 0 \le z \le 9-x^2.$$

   So the corresponding two iterated integrals are

   $$\int_{0}^{3}\int_{0}^{9-x^2}\int_{0}^{x} dydzdx = \int_{0}^{9}\int_{0}^{\sqrt{9-z}}\int_{0}^{x} dydxdz.$$



$38^{10} = 40^{9}$: Mass:

$$\iiint_Q \rho(x,y,z)dV = \iint_{R_{xy}} \int_0^{(15-3x-3y)/5} kydzdA$$

$$= \int_0^5 \int_0^{5-y} ky(15-3x-3y)/5dxdy = \frac{k}{5}\int_0^5 \left[3y(5-y)x - 3y\frac{x^2}{2}\right]_0^{5-y} dy$$

$$= \frac{k}{5}\int_0^5 \left(3y(5-y)(5-y) - 3y\frac{(5-y)^2}{2}\right) dy = \frac{3k}{10}\int_0^5 y(5-y)^2 dy$$

$$= \frac{3k}{10}\int_0^5 (25y - 10y^2 + y^3)dy = \frac{3k}{10}\left[\frac{25}{2}y^2 - \frac{10}{3}y^3 + \frac{y^4}{4}\right]_0^5$$

$$= \frac{3k}{10}(25)^2\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{125k}{8}.$$

The integral of y coordinate is:

$$\iiint_Q y\rho(x,y,z)dV = \iint_{R_{xy}} \int_0^{(15-3x-3y)/5} ykydzdA$$

$$= \int_0^5 \int_0^{5-y} ky^2(15-3x-3y)/5dxdy = \frac{k}{5}\int_0^5 \left[3y^2(5-y)x - 3y^2\frac{x^2}{2}\right]_0^{5-y} dx$$

$$= \frac{3k}{10}\int_0^5 y^2(5-y)^2 dy = \frac{3k}{10}\int_0^5 (25y^2 - 10y^3 + y^4)dy$$

$$= \frac{3k}{10}\left[25\frac{y^3}{3} - \frac{5}{2}y^4 + \frac{1}{5}y^5\right]_0^5$$

$$= \frac{3k}{10}(5^5)\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = \frac{125k}{4}.$$

So

$$\bar{y} = \frac{\iiint_Q y\rho(x,y,z)dV}{\iiint_Q \rho(x,y,z)dV} = 2.$$

2 11.7

$4^{10} = 4^{9}$: $x = r\cos\theta = 6\cos(-\pi/4) = 3\sqrt{2}$, $y = r\sin\theta = 6\sin(-\pi/4) = -3\sqrt{2}$, $z = 2$. So $(x,y,z) = (3\sqrt{2}, -3\sqrt{2}, 2)$.

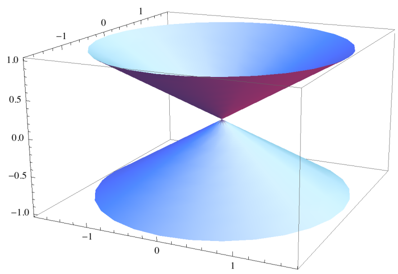
$\mathbf{8^{10} = 8^9}$: $r = \sqrt{x^2 + y^2} = 4$. $\tan\theta = y/x = -1$ & $x > 0 \Rightarrow \theta = -\pi/4$. So $(r, \theta, z) = (4, -\pi/4, 4)$.

$\mathbf{26^{10} = 26^9}$: $z = x^2$. This is a cylindrical surface parallel to the y-axis.

$\mathbf{32^{10} = 32^9}$: $\rho^2 = x^2 + y^2 + z^2 = 2^2 + 2^2 + (4\sqrt{2})^2 = 40 \Rightarrow \rho = 2\sqrt{10}$. $\tan\theta = y/x = 1$ & $x > 0 \to \theta = \pi/4$. $\cos\phi = \frac{z}{\rho} = \frac{4\sqrt{2}}{2\sqrt{10}} = 2/\sqrt{5} \Rightarrow \phi = \arccos(2/\sqrt{5})$. So $(\rho, \theta, \phi) = (2\sqrt{10}, \pi/4, \arccos(2/\sqrt{5}))$.

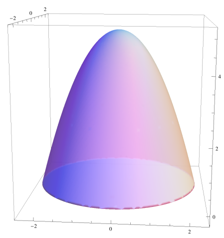
$\mathbf{38^{10} = 38^9}$: $x = \rho\sin\phi\cos\theta = 9\sin(\pi)\cos(\pi/4) = 0$, $y = \rho\sin\phi\sin\theta = 9\sin(\pi)\sin(\pi/4) = 0$, $z = \rho\cos\phi = 9\cos(\pi) = -9$. So $(x, y, z) = (0, 0, -9)$.

$\mathbf{44^{10} = 44^9}$: $x^2+y^2-3z^2 = 0 \iff \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta - 3\rho^2\cos^2\phi = 0 \iff \rho^2(\sin^2\phi - 3\cos^2\phi) = 0 \iff |\tan\phi| = \sqrt{3}, 0 \le \phi \le \pi \iff \phi = \frac{\pi}{3}$ or $\frac{2\pi}{3}$. Note that this surface is a circular cone.



3 14.7

$\mathbf{10^{10} = 10^9}$:



$$\begin{aligned} \int_0^{2\pi}\int_0^{\sqrt{5}}\int_0^{5-r^2} r\ dz\ dr\ d\theta &= 2\pi\int_0^{\sqrt{5}} r(5-r^2)dr \\ &= 2\pi\left[\frac{5}{2}r^2 - \frac{r^4}{4}\right]_0^{\sqrt{5}} = \frac{25\pi}{2}. \end{aligned}$$

$\mathbf{12^{10} = 12^9}$: The solid is between the sphere $x^2+y^2+z^2 = 25$ and $x^2+y^2+z^2 = 4$.

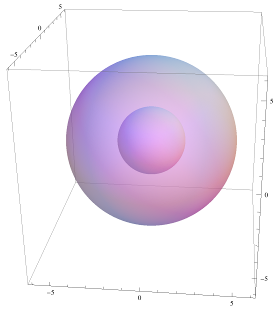


Figure 1: Problem 14.7.12

$$\int_0^{2\pi}\int_0^{\pi}\int_2^5 \rho^2 \sin\phi \; d\rho d\phi d\theta \;=\; 2\pi\,[-\cos\phi]_0^{\pi}\left[\frac{\rho^3}{3}\right]_2^5$$
$$= 156\pi.$$

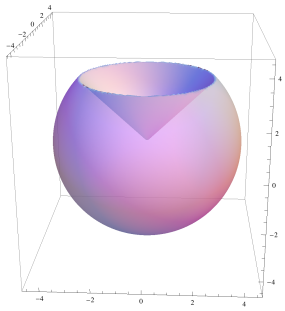
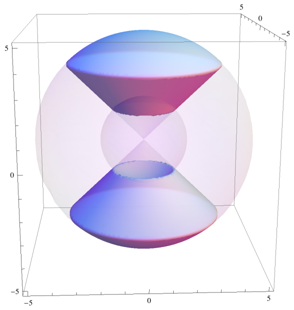


Figure 2: 14.7.18

$\mathbf{18^{10} = 18^9}$:

$$\begin{aligned} Vol &= \int_0^{2\pi}\int_{\pi/4}^{\pi}\int_0^4 \rho^2 \sin\phi \; d\rho d\phi d\theta = [\theta]_0^{2\pi}[-\cos\phi]_{\pi/4}^{\pi}\left[\frac{\rho^3}{3}\right]_0^4 \\ &= 2\pi\left(1+\frac{\sqrt{2}}{2}\right)\frac{64}{3} = \frac{64\pi}{3}(2+\sqrt{2}). \end{aligned}$$

$36^{10} = 36^{9}$:



$$
\begin{aligned}
Vol &= 2\int_0^{2\pi}\int_0^{\pi/4}\int_a^b \rho^2 \sin\phi \; d\rho d\phi d\theta = 2[\theta]_0^{2\pi}[-\cos\phi]_0^{\pi/4}\left[\frac{\rho^3}{3}\right]_a^b \\
&= 4\pi\left(1-\frac{\sqrt{2}}{2}\right)\frac{b^3-a^3}{3} = \frac{2\pi(2-\sqrt{2})(b^3-a^3)}{3}.
\end{aligned}
$$

(26^{10} **means problem 26 in 10-th edition textbook,** 26^{9} **means problem 26 in 9-th edition textbook**)

1 14.7

26^{10} $=$ **26^{9}**: The mass of the cone is given:

$$\begin{aligned}\iiint_Q dV &= \iint_{R_{xy}} \int_0^{h(1-r/r_0)} dz dA = \int_0^{2\pi} \int_0^{r_0} h\left(1-\frac{r}{r_0}\right) r\, dr d\theta \\ &= 2\pi h \left[\frac{r^2}{2} - \frac{r^3}{3r_0}\right]_0^{r_0} = \frac{\pi r_0^2 h}{3}.\end{aligned}$$

The integral of z coordinate is

$$\begin{aligned}\iiint_Q z dV &= \iint_{R_{xy}} \int_0^{h(1-r/r_0)} z dz dA = \int_0^{2\pi} \int_0^{r_0} \frac{1}{2} h^2 \left(1-\frac{r}{r_0}\right)^2 r dr d\theta \\ &= \pi h^2 \int_0^{r_0} \left(r - \frac{2r^2}{r_0} + \frac{r^3}{r_0^2}\right) dr = \pi h^2 \left[\frac{r^2}{2} - \frac{2r^3}{3r_0} + \frac{r^4}{4r_0^2}\right]_0^{r_0} \\ &= \pi h^2 r_0^2 \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{\pi h r_0^2}{12}.\end{aligned}$$

so

$$\bar{z} = \frac{\iiint z dV}{\iiint dV} = \frac{h}{4}.$$

By symmetry, $\bar{x} = \bar{y} = 0$. So the centroid is: $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, h/4)$.

34^{10} $=$ **34^{9}**: In spherical coordinate, the equation $x^2 + y^2 + z^2 = z$ is trans-

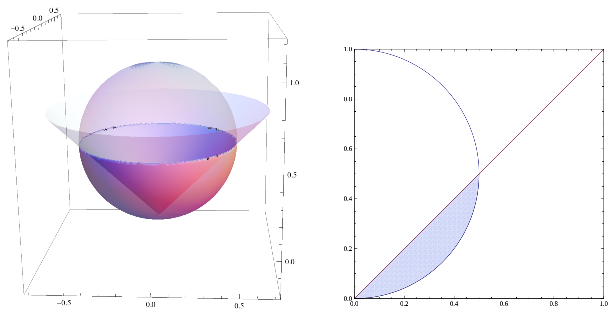


Figure 1: cross section

formed to the equation $\rho^2 = \rho \cos\phi \iff \rho = \cos\phi$. This is a sphere of radius $1/2$ centered at $(0, 0, 1/2)$. The equation $z = \sqrt{x^2 + y^2}$ is transformed to the equation $\phi = \pi/4$.

$$\begin{aligned}\iiint_Q dV &= \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{\cos\phi} \rho^2 \sin\phi d\rho d\phi d\theta \\ &= 2\pi \int_{\pi/4}^{\pi/2} \sin\phi \frac{\cos^3\phi}{3} d\phi = \frac{2\pi}{3}\int_{\pi/4}^{\pi/2} \cos^3\phi d(-\cos\phi) \\ &= -\frac{2\pi}{3}\left[\frac{\cos^4\phi}{4}\right]_{\pi/4}^{\pi/2} = \frac{\pi}{6}\left(\frac{1}{\sqrt{2}}\right)^4 \\ &= \frac{\pi}{24}.\end{aligned}$$

The volume can also be calculated as the difference of the volume of the half ball and the volume of the cone:

$$Vol = \frac{1}{2}\frac{4}{3}\pi\left(\frac{1}{2}\right)^3 - \frac{1}{3}\pi\left(\frac{1}{2}\right)^2\frac{1}{2} = \frac{\pi}{24}.$$

2 15.1

$1-4^{10}$ or **$1-6^{9}$**: See page 5 for figures.

$8^{10} = 9^{9}$: $\|\vec{F}\| = \sqrt{x^2+y^2} = r$. See page 5 figure 3 for visualization of the vector field.

$34^{10} = 40^{9}$: $M(x,y) = 3x^2y^2$, $N(x,y) = 2x^3y$. Because

$$\frac{\partial M}{\partial y} = 6xy^2 = \frac{\partial N}{\partial x},$$

$\vec{F}(x,y)$ is conservative. To find the potential function, we need to solve:

$$f_x = 3x^2y^2, \quad f_y = 2x^3y.$$

We integrate the first equation to find:

$$f(x,y) = \int f_x(x,y)dx = \int 3x^2y^2 dx = x^3y^2 + g(y).$$

Substitute this into the 2nd equation to get:

$$f_y = 2x^3y + g'(y) = 2x^3y.$$

So we get $g'(y) = 0$ which implies $g(y) = constant$. So we find a potential function $\vec{F}(x,y) = \nabla f$:

$$f(x,y) = x^3y^2.$$

$40^{10} = 46^{9}$: $M(x,y) = x/(x^2+y^2)$, $N(x,y) = y/(x^2+y^2)$. Because

$$\frac{\partial M}{\partial y} = -\frac{2xy}{(x^2+y^2)^2} = \frac{\partial N}{\partial x},$$

$\vec{F}(x,y)$ is conservative. To find a potential function, we need to solve:

$$f_x = \frac{x}{x^2+y^2}, \quad f_y = \frac{y}{x^2+y^2}.$$

We integrate the first equation to get:

$$f(x,y) = \int f_x(x,y)dx = \int \frac{x}{x^2+y^2}dx = \frac{1}{2}\ln(x^2+y^2) + g(y).$$

Substitute this into the 2nd equation to get:

$$f_y = \frac{y}{x^2+y^2} + g'(y) = \frac{y}{x^2+y^2}.$$

So $g'(y) = 0$ which implies $g(y) = constant$. So we find a potential function $\vec{F} = \nabla f$:

$$f(x,y) = \frac{1}{2}\ln(x^2+y^2).$$

$\mathbf{44^{10}} = \mathbf{50^9}$:

$$\begin{aligned} \text{curl}\vec{F} &= \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 z & -2xz & yz \end{vmatrix} \\ &= (z+2x)\vec{i} + x^2\vec{j} - 2z\vec{k}. \end{aligned}$$

So at point $(2,-1,3)$, the curl of $\vec{F}$ is equal to

$$\text{curl}\vec{F}\Big|_{(2,-1,3)} = 7\vec{i} + 4\vec{j} - 6\vec{k}.$$

$\mathbf{60^{10}} = \mathbf{66^9}$: $\vec{F} = \ln(x^2+y^2)\vec{i} + xy\vec{j} + \ln(y^2+z^2)\vec{k}$. The divergence is

$$\begin{aligned} div\vec{F} &= \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(\ln(x^2+y^2)) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(\ln(y^2+z^2)) \\ &= \frac{2x}{x^2+y^2} + x + \frac{2z}{y^2+z^2} \end{aligned}$$

3 15.2

$\mathbf{6^{10}} = \mathbf{6^9}$: $x = 4\cos\theta$, $y = 3\sin\theta$, $0 \le \theta \le 2\pi$.

$\mathbf{10^{10}} = \mathbf{10^9}$: $\vec{r}\,'(t) = 12\vec{i} + 5\vec{j} + 84\vec{k}$. So $\|\vec{r}\,'(t)\| = \sqrt{12^2+5^2+(84)^2} = \sqrt{7225} = 85$.

$$\begin{aligned} \int_C 2xyz\ ds &= 2\int_0^1 (12t)(5t)(84t)85dt = 2 \cdot 428400 \int_0^1 t^3 dt \\ &= \frac{428400}{2} = 214200. \end{aligned}$$

$\mathbf{16^{10}} = \mathbf{16^9}$: C is parametrized by $\vec{r}(y) = y\vec{j}$, $1 \le y \le 9$. $\|\vec{r}\,'(y)\| = 1$.

$$\int_C (x + 4\sqrt{y})ds \quad = \quad \int_1^9 4\sqrt{y}\ dy = \left[\frac{8}{3}y^{3/2}\right]_1^9 = \frac{208}{3}.$$

$\mathbf{20^{10}} = \mathbf{20^9}$: **(a):** Piecewise smooth parametrization:

1. $C_1 : \vec{r}_1(t) = t\vec{j}, 0 \le t \le 1$;
2. $C_2 : \vec{r}_2(t) = \vec{j} + t\vec{k}, 0 \le t \le 1$;
3. $C_3 : \vec{r}_3(t) = (1-t)\vec{j} + (1-t)\vec{j}, 0 \le t \le 1$.

$\|\vec{r}_1\,'(t)\| = 1$, $\|\vec{r}_2\,'(2)\| = 1$, $\|\vec{r}_3\,'(t)\| = \sqrt{2}$. So

$$\int_{C_1} (2x + y^2 - z)ds \quad = \quad \int_0^1 t^2 dt = \frac{1}{3}.$$

$$\int_{C_2} (2x + y^2 - z)ds \quad = \quad \int_0^1 (1-t)dt = \frac{1}{2}.$$

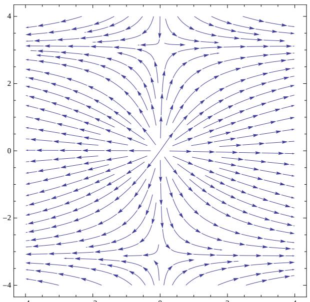
$$\begin{aligned} \int_{C_3} (3x + y^2 - z)ds \quad &= \quad \int_0^1 ((1-t)^2 - (1-t))\sqrt{2}dt = -\sqrt{2}\int_1^0 (u^2 - u)du \\ &= \quad -\sqrt{2}\left[\frac{u^3}{3} - \frac{u^2}{2}\right]_1^0 = -\frac{\sqrt{2}}{6}. \end{aligned}$$

S

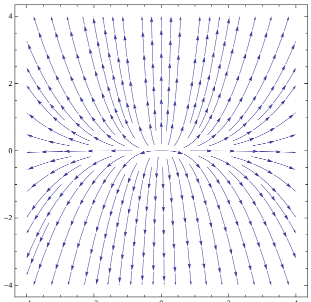
$$\int_C (2x + y^2 - z)\ ds = \int_{C_1} + \int_{C_2} + \int_{C_3} = \frac{5 - \sqrt{2}}{6}.$$

$\mathbf{22^{10}} = \mathbf{22^9}$: $\vec{r}(t) = 2\cos t\ \vec{i} + 2\sin t\ \vec{j} + t\ \vec{k}$. $\vec{r}\,'(t) = -2\sin t\ \vec{i} + 2\cos t\ \vec{j} + \vec{k}$. $\|\vec{r}\,'(t)\| = \sqrt{5}$.

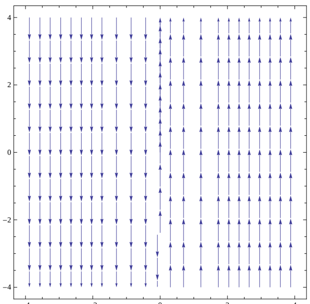
$$\begin{aligned} mass \quad &= \quad \int_C \rho(x,y,z)ds = \int_C zds = \int_0^{4\pi} t \cdot \sqrt{5}dt = \left[\frac{\sqrt{5}t^2}{2}\right]_0^{4\pi} \\ &= \quad 8\sqrt{5}\pi^2. \end{aligned}$$



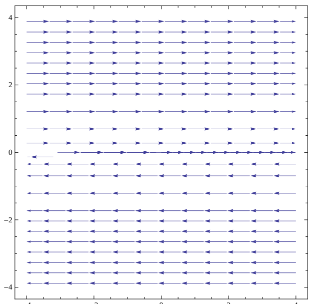
(a) $\langle x, \sin y\rangle$



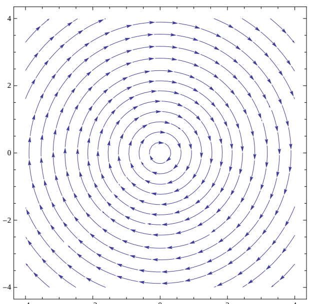
(b) $\vec{F} = x\vec{i} + 3y\vec{j}$



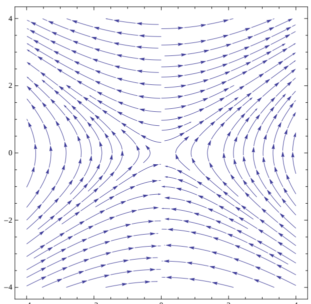
(c) $\vec{F} = x\vec{j}$



(d) $\vec{F} = y\vec{i}$



(e) $\vec{F} = y\vec{i} - x\vec{j}$



(f) $\vec{F} = \langle xy/2, x^2/4\rangle$

Figure 2: Vector fields

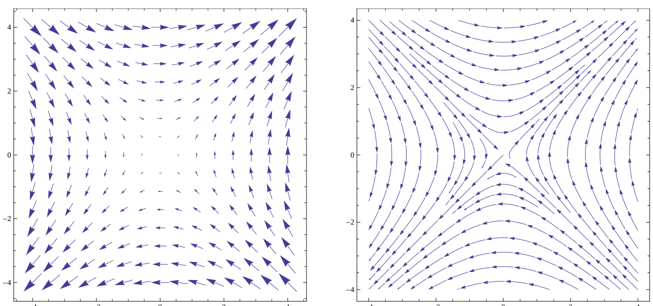


Figure 3: $\vec{F} = y\vec{i} + x\vec{j}$

1 15.2

30: $\vec{r}(t) = t\vec{i} + \sqrt{4-t^2}\vec{j}$, $-2 \le t \le 2$. $\vec{r}\,'(t) = \vec{i} - \frac{t}{\sqrt{4-t^2}}\vec{j}$. $\vec{F} = 3x\vec{i} + 4y\vec{j}$.

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_C \vec{F}\cdot\vec{r}\,'(t)dt = \int_{-2}^{2}\left(3t\cdot 1 + 4\sqrt{4-t^2}\cdot\frac{-t}{\sqrt{4-t^2}}\right)dt\\ &= \int_{-2}^{2} -tdt = 0.\end{aligned}$$

Note that $\vec{F}(x,y) = \nabla f$ is a conservative vector field with a potential $f = \frac{3x^2}{2} + 2y^2$. So by fundamental theorem of line integrals (for conservative vector fields), we have

$$\int_C \vec{F}\cdot d\vec{r} = f(\vec{r}(2)) - f(\vec{r}(-2)) = f(2,0) - f(-2,0) = 0.$$

38: Parametrize the curve C by $\vec{r}(t) = 2\cos t\ \vec{i} + 2\sin t\ \vec{j}, 0 \le t \le \pi$. $\vec{r}\,'(t) = -2\sin t\ \vec{i} + 2\cos t\ \vec{j}$. $\vec{F} = -y\vec{i} - x\vec{j}$. So

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_C \vec{F}\cdot\vec{r}\,'(t)dt = \int_0^{\pi}(-2\sin t\cdot(-2\sin t) - 2\cos t\cdot(2\cos t))\,dt\\ &= \int_0^{\pi}(4\sin^2 t - 4\cos^2 t)dt = -4\int_0^{\pi}\cos(2t)dt = -2\left[\sin(2t)\right]_0^{\pi} = 0.\end{aligned}$$

Note that $\vec{F} = \nabla f$ is a conservative vector field with a potential $f = -xy$. So by fundamental theorem of line integrals, we have

$$\int_C \vec{F}\cdot dr = f(\vec{r}(\pi)) - f(\vec{r}(0)) = f(-2,0) - f(2,0) = 0.$$

41 − 44: **41:** $\vec{F}\cdot\vec{r}\,'(t) > 0 \Rightarrow work > 0$. **42:** $\vec{F}\cdot\vec{r}\,'(t) < 0 \Rightarrow work < 0$. **43** : $\vec{F}\cdot\vec{r}\,'(t) = 0 \Rightarrow work = 0$. **44:** $\vec{F}\cdot\vec{r}\,'(t) = 0 \Rightarrow work = 0$.

54: $\vec{r}(t) = 2t\vec{i} + 10t\vec{j}$, $0 \le t \le 1$.

$$\begin{aligned}\int_C (3y-x)dx + y^2dy &= \int_0^1 (3\cdot 10t - 2t)\cdot 2dt + (10t)^2\cdot 10dt = \int_0^1 (56t + 1000t^2)dt\\ &= \left[28t^2 + \frac{1000}{3}t^3\right]_0^1 = \frac{1084}{3}.\end{aligned}$$

72: $\vec{r}(x) = x\vec{i} + x^{3/2}\vec{j}$. $\vec{r}\,'(x) = \vec{i} + \frac{3}{2}x^{1/2}\vec{j}$. $\|\vec{r}\,'(x\| = \sqrt{1 + \frac{9x}{4}}$.

$$\begin{aligned}Area &= \int_C z(x,y(x))ds = \int_0^{40}\left(20 + \frac{1}{4}x\right)\sqrt{1 + \frac{9x}{4}}dx = \int_1^{91}\left(20 + \frac{u-1}{9}\right)\sqrt{u}\frac{4}{9}du\\ &= \frac{4}{81}\int_1^{91}(179\sqrt{u} + u^{3/2})du = \frac{4}{81}\left[\frac{358u^{3/2}}{3} + \frac{2}{5}u^{5/2}\right]_1^{91} = \frac{8}{81}\left[\frac{179\cdot 5u^{3/2} + 3u^{5/2}}{15}\right]_1^{91}\\ &= \frac{16(-449 + 584\cdot 91\sqrt{91})}{1215} \sim 6670.12\ \text{ft}^2.\end{aligned}$$

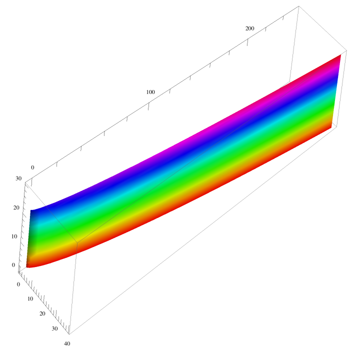


Figure 1: Problem 72: the wall

78: The work along the parabolic path $y = c(1-x^2)$ is:

$$\begin{aligned}
\int_{C_1} \vec{F}\cdot d\vec{r} &= \int_{C_1} 15[(4-x^2y)dx - xydy] \\
&= 15\int_{-1}^{1} (4-x^2(c(1-x^2)))dx - x(c(1-x^2))(-2cxdx) \\
&= 15\int_{-1}^{1} (4+(2c^2-c)x^2+(c-2c^2)x^4)dx = 30\int_{0}^{1} (4+(2c^2-c)x^2+(c-2c^2)x^4)dx \\
&= 120 + 10(2c^2-c) + 6(c-2c^2) = 8c^2-4c+120.
\end{aligned}$$

We want to minimize the quadratic function $f(c) = 8c^2 - 4c + 120$. $f'(c) = 16c - 4 = 0 \Rightarrow c = 1/4$. So the minimum is $f(1/4) = 119.5$.

If we move the object along a straight-line $x = t, y = 0$, $-1 \le t \le 1$. Then the work is

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_{-1}^{1} 15\cdot 4dt = 120 = f(0) > f(1/4) = 119.5.$$

2 15.3

2: $\vec{F}(t) = (x^2+y^2)\vec{i} - x\vec{j}$. **(a)** $\vec{r}_1(t) = t\vec{i} + \sqrt{t}\vec{j}$, $\vec{r}_1\,'(t) = \vec{i} + \frac{1}{2}t^{-1/2}\vec{j}$, $0 \le t \le 4$. So

$$\begin{aligned}
\int_C \vec{F}\cdot d\vec{r} &= \int_0^4 \vec{F}\cdot\vec{r}_1\,'(t)dt = \int_0^4 ((t^2+t)\cdot 1 + (-t)\cdot\frac{1}{2}t^{-1/2})dt \\
&= \left[\frac{t^3}{3} + \frac{t^2}{2} - \frac{1}{3}t^{3/2}\right]_0^4 = \frac{80}{3}.
\end{aligned}$$

(b) $\vec{r}_2(w) = w^2\vec{i} + w\vec{j}$, $\vec{r}_2\,'(w) = 2w\vec{i} + \vec{j}$, $0 \le w \le 2$. So

$$\begin{aligned}
\int_C \vec{F}\cdot d\vec{r} &= \int_0^2 \vec{F}\cdot\vec{r}\,'(w)dw = \int_0^2 ((w^4+w^2)\cdot 2w - w^2\cdot 1)dw \\
&= \left[\frac{1}{3}w^6 + \frac{1}{2}w^4 - \frac{1}{3}w^3\right]_0^2 = \frac{80}{3}.
\end{aligned}$$

16: $\vec{F} = M\vec{i} + N\vec{j} = (2x - 3y + 1)\vec{i} - (3x + y - 5)\vec{j}$. Test whether $\vec{F}$ is conservative or not:

$$\frac{\partial M}{\partial y} = -3 = \frac{\partial N}{\partial x},$$

$\vec{F}$ is a conservative vector field.

1. (**Method 1**) For conservative vector field, the line integral does not depend on the path connecting two fixed points. So

   (a) We can choose a point path to get $\int_C \vec{F} \cdot d\vec{r} = 0$.

   (b) We can choose a different path which is the straight line connecting $(0, -1)$ to $(0, 1)$, parametrized as $x = 0, y = t, -1 \le t \le 1$. So

   $$\int_C (2x - 3y + 1)dx - (3x + y - 5)dy = \int_{-1}^{1} -(t - 5)dt = 10.$$

   (c) We can calculate using the given path:

   $$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_0^2 (2x - 3e^x + 1)dx - (3x + e^x - 5)e^x dx \\ &= \int_0^2 (1 + 2x + 2e^x - e^{2x} - 3xe^x)dx \\ &= \left[x + x^2 + 2e^x - \frac{1}{2}e^{2x}\right]_0^2 - 3[xe^x]_0^2 + 3\int_0^2 e^x dx \\ &= (6 + 2e^2 - \frac{1}{2}e^4) - (3/2) - 6e^2 + 3(e^2 - 1) \\ &= \frac{3}{2} - e^2 - \frac{1}{2}e^4. \end{aligned}$$

   (d) We can choose a point path to get $\int_C \vec{F} \cdot d\vec{r} = 0$.

2. (**Method 2**) Since $\vec{F}$ is a conservative vector field, we can try to find a potential $f(x, y)$ for $\vec{F}$ by solving:

   $$f_x = 2x - 3y + 1, f_y = -3x - y + 5.$$

   Integrate the first equation to get $f(x, y) = x^2 - 3xy + x + g(y)$. Substitute this into the 2nd equation to get:

   $$f_y = -3x + g'(y) = -3x - y + 5 \Rightarrow g'(y) = -y + 5.$$

   Integrate this to get $g(y) = -\frac{y^2}{2} + 5y + C$ for some real constant C. So we find a potential function

   $$f(x, y) = x^2 - 3xy + x - \frac{1}{2}y^2 + 5y$$

   So

   (a) $\int_C \vec{F} \cdot d\vec{r} = f(0, 0) - f(0, 0) = 0$.

   (b) $\int_C \vec{F} \cdot d\vec{r} = f(0, 1) - f(0, -1) = 10$.

(c)

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= f(2,e^2)-f(0,1) = (4-6e^2+2-\frac{1}{2}e^4+5e^2)-(-\frac{1}{2}+5)\\ &= \frac{3}{2}-e^2-\frac{1}{2}e^4\end{aligned}$$

(d) $\int_C \vec{F}\cdot d\vec{r} = f(0,-1)-f(0,-1) = 0.$

26: $\vec{F} = M\vec{i}+N\vec{j} = 2(x+y)\vec{i}+2(x+y)\vec{j}$. Test whether $\vec{F}$ is conservative or not:

$$\frac{\partial M}{\partial y} = 2 = \frac{\partial N}{\partial x}.$$

So $\vec{F}$ is a conservative vector field. Find a potential:

$$f_x = 2(x+y), f_y = 2(x+y).$$

Integrate the first equation to get $f(x,y) = x^2+2xy+g(y)$. Substitute into then 2nd equation to get

$$f_y = 2x+g'(y) = 2x+2y \Rightarrow g'(y) = 2y \Rightarrow g(y) = y^2+C.$$

So we get a potential for $\vec{F}$: $f(x,y) = x^2+2xy+y^2 = (x+y)^2$. So

$$\int_C \vec{F}\cdot d\vec{r} = f(3,2)-f(-1,1) = 25-0 = 25.$$

36: $\vec{F} = M\vec{i}+N\vec{j} = \frac{2x}{y}\vec{i}-\frac{x^2}{y^2}\vec{j}$. Test whether $\vec{F}$ is conservative or not:

$$\frac{\partial M}{\partial y} = -\frac{2x}{y^2} = \frac{\partial N}{x}.$$

So $\vec{F}$ is a conservative vector field. Find a potential by solving:

$$f_x = \frac{2x}{y}, f_y = -\frac{x^2}{y^2}.$$

It's easy to get $f(x,y) = x^2/y$. So the work from P to Q is

$$\int_C \vec{F}\cdot d\vec{r} = f(3,2)-f(-1,1) = \frac{7}{2}.$$

$y = \sqrt{a^2 - x^2}$ and $y = 0$

s of $y = x^3$ and $y = x$,

$a, 0)$, and (b, c), where

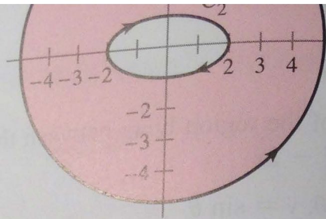
s of Exercise 32 to find

e graph of the polar

$) = \cos t\mathbf{i} + \sin t\mathbf{j}$,

$x + (27x - x^3)\, dy$,

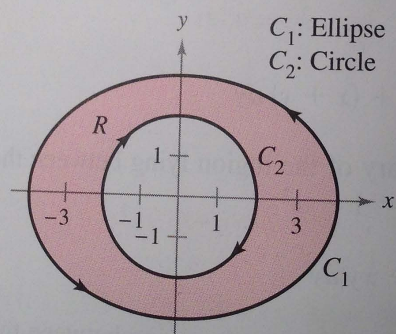
xy-plane, oriented



44. Green's Theorem: Region with a Hole Let R be the region inside the ellipse $x = 4 \cos \theta$, $y = 3 \sin \theta$ and outside the circle $x = 2 \cos \theta$, $y = 2 \sin \theta$. Evaluate the line integral

$$\int_C (3x^2y + 1)\, dx + (x^3 + 4x)\, dy$$

where $C = C_1 + C_2$ is the boundary of R, as shown in the figure.



Hyperbolic an

(a) Sketch the plane function $\mathbf{r}(t) =$ c
Show that the rect
hyperbola $x^2 - y$
utility to graph th

(b) Let $P =$ (cosh
corresponding to

$$A = \frac{1}{2}\int_C x\, dy -$$

to verify that the

(c) Show that the ar
integral

$$A = \int_0^{\sinh \phi} [\sqrt{}$$

Confirm your ar
this integral for

1 15.3

30: $\vec{F}(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j} = \frac{2x}{(x^2+y^2)^2}\vec{i} + \frac{2y}{(x^2+y^2)^2}\vec{j}$ satisfies the test for conservative vector field:

$$\frac{\partial M}{\partial y} = -\frac{8xy}{(x^2+y^2)^3} = \frac{\partial N}{\partial x}.$$

To find a potential f for $\vec{F}$, solve the equations:

$$f_x = \frac{2x}{(x^2+y^2)^2}, f_y = \frac{2y}{(x^2+y^2)^2}.$$

Integrate the 1st equation to get:

$$f(x,y) = -\frac{1}{x^2+y^2} + g(y).$$

Substitute this into the 2nd equation to get:

$$f_y = \frac{2y}{(x^2+y^2)^2} + g'(y) = \frac{2y}{(x^2+y^2)^2} \Rightarrow g'(y) = 0 \Rightarrow g(y) = C.$$

So we get a potential for $\vec{F}$: $f(x,y) = -1/(x^2+y^2)$. By the fundamental theorem of line integral, we get

$$\int_C \frac{2x}{(x^2+y^2)^2}dx + \frac{2y}{(x^2+y^2)^2}dy = \int_C \vec{F}\cdot d\vec{r} = f(1,5) - f(7,5) = -\frac{1}{26} + \frac{1}{74} = -\frac{12}{481}.$$

34: $\vec{F} = 6x\vec{i} - 4z\vec{j} - (4y - 20z)\vec{k}$. First test whether $\vec{F}$ is conservative or not by using the *curl*:

$$curl\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6x & -4z & -4y + 20z \end{vmatrix} = 0.$$

So $\vec{F}$ is conservative. We find a potential function for $\vec{F}$ by solving:

$$f_x = 6x, f_y = -4z, f_z = -4y + 20z.$$

Integrate the 1st equation to get $f(x,y,z) = 3x^2 + g(y,z)$. Substitute into the 2nd equation to get:

$$f_y = g_y = -4z \Rightarrow g(y,z) = -4yz + h(z) \Rightarrow f(x,y,z) = 3x^2 - 4yz + h(z).$$

Substitute into the 3rd equation to get:

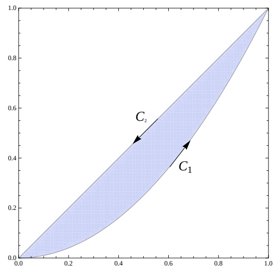
$$f_z = -4y + h'(z) = -4y + 20z \Rightarrow h'(z) = 20z \Rightarrow g(z) = 10z^2 + C.$$

So finally we get

$$f(x,y,z) = 3x^2 - 4yz + 10z^2 + C.$$

By fundamental theorem of line integrals for conservative vector fields, we get

$$\int_C 6xdx - 4zdy - (4y - 20z)dz = f(3,4,0) - f(0,0,0) = 27.$$



2 15.4

1: $C_1 : \vec{r}_1(t) = t\vec{i} + t^2\vec{j}, 0 \le t \le 1$, $C_2 : \vec{r}_2(t) = -t\vec{i} - t\vec{j}, -1 \le t \le 0$. (Note that we need the anti-clockwise orientation). First we calculate the line integral:

$$\begin{aligned}\int_{C_1} y^2dx + x^2dy &= \int_0^1 t^4dt + t^2 2tdt = \int_0^1 (t^4 + 2t^3)dt \\ &= \frac{1}{5} + \frac{2}{4} = \frac{7}{10}.\end{aligned}$$

$$\begin{aligned}\int_{C_2} y^2dx + x^2dy &= \int_{-1}^0 t^2(-dt) + t^2(-dt) = -\int_{-1}^0 2t^2dt = -\left[\frac{2}{3}t^3\right]_{-1}^0 \\ &= -\frac{2}{3}.\end{aligned}$$

So

$$\int_{C=C_1+C_2} y^2dx + x^2dy = \frac{7}{10} - \frac{2}{3} = \frac{1}{30}.$$

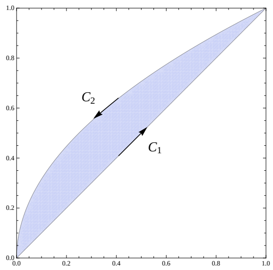
Next we calculate the double integral. Note that $N_x - M_y = 2x - 2y$. So

$$\begin{aligned}&\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_R (2x - 2y)dA \\ =\ &\int_0^1 \int_{x^2}^x 2(x-y)dydx = \int_0^1 \left[2xy - y^2\right]_{x^2}^x dx = \int_0^1 (x^2 - 2x^3 + x^4)dx \\ =\ &\left[\frac{x^3}{3} - \frac{1}{2}x^4 + \frac{1}{5}x^5\right]_0^1 = \frac{1}{3} - \frac{1}{2} + \frac{1}{5} \\ =\ &\frac{1}{30}.\end{aligned}$$

So the two sides are indeed equal to each other.

2: $C_1 : \vec{r}_1(t) = t\vec{i} + t\vec{j}, 0 \le t \le 1$, $-\mathbf{C_2} : \vec{r}_2(t) = t\vec{i} + \sqrt{t}\vec{j}, 0 \le t \le 1$. We first use the reverse direction for simplification of calculations. First we calculate the line integral:

$$\begin{aligned}\int_{C_1} y^2dx + x^2dy &= \int_0^1 t^2(dt) + t^2(dt) = \int_0^1 2t^2dt = \left[\frac{2}{3}t^3\right]_0^1 \\ &= \frac{2}{3}.\end{aligned}$$



$$\begin{aligned}\int_{-C_2} y^2dx + x^2dy &= \int_0^1 tdt + t^2 d(\sqrt{t}) = \int_0^1 \left(t + \frac{1}{2}t^{3/2}\right) dt \\ &= \frac{1}{2} + \frac{1}{5} = \frac{7}{10}.\end{aligned}$$

So

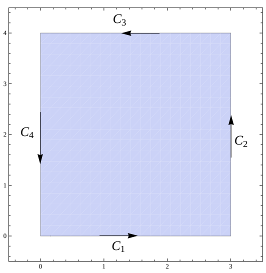
$$\int_{C=C_1+C_2} y^2dx + x^2dy = \int_{C_1} - \int_{-C_2} = \frac{2}{3} - \frac{7}{10} = -\frac{1}{30}.$$

Next we calculate the double integral. Note that $N_x - M_y = 2x - 2y$. So

$$\begin{aligned}&\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_R (2x-2y)dA \\ = &\int_0^1 \int_x^{\sqrt{x}} 2(x-y)dydx = \int_0^1 \left[2xy - y^2\right]_x^{\sqrt{x}} dx = \int_0^1 (2x^{3/2} - x - x^2)dx \\ = &\left[\frac{4}{5}x^{5/2} - \frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{4}{5} - \frac{1}{2} - \frac{1}{3} \\ = &-\frac{1}{30}.\end{aligned}$$

So the two sides are indeed equal to each other.

4: We first calculate the line integrals:



1. $C_1 : \vec{r}_1(t) = t\vec{i}, 0 \le t \le 3$.

$$\int_{C_1} y^2dx + x^2dy = \int_0^3 0^2dt + t^2d(0) = 0.$$

2. $C_2 : \vec{r}_2(t) = 3\vec{i} + t\vec{j}, 0 \le t \le 4$.

$$\int_{C_2} y^2dx + x^2dy = \int_0^4 t^2d(3) + 3^2dt = \int_0^4 9dt = 36.$$

3. $C_3 : \vec{r}_3(t) = (3-t)\vec{i} + 4\vec{j}, 0 \le t \le 3;$.

$$\int_{C_3} y^2dx + x^2dy \quad = \quad \int_0^3 4^2 d(3-t) + (3-t)^2 d(4) = -\int_0^3 16dt = -48.$$

4. $\vec{r}_4(t) = (4-t)\vec{j}, 0 \le t \le 4$.

$$\int_{C_4} y^2dx + x^2dy \quad = \quad \int_0^4 0^2 d(4-t) + (4-t)^2 d(0) = 0.$$

So

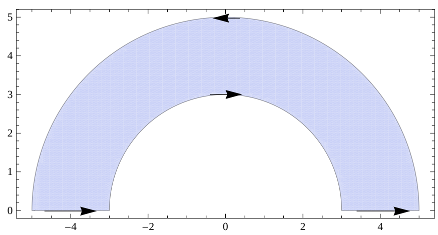
$$\int_C y^2dx + x^2dy = \sum_{i=1}^{4} \int_{C_i} (y^2dx + x^2dy) = 36 - 48 = -12.$$

Next we compute the double integral. Note that $N_x - M_y = 2x - 2y$. So

$$\begin{aligned} & \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \iint_R (2x - 2y) dA \\ = & \int_0^3 \int_0^4 (2x-2y)dydx = \int_0^3 [2xy - y^2]_0^4 dx = \int_0^3 (8x - 16)dx \\ = & \left[4x^2 - 16x\right]_0^3 = 36 - 48 = -12. \end{aligned}$$

So the two sides are indeed equal to each other.

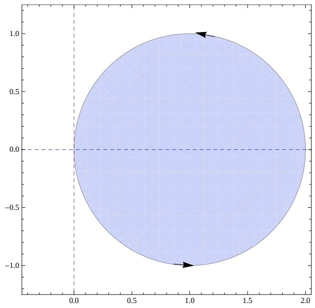
10:



$$\begin{aligned} \int_C (y-x)dx + (2x-y)dy \quad &= \quad \iint_R \left(\frac{\partial}{\partial x}(2x-y) - \frac{\partial}{\partial y}(y-x) \right) dA \\ &= \quad \iint_R dA = Area(R) \\ &= \quad \frac{\pi 25}{2} - \frac{\pi 9}{2} = 8\pi. \end{aligned}$$

22: We first transform the equation $r = 2\cos\theta$ into the rectangular coordinate system:

$$r = 2\cos\theta \Leftrightarrow r^2 = 2r\cos\theta \Leftrightarrow x^2 + y^2 = 2x.$$



So C is a circle: $(x-1)^2 + y^2 = 1$. By Green's theorem,

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_C (e^x - 3y)dx + (e^y + 6x)dy \\
&= \iint_R \left(\frac{\partial}{\partial x}(e^y + 6x) - \frac{\partial}{\partial y}(e^x - 3y) \right) dA \\
&= \iint_R 9dA = 9Area(R) \\
&= 9\pi.
\end{aligned}$$

38: Note that the curve $r = r(\theta)$ in polar coordinate is parametrized by:

$$\vec{r}(\theta) = (r(\theta)\cos\theta)\vec{i} + (r(\theta)\sin\theta)\vec{j}.$$

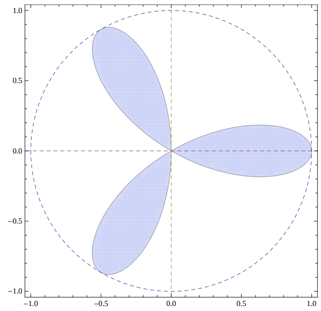
Let's first derive the following formula of Exercise 32 by using Green's formula:

$$\begin{aligned}
Area(R) &= \iint_R dA = \frac{1}{2}\oint_C xdy - ydx \quad \text{(by Green's formula)} \\
&= \frac{1}{2}\int_C (r(\theta)\cos(\theta)d(r(\theta)\sin\theta) - r(\theta)\sin(\theta)d(r(\theta)\cos\theta))d\theta \\
&= \frac{1}{2}\int_C [r(\theta)\cos\theta(r'(\theta)\sin\theta + r(\theta)\cos\theta) - r(\theta)\sin(\theta)(r'(\theta)\cos\theta - r(\theta)\sin\theta)]\, d\theta \\
&= \frac{1}{2}\int_C (rr'(\cos\theta\sin\theta - \sin\theta\cos\theta) + r^2(\cos^2\theta + \sin^2\theta))d\theta \\
&= \frac{1}{2}\int_C r(\theta)^2 d\theta.
\end{aligned}$$

Note that we get the same formula by using the polar coordinate to calculate double integral:

$$Area = \int_{\theta_1}^{\theta_2}\int_0^{r(\theta)} rdrd\theta = \frac{1}{2}\int_{\theta_1}^{\theta_2} r(\theta)^2 d\theta.$$

For the curve $r(\theta) = a\cos(3\theta)$, we calculate the area of branch of the region:



$$
\begin{aligned}
Area(R_1) &= \frac{1}{2}\int_{-\pi/6}^{\pi/6} a^2\cos^2(3\theta)d\theta = \frac{a^2}{2}\int_{-\pi/6}^{\pi/6}\frac{1+\cos(6\theta)}{2}d\theta \\
&= \frac{a^2}{4}\left[\theta + \frac{\sin(6\theta)}{6}\right]_{-\pi/6}^{\pi/6} \\
&= \frac{\pi a^2}{12}.
\end{aligned}
$$

The total area is equal to:

$$Area(R) = 3Area(R_1) = \frac{\pi a^2}{4}.$$

44$^{\mathbf{10}}$: Let R_1 denote the region enclosed ellipse and R_2 denote the region enclosed by the circle.

$$
\begin{aligned}
&\int_C (3x^2y+1)dx + (x^3+4x)dy = \iint_R \left(\frac{\partial}{\partial x}(x^3+4x) - \frac{\partial}{\partial y}(3x^2y+1)\right) dA \\
= &\iint_R 4dA = 4\int_{R_1} dA - 4Area(R_2) \\
= &\ 4\int_{-4}^{4}\int_{-3\sqrt{1-(x/4)^2}}^{3\sqrt{1-(x/4)^2}} dydx - 4\pi(2)^2 \\
= &\ 24\int_{-4}^{4}\sqrt{1-(x/4)^2}dx - 16\pi = 24\int_{-\pi/2}^{\pi/2}\cos\theta\cdot 4\cos\theta d\theta - 16\pi \\
= &\ 96\int_{-\pi/2}^{\pi/2}\frac{1+\cos(2\theta)}{2}d\theta - 16\pi \\
= &\ 48\pi - 16\pi = 32\pi.
\end{aligned}
$$

Note that in general, the area of the region enclosed by the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is equal to πab.

47$^{\mathbf{10}}$ = **44**$^{\mathbf{9}}$: **(a):** Parametrize the line segment $\overline{(x_1,y_1)(x_2,y_2)}$ by

$$C : \vec{r}(t) = ((1-t)x_1 + tx_2)\vec{i} + ((1-t)y_1 + ty_2)\vec{j}, 0 \le t \le 1.$$

So

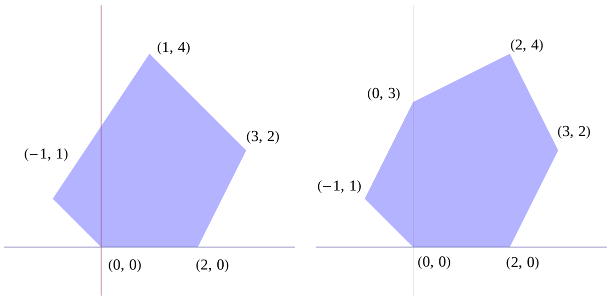
$$\begin{aligned}
\int_C (-ydx + xdy) &= \int_0^1 (-(1-t)y_1 - ty_2)d((1-t)x_1 + tx_2) + ((1-t)x_1 + tx_2)d((1-t)y_1 + ty_2) \\
&= \int_0^1 (-y_1 + t(y_1 - y_2))(x_2 - x_1)dt + (x_1 + t(x_2 - x_1))(y_2 - y_1)dt \\
&= -y_1(x_2 - x_1) + x_1(y_2 - y_1) = x_1y_2 - x_2y_1.
\end{aligned}$$

(b): By Green's theorem, the area can be calculated by:

$$\begin{aligned}
Area(R) &= \frac{1}{2}\int_{\partial R} xdy - ydx = \sum_{i=1}^{n} \int_{\overline{(x_i,y_i)(x_{i+1},y_{i+1})}} xdy - ydx, (denote(x_{n+1}, y_{n+1}) = (x_1, y_1)) \\
&= \frac{1}{2}((x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)).
\end{aligned}$$

The 2nd identity uses part (a).

$\mathbf{48^{10} = 45^9 - 46^9}$: $\mathbf{48^{10}.(a) = 45^9}$: By the above result, the area of the Pen-



tagon is:

$$\begin{aligned}
Area &= \frac{1}{2}((0\cdot 0 - 2\cdot 0) + (2\cdot 2 - 3\cdot 0) + (3\cdot 4 - 1\cdot 2) + (1\cdot 1 - (-1)\cdot 4) + ((-1)\cdot 0 - 0\cdot (1)) \\
&= \frac{19}{2} = 9.5.
\end{aligned}$$

$\mathbf{48^{10}.(b) = 46^9}$: The area of the Hexagon is:

$$\begin{aligned}
Area &= \frac{1}{2}((0\cdot 0 - 2\cdot 0) + (2\cdot 2 - 3\cdot 0) + (3\cdot 4 - 2\cdot 2) + (2\cdot 3 - 0\cdot 4) + (0\cdot 1 - (-1)\cdot 3) \\
&\quad +((-1\cdot 0) - 0\cdot 1)) \\
&= \frac{21}{2} = 10.5.
\end{aligned}$$

MAT 203 FALL 2013 Practice Final

NAME : ID :

RECITATION NUMBER:

THERE ARE TEN (10) PROBLEMS. THEY HAVE THE INDICATED VALUE.

SHOW YOUR WORK

DO NOT TEAR-OFF ANY PAGE

NO CALCULATORS NO CELLS ETC.

ON YOUR DESK: ONLY test, pen, pencil, eraser.

1		50pts
2		50pts
3		50pts
4		50pts
5		50pts
6		50pts
7		50pts
8		50pts
9		50pts
10		50pts
Total		500pts

!!! WRITE YOUR NAME, STUDENT ID AND LECTURE N. BELOW !!!

NAME : ID :

1. (50pts) Let $f(x, y, z) = 9x^2 + y^2 - z^2$.
(a): Sketch and describe the level surfaces $f(x, y, z) = 9$, $f(x, y, z) = 0$ and $f(x, y, z) = -4$.

(b): Find the tangent planes to the level surface $f(x, y, z) = 0$ at the points $(0, 5, 5)$ and $(0, 1, 1)$.

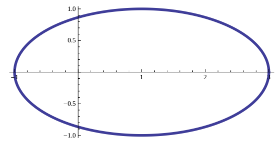
(c): Are the two planes in part (2) distinct? Explain how this relates to the geometry of the level surface.

2. (50pts) Consider the function $f(x, y) = \sin(x^2 + y^2)$.
(a): Compute ∇f.

(b): A particle is moving on the path $(-t, t^2)$. Sketch the path for $0 \leq t \leq 2$. At $t = 1$, is f increasing or decreasing along the path?

(c): Explain in words and draw the level curve $f = 0$.

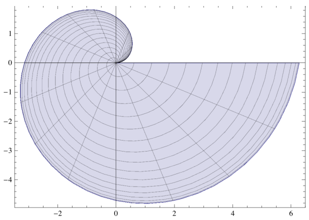
3. (50pts) Consider the curve C defined by the equation:



$$(x-1)^2 + 4y^2 = 4.$$

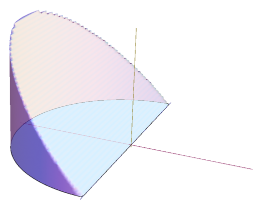
Find the point(s) on the curve C which are closest to/furthest from the origin.

4. (50pts) Find the center of mass of snail-shaped region bounded by the curve



$r = \theta^{1/3}$, $0 \le \theta \le 2\pi$. Assume the mass density $\rho \equiv 1$.

5. (50pts) Consider the solid Q defined by the inequalities



$$x^2 + y^2 = 1, z \leq -y, z \geq 0.$$

(a): Write the triple integral $\iiint_Q dV$ into the iterated integral using the order $dzdydx$ and $dydxdz$.

(b): Calculate the volume of Q.

6. (50pts)

(a): Sketch the 3-dimensional region over which the integral

$$\int_0^5 \int_0^{\sqrt{25-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{50-x^2-y^2}} z\,dz\,dy\,dx.$$

(b): Convert this integral into cylindrical and spherical coordinates.

(c): Compute the integral.

7. (50pts)

(a): Is the following fluid vector field compressible or not?

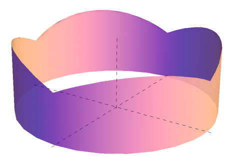
$$\vec{F} = e^x \sin(y+z)\vec{i} + e^x \cos(y+z)\vec{j} + (x^2+y^2)\vec{k}.$$

(b): Is the following vector field conservative or not?

$$\vec{F} = (y+z)\vec{i} + z\vec{j} + (x+y)\vec{k}.$$

8. (50pts)

Consider the circle $C : x^2 + y^2 = a^2$. **(a):** Find the area of lateral surface with height



function $z = |x| + |y|$ over the curve C.

(b): Consider the vector field:

$$\vec{F} = \frac{-y\vec{i} + x\vec{j}}{x^2 + y^2}, (x, y) \neq (0, 0).$$

Sketch the vector field, and calculate the circulation of $\vec{F}$ around the curve C : $x^2 + y^2 = a^2$ in the **counter-clockwise** direction. What's the circulation in the **clockwise** direction?

9. (50pts)

Let $\vec{F}$ be the vector field

$$\vec{F} = \langle 2x + y - 1, x + 2y \rangle .$$

(a): Is $\vec{F}$ conservative or not?

(b): If yes, find a potential function $f(x, y)$.

(c): A particle is moving from the point $(0, 0)$ to a point (p, q) in a straight line. For which values of p, q is the work

$$\int_{(0,0)}^{(p,q)} \vec{F} \cdot d\vec{r}.$$

minimal?

10. (50pts)

Use Green's theorem to calculate the work done by $\vec{F}$

$$\vec{F} = (4x^2 - 2y^2)\vec{i} + (2x^2 - 4y^2)\vec{j}.$$

in moving a particle once counterclockwise around the boundary of a triangle with vertex: $(0,0)$, $(1,0)$, $(0,1)$?

- Review the classification of quadratic surfaces.
- gradient vector is normal to the level set.
- lines and planes. tangent plane, tangent line

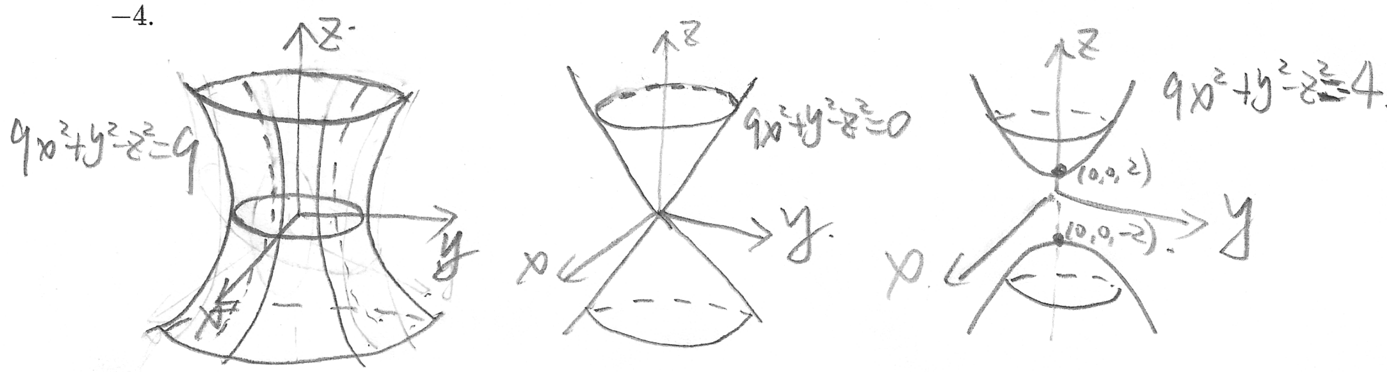
!!! WRITE YOUR NAME, STUDENT ID AND LECTURE N. BELOW !!!

NAME :

ID :

1. (50pts) Let $f(x, y, z) = 9x^2 + y^2 - z^2$.

(a): Sketch and describe the level surfaces $f(x, y, z) = 9$, $f(x, y, z) = 0$ and $f(x, y, z) = -4$.



(b): Find the tangent planes to the level surface $f(x, y, z) = 0$ at the points $(0, 5, 5)$ and $(0, 1, 1)$.

The gradient vector is normal to the level set.

$\nabla f(x,y,z) = \langle 18x, 2y, -2z \rangle$.

$\nabla f(0,5,5) = \langle 0, 10, -10 \rangle$, $\nabla f(0,1,1) = \langle 0, 2, -2 \rangle$.

$\Downarrow$

$0\cdot(x-0) + 10(y-5) - 10(z-5) = 0$

$\Updownarrow$

tangent plane at $(0,5,5)$ $\rightarrow$ $\boxed{y - z = 0}$

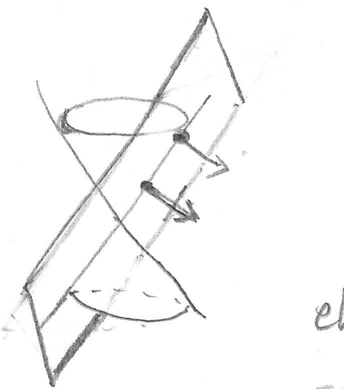
$\Downarrow$

$0\cdot(x-0) + 2\cdot(y-1) - 2(z-1) = 0$

$\Updownarrow$

$\boxed{y - z = 0}$ $\leftarrow$ tangent plane at $(0,1,1)$.

(c): Are the two planes in part (2) distinct? Explain how this relates to the geometry of the level surface.



The two planes in part (2) are the same:

$y - z = 0$.

This happens because $9x^2+y^2-z^2=0$ is an elliptical cone, and $(0,5,5)$ and $(0,1,1)$ lie on the same ruling passing through the vertex.

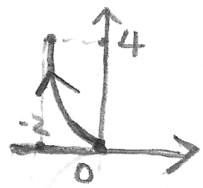
• Review chain rule and directional derivative.

2. (50pts) Consider the function $f(x, y) = \sin(x^2 + y^2)$.
(a): Compute ∇f.

$\nabla f = \langle f_x, f_y \rangle = \langle 2x \cdot \cos(x^2+y^2), 2y\cos(x^2+y^2) \rangle$

(b): A particle is moving on the path $(-t, t^2)$. Sketch the path for $0 \le t \le 2$. At $t = 1$, is f increasing or decreasing along the path?

$x = -t, y = t^2 \Rightarrow y = +x^2, -2 \le x \le 0$
$0 \le t \le +2$



$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x} \cdot x'(t) + \frac{\partial f}{\partial y} \cdot y'(t) (= \nabla f(x,t) \cdot \vec{r}'(t))$.

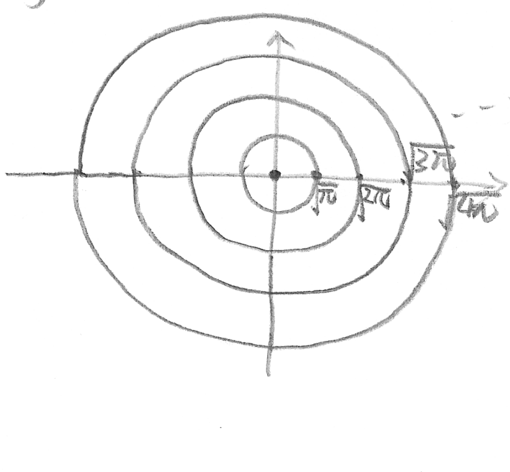
$\vec{r}(1) = (-1, 1)$, $\vec{r}'(1) = \langle -1, 2t \rangle|_{t=1} = \langle -1, 2 \rangle$.

$\nabla f(-1,1) = \langle -2\cos(2), 2\cos(2) \rangle$. so $\left.\frac{df}{dt}\right|_{t=1} = \langle -2\cos(2), 2\cos(2) \rangle \cdot \langle -1, 2 \rangle$
$= 6 \cdot \cos(2)$.

(c): Explain in words and draw the level curve $f = 0$.

$f(x,y) = \sin(x^2+y^2) = 0 \Leftrightarrow x^2+y^2 = n \cdot \pi,\ n \in \mathbb{Z}$, $n \geq 0$.

So the level curve consists of a family of circles centered at $(0,0)$: infinitely many



Note that $\frac{\pi}{2} < 2 < \pi$
$\frac{\pi}{2} \approx 1.57$, $\pi \approx 3.14$.

so $\cos(2) < 0$
$\Downarrow$
$\left.\frac{df}{dt}\right|_{t=1} < 0$.
So f is decreasing along the path at $t=1$.

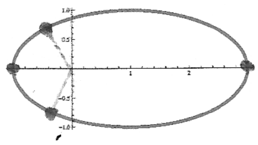
• Lagrange multiplier

• This question has another method by parametrizing the ellipse:

$$\begin{cases} x = 1 + 2\cos\theta \\ y = \sin\theta \end{cases} \quad 0 \le \theta \le 2\pi$$

see practice mid 2

3. (50pts) Consider the curve C defined by the equation:



$$(x-1)^2 + 4y^2 = 4.$$

Find the point(s) on the curve C which are closest to/furthest from the origin.

We need to find extrema of $f(x,y) = x^2 + y^2$ ($=$ distance^2)

under the constraint $g(x,y) = (x-1)^2 + 4y^2 - 4 = 0$.

We can use the Lagrange Multiplier method:

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 0 \end{cases} \Longleftrightarrow \begin{cases} 2x = \lambda\, 2(x-1) & ① \\ 2y = \lambda \cdot 8y & ② \\ (x-1)^2 + 4y^2 = 4 & ③ \end{cases}$$

$$② \Rightarrow 2y(1 - 4\lambda) = 0 \Rightarrow y = 0 \text{ or } \lambda = \frac{1}{4}.$$

i) $y = 0 \overset{③}{\Longrightarrow} (x-1)^2 = 4 \Rightarrow x - 1 = \pm 2 \Rightarrow x = 3$ or -1.

we get 2 candidates: $(3,0)$, $(-1,0)$.

ii). $\lambda = \frac{1}{4} \overset{①}{\Longrightarrow} 2x = \frac{1}{2}(x-1) \Rightarrow \frac{3}{2}x = -\frac{1}{2} \Rightarrow x = -\frac{1}{3} \overset{③}{\Longrightarrow} \left(\frac{4}{3}\right)^2 + 4y^2 = 4$

$\Rightarrow y^2 = \frac{5}{9} \Rightarrow y = \pm\frac{\sqrt{5}}{3}$. so we get 2 other candidates

$$\left(-\frac{1}{3}, \frac{\sqrt{5}}{3}\right), \left(-\frac{1}{3}, -\frac{\sqrt{5}}{3}\right)$$

compare the distances to origin:

$f(3,0) = 9$

$f(-1,0) = 1$

$f\left(-\frac{1}{3}, \pm\frac{\sqrt{5}}{3}\right) = \frac{6}{9} = \frac{2}{3}$

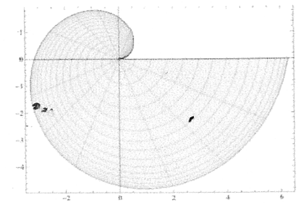
$\Longrightarrow$ furthest pt: $(3,0)$.

closest pts: $\left(-\frac{1}{3}, \frac{\sqrt{5}}{3}\right)$, $\left(-\frac{1}{3}, -\frac{\sqrt{5}}{3}\right)$.

• Double integral in polar coordinate • center of mass in both 2-dim and 3-dim situation.

• varying density cases.

4. **(50pts)** Find the center of mass of snail-shaped region bounded by the curve



$r = \theta^{1/3}$, $0 \le \theta \le 2\pi$. Assume the mass density $\rho \equiv 1$.

mass: $\iint_R dA = \int_0^{2\pi}\int_0^{\theta^{\frac13}} r\,dr\,d\theta = \int_0^{2\pi} \frac12 \theta^{\frac23}\,d\theta$

$= \frac12 \cdot \frac35 \theta^{\frac53}\Big]_0^{2\pi} = \frac{3}{10}\cdot(2\pi)^{\frac53} = \frac35 \cdot 2^{\frac23}\pi^{\frac53}.$

integral of x coordinate: $\iint_R x\,dA = \int_0^{2\pi}\int_0^{\theta^{\frac13}} r\cos\theta \cdot r\,dr\,d\theta = \int_0^{2\pi} \cos\theta \cdot \left[\frac{r^3}{3}\right]_0^{\theta^{\frac13}} d\theta$

$= \frac13 \int_0^{2\pi} (\cos\theta)\,\theta\,d\theta = \frac13 \int_0^{2\pi} \theta \cdot d(\sin\theta)$

$= \frac13 \left[\theta \cdot \sin\theta\right]_0^{2\pi} - \frac13 \cdot \int_0^{2\pi} \sin\theta \cdot d\theta = 0 + \frac13 \left[\cos\theta\right]_0^{2\pi} = 0.$

integral of y coordinate: $\iint_R y\,dA = \int_0^{2\pi}\int_0^{\theta^{\frac13}} r\sin\theta \cdot r\,dr\,d\theta = \int_0^{2\pi} \frac13 \cdot \theta \cdot \sin\theta\,d\theta$

$= -\frac13 \cdot \int_0^{2\pi} \theta \cdot d(\cos\theta) = -\frac13 \theta\cos\theta\Big]_0^{2\pi} + \frac13 \cdot \int_0^{2\pi} (\cos\theta)\,d\theta.$

$= -\frac13\, 2\pi + \frac13 \left[\sin\theta\right]_0^{2\pi} = -\frac23 \pi.$

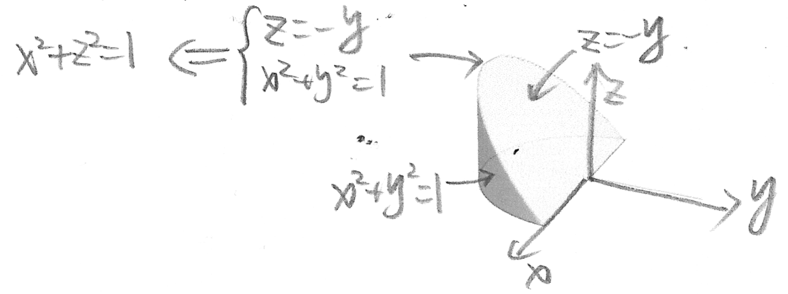
so the center of mass is:

$$(\bar{x}, \bar{y}) = \frac{1}{\text{mass}}\Big(\underbrace{\iint_R x\,dA}_{\iint_R x\rho(x,y)\,dA}, \underbrace{\iint_R y\,dA}_{\iint_R y\rho(x,y)\,dA}\Big) = \frac{1}{\frac35 2^{\frac23}\pi^{\frac53}}\left(0, -\frac23\pi\right) = \left(0, -\frac59 \frac{2^{\frac13}}{\pi^{\frac23}}\right).$$

• triple integral to iterated integral

• last 2 integration variables correspond to the choice of projection.

5. (50pts) Consider the solid Q defined by the inequalities



$$x^2+y^2=1, z \le -y, z \ge 0.$$

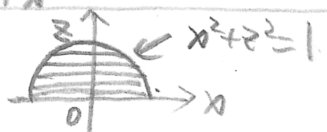
(a): Write the triple integral $\iiint_Q dV$ into the iterated integral using the order $dzdydx$ and $dydxdz$.

dzdydx: • project to xy-plane:

• dydx → use the vertical slice

$$\iiint_Q dV = \iint_{R_{xy}} \left(\int_0^{-y} dz\right) dA = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0}\int_0^{-y} dz\,dy\,dx.$$

dydxdz: • project to xz-plane



• dxdz → use horizontal slice

$$\iiint_Q dV = \iint_{R_{xz}} \int_{-\sqrt{1-x^2}}^{-z} dy\, dA = \int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-x^2}}^{-z} dy\,dx\,dz.$$

(b): Calculate the volume of Q.

Use projection to xy-plane and polar coordinate on xy-plane.
(that is, use cylindrical coordinate)

$$\iiint_Q dV = \iint_{R_{xy}} \left(\int_0^{-y} dz\right) dA = \int_{-\pi}^{0}\int_0^1 \left(\int_0^{-r\sin\theta} dz\right) r\,dr\,d\theta$$

$$= \int_{-\pi}^{0}\int_0^1 (-r^2\sin\theta)\, dr\,d\theta = \left[\frac{1}{3}r^3\right]_0^1 \left[\cos\theta\right]_{-\pi}^{0}$$

$$= \frac{2}{3}$$

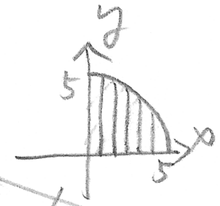
• cylindrical and spherical coordinate system
• Triple integrals with symmetry in these coordinates
• sketch the solid region with symmetry.

6. (50pts)

(a): Sketch the 3-dimensional region over which the integral

$$\int_0^5 \int_0^{\sqrt{25-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{50-x^2-y^2}} z\,dz\,dy\,dx.$$

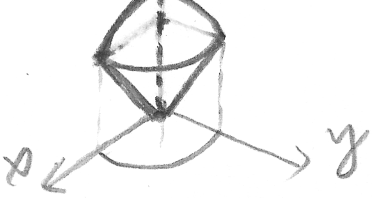
projection to the xy-plane: $0 \le x \le 5$, $0 \le y \le \sqrt{25-x^2}$



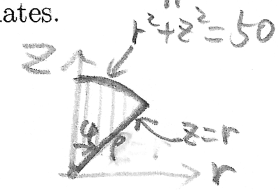
top: $z=\sqrt{50-x^2-y^2} \Leftrightarrow x^2+y^2+z^2=50$ sphere

bottom: $z=\sqrt{x^2+y^2} \Leftrightarrow x^2+y^2-z^2=0$ circular cone

↑ only in the 1st quadrant



(b): Convert this integral into cylindrical and spherical coordinates.



Cross section:

Cylindrical: $\int_0^{\frac{\pi}{2}} \int_0^{5} \left(\int_r^{\sqrt{50-r^2}} z\,dz \right) r\,dr\,d\theta$

spherical: $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{50}} (\rho\cos\varphi)\, \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$

(c): Compute the integral.

Use spherical coordinate:

$$Vol = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{50}} (\rho\cos\varphi)\, \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$= [\theta]_0^{\frac{\pi}{2}} \left(\int_0^{\frac{\pi}{4}} (\sin\varphi)\, d(\sin\varphi) \right) \left[\frac{\rho^4}{4}\right]_0^{\sqrt{50}} = \frac{\pi}{2} \cdot \frac{50^2}{4} \cdot \frac{1}{2} \left[(\sin\varphi)^2\right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi \cdot (25)^2 \cdot 2^2}{2^4} \cdot \frac{1}{2} = \frac{625\pi}{8}$$

• divergence

• curl ← test for conservativeness.

7. (50pts)

(a): Is the following fluid vector field compressible or not?

$$\vec{F} = e^{x}\sin(y+z)\vec{\imath} + e^{x}\cos(y+z)\vec{\jmath} + (x^2+y^2)\vec{k}.$$

Use divergence to test compressibility:

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{\partial}{\partial x}\left(e^{x}\sin(y+z)\right) + \frac{\partial}{\partial y}\left(e^{x}\cos(y+z)\right) + \frac{\partial}{\partial z}(x^2+y^2)$$

$$= e^{x}\sin(y+z) - e^{x}\sin(y+z) + 0$$

$$= 0 \quad \Rightarrow \text{ incompressible i.e. } \underline{\text{not}} \text{ compressible.}$$

(b): Is the following vector field conservative or not?

$$\vec{F} = (y+z)\vec{\imath} + z\vec{\jmath} + (x+y)\vec{k}.$$

Use curl to test conservativity:

$$\frac{\partial}{\partial x} z - \frac{\partial}{\partial y}(y+z)$$

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z & x+y \end{vmatrix} = 0\cdot\vec{i} + 0\cdot\vec{j} + \vec{k}(-1) \neq 0$$

$\Rightarrow \vec{F}$ is NOT conservative.

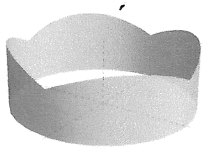
• line integral of function with respect to arclength

Application 1: mass of string (with varying density)

2. area of lateral surface

8. (50pts)

Consider the circle $C : x^2 + y^2 = a^2$. **(a):** Find the area of lateral surface with height



function $z = |x| + |y|$ over the curve C.

This is an application of integration of a function w.r.t. arclength parameter

By symmetry, we just need to calculate the area in the 1st quadrant

where $z = x + y$. Before calculating the line integral, we need to

parametrize the curve $C: \begin{cases} x = a\cos\theta \\ y = a\cdot\sin\theta \end{cases}$ $0 \le \theta \le 2\pi$.

$\vec{r}'(\theta) = -a\sin\theta\,\vec{i} + a\cos\theta\,\vec{j}$. $\|\vec{r}'(\theta)\| = a$

$$\underset{\substack{\| \\ z(x(s),y(s))}}{\int_C z(x,y)\,ds} = 4\int_0^{\frac{\pi}{2}} (a\cos\theta + a\sin\theta)\cdot a\,d\theta$$

$$= 4a^2\cdot[\sin\theta - \cos\theta]_0^{\frac{\pi}{2}}$$

$$= 8a^2.$$

- Definition of circulation: $\oint \vec{F} \cdot d\vec{r}$
- Calculation of $\int \vec{F} \cdot d\vec{r}$

- change orientation $\rightsquigarrow$ change sign (compare with directional derivative)

(b): Consider the vector field:

$$\vec{F} = \frac{-y\vec{i} + x\vec{j}}{x^2 + y^2}, (x, y) \neq (0, 0).$$

Sketch the vector field, and calculate the circulation of $\vec{F}$ around the curve C : $x^2 + y^2 = a^2$ in the **counter-clockwise** direction. What's the circulation in the **clockwise** direction?

in counter-clockwise direction: $\begin{cases} x = a\cos\theta \\ y = a\sin\theta \end{cases}$, $0 \le \theta \le 2\pi$.

(in clockwise direction: $\begin{cases} x = a\cos\theta \\ y = -a\sin\theta \end{cases}$, $0 \le \theta \le 2\pi$)

$\vec{r}'(\theta) = -a\sin\theta\,\vec{i} + a\cos\theta\,\vec{j}$. The circulation is calculated as:

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left(\vec{F} \cdot \frac{d\vec{r}}{d\theta}\right) d\theta = \int_0^{2\pi} \left(\frac{-a\sin\theta\,\vec{i} + a\cos\theta\,\vec{j}}{a^2}\right) \cdot \left(-a\sin\theta\,\vec{i} + a\cos\theta\,\vec{j}\right) d\theta$$

$$= \int_0^{2\pi} \frac{a^2}{a^2}\left(\sin^2\theta + \cos^2\theta\right) d\theta = \int_0^{2\pi} d\theta = 2\pi.$$

If we use the clockwise direction, then the circulation differs by a sign: $\oint \vec{F} \cdot d\vec{r} = -\oint \vec{F} \cdot d\vec{r} = -2\pi$. (does not depend on a).

$\vec{F}(x, y)$.

$$\|\vec{F}\|^2 = \frac{y^2 + x^2}{(x^2+y^2)^2} = \frac{1}{x^2+y^2}$$

(This example is important since the vector field satisfies

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial y} = \frac{y^2 - x^2}{(x^2+y^2)^2} \quad (x, y) \neq (0, 0).$$

but $\vec{F}$ is $\boxed{\text{not}}$ conservative. The problem arise because there is a hole $(0,0)$ where $\vec{F}$ is not defined)

- test for being conservative or not ← curl.
- Method to find potential function in both 2-dim and 3-dim situation.

9. (50pts)

Let $\vec{F}$ be the vector field

$$\vec{F} = \langle 2x + y - 1, x + 2y \rangle .$$

(a): Is $\vec{F}$ conservative or not?

$$\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x} \Rightarrow \vec{F} \text{ is conservative.}$$

(b): If yes, find a potential function $f(x, y)$.

need to solve:

$$\begin{cases} f_x = 2x+y-1 & ① \\ f_y = x+2y & ② \end{cases} \Rightarrow f = x^2+xy-x+g(y).$$

$$\Downarrow$$

$$f_y = x + g'(y) \overset{②}{=} x+2y$$

$$\Downarrow$$

$$g'(y) = 2y \Rightarrow g(y) = y^2 + C.$$

So $f(x,y) = x^2+xy-x+y^2$ is a potential function for $\vec{F}$.

• fundamental Theorem of line integrals for conservative vector fields ⇝ independence of paths

 • Classification of critical points for 2-variable function.

(c): A particle is moving from the point $(0,0)$ to a point (p,q) in a straight line. For which values of p, q is the work

$$\int_{(0,0)}^{(p,q)} \vec{F} \cdot d\vec{r}.$$

minimal?

Because $\vec{F}$ is conservative, by the fundamental Theorem of line integrals.

$$\int_{(0,0)}^{(P,q)} \underbrace{\vec{F}}_{\nabla f} \cdot d\vec{r} = f(P,q) - f(0,0)$$

$$= P^2 + Pq - P + q^2.$$

To find the minimal value, we calculate the critical pts of f by solving:

$$\begin{cases} f_P = 2P + q - 1 = 0 & ① \\ f_q = P + 2q = 0 & ② \Rightarrow P = -2q \overset{①}{\Rightarrow} 2\cdot(-2q) + q - 1 = 0 \end{cases}$$

$$\Downarrow$$

$$P = \frac{2}{3} \Leftarrow q = -\frac{1}{3}$$

so there is only one critical pt: $\left(+\frac{2}{3}, -\frac{1}{3}\right)$

we can classify this critical pt using the 2nd partial test:

$$Hess = \begin{pmatrix} f_{PP} & f_{Pq} \\ f_{qP} & f_{qq} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

- $f_{PP}f_{qq} - f_{Pq}^2 = 4 - 1 = 3 > 0$
- $f_{PP} = 2 > 0$

so $\left(\frac{2}{3}, -\frac{1}{3}\right)$ is a relative (local minimum). Because $\lim_{(P,q)\to\infty} f(P,q) = +\infty$, $\left(\frac{2}{3}, -\frac{1}{3}\right)$ is also an absolute minimum

In sum, $(P,q) = \left(\frac{2}{3}, -\frac{1}{3}\right)$ is the solution to the problem

- Green's theorem
- Double integral $\leftrightarrow$ line integral.

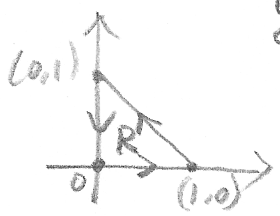
10. (50pts)

Use Green's theorem to calculate the work done by $\vec{F}$

$$\vec{F} = (4x^2 - 2y^2)\vec{i} + (2x^2 - 4y^2)\vec{j}.$$

in moving a particle once counterclockwise around the boundary of a triangle with vertex: $(0,0)$, $(1,0)$, $(0,1)$?

$R:\ x \geq 0,\ y \geq 0$
$y \leq 1-x.$



First, the work is calculated as the line integral of the vector field along the oriented curve.

$$\text{Work} = \oint_C \vec{F}\cdot d\vec{r} = \oint_C M\,dx + N\,dy. \quad \text{with } M(x,y) = 4x^2 - 2y^2$$
$$N(x,y) = 2x^2 - 4y^2$$

By Green's theorem,

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_R \left(\frac{\partial}{\partial x}(2x^2-4y^2) - \frac{\partial}{\partial y}(4x^2-2y^2)\right) dA$$

$$= \iint_R 4(x+y)\,dA = 4\int_0^1 \int_0^{1-x} (x+y)\,dy\,dx$$

$$= 4\int_0^1 \left[xy + \frac{y^2}{2}\right]_0^{1-x} dx = 4\int_0^1 \left(x(1-x) + \frac{(1-x)^2}{2}\right) dx$$

$$= 2\cdot\int_0^1 \left(2x - 2x^2 + 1 - 2x + x^2\right) dx = 2\int_0^1 (1-x^2)\,dx$$

$$= 2\cdot\left[x - \frac{x^3}{3}\right]_0^1 = \frac{4}{3}.$$