Stony Brook University Spring 2008

MAT203
Calculus III with Applications

10:40am-11:35am Heavy Engineering Lab 201

IMPORTANT ANNOUNCEMENTS:

- **BE SURE TO HIT THE RELOAD BUTTON ON YOUR BROWSER WHEN VIEWING THIS PAGE TO MAKE SURE YOU SEE THE MOST UPDATED INFORMATION.**
- A PRACTICE FINAL is now available,
- also solutions 1,2ab 2bc,3a 3b,4 5,6 7,8 9,10a 10ab
- FInal examination location - **Monday, May 19, 8:00AM-10:30AM - Old Engineering 145**
- **Review sessions!** Caner Koca will be holding review sessions May 6, 7, and 8 at 7pm in room P-131 in the mathematics building.
- **Extra Credit here.** Problems will be added to this link as they become available.
- Here are solutions to the practice midterm.
- On the practice midterm, number 11 has been adjusted to correct a typo. The new number 11 should be easier.
- A practice midterm is now available.
- Friday, Feb 22 - Attendance is optional. We will review.
- **Homework for section 11.7 has been pushed back to the following week. The two 11.6 homework problems are still due the week of Feb 18.**
- **A change of plans:** There will be only one midterm, not two, as stated earlier. The proportion of your grade corresponding to homework and exams has been updated.

MAT203 is a fun course! We'll begin by stepping out of that boring 2-dimensional blackboard and learn how to use mathematics to describe the 3-dimensional world we know and love. We'll apply what we already know about calculus to this new setting - nothing much new, just another dimension. All of a sudden we'll see that we can now describe a whole bunch of real world stuff like the best way to drive your math buggy accross wild rolling hills, compute the volume of a mineral to be mined underground, find out what's behind the capilary action of our veins, or predict how your rubber ducky will navigate a tub of turbulent water.

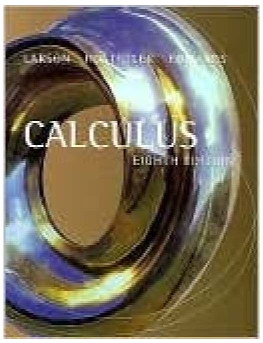
This is not a course about memorizing lots of tricks (like much of calculus II or calculus B). It's about being able to abstract what you already know to a new setting. Most solutions will come from your intuition, not a definite formula. If that intuition is ever missing or you ever become confused please seek help from your TA or myself. I genuinely want every student to get an A in this class, but you'll have to help me by getting us to help you.

Instructor: Andrew Bulawa
email: abulawa@math.sunysb.edu
Office: Math Buidling 2-121 - Monday 12:00-1:00
Math Learning Center: Math Building S240A - Wednesday 4:00-5:00, Friday 12:00-1:00

If you absolutely cannot meet me during office hours, send an email and we'll work something out.

Teaching Assistants:

- **Recitation 1, Tuesday 2:20-3:15:**
  Instructor: Luoying Wang
  email: lweng@math.sunysb.edu
  Office: Math Buidling 2-107. Hours posted shortly
- **Recitation 2, Wednesday 9:35-10:30:**
  Instructor: Caner Koca
  email: caner@math.sunysb.edu
  Office: Monday 1:00-2:00 Math 3-118, Tuesday-Thursday 11:20-12:20 in MLC (Math S-240A)



Text:
Multivariable Calculus by Larson, Hostetler, Edwards, 8th edition
The text is available at the university book store. Bring your textbook to class each day for lecture and recitation.

Grades:
35% Homework (posted below)
30% Midterm Exam
35% Final Exam

90-100 = A, 80-89 = B, 70-79 = C, 60-69 = D
There will be NO CURVE!

Class participation will have a positive effect on your grade!

Midterm Exam: Friday, March 14, 10:40AM-11:35AM in usual lecture room. The midterm will cover sections 11.1 through 13.4. Click here for a practice midterm.
Final Exam: Monday, May 19, 8:00AM-10:30AM - Old Engineering 145.

Homework:

- Homework is to be handed in to your recitation instructor.
- Your recitation instructor has been informed that **late homework is not to be accepted.** The only exception would be accompanied by a doctor's note and your instructor's permission.
- Homework spanning several sheets must be **stapled.**
- Homework must be organized and neat. Messy homework is not to be graded. The TAs have to grade a lot, so please make their lives as easy as possible. A happy grader is a kinder grader.
- Take pride in your work. I guarantee it will be reflected in your grade.

Homework listed below is subject to change. Please check often!
Deposit homework at your instructor's office.
Each assignment is **due the following week** unless otherwise specified.

Week	Topic	Notes	**Assignment:** Only turn in the **bold** exercises. Your TA will have instructions for handing them in.
1/28-2/1	11.1-11.2	It looks like a lot, but many are quick.	11.1 #5,21,22,33,39,41,49,53,63,**70**,75,83,85,**92** (for both parts assume the resultant force is zero) 11.2 #7,9,11,21,27,31,35,37,**40**,43,49,55,67,69,81,85,95,**98**,99,103-106,**114** (notice that |*BC*| is 18 ft, not 16 ft!)
2/4-2/8	11.3-11.5		11.3 #5,7,9,17,25,45,**50**,65,66,69,71,74,**80**,85 11.4 #7,13,17-20,**34**,37,45 11.5 #1,5,15,19,27,33,37,**42**,46,53,65,81,87,93,97,99,**112**
2/11-2/15	11.6-11.7	Only two problems due for this assignment.	11.6 #1-6,7,9,11,13,15,29,45,**50**,58,**62**
2/18-2/22	11.7, 12.2		11.7 #3,7,13,17,23,25,29,33,35,37,43,45,49,53,**54**,59,67,87-92,97,99,101,**102**,103,**114** 12.1 #1,9,13,15,**17-20**(1 pt each),33,46,59,72 12.2 #4,10,19,21,25,**28**,39,53,55,65
2/25-2/29	12.2-12.5	You are not required to study curvature in 12.5.	12.3 #1,5,15,17,**22**,28,35,49 12.4 #9,15,25,29,37,45-48,**50** 12.5 #9,**12**,17,19,**71(a)**,**86**,89,93-100 (be sure to convert pounds to mass in #86)
3/3-3/7	13.1-13.3,13.5		
3/10-3/14		**Midterm on Friday!** No homework will be collected this week.	
3/17-3/21		Spring Recess	
3/24-3/28	13.5-13.7	No homework will be collected this week. The following is for next week.	13.1 #17,45-48,55 13.2 #17,23 (consider what happens when (a)y=z=0 and (b)x=y and z=0),27,33 13.3 #13,23,35,57,61 13.5 #9,11,13,17,21,25,35,**54** 13.6 #1,5,11,15,21,23,37,39-46,**42** 13.7 #5,11,**16**,17,43,**46**,**47(a)**
3/31-4/4	13.8-13.10		13.8 #4,5,13,15,25,**28**,45,47,49,51,**56**,57 13.9 #7,9,19,**20** 13.10 #1,3,5,**10**,13,19,21,25,33,**39**
4/7-4/11	14.1-14.2		14.1 #1,5,9,13,21,23,33,51,**56**,**62** 14.2 #**2**,21,23,25,27,**28**,29,37,39,53,**61**
4/14-4/18	14.3-14.5		14.3 #5-8,9,**10**,13,15,17,21,23,**28**,**41** (Hint: you should find the area of a single petal and multiply by 3) 14.4 #7,11,**12**,**14**,15
4/21-4/25	14.6-14.8		14.6 #1,3,7,13,17,19,**22**,23,25,29,35,63,**66** 14.7 #1,3,5,9,11,**14**,19,**35** 14.8 #1,3,7,9,11,13,**20**

4/28-5/2	15.1-15.2		15.1 #1-6,9,13,15,23,27,**28**,35,**36**,41,**44**,53,55,57,**58**,63,75,76 15.2 #1,3,5,7,9,11,13,15,**16**,29,35,37,43
5/5-5/9	15.3,15.4		
5/12	Review		
5/19		Final Exam	

Disabilities:

If you have a physical, psychological, medical or learning disability that may impact on your ability to carry out assigned course work, please contact the staff in the Disabled Student Services office (DSS), Room 133 Humanities, 632-6748/TDD. DSS will review your concerns and determine, with you, what accommodations are necessary and appropriate. All information and documentation of disability is confidential.

MAT 203 SPRING 2008
PRACTICE FINAL EXAM

1. Consider the vector field $\vec{F}(x, y, z) = xy\vec{i} + z \ln x\vec{j} + xz^2\vec{k}$

(a) Compute Div $\vec{F}$.

(b) Compute Curl $\vec{F}$.

(c) Can you find a function f such that $\vec{\nabla} f = \vec{F}$?

2. Suppose a particle's postition in space is given by the vector valued function $\vec{r}(t) = 6t\vec{i} + 3\sqrt{2}t^2\vec{j} + 2t^3\vec{k}$. Assume the particle begins moving at time $t = 0$ and finishes at time $t = 2$.

(a) Compute the velocity, acceleration and speed of the particle at time t.

(b) Compute the distance the particle traveled.

(c) Suppose the particle travels through a force field given by $\vec{F}(x, y, z) = x\vec{i} + (1/\sqrt{2})\vec{j} + (x^2/z)\vec{k}$. Compute the work done by this field.

(d) Suppose the particle travels through a force field given by $\vec{G}(x, y, z) = y \sin z\vec{i} + x \sin z\vec{j} + xy \cos z\vec{k}$. Compute the work done by this field.

3. Find the points on the surface $z^2 = xy + 1$ which are closest to the origin

(a) without using lagrange mutlipliers.

(b) using the method of lagrange multipliers.

4. Evaluate $\iiint_R x\, dV$, where R is the region bounded by the paraboloid $x = 4y^2 + 4z^2$ and the plane $x = 4$.

5. Evaluate $\iiint_R x^2\, dV$, where R is the region bounded by the spheres $\rho = 1$, $\rho = 3$ and above the cone $\phi = \pi/4$.

6. Find the volume of the region bounded by the paraboloids $z = x^2 + y^2$ and $z = 36 - 3x^2 - 3y^2$.

7. Compute the surface area of the part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.

8. Evaluate $\iint_R xy\,dA$, where R is the region bounded by the lines $2x - y = 1$, $2x - y = -3$, $3x + y = 1$, and $3x + y = -2$.

9. Find the tangent plane to the surface $z = 3x^2 - 4y^2$ at the point $(1, 1, -1)$.

10. Consider the line integral $\int_C x\,dx - x^2y^2\,dy$, where C is the triangle with vertices (0,0), (1,1), and (0,1).

(a) Evaluate the integral directly.

(b) Evaluate the integral using Green's Theorem.

1 (a) $\operatorname{Div} \vec{F} = \vec{\nabla} \cdot \vec{F}$

$$= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(z \ln x) + \frac{\partial}{\partial z}(xz^2)$$

$$= y + 0 + 2xz = \underline{y + 2xz}$$

1 (b) $\operatorname{Curl} \vec{F} = \vec{\nabla} \times \vec{F}$

$$= \left[\frac{\partial}{\partial y}(xz^2) - \frac{\partial}{\partial z}(z \ln x)\right] \vec{i}$$

$$+ \left[\frac{\partial}{\partial z}(xy) - \frac{\partial}{\partial x}(xz^2)\right] \vec{j}$$

$$+ \left[\frac{\partial}{\partial x}(z \ln x) - \frac{\partial}{\partial y}(xy)\right] \vec{k}$$

$$= -\ln x \, \vec{i} - z^2 \vec{j} + \left(\frac{z}{x} - x\right) \vec{k}$$

1 (c) No such f can exist because $\operatorname{Curl} \vec{F} \neq 0$ in any open set.

2 (a) velocity $= r'(t) = 6\vec{i} + 6\sqrt{2}\,t\,\vec{j} + 6t^2\vec{k}$

acceleration $= r''(t) = 6\sqrt{2}\,\vec{j} + 12t\,\vec{k}$

speed $= \|r'(t)\| = (36 + 72t^2 + 36t^4)^{1/2}$

$= 6(1+t^2)$

2 (b) Distance = arclength $= \int_0^2 \|r'(t)\|\,dt$

$= \int_0^2 6(1+t^2)\,dt$

$= 28$

2(c) $\text{Work} = \int_0^2 \vec{F} \cdot d\vec{r}$

$$= \int_0^2 \vec{F}(\vec{r}(t)) \cdot r'(t)\, dt$$

$$= \int_0^2 \left\langle 6t, \frac{1}{\sqrt{2}}, \frac{18}{t} \right\rangle \cdot \left\langle 6, 6\sqrt{2}t, 6t^2 \right\rangle dt$$

$$= \int_0^2 36t + 6t + 108t \; dt = \int_0^2 150t\, dt$$

$$= \boxed{300}$$

2(d) $\vec{G} = \vec{\nabla} g$ where $g(x,y,z) = xy\sin z$.

$$\int_0^2 \vec{G} \cdot d\vec{r} = g(\vec{r}(2)) - g(\vec{r}(0))$$

$$= g(12, 12\sqrt{2}, 16) - g(0,0,0)$$

$$= \boxed{144\sqrt{2}\sin(16)}$$

3(a) We want to minimize the square of the distance to the origin, $f(x,y,z) = x^2+y^2+z^2$.

$$\left.\begin{aligned} f_x &= 2x+y \\ f_y &= 2y+x \end{aligned}\right\} \Rightarrow \text{critical points are } (0,0,\pm 1)$$

$f_{xx} = 2$, $f_{yy} = 2$, $f_{xy} = 1$

$d = f_{xx}f_{yy} - f_{xy}^2 = 3 > 0 \Rightarrow$ crit. pts are local mins.

The surface is closest to the origin at $(0,0,\pm 1)$.

3(b) The constraint is $g(x,y,z) = z^2 - xy = 1$

$$\vec{\nabla} f = 2x\,\vec{i} + 2y\,\vec{j} + 2z\,\vec{k}$$

$$\vec{\nabla} g = -y\,\vec{i} - x\,\vec{j} + 2z\,\vec{k}$$

$$\vec{\nabla} f = \lambda \vec{\nabla} g \implies \begin{aligned} 2x &= -\lambda y \\ 2y &= -\lambda x \\ 2z &= \lambda 2z \end{aligned}$$

So, $\lambda = 1$ and $x = y = 0$.

$z^2 = xy + 1 \implies z = \pm 1.$

Surface is closest to origin at $(0,0,\pm 1)$.

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4. You may either directly do the integral

$$\int_{-1}^{1} \int_{-\sqrt{\frac{1}{4}-z^2}}^{+\sqrt{\frac{1}{4}-z^2}} \int_{4y^2+4z^2}^{4} x \, dx\, dy\, dz$$

or notice that by rotating everything 90° this problem is equivalent to $\iiint_R z\,dV$ where R is bounded by $z = 4x^2+4y^2$ and the plane $z=4$.

Then

$$\iiint_R z\,dV = \int_0^1 \int_0^{2\pi} \int_{4r^2}^{4} rz\,dz\,d\theta\,dr$$

$$= \int_0^1 \int_0^{2\pi} 8r - 8r^5\,d\theta\,dr$$

$$= 16\pi \int_0^1 r - r^5\,dr = \boxed{\frac{16\pi}{3}}$$

5

$$\iiint_R x^2\,dV = \int_0^{2\pi}\int_0^{\pi/4}\int_1^3 (\rho\cos\theta\sin\phi)^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_1^3 \rho^4\cos^2\theta\sin^3\phi\,d\rho\,d\phi\,d\theta$$

$$= \frac{242}{5}\int_0^{2\pi}\int_0^{\pi/4}\cos^2\theta\sin^3\phi\,d\phi\,d\theta$$

$$= \frac{242}{5}\int_0^{2\pi}\int_0^{\pi/4}\cos^2\theta\,(1-\cos^2\phi)\sin\phi\,d\phi\,d\theta$$

$$= \frac{242}{5}\int_0^{2\pi}\cos^2\theta\left[\frac{1}{3}\cos^3\phi-\cos\phi\right]_0^{\pi/4}d\theta$$

$$= \frac{242}{5}\left(2^{-3/2}-2^{-1/2}\right)\int_0^{2\pi}\underbrace{\cos^2\theta}_{\frac{1}{2}(\cos 2\theta+1)}\,d\theta$$

$$= \frac{242}{5}\left(2^{-3/2}-2^{-1/2}\right)\pi$$

6

In cylindrical coords, these surfaces are $z = r^2$ and $z = 36 - 3r^2$. They intersect when $r = 3$.

$$V = \int_0^{2\pi} \int_0^{3} \int_{r^2}^{36-3r^2} r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^{3} 36r - 4r^3 \, dr \, d\theta$$

$$= \int_0^{2\pi} 81 \, d\theta$$

$$= 162\pi .$$

7

$$z_x = y$$
$$z_y = x$$

$$S = \iint_R \sqrt{1 + x^2 + y^2}\, dA \quad , \qquad R = \{(x,y) \mid x^2 + y^2 \le 1\}$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{1+r^2}\, r\, dr\, d\theta \quad = \int_0^{2\pi} \frac{2^{3/2} - 1}{3}\, d\theta$$

$$= \frac{2^{5/2} - 2}{3}\pi$$

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8 Put $u = 2x - y$, $v = 3x + y$ $\Longrightarrow$ $x = \frac{1}{5}(u + v)$, $y = \frac{1}{5}(2v - 3u)$

$$\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \frac{1}{5}\cdot\frac{2}{5} - \frac{1}{5}\cdot\left(\frac{-3}{5}\right) = \frac{1}{5}$$

$$\iint_R xy\, dA = \int_{-2} \int_{-3} \tfrac{1}{5}(u+v)\, \tfrac{1}{5}(2v-3u) \cdot \left|\tfrac{1}{5}\right| du\, dv$$

$$= \frac{1}{125} \int_{-2}^{1} \int_{-3}^{1} 2v^2 - 3u^2 - uv \; du\, dv$$

$$= \frac{1}{125} \int_{-2}^{1} 8v^2 + 4v - 28 \; dv$$

$$= \cancel{\frac{4}{125}} = \frac{-66}{125}$$

9 Put $f(x,y,z) = 3x^2 - 4y^2 - z$

$$\vec{\nabla} f = 6x\vec{i} - 8y\vec{j} - \vec{k}$$

$$\vec{\nabla} f(1,1,-1) = 6\vec{i} - 8\vec{j} - \vec{k} = \vec{N}$$

$(6,-8,-1)$

$$\vec{N}\cdot(x,y,z) = (1,1,-1)\cdot(6,-8,-1)$$

$$\boxed{6x - 8y - z = -1}$$

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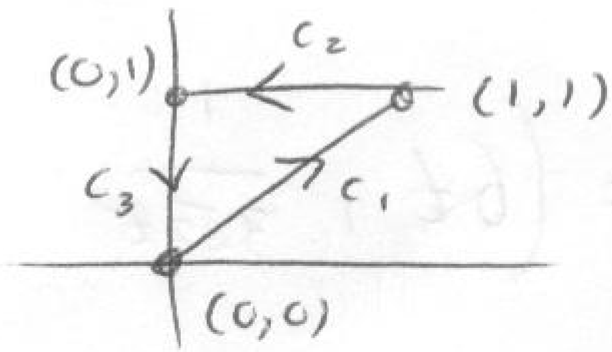
10 Parameterize the 3 segments:

$\vec{r}_1(t) = (t,t) \quad 0 \le t \le 1$

$\vec{r}_2(t) = (1-t,1) \quad 0 \le t \le 1$

$\vec{r}_3(t) = (0,1-t) \quad 0 \le t \le 1$



C_1 : $x = t \quad dx = dt$
$\ y = t \quad dy = dt$

$$\int x\,dx - x^2y^2\,dy = \int^1 t\,dt - t^2t^2\,dt$$

$$\int_{C_1} x\,dx - x^2y^2\,dy = \int_0^1$$

$$= \int_0^1 t - t^4\,dt = \frac{3}{10}$$

C_2: $x = 1 - t \qquad dx = -dt$

$y = 1 \qquad dy = 0\,dt$

$$\int_{C_2} x\,dx - x^2y^2\,dy = \int_0^1 (1-t)(-dt) = -\frac{1}{2}$$

C_3: $x = 0 \qquad dx = 0\,dt$

$y = 1 - t \qquad dy = -dt$

$$\int x\,dx - x^2y^2\,dy = 0$$

10 (a) $\displaystyle\int_C x\,dx - x^2y^2\,dy$

$$= \int_{C_1} + \int_{C_2} + \int_{C_3}$$

$$= \frac{3}{10} - \frac{1}{2} + 0 = \boxed{\frac{-1}{5}}$$

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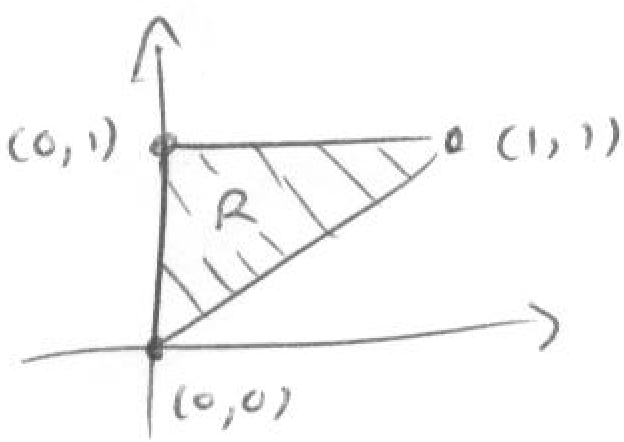
10 (b) $\displaystyle\int_C \overset{M}{\overset{\downarrow}{x}}\,dx \overset{N}{\overset{\downarrow}{-x^2y^2}}\,dy = \iint_R N_x - M_y \; dA$



$$= \iint_R -2xy^2 - 0 \; dA$$

$$= \int_0^1 \int_0^y -2xy^2 \; dx\,dy$$

$$= \int_0^1 -y^4 \; dy$$

$$\boxed{= -\frac{1}{5}}$$

Each extra credit problem will be worth one full homework.

1. (Due in lecture Friday, 4/4) no.71 on page 877, parts (a) and (b)

2. (Due in lecture Friday, 4/11) Suppose the altitude of a hill is given by $f(x, y) = x^2y - 2xy + 5$ corresponding to coordinates (x, y). If you begin at coordinates (2,1) and ascend the hill always hiking in the steepest direction, find a curve in the x,y-plane describing your path as seen on a map. Express the curve as an expression in x and y.

Hint: Temporarily express your path as a parametric curve given by $x(t)$ and $y(t)$. Use the chain rule

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

to find a differential equation involving y and x. Solve this differential equation.

3. (Due in lecture Friday, 5/2) section 14.4 no.54.

MAT 203 Partial Solutions to Practice Midterm

(1) Let $\vec{u} =< 1, 1, 1 >$ and $\vec{v} = -3\vec{i} + 2\vec{j} + 4\vec{k}$. Compute the following:
   (a) $\vec{u} \cdot \vec{v}$
   (b) $\vec{u} \times \vec{v}$
   (c) $||\vec{u}||$ and $||\vec{v}||$
   (d) Find the projection of $\vec{u}$ onto $\vec{v}$.
   (e) Compute the angle between $\vec{u}$ and $\vec{v}$.

   Answer: For (a),(b),(c), just apply definitions. For (d),

$$Pro_{\vec{v}}\vec{u} = \frac{\vec{u} \cdot \vec{v}}{||\vec{v}||^2}\vec{v},$$

   for (e), let θ be the angle, then

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{v}||||\vec{u}||}.$$

(2) How close does the line with direction vector $\vec{i} - \vec{j} + \vec{k}$ through the point $(4, 7, -2)$ come to the origin?

   Answer: It is about finding a distance between a point and a line. The length of projection of $< 4, 7, -2 >$ onto $< 1, -1, 1 >$ is $a = \frac{<1,-1,1>\cdot<4,7,-2>}{||<1,-1,1>||}$, set $b = || < 4, 7, -2 > ||$, then the distance is $\sqrt{b^2 - a^2}$.

(3) (a) Convert the point $(5/2, 4/3, -3/2)$ from rectangular to cylindrical and spherical coordinates.
   (b) Convert the point $(5, 3\pi/4, -5)$ from cylindrical to rectangular and spherical coordinates.
   (c) Convert the point $(4, 2\pi/3, \pi/4)$ from spherical to rectangular and cylindrical coordinates.

   Answer: (a)From rectangular coordinates to spherical coordinates:

$$r_s = \sqrt{x^2 + y^2 + z^2}, \tan\theta = \frac{y}{x}, \cos\phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

   where r_s is the distance between the point to the origin, which is also the first coordinate of spherical coordinates, ϕ is the angle with the positive z-axis, θ is angle with the positive x-axis. To cylindrical coordinates:

$$r_c = \sqrt{x^2 + y^2}, \tan\theta = \frac{y}{x}$$

   keeping in mind that the last coordinate of cylindrical coordinate is the same as that of rectangular coordinates.

   (b)From cylindrical to rectangular:

$$x = r_c \cos\theta, y = r_c \sin\theta.$$

where r_c is the first coordinate of cylindrical coordinates.
From cylindrical to spherical:

$$r_s = \sqrt{r_c^2 + z^2}, \cos\phi = \frac{z}{r_s}$$

keeping in mind their θ coordinates are the same.

(c)From spherical coordinates to rectangular coordinates, we can use the following relations

$$x = r_s \sin\phi\cos\theta, y = r_s \sin\phi\sin\theta, z = r_s\cos\phi.$$

From spherical coordinates to cylindrical:

$$r_c = \sqrt{r_s^2 - z^2}, z = r_s\cos\phi$$

(4) The lines given by

$$\vec{L_1}(t) = (t+1)\vec{i} + (t-1)\vec{j} + t\vec{k}$$

$$\vec{L_2}(t) = 2\vec{i} - 3t\vec{j} + (t+1)\vec{k}$$

lie in a plane. Write an equation for this plane.

Answer: Find the intersection point (x_0, y_0, z_0) of these two lines by $\vec{L_1}(t) = \vec{L_2}(t)$. Observe that $< 1, 1, 1 >$ is the vector parallel to $\vec{L_1}(t)$, and that $< 0, -3, 1 >$ is the one parallel to $\vec{L_2}(t)$, then $< 1, 1, 1 > \times < 0, -3, 1 >$ gives an orthogonal vector, say $\vec{n}$, to this two lines hence to the plane. Then the equation for this plane is given by

$$< x - x_0, y - y_0, z - z_0 > \cdot \vec{n} = 0$$

(5) Write the equation of a sphere with center $(4, -4, 12)$ and tangent to the plane $x = y + 1$.

Answer: Let $Q = (1, 0, 0)$ be a point on the plane, $P = (4, -4, 12)$, $\vec{n} =< 1, -1, 0 >$ be the normal vector the plane, then the distance between the center and the plane is given by

$$r = \frac{\vec{PQ} \cdot \vec{n}}{||\vec{n}||}$$

hence the equation of the sphere is given by

$$(x-4)^2 + (y+4)^2 + (z-12)^2 = r^2$$

(6) Omitted.
(7) Omitted.

(8) The surfaces given by

$$x^2 - 5y^2 + 2z^2 + 2y = 3$$
$$3x^2 - 15y^2 + 6z^2 + 4z = 7$$

intersects at a curve. Does this curve lie in a plane? Which plane?

Answer: To find the intersecting curve, we set the first equation by (1), and second by (2), then (2)-3(1)= $4z - 6y = -2$, which gives a relation between z and y, namely, $z = 3/2y - 1/2$. Now replace z in the equation (1) by $3/2y - 1/2$, you can get a equation only in x and y, namely,

$$2x^2 - y^2 - 2y = 5$$

which is a curve lying in $z = 3/2y - 1/2$ plane.

(9) A piece of driftwood floats in the ocean. It is carried by a current flowing due north at a speed of $1.8km/hr$. Just as the driftwood passes by a stationary buoy, a strong breeze begins blowing northwest accelerating the driftwood at a rate of $0.25m/s^2$. Find a vector valued function describing the location of the driftwood t seconds after it passes the buoy.

Answer: The location of the driftwood consists of two coordinates $x(t)$ and $y(t)$. To find these two coordinates, we need to consider the displacement of each direction, namely north and west. We know that there is an acceleration on northwest, it can be decomposed into two direction, with $\sqrt{2}/2 \times 0.25$ for each direction. So

$$x(t) = \int_0^t -\sqrt{2}/2 \times 0.25tdt = -\sqrt{2}/16t^2,$$

$$y(t) = \int_0^t \sqrt{2}/2 \times 0.25tdt - 1800/3600t = \sqrt{2}/16t^2 + 1800/3600t$$

$x(t)$ is negative since we consider east and north as positive. Hence the location of the driftwood at t is

$$r(t) = (-\sqrt{2}/16t^2, \sqrt{2}/16t^2 + 1800/3600t)$$

(10) Find the unit tangent $\vec{T}$ and unit normal $\vec{N}$ at $t = 0$ for the curve defined by the vector valued function $\vec{r}(t) = 5t\vec{i} + e^t\vec{j} + e^{-t}\vec{k}$.

Answer: $\vec{r'}(t) = 5\vec{i} + e^t\vec{j} - e^{-t}\vec{k}$, $||\vec{r'}(t)|| = \sqrt{25 + e^{2t} + e^{-2t}}$.

$$\vec{T}(t) = \frac{\vec{r'}(t)}{||\vec{r'}(t)||}$$

$$\vec{T} = \vec{T}(0) = \frac{< 5, 1, -1 >}{3\sqrt{3}}$$

$$\vec{N} = \frac{\vec{T}'(0)}{||\vec{T}'(0)||}.$$

(11) Find the arc length of the curve given by $\vec{r}(t) = t^3\vec{i} + \cos(6t^2)\vec{j} + \sin(6t^2)\vec{k}$ between $\vec{r}(0)$ and $\vec{r}(5)$.

Answer: $x(t) = t^3, y(t) = \cos(6t^2), z(t) = \sin(6t^2)$, then

$$\begin{aligned} L &= \int_0^5 \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}dt \\ &= \int_0^5 \sqrt{9t^4 + 144t^2}dt \\ &= \int_0^5 3t\sqrt{t^2+16}dt \\ &= \int_0^{25} 3/2\sqrt{u+16}du \\ &= 41^{3/2} - 64 \end{aligned}$$

Hence the arc length of the curve is $(41^{3/2} - 64)$.

(12) Find and plot the domain of the function

$$f(x,y) = \frac{\sqrt{x-3}}{x^2 - y}$$

Answer: The domain of this function consists of two parts:

$$x \leq 3, x^2 \neq y$$

hence is $\{x \leq 3, y \neq 9\}$.

(13) Consider the following functions
(a) $f(x,y) = \sqrt{x^2+y^2}$
(b) $f(x,y) = x^2 + y^2 - 3$
(c) $f(x,y) = \arctan x$
(d) $f(x,y) = -y^2e^{-x}$
Graph them and compute

$$f_x(x,y), f_y(x,y), f_{xy}(x,y)$$

. Where are these functions differentiable?

Answer: (a)

$$f_x = \frac{x}{\sqrt{x^2+y^2}}, f_y = \frac{y}{\sqrt{x^2+y^2}}, f_{xy} = \frac{1 + x \cdot y}{\sqrt{x^2+y^2}},$$

these functions are differential everywhere except the origin $(0,0)$, since only f_x, f_y, f_{xy} are continuous everywhere except the origin.

(b) Just take derivatives, and all these functions are differentiable everywhere.
(c)

$$f_x = \frac{1}{\sqrt{x^2+1}}, f_y = f_{xy} = 0.$$

All are differentiable everywhere.
(d)

$$f_x = y^2 e^{-x}, f_y = -2ye^{-x}, f_{xy} = 2ye^{-x},$$

all these functions are differentiable everywhere.

1. Let $\vec{u} =< 1, 1, 1 >$ and $\vec{v} = -3\vec{i} + 2\vec{j} + 4\vec{k}$. Compute the following:

(a) $\vec{u} \cdot \vec{v}$

(b) $\vec{u} \times \vec{v}$

(c) $\|\vec{u}\|$ and $\|\vec{v}\|$

(d) Find the projection of $\vec{u}$ onto $\vec{v}$.

(e) Compute the angle between $\vec{u}$ and $\vec{v}$

2. How close does the line with direction vector $\vec{i} - \vec{j} + \vec{k}$ through the point $(4, 7, -2)$ come to the origin?

3. (a) Convert the point $(5/2, 4/3, -3/2)$ from rectangular to cylindrical and spherical coordinates.

(b) Convert the point $(5, 3\pi/4, -5)$ from cylindrical to rectangular and spherical coordinates.

(c) Convert the point $(4, 2\pi/3, \pi/4)$ from spherical to rectangular and cylindrical coordinates.

4. The lines given by

$$\vec{L_1}(t) = (t+1)\vec{i} + (t-1)\vec{j} + t\vec{k}$$
$$\vec{L_2}(t) = 2\vec{i} - 3t\vec{j} + (t+1)\vec{k}$$

lie in a plane. Write an equation for this plane.

5. Write the equation of a sphere with center $(4, -4, 12)$ and tangent to the plane $x = y + 1$.

6. Graph the surfaces defined by the following equations.

(a) $4x^2 + 9y^2 - 36z = 36$

(b) $3x^2 - y^2 - z = 0$ (you might try setting $z = 1$)

(c) $x^2 + z^2 - e^{2y} = 0$

(d) $\theta = \frac{\pi}{3}$

7. Plot the images of the following vector valued functions.

(a) $\vec{r}(t) = 4\cos t\vec{i} + 4\sin t\vec{j} + t\vec{k}$

(b) $\vec{r}(t) = -2t\vec{i} + t\vec{j} + 3t\vec{k}$

(c) $\vec{r}(t) = \cos(2t)\vec{i} + \sin t\vec{j} + t\vec{k}$

(d) $\vec{r}(t) = t\vec{i} + \ln t\vec{j}$

8. The surfaces given by

$$x^2 - 5y^2 + 2z^2 + 2y = 3$$
$$3x^2 - 15y^2 + 6z^2 + 4z = 7$$

instersect at a curve. Does this curve lie in a plane? Which plane?

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10. Find the unit tangent $\vec{T}$ and unit normal $\vec{N}$ at $t = 0$ for the curve defined by the vector valued function

$$\vec{r}(t) = 5t\vec{i} + e^t\vec{j} + e^{-t}\vec{k}.$$

11. Find the arc length of the curve given by $\vec{r}(t) = t^3\vec{i} + \cos(6t^2)\vec{j} + \sin(6t^2)\vec{k}$ between $\vec{r}(0)$ and $\vec{r}(3)$.

12. Find and plot the domain of the function

$$f(x, y) = \frac{\sqrt{x-3}}{x^2 - y}$$

13. Consider the following functions

(a) $f(x, y) = \sqrt{x^2 + y^2}$

(b) $f(x, y) = x^2 + y^2 - 3$

(c) $f(x, y) = \arctan x$

(d) $f(x, y) = -y^2e^{-x}$

Graph them and compute $f_x(x, y)$, $f_y(x, y)$, $f_{xy}(x, y)$. Where are these functions differentiable?