MAT 200 Language, Logic, and Proof, Spring 2015

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- **Class meetings:** MoWe 4.00-5.20pm, Physics P113

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- **TA MLC hours:** Wednesday 5-6 and Thursday 2-3.
- **TA Office hours:** Wednesday and Thursday 3-3:30.

Sean Farrell and Gustavo Poscidonio will lead an informal MAT 200 recitation: Physics, P-113 on Fridays, Spring '15 from 12:00 PM to 1:30 PM. Contact email: sean.farell(at)stonybrook.edu.

Please note the room change to P113. To address the Monday cancellations Friday March 6th and Friday March 27th will follow a Monday schedule. All Monday classes will meet in their assigned rooms at the assigned times on these Fridays.

Here is a practice final by Julia Viro.

We have Midterm II in class on April 13. The midterm will cover Chapters 7-12 of the textbook.

We have Midterm I in class on March 9. See the Practice Midterm 1. There is going to be a review session on Saturday, March 7th from 10am till 12am in Library room W4550 by Julia Viro.

Homeworks:

Homework 1 due Feb 4. Reading assignment: pp 3-29 of the textbook.

Homework 2 due Feb 11. Reading assignment: pp 30-57 of the textbook.

Homework 3 due Feb 18. Reading assignment: pp 61-88 of the textbook.

Homework 4 due Feb 25. Reading assignment: pp 89-119 of the textbook.

Homework 5 due March 4. Reading assignment: pp 123-143 of the textbook.

Homework 6 due April 1. Reading assignment: pp 144-169 of the textbook.

Homework 7 due April 8. Reading assignment: pp 144-169 of the textbook.

Homework 8 due April 29. Reading assignment: pp 170-187 of the textbook.

Class Notes:

Class 1. We have discussed the overall structure of the course as well as the following problems. Reading assignment: pp 3-29 of the textbook. Highly recommended reading: The Game of Logic by Lewis Carroll.

Class 2. We have discussed the Pigeonhole Principle as well as the following problems. Highly recommended reading: Seven Puzzles You Think You Must Not Have Heard Correctly by Peter Winkler.

Class 3. We have solved the following problems using the Pigeonhole Principle .

Class 4. We have discussed Mathematical Induction and solved the following problem. Here is another example of a proof by induction.

Class 5. We have talked more on Mathematical Induction and solved the following problems.

Class 6. We have talked about quantifiers and solved the following problem.

Class 7. We have talked about Cartesian Products and solved the following problem.

Class 8. We have talked about Functions, Injections and Surjections and solved the following problem.

Classes 9-11. We have discussed Peano axioms. See section 9.4 of the textbook.

Classes 12-14. We have discussed properties of finite sets. See chapters 10 and 11 of the textbook.

Classes 15-17. We have discussed properties of infinite sets. See chapter 14 of the textbook.

Homework is a compulsory part of the course. Homework assignments are due each week at the beginning of the wednesday class. Under no circumstances will late homework be accepted.

Grading system: The final grade is the weighted average according the following weights: homework 10%, Midterm1 25%, Midterm2 25%, Final 40%.

Textbook: *An Introduction to Mathematical Reasoning,* Author: Peter J. Eccles

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Stony Brook University
Mathematics Department
Julia Viro

MAT 200, Lec 01
Logic, Language and Proof
Spring 2015

Practice Final Exam

1. The *Well-Ordering Principle* says that any non-empty subset of natural numbers has a least element. (This principle is equivalent to the principle of mathematical induction.) Give a symbolic form of this definition. Formulate its denial, both in words and symbols.

2. Let $X \subseteq \mathbb{R}$. The *supremum* (or the *least upper bound*) of X is the least real number that is greater or equal to every element of X. Notation: $\sup X$. Give a symbolic form of this definition. Formulate its denial, both in words and symbols.

3. Find the product

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\dots\left(1-\frac{1}{n^2}\right), \quad \text{where} \quad n \geq 2.$$

Hint: calculate several products for small values of n, guess the formula, and prove it by induction.

4. Prove that the Fibonacci numbers satisfy the inequality $a_n < \left(\frac{7}{4}\right)^{n-1}$ for all natural numbers $n \geq 2$.

5. Welcome to the club of the trigonometry lovers! Prove that $\cos x \cos 2x \dots \cos(2^n x) = \dfrac{\sin(2^{n+1}x)}{2^{n+1}\sin x}$ for any $n \in \mathbb{N}$. (If you are not a trig fan, the following formula may help: $\sin 2x = 2\sin x \cos x$.)

6. Prove that if $x + \dfrac{1}{x} \in \mathbb{Z}$ then $x^n + \dfrac{1}{x^n} \in \mathbb{Z}$ for any $n \in \mathbb{Z}$.

7. Prove that for any sets A, B and C,

$$A \setminus (B \cup C) = (A \setminus B) \setminus C \quad \text{and} \quad A \times (B \cap C) = (A \times B) \cap (A \times C).$$

8. Let $X = [-1, 2]$, $Y = [0, 1]$ and $Z = [2, 3] \cup [4, 5]$. What is $X \times Y \times Z$? Draw a picture!

9. Let $X = \{(x, y) \in \mathbb{R}^2 \mid |x| + |y| \leq 1\}$ and $Y = \{x \in \mathbb{R} \mid (x-2)(x-3)(x-4)(x-5) \leq 0\} \subseteq \mathbb{R}$. What is $X \times Y$? Draw a picture!

10. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by $f(x, y) = x + \sin y$. Is f injective (one-to-one)? Is f surjective (onto)? Is f bijective? Explain! Find the image of the set $[-3, 4] \times [\pi/6,\ 5\pi/6]$.

11. Construct a map $f : \mathbb{Z} \to \mathbb{Z}$ which is

a) injective, but not surjective
b) surjective, but not injective
c) bijective (but is not the identity map)

12. Let $g \circ f$ be an bijection. Show that f is injective and g is surjective.

13. Prove that if $g \circ f$ is injective and f is surjective then g is injective.

14. Prove that a composition of two decreasing functions is an increasing function.

15. Let $\sim$ be a relation on the set of all functions defined on the interval $[0,1]$ given by

$$f \sim g \iff f(0) = g(0) \text{ and } f(1) = g(1).$$

Is $\sim$ an equivalence relations? Justify you answer.

16. A relation on a set is called a *partial order* if it is reflexive, antisymmetric, and transitive. Show that the relation R on $\mathbb{N}$ given by aRb iff $b = 2^k a$ for some non-negative integer k is a partial order.

17. There are five clubs on a campus. Show that among 65 students in a class there are at least three students participating in the same variety of clubs.

18. Suppose 19 distinct integers are chosen from the set

$$\{1, 4, 7, \dots, 100\} = \{n \in \mathbb{Z} \mid n = 3k + 1 \text{ where } 0 \le k \le 33\}.$$

Prove that there exist two distinct integers among the chosen 19 whose sum is 104.

19. Prove that there exist distinct positive integers n and m such that $2^n - 2^m$ is divisible by 2015.

20. Find all values of a parameter a for which the function $f(x) = \sin(2x + a)$ is **a)** even **b)** odd. You may find the following formulas helpful: $\sin x \pm \sin y = 2 \sin \dfrac{x \pm y}{2} \cos \dfrac{x \mp y}{2}$.

21. Give the definition of a surjective function. Give the definition of a strictly decreasing function. Give the definition of an invertible function. Use these definitions to prove that a surjective strictly decreasing function is invertible.

22. Formulate Cantor-Schröder-Bernstein theorem. Use it to find the cardinality of the set $\mathbb{R} \setminus [-1, 1)$.

23. Find the cardinality of the set $\{n^3 - n \mid n \in \mathbb{Z}\}$. Justify your answer!

24. Let a set X has three elements and a Y has four elements. Let $\mathcal{M}$ be the set of all maps from X to Y. Find $|\mathcal{P}(\mathcal{M})|$.

25. Find the cardinalities of the following sets

$$\mathbb{Q} \cup \{\sqrt{2}, \sqrt{3}\},\ \ \mathbb{Q} \cup (0,1),\ \ \mathbb{R} \setminus \{\pi\},\ \ \mathbb{R} \setminus (0,1),\ \ \mathbb{R} \setminus \mathbb{Z},\ \ \mathbb{Z} \times \mathbb{Z}.$$

Justify your answers!

26. Let $A_n = \mathbb{R} \setminus [-n, n]$, where $n \in \mathbb{N}$. Find the cardinalities of the sets $\bigcup\limits_{n \in \mathbb{N}} A_n$ and $\bigcap\limits_{n \in \mathbb{N}} A_n$.

27. Arrange the following cardinal numbers in an ascending order:

$$|\{1,\ 2\}|,\ \ |(1,2)|,\ \ |[1,2]|,\ \ |\mathcal{P}(\mathbb{R})|,\ \ |\varnothing|,\ \ \aleph_{\mathbf{0}},\ \ |\mathbb{R} \setminus \mathbb{N}|,\ \ |\mathcal{P}(\mathbb{N})|,\ \ |\mathcal{P}(\mathcal{P}(\mathbb{R}))|,\ \ \mathbf{c}.$$

Stony Brook University
Mathematics Department
Nikita Selinger

MAT 200, Lec 02
Logic, Language and Proof
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Practice Midterm 1

This Practice Midterm is given to show what kind of problems you may expect on the exam. Solutions will **not** be given. Why? In order not to keep you busy with reading the specific solutions. The best way to effectively prepare for the exam is to read your lecture notes and the textbook, rather than the solutions. Make sure that you understand what you read. Revisit your homeworks. When you will feel comfortable with the material, do the practice problems. Actual Midterm 1 will be shorter and easier.

1. True or false? Explain! (a, x are real numbers).

a) $\forall a\, \exists x\ (x^2 + ax - 1 = 0)$ **b)** $\exists x\, \forall a\ (x^2 + ax - 1 = 0)$ **c)** $\forall x\, \exists a\ (x^2 + ax - 1 = 0)$.

2. Ken Olsen, CEO of *Digital Equipment*, claimed in 1977:

"There is no reason for any individual to have a computer in his home"

Give a symbolic writing of this phrase. You have to describe the universes, introduce appropriate notations and use quantifiers.

3. Use ε-δ definition to prove that $\lim\limits_{x \to 1}(5 - 2x) = 3$.

4. A real number L is called a *limit* of a sequence $\{a_n\}_{n=1}^{\infty}$ if for any positive real number ε there exists a positive integer N such that for all integers n greater than N the inequality $|a_n - L| < \varepsilon$ holds true. Notation: $L = \lim\limits_{n \to \infty} a_n$.

a) Rewrite this definition in a symbolic form.

b) Explain what it means that a number L is **not** the limit of the sequence $\{a_n\}_{n=1}^{\infty}$.

c) Use this definition to prove that $\lim\limits_{n \to \infty} \dfrac{1}{n} = 0$.

4. Give definition of a prime number, both in words and in symbols. Give definition of a composite number, both in words and in symbols. Remember that 1 is neither prime nor composite.

5. Give definition of an even function. Give definition of an odd function. What does it mean that a function is neither even nor odd? (Give a symbolic description.) For each of the following functions, use the definitions to determine whether it is even or odd or neither: $f(x) = x \sin x$, $g(x) = x^3 + 1$, $h(x) = e^x - e^{-x}$.

6. Let n be an integer. Prove that if $2015n$ is odd then n is odd.

7. Prove or give a counterexample:

a) There do not exist three consecutive integers a, b, c such that $a^2 + b^2 = c^2$.

b) There do not exist three consecutive even integers a, b, c such that $a^2 + b^2 = c^2$.

c) There do not exist three consecutive odd integers a, b, c such that $a^2 + b^2 = c^2$.

8. Prove that for all real numbers x, $\dfrac{3|x-2|}{x} \leq 4$ whenever $x \geq 1$.

9. Formulate and prove the triangle inequality.

10. In order to prove that $\sqrt{2013} + \sqrt{2015} < 2\sqrt{2014}$, you may think about more general statement, from which the inequality above follows. Formulate and prove this general statement (be careful with the details). Then get the inequality above as a special case.

11. For Fibonacci sequence a_n, prove the following:

a) $a_1 + a_3 + \cdots + a_{2n-1} = a_{2n}$

b) $\displaystyle\sum_{i=1}^{n} i a_i = n a_{n+2} - a_{n+3} + 2.$

12. Use induction to prove that

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 1 - \frac{1}{n} \qquad \text{for } n \geq 2.$$

Can you give a geometric interpretation of the inequality? Can you prove the inequality analytically (using calculus methods)?

13. Prove statements which are true and give counterexamples for those which are false.

a) $A = B \cup C$ is sufficient for $A \setminus B \subset C$.
b) $A = B \cup C$ is necessary for $A \setminus B \subset C$.

MAT200, Spring 2015. Homework 1.
Due on February 4, before class.

1. Formulate the contrapositive to the statements:

i. If you want to test a man's character give him power.

ii. Until you're ready to look foolish, you'll never have the possibility of being great.

iii. Whether you think you can, or think you can't, you're probably right.

2. By using truth tables prove that, for all statements P and Q, the following statements are logically equivalent.

(i) $P \Rightarrow Q$

(ii) $(P \text{ or } Q) \Leftrightarrow Q$

(ii) $(P \text{ and } Q) \Leftrightarrow P$

3. A grandmother prepared bowls with fruits for her family. All but five bowls are with cherries, all but four are with apricots, all but three are with red currants. How many bowls were at the dinner?

4. The four teletubbies organized a hugging competition with their guardian Noo-Noo. They reported the following results on the show:

Tinky Winky. (1) Po got the second place, (2) and I was third.
Dipsy. (3) I was the best hugger, (4) and Laa-Laa was second.
Laa-Laa. (5) I was third, (6) and Dipsy was the worst.
Po. (7) I was second, (8) and Noo-noo tried very hard but was only fourth.
Noo-noo. (9) I did better than only one teletubby, (10) and Tinky Winky won the competition.

All the kids were quite perplexed, so on the next episode the teletubbies confessed that each of them had made one true and one false statement. Let us help the kids figure out the results of the competition.

5. A donkey must transport 900 carrots to the market, which is 300 miles away. The donkey carries a maximum of 300 carrots, and eats 1 carrot every mile. What is the largest number of carrots that can be delivered to the market?

MAT200, Spring 2015. Homework 2.
Due on February 11, before class.

1. Prove by contradiction: $\sqrt{2}$ is an irrational number, i.e. it is not equal to a ratio of two integers.

2. Formulate negations to the following (series of) statements.

(i) How wonderful it is that nobody need wait a single moment before starting to improve the world.

(ii) We are all in the gutter, but some of us are looking at the stars.

(iii) The reasonable man adapts himself to the world: the unreasonable one persists in trying to adapt the world to himself. Therefore all progress depends on the unreasonable man.

(iv) If you look for perfection, you'll never be content.

(v) The man that hath no music in himself, nor is not moved with concord of sweet sounds, is fit for treasons, stratagems and spoils.

(vi) That which we call a rose by any other name would smell as sweet.

(vii) There's place and means for every man alive.

(viii) If there existed no external means for dimming their consciences, one-half of the men would at once shoot themselves, because to live contrary to one's reason is a most intolerable state, and all men of our time are in such a state.

(ix) All happy families resemble one another, each unhappy family is unhappy in its own way.

(x) For every good reason there is to lie, there is a better reason to tell the truth.

3. What is the maximal possible number of different Stony Brook ID numbers that can coexist at the same time? Every ID consists of 9 digits '0' to '9' and two different ID's have to be different in at least 2 digits.

4. Noo-noo had to clean a huge mess today. Someone broke a bowl of pudding in the teletubbies' kitchen. He tried to figure out what happened and this is what teletubbies had to say.

Tinky Winky.
(1) It was not me.
(2) When Dipsy came into the kitchen, Po was already there.
(3) Po came into the kitchen after me.

Dipsy.
(4) It was not me.
(5) When I came into the kitchen, Tinky Winky was already there.
(6) Po came into the kitchen the last.
Laa-Laa.
(7) It was not me.
(8) Both Tinky Winky and Dipsy are lying.
(9) When I came into the kitchen, Po was already there.
Po.
(10) It was not me.
(11) I was the last one to enter the kitchen.
(12) Dipsy came into the kitchen before me.

After some further investigation, Noo-noo figured out that only the last teletubby to enter the kitchen gave three truthful statements. The other three decided that it is okay to lie one time out of three. Who broke the bowl?

5. Prove that out of a sequence of n^2 pairwise different real numbers one can always chose a monotone (either increasing or decreasing) subsequence with at least n numbers. *Hint:* For every element of the sequence, consider a pair (x, y) where x is the length of the longest increasing sequence starting at this element and y is the length of the longest decreasing sequence starting at this element. Apply the Pigeonhole Principle.

MAT200, Spring 2015. Homework 3.
Due on February 18, before class.

1. Make a logical conclusion out of two premises whenever possible:

i.
- All my friends have colds;
- No one can sing who has a cold.

ii.
- Some oysters are silent;
- No silent creatures are amusing.

iii.
- No unhappy people chuckle;
- No happy people groan.

iv.
- All my friends have colds;
- No one can sing who has a cold.

v.
- No misers are generous;
- Some old men are not generous.

vi.
- No bride-cakes are wholesome;
- Unwholesome food should be avoided.

2. Use mathematical induction to prove Bernoulli's inequality

$$(1+x)^n \geq 1+nx$$

for all non-negative integers n and real numbers $x > -1$.

3. The lights when out again in the backroom of the "Seawolves Socks and Gloves" store. Good thing the box with 10 pairs of red and 10 pairs of blue gloves was delivered before that. How many left gloves and right gloves does one need to bring to the front to be surely able to satisfy the following orders. Naturally, we are looking for the least possible number of gloves in each case.

i. A pair of gloves.
ii. A pair of red gloves.
iii. Two pairs of gloves.
iv. Two pairs of red gloves.
v. Two pairs of gloves of the same color.
vi. Two pairs of gloves of different colors.
vii. Four pairs of blue gloves.
viii. Six pairs of gloves of the same color.
ix. Four pairs of blue gloves and six pairs of red gloves.

4. When a teletubby is full he walks upstairs to the kitchen taking one stair at a time. When a teletubby is starving he runs skipping one stair at a time. Depending on his/her hunger level, a teletubby skips at most one stair at a time. How many different ways are there for a teletubby to go upstairs if there are 12 stairs that lead to the kitchen.

5. A traveler decides to drive his car around a circular route to see the country. There is some number of gas stations along the way carrying at least as much gas in total as needed to complete the trip. Prove that you can start at one of those gas stations with an empty tank and cover the whole route filling up along the way. (We assume that the car can carry unlimited amount of gas.)

MAT200, Spring 2015. Homework 4.
Due on February 25, before class.

1. Let X be a set. Then, for subsets $A, B \in \mathcal{P}(X)$, define the symmetric difference of A and B to be

$$A \triangle B = (A \cup B) \setminus (A \cap B).$$

Observe that $A \triangle B = B \triangle A$.

i. Show that $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.

ii. Show that there exists a unique $E \in \mathcal{P}(X)$ such that, for all $A \in \mathcal{P}(X)$ we have $A \triangle E = A$.

iii. For E as above, show that for all $A \in \mathcal{P}(X)$ there exists a unique set $B \in \mathcal{P}(X)$ such that $A \triangle B = E$.

iv. Show that, for all $A, B \in \mathcal{P}(X)$, there exists a unique $C \in \mathcal{P}(X)$ such that $A \triangle C = B$.

v. Let $X = \mathbb{Z}$, A be the set of even integers and B be the set of multiples of 3. Describe the set $A \triangle B$.

2. Define the sequence of Fibonacci numbers by setting $F_0 = F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for all $n \in \mathbb{Z}^+$. Prove that the Fibonacci numbers follow the pattern "odd, odd, even": that is, show that for any positive integer m, F_{3m} and F_{3m+1} are odd, and F_{3m+2} is even.

3. Prove or give a counterexample to the following statements.

i. $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}, x + y > xy$.

ii. $\exists x \in \mathbb{R}\ \forall y \in \mathbb{R}, x + y > xy$.

iii. $\forall n \in \mathbb{Z}, (n \text{ is even})$ or $(n \text{ is odd})$.

iv. $\exists n \in \mathbb{Z}, (n \text{ is even})$ or $(n \text{ is odd})$.

v. $(\forall n \in \mathbb{Z}, n \text{ is even})$ or $(\forall n \in \mathbb{Z}, n \text{ is odd})$.

vi. $(\exists n \in \mathbb{Z}, n \text{ is even})$ or $(\exists n \in \mathbb{Z}, n \text{ is odd})$.

4. Teletubbies use coins of 3 and 7 telepennies to shop for new toys. Toys of what prices are possible to buy without getting change?

5. Let $a_n = n/(n^2 + n - 17)$ for all $n \in \mathbb{Z}^+$. Prove that

$$\forall \epsilon > 0\ \exists N \in \mathbb{Z}^+\ \forall n > N,\ a_n < \epsilon.$$

MAT200, Spring 2015. Homework 5.
Due on March 4, before class.

1. Prove that a composition of two injections is an injection.

2. Let $f\colon X \to Y$. Define $f_*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ by

$$f_*(A) = \{y \in Y \mid \exists x \in A\ f(x) = y\}$$

and $f^*\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ by

$$f^*(A) = \{x \in X \mid f(x) \in A\}.$$

Prove or give a counterexample:

i. $\forall X_1, X_2 \in \mathcal{P}(X),\ f_*(X_1 \cup X_2) = f_*(X_1) \cup f_*(X_2)$;

ii. $\forall X_1, X_2 \in \mathcal{P}(X),\ f_*(X_1 \cap X_2) = f_*(X_1) \cap f_*(X_2)$;

iii. $\forall Y_1, Y_2 \in \mathcal{P}(Y),\ f^*(Y_1 \cup Y_2) = f^*(Y_1) \cup f^*(Y_2)$.

iv. $\forall Y_1, Y_2 \in \mathcal{P}(Y),\ f^*(Y_1 \cap Y_2) = f^*(Y_1) \cap f^*(Y_2)$.

3. Let $f\colon X \to Y$ be a function. We say that $g\colon Y \to X$ is a right inverse of f if $f \circ g = \mathrm{Id}_Y$ and $g\colon Y \to X$ is a letf inverse of f if $g \circ f = \mathrm{Id}_X$.

i. Prove that f has a right inverse if and only if f is surjective.

ii. Prove that f has a left inverse if and only if f is injective.

iii. Give an example of a function $f\colon \mathbb{R} \to \mathbb{R}$ for which the right inverse exists and is not unique.

iv. Give an example of a function $f\colon \mathbb{R} \to \mathbb{R}$ for which the left inverse exists and is not unique.

4. The poll, conducted among 164 kids, about which of the three teletubbies they like (obviously, everyone loves Tinky Winky), showed the following. 110 kids liked Dipsy, 107 liked Laa-Laa and 94 liked Po. Also, 18 liked only Dipsy out of the three, 13 liked only Laa-Laa, and 4 liked only Po. There were 9 kids who liked only Tinky Winky. How many kids liked all teletubbies?

5. Calculate the number of ways to reach the point with positive integer coordinates $(5, 6)$ in the plane starting from the point $(0, 0)$ and using steps of either $(1, 0)$ or $(0, 1)$.

MAT200, Spring 2015. Homework 6.
Due on April 1, before class.

1. Prove that if X, Y are finite sets with $|X| < |Y|$, then there exists no surjection $f: X \to Y$.

2. Suppose that there exists an injection $f: \mathbb{Z}^+ \to X$. Prove that X is infinite.

3. Prove that if $X \subset Y$ and Y is a finite set, then X is also finite.

4. Let $f: X \to X$ is a map such that $f \circ f = \mathrm{Id}$. Prove that f is a bijection.

5. Calculate the number of ways to reach the point with positive integer coordinates (n, n) in the plane starting from the point $(0, 0)$ and using steps of either $(1, 0)$ or $(0, 1)$. Calculate how many of them stay on or below the straight line connecting $(0, 0)$ and (n, n).

MAT200, Spring 2015. Homework 7.
Due on April 8, before class.

1. See ex. 7 page 183 of the textbook.

2. Let $A_1, A_2, \ldots, A_n$ be finite sets. For $I \subset \mathbb{N}_n$ define

$$A_I = \bigcap_{i \in I} A_i.$$

Prove that

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{I \subseteq \mathbb{N}_n} (-1)^{|I|-1} |A_I|.$$

3. Let $|X| = n \in \mathbb{Z}^+$. Compute the number of bijections $f \colon X \to X$ which do not have fixed points (i.e. $\forall x \in X \;\; f(x) \neq x$).

4. Prove that there exists no injection from the powerset $\mathcal{P}(X)$ of any set X into X.

5. Prove that if there exist an injection $f \colon X \to Y$ and an injection $g \colon Y \to X$ then there exists a bijection between sets X and Y.

MAT200, Spring 2015. Homework 8.
Due on April 29, before class.

1. Prove that the $\mathcal{P}(\mathbb{N})$ is uncountable.

2. Construct a bijection between the open interval $(0,1)$ and the closed interval $[0,1]$.

3. Prove that a union of an infinite set and a finite set is infinite.

4. Let $n \in \mathbb{N}$. Suppose that $A \subset \mathbb{N}_{2n}$ and $|A| = n+1$. Show that A must contain a pair of integers a and b such that a divides b.

5. Prove that if there exist a surjection $f\colon X \to Y$ and a surjection $g\colon Y \to X$ then there exists a bijection between sets X and Y.

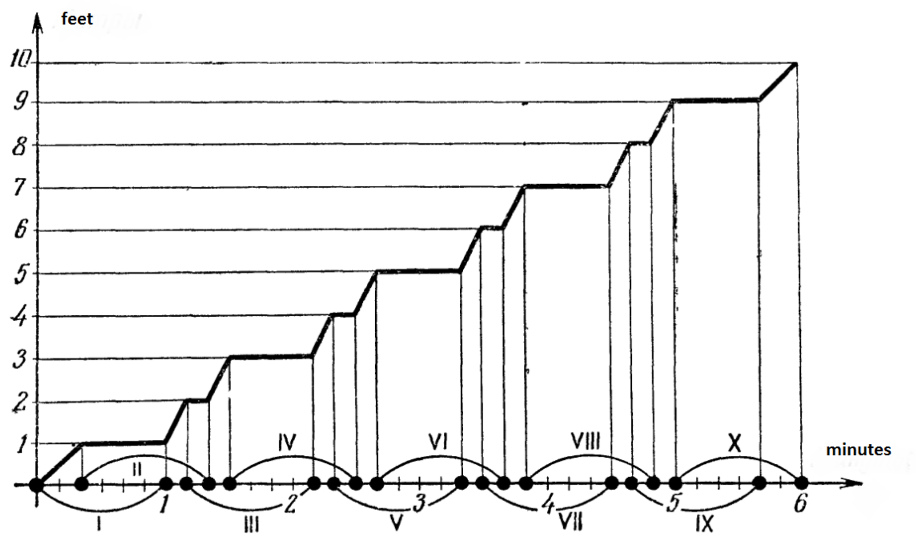
MAT200, Spring 2015. Class 1.

Problem 1. Mary and Michael decided to buy textbooks for our class. But when they got to the store it turned out that Mary is 7 cents short and Michael is 1 cent short. They decided to combine their money and buy one textbook to share, but still came short. How much did the textbook cost?

Solution. If Mary had any money, she would have at least one cent, and together with Michael, they would have been able to afford a textbook. Hence, Mary has no money at all and the textbook costs exactly 7 cents.

Problem 2. Several observers were watching a running lizard for an interval of 6 minutes. Each observer was watching the lizard for exactly 1 continuous minute of time and reported that the lizard ran exactly 1 foot during that time. The lizard has been watched neither before nor after the 6 minute interval, however during the interval it was constantly under surveillance. What is the maximal distance the lizard could possibly cover during these 6 minutes?

Solution. The answer is 10 feet. First we give an example.



The graph above shows an arrangement of 10 observes (the respective time intervals are marked on the time axis by roman numbers) in such a way that each of them has some, short as it may be, one-on-one time with the lizard. The lizard wisely uses that time to sprint 1 foot forwards, while when watched by at least two guys it stands still and pretends not to be in a hurry.

Now let us consider an arbitrary arrangement of observers. In order to prove that 10 is the maximal distance we will make the following two claims.

Claim 1. *The lizard ran exactly 1 foot during the initial and the final minute of the sprint.*

Proof. Indeed, since the lizard is watched for the whole period, there must be an observer that starts at time zero and watches for the whole initial minute. Hence, the lizard ran exactly 1 foot during that minute. Same argument applies to the last minute. □

Claim 2. *The lizard ran at most 2 feet during any continuous minute of the sprint.*

Proof. Let us say that our minute lasts from time x to time $x+1$. We pick the observer, call him A, who was the last to start his watch before or at the time x.

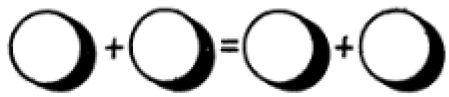
If A starts his watch exactly at time x, then he watches the lizard for exactly that minute, and the lizard ran for exactly 1 foot, which is less than 2.

If A starts his watch at time y with $y < x$, then by the choice of A there are no observers that start in the interval $[y, x]$. On the other hand, since the lizard is being watched at all times, there should exist an observer, call him B, which watches the lizard from time $y + 1$. Therefore B starts at some time z with $y < z \leq y+1$. By above z cannot be in the interval $[y, x]$ so $x \leq z \leq y+1$. This means that during the whole minute $[x, x+1]$ the lizard is observed by either A or B (or both at the same time, which only makes lizards life harder). We conclude that the lizard could not run for more than 2 feet during this minute, otherwise at least one of A and B would see it run more than 1 foot. □

Using these two claims we easily see that the lizard could not run for more than $1 + 2 + 2 + 2 + 2 + 1 = 10$ minutes.

MAT200, Spring 2015. Class 2.

Problem 1. Little Timmy got an 'F' again in his 2nd grade math class. His father was very disappointed. But little Timmy insisted it was not his fault at all. Each 2nd-grader was given a set of visual aids for the math class, including a set of 100 cards with numbers from 1 to 100 written on them. The teacher asked each student to pick 4 cards so that the sum of numbers on the first two cards is the same as the sum of numbers in the last two cards.



Timmy's problem was that during the recess he dropped his cards and by the time the bell rang he managed to pick up only 21 of them. Naturally, he tried to borrow some cards from his neighbor and got an 'F' for doing that. His father was very disappointed. You could have solved the problem with the cards you had, he said. But you don't even know which cards I had recovered, Timmy retorted. You could have solved it anyway, the father insisted, no matter what cards you had.

Prove that the father is right.

Solution. There are $(21 \cdot 20)/2 = 210$ ways to chose a pair of cards out of 21. The sum of the two numbers written on a pair of cards is, however, at least $1 + 2 = 3$ and at most $100 + 99 = 199$. Hence the sum can take at most 197 different values. By the Pigeonhole Principle, there will be two different pairs of cards, such that these sums are the same. Note that all four cards in these 2 pairs will be different.

Problem 2. There are 61 points marked inside a circle of radius 4. Show that among them there are two at distance at most $\sqrt{2}$ from each other.

Solution. Put down unit graph paper. The unit disk hits $15 \cdot 4 = 60$ unit squares.

MAT200, Spring 2015. Class 3.

Problem 1. Prove that every graph, with at least 2 vertices has 2 vertices of the same valency (i.e. with the same number of incident edges).

Solution. Let Γ be a graph with n vertices, where $n \geq 2$. Then every vertex has at least 0 and at most $n-1$ edges coming out. Also note that no graph has both a vertex of valency 0 (i.e. a vertex that is not connected to any other vertex) and a vertex of valency $n-1$ (i.e. a vertex that is connected to all other vertices) at the same time. Therefore, there are at most $n-1$ different valencies present in Γ, hence by the Pigeonhole Principle there are at least two vertices with the same valency.

Problem 2. Prove that any polyhedron has at least two faces with the same number of sides.

Solution. Consider the dual graph of the polyhedron: vertices of the graph correspond to faces of the polyhedron, two vertices are connected if and only if the corresponding faces have a common side. Then the number of sides of a given face is equal to the valency of the corresponding vertex, and the statement is reduced to Problem 1.

Problem 3. The lights when out in the backroom of the "Seawolves Socks and Gloves" store. Good thing the box with 10 pairs of red and 10 pairs of blue socks was delivered before that. How many socks does one need to bring to the front to be surely able to satisfy the following orders. Naturally, we are looking for the least possible number of socks in each case.

i. A pair of socks.
ii. A pair of red socks.
iii. Two pairs of socks.
iv. Two pairs of red socks.
v. Two pairs of socks of the same color.
vi. Two pairs of socks of different colors.
vii. Four pairs of blue socks.
viii. Four pairs of blue socks and six pairs of red socks.

Solution. Please find the answers below. The proofs are left to the reader.

i. 3.
ii. 22.
iii. 5.
iv. 24.
v. 7.
vi. 22.
vii. 28.
viii. 32.

Problem 1. An odd number of soldiers is located in the field such that pairwise distances between them are all distinct. Each soldier watches his closest neighbor. Prove that there exists at least one soldier that is not being watched.

Solution. Let $n = 2k + 1$ is the number of soldiers in the field. We will prove the statement for all $k \geq 0$ using induction.

1. Base of induction. Let $k = 0$. Then the statement is obvious since there is only one soldier in the field and no one is watching him.

2. Induction step. Suppose the statement is true for some k. Let us prove it for $k+1$. Suppose we have $2(k+1)+1 = 2k+3$ soldiers in the field. Consider the pair of soldiers that are at shortest possible distance from each other. These soldiers are necessarily watching each other. Imagine they leave their posts. Then the rest $2k + 1$ soldiers, possibly switching their targets, start watching each other. By the inductive assumption, there will be at least one soldier, call him Pvt. Lonely, that is not being watched among the $2k + 1$ soldiers that are left. Now when the first two guys come back to their posts and everything reverts back to the original disposition, still no one is watching Pvt. Lonely. Indeed, the two guys who came back are watching each other and everybody else is either watching the same target as before, or one of those two guys. This proves the statement for $k + 1$.

MAT200, Spring 2015. Class 5.

Problem 1. Let A be a finite set. Denote by $\mathcal{P}(A)$ the power set of A, that is the set of all subsets of A. Prove that if $\#A = n$ then $\#\mathcal{P}(A) = 2^n$.

Solution.

Let us prove the statement by induction on n. If $n = 0$ then $A = \emptyset$ and $\mathcal{P}(A) = \{\emptyset\}$ with $\#\mathcal{P}(A) = 1 = 2^0$ hence the statement is true.

Assume the statement is true for $n = k$. Let $n = k + 1 > 0$. Take any element $a \in A$ and consider $A' = A \setminus \{a\}$. Then $\#A' = \#A - 1 = k$. Define a function $f \colon \mathcal{P}(A) \to \mathcal{P}(A')$ by setting $f(B) = B \cap A', \forall B \subset A$. This function is surjective and two-to-one. Indeed, $\forall C \subset A', f^{-1}(C) = \{C, C \cup \{a\}\}$. Therefore $\#\mathcal{P}(A) = 2 \cdot \#\mathcal{P}(A') = 2 \cdot 2^k = 2^{k+1}$ as required.

Problem 2. (Well-ordering principle.) Every non-empty subset of $\mathbb{Z}^+$ contains a minimal element.

Note: this statement is logically equivalent to the mathematical induction axiom. We are proving one of the implications here.

Solution. Suppose $A \subset \mathbb{Z}^+$ has no minimal element. Let us prove that A is empty. We prove using strong induction that $n \notin A, \forall n \in \mathbb{Z}^+$.

Base of induction: $1 \notin A$. Indeed if $1 \in A$ then it is clearly the minimal element of A which contradicts our assumptions.

Induction step: Assume $n \notin A, \forall n \leq k$. Then $k + 1 \notin A$ because otherwise it would be the minimal element of A.

MAT200, Spring 2015. Class 6.

Definition. A function $f\colon \mathbb{R} \to \mathbb{R}$ is called *continuous* if

$$\forall \epsilon > 0\ \forall x \in \mathbb{R}\ \exists \delta > 0 \mid |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Problem. Prove that $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5x$ is continuous.
Solution. For a given ϵ chose $\delta = \epsilon/5$. Then the statement

$$\forall x \in \mathbb{R}\ |x - y| < \epsilon/5 \implies |5x - 5y| < \epsilon.$$

is obviously true.

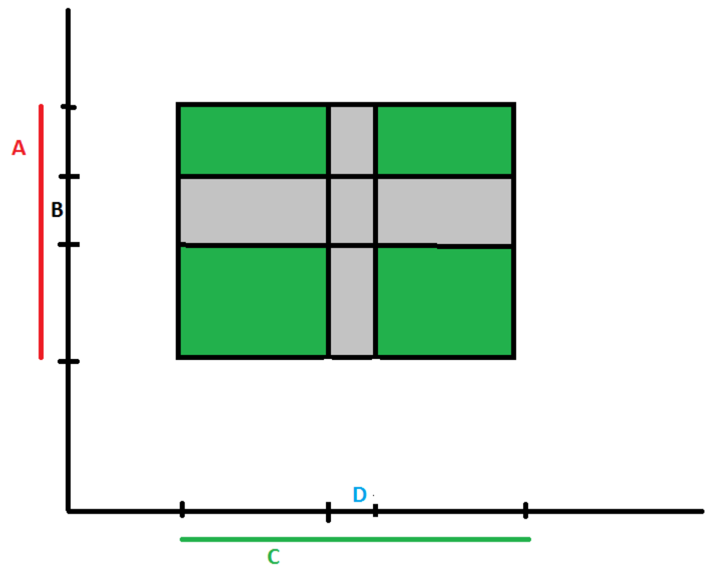
MAT200, Spring 2015. Class 7.

Problem. Prove that for any sets A, B, C, D the following is true:

$$(A - B) \times (C - D) = (A \times C - B \times C) - A \times D.$$

Solution.

One can easily convince oneself that the statement is true by looking at the following sketch.



Let us write down a formal proof.

Suppose $(x, y) \in (A - B) \times (C - D)$. That means that $x \in (A - B)$ hence, by the definition of set difference, $x \in A$ and $x \notin B$. Similarly, $y \in (C - D)$ hence $y \in C$ and $y \notin D$. But then $(x, y) \in A \times C$, $(x, y) \notin B \times C$ and $(x, y) \notin A \times D$. We see that $(x, y) \in (A \times C - B \times C) - A \times D$.

On the other hand, suppose $(x, y) \in (A \times C - B \times C) - A \times D$. Then $(x, y) \in A \times C$, $(x, y) \notin B \times C$ and $(x, y) \notin A \times D$. These three statements imply respectively:

- $(x, y) \in A \times C$ implies $x \in A$ and $y \in C$.
- $(x, y) \notin B \times C$ implies $x \notin B$ or $y \notin C$. Since we already know from the first item that $y \in C$, we conclude $x \notin B$.
- $(x, y) \notin A \times D$ implies $x \notin A$ or $y \notin D$. Since we already know from the first item that $x \in A$, we conclude $y \notin D$.

We see that $x \in (A-B)$ and $y \in (C-D)$, therefore $(x, y) \in (A-B)\times(C-D)$.

We have shown that every element of the set on the left hand side of the equation is also an element of the right hand side and vice versa. Therefore the identity holds.

MAT200, Spring 2015. Class 8.

Problem. A 100×100 board is divided into unit squares. In every square there is an arrow that points up, down, left or right. The board is surrounded by a wall, except for the right side of the top right corner square. A ladybug is placed in one of the squares. Each second, the ladybug moves one unit in the direction of the arrow in its square. When the ladybug moves, the arrow of the square it was in turns 90 degrees clockwise. If the indicated movement cannot be done, the ladybug does not move that second, but the arrow in the square does turn. Is it possible for the ladybug to never leave the board?

Solution. We are going to prove that regardless of the initial position of the arrows and the ladybug, it always leaves the board. Suppose this is not true, i.e. the ladybug is trapped. In this case, the ladybug makes an infinite number of steps on the board. Since there are only 100^2 squares, by the infinite pigeonhole principle, there is a square that is visited an infinite number of times. Each time the ladybug goes through this square, the arrow in there turns. Thus, the ladybug was also an infinite number of times in each of the neighboring squares. By repeating this argument, we see that the ladybug also has visited an infinite number of times each of the neighbors of those squares. In this way we conclude that the ladybug has visited an infinite number of times each square on the board, in particular the top right corner. This is impossible, because when the arrow in that corner points to the right the ladybug leaves the board.