MAT132 Calculus II

The goal of this course is to develop your understanding of the concepts of integral calculus, infinite series, and differential equations and your ability to apply them to problems both within and outside of mathematics.

Textbook: *Single Variable Calculus (Stony Brook Edition 4), by James Stewart*. This is the same book as Stewart's Single Variable Calculus: Concepts and Contexts, 4th ed, but with a different cover. However, the Stony Brook Edition comes with an access code for WebAssign. The Stony Brook edition of the text is available at the campus bookstore. There is also an electronic-only version of the text available at WebAssign. In addition, you can buy a new printed copy of the text online from the publisher, including webassign access.

	Lecture 1	**Lecture 2**
Schedule	MWF 10.00-10.53 (Engineering 143)	MW 4.00-5.20 (Engineering 145)
Instructor	Jaepil Lee	Thomas Sharland*
Office hours	• Monday 1-2 in 3-117 • Tuesday 10-12 in 3-117	• Monday 2-3 in 5D148 • Tuesday 4-5 in 5D148 • Thursday 2.30-3.30 in MLC
Recitations	• R01: MW 12.00-12.53 (Library E4315) • R02: TuTh 10:00-10:53 (Lgt Engr Lab 152) • R03: TuTh 4.00-4:53 (Lgt Engr Lab 152) • R05: MW 11:00-11:53 (Math P131)	• R06: TuTh 5:30-6:23 (Lgt Engr Lab 152) • R07: MW 10:00-10:53 (Library E4315) • R09: TuTh 11:30-12:23 (Library N3074)

Homework policy

There will be weekly assignments on WebAssign - WebAssign should be accessed **through Blackboard**. In addition to this, on alternate weeks there will be (short) written assignments, which are due in during your second recitation in the week after it was set (giving you around a week to solve it). This is to ensure you are used to writing solutions to mathematics problems, so the exams don't came as too much of a shock! To receive full credit for the written homework, you should show your working. On weeks where there are written homeworks, the WebAssign assignments will be shortened. Late homework will never be accepted!

Syllabus

The textbook is intended to be read. Below is a syllabus for the course that shows what will be covered each week. It will greatly improve your comprehension if you read the relevant sections **before** class. Furthermore, this will allow you to ask good questions on anything you didn't understand when you read through it.

Week Commencing	Sections Covered	Notes/Homework etc.
1/27	• Administration • Review of Sections 5.1, 5.2, 5.3, 5.4 and 5.5.	Homework 1 on WebAssign will appear on 1/29. It is due on 2/5 at 2pm.
2/3	• ~~Build a snowman.~~ • ~~Buy an umbrella~~	Classes cancelled 2/3, 2/5. Homework 2 on WebAssign will appear on 2/5. It is due on 2/19 (**note change of date**) at 2pm. There is also a paper homework, here. It is is due in recitation in two weeks' time. Here are some solutions.
2/10	• 5.6:Integration by Parts • 5.7: Additional Techniques of Integration	No homework this week due to week 2 being a write-off.
2/17	• 5.9: Approximate Integration • 5.9: Simpson's Rule • 5.10: Improper Integrals	No homework this week, due to **Midterm I taking place on 2/25**. It will cover sections 5.1-5.7. A practice exam is here with some sample solutions here
2/24	• 5.10: Improper Integrals (continued) • 6.1: More about Area	Homework 3 on WebAssign will appear on 2/26. It is due on 3/5 at 2pm.
3/3	• 6.2: Volumes • 6.3: Cylindrical Shells • 6.4: Arc Length	Homework 4 on WebAssign will appear on 3/5. It is due on 3/12 at 2pm. There is also a paper homework, here. It is is due in recitation on 3/12 or 3/13. Here are some solutions.
3/10	• 6.5: Average Value of a Function • Appendix H: Polar Coordinates • 8.1: Sequences	Homework 5 on WebAssign will appear on 3/12. It is due on 3/26 at 2pm.
3/17	• **Spring Break!** • Have fun!	No homework!
3/24	• 8.1: Sequences • 8.2: Infinite Series	Homework 6 on WebAssign will appear on 3/26. It is due on 4/2 at 2pm. There is also a paper homework, here. It is is due in recitation on 4/2 or 4/3. Here are some solutions.

3/31	• 8.2: Series • 8.3: Convergence Tests	No homework this week, due to **Midterm II taking place on 4/8**. It will cover sections 5.9, 5.10, 6.1-6.5, 8.1, 8.2 and Appendix H. A practice exam is here with some sample solutions here.
4/7	• 8.4: More Convergence Tests • 8.5: Power Series • 8.6: Representation of Functions By Power Series	Homework 7 on WebAssign will appear on 4/9. It is due on 4/16 at 2pm. There is also a paper homework, here. It is is due in recitation on 4/16 or 4/17. Here are some solutions.
4/14	• 8.7: Taylor Series	Homework 8 on WebAssign will appear on 4/16. It is due on 4/23 at 2pm.
4/21	• 8.8: Approximating Functions by Polynomials • Appendix I: Complex Numbers	Homework 9 on WebAssign will appear on 4/23. It is due on 4/30 at 2pm. There is also a paper homework, here. It is is due in recitation on 4/30 or 5/1. Here are some solutions.
4/28	• 7.1: Modeling with Differential Equations • 7.2: Direction Fields & Euler's Method • 7.3: Separable Equations	Homework 9 on WebAssign will appear on 4/30. It is due on 5/7 at 2pm.
5/5	• 7.4: Exponential Growth • Notes on Second Order Differential Equations	No homework, so start preparing for the final!
5/12	• 7.5: Logistic Equation	Here are some practice problems for the final. There are a lot of questions here (far more than on the exam). Here are some solutions.

Grade Boundaries

Here are the grade boundaries for the course:

Grade	Percentage

A	82%+
A-	74%+
B+	66%+
B	58%+
B-	51%+
C+	44%+
C	38%+
D+	30%+
D	21%+

If you have any questions about your final grade, send a letter (**not an e-mail**) to <*Instructor's name*>, Department of Mathematics, SUNY Stony Brook, Stony Brook N.Y. 11794-3651. If appropriate, you will receive a written reply. These matters will be dealt with in writing only; that way, we have a written record of what the student says, and what we reply. Grades are not subject to change unless we made a clerical mistake in adding up the numerical grades and/or in converting them into a letter grade. In this case contact your instructor as soon as possible. We cannot change grades for any other reason, as this would be inappropriate. Remember the grades are calculated with the following recipe:

Type	Weight
Final	40%
Midterm I	25%
Midterm II	25%
Homework	10%

Disability Support Services: If you have a physical, psychological, medical, or learning disability that may affect your course work, please contact Disability Support Services (DSS) office: ECC (Educational Communications Center) Building, room 128, telephone (631) 632-6748/TDD. DSS will determine with you what accommodations are necessary and appropriate. Arrangements should be made early in the semester (before the first exam) so that your needs can be accommodated. All information and documentation of disability is confidential. Students requiring emergency evacuation are encouraged to discuss their needs with their professors and DSS. For procedures and information, go to the following web site http://www.ehs.sunysb.edu and search Fire safety and Evacuation and Disabilities.

Academic Integrity: Each student must pursue his or her academic goals honestly and be personally accountable for all submitted work. Representing another person's work as your own is always wrong. Faculty are required to report any suspected instance of academic dishonesty to the Academic Judiciary. For more comprehensive information on academic integrity, including categories of academic dishonesty, please refer to the academic judiciary website at http://www.stonybrook.edu/uaa/academicjudiciary/.

Critical Incident Management: Stony Brook University expects students to respect the rights, privileges, and property of other people. Faculty are required to report to the Office of Judicial Affairs any disruptive behavior that interrupts their ability to teach, compromises the safety of the learning environment, and/or inhibits students' ability to learn.

MAT132 PAPER HOMEWORK 1

DUE IN RECITATION ON 2/19 OR 2/20

Problem 1. *The Lee-Sharland Company have invented a new machine that solves Calculus problems. To speed up manufacture, they set up a production line. The machine is quite complicated, so initially the workers find it difficult to construct it quickly, but as they get more used to the design, the process speeds up. In fact, the company notices that the* ***rate*** *of production after t weeks has elapsed is given by*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1600\left(1 - \frac{50}{(t+5)^3}\right).$$

How many machines are produced between the start of the fourth week and the end of the fifth? (Be very careful which limits you choose in your integral!)

Problem 2.

(i) *Use integration by parts to show that*

$$\int_a^b f(x)\,\mathrm{d}x = xf(x)\big]_a^b - \int_a^b xf'(x)\,\mathrm{d}x.$$

(ii) *Suppose g is the inverse of f and that f' is a continuous function. Using part (i) and the substitution $y = f(x)$ (or otherwise), show that*

$$\int_a^b f(x)\,\mathrm{d}x = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)\,\mathrm{d}y.$$

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$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1600\left(1 - \frac{50}{(t+5)^3}\right).$$

How many machines are produced between the start of the fourth week and the end of the fifth? (Be very careful which limits you choose in your integral!)

Solution 1. Clearly, the number of machines produced between t_1 having elapsed and t_2 weeks having elapsed is

$$\int_{t_1}^{t_2} \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = \int_{t_1}^{t_2} 1600\left(1 - \frac{50}{(t+5)^3}\right)\mathrm{d}t.$$

Since we are seeing how many are made between the start of the fourth week and the end of the fifth, this means we need to calculate

$$\int_3^5 1600\left(1 - \frac{50}{(t+5)^3}\right)\mathrm{d}t$$

since the start of the fourth week occurs when the third week has finished (this was why the question contained a warning to be careful with which limits are chosen in the integral). So now we can carry out the integral, and get

$$\begin{aligned}
\int_3^5 1600\left(1 - \frac{50}{(t+5)^3}\right)\mathrm{d}t &= \int_3^5 1600\,\mathrm{d}t - 1600\int_3^5 \frac{50}{(t+5)^3}\,\mathrm{d}t \\
&= 3200 + 1600\left[\frac{25}{(t+5)^2}\right]_{t=3}^{t=5} \\
&= 3200 + 1600\left(\frac{25}{100} - \frac{25}{64}\right) \\
&= 3200 - 1600\cdot\frac{9}{64} \\
&= 3200 - 225 = 2975
\end{aligned}$$

So **2975** machines are built between the start of the fourth week and the end of the fifth.

Problem 2.

(i) *Use integration by parts to show that*

$$\int_a^b f(x)\,\mathrm{d}x = xf(x)]_a^b - \int_a^b xf'(x)\,\mathrm{d}x.$$

(ii) *Suppose g is the inverse of f and that f' is a continuous function. Using part (i) and the substitution $y = f(x)$ (or otherwise), show that*

$$\int_a^b f(x)\,\mathrm{d}x = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)\,\mathrm{d}y.$$

Solution 2.

(i) In the integration by parts formula

$$\int_a^b u\,\mathrm{d}v = uv]_a^b - \int_a^b v\,\mathrm{d}u$$

take $u = f(x)$ and $\mathrm{d}v = \mathrm{d}x$, so that $\mathrm{d}u = f'(x)\,\mathrm{d}x$ and $v = x$. Then we get

$$\int_a^b f(x)\,\mathrm{d}x = xf(x)]_a^b - \int_a^b xf'(x)\,\mathrm{d}x$$

as required.

(ii) Using part (i), we get

$$\int_a^b f(x)\,\mathrm{d}x = xf(x)]_a^b - \int_a^b xf'(x)\,\mathrm{d}x = bf(b) - af(a) - \int_a^b xf'(x)\,\mathrm{d}x. \tag{1}$$

We now tackle the final integral, by taking the suggested substitution $y = f(x)$, so that $\mathrm{d}y = f'(x)\,\mathrm{d}x$. Since g is the inverse of f, we also have $g(y) = g(f(x)) = x$. With regards to the limits of the integral, when $x = a$ we have $y = f(a)$ and when $x = b$ we have $y = f(b)$. Hence we have shown that

$$\int_a^b xf'(x)\,\mathrm{d}x = \int_{f(a)}^{f(b)} \underbrace{g(y)}_{x}\ \underbrace{\mathrm{d}y}_{f'(x)\,\mathrm{d}x}$$

and the solution follows by inserting this into equation (1).

Here are some practice questions for the first midterm. Some questions may be useful on the midterm, some may not... Indeed, one of the (sub)-questions will appear on the midterm itself!

Question 1. Let

$$f(x) = \begin{cases} -(x+1)^2 + 1 & \text{for } -2 \le x \le 0 \\ (x-1)^2 - 1 & \text{for } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

and define

$$g(x) = \int_{-2}^{x} f(t)\,\mathrm{d}t.$$

a) Where is g differentiable?
b) Where does g attain its maximum?
c) Find

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{x^2}^{x^3} \sqrt{1 + \sin t}\,\mathrm{d}t$$

Question 2. In this question, we will find an antiderivative for $e^x \cos x$.

a) Carry out integration by parts twice on the integral

$$\int e^x \cos x\,\mathrm{d}x.$$

b) Use your answer to part (a) to give a closed form (in other words, a solution not containing an integral) for $\int e^x \cos x\,\mathrm{d}x$.
c) Check your answer to part (b) by differentiating.

Question 3.

a) Evaluate

$$\int_1^{\sqrt{2}} x^3 (x^4 - 2)^5\,\mathrm{d}x$$

b) (i) Use trigonometric identities to show that

$$\sin^4 x = \frac{3}{8} - \frac{\cos 2x}{2} + \frac{\cos 4x}{8}$$

(Hint: First write $\sin^4 x = (\sin^2 x)^2$)

(ii) Compute

$$\int_0^{\frac{\pi}{2}} \sin^4 x\,\mathrm{d}x$$

Question 4. Integrate the following rational function (you will have to carry out polynomial division first!)

$$\int \frac{x^3 + 4x + 2}{x^3 + 4x}\mathrm{d}x$$

Here are some practice questions for the first midterm. Some questions may be useful on the midterm, some may not... Indeed, one of the (sub)-questions will appear on the midterm itself!

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$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{x^2}^{x^3} \sqrt{1+\sin t}\,\mathrm{d}t$$

Solution 1.

a) We see that f is continuous everywhere and so g is differentiable everywhere by the Fundamental Theorem of Calculus.

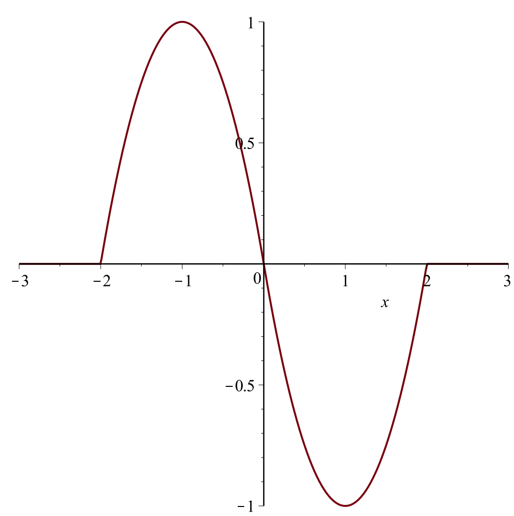


FIGURE 1. The graph of f.

b) By the Fundamental Theorem of Calculus

$$\frac{\mathrm{d}}{\mathrm{d}x} g(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_{-2}^{x} f(t)\,\mathrm{d}t = f(x).$$

So the critical numbers of g are the points where $f(x) = 0$, namely -2, 0 and 2. By the closed interval method, we only need to check the value taken by g at the critical numbers and the endpoints, which gives $g(-2) = 0$, $g(0) = \frac{4}{3}$ and $g(2) = 0$. So g attains its maximum at $x = 0$.

c) By the Fundamental Theorem of Calculus (and use of the chain rule)

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{x^2}^{x^3} \sqrt{1 + \sin t}\,\mathrm{d}t = 3x^2\sqrt{1 + \sin(x^3)} - 2x\sqrt{1 + \sin(x^2)}.$$

Question 2. In this question, we will find an antiderivative for $e^x \cos x$.

a) Carry out integration by parts twice on the integral
$$\int e^x \cos x \,\mathrm{d}x.$$

b) Use your answer to part (a) to give a closed form (in other words, a solution not containing an integral) for $\int e^x \cos x \,\mathrm{d}x$.

c) Check your answer to part (b) by differentiating.

Solution 2.

a) We take $u = e^x$ and $\mathrm{d}v = \cos x \,\mathrm{d}x$ in the integration by parts formula, so that $\mathrm{d}u = e^x \,\mathrm{d}x$ and $v = \sin x$. Thus
$$\int e^x \cos x \,\mathrm{d}x = e^x \sin x - \int e^x \sin x \,\mathrm{d}x. \tag{1}$$
We now apply integration by parts again on $\int e^x \sin x \,\mathrm{d}x$, taking $u = e^x$, $\mathrm{d}v = \sin x \,\mathrm{d}x$, which gives $\mathrm{d}u = e^x \,\mathrm{d}x$ and $v = -\cos x$. This yields
$$\int e^x \sin x \,\mathrm{d}x = -e^x \cos x + \int e^x \cos x \,\mathrm{d}x.$$

b) Substituting this second formula into (1), we get
$$\begin{aligned}
\int e^x \cos x \,\mathrm{d}x &= e^x \sin x - \int e^x \sin x \,\mathrm{d}x \\
&= e^x \sin x - \left(-e^x \cos x + \int e^x \cos x \,\mathrm{d}x\right) \\
&= e^x(\sin x + \cos x) - \int e^x \cos x \,\mathrm{d}x.
\end{aligned}$$
Rearranging this gives
$$\int e^x \cos x \,\mathrm{d}x = \frac{e^x(\sin x + \cos x)}{2}.$$

c) It is simple to check that
$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{e^x(\sin x + \cos x)}{2}\right) &= \frac{1}{2}e^x(\sin x + \cos x) + e^x(\cos x - \sin x) \\
&= \frac{1}{2}e^x(2\cos x) \\
&= e^x \cos x
\end{aligned}$$
as required.

Question 3.

a) Evaluate

$$\int_1^{\sqrt{2}} x^3(x^4-2)^5 \, \mathrm{d}x$$

b) (i) Use trigonometric identities to show that

$$\sin^4 x = \frac{3}{8} - \frac{\cos 2x}{2} + \frac{\cos 4x}{8}$$

(Hint: First write $\sin^4 x = (\sin^2 x)^2$)

(ii) Compute

$$\int_0^{\frac{\pi}{2}} \sin^4 x \, \mathrm{d}x$$

Solution 3.

a) We use the substitution $u = x^4 - 2$ so that $\mathrm{d}u = 4x^3 \, \mathrm{d}x$. When $x = 1$, we have $u = -1$ and when $x = \sqrt{2}$ have $u = 2$. Thus the integral becomes

$$\begin{aligned}
\int_1^{\sqrt{2}} x^3(x^4-2)^5 \, \mathrm{d}x &= \int_{-1}^{2} \frac{1}{4} u^5 \, \mathrm{d}u \\
&= \frac{1}{24} u^6 \Big]_{u=-1}^{u=2} \\
&= \frac{1}{24}(64-1) \\
&= \frac{63}{24} = \frac{21}{8}.
\end{aligned}$$

b) (i) We compute

$$\begin{aligned}
\sin^4 x &= (\sin^2 x)^2 \\
&= \left(\frac{1}{2}(1-\cos 2x)\right)^2 \\
&= \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x) \\
&= \frac{1}{4}\left(1 - 2\cos 2x + \left(\frac{1+\cos 4x}{2}\right)\right) \\
&= \frac{3}{8} - \frac{\cos 2x}{2} + \frac{\cos 4x}{8}
\end{aligned}$$

as required.

b) (ii) Now we can calculate

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \sin^4 x \, \mathrm{d}x &= \int_0^{\frac{\pi}{2}} \left(\frac{3}{8} - \frac{\cos 2x}{2} + \frac{\cos 4x}{8}\right) \mathrm{d}x \\
&= \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} \Big]_{x=0}^{x=\frac{\pi}{2}} \\
&= \left(\frac{3\pi}{16} - 0 + 0\right) - (0 - 0 - 0) = \frac{3}{16}\pi
\end{aligned}$$

Question 4. Integrate the following rational function (you will have to carry out polynomial division first!)

$$\int \frac{x^3+4x+2}{x^3+4x}\mathrm{d}x$$

Solution 4. We simplify the expression and get

$$\frac{x^3+4x+2}{x^3+4x} = \frac{x^3+4x}{x^3+4x} + \frac{2}{x^3+4x} = 1 + \frac{2}{x(x^2+4)}$$

We now use the partial fractions technique to compute the integral of the right-hand term:

$$\frac{2}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

which we can solve (by comparing coefficients on both sides of the equation and using standard techniques for solving simultaneous equations) by

$$A = \frac{1}{2}, \quad B = -\frac{1}{2} \quad C = 0.$$

Thus we get

$$\begin{aligned}\int \frac{x^3+4x+2}{x^3+4x}\mathrm{d}x &= \int \left(1 + \left(\frac{\frac{1}{2}}{x} - \frac{\frac{1}{2}x}{x^2+4}\right)\right)\mathrm{d}x \\ &= x + \frac{1}{2}\ln|x| - \frac{1}{4}\ln|x^2+4| + C.\end{aligned}$$

MAT132 PAPER HOMEWORK 2

DUE IN RECITATION ON 3/12 OR 3/19

Problem 1. An ellipse with major axis of length $2a$ and minor axis of length $2b$ is the set of all (x, y) satisfying

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Show that the area enclosed by the ellipse is πab.

Problem 2. Professor Lee and Professor Sharland go on a fishing trip on a sufficiently large lake. The boat begins sailing with constant velocity $v_0 = 5m/s$. Unfortunately, since they are absent minded, they forgot to put enough fuel in the boat, and so after 5 seconds the boat lost its power and began to drift on the lake. The velocity of the boat t seconds after the fuel ran out can be written as $v(t) = 5e^{-0.2t}m/s$. Set up an improper integral to represent the total distance traveled by the boat, and then compute this distance.

Paper Homework 2 Solution

1. Solving $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for y, we get

$$y = b\sqrt{1 - \frac{x^2}{a^2}}.$$

Integrating y from 0 to a will give the area of the ellipse in first quadrant. So desired area is

$$\begin{aligned}
4\int_0^a b\sqrt{1+\frac{x^2}{a^2}}\,dx &= 4\int_0^{\pi/2} ab\sqrt{1-\sin^2\theta}\cos\theta d\theta \quad (\text{put } x = a\sin\theta) \\
&= 4ab\int_0^{\pi/2} \cos^2\theta\ d\theta \\
&= 4ab\int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\ d\theta \\
&= 4ab\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = \pi ab
\end{aligned}$$

2. For first 5 seconds they have traveled 25 meters, and after they ran out of fuel the distance they traveled is,

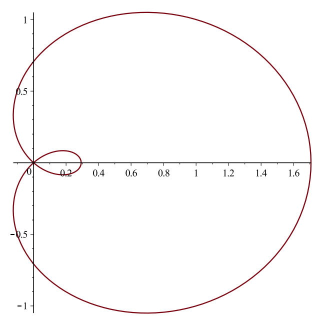
$$\begin{aligned}
\int_0^\infty 5e^{-0.2t}\ dt &= \lim_{b\to\infty}\int_0^b 5e^{-0.2t}\ dt = \lim_{b\to\infty}\left[\frac{5}{-0.2}e^{-0.2t}\right]_0^b \\
& \lim_{b\to\infty} 25 - 25e^{-0.2b} = 25.
\end{aligned}$$

The total distance traveled is 25m + 25m = 50m.

MAT132 PAPER HOMEWORK 3

DUE IN RECITATION ON 4/2 OR 4/3

Problem 1. Below is a picture of a *limaçon* with polar equation $r = \frac{1}{\sqrt{2}} + \cos\theta$. Find the area enclosed by the inner and outer loops (you will have to find the relevant values of θ to compute the area).



Problem 2. After the failed fishing trip, Professor Lee and Professor Sharland decide to fly a kite. Unfortunately the string breaks and the kite is blown away by the wind, with its height above the ground given by $y = 100 - \frac{1}{40}(x-50)^2$ (measured in feet), where x measures the horizontal distance traveled in feet.

(i) Using the substitution $u = x - 50$, show that the distance traveled by the kite from when the string snapped to when the kite is a horizontal distance of 100 ft away from the hapless professors is given by the integral

$$\frac{1}{10}\int_0^{50} \sqrt{400 + u^2}\, \mathrm{d}u.$$

(ii) Using the formula

$$\int \sqrt{a^2 + u^2}\, \mathrm{d}u = \frac{u}{2}\sqrt{a^2 + u^2} + \frac{a^2}{2}\ln(u + \sqrt{a^2 + u^2}) + C,$$

compute the distance traveled by the kite.

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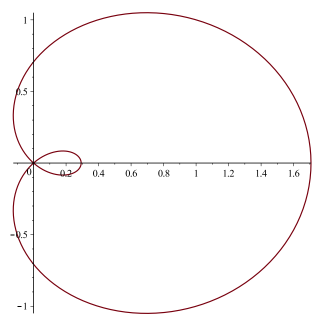


Solution 1. The area inside the outer loop is twice the area inside half the loop, which is

$$\int_0^{\frac{3\pi}{4}} \frac{1}{2} r^2 \, \mathrm{d}\theta = \frac{1}{2}\int_0^{\frac{3\pi}{4}} \left(\frac{1}{\sqrt{2}} + \cos\theta\right)^2 \mathrm{d}\theta = \frac{1}{2}\int_0^{\frac{3\pi}{4}} \left(\frac{1}{2} + \sqrt{2}\cos\theta + \cos^2\theta\right) \mathrm{d}\theta.$$

since the point at radius 0 is attained when $\theta = \frac{3\pi}{4}$. So the area inside the outer loop is

$$\begin{aligned}
\int_0^{\frac{3\pi}{4}} \left(\frac{1}{2} + \sqrt{2}\cos\theta + \cos^2\theta\right) \mathrm{d}\theta &= \int_0^{\frac{3\pi}{4}} \left(\frac{1}{2} + \sqrt{2}\cos\theta + \left(\frac{1+\cos 2\theta}{2}\right)\right) \mathrm{d}\theta \\
&= \int_0^{\frac{3\pi}{4}} \left(1 + \sqrt{2}\cos\theta + \frac{\cos 2\theta}{2}\right) \mathrm{d}\theta \\
&= \theta + \sqrt{2}\sin\theta + \frac{\sin 2\theta}{4}\Bigg]_{\theta=0}^{\theta=\frac{3\pi}{4}} \\
&= \left[\frac{3\pi}{4} + \sqrt{2}\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{4}(-1)\right] - [0] = \frac{1}{4}(3\pi + 3).
\end{aligned}$$

Similarly, the area inside the smaller loop is

$$\begin{aligned}\int_{\frac{3\pi}{4}}^{\pi}\left(\frac{1}{2}+\sqrt{2}\cos\theta+\cos^2\theta\right)\,\mathrm{d}\theta &= \theta+\sqrt{2}\sin\theta+\frac{\sin 2\theta}{4}\Bigg]_{\theta=\frac{3\pi}{4}}^{\theta=\pi} \\ &= [\pi+0+0]-\left[\frac{3\pi}{4}+\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)+\frac{1}{4}\,(-1)\right]=\frac{1}{4}(\pi-3).\end{aligned}$$

Hence the total area is the difference between these two areas, namely

$$\frac{1}{4}(3\pi+3)-\frac{1}{4}(\pi-3)=\frac{1}{4}(2\pi+6)=\frac{1}{2}(\pi+3).$$

Problem 2. After the failed fishing trip, Professor Lee and Professor Sharland decide to fly a kite. Unfortunately the string breaks and the kite is blown away by the wind, with its height above the ground given by $y = 100 - \frac{1}{40}(x-50)^2$ (measured in feet), where x measures the horizontal distance traveled in feet.

(i) Using the substitution $u = x - 50$, show that the distance traveled by the kite from when the string snapped to when the kite is a horizontal distance of 100 ft away from the hapless professors is given by the integral

$$\frac{1}{10}\int_0^{50} \sqrt{400+u^2}\,\mathrm{d}u.$$

(ii) Using the formula

$$\int \sqrt{a^2+u^2}\,\mathrm{d}u = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln(u+\sqrt{a^2+u^2}) + C,$$

compute the distance traveled by the kite.

Solution 2.

(i) The distance traveled is

$$\begin{aligned}
\int_0^{100} \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x &= \int_0^{100} \sqrt{1+\left(\frac{1}{20}(x-50)\right)^2}\,\mathrm{d}x \\
&= \int_0^{100} \sqrt{1+\frac{1}{400}(x-50)^2\,\mathrm{d}x} \\
&= \frac{1}{20}\int_0^{100} \sqrt{400+(x-50)^2}\,\mathrm{d}x \\
&= \frac{1}{20}\int_{-50}^{50} \sqrt{400+u^2}\,\mathrm{d}u \qquad (u = x-50,\ \mathrm{d}u = \mathrm{d}x) \\
&= \frac{1}{10}\int_0^{50} \sqrt{400+u^2}\,\mathrm{d}u \qquad \text{(The integrand is even.)}
\end{aligned}$$

(ii) By the given formula with $a = 20$ and taking the antiderivative with $C = 0$, we get

$$\begin{aligned}
\frac{1}{10}\int_0^{50} \sqrt{400+u^2}\,\mathrm{d}u &= \frac{1}{10}\left(\frac{u}{2}\sqrt{400+u^2} + 200\ln(u+\sqrt{400+u^2})\Big]_{u=0}^{u=50}\right) \\
&= \frac{1}{10}\left(\left[25\sqrt{2900}+200\ln(50+\sqrt{2900})\right] - \left[200\ln(\sqrt{400})\right]\right) \\
&= \frac{1}{10}\left(250\sqrt{29}+200(\ln 10+\ln(5+\sqrt{29}) - 200(\ln 10+\ln 2)\right) \\
&= 25\sqrt{29}+20\ln(5+\sqrt{29}) - 20\ln 2 \\
&= 25\sqrt{29}+20\ln\left(\frac{5+\sqrt{29}}{2}\right).
\end{aligned}$$

Question 1.

a) Use the trapezoidal rule with $n = 6$ to approximate

$$\int_0^\pi \sin x \,\mathrm{d}x$$

b) According to the error bounds formula, what is the maximum error in the above approximation?

c) Sketch the region $S = \{(x, y) \mid x \geq 1,\ 0 \leq y \leq e^{-x}\}$. Express the area of S as an improper integral and compute its area.

Question 2.

a) Verify by differentiation that

$$\int \sec^3 \theta \,\mathrm{d}\theta = \frac{1}{2}(\sec\theta \tan\theta + \ln|\sec\theta + \tan\theta|) + C.$$

b) Evaluate the arc length of the curve $y = \frac{1}{2}x^2$ from $x = 0$ to $x = 1$.

Question 3.

a) Find the volume of the solid generated by rotating the region enclosed by $y = \frac{1}{1+x^2}$, $x = -1$, $x = 1$ and $y = 0$ about the line $y = 2$.

b) Find the volume of the solid generated by rotating the region enclosed by $y = \frac{1}{1+x^2}$, $x = -1$, $x = 1$ and $y = 0$ about the line $x = 2$.

c) (i) Sketch the region bounded by the curves $y = x^2$ and $x = y^2$.

(ii) Using the method of cylindrical shells, find the volume of the solid generated by rotating the above region around the line $x = -1$.

Question 4.

a) Sketch the curve defined in polar coordinates by $r = 2 - \cos\theta$. What is the area bounded by this curve?

b) Find the points on the line $r = 1 + \cos\theta$ where the tangent line is horizontal (there are three such points...).

c) Find the length of the curve $r = 1 + \cos\theta$ for $0 \leq \theta \leq 2\pi$ (this is quite tricky, but doable by manipulating the integrand and using a substitution).

Question 5.

a) Show that the sequence defined recursively by

$$a_1 = 1, \quad a_{n+1} = 3 - \frac{1}{a_n}$$

is convergent and find the limit.

b) Decide whether the following series converge, and if so, find their limits.

(i) $\sum_{n=1}^{\infty} \frac{1}{\pi^n} + \frac{1}{n(n+1)}$

(ii) $\sum_{n=1}^{\infty} (0.7)^{n+2} - (0.2)^{n+1}$

c) Express $0.\overline{17} = 0.17171717\ldots$ as a ratio of integers.

Question 1.

a) Use the trapezoidal rule with $n = 6$ to approximate

$$\int_0^\pi \sin x \, \mathrm{d}x$$

b) According to the error bounds formula, what is the maximum error in the above approximation?

c) Sketch the region $S = \{(x, y) \mid x \geq 1,\ 0 \leq y \leq e^{-x}\}$. Express the area of S as an improper integral and compute its area.

Solution 1.

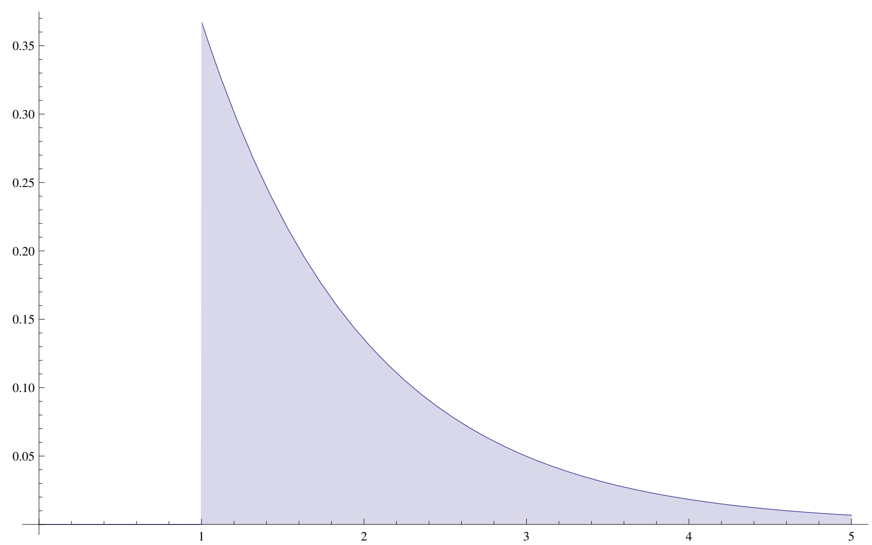
a) The trapezoid rule says that

$$\begin{aligned}
\int_0^\pi \sin x \, \mathrm{d}x &\approx \frac{\frac{\pi}{6}}{2} \left(\sin(0) + 2\sin\left(\frac{\pi}{6}\right) + 2\sin\left(\frac{\pi}{3}\right) + 2\sin\left(\frac{\pi}{2}\right) + 2\sin\left(\frac{2\pi}{3}\right) + 2\sin\left(\frac{5\pi}{6}\right) + \sin(\pi) \right) \\
&= \frac{\pi}{12} \left(0 + 2\left(\frac{1}{2}\right) + 2\left(\frac{\sqrt{3}}{2}\right) + 2 + 2\left(\frac{\sqrt{3}}{2}\right) + 2\left(\frac{1}{2}\right) + 0 \right) \\
&= \frac{\pi}{12} \left(4 + 2\sqrt{3} \right).
\end{aligned}$$

b) Since $\left| \frac{\mathrm{d}^2}{\mathrm{d}x^2} \sin(x) \right| = |-\sin(x)| \leq 1$, the error bound is

$$|E_T| \leq \frac{\pi^3}{12(6)^2} = \frac{\pi^3}{12 \times 36}.$$

c) We sketch the region S.



Now the area of this region is

$$\begin{aligned}
\int_1^\infty e^{-x}\,\mathrm{d}x &= \lim_{t\to\infty}\int_1^t e^{-x}\,\mathrm{d}x \\
&= \lim_{t\to\infty}\left(-e^{-x}\Big|_{x=1}^{x=t}\right) \\
&= \lim_{t\to\infty}\left(-e^{-t}+\frac{1}{e}\right) = \frac{1}{e}.
\end{aligned}$$

Question 2.

a) Verify by differentiation that

$$\int \sec^3\theta \, \mathrm{d}\theta = \frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|) + C.$$

b) Evaluate the arc length of the curve $y = \frac{1}{2}x^2$ from $x = 0$ to $x = 1$.

Solution 2.

a) We compute

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|) + C\right) \\
&\qquad = \frac{1}{2}\left((\sec\theta(\sec^2\theta) + \tan\theta(\sec\theta\tan\theta)) + \frac{\sec\theta\tan\theta + \sec^2\theta}{\sec\theta + \tan\theta}\right) \\
&\qquad = \frac{1}{2}\left((\sec^3\theta) + \tan^2\theta\sec\theta + \sec\theta\frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}\right) \\
&\qquad = \frac{1}{2}\left(\sec^3\theta + (\sec^2\theta - 1)\sec\theta + \sec\theta\right) \\
&\qquad = \frac{1}{2}\left(2\sec^3\theta\right) = \sec^3\theta.
\end{aligned}$$

b) The length of the curve is

$$\begin{aligned}
L &= \int_0^1 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x \\
&= \int_0^1 \sqrt{1 + x^2}\, \mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \left(\sqrt{1 + \tan^2 u}\right)(\sec^2 u \, \mathrm{d}u) \qquad (\text{using } x = \tan u) \\
&= \int_0^{\frac{\pi}{4}} \sec^3 u \, \mathrm{d}u \\
&= \frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\Big]_{u=0}^{u=\frac{\pi}{4}} \\
&= \frac{1}{2}\left(\left[\sqrt{2}\cdot 1 + \ln|\sqrt{2} + 1|\right] - [1 \cdot 0 + \ln|1 + 0|]\right) \\
&= \frac{1}{2}\left(\sqrt{2} + \ln(\sqrt{2} + 1)\right)
\end{aligned}$$

Question 3.

a) Find the volume of the solid generated by rotating the region enclosed by $y = \frac{1}{1+x^2}$, $x = -1$, $x = 1$ and $y = 0$ about the line $y = 2$.

b) Find the volume of the solid generated by rotating the region enclosed by $y = \frac{1}{1+x^2}$, $x = -1$, $x = 1$ and $y = 0$ about the line $x = 2$.

c) (i) Sketch the region bounded by the curves $y = x^2$ and $x = y^2$.

(ii) Using the method of cylindrical shells, find the volume of the solid generated by rotating the above region around the line $x = -1$.

Solution 3.

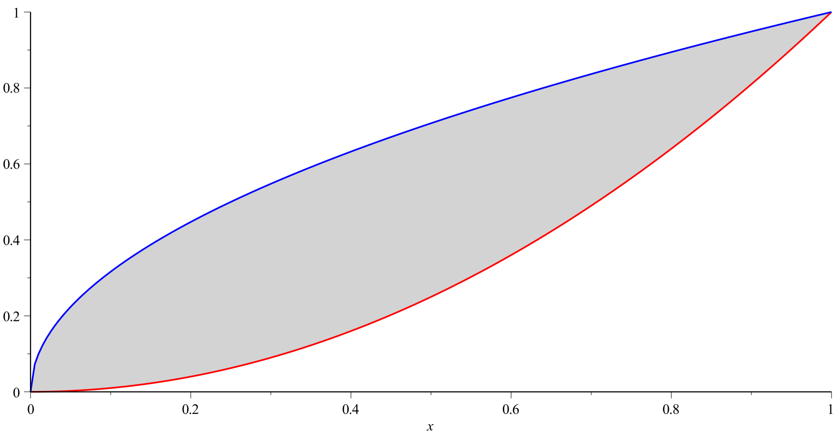
a) We use the washer method with inner radius $2 - \frac{1}{1+x^2}$ and outer radius 2. This gives the volume of the solid as

$$\begin{aligned}
\int_{-1}^{1} \pi \left(2^2 - \left(2 - \frac{1}{1+x^2}\right)^2\right) \mathrm{d}x &= \pi \int_{-1}^{1} \frac{4}{1+x^2} - \frac{1}{(1+x^2)^2}\,\mathrm{d}x \\
&= \pi \left(4 \int_{-1}^{1} \frac{1}{1+x^2}\,\mathrm{d}x - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2 u}{(1+\tan^2 u)^2}\,\mathrm{d}u\right) \\
&= \pi \left(4 \int_{-1}^{1} \frac{1}{1+x^2}\,\mathrm{d}x - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 u\,\mathrm{d}u\right) \\
&= \pi \left(4 \tan^{-1}(x)\Big]_{x=-1}^{x=1} - \frac{2u + \sin 2u}{4}\Big]_{u=-\frac{\pi}{4}}^{u=\frac{\pi}{4}}\right) \\
&= \pi \left(4\left(\left[\frac{\pi}{4}\right] - \left[-\frac{\pi}{4}\right]\right) - \frac{1}{4}\left(\left[\frac{\pi}{2}+1\right] - \left[-\frac{\pi}{2}-1\right]\right)\right) \\
&= \frac{\pi}{4}(7\pi - 2).
\end{aligned}$$

b) This time we have to use the cylindrical shells method. The typical shell has height $h = \frac{1}{1+x^2}$ and radius $r = 2 - x$. Thus the volume of the solid is

$$\begin{aligned}
\int_{-1}^{1} 2\pi h r\,\mathrm{d}r &= \int_{-1}^{1} 2\pi \left(\frac{1}{1+x^2}\right)(2-x)\,\mathrm{d}x \\
&= 2\pi \int_{-1}^{1} \frac{2}{1+x^2} - \frac{x}{1+x^2}\,\mathrm{d}x \\
&= 2\pi \left(\int_{-1}^{1} \frac{2}{1+x^2}\,\mathrm{d}x - \int_{-1}^{1} \frac{x}{1+x^2}\,\mathrm{d}x\right) \\
&= 2\pi \left(2 \int_{0}^{1} \frac{2}{1+x^2}\,\mathrm{d}x + 0\right) \\
&= 8\pi \tan^{-1}(x)\Big]_{x=0}^{x=1} \\
&= 8\pi \left(\frac{\pi}{4}\right) = 2\pi^2.
\end{aligned}$$

c) (i) The region is shown on the next page.



(ii) The height of the cylinder is $h = \sqrt{x} - x^2$ and the radius is $x + 1$. Thus the volume is

$$\begin{aligned}
\int_0^1 2\pi h r \, \mathrm{d}r &= 2\pi \int_0^1 (\sqrt{x} - x^2)(x+1) \, \mathrm{d}x \\
&= 2\pi \int_0^1 x^{\frac{3}{2}} - x^3 + \sqrt{x} - x^2 \, \mathrm{d}x \\
&= 2\pi \left(\frac{2}{5} x^{\frac{5}{2}} - \frac{1}{4} x^4 + \frac{3}{2} x^{\frac{3}{2}} - \frac{1}{3} x^3 \Big]_{x=0}^{x=1} \right) \\
&= 2\pi \left(\frac{2}{5} - \frac{1}{4} + \frac{2}{3} - \frac{1}{3} \right) \\
&= 2\pi \left(\frac{29}{60} \right) = \frac{29}{30}\pi.
\end{aligned}$$

Question 4.

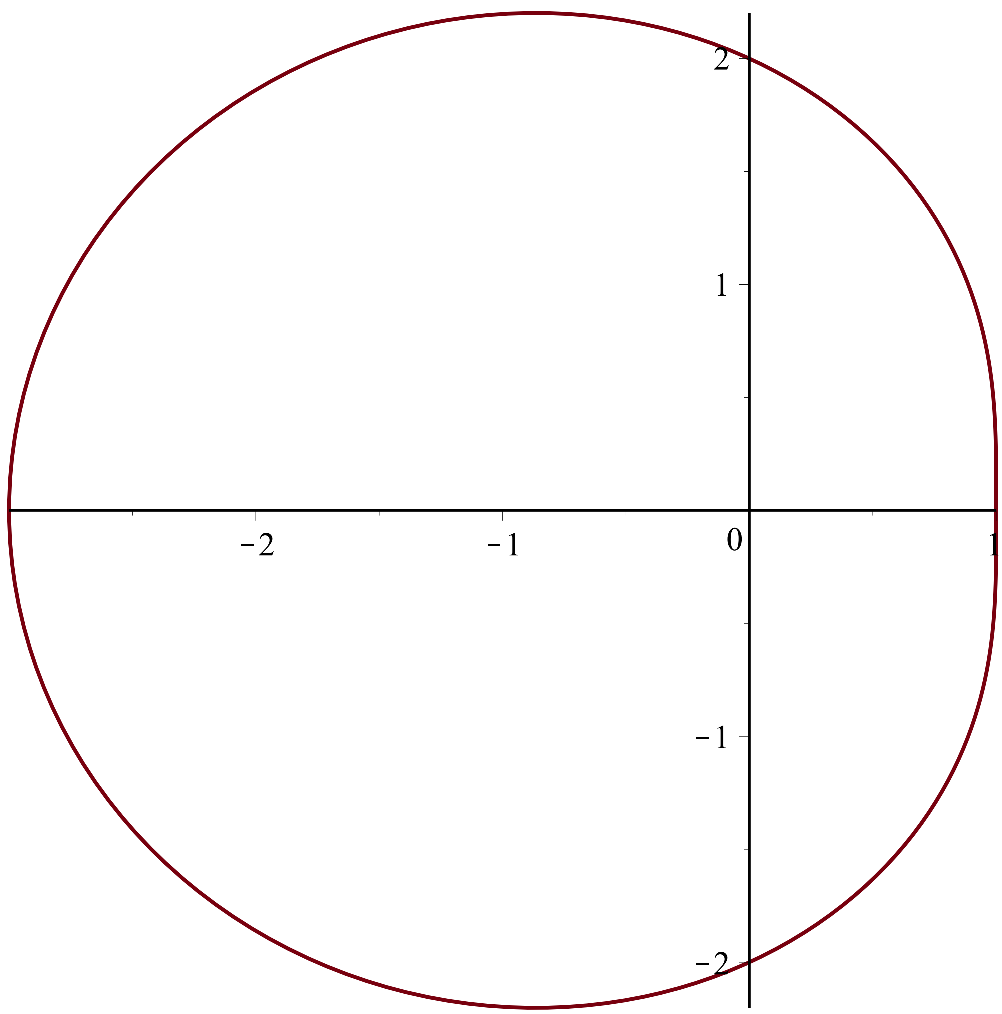
a) Sketch the curve defined in polar coordinates by $r = 2 - \cos\theta$. What is the area bounded by this curve?

b) Find the points on the line $r = 1 + \cos\theta$ where the tangent line is horizontal (there are three such points. . .).

c) Find the length of the curve $r = 1 + \cos\theta$ for $0 \leq \theta \leq 2\pi$ (this is quite tricky, but doable by manipulating the integrand and using a substitution).

Solution 4.

a) The curve is sketched below.



The area is given by

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{2} r^2 \, \mathrm{d}\theta &= \frac{1}{2} \int_0^{2\pi} (2-\cos\theta)^2 \, \mathrm{d}\theta \\
&= \frac{1}{2} \int_0^{2\pi} 4 - 4\cos\theta + \cos^2\theta \, \mathrm{d}\theta \\
&= \frac{1}{2} \int_0^{2\pi} 4 - 4\cos\theta + \frac{1+\cos 2\theta}{2} \, \mathrm{d}\theta \\
&= \frac{1}{2} \left(4\theta - 4\sin\theta + \frac{2\theta + \sin 2\theta}{4} \right]_{\theta=0}^{\theta=2\pi} \Bigg) \\
&= \frac{1}{2} \left(\left[8\pi - 0 + \frac{4\pi + 0}{4} \right] - [0] \right) \\
&= \frac{9\pi}{2}.
\end{aligned}$$

b) We compute $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}} \\
&= \frac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta} \\
&= \frac{(-\sin\theta)\sin\theta + (1+\cos\theta)\cos\theta}{(-\sin\theta)\cos\theta - (1+\cos\theta)\sin\theta} \\
&= -\frac{-\sin^2\theta + \cos\theta + \cos^2\theta}{\sin\theta + 2\sin\theta\cos\theta} \\
&= -\frac{2\cos^2\theta + \cos\theta - 1}{\sin\theta + \sin 2\theta}.
\end{aligned}$$

The tangent line is horizontal when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, which is when $\frac{\mathrm{d}y}{\mathrm{d}\theta} = 0$ (assuming $\frac{\mathrm{d}x}{\mathrm{d}\theta} \neq 0$). So to solve $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ we need to solve

$$2\cos^2\theta + \cos\theta - 1 = 0$$

which yields

$$\cos\theta = -1 \quad \text{ or } \cos t = \frac{1}{2}.$$

Hence the relevant values of θ are π, $\frac{\pi}{3}$ and $\frac{5\pi}{3}$. When θ is $\frac{\pi}{3}$ or $\frac{5\pi}{3}$, $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 0$, so the tangent is horizontal at $\left(\frac{3}{2}, \frac{\pi}{3}\right)$ and at $\left(\frac{3}{2}, \frac{5\pi}{3}\right)$. However, when $\theta = \pi$, we see that $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 0$, so we need to be more careful. We use L'Hospital's rule and get

$$\begin{aligned}
\lim_{\theta\to\pi} \frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{\theta\to\pi} -\left(\frac{4\cos\theta\sin\theta - \sin\theta}{\cos\theta + 2\cos\theta} \right) \\
&= \frac{0}{-1+2} = 0,
\end{aligned}$$

so there is a horizontal tangent at $(0,0)$.

c) The length is given by twice the length of half the loop, which is

$$\begin{aligned}
2\int_0^\pi \sqrt{r^2+\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta &= \int_0^\pi \sqrt{(1+\cos\theta)^2+\sin^2\theta}\,\mathrm{d}\theta \\
&= 2\int_0^\pi \sqrt{2+2\cos\theta}\,\mathrm{d}\theta \\
&= 2\int_0^\pi \sqrt{2+2\cos\theta}\frac{\sqrt{2-2\cos\theta}}{\sqrt{2-2\cos\theta}}\,\mathrm{d}\theta \\
&= 2\int_0^\pi \frac{\sqrt{4-4\cos^2\theta}}{\sqrt{2-2\cos\theta}}\,\mathrm{d}\theta \\
&= 2\int_0^\pi \frac{2\sin\theta}{\sqrt{2-2\cos\theta}}\,\mathrm{d}\theta \\
&= 2\int_0^4 \frac{1}{\sqrt{u}}\,\mathrm{d}u \qquad (u=2-2\cos\theta) \\
&= 2\left(2\sqrt{u}\Big]_{u=0}^{u=4}\right) = 8
\end{aligned}$$

Question 5.

a) Show that the sequence defined recursively by

$$a_1 = 1, \quad a_{n+1} = 3 - \frac{1}{a_n}$$

is convergent and find the limit.

b) Decide whether the following series converge, and if so, find their limits.

(i) $\sum_{n=1}^{\infty} \frac{1}{\pi^n} + \frac{1}{n(n+1)}$

(ii) $\sum_{n=1}^{\infty} (0.7)^{n+2} - (0.2)^{n+1}$

c) Express $0.\overline{17} = 0.17171717\ldots$ as a ratio of integers.

Solution 5.

a) We use induction to show that (a_n) is increasing and bounded.

Bounded. We will show for all n that $1 \le a_n \le 3$. The statement trivially holds when $n = 1$, so assume it holds for $n = k$. Then we have $1 \le a_k \le 3$, which means

$$1 \ge \frac{1}{a_k} \ge \frac{1}{3}$$

and so

$$1 \le 2 \le 3 - \frac{1}{a_k} = a_{k+1} 2\frac{2}{3} \le 3.$$

Hence, by the principle of mathematical induction, the sequence (a_n) is bounded.

Increasing. We will show that $a_{n+1} > a_n$ for all n. Clearly it is true for $n = 1$, since $a_2 = 2 > 1 = a_1$. So suppose it is true for $n = k$, so that $a_k + 1 > a_k$. This mean

$$\frac{1}{a_{k+1}} < \frac{1}{a_k},$$

which means

$$a_{k+2} = 3 - \frac{1}{a_{k+1}} > 3 - \frac{1}{a_k} = a_{k+1}.$$

Hence, by the principle of mathematical induction, the sequence (a_n) is increasing.

Since it is increasing and bounded, the sequence must converge by the monotonic sequence theorem. Call this limit L. Then we have

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \left(3 - \frac{1}{a_n}\right) = 3 - \frac{1}{L}.$$

Hence $L^2 - 3L + 1 = 0$, and so $L = \frac{3\pm\sqrt{9-4}}{2} = \frac{3\pm\sqrt{5}}{2}$. Since $\frac{3-\sqrt{5}}{2} < 1$, this cannot be the solution, so we deduce that $L = \frac{3+\sqrt{5}}{2}$.

b) (i) We use the sum law for convergent series. We see that

$$\sum_{n=1}^{\infty} \frac{1}{\pi^n} = \frac{\frac{1}{\pi}}{1 - \frac{1}{\pi}} = \frac{1}{\pi - 1} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1,$$

and so

$$\sum_{n=1}^{\infty} \frac{1}{\pi^n} + \frac{1}{n(n+1)} = \frac{1}{\pi - 1} + 1.$$

(ii) This time we use the difference law for convergent series. We have

$$\sum_{n=1}^{\infty}(0.7)^{n+2} = \frac{(0.7)^3}{1-0.7} = \frac{\frac{343}{1000}}{\frac{3}{10}} = \frac{343}{300}$$

and

$$\sum_{n=1}^{\infty}(0.2)^{n+1} = \frac{(0.2)^2}{1-0.2} = \frac{\frac{4}{100}}{\frac{8}{10}} = \frac{1}{20}.$$

Hence

$$\sum_{n=1}^{\infty}(0.7)^{n+2} - (0.2)^{n+1} = \frac{343}{300} - \frac{1}{20} = \frac{328}{300}.$$

c) We see that

$$\begin{aligned}
0.\overline{17} &= 0.17\,17\,17\,17\,17\,17\ldots \\
&= 0.17 + 0.0017 + 0.000017 + 0.00000017 + \cdots \\
&= \frac{17}{100} + \frac{17}{10000} + \frac{17}{1000000} + \cdots \\
&= \sum_{n=1}^{\infty} \frac{17}{100}\left(\frac{1}{100}\right)^n \\
&= \frac{\frac{17}{100}}{1-\frac{1}{100}} \\
&= \frac{\frac{17}{100}}{\frac{99}{100}} = \frac{17}{99}.
\end{aligned}$$

MAT132 PAPER HOMEWORK 4

DUE IN RECITATION ON 4/16 OR 4/17

Problem 1. Professors Lee and Sharland cannot decide who should buy the new kite. In order to decide, they play a game. They take it is turns to throw a (standard, six-sided) die, with Professor Lee going first. The winner of the game is the first one to throw a 4. For example, Professor Lee wins if he throws a 4 immediately or the results are

non-4 for Lee, non-4 for Sharland, 4 for Lee

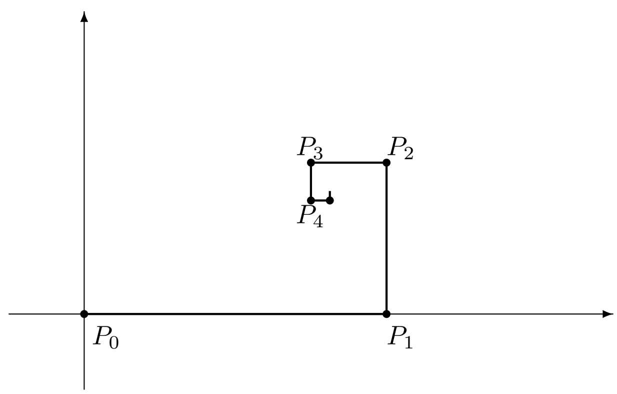
or

non-4 for Lee, non-4 for Sharland, non-4 for Lee, non-4 for Sharland, 4 for Lee

and so on.

Find the probability that Professor Lee wins. (Hint: the calculation requires a certain type of series).

Problem 2. Let $\{P_i\}_{i=0}^{\infty}$ be a sequence of points on a plane. Suppose P_is are placed as on the picture below, so that $|P_0P_1| = 2$, $|P_1P_2| = 1$, $|P_2P_3| = 0.5$, $|P_3P_4| = 0.25, \cdots$. Find the coordinate of the point $P = \lim_{i\to\infty} P_i$.



Paper Homework 4 Solution

1. Consider all possible cases and their probabilities that Professor Lee wins.

   - 4 for Lee : $\dfrac{1}{6}$.
   - non-4 for Lee, non-4 for Sharland, 4 for Lee : $\dfrac{5}{6}\cdot\dfrac{5}{6}\cdot\dfrac{1}{6}$.
   - non-4 for Lee, non-4 for Sharland, non-4 for Lee, non-4 for Sharland, 4 for Lee : $\dfrac{5}{6}\cdot\dfrac{5}{6}\cdot\dfrac{5}{6}\cdot\dfrac{5}{6}\cdot\dfrac{1}{6}$.
   - non-4 for Lee, non-4 for Sharland, non-4 for Lee, non-4 for Sharland, non-4 for Lee, non-4 for Sharland, 4 for Lee : $\dfrac{5}{6}\cdot\dfrac{5}{6}\cdot\dfrac{5}{6}\cdot\dfrac{5}{6}\cdot\dfrac{5}{6}\cdot\dfrac{5}{6}\cdot\dfrac{1}{6}$.

   and so on.
   If we sum up these probabilities, then we can write the sum as an infinite geometric series.

   $$\frac{1}{6}+\left(\frac{5}{6}\right)^2\cdot\frac{1}{6}+\left(\frac{5}{6}\right)^4\cdot\frac{1}{6}+\left(\frac{5}{6}\right)^6\cdot\frac{1}{6}+\cdots=\sum_{i=1}^{\infty}\frac{1}{6}\left(\frac{25}{36}\right)^{i-1}$$

   The sum is $\dfrac{\frac{1}{6}}{1-\frac{25}{36}}=\dfrac{6}{11}$.

2. We first compute x-coordinates of points $\{P_i\}_0^\infty$.

   - x-coordinate of P_0 is : 0.
   - x-coordinate of P_1 is : $0+2$.
   - x-coordinate of P_2 is : $0+2$.
   - x-coordinate of P_3 is : $0+2-\dfrac{1}{2}$.
   - x-coordinate of P_4 is : $0+2-\dfrac{1}{2}$.
   - x-coordinate of P_5 is : $0+2-\dfrac{1}{2}+\dfrac{1}{8}$.
   - x-coordinate of P_6 is : $0+2-\dfrac{1}{2}+\dfrac{1}{8}$.
   - x-coordinate of P_7 is : $0+2-\dfrac{1}{2}+\dfrac{1}{8}-\dfrac{1}{32}$.

   and so on. Then the x-coordinate of $P_{2n-1}=\displaystyle\sum_{i=1}^{n}\left(-\frac{1}{2}\right)\left(-\frac{1}{4}\right)^{i-2}$. As $n\to\infty$, $P_{2n-1}\to\dfrac{2}{1-(-\frac{1}{4})}=\dfrac{8}{5}$.

   Similarly, the y-coordinate of $P_{2n}=\displaystyle\sum_{i=1}^{n}\left(\frac{1}{4}\right)^{i-1}$. So, as $n\to\infty$, $P_{2n}\to\dfrac{1}{1-(-\frac{1}{4})}=\dfrac{4}{5}$.
   Therefore, $P_i\to(\frac{8}{5},\frac{4}{5})$ as $n\to\infty$.

MAT132 PAPER HOMEWORK 5

DUE IN RECITATION ON 4/30 OR 5/1

Problem 1. We will prove the Generalized Binomial Theorem; that is, for any k and for $|x| < 1$

$$(1+x)^k = 1 + kx + \cdots + \frac{k(k-1)\cdots(k-(n-1))}{n!}x^n + \cdots = \sum_{n=0}^{\infty} \binom{k}{n} x^n.$$

(i) Show that $\binom{k-1}{n} + \binom{k-1}{n-1} = \binom{k}{n}$

(ii) Let $g(x) = \sum_{n=0}^{\infty} \binom{k}{n} x^n$ and show that $g'(x) = k \sum_{n=0}^{\infty} \binom{k-1}{n} x^n$.

(iii) Use parts (i) and (ii) to show that $(1+x)g'(x) = kg(x)$.

(iv) Let $f(x) = (1+x)^{-k} g(x)$ and, by differentiating, deduce that f is a constant function.

(v) Using the fact that $g(0) = 1$ and part (iv), show that $g(x) = (1+x)^k$.

Problem 2. Professors Lee and Sharland are driving to the store to buy their new kite. At one point they are traveling at $12\,\mathrm{m/s}$ and accelerating at $2\,\mathrm{m/s^2}$.

(i) Write down the second-degree Taylor polynomial that approximates the distance traveled from this point after t seconds.

(ii) Use this approximation to compute how far they travel in the following 2 seconds.

(iii) Explain why this approximation does not give a good estimate for how far they travel in the next minute.

MAT132 PAPER HOMEWORK 3

DUE IN RECITATION ON 4/2 OR 4/3

Problem 1. We will prove the Generalized Binomial Theorem; that is, for any k and for $|x| < 1$

$$(1+x)^k = 1 + kx + \cdots + \frac{k(k-1)\cdots(k-(n-1))}{n!}x^n + \cdots = \sum_{n=0}^{\infty} \binom{k}{n} x^n.$$

(i) Show that $\binom{k-1}{n} + \binom{k-1}{n-1} = \binom{k}{n}$

(ii) Let $g(x) = \sum_{n=0}^{\infty} \binom{k}{n} x^n$ and show that $g'(x) = k \sum_{n=0}^{\infty} \binom{k-1}{n} x^n$.

(iii) Use parts (i) and (ii) to show that $(1+x)g'(x) = kg(x)$.

(iv) Let $f(x) = (1+x)^{-k} g(x)$ and, by differentiating, deduce that f is a constant function.

(v) Using the fact that $g(0) = 1$ and part (iv), show that $g(x) = (1+x)^k$.

Solution 1.

(i) We get

$$\begin{aligned}
\binom{k-1}{n} + \binom{k-1}{n-1} &= \frac{(k-1)(k-2)\cdots(k-(n-1))(k-n)}{n!} + \frac{(k-1)(k-2)\cdots(k-(n-1))}{(n-1)!} \\
&= \frac{(k-1)(k-2)\cdots(k-(n-1))}{(n-1)!}\left(\frac{k-n}{n} + 1\right) \\
&= \frac{(k-1)(k-2)\cdots(k-(n-1))}{(n-1)!}\left(\frac{(k-n)+n}{n}\right) \\
&= \frac{k(k-1)(k-2)\cdots(k-(n-1))}{n!} = \binom{k}{n}.
\end{aligned}$$

(ii) Differentiating $g(x) = \sum_{n=0}^{\infty} \binom{k}{n} x^n$ term by term gives

$$\begin{aligned}
g'(x) &= \sum_{n=1}^{\infty} n\binom{k}{n} x^{n-1} \\
&= \sum_{n=0}^{\infty} (n+1)\binom{k}{n+1} x^n \\
&= \sum_{n=0}^{\infty} (n+1)\frac{k(k-1)\cdots(k-(n-1))(k-n)}{(n+1)!} x^n \\
&= \sum_{n=0}^{\infty} k\frac{(k-1)\cdots(k-(n-1))(k-n)}{n!} x^n \\
&= k\sum_{n=0}^{\infty} \binom{k-1}{n} x^n.
\end{aligned}$$

(iii) Now we get

$$\begin{aligned}
(1+x)g'(x) &= (1+x)k\sum_{n=0}^{\infty}\binom{k-1}{n}x^n \\
&= k\left(\sum_{n=0}^{\infty}\binom{k-1}{n}x^n + \sum_{n=0}^{\infty}\binom{k-1}{n}x^{n+1}\right) \\
&= k\left(\sum_{n=0}^{\infty}\binom{k-1}{n}x^n + \sum_{n=0}^{\infty}\binom{k-1}{n-1}x^{n}\right) \\
&= k\left(\sum_{n=0}^{\infty}\left(\binom{k-1}{n} + \binom{k-1}{n-1}\right)x^{n}\right) \\
&= k\sum_{n=0}^{\infty}\binom{k}{n}x^n \qquad \text{by part (i).} \\
&= kg(x).
\end{aligned}$$

(iv) The derivative of f is found by the product rule:

$$\begin{aligned}
f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left((1+x)^{-k}g(x)\right) \\
&= (1+x)^{-k}g'(x) - k(1+x)^{-k-1}g(x) \\
&= (1+x)^{-k-1}((1+x)g'(x) - kg(x)) \;=\; 0,
\end{aligned}$$

this last fact following from the fact that the term in parantheses is 0 by part (iii). Since $f'(x) = 0$, it follows (from the Mean Value Theorem) that f is constant.

(v) Since $g(0) = 1$, we see that $f(0) = (1+0)^{-k}g(0) = 1$. Hence $f(x) = 1$ for all x. Then since $1 = f(x) = (1+x)^{-k}g(x)$ we get $g(x) = (1+x)^k$ as required.

Problem 2. Professors Lee and Sharland are driving to the store to buy their new kite. At one point they are traveling at $12\,\mathrm{m/s}$ and accelerating at $2\,\mathrm{m/s^2}$.

(i) Write down the second-degree Taylor polynomial that approximates the distance traveled from this point after t seconds.

(ii) Use this approximation to compute how far they travel in the following 2 seconds.

(iii) Explain why this approximation does not give a good estimate for how far they travel in the next minute.

Solution 2.

(i) Let $s(t)$ be the distance from the starting point after t seconds. Then the given information tells us that $s(0) = 0$, $s'(0) = 12$ and $s''(0) = 2$. Hence

$$T_2(x) = 0 + 12t + t^2.$$

(ii) $s(2) \approx T_2(2) = 0 + 24 + 4 = 28$.

(iii) The approximation will only work for small values of t. For example, this model supposes that the acceleration is $2\,\mathrm{m/s^2}$, which means that after one minute they will be traveling at $132\,\mathrm{m/s}$, which is about 295 mph (and they are not in that much of a rush to buy a new kite)!

Notes on Second Order Linear Differential Equations

Stony Brook University Mathematics Department

1. The general second order homogeneous linear differential equation with constant coefficients looks like

$$Ay'' + By' + Cy = 0,$$

where y is an unknown function of the variable x, and A, B, and C are constants. If $A = 0$ this becomes a first order linear equation, which we already know how to solve. So we will consider the case $A \neq 0$. We can divide through by A and obtain the equivalent equation

$$y'' + by' + cy = 0$$

where $b = B/A$ and $c = C/A$.

"Linear with constant coefficients" means that each term in the equation is a constant times y or a derivative of y. "Homogeneous" excludes equations like $y'' + by' + cy = f(x)$ which can be solved, in certain important cases, by an extension of the methods we will study here.

2. In order to solve this equation, we guess that there is a solution of the form

$$y = e^{\lambda x},$$

where λ is an unknown constant. Why? Because it works!

We substitute $y = e^{\lambda x}$ in our equation. This gives

$$\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = 0.$$

Since $e^{\lambda x}$ is never zero, we can divide through and get the equation

$$\lambda^2 + b\lambda + c = 0.$$

Whenever λ is a solution of this equation, $y = e^{\lambda x}$ will automatically be a solution of our original differential equation, and if λ is not a solution, then $y = e^{\lambda x}$ cannot solve the differential equation. So the substitution $y = e^{\lambda x}$ transforms the differential equation into an algebraic equation!

Example 1. Consider the differential equation

$$y'' - y = 0.$$

Plugging in $y = e^{\lambda x}$ give us the associated equation

$$\lambda^2 - 1 = 0,$$

which factors as

$$(\lambda+1)(\lambda-1)=0;$$

this equation has $\lambda = 1$ and $\lambda = -1$ as solutions. Both $y = e^x$ and $y = e^{-x}$ are solutions to the differential equation $y'' - y = 0$. (You should check this for yourself!)

Example 2. For the differential equation

$$y'' + y' - 2y = 0,$$

we look for the roots of the associated algebraic equation

$$\lambda^2 + \lambda - 2 = 0.$$

Since this factors as $(\lambda-1)(\lambda+2) = 0$, we get both $y = e^x$ and $y = e^{-2x}$ as solutions to the differential equation. Again, you should check that these are solutions.

3. For the general equation of the form

$$y'' + by' + cy = 0,$$

we need to find the roots of $\lambda^2 + b\lambda + c = 0$, which we can do using the quadratic formula to get

$$\lambda = \frac{-b \pm \sqrt{b^2-4c}}{2}.$$

If the *discriminant* $b^2 - 4c$ is positive, then there are two solutions, one for the plus sign and one for the minus.

This is what we saw in the two examples above.

Now here is a useful fact about linear differential equations: if y_1 and y_2 are solutions of the homogeneous differential equation $y'' + by' + cy = 0$, then so is the linear combination $py_1 + qy_2$ for any numbers p and q. This fact is easy to check (just plug $py_1 + qy_2$ into the equation and regroup terms; note that the coefficients b and c do not need to be constant for this to work. This means that for the differential equation in Example 1 ($y'' - y = 0$), any function of the form

$$pe^x + qe^{-x} \qquad \text{where } p \text{ and } q \text{ are any constants}$$

is a solution. Indeed, while we can't justify it here, *all* solutions are of this form. Similarly, in Example 2, the general solution of

$$y'' + y' - 2y = 0$$

is

$$y = pe^x + qe^{-2x}, \qquad \text{where } p \text{ and } q \text{ are constants.}$$

4. If the discriminant $b^2 - 4c$ is negative, then the equation $\lambda^2 + b\lambda + c = 0$ has no solutions, unless we enlarge the number field to include $i = \sqrt{-1}$, i.e. unless we work with complex numbers. If $b^2 - 4c < 0$, then since we can write any positive number as a square k^2, we let $k^2 = -(b^2 - 4c)$.

Then ik will be a square root of b^2-4c, since $(ik)^2 = i^2k^2 = (-1)k^2 = -k^2 = b^2-4c$. The solutions of the associated algebraic equation are then

$$\lambda_1 = \frac{-b+ik}{2}, \quad \lambda_2 = \frac{-b-ik}{2}.$$

Example 3. If we start with the differential equation $y''+y=0$ (so $b=0$ and $c=1$) the discriminant is $b^2-4c=-4$, so $2i$ is a square root of the discriminant and the solutions of the associated algebraic equation are $\lambda_1 = i$ and $\lambda_2 = -i$.

Example 4. If the differential equation is $y''+2y'+2y=0$ (so $b=2$ and $c=2$ and $b^2-4c = 4-8=-4$). In this case the solutions of the associated algebraic equation are $\lambda = (-2\pm 2i)/2$, i.e. $\lambda_1 = -1+i$ and $\lambda_2 = -1-i$.

5. Going from the solutions of the associated algebraic equation to the solutions of the differential equation involves interpreting $e^{\lambda x}$ as a function of x when λ is a complex number. Suppose λ has real part a and imaginary part ib, so that $\lambda = a+ib$ with a and b real numbers. Then

$$e^{\lambda x} = e^{(a+ib)x} = e^{ax}e^{ibx}$$

assuming for the moment that complex numbers can be exponentiated so as to satisfy the law of exponents. The factor e^{ax} does not cause a problem, but what is e^{ibx}? Everything will work out if we take

$$e^{ibx} = \cos(bx) + i\sin(bx),$$

and we will see later that this formula is a necessary consequence of the elementary properties of the exponential, sine and cosine functions.

6. Let us try this formula with our examples.

Example 3. For $y''+y=0$ we found $\lambda_1 = i$ and $\lambda_2 = -i$, so the solutions are $y_1 = e^{ix}$ and $y_2 = e^{-ix}$. The formula gives us $y_1 = \cos x + i\sin x$ and $y_2 = \cos x - i\sin x$.

Our earlier observation that if y_1 and y_2 are solutions of the linear differential equation, then so is the combination py_1+qy_2 for any numbers p and q holds even if p and q are complex constants.

Using this fact with the solutions from our example, we notice that $\frac{1}{2}(y_1+y_2) = \cos x$ and $\frac{1}{2i}(y_1-y_2) = \sin x$ are both solutions. *When we are given a problem with real coefficients it is customary, and always possible, to exhibit real solutions.* Using the fact about linear combinations again, we can say that $y = p\cos x + q\sin x$ is a solution for any p and q. This is the general solution. (It is also correct to call $y = pe^{ix}+qe^{-ix}$ the general solution; which one you use depends on the context.)

Example 4. $y''+2y'+2y=0$. We found $\lambda_1 = -1+i$ and $\lambda_2 = -1-i$. Using the formula we have

$$y_1 = e^{\lambda_1 x} = e^{(-1+i)x} = e^{-x}e^{ix} = e^{-x}(\cos x + i\sin x),$$

$$y_2 = e^{\lambda_2 x} = e^{(-1-i)x} = e^{-x}e^{-ix} = e^{-x}(\cos x - i\sin x).$$

Exactly as before we can take $\frac{1}{2}(y_1+y_2)$ and $\frac{1}{2i}(y_1-y_2)$ to get the real solutions $e^{-x}\cos x$ and $e^{-x}\sin x$. (Check that these functions both satisfy the differential equation!) The general solution will be $y = pe^{-x}\cos x + qe^{-x}\sin x$.

7. *Repeated roots.* Suppose the discriminant is zero: $b^2-4c=0$. Then the "characteristic equation" $\lambda^2+b\lambda+c=0$ has one root. In this case both $e^{\lambda x}$ **and** $xe^{\lambda x}$ are solutions of the differential equation.

Example 5. Consider the equation $y''+4y'+4y=0$. Here $b=c=4$. The discriminant is $b^2-4c=4^2-4\times 4=0$. The only root is $\lambda=-2$. Check that **both** e^{-2x} and xe^{-2x} are solutions. The general solution is then $y = pe^{-2x}+qxe^{-2x}$.

8. *Initial Conditions.* For a first-order differential equation the undetermined constant can be adjusted to make the solution satisfy the initial condition $y(0)=y_0$; in the same way the p and the q in the general solution of a second order differential equation can be adjusted to satisfy initial conditions. Now there are two: we can specify both the value and the first derivative of the solution for some "initial" value of x.

Example 5. Suppose that for the differential equation of Example 2, $y''+y'-2y=0$, we want a solution with $y(0)=1$ and $y'(0)=-1$. The general solution is $y=pe^x+qe^{-2x}$, since the two roots of the characteristic equation are 1 and -2. The method is to write down what the initial conditions mean in terms of the general solution, and then to solve for p and q. In this case we have

$$1 = y(0) = pe^0 + qe^{-2\times 0} = p+q$$

$$-1 = y'(0) = pe^0 - 2qe^{-2\times 0} = p-2q.$$

This leads to the set of linear equations $p+q=1, p-2q=-1$ with solution $q=2/3, p=1/3$. You should check that the solution

$$y = \frac{1}{3}e^x + \frac{2}{3}e^{-2x}$$

satisfies the initial conditions.

Example 6. For the differential equation of Example 4, $y''+2y'+2y=0$, we found the general solution $y=pe^{-x}\cos x+qe^{-x}\sin x$. To find a solution satisfying the initial conditions $y(0)=-2$ and $y'(0)=1$ we proceed as in the last example:

$$-2 = y(0) = pe^{-0}\cos 0 + qe^{-0}\sin 0 = p$$

$$1 = y'(0) = -pe^{-0}\cos 0 - pe^{-0}\sin 0 - qe^{-0}\sin 0 + qe^{-0}\cos 0 = -p+q.$$

So $p=-2$ and $q=-1$. Again check that the solution

$$y = -2e^{-x}\cos x - e^{-x}\sin x$$

satisfies the initial conditions.

PRACTICE PROBLEMS FOR MAT132 FINAL

Please note that the first question in the exam will cover the fundamental concepts: a good performance on this question will guarantee you at least a C grade. Below are some practice problems (a lot more than you will find on the exam!).

Question 1.

a) Evaluate

$$\int_0^1 e^{3x} \cos(9x)\, \mathrm{d}x$$

b) Evaluate

$$\int_1^2 x^2 \ln(x)\, \mathrm{d}x$$

c) Compute

$$\int \frac{\ln(x)}{\sqrt{x}}\, \mathrm{d}x$$

d) Compute

$$\int \cos^2\left(\frac{2\theta}{3}\right) d\theta$$

e) Compute

$$\int \frac{t+4}{t^2+5}\, \mathrm{d}t$$

f) Is the following improper integral convergent or divergent?

$$\int_0^1 \frac{e^{1/x}}{x^2}\, \mathrm{d}x$$

g) Is the following improper integral convergent or divergent?

$$\int_0^1 \frac{\ln(x)}{\sqrt{x}}\, \mathrm{d}x$$

h) Compute the area enclosed by curves given below.

$$y = 1 - 2x^2, \quad y = x, \quad x = 1, \quad x = 0$$

(i) Compute the length of the curve given by

$$y = \frac{x^2}{2} + \ln\left(\frac{1}{\sqrt[4]{x}}\right), \qquad 1 \le x \le 2.$$

j) The region $\{(x, y) \in \mathbb{R}^2 |\ 0 \leq y \leq \sin x,\ 0 \leq x \leq \pi\}$ is rotated about the x-axis. Compute the volume of the solid.

k) The region enclosed by the curves $y = x$ and $y = x^2$ is rotated about the y-axis. Compute the volume of the solid.

l) Is the series given by
$$\sum_{n=0}^{\infty} \frac{\sin(e^2 n)}{\pi^n}$$
convergent? Justify your answer.

m) Compute the infinite sum, if it exists:
$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right).$$

n) Solve the initial value problem
$$y' = \frac{\ln x}{xy}, \qquad y(1) = 4$$

o) Use Euler's method with step size 0.1 to estimate $y(0.2)$, where y is the solution to the initial value problem $y' = x + y^2$, $y(0) = 0$.

The following questions are (in our opinion) a little harder than those for Question 1, but should be a guide to the difficulty of the other questions in the final. Again, there are more questions here than on the actual exam, but practice makes perfect! All of the following questions are useful, but some may be more useful than others...

Question 2.

a) Compute
$$\int \frac{3x^2 - 4x + 3}{x^3 + x}\, \mathrm{d}x$$

b) Compute
$$\int \frac{3x^2 - 3x + 2}{x^3 - 2x^2}\, \mathrm{d}x$$

c) Compute
$$\int_{\sqrt{2}}^{2} \frac{\mathrm{d}x}{x^3\sqrt{x^2 - 1}}$$

d) Use Simpson's rule with $n = 8$ to estimate
$$\int_0^2 (x+1)^3$$

Question 3. Let $f(x) = \dfrac{1}{(1-x)^2}$

a) Find (or write down) power series for $f(x)$ and $f'(x)$ about $x = 0$.

b) Compute the sum of the infinite series

$$\sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2^n}$$

c) Decide whether the following series is convergent or divergent.

$$\sum_{n=0}^{\infty} \frac{\sin(n) + 2(-1)^n}{n^3}$$

d) (i) Find first four nonzero terms of the binomial series of

$$f(x) = \frac{1}{\sqrt{1+4x^2}}.$$

What is its radius of convergence?

(ii) We approximate

$$\frac{1}{\sqrt{1+4x^2}}$$

by the Taylor polynomial $T_4(x)$, where $0 \leq x \leq 0.2$, $a = 0$. Find the maximum possible error of this approximation.

e) Prove that $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent. Moreover, using the inequality $R_n \leq \int_n^{\infty} \frac{1}{x^3}\, dx$, find the least n value so that error of approximation by partial sum $s_n = \sum_{i=1}^{n} \frac{1}{i^3}$ is less than 0.01.

f) For which values of p does the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

converge?

Question 4.

a) Find the volume of the region that lies inside $r = 2\sin\theta$ and outside $r = 1$.

b) Find the length of the polar curve $r = \theta^2$ for $0 \leq \theta \leq 2\pi$.

c) Find the slope of the tangent line to the polar curve $r = 2\sin 2\theta$ when $\theta = \frac{\pi}{4}$.

d) Let $z = -\sqrt{2} + \sqrt{2}i$ and $w = \sqrt{3} + i$. Calculate zw, $\frac{z}{w}$ and $\frac{1}{w}$.

e) Find $(\sqrt{3} + i)^3$.

f) Find all solutions to $z^6 = -\sqrt{2} + \sqrt{2}i$.

Question 5.

a) Compute
$$\int_0^\pi \sin^9 x \, \mathrm{d}x$$
b) Compute
$$\int \sec^4 x \tan^2 x \, \mathrm{d}x$$
c) The region R is bounded by the curves $y = x^2$ and $y = 2 - x^2$ is rotated about the line $x = 1$. Use the method of cylindrical shells to find the volume of this solid.

d) Find the volume of a right circular cone (a cone whose height is perpendicular to its base) with height h and base radius r.

e) The velocity v of an object of mass m, falling in the gravitational field with air resistance is modeled by
$$\frac{\mathrm{d}v}{\mathrm{d}t} = g - \frac{k}{m} v$$
where $g = 9.8\,\mathrm{m/s^2}$ is the gravitational acceleration constant, and k is the air resistance parameter. Compute the terminal velocity $\lim_{t\to\infty} v(t)$ for the case where $m = 1\,\mathrm{kg}$ and $k = 0.2\,\mathrm{kg/s}$.

f) Solve the initial value problem
$$y'' - y' + y = 0 \quad y(0) = 2, \quad y'(0) = 0$$
g) Show that $y = x^2 + 2x$ is a solution to
$$y'' - y' + y = x^2$$

BEST OF LUCK!

PRACTICE PROBLEMS FOR MAT132 FINAL

Here are some sketch solutions for the practice problems. Most of the solutions just show the computation required, without discussing how to set up the integral (unless I felt that this construction is difficult). Also, I do not state which substitutions I use or how I split the product in integration by parts, but these are usually clear from the context.

Question 1.

a) Evaluate
$$\int_0^1 e^{3x}\cos(9x)\,\mathrm{d}x$$

b) Evaluate
$$\int_1^2 x^2\ln(x)\,\mathrm{d}x$$

c) Compute
$$\int \frac{\ln(x)}{\sqrt{x}}\,\mathrm{d}x$$

d) Compute
$$\int \cos^2\left(\frac{2\theta}{3}\right)d\theta$$

e) Compute
$$\int \frac{t+4}{t^2+5}\,\mathrm{d}t$$

f) Is the following improper integral convergent or divergent?
$$\int_0^1 \frac{e^{1/x}}{x^2}\,\mathrm{d}x$$

g) Is the following improper integral convergent or divergent?
$$\int_0^1 \frac{\ln(x)}{\sqrt{x}}\,\mathrm{d}x$$

h) Compute the area enclosed by curves given below.
$$y = 1-2x^2, \quad y = x, \quad x = 1, \quad x = 0$$

(i) Compute the length of the curve given by
$$y = \frac{x^2}{2} + \ln\left(\frac{1}{\sqrt[4]{x}}\right), \qquad 1 \le x \le 2.$$

j) The region $\{(x,y)\in\mathbb{R}^2 |\ 0\le y\le \sin x,\ 0\le x\le\pi\}$ is rotated about the x-axis. Compute the volume of the solid.

k) The region enclosed by the curves $y = x$ and $y = x^2$ is rotated about the y-axis. Compute the volume of the solid.

l) Is the series given by

$$\sum_{n=0}^{\infty} \frac{\sin(e^2 n)}{\pi^n}$$

convergent? Justify your answer.

m) Compute the infinite sum, if it exists:

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right).$$

n) Solve the initial value problem

$$y' = \frac{\ln x}{xy}, \qquad y(1) = 4$$

o) Use Euler's method with step size 0.1 to estimate $y(0.2)$, where y is the solution to the initial value problem $y' = x + y^2$, $y(0) = 0$.

Solution 1.

a) Using integration by parts:

$$\begin{aligned}\int_0^1 e^{3x}\cos(9x)\,\mathrm{d}x &= \left[\frac{e^3 x}{9}\sin(9x)\right]_{x=0}^{x=1} - \frac{1}{3}\int_0^1 e^{3x}\sin(9x)\,\mathrm{d}x \\ &= \frac{e^3\sin(9)}{9} - \frac{1}{3}\left(\left[-\frac{e^3 x}{9}\cos(9x)\right] + \frac{1}{3}\int_0^1 e^{3x}\cos(9x)\,\mathrm{d}x\right) \\ &= \frac{e^3\sin(9)}{9} + \frac{e^3\cos(9)-1}{27} - \frac{1}{9}\int_0^1 e^{3x}\cos(9x)\,\mathrm{d}x,\end{aligned}$$

which after rearranging gives

$$\int_0^1 e^{3x}\cos(9x)\,\mathrm{d}x = \frac{1}{10}\left(e^3\sin(9) + \frac{1}{3}(e^3\cos(9)-1)\right).$$

b)

$$\int_1^2 x^2\ln(x)\,\mathrm{d}x = \left[\frac{x^3}{3}\ln(x)\right]_{x=1}^{x=2} - \frac{1}{3}\int_1^2 x^2\,\mathrm{d}x = \frac{8\ln(2)}{3} - \frac{7}{9}.$$

c)

$$\int x^{-\frac{1}{2}}\ln(x)\,\mathrm{d}x = \left[2x^{\frac{1}{2}}\ln(x)\right] - 2\int x^{-\frac{1}{2}}\,\mathrm{d}x = 2\sqrt{x}\ln(x) - 4\sqrt{x} + C.$$

d)

$$\int \cos^2\left(\frac{2\theta}{3}\right)d\theta = \int \frac{1}{2}\left(1 + 2\cos\left(\frac{4\theta}{3}\right)\right)d\theta = \frac{\theta}{2} + \frac{3}{8}\sin\left(\frac{4\theta}{3}\right) + C.$$

e)

$$\int \frac{t+4}{t^2+5}\,\mathrm{d}t = \int \frac{t}{t^2+5}\,\mathrm{d}t + \int \frac{4}{t^2+5}\,\mathrm{d}t = \frac{1}{2}\ln|t^2+5| + \frac{4\sqrt{5}}{5}\tan^{-1}\left(\frac{t}{\sqrt{5}}\right) + C.$$

f)

$$\int_0^1 \frac{e^{1/x}}{x^2}\,\mathrm{d}x = \lim_{t\to 0^+}\int_t^1 \frac{e^{1/x}}{x^2}\,\mathrm{d}x = \lim_{t\to 0^+}\int_t^1 -e^u\,\mathrm{d}u = \lim_{t\to 0^+}(e^{\frac{1}{t}} - e).$$

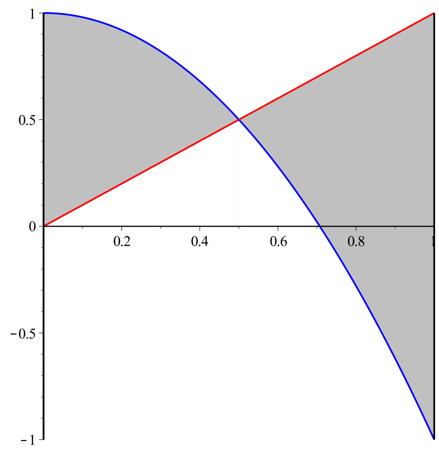
Since this limit is infinite, the integral is divergent.

g)

$$\int_0^1 \frac{\ln(x)}{\sqrt{x}}\,\mathrm{d}x = \lim_{t\to 0^+}\int_t^1 \frac{\ln(x)}{\sqrt{x}}\,\mathrm{d}x = \lim_{t\to 0^+}(-4-(2\sqrt{t}\ln(t)-4\sqrt{t})) = -4.$$

The last fact following from L'Hospital's rule, so the integral is convergent.

h) Notice the region is made up of two subregions.



$$\begin{aligned}
\text{Area} &= \int_0^{\frac{1}{2}} (1-2t^2-t)\,\mathrm{d}t + \int_{\frac{1}{2}}^1 (t-(1-2t^2))\,\mathrm{d}t \\
&= \left(t-\frac{2t^3}{3}-\frac{t^2}{2}\right)\Bigg]_{t=0}^{t=\frac{1}{2}} + \left(\frac{t^2}{2}-t+\frac{2t^3}{3}\right)\Bigg]_{t=\frac{1}{2}}^{t=1} \\
&= \left[\frac{1}{2}-\frac{1}{12}-\frac{1}{8}\right] + \left(\left[\frac{1}{2}-1+\frac{2}{3}\right]-\left[\frac{1}{8}-\frac{1}{2}+\frac{1}{12}\right]\right) \\
&= \frac{7}{24}+\frac{11}{24} = \frac{18}{24} = \frac{3}{4}.
\end{aligned}$$

i) First we compute

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2}{2}-\frac{1}{4}\ln x\right) \\
&= x-\frac{1}{4x}.
\end{aligned}$$

So

$$\begin{aligned}\text{Length} &= \int_1^2 \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \\ &= \int_1^2 \sqrt{1+\left(x-\frac{1}{4x}\right)^2}\,\mathrm{d}x \\ &= \int_1^2 x+\frac{1}{4x}\,\mathrm{d}x = \frac{6+\ln 2}{4}.\end{aligned}$$

j)

$$\text{Volume} = \int_0^\pi \pi\sin^2 x\,\mathrm{d}x = \pi\left[\frac{1}{2}-\frac{\sin 2x}{2}\right]_{x=0}^{x=\pi} = \frac{\pi^2}{2}.$$

k) Using the cylindrical shells method:

$$\text{Volume} = \int_0^1 2\pi(x^2-x^3)\,\mathrm{d}x = \frac{\pi}{6}.$$

l) Since $0 \le \left|\frac{\sin(e^2 n)}{\pi^n}\right| \le \frac{1}{\pi^n}$ and $\sum \frac{1}{\pi^n}$ is convergent, the series is absolutely convergent by the comparison test and so convergent.

m) The partial sum s_n is

$$\begin{aligned}s_n &= \sum_{k=1}^n \ln\left(\frac{k}{k+1}\right) = \sum_{k=1}^n \ln(k)-\ln(k+1) \\ &= (\ln 1-\ln 2)+(\ln 2-\ln 3)+\cdots+(\ln n-\ln(n+1)) \qquad = -\ln(n+1)\end{aligned}$$

which diverges, so the series diverges.

n) This is separable:

$$\begin{aligned}\int y\,\mathrm{d}y &= \int \frac{\ln x}{x}\mathrm{d}x \\ \frac{y^2}{2} &= \frac{1}{2}(\ln x)^2+C\end{aligned}$$

and the initial condition gives $C=16$, so the solution is

$$y=\sqrt{(\ln x)^2+16}$$

o) Set $F(x,y)=x+y^2$, $y_0=0$ and $x_n=(0.1)n$. Then

$$y_1 = y_0+0.1F(x_0,y_0) = 0+0 = 0 y_2 \qquad = y_1+0.1F(x_1,y_1) =)+0.01=0.01.$$

So the estimate is $y(0.2)\approx 0.01$.

Question 2.

a) Compute

$$\int \frac{3x^2-4x+3}{x^3+x}\,\mathrm{d}x$$

b) Compute

$$\int \frac{3x^2-3x+2}{x^3-2x^2}\,\mathrm{d}x$$

c) Compute
$$\int_{\sqrt{2}}^{2} \frac{\mathrm{d}x}{x^3\sqrt{x^2-1}}$$
d) Use Simpson's rule with $n = 8$ to estimate
$$\int_0^2 (x+1)^3\,\mathrm{d}x$$

Solution 2.

a)
$$\int \frac{3x^2-4x+3}{x^3+x}\,\mathrm{d}x = \int \frac{3}{x} - \frac{4}{x^2+1}\,\mathrm{d}x$$
$$= 3\ln x - 4\tan^{-1} x + C.$$

b)
$$\int \frac{3x^2-3x+2}{x^3-2x^2}\,\mathrm{d}x = \int \frac{2}{x-2} + \frac{1}{x} - \frac{1}{x^2}\,\mathrm{d}x$$
$$= \ln(x-2) + \ln x + \frac{1}{x} + C.$$

c)
$$\int_{\sqrt{2}}^{2} \frac{\mathrm{d}x}{x^3\sqrt{x^2-1}} = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec u \tan u}{\sec^3 u\sqrt{\sec^2 u - 1}}\,\mathrm{d}u$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^2 u\,\mathrm{d}u$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{2}(1+\cos 2u)\,\mathrm{d}u = \frac{\pi}{24} - \frac{2-\sqrt{3}}{8}$$

(d)
$$\int_0^2 (x+1)^3\,\mathrm{d}x \approx S_8$$
$$= \frac{1}{12}\left(1^3 + 4\left(\frac{5}{4}\right)^3 + 2\left(\frac{3}{2}\right)^3 + \cdots + 2\left(\frac{5}{2}\right)^3 + 4\left(\frac{7}{4}\right)^3 + 3^3\right) = 20.$$

Question 3. Let $f(x) = \dfrac{1}{(1-x)^2}$

a) Find (or write down) power series for $f(x)$ and $f'(x)$ about $x = 0$.
b) Compute the sum of the infinite series
$$\sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2^n}$$
c) Decide whether the following series is convergent or divergent.
$$\sum_{n=0}^{\infty} \frac{\sin(n) + 2(-1)^n}{n^3}$$

d) (i) Find first four nonzero terms of the binomial series of

$$f(x) = \frac{1}{\sqrt{1+4x^2}}.$$

What is its radius of convergence?

(ii) We approximate

$$\frac{1}{\sqrt{1+4x^2}}$$

by the Taylor polynomial $T_4(x)$, where $0 \le x \le 0.2$, $a = 0$. Find the maximum possible error of this approximation.

e) Prove that $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent. Moreover, using the inequality $R_n \le \int_n^{\infty} \frac{1}{x^3}\, dx$, find the least n value so that error of approximation by partial sum $s_n = \sum_{i=1}^{n} \frac{1}{i^3}$ is less than 0.01.

f) For which values of p does the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

converge?

Solution 3.

a)

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$f'(x) = \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n$$

b)

$$f''(x) = \frac{2}{(1-x)^3} = \sum_{n=0}^{\infty} (n+1)(n+2) x^n$$

from which it follows that

$$\sum_{n=0}^{\infty} (n+1)(n+2) \left(\frac{1}{2}\right)^n = f''\left(\frac{1}{2}\right) = \frac{2}{\left(\frac{1}{2}\right)^3} = 16.$$

c)

$$\sum_{n=0}^{\infty} \frac{\sin(n) + 2(-1)^n}{n^3} = \sum_{n=0}^{\infty} \frac{\sin(n)}{n^3} + 2\sum_{n=0}^{\infty} \frac{(-1)^n}{n^3}.$$

The left sum is absolutely convergent and the right sum is convergent by the alternating series test, hence the series is convergent.

d) (i)

$$\begin{aligned}
\frac{1}{\sqrt{1+4x^2}} &= (1+4x^2)^{-\frac{1}{2}} \\
&= \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (4x^2)^n \\
&= 1 - 2x^2 + 6x^4 - 20x^6 + \cdots
\end{aligned}$$

with radius of convergence $R = \frac{1}{2}$.

(ii) This is an alternating series, so the maximum error is $20(0.2)^6 = \frac{4}{3125}$.

e) The series is convergent by the integral test. Now we want

$$0.01 \geq R_n \geq \int_n^\infty \frac{1}{x^3}\,\mathrm{d}x = \frac{1}{2n^2}$$

which is solved by

$$n \geq \sqrt{50}$$

and so we take $n \geq 8$.

f) We use the integral test

$$\int_2^\infty \frac{1}{x(\ln x)^p}\,\mathrm{d}x = \lim_{t\to\infty} \frac{\ln(t)^{p-1} - \ln(2)}{1-p}$$

and this is finite for $p > 1$, hence the series is convergent for $p > 1$.

Question 4.

a) Find the volume of the region that lies inside $r = 2\sin\theta$ and outside $r = 1$.

b) Find the length of the polar curve $r = \theta^2$ for $0 \leq \theta \leq 2\pi$.

c) Find the slope of the tangent line to the polar curve $r = 2\sin 2\theta$ when $\theta = \frac{\pi}{4}$.

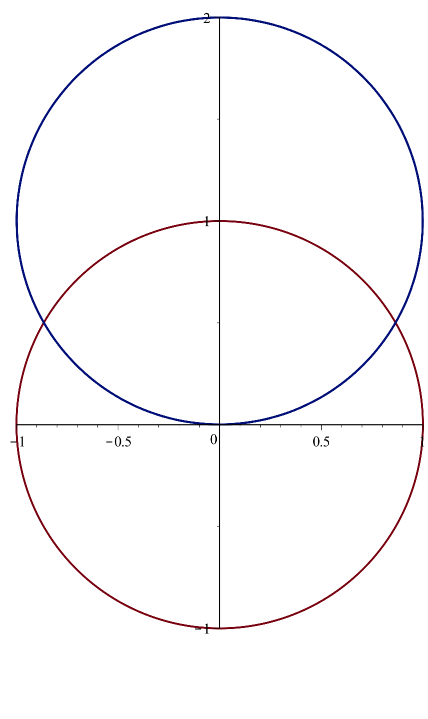
d) Let $z = -\sqrt{2} + \sqrt{2}i$ and $w = \sqrt{3} + i$. Calculate zw, $\frac{z}{w}$ and $\frac{1}{w}$.

e) Find $(\sqrt{3} + i)^3$.

f) Find all solutions to $z^6 = -\sqrt{2} + \sqrt{2}i$.

Solution 4.

a) The curves intersect at $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$.



$$\begin{aligned}
\text{Volume} &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (2\sin\theta)^2 - 1^2 \,\mathrm{d}\theta \\
&= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 4\sin^2\theta - 1 \,\mathrm{d}\theta \\
&= \frac{\pi}{3} + \frac{\sqrt{3}}{2}
\end{aligned}$$

b)

$$\begin{aligned}
\text{Length} &= \int_0^{2\pi} \sqrt{\theta^4 + (2\theta)^2} \,\mathrm{d}\theta \\
&= \int_0^{2\pi} \theta\sqrt{\theta^2 + 4} \,\mathrm{d}\theta \\
&= \int_4^{4\pi^2+4} \frac{1}{2}\sqrt{u} \,\mathrm{d}u \\
&= \frac{8}{3}\left((\pi^2+1)^{\frac{3}{2}} - 1\right).
\end{aligned}$$

c)

$$\begin{aligned}
\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\theta=\frac{\pi}{4}} &= \left.\frac{r\cos\theta + \frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta}{-r\sin\theta + \frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta}\right|_{\theta=\frac{\pi}{4}} \\
&= \left.\frac{(2\sin 2\theta)\cos\theta + (4\cos 2\theta)\sin\theta}{-(2\sin 2\theta)\sin\theta + (4\cos 2\theta)\cos\theta}\right|_{\theta=\frac{\pi}{4}} \\
&= -1.
\end{aligned}$$

d) It is easier to work in polar form here.

$$z = -\sqrt{2} + \sqrt{2}i = 2\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right) \qquad w = \sqrt{3} + i = 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right).$$

Which then gives

$$\begin{aligned}
zw &= 4\left(\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right) \\
\frac{z}{w} &= \left(\cos\left(\frac{7\pi}{12}\right) + i\sin\left(\frac{7\pi}{12}\right)\right) \\
\frac{1}{w} &= \frac{1}{2}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)
\end{aligned}$$

e) $w^3 = 8\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) = 8i.$

f) The sixth roots of z are

$$w_1 = \sqrt[6]{2}\left(\cos\left(\frac{3\pi}{24}\right) + i\sin\left(\frac{3\pi}{24}\right)\right)$$
$$w_2 = \sqrt[6]{2}\left(\cos\left(\frac{11\pi}{24}\right) + i\sin\left(\frac{11\pi}{24}\right)\right)$$
$$w_3 = \sqrt[6]{2}\left(\cos\left(\frac{19\pi}{24}\right) + i\sin\left(\frac{19\pi}{24}\right)\right)$$
$$w_4 = \sqrt[6]{2}\left(\cos\left(\frac{27\pi}{24}\right) + i\sin\left(\frac{27\pi}{24}\right)\right)$$
$$w_5 = \sqrt[6]{2}\left(\cos\left(\frac{35\pi}{24}\right) + i\sin\left(\frac{35\pi}{24}\right)\right)$$
$$w_6 = \sqrt[6]{2}\left(\cos\left(\frac{43\pi}{24}\right) + i\sin\left(\frac{43\pi}{24}\right)\right).$$

Question 5.

a) Compute

$$\int_0^{\pi} \sin^9 x \, \mathrm{d}x$$

b) Compute

$$\int \sec^4 x \tan^2 x \, \mathrm{d}x$$

c) The region R is bounded by the curves $y = x^2$ and $y = 2 - x^2$ is rotated about the line $x = 1$. Use the method of cylindrical shells to find the volume of this solid.

d) Find the volume of a right circular cone (a cone whose height is perpendicular to its base) with height h and base radius r.

e) The velocity v of an object of mass m, falling in the gravitational field with air resistance is modeled by

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g - \frac{k}{m}v$$

where $g = 9.8\,\mathrm{m/s^2}$ is the gravitational acceleration constant, and k is the air resistance parameter. Compute the terminal velocity $\lim_{t\to\infty} v(t)$ for the case where $m = 1\,\mathrm{kg}$ and $k = 0.2\,\mathrm{kg/s}$.

f) Solve the initial value problem

$$y'' - y' + y = 0 \quad y(0) = 2, \quad y'(0) = 0$$

g) Show that $y = x^2 + 2x$ is a solution to

$$y'' - y' + y = x^2$$

Solution 5.

a)

$$\int_0^{\pi} \sin^9 x \, \mathrm{d}x = \int_0^{\pi} (1 - \cos^2 x) \sin x \, \mathrm{d}x = \int_{-1}^{1} (1 - u^2)^4 \, \mathrm{d}u = \frac{256}{315}.$$

b)

$$\int \sec^4 x \tan^2 x \, \mathrm{d}x = \int \sec^2 x (1 + \tan^2 x) \tan^2 x \, \mathrm{d}x = \int (1+u^2)u^2 \, \mathrm{d}u = \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C$$

c)

$$\text{Volume} = 2\pi \int_0^1 ((2-x^2) - x^2)(1-x) \, \mathrm{d}x = 4\pi \int_0^1 (1 - x - x^2 + x^3) \, \mathrm{d}x = \frac{16\pi}{3}.$$

d) Putting the apex at the origin, we see that each cross-section is a disc of radius $R = \frac{rx}{h}$. Hence

$$\text{Volume} = \int_0^h \pi \left(\frac{r}{h}x\right)^2 \mathrm{d}x = \frac{1}{3}\pi r^2 h.$$

e) This is a separable equation, so we need to solve

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g - \frac{k}{m}v = 9.8 - 0.2v$$
$$\int \frac{\mathrm{d}v}{9.8 - 0.2v} = \int \mathrm{d}t$$
$$5\ln(9.8 - 0.2v) = t + C$$
$$v(t) = 49 + Ae^{-\frac{t}{5}} \qquad (A = e^C)$$

and so the terminal velocity is $49\,\mathrm{m/s}$.

f) The characterisitic equation is

$$\lambda^2 - \lambda + 1 = 0$$

which is solved by

$$\lambda_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i \qquad \lambda_2 = \frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

Hence the general solution is

$$y(x) = Ae^{\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right) + Be^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right).$$

The initial values give $A = 2$ and $B = -\frac{2\sqrt{3}}{3}$, so the solution is

$$2e^{\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right) - \frac{2\sqrt{3}}{3}e^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right).$$

g)

$$y = x^2 + 2x \qquad y' = 2x + 2 \qquad y'' = 2$$

and so

$$y'' - y' + y = 2 - (2x+2) + (x^2 + 2x) = x^2$$

as required.

BEST OF LUCK!