MAT 127: Calculus C
Fall 2018

Welcome to MAT 127

A continuation of MAT 126, covering: sequences, series, Taylor series, differential equations and modeling. Detailed syllabus is available **here**

For more information, see the **General information** page.

Announcements

Midterm 1 is on October 2. More details **here**.

Midterm 2 is on October 29. More details **here**.

Final Exam is on December 17. More details **here**.

MAT 127: Calculus C
Fall 2018

General Information

Schedule: Course schedule can be found **here**. Information about instructors' office hours and contact information, can be found **here**.

Course description: A continuation of MAT 126, covering sequences, series, Taylor series, differential equations and modeling. Detailed syllabus is available **here**.

Textbook: Single Variable Calculus (Stony Brook Edition 4), by James Stewart. You would also need WebAssign access code. For a detalied discussion of buying options, please see this **page**.

Calculators use: Some homework problems may require use of calculator. Graphing calculator will not be necessary. Calculators will not be allowed in the final exam or the midterms.

Homeworks and exams: There will be weekly homework assignments, consisting of online homework (WebAssign) and some paper homeworks. For more details, go **here**.

In addition, there will be two midterm exams and the final exam. The time of these exams are as follows:

Midterm 1: 8:45pm-10:15pm, 10/2

Midterm 2: 8:45pm-10:15pm, 10/29

Final exam: 11:15am-1:45pm, 12/17

Final Grade: Your final grade will be determined by the weighted average of grades for homeworks, midterms, and the final exam:

Homeworks: 15%
Midterms: 25% each
Final exam: 35%

Information for students with disabilities
If you have a physical, psychological, medical, or learning disability that may impact your course work, please contact Disability Support Services at (631) 632-6748 or http://studentaffairs.stonybrook.edu/dss/. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential.

Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to the following website: http://www.sunysb.edu/ehs/fire/disabilities.shtml

Disruptive Behavior: Stony Brook University expects students to maintain standards of personal integrity that are in harmony with the educational goals of the institution; to observe national, state, and local laws and University regulations; and to respect the rights, privileges, and property of other people. Faculty are required to report to the Office of Judicial Affairs any disruptive behavior that interrupts their ability to teach, compromises the safety of the learning environment, and/or inhibits students' ability to learn.

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MLC stands for Math Learning Center, located in the basement of the math tower (room S-235); P-143 is the undegraduate math office located on the ground floor (P level) of the math tower, across the hall from the elevators.

Section	Room	Instructor	email	Office	Office hours
Lecture 1 (MWF 10:00am-10:53am)	Library E4320	Chuanhao Wei	chuanhao.wei@stonybrook.edu	Math Tower 3-116	Mon 11:00am-12:00pm, Wed 11:00am-12:00pm (Math Tower 3-116); Tu 10:00am-11:00am (MLC)
Lecture 2 (MF 1:00pm-2:20pm)	Library E4320	Jingrui Cheng	Jingrui.Cheng@stonybrook.edu	Simons Center 509	Tu 1:30pm-2:30pm, Th 10:00am-11:00am (Simons Center 509); Wed 6:00pm-7:00pm (MLC)
					Wed 1:00pm -

Lecture 3 (TuTh 10:00am-11:20am)	Library E4330	Sabya Mukherjee (course coordinator)	sabya@math.stonybrook.edu	Math Tower 4-115	3:00pm (Math Tower 4-115); Th 12:00pm - 1:00pm (MLC)
Lecture 4 (TuTh 5:30pm-6:50pm)	Library W4540	Babak Modami	babak.modami@stonybrook.edu	Math Tower 3-114	Tu 1:00pm-2:00pm, Th 1:00pm-2:00pm (Math Tower 3-114); Th 10:00am-11:00am (MLC)

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Syllabus and Weekly Plan

Week	Dates	Notes	Topics	HW	Due Date
1	8/27-8/31		8.1	WebAssign assigment 1	9/7
2	9/3-9/7	9/3: no classes labor day	8.2	WebAssign assigment 2	9/14
3	9/10-9/14		8.3	WebAssign assigment 3	9/21
4	9/17-9/21		8.3, 8.4	WebAssign assigment 4	9/28
5	9/24-9/28		8.5, 8.6	WebAssign assigment 5	10/13
6	10/1-10/5	10/2: Midterm 1	8.7	WebAssign assigment 6	10/21
7	10/8-10/12		8.7, 8.8		
8	10/15-10/19		8.8, 7.1	WebAssign assigment 7	10/26
9	10/22-10/26		7.2	WebAssign assigment 8	11/2
10	10/29-	10/29:	7.3	WebAssign assignment	11/19

	11/2	Midterm 2		9	
11	11/5-11/9		7.4		
12	11/12-11/16		7.4, 7.5	WebAssign assignment 10	12/15
13	11/19-11/23	11/21-11/25: Thanksgiving break	7.6	WebAssign assignment 11	12/15
14	11/26-11/30		**Second Order Linear Differential Equations**	Problems 4, 6, 14, and 15 of **this set**	12/7
15	12/3-12/7		Review		
	12/17	12/17: **Final Exam**			

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The best way to access WebAssign is through **Blackboard**. From within Blackboard, click on the link for your MAT 127 section. Click on "Tools" in the menu on the left, and then on "Access Webassign". That should automatically log you into your WebAssign account. You do not need a course key or any other information.

At the beginning of the semester there is a 2 week "grace period" during which you may access WebAssign without an access code. But within the first 2 weeks you are required to purchase a WebAssign access code (either bundled with a textbook, or as a stand-alone access code, whichever you prefer). Without a WebAssign access code, you will not be able to continue accessing WebAssign. That means you will not be able to complete the WebAssign assignments.

You can access the web interface from any computer with Internet access and a recent web browser (the computers in the **SINC sites**, for instance).

When you first access the WebAssign account, please go to the My Options page (in the upper right of the screen) and put in your email address.

After they are assigned, the online problems may be completed anytime before the assigned deadline. You can look at problems online, print them out, work on them as long as you like, and then answer them in a later Internet session (before the deadline). The online problems are automatically graded with instant feedback.

The online problems are of different types; some are short answer and some are multiple choice. There are different problems and different variants of the same problem. Different students will be assigned different problems. So do not try to compare your answers to another student's answers.

If you got the answer wrong, you can retry it. However, you with each wrong answer you are losing points: getting the answer right on the 1st try gives you full credit, getting it right on the second try gives you 1/2 of the credit, getting it right on the 3rd try gives 1/3 of the credit, etc.

Doing the Assignment

WebAssign has a variety of different question types, ranging from multiple choice to questions that require you to type the formula as an answer. Here are some things to keep in mind as you work through your assignments:

- Some questions require entering formulas. You must use the exact variables specified in the questions; the system is **case sensitive**: variable a is not the same as A. The order is not important as long as it is mathematically correct. Clicking on the eye button previews the expression you enter in proper mathematical notation. Clicking on the symbolic formatting help button provides tips for using the correct keystrokes.
- When you click on some WebAssign math questions an input palette will open. This palette, called mathPad, will help you enter your answer in proper notation.
- You can save your work without grading by selecting the Save Work button at the end of the question. After you save your work, it will be available to you the next time you click the assignment.
- Please note that WebAssign will not automatically submit your answers for scoring if you only **Save** your work. Please be sure to **Submit** prior to the due date and time.
- You can submit answers by question part or for the entire assignment. To submit an individual question answer for grading, click the "Submit New Answers to Question" button at the bottom of each question. To submit the entire assignment for grading, click the "Submit All New Answers" button at the end of the assignment.
- Some WebAssign questions check the number of significant figures in your answer. If you enter the

correct value with the wrong number of significant figures, you will not receive credit, but you will receive a hint that your number does not have the correct number of significant figures.

- While different students may get slightly different versions of the assignment, your questions will be the same every time you return. This means you can print out the assignment, work the problems, and then come back later and put in your answers. Since you get multiple attempts to get the question correct, be sure to leave yourself enough time to rework the problems that you did wrong.
- Each question is (typically) worth one point. If a question has multiple answers, each of those are worth a fraction of a point.
- If you put in a wrong answer for a question and ask to have it graded, you will be told it is wrong and be ably to try again. However, if you put in the correct answer on the second try, you get half credit. On the third try, you get 1/3 credit, and so on.
- If you have issues with the assignment, you can use the "Ask your Teacher" button to send a message to your TA and/or lecturer. You should make it clear which problem you are talking about, and what, specifically, your issue is. Using "Ask your teacher" is prefered to sending an email because your question gets saved with your assignment.

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Exams

There will be two midterm exams and the final exam. The time of these exams are as follows:

Midterm 1: 8:45pm-10:15pm, 10/2

Midterm 2: 8:45pm-10:15pm, 10/29

Final exam: 11:15am-1:45pm, 12/17

Midterm 1

- **Date and place**: Tuesday, 10/2, 8:45-10:15pm. Lecture 1 (MWF 10:00-10:53am) and Lecture 4 (TuTh 5:30-6:50pm) will be in Humanities 1006. Lecture 2 (MF 1:00-2:20pm) will be in Javits 101. Lecture 3 (TuTh 10:00-11:20am) will be in Humanities 1003.
- **Material covered**: 8.1-8.6 of the textbook.
- **Review session**: 10/1, 7:00pm, Harriman Hall 137.
- **Policies**: no books, notes, or calculators. You must bring your ID with you. When writing solutions, you must show your reasoning, not just the answer. Answers without justification will get only partial credit. Your solutions should be written so that the grader is able to follow your reasoning and computations.
- **Practice problems**: available **here**, with **solutions**.

Midterm 2

- **Date and place**: Monday, 10/29, 8:45-10:15pm. Lecture 1 (MWF 10:00-10:53am) and Lecture 4 (TuTh 5:30-6:50pm) will be in Humanities 1006. Lecture 2 (MF 1:00-2:20pm) will be in Javits 101. Lecture 3

(TuTh 10:00-11:20am) will be in Humanities 1003.

- **Material covered**: 8.7, 8.8, 7.1, 7.2 of the textbook.
- **Review session**: 10/26, 4:00pm, Earth and Space 001.
- **Policies**: no books, notes, or calculators. You must bring your ID with you. When writing solutions, you must show your reasoning, not just the answer. Answers without justification will get only partial credit. Your solutions should be written so that the grader is able to follow your reasoning and computations.
- **Practice problems**: available **here**, with **solutions**.

Final Exam

- **Date and place**: Monday, 12/17, 11:15am-1:45pm. Frey Hall 100 for all sections.
- **Material covered**: Chapters 7 and 8 of the textbook, and Second Order Linear Differential Equations.
- **Review session**: 12/12, 4:00pm, Engineering 145.
- **Policies**: no books, notes, or calculators. You must bring your ID with you. When writing solutions, you must show your reasoning, not just the answer. Answers without justification will get only partial credit. Your solutions should be written so that the grader is able to follow your reasoning and computations.
- **Practice problems**: **differential equations** with **solutions**; **sequences and series** with **solutions**.

Textbook for MAT125, 126, 127, 131, and 132

You have several options regarding the textbook for your calculus course.

The text for MAT125, 126, 127, 131, and 132 (and also AMS151 and AMS161) is Single Variable Calculus(Stony Brook Edition 4), by James Stewart.
These courses also use WebAssign for homework problems and access to **WebAssign is required** in these courses.

Note that Webassign access is **free for the first two weeks** and **includes an electronic text**. So you don't have to do anything right away.

You can purchase the textbook and webassign together or separately; buying them together tends to be less expensive, but not always.
You have several options:

- **Purchase *Cengage Unlimited***. [cost: $119.99]
  This is a digital subscription service, which gives you access to the (electronic) textbook for this course, and also WebAssign. It also gives you access to the text in any other course using a Cengage text (these include courses in many subjects-- see an explicit list here. But not all courses use Cengage texts, so check first!)
  You have the option of renting or purchasing a physical text. Furthermore, you can select up to 6 texts to use for another year, which also gives you WebAssign access. Go to the publisher's site to purchase.
- **Buy a physical text and webassign access for several semesters**. [cost: $169]
  This option is called an "ePack" on the publisher's site.
- **Buy several semesters of Webassign access with an electronic text**. [Cost: $125]
  This is called "Webassign Instant Access" on the publisher's site You can also buy this from within WebAssign for the same price. I don't know why anyone would choose this option over the Unlimited one.
- **Buy only one semster of access to Webassign, with an electronic text**. [cost: $100]
  This can only be done within Webassign. As long as none of your other courses use Cengage texts, this could be a good option for students who

are taking MAT132 and MAT127, but is typically not a good idea for students in MAT125 and MAT131 (because they usually also take MAT126 or MAT132 afterwards). Many students in MAT126 also take MAT127, in which case it would be better to purchase multi-term access; others stop with MAT126, and should buy only one semester.

- **Buy webassign without an e-book.** [cost: $60-80]
  Students can buy the webassign-only access direct from within WebAssign. In this case, the cost is $60 for a single term, or $80 for multiple terms. But you don't get access to a textbook (but see below).

If you don't buy the e-book with webassign, or a physical book from the publisher, the Stony Brook edition of the text is **identical** to the book (Stewart's Single Variable Calculus: Concepts, 4th ed, with WebAssign) **except for the cover** because we made a deal with the publisher to reduce the price for Stony Brook students. This is (or was) the same textbook used in Calculus courses at Suffolk County Community College, just with the different cover.
This might be helpful if you can find a used copy of the text, but these seem to cost about $40-60 online (without webassign). So this may not be the most cost-effective solution.
However, the 3rd edition is very similar to the 4th edition (although the homework problems have different numbers, and some sections have changed a little). Used copies of the 3rd edition can be found for about $5. This might be a good option if you want a paper text for reference, and purchase WebAssign separately.

Notes on Second Order Linear Differential Equations

Stony Brook University Mathematics Department

1. The general second order homogeneous linear differential equation with constant coefficients looks like

$$Ay'' + By' + Cy = 0,$$

where y is an unknown function of the variable x, and A, B, and C are constants. If $A = 0$ this becomes a first order linear equation, which we already know how to solve. So we will consider the case $A \neq 0$. We can divide through by A and obtain the equivalent equation

$$y'' + by' + cy = 0$$

where $b = B/A$ and $c = C/A$.

"Linear with constant coefficients" means that each term in the equation is a constant times y or a derivative of y. "Homogeneous" excludes equations like $y'' + by' + cy = f(x)$ which can be solved, in certain important cases, by an extension of the methods we will study here.

2. In order to solve this equation, we guess that there is a solution of the form

$$y = e^{\lambda x},$$

where λ is an unknown constant. Why? Because it works!

We substitute $y = e^{\lambda x}$ in our equation. This gives

$$\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = 0.$$

Since $e^{\lambda x}$ is never zero, we can divide through and get the equation

$$\lambda^2 + b\lambda + c = 0.$$

Whenever λ is a solution of this equation, $y = e^{\lambda x}$ will automatically be a solution of our original differential equation, and if λ is not a solution, then $y = e^{\lambda x}$ cannot solve the differential equation. So the substitution $y = e^{\lambda x}$ transforms the differential equation into an algebraic equation!

Example 1. Consider the differential equation

$$y'' - y = 0.$$

Plugging in $y = e^{\lambda x}$ give us the associated equation

$$\lambda^2 - 1 = 0,$$

which factors as

$$(\lambda + 1)(\lambda - 1) = 0;$$

this equation has $\lambda = 1$ and $\lambda = -1$ as solutions. Both $y = e^x$ and $y = e^{-x}$ are solutions to the differential equation $y'' - y = 0$. (You should check this for yourself!)

Example 2. For the differential equation

$$y'' + y' - 2y = 0,$$

we look for the roots of the associated algebraic equation

$$\lambda^2 + \lambda - 2 = 0.$$

Since this factors as $(\lambda - 1)(\lambda + 2) = 0$, we get both $y = e^x$ and $y = e^{-2x}$ as solutions to the differential equation. Again, you should check that these are solutions.

3. For the general equation of the form

$$y'' + by' + cy = 0,$$

we need to find the roots of $\lambda^2 + b\lambda + c = 0$, which we can do using the quadratic formula to get

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

If the *discriminant* $b^2 - 4c$ is positive, then there are two solutions, one for the plus sign and one for the minus.

This is what we saw in the two examples above.

Now here is a useful fact about linear differential equations: if y_1 and y_2 are solutions of the homogeneous differential equation $y'' + by' + cy = 0$, then so is the linear combination $py_1 + qy_2$ for any numbers p and q. This fact is easy to check (just plug $py_1 + qy_2$ into the equation and regroup terms; note that the coefficients b and c do not need to be constant for this to work. This means that for the differential equation in Example 1 ($y'' - y = 0$), any function of the form

$$pe^x + qe^{-x} \qquad \text{where } p \text{ and } q \text{ are any constants}$$

is a solution. Indeed, while we can't justify it here, *all* solutions are of this form. Similarly, in Example 2, the general solution of

$$y'' + y' - 2y = 0$$

is

$$y = pe^x + qe^{-2x}, \qquad \text{where } p \text{ and } q \text{ are constants.}$$

4. If the discriminant $b^2 - 4c$ is negative, then the equation $\lambda^2 + b\lambda + c = 0$ has no solutions, unless we enlarge the number field to include $i = \sqrt{-1}$, i.e. unless we work with complex numbers. If $b^2 - 4c < 0$, then since we can write any positive number as a square k^2, we let $k^2 = -(b^2 - 4c)$. Then ik will be a square root of $b^2 - 4c$, since $(ik)^2 = i^2k^2 = (-1)k^2 = -k^2 = b^2 - 4c$. The solutions of the associated algebraic equation are then

$$\lambda_1 = \frac{-b + ik}{2}, \ \ \lambda_2 = \frac{-b - ik}{2}.$$

Example 3. If we start with the differential equation $y'' + y = 0$ (so $b = 0$ and $c = 1$) the discriminant is $b^2 - 4c = -4$, so $2i$ is a square root of the discriminant and the solutions of the associated algebraic equation are $\lambda_1 = i$ and $\lambda_2 = -i$.

Example 4. If the differential equation is $y'' + 2y' + 2y = 0$ (so $b = 2$ and $c = 2$ and $b^2 - 4c = 4 - 8 = -4$). In this case the solutions of the associated algebraic equation are $\lambda = (-2 \pm 2i)/2$, i.e. $\lambda_1 = -1 + i$ and $\lambda_2 = -1 - i$.

5. Going from the solutions of the associated algebraic equation to the solutions of the differential equation involves interpreting $e^{\lambda x}$ as a function of x when λ is a complex number. Suppose λ has real part a and imaginary part ib, so that $\lambda = a + ib$ with a and b real numbers. Then

$$e^{\lambda x} = e^{(a+ib)x} = e^{ax}e^{ibx}$$

assuming for the moment that complex numbers can be exponentiated so as to satisfy the law of exponents. The factor e^{ax} does not cause a problem, but what is e^{ibx}? Everything will work out if we take

$$e^{ibx} = \cos(bx) + i\sin(bx),$$

and we will see later that this formula is a necessary consequence of the elementary properties of the exponential, sine and cosine functions.

6. Let us try this formula with our examples.

Example 3. For $y'' + y = 0$ we found $\lambda_1 = i$ and $\lambda_2 = -i$, so the solutions are $y_1 = e^{ix}$ and $y_2 = e^{-ix}$. The formula gives us $y_1 = \cos x + i\sin x$ and $y_2 = \cos x - i\sin x$.

Our earlier observation that if y_1 and y_2 are solutions of the linear differential equation, then so is the combination $py_1 + qy_2$ for any numbers p and q holds even if p and q are complex constants.

Using this fact with the solutions from our example, we notice that $\frac{1}{2}(y_1 + y_2) = \cos x$ and $\frac{1}{2i}(y_1 - y_2) = \sin x$ are both solutions. *When we are given a problem with real coefficients it is customary, and always possible, to exhibit real solutions.* Using the fact about linear combinations again, we can say that $y = p\cos x + q\sin x$ is a solution for any p and q. This is the general solution. (It is also correct to call $y = pe^{ix} + qe^{-ix}$ the general solution; which one you use depends on the context.)

Example 4. $y'' + 2y' + 2y = 0$. We found $\lambda_1 = -1 + i$ and $\lambda_2 = -1 - i$. Using the formula we have

$$y_1 = e^{\lambda_1 x} = e^{(-1+i)x} = e^{-x}e^{ix} = e^{-x}(\cos x + i\sin x),$$
$$y_2 = e^{\lambda_2 x} = e^{(-1-i)x} = e^{-x}e^{-ix} = e^{-x}(\cos x - i\sin x).$$

Exactly as before we can take $\frac{1}{2}(y_1 + y_2)$ and $\frac{1}{2i}(y_1 - y_2)$ to get the real solutions $e^{-x}\cos x$ and $e^{-x}\sin x$. (Check that these functions both satisfy the differential equation!) The general solution will be $y = pe^{-x}\cos x + qe^{-x}\sin x$.

7. *Repeated roots.* Suppose the discriminant is zero: $b^2 - 4c = 0$. Then the "characteristic equation" $\lambda^2 + b\lambda + c = 0$ has one root. In this case both $e^{\lambda x}$ **and** $xe^{\lambda x}$ are solutions of the differential equation.

Example 5. Consider the equation $y'' + 4y' + 4y = 0$. Here $b = c = 4$. The discriminant is $b^2 - 4c = 4^2 - 4 \times 4 = 0$. The only root is $\lambda = -2$. Check that **both** e^{-2x} and xe^{-2x} are solutions. The general solution is then $y = pe^{-2x} + qxe^{-2x}$.

8. *Initial Conditions.* For a first-order differential equation the undetermined constant can be adjusted to make the solution satisfy the initial condition $y(0) = y_0$; in the same way the p and the q in the general solution of a second order differential equation can be adjusted to satisfy initial conditions. Now there are two: we can specify both the value and the first derivative of the solution for some "initial" value of x.

Example 5. Suppose that for the differential equation of Example 2, $y'' + y' - 2y = 0$, we want a solution with $y(0) = 1$ and $y'(0) = -1$. The general solution is $y = pe^x + qe^{-2x}$, since the two roots of the characteristic equation are 1 and -2. The method is to write down what the initial conditions mean in terms of the general solution, and then to solve for p and q. In this case we have

$$1 = y(0) = pe0 + qe^{-2\times 0} = p + q$$
$$-1 = y'(0) = pe0 - 2qe^{-2\times 0} = p - 2q.$$

This leads to the set of linear equations $p + q = 1, p - 2q = -1$ with solution $q = 2/3, p = 1/3$. You should check that the solution

$$y = \frac{1}{3}e^x + \frac{2}{3}e^{-2x}$$

satisfies the initial conditions.

Example 6. For the differential equation of Example 4, $y'' + 2y' + 2y = 0$, we found the general solution $y = pe^{-x}\cos x + qe^{-x}\sin x$. To find a solution satisfying the initial conditions $y(0) = -2$ and $y'(0) = 1$ we proceed as in the last example:

$$-2 = y(0) = pe^{-0}\cos 0 + qe^{-0}\sin 0 = p$$

$$1 = y'(0) = -pe^{-0}\cos 0 - pe^{-0}\sin 0 - qe^{-0}\sin 0 + qe^{-0}\cos 0 = -p + q.$$

So $p = -2$ and $q = -1$. Again check that the solution

$$y = -2e^{-x}\cos x - e^{-x}\sin x$$

satisfies the initial conditions.

Problems cribbed from Salas-Hille-Etgen, page 1133

In exercises 1-10, find the general solution. Give the real form.

1. $y'' - 13y' + 42y = 0.$
2. $y'' + 7y' + 3y = 0.$
3. $y'' - 3y' + 8y = 0.$
4. $y'' - 12y = 0.$
5. $y'' + 12y = 0.$
6. $y'' - 3y' + \frac{9}{4}y = 0.$
7. $2y'' + 3y' = 0.$
8. $y'' - y' - 30y = 0.$
9. $y'' - 4y' + 4y = 0.$
10. $5y'' - 2y' + y = 0.$

In exercises 11-16, solve the given initial-value problem.

11. $y'' - 5y' + 6y = 0, \;\; y(0) = 1, \;\; y'(0) = 1$
12. $y'' + 2y' + y = 0, \;\; y(2) = 1, \;\; y'(2) = 2$
13. $y'' + \frac{1}{4}y = 0, \;\; y(\pi) = 1, \;\; y'(\pi) = -1$
14. $y'' - 2y' + 2y = 0, \;\; y(0) = -1, \;\; y'(0) = -1$
15. $y'' + 4y' + 4y = 0, \;\; y(-1) = 2, \;\; y'(-1) = 1$
16. $y'' - 2y' + 5y = 0, \;\; y(\pi/2) = 0, \;\; y'(\pi/2) = 2$

Problems cribbed from Salas-Hille-Etgen, page 1133

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2. $y'' + 7y' + 3y = 0.$
3. $y'' - 3y' + 8y = 0.$
4. $y'' - 12y = 0.$
5. $y'' + 12y = 0.$
6. $y'' - 3y' + \frac{9}{4}y = 0.$
7. $2y'' + 3y' = 0.$
8. $y'' - y' - 30y = 0.$
9. $y'' - 4y' + 4y = 0.$
10. $5y'' - 2y' + y = 0.$

In exercises 11-16, solve the given initial-value problem.

11. $y'' - 5y' + 6y = 0,\ \ y(0) = 1,\ \ y'(0) = 1$
12. $y'' + 2y' + y = 0,\ \ y(2) = 1,\ \ y'(2) = 2$
13. $y'' + \frac{1}{4}y = 0,\ \ y(\pi) = 1,\ \ y'(\pi) = -1$
14. $y'' - 2y' + 2y = 0,\ \ y(0) = -1,\ \ y'(0) = -1$
15. $y'' + 4y' + 4y = 0,\ \ y(-1) = 2,\ \ y'(-1) = 1$
16. $y'' - 2y' + 5y = 0,\ \ y(\pi/2) = 0,\ \ y'(\pi/2) = 2$

MAT 127, MIDTERM 1
PRACTICE PROBLEMS

The midterm covers chapters 8.1 — 8.6 in the textbook. The actual exam will contain 4 problems (some multipart), so it will be shorter than this practice exam.

1. Determine whether the following sequence converges. If it converges, find the limit

$$a_n = (-1)^n \frac{n+4}{n^3 - 2n^2 + 4}$$

2. Determine whether the following sequence converges. If it converges, find the limit

$$a_n = \frac{3^n + 1}{n!}$$

3. Determine whether the following sequence converges. If it converges, find the limit

$$a_n = \frac{(\ln n)^2}{n}$$

4. Let the sequence a_n be defined by $a_1 = 1$, $a_{n+1} = = \dfrac{3 + a_n}{2}$ for $n \geq 1$.

(a) Show that this sequence is bounded: $a_n \leq 3$ for all n.

(b) Explain why this sequence is convergent and find the limit.

5. If $\displaystyle\sum_{n=1}^{\infty} a_n$ is a convergent series with positive terms, what can you say about the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \sin(a_n)$? Does it converge? Does it converge absolutely?

6. For which values of p is the series $\displaystyle\sum_{n=1}^{\infty} p^n \frac{n!}{(2n)!}$ convergent?

7. If the series $\displaystyle\sum_{n=1}^{\infty} c_n 4^n$ is divergent, what can you say about the following series:

$$\sum_{n=1}^{\infty} c_n 2^n, \qquad \sum_{n=1}^{\infty} c_n (-8)^n, \qquad \sum_{n=1}^{\infty} c_n (-4)^n.$$

8. Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{1 + (-2)^n}{3^n}$$

9. Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+3)}$$

10. Write the number $1.1\overline{009} = 1.1009009009\ldots$ as a fraction

11. Determine whether the following series converges or diverges

$$\sum_{n=0}^{\infty} \frac{\sin(3\pi n/7)}{n^2+1}$$

12. Consider the power series

$$\sum_{n=1}^{\infty} \frac{(2x-1)^n}{n \cdot 3^n}$$

Find the radius of convergence and the interval of convergence. You are not required to determine whether the series is convergent at the endpoints of the interval of convergence.

13. Write the function $f(x) = \ln(1+2x)/x$ as a power series in x. Find the radius of convergence of this series.

1) $a_n = (-1)^n \dfrac{n+4}{n^3-2n^2+4}$

$\Rightarrow |a_n| = \dfrac{n+4}{n^3-2n^2+4}$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n+4}{n^3-2n^2+4}$$

$$= \lim_{n\to\infty} \frac{\frac{n+4}{n^3}}{\frac{n^3-2n^2+4}{n^3}}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n^2}+\frac{4}{n^3}}{1-\frac{2}{n}+\frac{4}{n^3}}$$

$$= \frac{0+0}{1-0+0} \quad \left(\text{Since } \lim_{n\to\infty} \frac{1}{n^p} = 0, \text{ for } p>0\right)$$

So, $\lim_{n\to\infty} |a_n| = 0$.

But, $-|a_n| \le a_n \le |a_n|$

Since $\lim_{n\to\infty} -|a_n| = 0 = \lim_{n\to\infty} |a_n|$, by the squeeze theorem, we conclude that

$$\lim_{n\to\infty} a_n = 0.$$

2) $\frac{1}{n!} = \frac{1}{1.2.3.\cdots.n} < \frac{1}{n}$

So, $0 < \frac{1}{n!} < \frac{1}{n}$.

But $\lim_{n\to\infty} \frac{1}{n} = 0$.

By the squeeze theorem, $\lim_{n\to\infty} \frac{1}{n!} = 0$.

$$\frac{3^n}{n!} = \frac{\overbrace{3.3.3.\cdots 3}^{n \text{ times}}}{1.2.3.\cdots.n}$$

$$= \underbrace{\left(\frac{3}{1}\right)\left(\frac{3}{2}\right)\left(\frac{3}{3}\right)}\underbrace{\left(\frac{3}{4}\right)\left(\frac{3}{5}\right)\cdots\left(\frac{3}{n}\right)}$$

$$= \left(\frac{9}{2}\right)\left(\frac{3}{4}\right)\left(\frac{3}{5}\right)\cdots\left(\frac{3}{n}\right)$$

Note that $\frac{3}{n} \le \frac{3}{4}$ for $n \ge 4$.

So, $0 \le \frac{3^n}{n!} \le \left(\frac{9}{2}\right)\left(\frac{3}{4}\right)^{n-3}$

But $\lim_{n\to\infty} \left(\frac{3}{4}\right)^{n-3} = 0$. [because $\frac{3}{4} < 1$]

Again by the squeeze theorem.

$$\lim_{n\to\infty} \frac{3^n}{n!} = 0$$

Therefore, $\lim_{n\to\infty} \left(\frac{3^n+1}{n!}\right) = \left(\lim_{n\to\infty} \frac{3^n}{n!} + \lim_{n\to\infty} \frac{1}{n!}\right) = 0.$

3) $a_n = \dfrac{(\ln n)^2}{n}$

We know that,

$$\lim_{n\to\infty} \frac{(\ln n)^2}{n} = \lim_{x\to\infty} \frac{(\ln x)^2}{x}$$

[Note that $\left(\dfrac{\ln x}{x}\right)^2$ is a differentiable function]

So, $\displaystyle\lim_{x\to\infty} \frac{(\ln x)^2}{x}$

$$= \lim_{x\to\infty} \frac{2\ln x \cdot \frac{1}{x}}{1} \quad \text{(L'Hospital's rule)}$$

$$= \lim_{x\to\infty} \frac{2\ln x}{x}$$

$$= \lim_{x\to\infty} \frac{2 \cdot \frac{1}{x}}{1} \quad \text{(L'Hospital again!)}$$

$$= \lim_{x\to\infty} \frac{2}{x} = 0$$

So, $\displaystyle\lim_{n\to\infty} a_n = 0$.

4) $a_1 = 1$, $a_{n+1} = \dfrac{3+a_n}{2}$ for $n \geq 1$.

a) We'll show by induction that $a_n \leq 3$ for all n.

This is clearly true for $n=1$.

We assume that $a_m \leq 3$

Then, $3 + a_m \leq 3+3$

$$\Rightarrow \frac{3+a_m}{2} \leq \frac{6}{2} = 3$$

$$\Rightarrow a_{m+1} \leq 3.$$

Therefore by the principle of mathematical induction, $a_n \leq 3$ for all n.

b) We'll now show ~~by induction~~ that the sequence $\{a_n\}$ is increasing.

$a_1 = 1$, $a_2 = \dfrac{3+a_1}{2} = \dfrac{3+1}{2} = 2 > a_1$

~~[illegible]~~

$$a_{n+1} - a_n = \frac{a_n+3}{2} - a_n$$

$$= \frac{a_n+3-2a_n}{2} = \frac{3-a_n}{2} \geq 0$$

[as $a_n \leq 3$]

So, $a_{n+1} \geq a_n$ for all n.

We know that a bounded (above) increasing sequence converges.

Let, $\lim_{n \to \infty} a_n = L$.

We have, $a_{n+1} = \dfrac{3+a_n}{2}$

$$\Rightarrow \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \frac{3+a_n}{2}$$

$$\Rightarrow \lim_{n \to \infty} a_{n+1} = \frac{3+\lim_{n \to \infty} a_n}{2}$$

$$\Rightarrow L = \frac{3+L}{2}$$

$$\Rightarrow 2L = L+3$$

$$\Rightarrow 2L-L = 3$$

$$\Rightarrow \boxed{L=3}.$$

Therefore, $\lim_{n \to \infty} a_n = 3$.

5) Since $\sum_{n=1}^{\infty} a_n$ is a convergent series with positive terms, we know that

$$\lim_{n\to\infty} a_n = 0.$$

We'll show that the series $\sum_{n=1}^{\infty} \sin(a_n)$ converges absolutely. We'll use the limit test. (We are using the fact that $\lim_{x\to 0} \frac{\sin x}{x} = 1$)

Note that $\lim_{n\to\infty} \frac{|\sin(a_n)|}{|a_n|} = 1 \neq 0.$

Since the series $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} a_n$ $(a_n > 0)$ converges, the limit test implies that $\sum_{n=1}^{\infty} |\sin(a_n)|$ is convergent too.

Therefore, $\sum_{n=1}^{\infty} \sin(a_n)$ is <u>absolutely convergent</u>.

Hence, $\sum_{n=1}^{\infty} \sin(a_n)$ is convergent.

6) Define $a_n = p^n \dfrac{n!}{(2n)!}$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

$$= \lim_{n\to\infty} \frac{|p|^{n+1}(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{|p|^n \cdot n!}$$

$$= \lim_{n\to\infty} \frac{|p| \cdot \overbrace{(n! \cdot (n+1))}^{(n+1)!}\,(2n)!}{\underbrace{((2n)! \cdot (2n+1) \cdot (2n+2))}_{(2n+2)!}\, n!}$$

$$= \lim_{n\to\infty} \frac{|p| \cdot (n+1)}{(2n+1) \cdot 2(n+1)}$$

$$= \lim_{n\to\infty} \frac{|p|}{2(2n+1)} = 0 < 1, \text{ for every value of } p.$$

By the ratio test, the series $\displaystyle\sum_{n=1}^{\infty} p^n \frac{n!}{(2n)!}$ converges for <u>every</u> value of p.

7) Consider the power series

$$\sum_{n=1}^{\infty} c_n x^n .$$

If $\sum_{n=1}^{\infty} c_n 4^n$ is divergent, we can conclude that $\boxed{R \leq 4}$ (where R is the radius of convergence)

a) Can't say anything about $\sum_{n=1}^{\infty} c_n 2^n$ because we don't know whether R is bigger or smaller than 2.

b) $\sum_{n=1}^{\infty} c_n (-8)^n$ diverges because $R \leq 4$.

(so, $\sum_{n=1}^{\infty} c_n x^n$ will diverge if $|x| > 4$)

c) Can't say anything about $\sum_{n=1}^{\infty} c_n (-4)^n$ because we don't know if R is strictly smaller than 4.

8) $\sum_{n=0}^{\infty} \frac{1+(-2)^n}{3^n}$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n + \sum_{n=0}^{\infty} \left(\frac{-2}{3}\right)^n$$

Geometric series

The sum of the first series

$$= \frac{1}{1-\frac{1}{3}} = \frac{1}{\frac{2}{3}} = \frac{3}{2}.$$

The sum of the second series

$$= \frac{1}{1-\left(-\frac{2}{3}\right)} = \frac{1}{1+\frac{2}{3}} = \frac{3}{5}$$

Answer : $\frac{3}{2} + \frac{3}{5}$ = Please add!

9) $$\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+3)} = ?$$

Note that,

$$\frac{2}{(n+1)(n+3)} = \frac{1}{n+1} - \frac{1}{n+3}$$

We can write,

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+3)} = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{2}{(n+1)(n+3)}\right)$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+3}\right)\right)$$

Now, S_n

$$= \frac{1}{2}\left(1 - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \frac{1}{5} + \cancel{\frac{1}{4}} - \frac{1}{6} + \cdots + \cdots + \frac{1}{n} - \frac{1}{n+2} + \frac{1}{n+1} - \frac{1}{n+3}\right)$$

$$= \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n+2} - \frac{1}{n+3}\right)$$

Therefore, $\lim_{n\to\infty} S_n = \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4}$

The sum of the series $= \boxed{\frac{3}{4}}$.

10) $1.1\overline{009}$

$= 1.1009009009\ldots$

$= 1.1 + 0.009\times(0.1) + 0.009\times(0.1)^4 +$

$0.009\times(0.1)^7 + \ldots$

$= 1.1 + 0.009\left(\frac{1}{10} + \left(\frac{1}{10}\right)^4 + \left(\frac{1}{10}\right)^7 + \ldots\right)$

$= 1.1 + 0.009\left(\dfrac{\frac{1}{10}}{1-\left(\frac{1}{10}\right)^3}\right)$

$= \frac{11}{10} + \frac{9}{1000}\left(\dfrac{\frac{1}{10}}{\frac{999}{1000}}\right)$

$= \frac{11}{10} + \frac{9}{\cancel{1000}} \cdot \frac{1}{10} \cdot \frac{\cancel{1000}}{999}$

$= \frac{11}{10} + \frac{1}{10} \cdot \frac{1}{111}$

$= \boxed{\frac{1222}{1110}}.$

11) $\displaystyle\sum_{n=0}^{\infty} \frac{\sin(3\pi n/7)}{n^2+1}$

Put $a_n = \dfrac{\sin(3\pi n/7)}{n^2+1}$

Now, ~~we~~ ~~[illegible]~~ ~~[illegible]~~ This is neither a positive series nor an alternating series. So we have to resort to the absolute convergence test.

In fact, we'll show that $\displaystyle\sum_{n=0}^{\infty} |a_n|$ is convergent.

Note that, $|a_n|$

$$= \frac{|\sin(3\pi n/7)|}{n^2+1} \leq \frac{1}{n^2+1}.$$

But we know that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ is convergent! (use limit comparison test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$)

Therefore, by the comparison test, $\displaystyle\sum_{n=0}^{\infty} |a_n|$ converges.

So, $\sum_{n=0}^{\infty} a_n$ is absolutely convergent.

Hence, $\sum_{n=0}^{\infty} a_n$ is convergent.

---

12) $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{n \cdot 3^n}$

$= \sum_{n=1}^{\infty} \frac{\left(2\left(x-\frac{1}{2}\right)\right)^n}{n \cdot 3^n}$

$= \sum_{n=0}^{\infty} \frac{2^n \cdot \left(x-\frac{1}{2}\right)^n}{n \cdot 3^n}$

So the center of the power series is $\frac{1}{2}$.

Put $a_n = \frac{2^n \cdot \left(x-\frac{1}{2}\right)^n}{n \cdot 3^n}$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

$$= \lim_{n\to\infty} \frac{2^{n+1}\left|x-\frac{1}{2}\right|^{n+1}}{(n+1)\cdot 3^{n+1}} \cdot \frac{n\cdot 3^{n}}{2^{n}\left|x-\frac{1}{2}\right|^{n}}$$

$$= \lim_{n\to\infty} \left|x-\frac{1}{2}\right| \cdot \frac{n}{n+1} \cdot \frac{2}{3}$$

$$= \frac{2}{3}\left|x-\frac{1}{2}\right| \lim_{n\to\infty} \frac{n}{n+1}$$

$$= \frac{2}{3}\left|x-\frac{1}{2}\right| \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{n+1}{n}}$$

$$= \frac{2}{3}\left|x-\frac{1}{2}\right| \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}$$

$$= \frac{2}{3}\left|x-\frac{1}{2}\right| \cdot \frac{1}{1+0}$$

$$= \frac{2}{3}\left|x-\frac{1}{2}\right|$$

The power series will converge for all those values of x for which the limit $\frac{2}{3}\left|x-\frac{1}{2}\right|$ is less than 1.

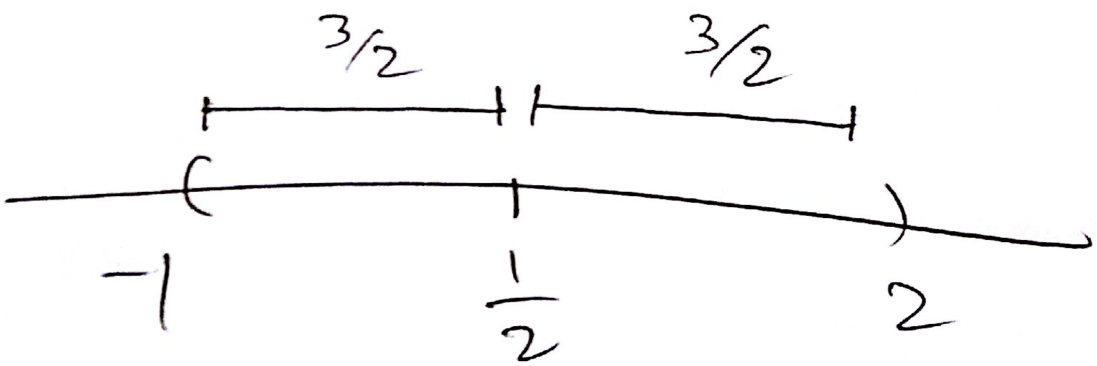
Now, $$\frac{2}{3}\left|x-\frac{1}{2}\right| < 1$$

$$\Leftrightarrow \left|x-\frac{1}{2}\right| < \frac{3}{2}$$

$$\Leftrightarrow -\frac{3}{2} < x-\frac{1}{2} < \frac{3}{2}$$

$$\Leftrightarrow \frac{1}{2}-\frac{3}{2} < x < \frac{1}{2}+\frac{3}{2}$$

$$\Leftrightarrow -1 < x < 2.$$



So, the radius of convergence $= \boxed{\frac{3}{2}}$,

& the interval of convergence $= (-1, 2)$.

13) We know that

$$\frac{1}{1-t} = 1 + t + t^2 + \cdots + t^n + \cdots,$$

for $|t| < 1$.

Put, $t = -2x$

Then, $$\frac{1}{1+2x} = 1 - 2x + 4x^2 + \cdots + (-2x)^n + \cdots,$$

for $|x| < \frac{1}{2}$.

Integrating this equation, we obtain

$$\frac{1}{2}\ln(1+2x) = x - x^2 + \frac{4}{3}x^3 - \cdots + \frac{(-2)^n x^{n+1}}{n+1} + \cdots + C,$$

for $|x| < \frac{1}{2}$.

Putting $x = 0$ on both sides, we see that $0 = 0 + C$

$\Rightarrow C = 0$.

So, $\ln(1+2x) = 2x - 2x^2 + \frac{8x^3}{3} - \cdots + \frac{(-1)^n 2^{n+1} x^{n+1}}{n+1} + \cdots$, for $|x| < \frac{1}{2}$

Dividing out by x, we get

$$\frac{\ln(1+2x)}{x} = 2 - 2x + \frac{8x^2}{3} - \cdots + \frac{(-1)^n 2^{n+1} x^n}{n+1} + \cdots,$$

for $|x| < \frac{1}{2}$.

MAT 127, MIDTERM 2
PRACTICE PROBLEMS

The midterm covers chapters 7.1-7.2 and 8.7-8.8 of the textbook.

1. Calculate the second degree Taylor polynomial $T_2(x)$ about a for the following functions.
   (a) $f(x) = \sin(x)$ where $a = \pi$.
   (b) $f(x) = \sqrt[3]{x}$ where $a = 1$.
   (c) $f(x) = e^{2x}$ around $x = 1$.

2. (a) Approximate $\sin(\frac{7\pi}{6})$ using the second degree Taylor polynomial $T_2(x)$ of $f(x) = \sin(x)$ (at $a = \pi$) found in Problem 1(a).
   (b) Approximate $\sqrt[3]{2}$ using the second degree Taylor polynomial $T_2(x)$ of $f(x) = \sqrt[3]{x}$ (at $a = 1$) found in Problem 1(b).

3. Find the first three non-zero terms of the Maclaurin series of
   (a) $f(x) = e^{2x}\sin(x)$, and
   (b) $f(x) = \cos(x)\ln(1+x)$,

using the Maclaurin series listed below.

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$R = \infty$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$R = \infty$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$R = \infty$
$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$R = 1$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$R = 1$

4. (a) Find a constant c such that $y = e^{cx}$ is a solution of $y'' - 4y' + 3y = 0$.
   (b) Find a constant c such that $y = xe^{cx}$ is a solution of $y'' - 4y' + 4y = 0$.
   (c) Find constants a and b such that $y = ax + b$ is a solution of the differential equation $y'' = y^2 - 2yx + x^2$.

5. Draw direction fields for the following differential equations.
   (a) $y' = 1$;
   (b) $y' = y^2 - 5y + 6$;
   (c) $y' = y^2 - 4$;
   (d) $y' = x - y$.

6. Use Euler's Method with step size 0.01 to estimate $y(0.02)$ where y satisfies:
   (a) $y' = y, \quad y(0) = 1$.
   (b) $y' = xy, \quad y(0) = 3$.

(1)

1. For a function $f(x)$ the 2^{nd} degree Taylor polynomial at a is $T_2(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2$.

a) $f(x) = \sin x$ and $a = \pi$

$f(x) = \sin x \qquad f(\pi) = 0$

$f'(x) = \cos x \qquad f'(\pi) = -1$

$f''(x) = -\sin x \qquad f''(\pi) = 0$

so $T_2(x) = 0 + \frac{-1}{1!}(x-\pi) + \frac{0}{2!}(x-\pi)^2$

$= -(x-\pi) = \pi - x$

b) $f(x) = x^{1/3}$ and $a = 1$.

$f(x) = (x)^{1/3} = 1 \qquad f(1) = 1$

$f'(x) = \frac{1}{3}x^{-2/3} \qquad f'(1) = \frac{1}{3}$

$f''(x) = \frac{-2}{9}x^{-5/3} \qquad f''(1) = \frac{-2}{9}$

so $T_2(x) = 1 + \frac{1/3}{1!}(x-1) + \frac{-2/9}{2!}(x-1)^2$

$= 1 + \frac{1}{3}(x-1) - \frac{1}{9}(x-1)^2$.

(2)

c) $f(x) = e^{2x}$ and $a = 1$

$f(x) = e^{2x}$ $\quad f(1) = e^2$

$f'(x) = 2e^{2x}$ $\quad f'(1) = 2e^2$

$f''(x) = 4e^{2x}$ $\quad f''(1) = 4e^2$

so $T_2(x) = e^2 + \frac{2e^2}{1!}(x-1) + \frac{4e^2}{2!}(x-1)^2$

$= e^2 + 2e^2(x-1) + 2e^2(x-1)^2$

2. a) From problem 1(a) we have

$T_2(x) = \pi - x$, so $T_2\left(\frac{7\pi}{6}\right) = \pi - \frac{7\pi}{6} = \frac{-\pi}{6}$

is the approximation for $\sin\left(\frac{7\pi}{6}\right)$.

b) From problem 1(b) we have

$T_2(x) = 1 + \frac{1}{3}(x-1) - \frac{1}{9}(x-1)^2$,

so $T_2(2) = 1 + \frac{1}{3} - \frac{1}{9} = \frac{11}{9}$

is the approximation for $\sqrt[3]{2}$.

③

3. a) Using the table of Maclaurin series, we have

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} \quad \text{and} \quad \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1!},$$

Thus

$$e^{2x} \sin x = \left(1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots\right)$$

$$\times \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

$$= x + \frac{2x^2}{1!} + \left(-\frac{x^3}{3!} + \frac{4}{2!}x^3\right) + \cdots$$

$$= x + 2x^2 + \frac{11}{6}x^3 + \cdots$$

So the first three nozero terms of the Maclaurin series of $e^{2x} \sin x$ are $x + 2x^2 + \frac{11}{6}x^3$.

b) Again using the table we have

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!} \quad \text{and} \quad \ln 1+x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

Thus

$$\cos x \ln 1+x = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

$$\times \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)$$

(4)

$$= x - \frac{x^2}{2} + \left(\frac{x^3}{3} - \frac{x^3}{2!}\right) + \cdots$$

$$= \boxed{x - \frac{x^2}{2} - \frac{x^3}{6}} + \cdots$$

$\hookrightarrow$ the first 3 non zero terms

* Note that we should put all terms which have the same power of x together in this problem.

4. a) $y = e^{cx}$ is a solution of the differential equation $y'' - 4y' + 3y = 0$, so

$$(e^{cx})'' - 4(e^{cx})' + 3e^{cx} = 0$$

$$\Rightarrow c^2 e^{cx} - 4ce^{cx} + 3e^{cx} = 0$$

$$\Rightarrow (c^2 - 4c + 3)\, e^{cx} = 0 \quad \text{for all } x$$

$$\Rightarrow c^2 - 4c + 3 = 0 \Rightarrow c = \frac{4 \pm \sqrt{16-12}}{2} = \frac{4 \pm 2}{2} = \boxed{1, 3}$$

b) $y = xe^{cx}$ is a solution of $y'' - 4y' + 4y = 0$, so

$$(xe^{cx})'' - 4(xe^{cx})' + 4(xe^{cx}) = 0$$

$$\Rightarrow (2ce^{cx} + c^2xe^{cx}) - 4(e^{cx} + cxe^{cx}) + 4xe^{cx} = 0$$

$$\Rightarrow e^{cx}(2c + c^2x - 4 - 4cx + 4x) = 0 \text{ for all } x$$

$$\Rightarrow e^{cx}\left[(c^2 - 4c + 4)x + (2c - 4)\right] = 0 \text{ for all } x$$

$$\Rightarrow c^2 - 4c + 4 = 0 \text{ and } 2c - 4 = 0 \Rightarrow \boxed{c = 2}$$

c) $y = ax + b$ satisfies $y'' = y^2 - 2yx + x^2$, so

$$(ax+b)'' = (ax+b)^2 - 2(ax+b)x + x^2$$

$$\Rightarrow 0 = a^2x^2 + 2abx + b^2 - 2ax^2 - 2bx + x^2 \text{ for all } x$$

$$\Rightarrow (a^2 - 2a + 1)x^2 + (2ab - 2b)x + b^2 = 0 \text{ for all } x$$

$$\Rightarrow (a-1)^2 = 0 \Rightarrow a = 1$$

$$2ab - 2b = 0 \Rightarrow a = 1 \text{ or } b = 0$$

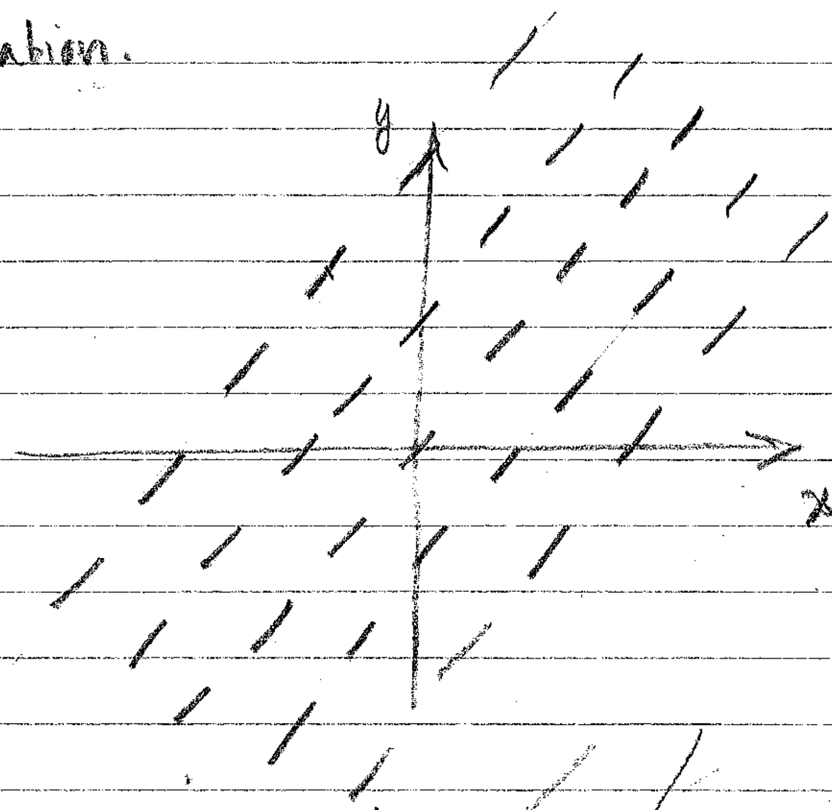
$$b^2 = 0 \Rightarrow b = 0$$

For $a = 1$ and $b = 0$ all three equations are satisfied and hence $y = x$ is a solution of the differential equation.

(6)

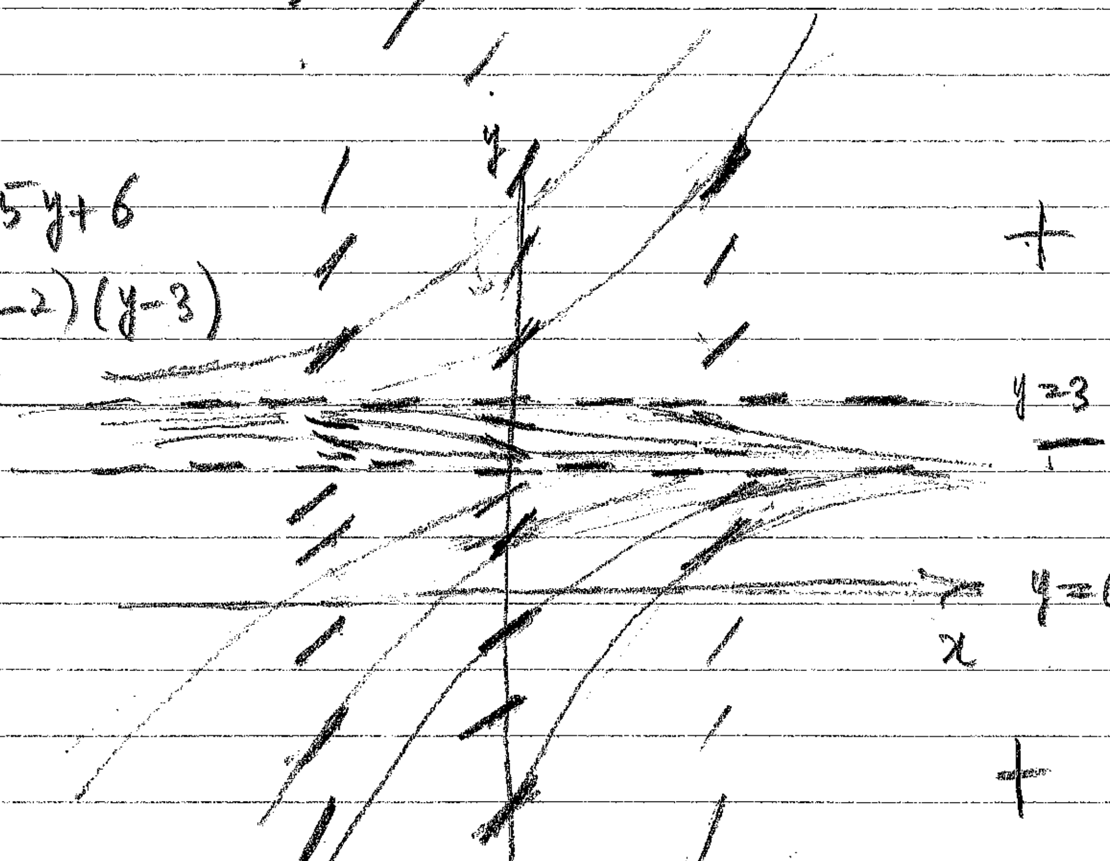
5. We draw lines whose slopes are given by the right-hand of the equation.

a) $y' = 1$



b) $y' = y^2 - 5y + 6$

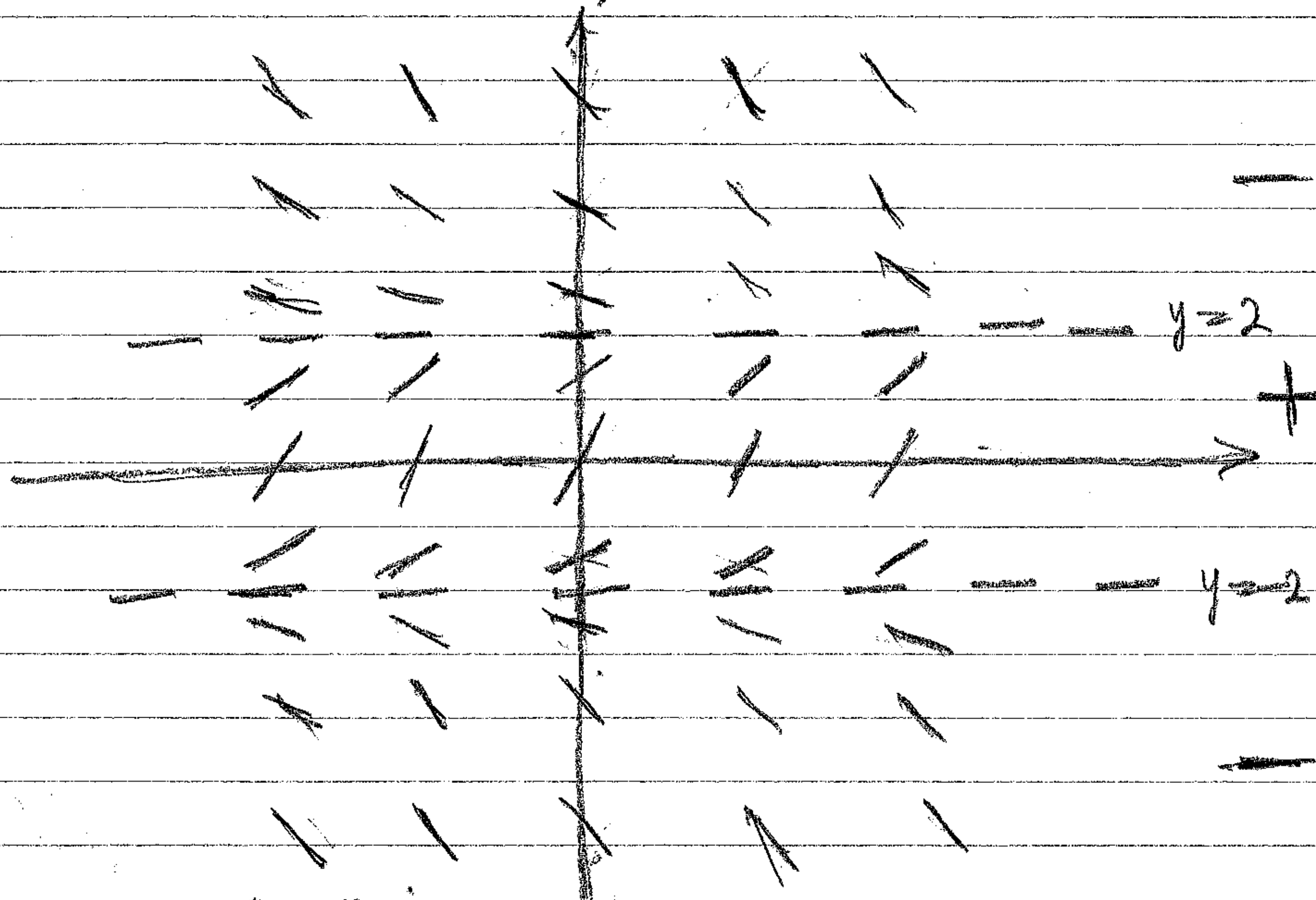
$= (y-2)(y-3)$



$$(y-2)(y-3) = \begin{cases} \text{negative} & 2 < y < 3 \\ \text{positive} & y < 2 \text{ or } y > 3 \\ 0 & y = 2 \text{ or } y = 3 \end{cases}$$

7

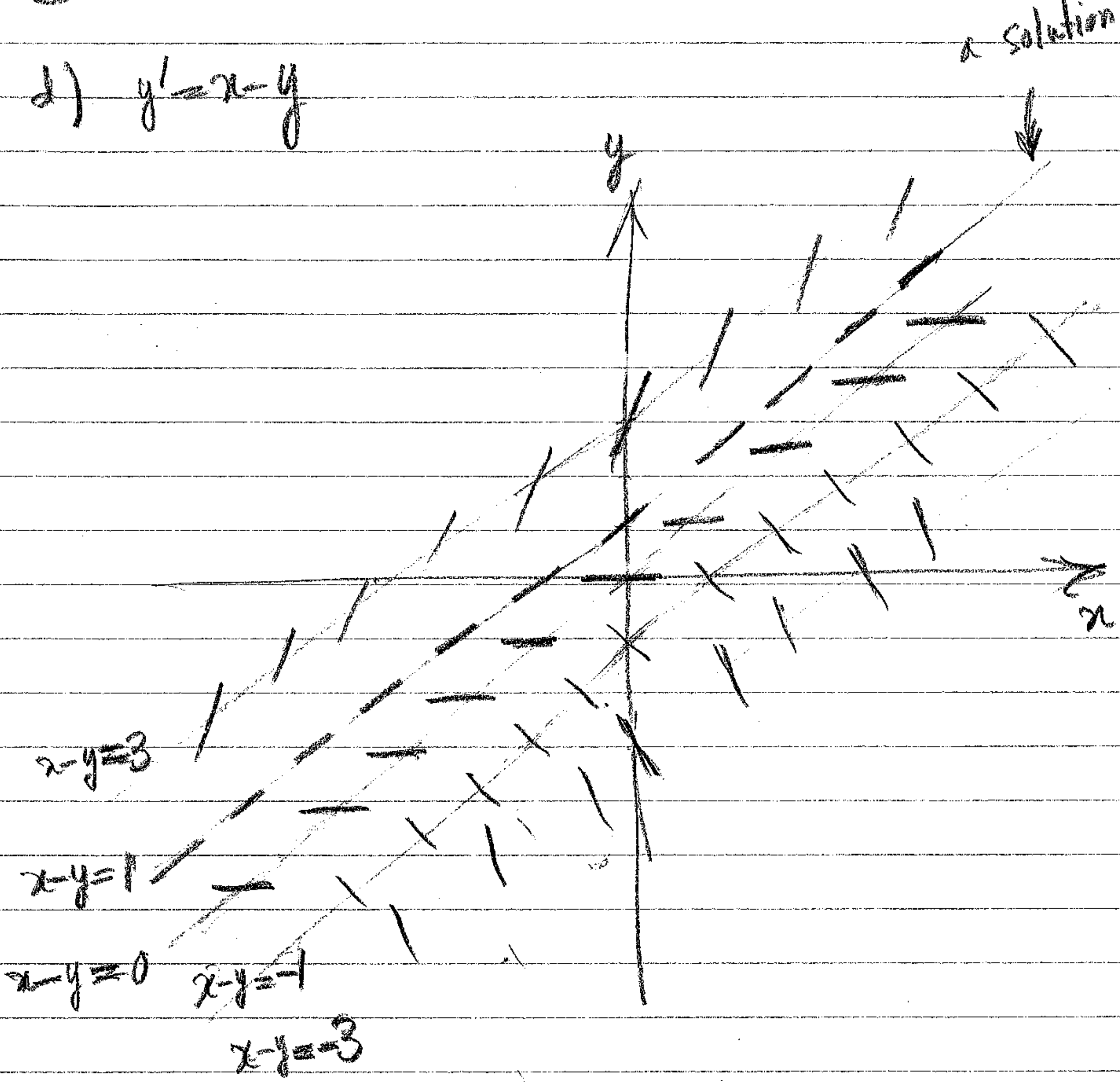
c) $y' = 4 - y^2 = (2-y)(2+y)$



$$(2-y)(2+y) = \begin{cases} \text{negative} & y<-2 \text{ or } y>2 \\ \text{positive} & -2<y<2 \\ 0 & y=2 \text{ or } y=-2 \end{cases}$$

(8)

d) $y' = x - y$



(9)

6. Suppose that $y(x)$ is a solution of the intial-value problem: $\begin{cases} y' = F(x,y) \\ y(x_0) = y_0 \end{cases}$. Let step size be h and

$x_1 = x_0 + h$, $x_2 = x_1 + h$, ..., $x_n = x_{n-1} + h$

then by Euler's method we have:

$y_1 = y_0 + hF(x_0, y_0)$ is an approximation for $y(x_1)$

$y_2 = y_1 + hF(x_1, y_1)$ is an approximation for $y(x_2)$

$\vdots$

$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$ is an approximation for $y(x_n)$.

a) $\begin{cases} y' = y \\ y(0) = 1 \end{cases}$ and $h = 0.01$

$x_0 = 0$ $\quad y_0 = 1$

$x_1 = 0.01$ $\quad y_1 = 1 + 0.01 \times 1 = 1.01$

$x_2 = 0.02$ $\quad y_2 = 1.01 + 0.01 \times 1.01 = 1.0201$

is an approx for $y(0.02)$

b) $\begin{cases} y' = xy \\ y(0) = 3 \end{cases}$ & $h = 0.01$

$x_0 = 0$ $\quad y_0 = 3$

$x_1 = 0.01$ $\quad y_1 = 3 + 0.01 \times 0 = 3$

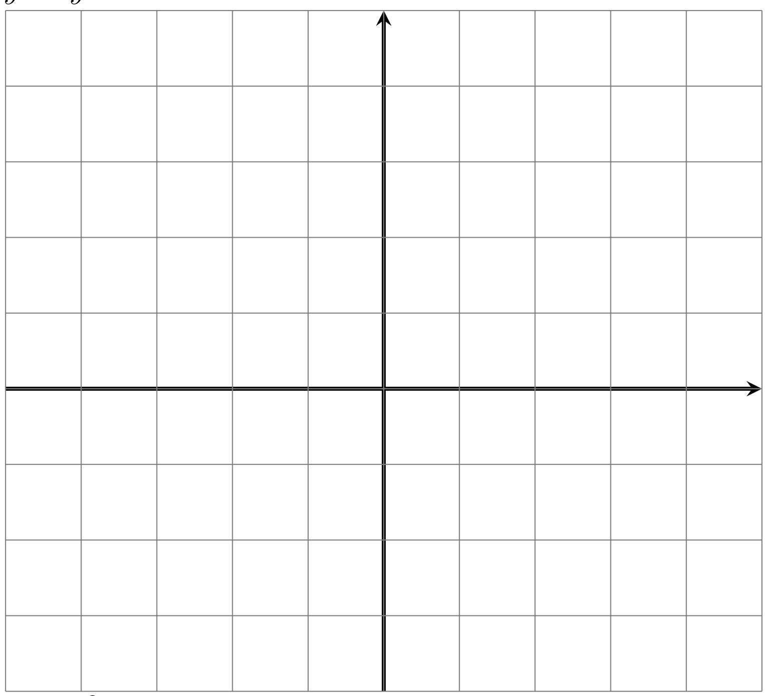
$x_2 = 0.02$ $\quad y_2 = 3 + 0.01 \times 0.03 = 3.0003$

is an approx for $y(0.02)$

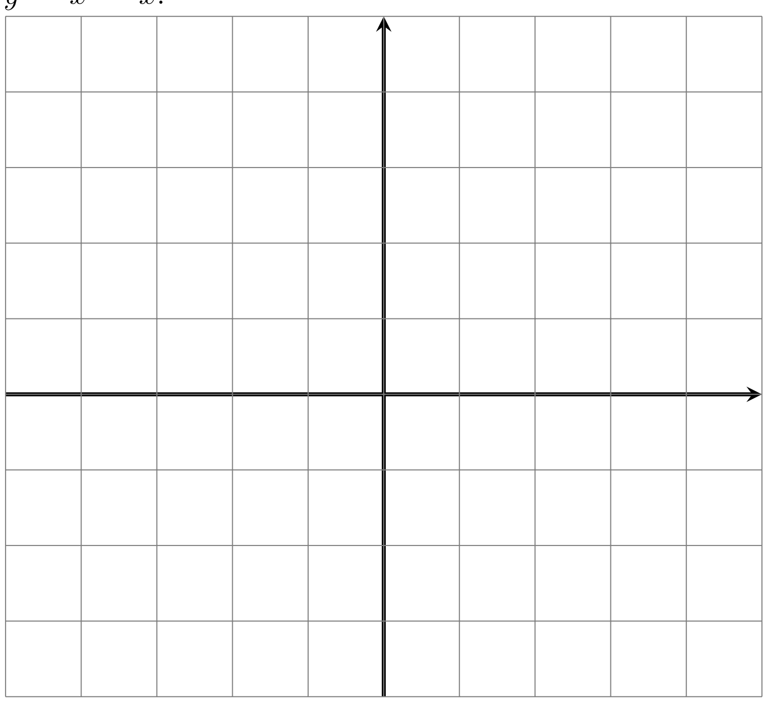
MAT 127 FINAL
DIFF. EQN. PRACTICE PROBLEMS

1. Draw direction fields for the following differential equations.

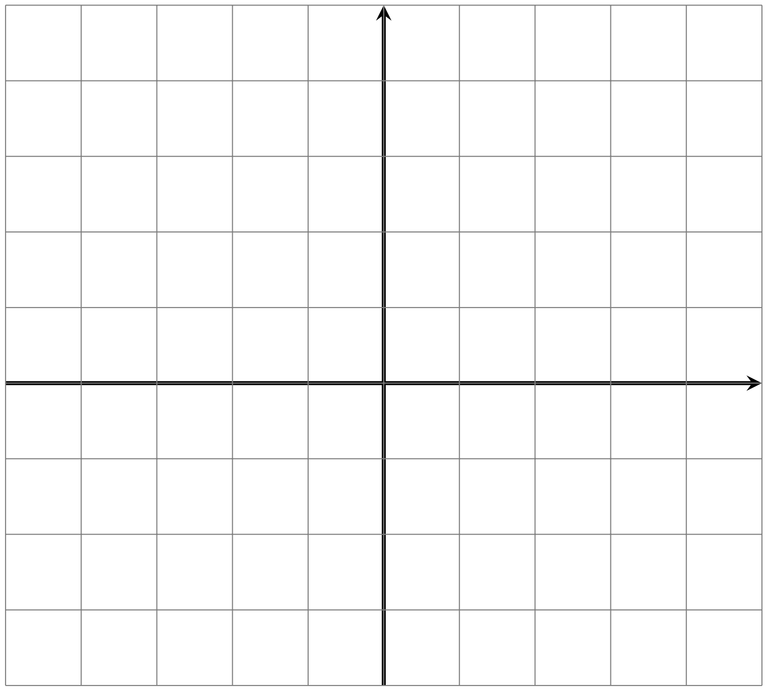
(a) $y' = y^2 - 1$



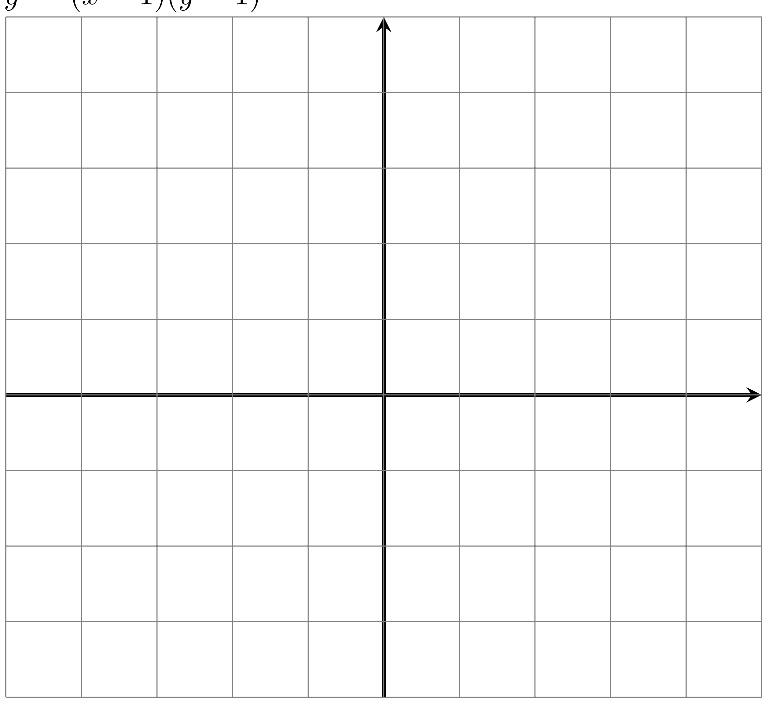
(b) $y' = x^2 - x$.



(c) $y' = x + y$.



(d) $y' = (x-1)(y-1)$



2. Use Eulers Method with step size 0.01 to estimate $y(0.02)$ where y satisfies:
$y' = 3y, \quad y(0) = 2.$

3. Find the orthogonal trajectories of the family of curves $y^2 = kx^3$.

4. Solve the following differential equations:
(a) $y' = y^2 - 1, \quad y(0) = 2.$

(b) $y' = \frac{1}{y(\sqrt{1-x^2})}, y(0) = 1.$

5. Solve the following initial value problems:

(a) $\frac{dy}{dx} = (x^2 + x - 2)(y^2 + 2y - 3),\ y(0) = 1.$

(b) $(1 + \cos(x))\frac{dy}{dx} = (1 + e^{-y})\sin(x),\ y(0) = 0.$

(c) $y'' - 2y' + 5y = 0,\ y(\pi/2) = 0,\ y'(\pi/2) = 2.$

(d) $y'' + 4y' + 4y = 0,\ y(-1) = 2,\ y'(-1) = 1.$

6. A radioactive material takes 300 years to loose 10% of its mass. How long does it take to loose 15% of its mass?

7. A tank contains 100 L of pure water. Brine that contains 0.1 kg of salt per liter enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank after 6 minutes?

8. A cup of hot chocolate has temperature 80°C in a room kept at 20°C. After half an hour the hot chocolate cools to 60°C.
(a) What is the temperature of the chocolate after another half hour?
(b) When will the chocolate have cooled to 40°C?

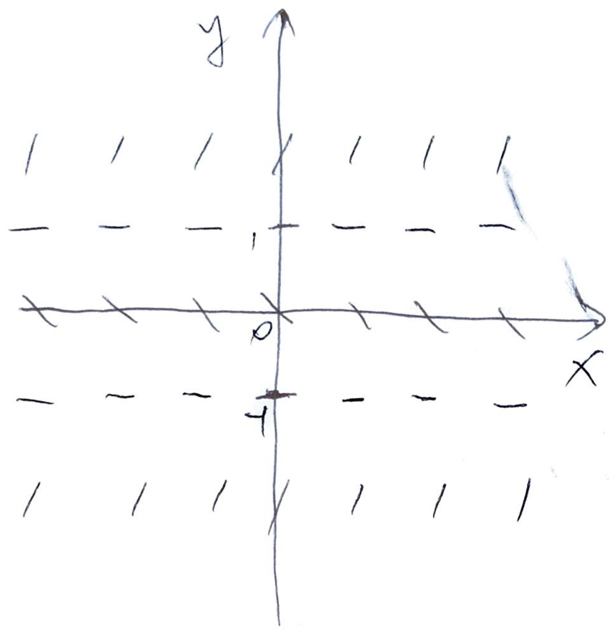
9. A population of bees in a particular region satisfies the logistic equation with carrying capacity 10000. Suppose that there are only 1000 bees initially and 2000 bees after 2-years. How many bees are there after 3-years?

10. Find the equilibrium solutions to the following equations:

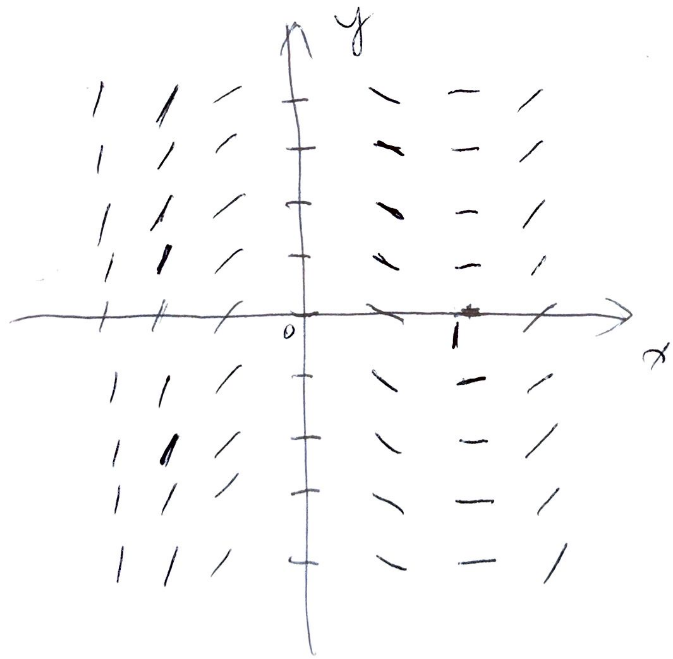
$$\frac{dW}{dt} = R^2 + RW$$
$$\frac{dR}{dt} = W^2 - R.$$

1. (a) $y' = y^2 - 1$

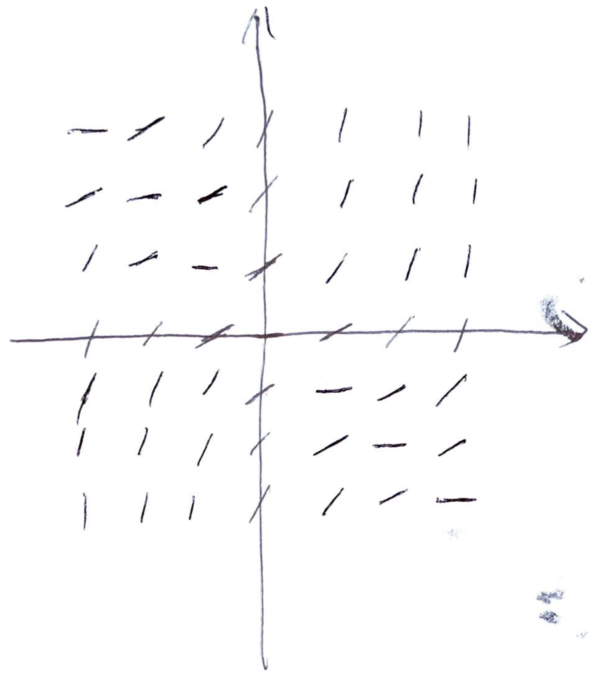
$y^2 - 1 = (y+1)(y-1)$.



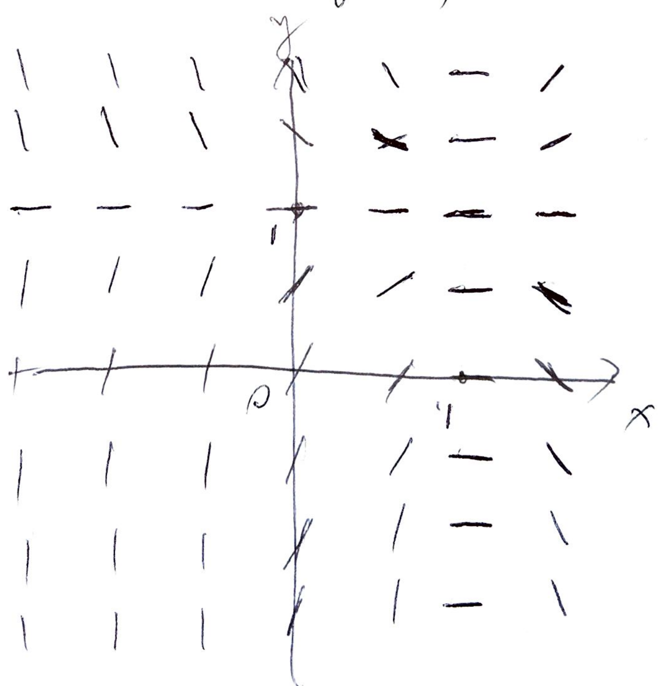
(b) $y' = x^2 - x \quad = x(x-1)$



(c) $y' = x + y$



(d) $y' = (x-1)(y-1)$



2. $x_0=0$ $x_1=0.01$ $x_2=0.02$

$y_0=2$ $y_1=\cancel{y_0}+0.01\cdot(y_0^2-1)$ $y_2=y_1+0.01\cdot(y_1^2-1)$

$=2+0.01\cdot(2^2-1)$ $=2.02+0.01\cdot(2.03^2-1)$

$=2.03$ $=2.051209.$

$y(0.02)\approx 2.051209.$

3. $y^2=kx^3$ $2y\cdot y'=3kx^2$ $k=\frac{y^2}{x^3}$

$2y\cdot y'=\frac{3y^2}{x}$ $y'=\frac{3y}{2x}$

For the orthogonal traj: $y'=-\frac{2x}{3y}.$

$\int 3y\,dy=\int -2x\,dx$ $\frac{3}{2}y^2=-x^2+k.$

4 $y'=y^2-1$ $y(0)=2$

$\frac{dy}{y^2-1}=dx$ $\left(\frac{1}{y^2-1}=\frac{1}{2}\left(\frac{1}{y-1}-\frac{1}{y+1}\right)\right)$

$\frac{1}{2}\left(\int\frac{dy}{y-1}-\int\frac{dy}{y+1}\right)=x+C$ $\frac{1}{2}(\ln|y-1|-\ln|y+1|)=x+C$

$\ln\left|\frac{y-1}{y+1}\right|=2x+C$ $\frac{y-1}{y+1}=Ae^{2x}$

$y-1=y\cdot Ae^{2x}+A\cdot e^{2x}$ $y=\frac{A\cdot e^{2x}+1}{1-Ae^{2x}}$

$y(0)=2 \Rightarrow A=\frac{1}{3}$

4 (b) $y' = \frac{1}{y(\sqrt{1-x^2})}$ $y(0)=1$.

$$\int y\,dy = \int \frac{dx}{\sqrt{1-x^2}} \quad \left(\begin{array}{l} x = \sin t \\ \int \frac{d\sin t}{\cos t} = t + C \\ t = \sin^{-1} x. \end{array} \right)$$

$$\frac{1}{2}y^2 = \sin^{-1}(x) + C.$$

$$y = (2\sin^{-1}(x) + A)^{\frac{1}{2}}$$

$$y(0) = 1 \Rightarrow A = 1.$$

5. (a) $\frac{dy}{dx} = (x^2+x-2)(y^2+2y-3)$ $y(0)=1$.

$$\int \frac{dy}{y^2+2y-3} = \int (x^2+x-2)\,dx \quad \left(\frac{1}{y^2+2y-3} = \frac{\frac{1}{4}}{y-1} - \frac{\frac{1}{4}}{y+3} \right)$$

$$\frac{1}{4}\left(\int \frac{dy}{y-1} - \int \frac{dy}{y+3}\right) = \int (x^2+x-2)\,dx.$$

$$\frac{1}{4}(\ln|y-1| - \ln|y+3|) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + C$$

$$\frac{1}{4}\ln\left|\frac{y-1}{y+3}\right| = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + C.$$

$$y(0) = 1 \Rightarrow C = \frac{1}{4}.$$

$$\cancel{\ln}\ \frac{y-1}{y+3} = e^{\frac{4}{3}x^3 + \frac{4}{2}x^2 - 8x + 1}.$$

$$y = \frac{1 + 3e^{\frac{4}{3}x^3 + 2x^2 - 8x + 1}}{1 - e^{\frac{4}{3}x^3 + 2x^2 - 8x + 1}}$$

5(b) $(1+\cos(x))\frac{dy}{dx}=(1+e^{-y})\sin(x) \quad y(0)=0.$

$$\int\frac{dy}{(1+e^{-y})}=\int\frac{\sin x\,dx}{1+\cos x}.$$

$$\int\frac{dy}{1+e^{-y}}=\int\frac{e^{y}\,dy}{e^{y}+1}=\int\frac{de^{y}}{e^{y}+1}=\ln|e^{y}+1|$$

$$\int\frac{\sin x\,dx}{1+\cos x}=\int\frac{-d\cos x}{1+\cos x}=-\ln|1+\cos x|$$

$$e^{y}+1=\frac{C}{1+\cos x} \qquad y=\ln\left(\frac{C}{1+\cos x}-1\right)$$

$y(0)=0 \Rightarrow C=4.$

5(c) $y''-2y'+5y=0$

$\lambda^2-2\lambda+5=0 \quad \lambda=1+2i \text{ or } 1-2i.$

$y=Ae^{x}\cdot\cos 2x+Be^{x}\cdot\sin 2x$

$y(\frac{\pi}{2})=0 \Rightarrow A=0.$

$y'(\frac{\pi}{2})=2 \Rightarrow B=-e^{-\frac{\pi}{2}}$

5(d) $y''+4y'+4y=0$

$\lambda^2+4\lambda+4=0 \quad \lambda=-2.$

$y=Ax\cdot e^{-2x}+Be^{-2x}.$

$$\begin{cases} y(-1)=2 \\ y'(-1)=1 \end{cases} \Rightarrow \begin{cases} -e^{2}A+e^{2}B=2 \\ (e^{2}+2\cdot e^{2})A-2e^{2}B=1 \end{cases} \Rightarrow \begin{cases} A=\frac{5}{e^{2}} \\ B=\frac{7}{e^{2}} \end{cases}$$

6. $y = C \cdot e^{-kt}$

$y_{(0)} = C \quad y_{(300)} = 0.9C$

$C \cdot e^{-k(300)} = 0.9C$

$k = -\frac{1}{300} \cdot \ln(0.9)$

$y_{(t)} = 0.\cancel{8}5C$

$e^{kt} = 0.85 \Rightarrow -\frac{1}{300} \cdot \ln(0.9) \cdot t = \ln 0.85$

$t = 300 \cdot \left(\ln \frac{0.9}{0.85}\right)$

7. y (kg) salt in tank

Rate in: $0.1 \cdot 10 = 1$ kg/min

Rate out: $\frac{y}{100} \cdot 10 = \frac{y}{10}$ kg/min

$y' = 1 - \frac{y}{10} \Rightarrow \int \frac{dy}{10-y} = \int \frac{1}{10} dx$

$-\ln|10-y| = \frac{1}{10}x + C$

$10 - y = A \cdot e^{-\frac{1}{10}x}$

$y = 10 - A \cdot e^{-\frac{1}{10}x}$

$y_{(0)} = 0 \Rightarrow A = 10. \quad y_{(6)} = 10 - 10 \cdot e^{-\frac{3}{5}}$

8. $T_{(t)} - 20 = Ae^{-kt}$

$T_{(0)} = 80 \Rightarrow A = 60.$

$T_{(0.5)} = 60 \Rightarrow 40 = 60 \cdot e^{-k(0.5)}$

$\Rightarrow k = -2 \cdot \ln(\frac{2}{3}).$

(a) $T_{(1)} = 60e^{-2\ln(\frac{3}{2})} + 20.$

(b) $T_{(x)} = 40 \Rightarrow 20 = 60 \cdot e^{-2\ln(\frac{3}{2}) \cdot x}$

$\Rightarrow x = \frac{-(\ln(\frac{1}{3})}{2\ln(\frac{3}{2})}$

9. $y' = ky \cdot (1 - \frac{y}{10000})$

$y = \frac{10000}{1 + A \cdot e^{-kx}}$ $\quad y_{(0)} = 1000 \Rightarrow A = 9.$

$y_{(2)} = 2000 \Rightarrow 2000 = \frac{10000}{1 + 9 \cdot e^{-k2}}$

$\Rightarrow 9e^{-k2} = 4 \quad k = \frac{1}{2}\ln(\frac{9}{4}) = \ln\frac{3}{2}$

$y_{(3)} = \frac{10000}{1 + 9 \cdot e^{-\ln(\frac{3}{2}) \cdot 3}}$

10 $\begin{cases} R^2 + RW = 0 \\ W^2 - R = 0 \end{cases} \Rightarrow \begin{cases} R = 0 \\ W = 0 \end{cases}$ or $\begin{cases} R + W = 0 \\ W^2 - R = 0 \end{cases} \Rightarrow \begin{cases} R = 0 \\ W = 0 \end{cases}$ or $\begin{cases} R = 1 \\ W = -1 \end{cases}$

MAT 127 FINAL
SEQN. AND SERIES PRACTICE PROBLEMS

For each of the following sequence, determine whether it converges. If it converges, find the limit.

(a) $a_n = \frac{n^3+2}{2n^4+1}$

(b) $a_n = n\sin(\pi/n)$

(c) $a_n = \sqrt{2n+1} - \sqrt{2n-1}$

(d) $a_n = \frac{2^n}{n!}$

(e) $a_n = \frac{5n^2 - \sin(3n)}{n^2+10}$

For each of the series below, determine whether the series converges.

(a)

$$\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+4}$$

(b)

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{2n+3}}$$

(c)

$$\sum_{n=0}^{\infty} \frac{n^2+4n}{n^3+1}$$

(d)

$$\sum_{n=0}^{\infty} \frac{ne^{2n}}{(2n)!}$$

(e)

$$\sum_{n=0}^{\infty} \frac{1+e^n}{\pi^n}$$

(f)

$$\sum_{n=0}^{\infty} ne^{-n^2}$$

(a) Find the radius of convergence of the following power series.

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)x^n}{3^n}$$

(b) Write the indefinite integral

$$F(x) = \int f(x)\,dx$$

as a power series (inside the interval of convergence of the power series for $f(x)$.)

(c) The product $\cos(x)f(x)$ can also be written by a power series. Write the first three terms of this power series, up to x^2 term.

Let $f(x) = \tan^{-1}(2x) - 2x$.

Write the Taylor polynomial $T_5(x)$ of degree 5 for $f(x)$, centered at $x = 0$.

(a). $a_n = \dfrac{n^3+2}{2n^4+1}$.

a_n converges, and $\lim\limits_{n\to\infty} a_n = 0$.

$$a_n = \frac{n^3+2}{2n^4+1} = \frac{\frac{1}{n}+\frac{2}{n^4}}{2+\frac{1}{n^4}} \xrightarrow[n\to\infty]{} \frac{0}{2} = 0$$

(b). $a_n = n\sin\left(\frac{\pi}{n}\right)$.

a_n converges. $\lim\limits_{n\to\infty} a_n = \pi$.

$$a_n = \frac{\sin\left(\frac{\pi}{n}\right)}{\left(\frac{1}{n}\right)} = \pi\cdot\frac{\sin\left(\frac{\pi}{n}\right)}{\left(\frac{\pi}{n}\right)} \xrightarrow[n\to\infty]{} \pi\cdot 1 = \pi.$$

Here it uses $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$, with $x = \frac{\pi}{n}$.

(c) $a_n = \sqrt{2n+1} - \sqrt{2n-1}$.

a_n converges. $\lim\limits_{n\to\infty} a_n = 0$.

$$a_n = \frac{\left(\sqrt{2n+1}-\sqrt{2n-1}\right)\left(\sqrt{2n+1}-\sqrt{2n-1}\right)}{\sqrt{2n+1}+\sqrt{2n-1}} = \frac{2}{\sqrt{2n+1}+\sqrt{2n-1}} \xrightarrow[n\to\infty]{} 0$$

(d). $a_n = \dfrac{2^n}{n!}$

a_n converges, $\lim\limits_{n\to\infty} a_n = 0$.

$$a_n = \frac{\underbrace{2\times\cdots\times 2}_{n\text{ times}}}{1\times 2\times\cdots\times n} = \left(\frac{2}{1}\right)\times\left(\frac{2}{2}\right)\times\left(\frac{2}{3}\right)\times\cdots\times\left(\frac{2}{n}\right)$$

$$\le 2\times\frac{2}{n} \xrightarrow[n\to\infty]{} 0$$

(e). $a_n = \dfrac{5n^2 - \sin(3n)}{n^2+10}$

a_n converges. $\lim_{n\to\infty} a_n = 5$.

$$a_n = \frac{5n^2 - \sin(3n)}{n^2+10} = \frac{5 - \frac{\sin(3n)}{n^2}}{1 + \frac{10}{n^2}} \xrightarrow[n\to\infty]{} \frac{5}{1} = 5.$$

---

a). $\sum_{n=1}^{\infty} \dfrac{n^2-1}{n^2+4}$.

Diverges since $\lim_{n\to\infty} \dfrac{n^2-1}{n^2+4} = 1 \neq 0$.

Recall $\sum_{n=1}^{\infty} a_n$ converges implies $\lim_{n\to\infty} a_n = 0$.

(b). $\sum_{n=1}^{\infty} (-1)^n \dfrac{1}{\sqrt{2n+3}}$.

Converges. let $a_n = \dfrac{1}{\sqrt{2n+3}}$, then $\lim_{n\to\infty} a_n = 0$.

Also since $2n+3$ is increasing in n, a_n is decreasing. Hence convergence follows from Alternating series test.

c). $\sum_{n=0}^{\infty} \dfrac{n^2+4n}{n^3+1}$.

Diverges. since $\dfrac{n^2+4n}{n^3+1} \geq \dfrac{n^2}{n^3+1} \geq \dfrac{n^2}{2n^3} = \dfrac{1}{2n}$, for $n \geq 1$.

But $\sum_{n=1}^{\infty} \dfrac{1}{2n}$ diverges.

Hence it diverges from Comparison test.

d). $\displaystyle\sum_{n=0}^{\infty} \frac{n e^{2n}}{(2n)!}$

Converges. Indeed, let $a_n = \dfrac{n e^{2n}}{(2n)!}$

$a_{n+1} = \dfrac{(n+1) e^{2(n+1)}}{(2(n+1))!}$, then.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)\, e^{2(n+1)}}{(2(n+1))!} \times \frac{(2n)!}{n e^{2n}} = \frac{e^2}{(2n+2)(2n+1)} \times \frac{n+1}{n} \xrightarrow[n\to\infty]{} 0$$

i.e $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$. So we get convergence by Ratio test.

e). $\displaystyle\sum_{n=0}^{\infty} \frac{1+e^n}{\pi^n}$.

Converges. Indeed, let $a_n = \dfrac{1+e^n}{\pi^n}$.

$a_{n+1} = \dfrac{1+e^{n+1}}{\pi^{n+1}}$, then

$$\frac{a_{n+1}}{a_n} = \frac{1+e^{n+1}}{\pi^{n+1}} \times \frac{\pi^n}{1+e^n} = \frac{1}{\pi} \cdot \frac{1+e^{n+1}}{1+e^n}$$

$$= \frac{1}{\pi} \times \frac{\frac{1}{e^n}+e}{\frac{1}{e^n}+1} \xrightarrow[n\to\infty]{} \frac{1}{\pi} \times e < 1.$$

i.e $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{e}{\pi} < 1$. So converge by Ratio test.

$\boxed{3}$

f). $\sum_{n=0}^{\infty} n e^{-n^2}$

~~First~~ Converges.

First observe that for $n \geq 0$, $ne^{-n^2} \leq ne^{-n}$.

So we just need to show $\sum_{n=0}^{\infty} ne^{-n}$ converges, then it will gives $\sum_{n=0}^{\infty} ne^{-n^2}$ converges, by comparison test.

To see $\sum_{n=0}^{\infty} ne^{-n}$ converges, let $a_n = ne^{-n}$.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)e^{-(n+1)}}{ne^{-n}} = e^{-1} \cdot \frac{n+1}{n} \xrightarrow[n\to\infty]{} e^{-1} < 1.$$

i.e $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = e^{-1} < 1$, so $\sum_{n=0}^{\infty} ne^{-n}$ converges by Ratio test.

---

a). $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)x^n}{3^n}$.

Put $a_n = (-1)^n \frac{(2n+1)}{3^n}$ we compute $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

Indeed, $a_{n+1} = (-1)^{n+1} \frac{(2(n+1)+1)}{3^{n+1}}$, So.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2(n+1)+1}{3^{n+1}} \times \frac{3^n}{2n+1} = \frac{1}{3} \times \frac{2n+3}{2n+1} \xrightarrow[n\to\infty]{} \frac{1}{3}$$

Hence $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3}$. So radius of convergence $= \frac{1}{\left(\frac{1}{3}\right)} = 3$.

$\boxed{4.}$

b).

$$F(x) = \int \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)x^n}{3^n} \, dx.$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (2n+1)}{3^n} \int x^n dx.$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{3^n} \times \frac{x^{n+1}}{(n+1)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{3^n (n+1)} x^{n+1}.$$

c). we can write

$$f(x) = 1 - x + \frac{5}{9}x^2 + \dots$$

$$\cos(x) = 1 - \frac{x^2}{2} + \dots$$

So

$$\cos(x)f(x) = \left(1 - \frac{x^2}{2} + \dots\right)\left(1 - x + \frac{5}{9}x^2 + \dots\right)$$

$$= 1 - x + \left(\frac{5}{9} - \frac{1}{2}\right)x^2 + \dots$$

$$= 1 - x + \frac{1}{18}x^2 + \dots$$

$\boxed{5}$

$f(x) = \tan^{-1}(2x) - 2x.$

Recall $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \Rightarrow \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$

$\Rightarrow \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n (x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$

$\tan^{-1}(x) = \int \frac{1}{1+x^2}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx$

$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$

$x \rightarrow 2x.$

$\tan^{-1}(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{2n+1}.$

$= 2x - \frac{(2x)^3}{3} + \frac{(2x)^5}{5} + \cdots$

i.e $f(x) = \tan^{-1}(2x) - 2x$

$= \frac{-(2x)^3}{3} + \frac{(2x)^5}{5} = \frac{-8}{3}x^3 + \frac{32}{5}x^5.$

i.e $T_5(x) = \frac{-8}{3}x^3 + \frac{32}{5}x^5.$

$\boxed{6}$