MAT 127: Calculus C, Lec 01
Fall 2013

General Information

Instructor:
Artem Dudko, artem.dudko@stonybrook.edu
Lectures: MWF 10:00-10:53, Lgt Engr Lab 102
Office hours: MW 12:00-12:53 (Math Tower 3114) and F 11:00-11:53 (Math Learning Center, Math Tower S-240A)

Textbook: Single Variable Calculus (Stony Brook University 4th Edition) by James Stewart.

Course coordinator: Marie-Louise Michelsohn, mlm@math.sunysb.edu

Tests:

Midterm Test I: Monday, September 23, 8:45pm, covering Chapter 8 Sections 1 through 4.
Midterm Test II: Tuesday, October 29, 8:45pm, covering Chapter 8.
Final Exam: Wednesday, December 11, 11:15am-1:45pm, Javits 100, covering Chapters 7 and 8 and a section on second order linear equations (separate set of notes will be given).
Last day of classes: Friday, December 7.

Course grade is computed by the following scheme:
Midterm Test I: 30%
Midterm Test II: 30%
Final Exam: 40%

The grades will NOT be curved

Final grade cutoffs:

Weighted total	<50	50-60	60-70	70-75	75-80	80-85	85-90	90-100
Grade	F	C	C+	B-	B	B+	A-	A

Information for students with disabilities
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Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to the following website: http://www.sunysb.edu/ehs/fire/disabilities.shtml

Academic integrity
Each student must pursue his or her academic goals honestly and be personally accountable for all submitted work. Representing another person's work as your own is always wrong. Faculty are required to report any suspected instances of academic dishonesty to the Academic Judiciary. Faculty in the Health Sciences Center (School of Health Technology & Management, Nursing, Social Welfare, Dental Medicine) and School of Medicine are required to follow their school-specific procedures. For more comprehensive information on academic integrity, including categories of academic dishonesty, please refer to the academic judiciary website
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Homework assignments.

Each assignment consists of two parts: "To hand in" and "To do". The first part is to be handed in at the end of the class on the due date and will be graded. The second part consists of an additional list of recommended problems (not to be handed in). There will be no partial credit marks, so you get 0 for the corresponding problem if your solution is not complete. Your marks for the assignments will not affect the final grade, but it is highly recommended that you do all the assignments (this will be very helpful for passing the tests).

Assignment 1.
To Hand In: 8.1.4, 8, 16, 30, 31.
To Do: 8.1.10, 12, 14, 25, 38.
Due on Wednesday, September 4.

Assignment 2.
To Hand In: 8.1.22, 48, 52.
Problem 1. Verify whether the sequence $\left\{\frac{n^3+7}{2^{2n}}\right\}$ has a limit. If yes, find the limit.
Problem 2. Show that the sequence given by the recurrent relation

$$a_1 = 2, \;\; a_{n+1} = 2 - \tfrac{1}{a_n}$$

has a limit. Find the limit.
To Do: 8.1.17, 35, 40, 50, 51.
Due on Wednesday, September 11.

Assignment 3.
To Hand In: 8.2.9, 12, 29, 8.3.8, 19.
To Do: 8.2.21, 30, 33, 8.3.12, 26.
Due on Monday, September 16.

Assignment 4.
To Do: 8.3.28, 29, 34, 38, 8.4.3, 10, 14, 22, 30, 35.

Assignment 5.
To Hand In: 8.5.13, 16, 19, 20.

To Do: 8.5.8, 22, 26, 31.
Due on Friday, October 4.

Assignment 6.
To Hand In: 8.6.4, 9, 13, 14, 23.
To Do: 8.6.5, 8, 17, 26, 27.
Due on Monday, October 14.

Assignment 7.
To Hand In: 8.7.5, 16, 24, 39, 61.
To Do: 8.7.4, 10, 17, 30, 45.
Due on Monday, October 21.

Assignment 8.
To Hand In: 7.1.1, 5, 6 (a) and (c), 7 (b)-(d), 11.
To Do: 7.1.2, 3, 10, 12.
Due on Monday, November 11.

Assignment 9.
To Hand In: 7.3.1, 9, 12, 21, 22.
To Do: 7.3.3, 17, 20, 23, 45.
Due on Monday, November 18.

Assignment 10.
To Hand In: 7.2.3-6, 7.2.22, 7.3.6,
Problem 1. Solve: $y' = \frac{1}{x-y} + 1$.
To Do: 7.2.12 (you may use a computer algebra system like maple or mathematica), 7.2.24, 7.3.14,
Problem 2. Solve: $y' = \frac{y^2+xy}{x^2},\ y(e) = e$.
Due on Monday, November 25.

MAT 127: Calculus C, Lec 01
Fall 2013

Solutions

Assignment 1 solutions

Assignment 2 solutions

Assignment 3 solutions

Assignment 4 solutions

Assignment 5 solutions

Assignment 6 solutions

Assignment 7 solutions

Assignment 8 solutions

Assignment 9 solutions

Assignment 10 solutions

MAT 127: Calculus C, Lec 01
Fall 2013

Examples

Sequences and series

Differential equations

N8.1.4

$\{1, \frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, -\frac{1}{2}, \ldots\}$

sequence does not approach one number => has no limit. Let us prove this by contradiction. Assume that this sequence has a limit. Let $\lim_{n\to\infty} \cos(\pi n/3) = L$. Take a small $\varepsilon > 0$, say, $\varepsilon = \frac{1}{2}$. By definition of the limit, for all indexes n starting from some index N we have: $|\cos(\pi n/3) - L| < \varepsilon = \frac{1}{2}$. Take $n > N$ of the form $6k$. Then $\cos(\pi n/3) = \cos(2\pi k) = 1$. Thus, $|1 - L| < \frac{1}{2}$. Take $n > N$ of the form $6k+3$. Then $\cos(\pi n/3) = \cos(2\pi k + \pi) = -1$. Thus, $|-1-L| < \frac{1}{2}$. But then $|(1-L)-(-1-L)| \leq |1-L| + |-1-L|$ and $2 < \frac{1}{2} + \frac{1}{2} = 1$. This is a contradiction.

It shows that the sequence $\{\cos(\pi n/3)\}$ has no limit.

N8.1.8. $a_n = \dfrac{(-1)^n n}{(n+1)^2}$

N 8.1.16

$a_n = \frac{3^{n+2}}{5^n} = 9 \cdot \left(\frac{3}{5}\right)^n$. Observe that $\left(\frac{3}{5}\right)^n$ converges to 0, since it is of the form r^n with $|r| < 1$. By product rule,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 9 \cdot \left(\frac{3}{5}\right)^n = 9 \cdot \lim_{n\to\infty}\left(\frac{3}{5}\right)^n = 9 \cdot 0 = 0$$

Answer: $\lim_{n\to\infty} \frac{3^{n+2}}{5^n} = 0$.

N 8.1.30

$$-\frac{1}{\sqrt{n}} \le \frac{-1}{1+\sqrt{n}} \le \frac{\sin 2n}{1+\sqrt{n}} \le \frac{1}{1+\sqrt{n}} \le \frac{1}{\sqrt{n}}.$$

Thus, $\frac{-1}{\sqrt{n}} \le \frac{\sin 2n}{1+\sqrt{n}} \le \frac{1}{\sqrt{n}}$

Observe that $\{\frac{1}{\sqrt{n}}\}$ converges to 0 since it is of the form $\{n^r\}$ with $r = -\frac{1}{2} < 0$. Then also

$$\lim_{n\to\infty} \frac{-1}{\sqrt{n}} = -1 \cdot \lim_{n\to\infty} \frac{1}{\sqrt{n}} = -1 \cdot 0 = 0.$$

By squeeze Theorem, $\lim_{n\to\infty} \frac{\sin 2n}{1+\sqrt{n}} = 0$.

N 8.1.31.

The sequence does not approach one number, so it does not have a limit. prove it using contradiction. Assume that it has limit L. Then for all $\varepsilon > 0$ ~~starting~~ we can find index N starting from which $|a_n - L| < \varepsilon$. Take $\varepsilon = \frac{1}{10}$. There are infinitely many terms a_n equal to 0, so there is an index $n > N$ with $a_n = 0$. Thus, $|0 - L| < \varepsilon$, that is $|L| < \frac{1}{10}$. But also there are infinitely many indexes n for which $a_n = 1$ Therefore, we can find $n > N$ with $a_n = 1$. We obtain $|1 - L| < \varepsilon = \frac{1}{10}$

But $|1 - L| \geq 1 - |L| \geq 1 - \frac{1}{10} = \frac{9}{10} > \frac{1}{10}$. This contradiction shows that this sequence cannot have a limit.

Answer No limit.

N8.1.10 [To do]

To find the formula observe that 5 and 1 are on the same distance from their mean $\frac{5+1}{2}=3$: $5=3+2$, $1=3-2$. Thus,

$$a_n = 3+(-1)^{n-1}\cdot 2$$

N8.1.12

We have $a_n = \frac{n^3}{n^3+1} = \frac{1}{1+\frac{1}{n^3}}$. By the Sum and the Quotient Laws,

$$\lim_{n\to\infty} a_n = \frac{\lim\limits_{n\to\infty} 1}{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty}\frac{1}{n^3}} = \frac{1}{1+0} = 1.$$

N8.1.14

$a_n = \frac{n^3}{n+1} = \frac{n^2}{1+\frac{1}{n}}$. Observe that

$\frac{1}{n}\le 1 \Rightarrow 1+\frac{1}{n}\le 2 \Rightarrow \frac{n^2}{1+\frac{1}{n}} \ge \frac{n^2}{2}.$

In particular, a_n becomes larger than any number when n grows. Therefor, a_n diverges to ∞.

N8.1.25 (5

Since $0 \le \cos^2 n \le 1$ we have:

$0 \le a_n \le \frac{1}{2^n}$. The sequence $\{\frac{1}{2^n}\}_{n=1}^{\infty}$ is the geometric sequence with $r = \frac{1}{2}$, $|r|<1$.

Therefor, $\lim_{n\to\infty} \frac{1}{2^n} = 0$. By the Squeeze Theorem, $\underline{\lim_{n\to\infty} a_n = 0}$.

N8.1.38

The main term under the root sign is 5^n (3^n is comparably small: $\lim_{n\to\infty} \frac{3^n}{5^n} = \lim_{n\to\infty} \left(\frac{3}{5}\right)^n = 0$).

It is convenient to rewrite a_n as follows:

$a_n = \sqrt[n]{5^n \cdot \left(1+\left(\frac{3}{5}\right)^n\right)} = 5 \cdot \sqrt[n]{1+\left(\frac{3}{5}\right)^n}$.

Observe that $1 \le \sqrt[n]{x} \le x$ for all $x \ge 1$.

Therefor, $5 \le a_n \le 5 \cdot \left(1+\left(\frac{3}{5}\right)^n\right)$.

We have: $\lim_{n\to\infty} 5\left(1+\left(\frac{3}{5}\right)^n\right) = 5 + 5 \cdot \lim_{n\to\infty}\left(\frac{3}{5}\right)^n = 5 + 5 \cdot 0 = 5$. By the Squeeze Theorem, a_n is convergent to 5:

$\underline{\lim_{n\to\infty} a_n = 5}$.

N8.1.22.

The sequence is given by $a_n = f(\frac{2}{n})$, where $f(x) = \cos x$. Observe that

$\lim_{n\to\infty} \frac{2}{n} = 2 \lim_{n\to\infty} \frac{1}{n} = 2 \cdot 0 = 0$. Since $f(x)$ is continuous at $L = 0$, we obtain:

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(\frac{2}{n}) = f(0) = \cos 0 = 1$.

<u>Answer</u> $\lim_{n\to\infty} \cos(\frac{2}{n}) = 1$.

N 8.1.48

Observe that $\sqrt{2} = 2^{\frac{1}{2}}$, $\sqrt{2\sqrt{2}} = \sqrt{2} \cdot \sqrt[4]{2} = 2^{\frac{1}{2}+\frac{1}{4}} = 2^{\frac{3}{4}}$,

$\sqrt{2\sqrt{2\sqrt{2}}} = \sqrt{2} \cdot \sqrt[4]{2} \cdot \sqrt[8]{2} = 2^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}} = 2^{\frac{7}{8}}$, etc.

Thus, the sequence can be shortly written as

$a_n = 2^{1-\frac{1}{2^n}} = \frac{2}{2^{\frac{1}{2^n}}}$, or as a composition of the function ~~...~~ 2^{1-x} and the sequence $\{\frac{1}{2^n}\}$. We have: $\lim_{n\to\infty} \frac{1}{2^n} = 0$ (geometric sequence with $r = \frac{1}{2}$). The function $f(x)$ is continuous. Therefor,

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(\frac{1}{2^n}) = f(0) = 2^{1-0} = 2$.

<u>Answer</u> The limit is 2.

<u>Remark</u> This question can be as well done using Monotonic Sequence Theorem.

N 8.1 52 HW 2

To determine whether the sequence is monotonic we need to compare a_n and a_{n+1}. We have:

$$a_{n+1} - a_n = n+1 + \frac{1}{n+1} - \left(n + \frac{1}{n}\right) =$$

$$= 1 + \frac{1}{n+1} - \frac{1}{n} \geq 1 + \frac{1}{n+1} - 1 > 0$$, since

$n \geq 1$. Thus, for all n $a_{n+1} > a_n$.

Therefor, the sequence is increasing.

Since $a_n = n + \frac{1}{n} > n$ diverges to ∞, it is unbounded.

Problem 1 $a_n = \frac{n^3+7}{2^{2n}}$ is a ratio of two sequences diverging to ∞ $\Rightarrow$ it is reasonable to try the L'Hospital's rule.

Let $f(x) = x^3+7$, $g(x) = 2^{2x} = 4^x$. Then

$$\frac{f'(x)}{g'(x)} = \frac{3x^2}{\ln 4 \cdot 4^x}$$ still $\frac{\infty}{\infty}$ $\Rightarrow$

differentiate further

$$\frac{f''(x)}{g''(x)} = \frac{6x}{(\ln 4)^2 \cdot 4^x}$$ (still $\frac{\infty}{\infty}$)

$$\frac{f'''(x)}{g'''(x)} = \frac{6}{(\ln 4)^3 \cdot 4^x}$$. Since $f'''(x) = \text{const}$,

$g'''(x) \longrightarrow \infty$ when $x \to \infty$, we get

$$\frac{f'''(x)}{g'''(x)} \longrightarrow 0$$ when $x \to \infty$

By L'Hospital's Rule, we obtain

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{f''(x)}{g''(x)} =$$

$$= \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = 0. \quad \text{Thus, } \lim_{x\to\infty} \frac{x^3+7}{2^{2x}} = 0.$$

Therefore, $\lim_{n\to\infty} \frac{n^3+7}{2^{2n}} = 0.$

Problem 2 The sequence has first terms equal to:

$a_1 = 2, \; a_2 = \frac{3}{2}, \; a_3 = 2 - \frac{2}{3} = \frac{4}{3}, \; a_4 = 2 - \frac{3}{4} = \frac{5}{4},$

$a_5 = \frac{6}{5}.$

Solution 1. Notice that $a_n = \frac{n+1}{n}$ at least for $n = 1, \dots 5$. Hypothesis: $a_n = \frac{n+1}{n}$ for all n. Since the sequence is given by recursive formula, it is reasonable to try to prove the hypothesis by induction.

Base $n=1$, $a_1 = 2 = \frac{2}{1}$ true.

Step of induction assume that $a_n = \frac{n+1}{n}$.

Then $a_{n+1} = 2 - \frac{1}{a_n} = 2 - \frac{n}{n+1} = \frac{2n+2-n}{n+1} =$

$= \frac{n+2}{n+1} = \frac{(n+1)+1}{n+1} \Rightarrow$ the step of induction holds.

Thus, $a_n = \frac{n+1}{n}$ for all n.

We get:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1.$$

Solution 2 From the first terms we guess that $\{a_n\}$ should be decreasing and bounded below by 1. Let's prove by induction that $a_n > 1$ for all a_n.

Base $n=1$: $a_1 = 2 > 1$ true

Step of induction: let $a_n > 1$; then $\frac{1}{a_n} < 1$,

$-\frac{1}{a_n} > -1 \Rightarrow a_{n+1} = 2 - \frac{1}{a_n} > 2 - 1 = 1.$

Thus, step of induction holds. We conclude that $a_n > 1$ for all n.

Now, let's show that a_n is decreasing.

We have: $a_n - a_{n+1} = a_n - 2 + \frac{1}{a_n} =$

$= a_n + \frac{1}{a_n} - 2 \geq 0$ by the inequality between the arithmetic and geometric means (or we can say:

$$a_n + \frac{1}{a_n} - 2 = \frac{a_n^2 - 2a_n + 1}{a_n} = \frac{(a_n-1)^2}{a_n} \geq 0$$

since $a_n > 0$). Thus, $a_n \geq a_{n+1}$ for all n.

Since $\{a_n\}_{n=1}^{\infty}$ is monotonic and bounded it has a limit.

Let $L = \lim_{n\to\infty} a_n$. We have: $L \geq 1$ since $a_n > 1 \; \forall n$. (5

Further, $a_{n+1} = 2 - \frac{1}{a_n}$. When n goes to ∞ the left hand side has limit L and the right hand side has limit $2 - \frac{1}{L}$ (by the difference and quotient rule). Therefor,

$L = 2 - \frac{1}{L}$, $L^2 = 2L - 1$, $L^2 - 2L + 1 = 0$,

$(L-1)^2 = 0$ and $L = 1$. Thus,

$\lim_{n\to\infty} a_n = 1$.

N8.1.17 $\boxed{\text{To do}}$ HW2 (6)

a_n can be represented as $f(b_n)$ where $b_n = \frac{2\pi n}{1+8n}$ and $f(x) = \tan x$.

We have: $b_n = \dfrac{2\pi}{\frac{1}{n}+8}$. By the Sum and the Quotient Laws, $b_n \to \frac{2\pi}{8} = \frac{\pi}{4}$ when $n \to \infty$.

The function $\tan x$ is continuous at $x = \frac{\pi}{4}$.

Therefore, $\tan \frac{2\pi n}{1+8n} \to \tan\frac{\pi}{4} = 1$ when $n\to\infty$.

<u>Answer</u> $\lim\limits_{n\to\infty} \tan\frac{2\pi n}{1+8n} = 1$.

N8.1.35

Observe that $\left(-\frac{2}{e}\right)^n$ is a geometric sequence with $r = -\frac{2}{e}$, $|r| = \frac{2}{e} < 1$. Therefore,

$\lim\limits_{n\to\infty}\left(-\frac{2}{e}\right)^n = 0$. By the Sum Law,

$\lim\limits_{n\to\infty}\left(1+\left(-\frac{2}{e}\right)^n\right) = 1$.

N8.1.40

Each term of the numerator of a_n is less than $2n$. There are n terms in the numerator. If we replace $n-1$ of them by $2n$ we only increase a_n. Thus,

$a_n = \dfrac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{(2n)^n} \le \dfrac{1\cdot 2n\cdot 2n\cdot\ldots\cdot 2n}{(2n)^n} \le$

$$\frac{1\cdot(2n)^{n-1}}{(2n)^n} = \frac{1}{2n}.$$ So

$0 \le a_n \le \frac{1}{2n}$. We have $\lim\limits_{n\to\infty} \frac{1}{2n} = 0$.

By the Squeeze Theorem, $a_n \to 0$ when $n\to\infty$.

<u>Answer</u> $\lim\limits_{n\to\infty} a_n = 0$.

N 8.1 50

$$a_n = \frac{2n-3}{3n+4}.$$

To see if the sequence is monotonic compare a_n and a_{n+1}:

$$a_{n+1} - a_n = \frac{2(n+1)-3}{3(n+1)+4} - \frac{2n-3}{3n+4} =$$

$$= \frac{2n-1}{3n+7} - \frac{2n-3}{3n+4} = \frac{(2n-1)(3n+4)-(2n-3)(3n+7)}{(3n+7)(3n+4)} =$$

$$= \frac{6n^2+5n-4-(6n^2+5n-21)}{(3n+7)(3n+4)} = \frac{17}{(3n+7)(3n+4)}$$

Thus, $a_{n+1} - a_n > 0$ for all $n \in \mathbb{N} \Rightarrow$

$a_{n+1} > a_n$ and $\{a_n\}$ is <u>increasing</u>.

We have: $a_1 = -\frac{1}{7}$, and $a_n > 0$ for all $n \ge 2$, since $2n-3>0$. Thus, $a_n \ge -\frac{1}{7}$ for all n.

On the other hand, $a_n = \frac{2n-3}{3n+4} < \frac{2n}{3n} = \frac{2}{3}$.

$\{a_n\}$ is bounded from above and from below $\Rightarrow$ is <u>bounded</u>.

<u>Answer</u>: increasing and bounded

8.1.51

$a_n = n(-1)^n$.

$a_1 = -1,\ a_2 = 2,\ a_3 = -3,\ a_4 = 4$.

We have $a_1 < a_2$ but $a_2 > a_3 \Rightarrow$ the sequence is neither increasing nor decreasing. Thus, it is <u>not monotonic</u>.

$|a_n| = n$ grows without bound $\Rightarrow$ the sequence is <u>unbounded</u>

<u>Answer</u>: not monotonic, unbounded.

N 8.2.9

a) We have:
$$a_n = \frac{2n}{3n+1} = \frac{2}{3+\frac{1}{n}}.$$

By Quotient and Sum Laws, we get
$$\lim_{n\to\infty} a_n = \frac{\lim_{n\to\infty} 2}{\lim_{n\to\infty} 3 + \lim_{n\to\infty}\frac{1}{n}} = \frac{2}{3+0} = \underline{\frac{2}{3}}.$$

b) Since $\lim_{n\to\infty} a_n \neq 0$, by The Divergence Test, $\sum_{n=1}^{\infty} a_n$ is divergent.

N 8.2.12

$a_n = 4\cdot\left(\frac{3}{4}\right)^{n-1}$. Since $r=\frac{3}{4}$, $|r|<1$, it is convergent
$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 4\cdot\left(\frac{3}{4}\right)^{n-1} = \frac{4}{1-\frac{3}{4}} = \frac{4}{\frac{1}{4}} = \underline{16}.$$

N 8.2.29

Consider the series formed by the terms $\frac{1}{e^n}$ and $\frac{1}{n(n+1)}$ separately.

$\sum_{n=1}^{\infty} \frac{1}{e^n} = \sum_{n=1}^{\infty}\left(\frac{1}{e}\right)^n$ is a geometric series with $r=\frac{1}{e}$, $|r|<1 \Rightarrow$ it is convergent.
$$\sum_{n=1}^{\infty}\left(\frac{1}{e}\right)^n = \frac{\frac{1}{e}}{1-\frac{1}{e}} = \frac{1}{e-1}$$

(here we use either $\sum_{n=1}^{\infty} ar^n = \frac{ar}{1-r}$ with $a=1$ or
$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ with $a=\frac{1}{e}$)

$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent (see Example 6 on page 568 of the course book),

$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$

Since both series are convergent, by the "sum law" for the series, we get the convergence of

$$\sum_{n=1}^{\infty} \left(\frac{1}{e^n} + \frac{1}{n(n+1)} \right) = \sum_{n=1}^{\infty} \frac{1}{e^n} + \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \underline{\frac{1}{e-1} + 1}.$$

N 8.3.8

Let $f(x) = \frac{1}{\sqrt{x+4}}$. Then $f(x)$ is positive, continuous and decreasing on $[1, \infty)$. By The Integral Test, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$ is convergent if and only if $\int_1^{\infty} \frac{1}{\sqrt{x+4}}\, dx$ is convergent. We have

$\int_1^t \frac{1}{\sqrt{x+4}}\, dx = 2\sqrt{x+4}\,\Big|_1^t = 2\sqrt{t+4} - 2\sqrt{5} \to +\infty$ when $t \to +\infty$. Therefor, $\int_1^{\infty} \frac{1}{\sqrt{x+4}}\, dx$ is divergent.

Thus, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$ is <u>divergent.</u>

N 8.3.19. Observe that $0 \le \cos^2 n \le 1$ and

$\frac{1}{n^2+1} \le \frac{1}{n^2}$. Thus, $0 \le \frac{\cos^2 n}{n^2+1} \le \frac{1}{n^2}$.

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is the p-series with $p = 2 > 1$ $\Rightarrow$ is convergent. By the Comparison Test,

$\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2+1}$ is also <u>convergent.</u>

Todo

HW 3 (3

№ 8.2.21.

$a_k = \frac{k^2}{k^2-1} = \frac{1}{1-\frac{1}{k^2}}$. By The Quotient and The Difference Limits for sequences,

$\lim_{k\to\infty} a_k = \frac{\lim_{k\to\infty} 1}{\lim_{k\to\infty} 1 - \lim_{k\to\infty} \frac{1}{k^2}} = 1$ Thus,

$\lim_{k\to\infty} a_k \neq 0$. By the Divergence Test,

$\sum_{k\geq 1}^{\infty} a_k$ is <u>divergent</u>

<u>answer</u>: divergent.

№ 8.2.30

Consider the two series with the terms $\frac{3}{5^n}$ and $\frac{2}{n}$ separately.

$\sum_{n=1}^{\infty} \frac{3}{5^n}$ is a geometric series with $r = \frac{1}{5} \Rightarrow$

<u>convergent</u>.

$\sum_{n=1}^{\infty} \frac{3}{5^n} = \frac{3 \cdot \frac{1}{5}}{1-\frac{1}{5}} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$.

$\sum_{n=1}^{\infty} \frac{2}{n} = 2 \cdot \sum_{n=1}^{\infty} \frac{1}{n}$ is <u>divergent</u> (p-series with p=1).

Therefore, $\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n}\right)$ is <u>divergent</u>

<u>Answer</u>: divergent.

<u>Remark</u> If $\sum_{n=1}^{\infty} a_n$ is <u>convergent</u> and $\sum_{n=1}^{\infty} b_n$ is <u>divergent</u>, then $\sum_{n=1}^{\infty}(a_n+b_n)$ is <u>divergent</u> (otherwise using the Difference Law we would obtain that $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}((a_n+b_n)-a_n)$ is convergent).

However, it is <u>NOT</u> true that given $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ divergent implies $\sum_{n=1}^{\infty}(a_n+b_n)$ is divergent.

<u>Trivial example</u>: $a_n = 1$, $b_n = -1$ for all n.

Then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ diverge, but

$$\sum_{n=1}^{\infty}(a_n+b_n) = \sum_{n=1}^{\infty} 0 = 0 \text{ converges.}$$

N 8.2.33 Useful formula to remember:

$$\frac{b-a}{(n+a)(n+b)} = \frac{1}{n+a} - \frac{1}{n+b}.$$

For $a=0, b=3$ we get: $\frac{3}{n(n+3)} = \frac{1}{n} - \frac{1}{n+3}$.

Thus, S_n can be expressed as follows:

$$S_n = \left(1-\frac{1}{4}\right)+\left(\frac{1}{2}-\frac{1}{5}\right)+\left(\frac{1}{3}-\frac{1}{6}\right)+\left(\frac{1}{4}-\frac{1}{7}\right)+\left(\frac{1}{5}-\frac{1}{8}\right)+\left(\frac{1}{6}-\frac{1}{9}\right)+$$
$$\ldots+\left(\frac{1}{n-3}-\frac{1}{n}\right)+\left(\frac{1}{n-2}-\frac{1}{n+1}\right)+\left(\frac{1}{n-1}-\frac{1}{n+2}\right)+\left(\frac{1}{n}-\frac{1}{n+3}\right)$$

(for large enough n). We see that for large n most of the terms cancel. In fact, only 6 terms stay: $S_n = 1+\frac{1}{2}+\frac{1}{3}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3} =$

$$= 1\frac{5}{6} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}$$

By the Sum and the Difference Laws we get:

$$\lim_{n\to\infty} S_n = 1\frac{5}{6} - \lim_{n\to\infty}\frac{1}{n+1} - \lim_{n\to\infty}\frac{1}{n+2} - \lim_{n\to\infty}\frac{1}{n+3} =$$

$= 1\frac{5}{6}$. By definition of the sum of a series,

$$\sum_{n=1}^{\infty}\frac{3}{n(n+3)} = 1\frac{5}{6}.$$

Answer: convergent to $1\frac{5}{6}$

N8.3.12.

$\sum_{n=1}^{\infty} n^{-1.4}$ is a p-series with $p = 1.4 > 1$

$\left(n^{-1.4} = \frac{1}{n^{1.4}}\right)$. Therefore, it is convergent.

$\sum_{n=1}^{\infty} n^{-1.2}$ is a p-series with $p = 1.2 > 1 \Rightarrow$

it is also convergent. It follows that

$$\sum_{n=1}^{\infty}\left(n^{-1.4} + 3\cdot n^{-1.2}\right) = \sum_{n=1}^{\infty} n^{-1.4} + 3\cdot\sum_{n=1}^{\infty} n^{-1.2}$$

is convergent.

Answer: convergent.

N8.3.26.

We have: $\frac{1}{\sqrt{n^3+1}} > 0$ and $\frac{1}{\sqrt{n^3+1}} < \frac{1}{\sqrt{n^3}} =$

$= \frac{1}{n^{3/2}}$ for all n. $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ is p-series with

$p = 3/2 \Rightarrow$ convergent. By Comparison Test

$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^3+1}}$ is also convergent. Answer: convergent

(To do)

N8.3.28

$a_n = \frac{1+\sin n}{10^n}$. We have $0 \le 1+\sin n \le 2$ for all n. Thus, $0 \le a_n \le \frac{2}{10^n}$.

The series $\sum_{n=1}^{\infty} \frac{2}{10^n}$ is the geometric series, $r = \frac{1}{10}$, $|r|<1 \Rightarrow$ it is convergent. By the Comparison Test, $\sum_{n=1}^{\infty} \frac{1+\sin n}{10^n}$ is convergent.

N8.3.29 When $n \to \infty$ sequence $\{\frac{1}{n}\}$ converges to 0. Near 0 $\sin x$ behaves like x. More precisely, $\lim_{x\to 0} \frac{\sin x}{x} = 1$. Therefore,

$\lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = 1$. Since $\sin\frac{1}{n} > 0$ and $\frac{1}{n} > 0$ for all n, we can use the Limit Comparison Test. We have: $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. Therefore, $\sum_{n=1}^{\infty} \sin(\frac{1}{n})$ is also divergent.

N8.3.34 Let $S_n = \sum_{k=1}^{n} \frac{1}{k^5}$ be the n-th partial sum and $R_n = \sum_{k=n+1}^{\infty} \frac{1}{k^5}$ be the n-th remainder of the series. Let $f(x) = \frac{1}{x^5}$ so that $a_n = \frac{1}{n^5} = f(n)$. By the Remainder Estimate for the Integral Test,

$$\int_{n+1}^{\infty} \frac{1}{x^5}\,dx \le R_n \le \int_{n}^{\infty} \frac{1}{x^5}\,dx$$

(since $\frac{1}{x^5}$ is positive, continuous and decreasing on $[1,+\infty)$)

Thus, $R_n \le -\frac{1}{4x^4}\Big|_n^\infty =$

$0-\left(-\frac{1}{4n^4}\right)=\frac{1}{4n^4}$ and $R_n \ge -\frac{1}{4x^4}\Big|_{n+1}^\infty=\frac{1}{4(n+1)^4}$.

To estimate s correct to three digits lets take n so that $R_n < 10^{-3}$. Solving $\frac{1}{4n^4}<10^{-3} \Leftrightarrow$ $n^4 > \frac{10^3}{4}=250$ we see that $n=4$ is sufficient. We have: $S_4 = 1+\frac{1}{32}+\frac{1}{243}+\frac{1}{1024} \approx$ 1.03634 and;

$$S_4+\frac{1}{4\cdot 5^4} \le S = S_4+R_4 \le S_4+\frac{1}{4\cdot 4^4}$$

$$1.03674 \le S \le 1.03732.$$

Therefor, round of S to three digits is 1.037.

<u>Answer</u>; 1.037.

N 8.3.38

$$S_{10}=\sum_{n=1}^{10}\frac{\sin^2 n}{n^3}\approx 0.83253$$

$$R_{10}=\sum_{n=11}^{\infty}\frac{\sin^2 n}{n^3}.$$

We cannot use the integral test directly to $f(x)=\frac{\sin^2 x}{x^3}$ since there is no formula for $\int f(x)\,dx$. Instead first use the comparison test.

We have:

$$0 \le \frac{\sin^2 n}{n^3} \le \frac{1}{n^3}, \quad \sum_{n=11}^{\infty} \frac{1}{n^3} \text{ is}$$

convergent (p-series, $p = 3 > 1$). Therefore,

$$\sum_{n=11}^{\infty} \frac{\sin^2 n}{n^3} \le \sum_{n=11}^{\infty} \frac{1}{n^3}.$$

For the latter sum use the integral test to estimate the remainder. Set $f(x) = \frac{1}{x^3}$. Then $f(x)$ is decreasing, positive, continuous. Therefore,

$$\sum_{n=11}^{\infty} \frac{1}{n^3} \le \int_{10}^{\infty} \frac{1}{x^3}\,dx = -\frac{1}{2x^2}\Big|_{10}^{\infty} = 0 - \left(-\frac{1}{2\cdot 10^2}\right) = \frac{1}{200} = 0.005.$$

Thus, $0 \le R_n \le 0.005$.

Answer The error is less or equal to 0.005.

$S \approx S_{10} \approx 0.83253$.

Remark Here we could not use improved estimate, because $R_{10} = \sum_{n=11}^{\infty} \frac{\sin^2 n}{n^3} \le \sum_{n=11}^{\infty} \frac{1}{n^3}$ and saying that $\sum_{n=11}^{\infty} \frac{1}{n^3} \ge \int_{11}^{\infty} \frac{1}{x^3}\,dx$ would not give any information about R_{10}

~~N8.3.38~~ HW4 (4)

N8.4.3

$$a_n = \frac{4\cdot(-1)^{n-1}}{n+6}$$

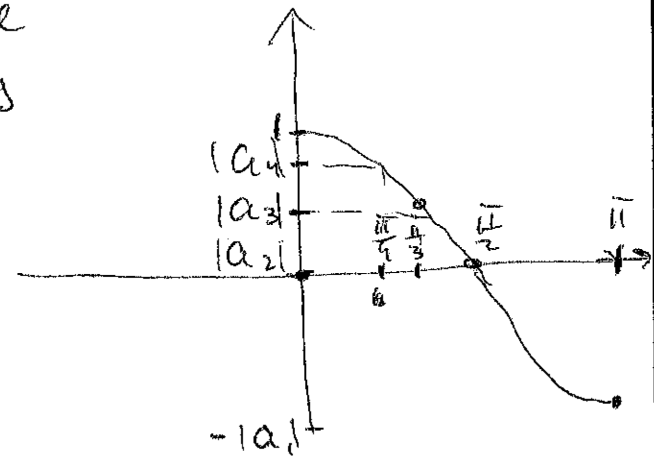
$\sum_{n=1}^{\infty} a_n$ is an alternating series.

$|a_n| = \frac{4}{n+6}$ is decreasing and convergent to 0. By the alternating series test, $\sum_{n=1}^{\infty} a_n$ is convergent.

N8.4.10

$a_n = (-1)^n \cos\left(\frac{\pi}{n}\right)$, $|a_n| = |\cos\left(\frac{\pi}{n}\right)|$

From the picture we can see that $|a_n|$ approache 1. Let's prove this. For $n \geq 2$



$|a_n| = \cos\frac{\pi}{n}$, since $\cos\frac{\pi}{n} \geq 0$ for $n \geq 2$.

Let $f(x) = \cos x$, $b_n = \frac{\pi}{n}$.

Then $b_n \to 0$ when $n \to \infty$. Since $\cos x$ is continuous everywhere (in particular at 0) we have: $\lim_{n\to\infty} |a_n| = \cos 0 = 1$.

Thus, $|a_n|$ does not converge to 0 $\Rightarrow$ a_n does not converge to 0. By the Divergence Test, $\sum_{n=1}^{\infty} a_n$ is divergent.

N8.4.14. HW4 (5

$a_n = (-1)^n b_n$ where $b_n = \frac{1}{n\cdot 5^n}$.

$\{b_n\}$ is decreasing, since $n\cdot 5^n$ is increasing. $\lim_{n\to\infty} b_n = 0$. By the alternating series Test, $\sum_{n=1}^{\infty}(-1)^n b_n$ is convergent. Moreover, the remainder $R_n = S - S_n$ satisfies

$|R_n| \le b_{n+1} = \frac{1}{(n+1)5^{n+1}}$. To make the error < 0.0001 choose n so that $b_{n+1} < 0.0001$. We have

$\frac{1}{(n+1)5^{n+1}} < 0.0001 \Leftrightarrow (n+1)5^{n+1} > 10^4$. By try and error we find that $n = 4$ is sufficient. Thus, $S_4 = \sum_{n=1}^{4} \frac{(-1)^n}{n\cdot 5^n} = -0.1922\overline{6}$ approximate S with an error < 0.0001.

N8.4.22.

Use the Ratio Test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{(n+1)!}{100^{n+1}}}{\frac{n!}{100^n}} = \frac{n+1}{100}$$ diverges to ∞,

since grows without a bound. By the Ratio Test, $\sum_{n=1}^{\infty}|a_n|$ is divergent $\Rightarrow$ not absolutely convergent

Remark This problem can be also solved by the Divergence Test by showing that $\lim_{n\to\infty} a_n = \infty$

N8.4.30

$$a_n = \frac{\sin 4n}{4^n}, \quad |a_n| = \frac{|\sin 4n|}{4^n}$$

We have: $0 \le \frac{|\sin 4n|}{4^n} \le \frac{1}{4^n}$.

The series $\sum_{n=1}^{\infty} \frac{1}{4^n}$ is the geometric series with $r = \frac{1}{4}$, $|r| < 1 \Rightarrow$ convergent. By the Comparison Test $\sum_{n=1}^{\infty} \frac{|\sin 4n|}{4^n}$ is convergent. Thus, $\sum_{n=1}^{\infty} \frac{\sin 4n}{4^n}$ is <u>absolutely convergent</u>.

N8.4.35

Clearly, $a_n > 0$ for all n. Use the Ratio Test. We have: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{5n+1}{4n+3} = \frac{5+\frac{1}{n}}{4+\frac{3}{n}} \longrightarrow \frac{5}{4} > 1$ when $n \to \infty$. Therefore, $\sum_{n=1}^{\infty} a_n$ is <u>divergent</u>.

N8.5.13

$$a_n = \frac{(x-2)^n}{n^2+1}, \quad \left|\frac{a_{n+1}}{a_n}\right| = \frac{\left|\frac{(x-2)^{n+1}}{(n+1)^2+1}\right|}{\left|\frac{(x-2)^n}{n^2+1}\right|} =$$

$$= |x-2| \cdot \frac{n^2+1}{n^2+2n+2} = |x-2| \cdot \frac{1+\frac{1}{n^2}}{1+\frac{2}{n}+\frac{2}{n^2}} \longrightarrow |x-2|.$$

By the Ratio Test, if $|x-2|<1$ the series is convergent, if $|x-2|>1$ the series is divergent. Thus, $R=1$.

Let's check the boundary points: $|x-2|=1$, $x=1$ or $x=3$. Then $|a_n| = \frac{1}{n^2+1} < \frac{1}{n^2}$.

By The Comparison Test, $\sum |a_n|$ is convergent, since $\sum \frac{1}{n^2}$ is convergent. Thus, $\sum a_n$ is absolutely convergent $\Rightarrow$ convergent.

Answer: $R=1$, $[1,3]$

N 8.5.16 $\quad a_n = \frac{n}{4^n}\cdot(x+1)^n$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{n+1}{4^{n+1}}\cdot|x+1|^{n+1}}{\frac{n}{4^n}\cdot|x+1|^n} = \frac{n+1}{n}\cdot\frac{|x+1|}{4} \longrightarrow \frac{|x+1|}{4}.$$

By The Ratio Test, if $\frac{|x+1|}{4}<1$ the series is convergent, if $\frac{|x+1|}{4}>1$ it is divergent. $\frac{|x+1|}{4}<1 \Leftrightarrow$ $|x+1|<4 \Rightarrow R=4$. When $|x+1|=4$, $x=-5$ or $x=3$.

We have: $|a_n| = \frac{n\cdot|x+1|^n}{4^n} = \frac{n\cdot 4^n}{4^n} = n \longrightarrow \infty \Rightarrow$ a_n does not converge to 0. By the divergence test $\sum a_n$ is divergent. Answer $R=4$, $(-5,3)$.

N8.5.19 (2

$a_n = n!\,(2x-1)^n$, $\left|\frac{a_{n+1}}{a_n}\right| = \frac{|(n+1)!\cdot(2x-1)^{n+1}|}{|n!\cdot(2x-1)^n|} =$

$= (n+1)\cdot|2x-1| \to \infty$ when $2x-1 \neq 0$.

Thus, $R = \infty$, $\sum_{n=1}^{\infty} n!\,(2x-1)^n$ converges

only for $x = \frac{1}{2}$.

__Answer__: $R = \infty$, $\{\frac{1}{2}\}$.

N8.5.20 $a_n = \frac{(3x-2)^n}{n\cdot 3^n}$, $\left|\frac{a_{n+1}}{a_n}\right| = \frac{\left|\frac{(3x-2)^{n+1}}{(n+1)\cdot 3^{n+1}}\right|}{\left|\frac{(3x-2)^n}{n\cdot 3^n}\right|} =$

$= |3x-2|\cdot\frac{n}{n+1}\cdot\frac{1}{3} \to \frac{|3x-2|}{3} = |x-\frac{2}{3}|$. Ratio Test

$\Rightarrow$ If $|x-\frac{2}{3}|<1$ the series is convergent, if $|x-\frac{2}{3}|>1$ the series is divergent. Thus, $R=1$

Let's check the boundary points: $|x-\frac{2}{3}|=1$,

$x = \frac{5}{3}$ and $x = -\frac{1}{3}$.

a) $x = \frac{5}{3}$. Then $a_n = \frac{(3\cdot\frac{5}{3}-2)^n}{n\cdot 3^n} = \frac{1}{n}$,

$\sum_{n=1}^{\infty} a_n$ is the harmonic series, divergent.

b) $x = -\frac{1}{3}$, then $a_n = \frac{(-\frac{1}{3}\cdot 3-2)^n}{n\cdot 3^n} = \frac{(-1)^n}{n}$.

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ is alternating series.

$|a_n| = \frac{1}{n}$ is decreasing, convergent to 0.

By the alternating series Test, $\sum_{n=1}^{\infty} a_n$ is convergent.

__Answer__ $R=1$, $[-\frac{1}{3}, \frac{5}{3})$.

To Do

N 8.5.8

$$a_n = \frac{10^n x^n}{n^3}, \quad \left|\frac{a_{n+1}}{a_n}\right| = \frac{\left|\frac{10^{n+1} x^{n+1}}{(n+1)^3}\right|}{\left|\frac{10^n x^n}{n^3}\right|} =$$

$$= 10|x| \cdot \frac{n^3}{(n+1)^3} = 10|x| \cdot \frac{1}{\left(1+\frac{1}{n}\right)^3} \to 10|x|.$$

By the Ratio Test, if $10|x|<1$ then the series is convergent, if $10|x|>1$ then the series is divergent. $10|x|<1 \iff |x|<\frac{1}{10}$. Therefore, $R=\frac{1}{10}$. Boundary points: $x=-\frac{1}{10}$ and $x=\frac{1}{10}$.

In both cases we have:

$|a_n| = \frac{10^n|x|^n}{n^3} = \frac{1}{n^3}$, $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{1}{n^3}$ is a p-series with $p=3>1 \Rightarrow$ is convergent. By the absolute convergence Test, $\sum_{n=1}^{\infty} a_n$ is convergent as well.

Answer $R=\frac{1}{10}$, $\left[-\frac{1}{10}, \frac{1}{10}\right]$.

N 8.5.22.

$$a_n = \frac{x^{2n}}{n(\ln n)^2}, \quad \frac{|a_{n+1}|}{|a_n|} = \frac{\frac{|x^{2n+2}|}{(n+1)(\ln(n+1))^2}}{\frac{|x^{2n}|}{n \ln n^2}} =$$

$$= |x^2| \cdot \frac{n}{n+1} \cdot \left(\frac{\ln(n)}{\ln(n+1)}\right)^2.$$ We have: $\frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \to 1$ when $n \to \infty$. To find $\lim_{n\to\infty} \frac{\ln(n)}{\ln(n+1)}$ use the L'Hospital's Rule. Let $f(x)=\ln(x)$, $g(x)=\ln(x+1)$. Then $f(x)\to\infty$ and $g(x)\to\infty$ when $x\to\infty \Rightarrow$

$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim\limits_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{x+1}} =$ (4

$= \lim\limits_{x\to\infty} \left(1+\frac{1}{x}\right) = 1$. Thus, by the Limit Laws,

$\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x|^2$. By The Ratio Test, when $|x|^2 < 1$ the series is convergent. When $|x|^2 > 1$ the series is divergent.

$|x|^2 < 1 \Leftrightarrow |x| < 1 \Rightarrow R = 1$.

Boundary points: $|x| = 1$, $x = -1$ and $x = 1$.

In both cases we have $|a_n| = \frac{1}{n(\ln n)^2}$.

The series $\sum\limits_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ is convergent (see Example 9 on the course webpage). ~~Thus~~,

~~$\sum\limits_{n=2}^{\infty} |a_n|$ converges. By the Absolute Convergence Test, $\sum\limits_{n=2}^{\infty} a_n$ is convergent.~~

Answer $R = 1$, $[-1, 1]$.

N 8.5.26 Let R be the radius of convergence. Then the series is convergent when $|x| < R$ and divergent when $|x| > R$. Convergence for $x = -4$ implies that $R \geq |x| = 4$. Divergence for $x = 6$ implies that $R \leq |x| = 6$. Thus, $4 \leq R \leq 6$.

(a) $\sum\limits_{n=0}^{\infty} c_n = \sum\limits_{n=0}^{\infty} c_n (1)^n$ is convergent since

$1 < 4 \leq R \Rightarrow 1 < R$

(b) $\sum\limits_{n=0}^{\infty} c_n 8^n$ is divergent since $8 > 6 \geq R \Rightarrow$

$8 > R$.

(c) $\sum_{n=0}^{\infty} c_n(-3)^n$ is <u>convergent</u> since

$|-3| = 3 < 4 \leq R$

(d) $\sum_{n=0}^{\infty} (-1)^n c_n 9^n = \sum_{n=0}^{\infty} c_n \cdot (-9)^n$ is <u>divergent</u>

since $|-9| = 9 > 6 \geq R$.

N 8.5.31

$f(x) = (1 + x^2 + x^4 + x^6 + \dots) + (2x + 2x^3 + 2x^5 + \dots) =$
$= \sum_{n=0}^{\infty} x^{2n} + 2\sum_{n=0}^{\infty} x^{2n+1}$. Using the Ratio test we obtain that the radius of convergence for both series is 1 $\Rightarrow$ by the Sum Law for series $f(x)$ is convergent for $|x| < 1$. Notice, the Sum Law <u>DOES NOT</u> imply that $f(x)$ is divergent for $|x| > 1$. However, if $|x| \geq 1$ the terms of $f(x)$ do not converge to 0 $\Rightarrow$ By the Divergence Test $f(x)$ is divergent. Thus, the interval of convergence is $|x| < 1$: $(-1, 1)$, and so $R = 1$.

We have: $\sum_{n=0}^{\infty} x^{2n} = \frac{1}{1-x^2}$, $\sum_{n=0}^{\infty} x^{2n+1} = \frac{x}{1-x^2}$,

by geometric series. Thus,

$$f(x) = \frac{1}{1-x^2} + \frac{2x}{1-x^2} = \frac{2x+1}{1-x^2} \quad \text{for } |x| < 1$$

<u>Answer</u> $R = 1$, $(-1, 1)$, $f(x) = \frac{2x+1}{1-x^2}$

To Hand in HW6

N8.6.4 $\frac{3}{1-x^4} = 3 \cdot \frac{1}{1-x^4} = 3 \cdot \sum_{n=0}^{\infty} x^{4n} =$

$= 3 + 3x^4 + 3x^8 + \dots$

Since the geometric series converges for $|r| < 1$ we have: $|x^4| < 1 \Leftrightarrow |x| < 1 \Rightarrow R = 1$ and the interval of convergence is $(-1, 1)$.

N8.6.9

$\frac{1+x}{1-x} = \frac{1}{1-x} + x \cdot \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n + x \cdot \sum_{n=0}^{\infty} x^n =$

$= (1 + x + x^2 + x^3 + \dots) + (x + x^2 + x^3 + \dots) = 1 + 2x + 2x^2 + 2x^3 + \dots =$

$= 1 + 2 \sum_{n=1}^{\infty} x^n$. To find R use the ratio test:

$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{2x^{n+1}}{2x^n} \right| = |x|$ for $n \geq 1 \Rightarrow$

$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = |x|$. The series converges when $|x| < 1$ and diverges when $|x| > 1$. Thus, $R = 1$.

If $|x| = 1$, then $|a_n| = 2|x^n| = 2$ for $n \geq 1 \Rightarrow$ By the Divergence Test $\sum a_n$ diverges.

<u>Answer</u> $\frac{1+x}{1-x} = 1 + 2 \sum_{n=1}^{\infty} x^n$, $R = 1$, $(-1, 1)$.

N8.6.13 By the properties of the logarith,

$$\ln(5-x) = \ln\left(5\cdot\left(1-\frac{x}{5}\right)\right) = \ln 5 + \ln\left(1-\frac{x}{5}\right).$$

By the power series representation for $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}$ we have:

$$\ln\left(1-\frac{x}{5}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cdot\left(-\frac{x}{5}\right)^n}{n} = -\sum_{n=1}^{\infty} \frac{x^n}{n\cdot 5^n}.$$

Since the power series for $\ln(1+x)$ converges with $R=1$, the power series for $\ln\left(1-\frac{x}{5}\right)$ converges when $\left|-\frac{x}{5}\right| < 1$ (that is, $|x|<5$) and diverges when $\left|-\frac{x}{5}\right| > 1$ (that is, $|x|>5$). Therefore, $R=5$.

<u>Answer</u> $\ln(5-x) = \ln 5 - \sum_{n=1}^{\infty} \frac{x^n}{n\cdot 5^n}$, $R=5$.

N8.6.14 Using the power series representation for $\tan^{-1} x$ we obtain:

$$f(x) = x^2 \tan^{-1}(x^3) = x^2 \cdot \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n+1}}{2n+1} =$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{6n+5}}{2n+1} = x^5 - \frac{x^{11}}{3} + \frac{x^{17}}{5} - \frac{x^{23}}{7} + \ldots$$

Since the radius of convergence for $\tan^{-1} x$ is 1, the series for $f(x)$ converges when $|x^3|<1$ (that is, $|x|<1$) and diverges when $|x^3|>1$ (that is, $|x|>1$). Therefore, $R=1$

<u>Answer</u> $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{6n+5}}{2n+1}$, $R=1$.

N8.6.23 By geometric series

$$\int \frac{t}{1-t^8}\,dt = \int t \cdot \sum_{n=0}^{\infty} t^{8n}\,dt =$$

$$= \int \sum_{n=0}^{\infty} t^{8n+1}\,dt = \sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2} + C =$$

$$= C + \frac{t^2}{2} + \frac{t^{10}}{10} + \frac{t^{18}}{18} + \ldots$$

Since the series $\frac{t}{1-t^8} = \sum_{n=0}^{\infty} t^{8n+1}$ has radius of convergence 1, the integral has also radius of convergence 1.

<u>Answer</u> $\int \frac{t}{1-t^8}\,dt = C + \sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2}$, $R = 1$.

<u>To Do</u>

N8.6.5 Using the geometric series formula we obtain:

$$\frac{2}{3-x} = \frac{2}{3} \cdot \frac{1}{1-\frac{x}{3}} = \frac{2}{3} \cdot \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty} \frac{2x^n}{3^{n+1}}.$$

The representation is valid when $\left|\frac{x}{3}\right| < 1$, that is $|x| < 3$.

<u>Answer</u> $\sum_{n=0}^{\infty} \frac{2x^n}{3^{n+1}}$, $(-3, 3)$.

N8.6.8 Replacing x by $-2x^2$ in the formula $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ we obtain: $\frac{x}{2x^2+1} = x \cdot \frac{1}{1-(-2x^2)} = x \cdot \sum_{n=0}^{\infty}(-2x^2)^n =$

$= \sum_{n=0}^{\infty}(-2)^n x^{2n+1}$. The representation is valid if $|-2x^2| < 1$, that is $|x| < \frac{1}{\sqrt{2}}$

Answer $\sum_{n=0}^{\infty}(-2)^n x^{2n+1}$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

N8.6.17.

Using the formula $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n$ $(|x|<1)$

we get:

$\frac{1+x}{(1-x)^2} = \frac{1}{(1-x)^2} + \frac{x}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n + \sum_{n=0}^{\infty}(n+1)x^{n+1} =$

$= (1+2x+3x^2+4x^3+\ldots) + (x+2x^2+3x^3+4x^4+\ldots) =$

$= 1+3x+5x^2+7x^3+\ldots = \sum_{n=0}^{\infty}(2n+1)x^n$. Find the radius of convergence using the Ratio Test:

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(2(n+1)+1)x^{n+1}}{(2n+1)x^n}\right| = \frac{2+\frac{3}{n}}{2+\frac{1}{n}} \cdot |x| \to |x|$ when $n \to \infty$.

Therefore, $R = 1$

Answer $\sum_{n=0}^{\infty}(2n+1)x^n$, $R = 1$.

N8.6.26

$$\int \tan^{-1}(x^2)\,dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1}\,dx =$$

$$= \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{2n+1}\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)\cdot(4n+3)}.$$

Since the power series representation for $\tan^{-1}(x^2)$ converges with $|x^2|<1$ and diverges with $|x^2|>1$, the radius of convergence for $\tan^{-1}(x^2)$ is 1, therefore the radius of convergence for $\int \tan^{-1}(x^2)dx$ is also 1

<u>answer</u> $C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)(4n+3)}$, $R = 1$.

N8.6.27

$$\frac{1}{1+x^5} = \sum_{n=0}^{\infty} (-x^5)^n = \sum_{n=0}^{\infty} (-1)^n x^{5n}.$$

$$\int_0^{0.2} \frac{1}{1+x^5}\,dx = \int_0^{0.2} \sum_{n=0}^{\infty} (-1)^n x^{5n}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n+1}}{5n+1}\Big|_0^{0.2} =$$

$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (0.2)^{5n+1}}{5n+1}$. By the alternating series remainder estimation $|R_n| \le B_{n+1} = \frac{(0.2)^{5(n+1)+1}}{5(n+1)+1} = \frac{(0.2)^{5n+6}}{5n+6}$. To make $b_{n+1} < 10^{-6}$ we can take $\underline{n=1}$, since $b_2 = \frac{(0.2)^{11}}{11} = \frac{2{,}048 \cdot 10^{-8}}{11} < 10^{-8}$.

Thus, $\int_0^{0.2} \frac{1}{1+x^5}\,dx \approx \sum_{n=0}^{1} \frac{(-1)^n (0.2)^{5n+1}}{5n+1} = 0.2 - \frac{64}{6\cdot 10^6} =$

$= 0.1999893\overline{3}$ with an error $< 10^{-8}$.
Therefore, the first six digits are:
0.199989.

<u>Answer</u> 0.199989.

N8.7.5

$f(x) = (1-x)^{-2}$, $f'(x) = (-2)\cdot(1-x)^{-3} = -\frac{2}{(1-x)^3}$,

$f''(x) = (-3)\cdot\frac{-2}{(1-x)^4} = \frac{6}{(1-x)^4}$, etc. By induction

we can show that $f^{(n)}(x) = \frac{(-1)^n(n+1)!}{(1-x)^{n+2}}$ for

all n. Indeed, the base $n=0$:
$f(x) = \frac{1}{(1-x)^2}$ is true. The step of induction:

if $f^{(n)}(x) = \frac{(-1)^n(n+1)!}{(1-x)^{n+2}}$, then $f^{(n+1)}(x) = \left(f^{(n)}(x)\right)' =$

$= \frac{(-1)^n(n+1)!\cdot(-1)\cdot(n+2)}{(1-x)^{n+3}} = \frac{(-1)^{n+1}(n+2)!}{(1-x)^{n+3}} \Rightarrow$ the formula

is true.

Thus, $f^{(n)}(0) = \frac{(-1)^n(n+1)!}{(1-0)^{n+2}} = (-1)^n\cdot(n+1)!$

The Maclaurin series for $f(x)$ is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\cdot x^n = \sum_{n=0}^{\infty}\frac{(-1)^n\cdot(n+1)!}{n!}x^n = \sum_{n=0}^{\infty}(-1)^n(n+1)x^n.$$

We have: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)|x^{n+1}|}{n\,|x^n|} = \left(1+\frac{1}{n}\right)|x| \to |x|.$

Therefore, $R = 1$.

<u>Answer</u> $\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$, $R = 1$.

N 8.7.16

The derivatives of $f(x) = \sin x$ are:

$f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$,
$f^{(4)}(x) = \sin x$, etc., repeating with period 4.

Thus, $f^{(n)}(a) = f^{(n)}\left(\frac{\pi}{2}\right) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{\frac{n}{2}}, & \text{if } n \text{ is even.} \end{cases}$

The Taylor series for $\sin x$ at $a = \frac{\pi}{2}$ is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}\left(\frac{\pi}{2}\right)}{n!} \cdot \left(x - \frac{\pi}{2}\right)^n = 1 - \frac{\left(x-\frac{\pi}{2}\right)^2}{2!} + \frac{\left(x-\frac{\pi}{2}\right)^4}{4!} - \frac{\left(x-\frac{\pi}{2}\right)^6}{6!} + \dots =$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(x - \frac{\pi}{2}\right)^{2n}}{(2n)!}$$

Answer: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \cdot \left(x - \frac{\pi}{2}\right)^{2n}}{(2n)!}$

N 8.7.24

$$(1-x)^{\frac{2}{3}} = \sum_{n=0}^{\infty} \binom{\frac{2}{3}}{n} (-x)^n = \sum_{n=0}^{\infty} (-1)^n \binom{\frac{2}{3}}{n} \cdot x^n.$$

The series is convergent when $|-x| < 1$. Thus,

$R = 1$.

N8.7.39 HW7 (3

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-0.2} = \sum_{n=0}^{\infty} \frac{(-0.2)^n}{n!}$. This is an alternating series, $b_n = \frac{(0.2)^n}{n!}$ is decreasing, convergent to 0 $\Rightarrow |R_n| \le b_{n+1}$ for all n.

By try and error we can find that to make $b_{n+1} < 10^{-5}$ it is sufficient to take $n=4$:

$b_5 = \frac{(0.2)^5}{5!} = \frac{32}{120} \cdot 10^{-5}$. Thus

$e^{-0.2} \approx \sum_{n=0}^{4} \frac{(-0.2)^n}{n!} = 1 - 0.2 + \frac{0.04}{2} - \frac{0.008}{6} + \frac{0.0016}{24} =$

$= 0.8187333\ldots$ Since $|R_4| \le b_5 = \frac{32}{120} \cdot 10^{-5} < \frac{10^{-5}}{3}$, the error does not affect first 5 digits $\Rightarrow$ the first 5 digits of $e^{-0.2}$ are:

0.81873.

Answer 0.81873

N8.7.61

$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{n \cdot 5^n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\left(\frac{3}{5}\right)^n}{n}$. Since

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = \ln(1+x)$ for $|x|<1$, we get:

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \left(\frac{3}{5}\right)^n}{n} = \ln\left(1+\frac{3}{5}\right) = \ln\frac{8}{5}$. Answer $\ln\left(\frac{8}{5}\right)$

To Do

N8.7.4.

The Taylor series at $a=4$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(4)}{n!}\cdot(x-4)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(n+1)}(x-4)^n.$$

We have: $$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{|x-4|^{n+1}}{3^{n+1}(n+2)}}{\frac{|x-4|^n}{3^n\cdot(n+1)}} =$$

$$= \frac{|x-4|}{3}\cdot\frac{n+2}{n+1} \to \frac{|x-4|}{3}.$$

$\frac{|x-4|}{3}<1 \Rightarrow$ convergent, $\frac{|x-4|}{3}>1 \Rightarrow$ divergent.

$\frac{|x-4|}{3}<1 \Leftrightarrow |x-4|<3 \Rightarrow$ the radius of convergence is 3.

<u>Answer</u> $\sum_{n=0}^{\infty} \frac{(-1)^n(x-4)^n}{3^n\cdot(n+1)}$, $R=3$.

N8.7.10 Using the product rule for differentiation, we get:

$f(x)=xe^x$, $f'(x)=xe^x+e^x=(x+1)e^x$,

$f''(x)=(x+1)e^x+e^x=(x+2)e^x$, etc. By induction,

$f^{(n)}(x)=(x+n)e^x$. Indeed, the base: $n=0$

$f(x)=xe^x$, true. The step of induction:

if $f^{(n)}(x)=(x+n)e^x$, then $f^{(n+1)}(x)=(f^{(n)}(x))'=$

$=(x+n)e^x+e^x=(x+n+1)e^x$. Thus, the formula is true.

we get: $f^{(n)}(0)=n$. The Maclaurin series for xe^x:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{n x^n}{n!} = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}$$

(we start from $n=1$ since for $n=0$: $\frac{0 \cdot x^0}{0!} = 0$).

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{|x^{n+1}|}{n!}}{\frac{|x|^n}{(n-1)!}} = \frac{|x|}{n} \to 0 \text{ for all } x \Rightarrow R=\infty.$$

Answer $\sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}$, $R=\infty$.

N 8.7.17. $f(x) = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$. Same way as we calculated $((1+x)^k)^{(n)}$, we get:

$$(x^k)^{(n)} = k(k-1)\cdot\ldots\cdot(k-n+1)\,x^{k-n}.$$ In particular

$$\left(x^{-\frac{1}{2}}\right)^{(n)} = \left(-\tfrac{1}{2}\right)\cdot\left(-\tfrac{1}{2}-1\right)\cdot\ldots\left(-\tfrac{1}{2}-n+1\right)x^{-\frac{1}{2}-n}$$

and thus, the Taylor series for $f(x)$ at $a=9$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(9)}{n!}(x-9)^n = \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{1}{2}-1\right)\cdot\ldots\cdot\left(-\frac{1}{2}-n+1\right)}{n!}\cdot 9^{-\frac{1}{2}-n} \times$$

$$(x-9)^n = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n}\cdot 3^{-2n-1}\cdot(x-9)^n.$$

Answer $\sum_{n=0}^{\infty} \frac{\binom{-\frac{1}{2}}{n}\cdot(x-9)^n}{3^{2n+1}}$.

N8.7.30 HW7 6

$x^2 \ln(1+x^3) = x^2 \cdot \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x^3)^n}{n} =$

$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{3n+2}}{n}$, where the latter

series converges when $|x^3| < 1$, that is $|x| < 1$. Since $x^2 \ln(1+x^3)$ has a power series representation at 0, it coincides with it's Maclaurin series.

<u>Answer</u> $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{3n+2}}{n}$.

N8.7.45

$$\int \frac{\cos x - 1}{x} dx = \int \frac{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) - 1}{x} dx =$$

$$= \int \frac{-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots}{x} dx =$$

$$= \int \left(-\frac{x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} + \dots\right) dx =$$

$$= C - \frac{x^2}{2 \cdot 2!} + \frac{x^4}{4 \cdot 4!} - \frac{x^6}{6 \cdot 6!} + \dots = C + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2n \cdot (2n)!}.$$

Assignment 8 HW 8 (1

To Hand in.

N 7.1.1.

$y' = \frac{2}{3}e^{x} - 2e^{-2x}$. plugging y and y' in the equation we get:

$y' + 2y = \frac{2}{3}e^{x} - \underset{\sim}{2e^{-2x}} + 2\cdot\left(\frac{2}{3}e^{x} + \underset{\sim}{e^{-2x}}\right) =$

$= \frac{2}{3}e^{x} + \frac{4}{3}e^{x} = 2e^{x} \Rightarrow y$ is a solution.

N 7.1.5

(a) $(\sin x)'' = (\cos x)' = -\sin x \Rightarrow$
$(\sin x)'' + \sin x = 0 \neq \sin x$. Thus, $\sin x$ is not a solution.

(b) $(\cos x)'' = (-\sin x)' = -\cos x \Rightarrow$
$(\cos x)'' + \cos x = 0 \neq \sin x$. Thus, $\cos x$ is not a solution.

(c) $\left(\frac{1}{2}x\sin x\right)'' = \left(\frac{1}{2}\sin x + \frac{1}{2}x\cos x\right)' =$

$= \frac{1}{2}\cos x + \frac{1}{2}\cos x - \frac{1}{2}x\sin x. \Rightarrow$
$\left(\frac{1}{2}x\sin x\right)'' + \frac{1}{2}x\sin x = \cos x \neq \sin x \Rightarrow$

$\frac{1}{2}x\sin x$ is not a solution

(d) $\left(-\frac{1}{2}x\cos x\right)'' = \left(-\frac{1}{2}\cos x + \frac{1}{2}x\sin x\right)' =$
$= \frac{1}{2}\sin x + \frac{1}{2}\sin x + \frac{1}{2}x\cos x \Rightarrow$
$\left(-\frac{1}{2}x\cos x\right)'' + \left(-\frac{1}{2}x\cos x\right) = \sin x \Rightarrow$

$-\frac{1}{2}x\cos x$ is a solution.

N 7.1.6 (a) (2

Let $y = (\ln x + C)/x$. Then

$y' = \frac{1}{x^2} - \frac{\ln x}{x^2} - \frac{C}{x^2}$

$x^2 y' + x\, y = 1 - \ln x - C + \ln x + C = 1 \Rightarrow$

$y = (\ln x + C)/x$ is a solution for all C.

(c) plug $x = 1$ in the formula $y(x) = (\ln x + C)/x$:

$y(1) = 2 \Rightarrow (\ln 1 + C)/1 = 2 \Rightarrow C = 2$.

Thus, $y(x) = (\ln x + 2)/x$ satisfies the condition $y(1) = 2$.

N 7.1.7

(b) $y(x) = \frac{1}{x+C}$. Then $y' = -\frac{1}{(x+C)^2} = -y^2$.

(c) $y = 0$ is a solution which is not equal to $\frac{1}{x+C}$ for any C.

(d) plug $x = 0$ in $y(x) = \frac{1}{x+C}$:

$y(0) = 0.5 \Rightarrow \frac{1}{C} = 0.5 \Rightarrow C = 2$.

Thus, $y(x) = \frac{1}{x+2}$ is a solution with $y(0) = 0.5$.

N 7.1.11

(a) $\frac{dy}{dt} = e^t (y-1)^2 \geq 0$ for all y and t $\Rightarrow$

any solution $y(t)$ is increasing for all t $\Rightarrow$

the graph on the picture cannot be a solution curve.

(b) If $y(t)$ is a solution and $y(t_0) = 1$

then $\frac{dy}{dt}(t_0) = e^{t_0}(1-1)^2 = 0$ $\Rightarrow$ the slope is 0

at points where $y(t_0) = 1$ But on the graph

there is a point at which $y = 0$ but the slope is $\neq 0$

$\Rightarrow$ it cannot be a solution curve.

To Do

N 7.1.2

$$\frac{dy}{dt} = \frac{d(-t\cos t - t)}{dt} = -\cos t + t\sin t - 1 \Rightarrow$$

$$t\frac{dy}{dt} = -t\cos t + t^2 \sin t - t = y + t^2 \sin t.$$

since $y(\pi) = -\pi\cos\pi - \pi = \pi - \pi = 0$,

this is a solution of the given initial value problem.

N 7.1.3 a) Let $y = e^{rx}$. Then (4

$y' = re^{rx}$, $y'' = r^2 e^{rx}$.

$2y'' + y' - y = 0 \iff$

$2r^2 e^{rx} + re^{rx} - e^{rx} = 0 \iff$ (*)

$2r^2 + r - 1 = 0$. The roots of this quadratic equation are:

$$r = \frac{-1 \pm \sqrt{1 - 4\cdot(-1)\cdot 2}}{2\cdot 2} = \frac{-1 \pm 3}{4}, \quad r_1 = -1, \; r_2 = \frac{1}{2}$$

Thus, $y = e^{rx}$ is a solution for $r = -1$ and $r = \frac{1}{2}$

b) $y = ae^{r_1 x} + be^{r_2 x}$ then

$y' = ar_1 e^{r_1 x} + br_2 e^{r_2 x}$, $y'' = ar_1^2 e^{r_1 x} + br_2^2 e^{r_2 x}$,

$2y'' + y' - y = 2(ar_1^2 e^{r_1 x} + br_2^2 e^{r_2 x}) +$

$(ar_1 e^{r_1 x} + br_2 e^{r_2 x}) - (ae^{r_1 x} + be^{r_2 x}) =$

$= a(2r_1^2 e^{r_1 x} + r_1 e^{r_1 x} - e^{r_1 x}) +$

$b(2r_2^2 e^{r_2 x} + r_2 e^{r_2 x} - e^{r_2 x}) = a\cdot 0 + b\cdot 0 = 0$

since $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are solutions

(see formula (*) above).

N7.L.10. (5

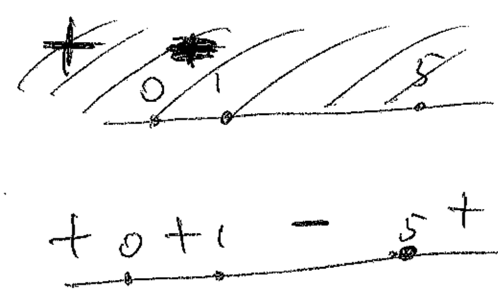
(a) plug $y = c$ in the equation:

$0 = c^4 - 6c^3 + 5c^2$

$c^2(c^2 - 6c + 5) = 0$. The roots are: $c = 0$, 1 and 5. Thus, the constant solutions are: $y = 0$, $y = 1$ and $y = 5$.

(b,c) $\frac{dy}{dt} = y^2(y^2 - 6y + 5) = \overbrace{y^2(y-1)(y-5)}^{p(y)}$

This polynomial $p(y)$ changes it sign at points 0, 1 and 5 only. We have:



$p(-1) = 2 \cdot 6 > 0 \Rightarrow y^2(y-1)(y-5) > 0$ for $y < 0$

$p(\frac{1}{2}) = \frac{1}{4} \cdot (-\frac{1}{2}) \cdot (-4\frac{1}{2}) > 0$

$\Rightarrow p(y) > 0$ for $0 < y < 1$

$p(2) = 4 \cdot (-3) = -12 < 0$

$\Rightarrow p(y) < 0$ for $1 < y < 5$

$p(6) = 36 \cdot 5 > 0 \Rightarrow p(y) > 0$ for $y > 5$.

Therefore, $y(t)$ is increasing when $y < 0$, $0 < y < 1$ and $y > 5$, and decreasing, when $1 < y < 5$

№7.1.12. (6

Notice, when $x, y > 0$ in A we should have $y' = 1 + xy > 0$, but on the graph the function is decreasing for some values $x > 0$ with $y > 0 \Rightarrow$ it cannot be A.

When $x = 0$ in B we should have: $y' = 0$.
But on the graph at $x = 0$ the slope is $> 0 \Rightarrow$ it cannot be B.

Thus, it can be only $\boxed{C}$.

N 7.3.1

$$\frac{dy}{dx} = xy^2$$

$$\int \frac{dy}{y^2} = \int x \, dx$$

$$-\frac{1}{y} = \frac{x^2}{2} + C \Rightarrow y(x) = -\frac{1}{\frac{x^2}{2}+C} = -\frac{2}{x^2+2C}$$

N 7.3.9

$$\frac{du}{dt} = 2 + 2u + t + tu = 2(1+u) + t(1+u) = (2+t)(1+u)$$

$$\frac{du}{1+u} = dt \cdot (2+t)$$

$$\int \frac{du}{1+u} = \int (2+t)\,dt$$

$$\ln|1+u| = 2t + \frac{t^2}{2} + C$$

$$1+u = \pm e^{2t+\frac{t^2}{2}+C} = \pm e^{C} \cdot e^{2t+\frac{t^2}{2}} = A \cdot e^{2t+\frac{t^2}{2}}$$

$$u = A \cdot e^{2t+\frac{t^2}{2}} - 1.$$

N 7.3.12

$$\frac{dy}{dx} = \frac{\ln x}{xy}, \quad y(1) = 2.$$

$$\cancel{\frac{dy}{y}} \int y \, dy = \int \frac{\ln x}{x} \, dx, \quad u = \ln x, \; du = \frac{dx}{x}$$

$$\frac{y^2}{2} = \int u \, du = \frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C$$

$$y = \pm\sqrt{(\ln x)^2 + 2C}. \quad \text{For } x = 1:$$

$$2 = y(1) = \pm\sqrt{(\ln 1)^2 + 2C} = \pm\sqrt{2C} \Rightarrow$$

"+" and $C = 2$. Thus, $y(x) = \sqrt{(\ln x)^2 + 4}$

N 7.3.21

$y' = x + y$, $u = x + y$. Then $u' = 1 + y' = 1 + x + y = 1 + u$.

$\frac{du}{1+u} = dx$, $\int \frac{du}{1+u} = \int dx$

$\ln|1+u| = x + C$,

$1 + u = \pm e^{x+C} = \pm e^{C} \cdot e^{x} = A \cdot e^{x}$. Thus

$1 + x + y = A e^{x} \Rightarrow y = A e^{x} - x - 1$.

N 7.3.22

$xy' = y + x e^{y/x}$, $v = y/x$. Then

$v' = \frac{y'}{x} - \frac{y}{x^2}$ We have! $\frac{y'}{x} = \frac{y' x}{x^2} = \frac{y}{x^2} + \frac{e^{y/x}}{x}$

Thus, $v' = \frac{e^{y/x}}{x} = \frac{e^{v}}{x}$.

$e^{-v} dv = \frac{1}{x} dx$, $\int e^{-v} dv = \int \frac{dx}{x}$

$-e^{-v} = \ln|x| + C$

$v = -\ln(-\ln|x| - C)$. So,

$\frac{y}{x} = -\ln(-\ln|x| - C)$ and

$y = -x \ln(-\ln|x| - C)$.

To do

N7.3.3

$(x^2+1)\,y' = xy$

$\frac{dy}{y} = \frac{x\,dx}{x^2+1}$

$\int \frac{dy}{y} = \int \frac{x\,dx}{x^2+1}$, $u = x^2+1$, $du = 2x\,dx$

$\ln|y| = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+1) + C$

$y = \pm e^{\frac{1}{2}\ln(x^2+1)+C} = \pm e^{C}\cdot\left(e^{\ln(x^2+1)}\right)^{\frac{1}{2}} = A\,(x^2+1)^{\frac{1}{2}}.$

Thus, $y = A\,(x^2+1)^{\frac{1}{2}}$

N7.3.17

$y'\tan x = a + y$, $y(\frac{\pi}{3}) = a$, $0 < x < \frac{\pi}{2}$.

$\frac{dy}{a+y} = \cot x\,dx$

$\int \frac{dy}{a+y} = \int \cot x\,dx$ table integrals =>

$\ln|a+y| = \ln|\sin x| + C.$

$a+y = \pm e^{\ln|\sin x|+C} = \pm e^{C}\cdot e^{\ln|\sin x|} = A\sin x,$

since $\sin x > 0$ on $0 < x < \frac{\pi}{2}$.

For $x = \frac{\pi}{3}$:

$a = y(\frac{\pi}{3}) = A\sin\frac{\pi}{3} - a = \frac{\sqrt{3}}{2}A - a$ => $A = \frac{4}{\sqrt{3}}a.$

Thus, $y = \frac{4}{\sqrt{3}}a\sin x - a$

N 7.3.20

$$f'(x) = f(x)(1-f(x)), \quad f(0) = \frac{1}{2}.$$

$\int \frac{df}{f(1-f)} = \int dx$. using partial fractions

we have: $\frac{1}{f(1-f)} = \frac{1}{f} + \frac{1}{1-f} \Rightarrow$

$$\int \frac{df}{f(1-f)} = \int \left(\frac{1}{f} + \frac{1}{1-f}\right) df = \ln|f| - \ln|1-f| + C_1 =$$

$$= \ln\left|\frac{f}{1-f}\right| + C_1$$

Thus, $\ln\left|\frac{f}{1-f}\right| = x + C$

$$\frac{f}{1-f} = \pm e^{x+C} = Ae^{x}$$

$$f = Ae^{x} - Ae^{x}f \Rightarrow f(x) = \frac{Ae^{x}}{1+Ae^{x}}.$$

For $x = 0$:

$$\frac{1}{2} = f(0) = \frac{A}{1+A} \Rightarrow \frac{1+A}{2} = A \Rightarrow A = 1.$$

Thus, $f(x) = \frac{e^{x}}{1+e^{x}}$

N 7.3.23

a) $\frac{dy}{dx} = 2x\sqrt{1-y^2}$

$\int \frac{dy}{\sqrt{1-y^2}} = \int 2x\,dx$. By table integrals we get

$\sin^{-1} y = x^2 + C$. applying sin we get:

$y = \sin(x^2 + C)$. b) ~~For x=0~~:

~~$0 = y(0) = \sin C \Rightarrow C = \pi n, n \in \mathbb{Z}$~~

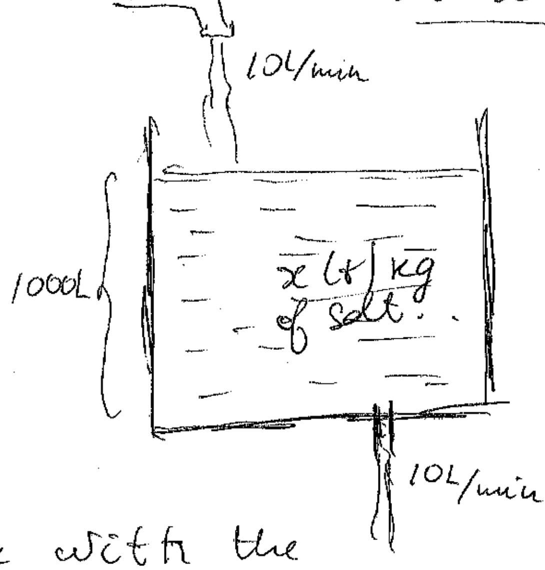
b) plug $x=0, y=0$ in $\sin^{-1} y = x^2 + C$; HW9

$\sin^{-1} 0 = 0 + C \Rightarrow C = 0$. Thus,

$y(x) = \sin x^2$ (see the graph on the next page).

c) The equation $y' = 2x\sqrt{1-y^2}$ makes sense only when $|y| \leq 1$, therefore the initial value problem $y(0) = 2$ cannot have a solution. Another way to see this: the general solution is $y(x) = \sin(x^2 + C) \Rightarrow y(0) = \sin C$ cannot be equal to 2.

$x_0 = x(0) = 15$



N 7.3.45.

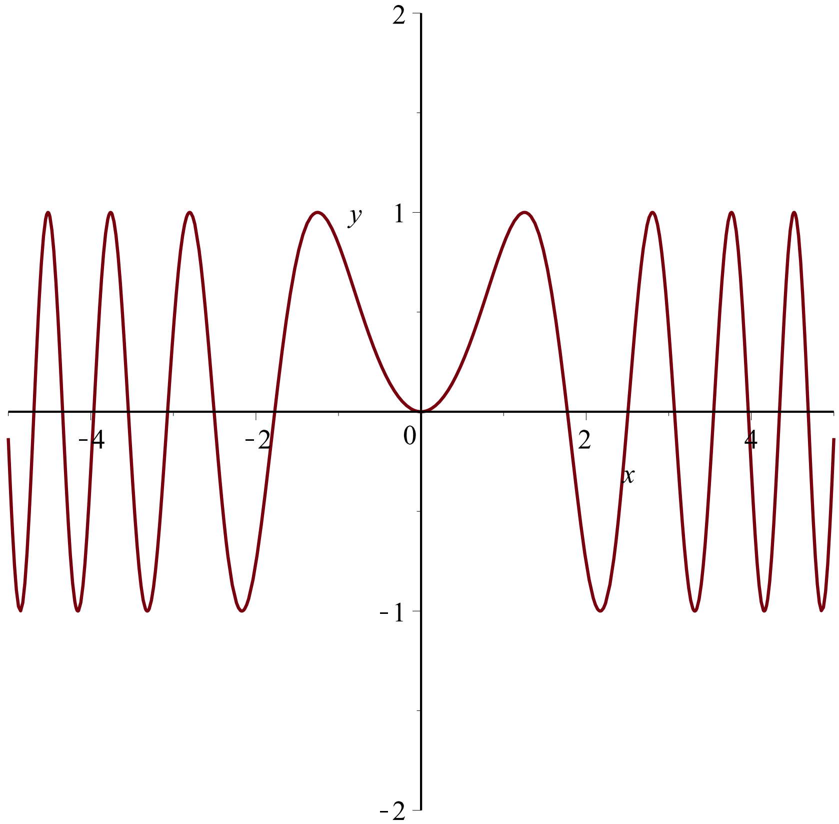
Let $x(t)$ be the amount of salt in the tank at time t. Then the density of salt is $\frac{x(t)}{1000}$. The salt leaves the tank with the solution, which drains with the rate 10L/min. Thus, the salt leaves the tank with the rate $\frac{x(t)}{1000} \cdot 10 = \frac{x(t)}{100}$. We obtain:

$\frac{dx}{dt} = -\frac{x}{100}$. This is the exp. growth/decay equation with $k = -\frac{1}{100} \Rightarrow x(t) = x_0 e^{-\frac{t}{100}} = 15 e^{-\frac{t}{100}}$

So after t minutes there are $15 e^{-\frac{t}{100}}$ kg of salt in the tank. After 20 minutes:

$x(20) = 15 e^{-\frac{20}{100}} \approx 12.3$ kg.

> $with(plots) : plot(\sin(x^2), x = -5 ..5, y = -2 ..2);$



>

To hand in HW10

N 7.2.3–6.

3. In this example the slope $\frac{dy}{dx} = 2-y$ is a function of y and does not depend on x. On all pictures except III we see that the slope for a fixed level y changes with x. Thus only III can match 3.

4. When $x=0$ or $y=2$ the slope $\frac{dy}{dx} = x(2-y)$ of any solution curve $y(x)$ is 0. Only the slope field I satisfies this property.

5. When $x=-y$ the slope $\frac{dy}{dx} = x+y-1$ is -1. Only picture IV satisfies this property.

6. When $y=\pi$ or $x=\pi$ the slope $y' = \sin x \sin y$ is 0. Only the slope field II satisfies this property

Answer 3. III, 4. I, 5. IV, 6. II.

N 7.2.22

$h = 0.2$, $x_0 = 0$, $y_0 = 1$, $f(x,y) = xy - x^2$.

We have; $x_n = x_0 + n \cdot h = 0.2 \cdot n$,

$y_{n+1} = y_n + h \cdot f(x_n, y_n) = y_n + 0.2 \cdot (x_n y_n - x_n^2) =$

$= y_n (1 + 0.04n) - 0.008 n^2$.

$x_5 = 1$. Need to find y_5.

$y_1 = y_0 (1 + 0.04 \cdot 0) - 0.008 \cdot 0^2 = y_0 = 1$.

$y_2 = y_1 (1 + 0.04 \cdot 1) - 0.008 \cdot 1^2 = 1.032$ etc.

$y_3 = 1.083$

$y_4 = 1.140$

$y_5 = 1.195$

Answer $y(1) \approx 1.195$.

N 7.3.6

$$\frac{du}{dr} = \frac{1+\sqrt{r}}{1+\sqrt{u}} \quad \text{separable.}$$

$$\int (1+\sqrt{u})\,du = \int (1+\sqrt{r})\,dr$$

$$u + \frac{2}{3}u^{\frac{3}{2}} = r + \frac{2}{3}r^{\frac{3}{2}} + C.$$

For this equation there is no simple formula for u in terms of x $\Rightarrow$ leave the solution in the implicit form.

Answer $u + \frac{2}{3}u^{\frac{3}{2}} = r + \frac{2}{3}r^{\frac{3}{2}} + C$.

problem 1 $\quad y' = \frac{1}{x-y} + 1.$

Change variable : substitution $u = x-y$.

Then $u' = 1 - y' = 1 - \left(\frac{1}{x-y} + 1\right) = -\frac{1}{u}$.

separable :

$$\frac{du}{dx} = -\frac{1}{u}, \qquad \int u\,du = -\int dx$$

$$\frac{u^2}{2} = -x + C$$

$$u = \sqrt{-2x + 2C}.$$

Thus, $\quad x - y = \sqrt{-2x + C} \Rightarrow$

$$y = x - \sqrt{-2x + C}.$$

answer $\quad y = x - \sqrt{-2x + C}.$

To do

HW 10 (4

N 7.2.24

a) $h = 0.2$, $x_0 = 0$, $y_0 = 0$, $x_n = n \cdot h + x_0 = 0.2 n$.
$x_2 = 0.4$. Need to find y_2. $f(x, y) = x + y^2$

$y_{n+1} = y_n + h \cdot f(x_n, y_n) = y_n + 0.2 \cdot (x_n + y_n^2) =$

$= y_n + 0.04 n + 0.2 y_n^2$. Thus,

$y_1 = y_0 + 0.04 \cdot 0 + 0.2 \cdot y_0^2 = 0$

$y_2 = y_1 + 0.04 \cdot 1 + 0.2 \cdot y_1^2 = 0.04$.

Answer $y(0.4) \approx 0.04$

b) $h = 0.1$, $y_{n+1} = y_n + 0.1 \cdot (x_n + y_n^2) =$
$= y_n + 0.01 \cdot n + 0.1 y_n^2$. $x_4 = 0.4$. Need to find y_4.

We have:

$y_1 = y_0 + 0.01 \cdot 0 + 0.1 \cdot y_0^2 = 0$

$y_2 = y_1 + 0.01 \cdot 1 + 0.1 \cdot y_1^2 = 0.01$ etc.

$y_3 \approx 0.03$

$y_4 \approx 0.06$.

Answer $y(0.4) \approx 0.06$

Remark for true solution $y(x)$ the value $y(0.4) \approx 0.08$
we obtain bad approximations (0.04 and 0.06) because
h is not small enough. For example, for $h = 0.01$ we
would have $y(0.4) \approx y_{40} \approx 0.078$

N7.3.14 HW10 (5

$$y' = \frac{xy\sin x}{y+1}, \quad y(0) = 1.$$

Separable:

$$\int \frac{y+1}{y}\,dy = \int x\sin x\,dx$$

1) $\int \frac{y+1}{y}\,dy = \int\left(1+\frac{1}{y}\right)dy = y + \ln y + C_1$

2) To find the second integral use integration by parts:

$u = x, \quad dv = \sin x\,dx, \quad v = -\cos x$ Then

$$\int x\sin x\,dx = \int u\,dv = uv - \int v\,du =$$

$$= -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C_2.$$

Thus, $y + \ln y = -x\cos x + \sin x + C$. This equation can't be solved explicitely => leave it in the implicit form. To solve the initial value problem plug the initial condition $y(0) = 1$:

$$1 + \ln 1 = -0\cdot\cos 0 + \sin 0 + C \Rightarrow \quad C = 1$$

Answer $y + \ln y = -x\cos x + \sin x + 1.$

problem 2 $\quad y' = \dfrac{y^2 + xy}{x^2}, \quad y(e) = e.$ HW10 C6

We have: $\quad y' = \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)$. Use substitution $v = \frac{y}{x}$. Then $y = vx$ and $y' = v'x + v$. Thus

$$v'x + v = \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right) = v^2 + v$$

$v'x = v^2$. separable:

$$\int \frac{dv}{v^2} = \int \frac{dx}{x}, \qquad -\frac{1}{v} = \ln x + C$$

$$v = -\frac{1}{\ln x + C} \Rightarrow$$

$$\frac{y}{x} = -\frac{1}{\ln x + C}, \qquad y = -\frac{x}{\ln x + C}.$$

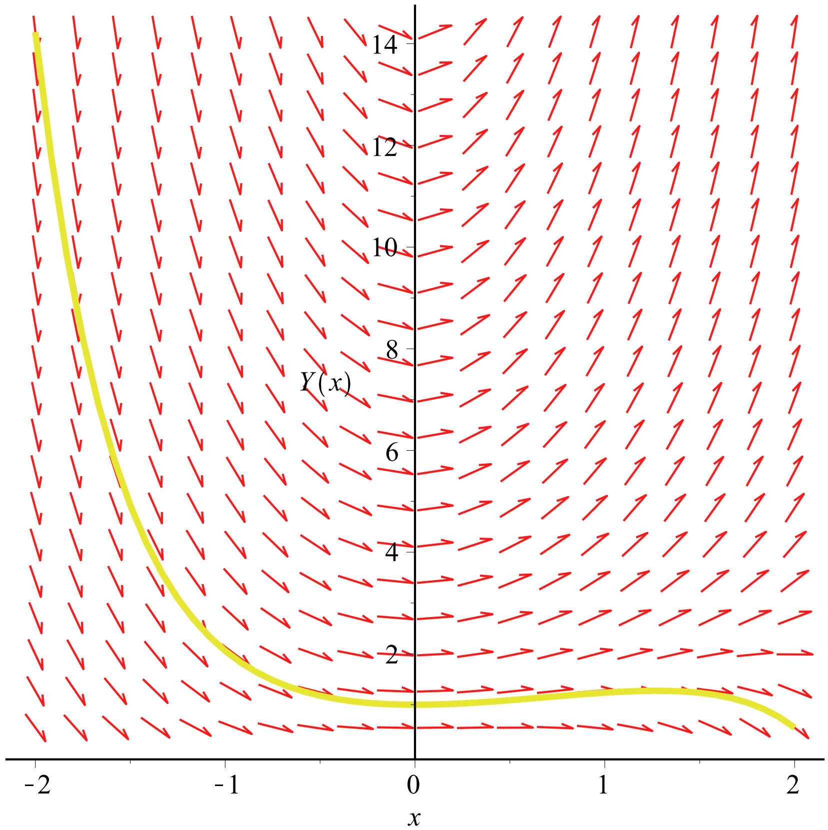
plug $x = e, y = e$:

$$e = -\frac{e}{\ln e + C} = -\frac{e}{C+1} \Rightarrow C = -2.$$

Answer $\quad y(x) = -\dfrac{x}{\ln x - 2}.$

> with(DEtools) : with(plots) : eq := {diff(Y(x), x) = x * Y(x) - x^2};   DEplot(eq, [Y(x)], -2..2, {[0, 1]});

$$eq := \left\{ \frac{d}{dx} Y(x) = x\, Y(x) - x^2 \right\}$$



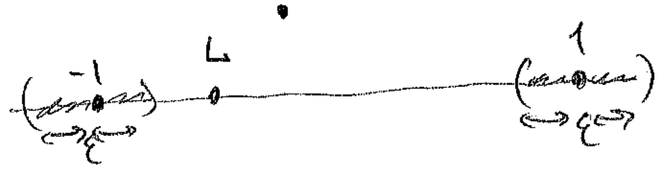
>

__Example__ Show that the sequence $\{(-1)^n\}_{n=1}^{\infty}$ does not have a limit.

__Proof__ Assume that it does have a limit L. Then for any $\varepsilon > 0$ for all $n \geq N$ (where N depends on ε) one has: $|a_n - L| < \varepsilon$, where $a_n = (-1)^n$. Take $\varepsilon = \frac{1}{2}$. Let $n \geq N$ be even. Then we have: $|(-1)^n - L| = |1 - L| < \frac{1}{2}$.

If we take $n \geq N$ to be odd, we obtain: $|(-1)^n - L| = |-1 - L| < \frac{1}{2}$. But then by triangle inequality $|1 - (-1)| = |1 - L - (-1 - L)| \leq$ $|1 - L| + |-1 - L| < \frac{1}{2} + \frac{1}{2} = 1$, which is false: $|1 - (-1)| = 2 > 1$. This contradiction shows that the sequence $\{(-1)^n\}$ cannot have a limit. $\square$

Illustration:



The essential idea of the proof:
L cannot be simultaneously close to (-1) and to 1.

Example 2 Find the limit of the sequence $\{e^{\frac{n^2+1}{2n^2-5}}\}_{n=1}^{\infty}$.

Solution The sequence can be written as $\{f(a_n)\}_{n=1}^{\infty}$ where $f(x) = e^x$ and $a_n = \frac{n^2+1}{2n^2-5}$. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+1}{2n^2-5} = \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{2-\frac{5}{n^2}} =$$

$$= \frac{\lim_{n\to\infty}(1+\frac{1}{n^2})}{\lim_{n\to\infty}(2-\frac{5}{n^2})} = \frac{1}{2}$$, by quotient and sum laws. Since $f(x)=e^x$ is continuous everywhere (in particular, at $L=\frac{1}{2}$), we obtain:

$$\lim_{n\to\infty} f(a_n) = f(L) = e^{\frac{1}{2}}.$$

Answer: $e^{\frac{1}{2}}$.

Example 3: Determine whether the sequence is monotonic and whether it is bounded: $a_n = \frac{\cos n}{n}$.

Solution $a_1 \approx 0.54$, $a_2 \approx -0.42$, $a_3 \approx -0.99$, $a_4 \approx -0.65$. Thus $a_1 > a_2 > a_3$, but $a_3 < a_4$. Therefore, $\{a_n\}$ is not monotonic. Since

$$|a_n| = \frac{|\cos n|}{n} \le \frac{1}{n} \le 1$$, that is $-1 \le a_n \le 1$, the sequence is bounded.

Example 4 A sequence is given by recursive relation: $a_1 = 1$, $a_{n+1} = 1 + \frac{a_n}{a_n+2}$ for $n \geq 1$. Show that $\{a_n\}$ has a limit; find the limit.

Solution Let's calculate approximate values of the first few terms:

$a_2 = 1 + \frac{1}{3} = \frac{4}{3} \approx 1.333$, $a_3 = 1 + \frac{\frac{4}{3}}{\frac{4}{3}+2} = 1\frac{2}{5} = 1.4$

$a_4 = 1 + \frac{\frac{7}{5}}{\frac{7}{5}+2} = 1\frac{7}{17} \approx 1.4117$,

$a_5 = 1 + \frac{\frac{24}{17}}{\frac{24}{17}+2} = 1\frac{12}{29} \approx 1.4137$. Thus,

$a_1 < a_2 < a_3 < a_4 < a_5 < \sqrt{2} = 1.4142\ldots$

A natural guess is that $\{a_n\}$ is increasing and bounded above by $\sqrt{2}$. Let's prove this (to be able to use the Monotonic sequence theorem).

1) Let's show that $a_n < \sqrt{2}$ for all by induction.

Base of induction: $a_1 = 1 < \sqrt{2}$.

Step of induction: assume that $a_n < \sqrt{2}$. What can we say about a_{n+1}? We have:

$a_{n+1} = 1 + \frac{a_n}{a_n+2} = 1 + \frac{a_n+2-2}{a_n+2} = 2 - \frac{2}{a_n+2}$

Since $a_n < \sqrt{2}$, $a_n + 2 < 2 + \sqrt{2}$, $\frac{1}{a_n+2} > \frac{1}{2+\sqrt{2}}$,

we obtain: $a_{n+1} < 2 - \frac{2}{2+\sqrt{2}} = 2 - \frac{2\cdot(2-\sqrt{2})}{(2+\sqrt{2})(2-\sqrt{2})} =$

$= 2 - \frac{2(2-\sqrt{2})}{4-2} = 2 - 2 + \sqrt{2} = \sqrt{2}$. Thus, $a_{n+1} < \sqrt{2}$.

The step of induction holds, therefore, $a_n < \sqrt{2}$ for all n.

2) Let's show that $a_{n+1} \geq a_n$. We have:

$$1 + \frac{a_n}{a_n+2} \geq a_n \iff \quad \text{(multiply by } a_n+2)$$

$$(a_n+2) + a_n > a_n(a_n+2) \iff$$

$$2 > a_n^2 .$$

The last inequality is true, since $0 < a_n < \sqrt{2}$. Therefore, $a_{n+1} > a_n$.

Thus, by Monotonic Sequence Theorem, $\{a_n\}$ converges. To find the limit, let n go to ∞ in the formula $a_{n+1} = 1 + \frac{a_n}{a_n+2}$.

We obtain: $\lim\limits_{n\to\infty} a_{n+1} = L$,

$$\lim_{n\to\infty}\left(1+\frac{a_n}{a_n+2}\right) = 1 + \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty}(a_n+2)} = 1 + \frac{L}{L+2},$$

by the quotient and the sum laws. It follows that $L = 1 + \frac{L}{L+2}$,

$$L^2 + 2 = (L+2) + L, \quad L^2 = 2, \quad L = \pm\sqrt{2}.$$

Since $a_n > 0$ for all n, L cannot be negative. Therefore, $L = \sqrt{2}$.

<u>Answer</u> $\lim\limits_{n\to\infty} a_n = \sqrt{2}$.

Example 5 (Geometric series). Determine whether the following series are convergent or divergent. If convergent, find the ~~limit~~ sum of series.

a) $2 - \frac{6}{5} + \frac{18}{25} - \frac{54}{125} + \dots$

b) $\sum_{n=1}^{\infty} e^{\frac{n}{10}}$

Solution a) Observe that each following term is obtained by multiplying by $-\frac{3}{5}$:

$-\frac{6}{5} = 2\cdot\left(-\frac{3}{5}\right)$, $\frac{18}{25} = 2\cdot\left(-\frac{3}{5}\right)^2$, $-\frac{54}{125} = 2\cdot\left(\frac{-3}{5}\right)^3$.

Continuing the pattern we get $a_n = 2\cdot\left(-\frac{3}{5}\right)^n$.

Thus, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 2\cdot\left(-\frac{3}{5}\right)^{n-1}$ is a geometric series. Since $r = -\frac{3}{5}$, $|r| < 1$, it is convergent and $\sum_{n=1}^{\infty} 2\cdot\left(-\frac{3}{5}\right)^{n-1} = \frac{2}{1-\left(-\frac{3}{5}\right)} = \frac{2}{\frac{8}{5}} = \frac{5}{4}$.

Answer convergent, $\frac{5}{4}$.

b) $e^{\frac{n}{10}} = \left(e^{\frac{1}{10}}\right)^n \Rightarrow \sum_{n=1}^{\infty} e^{\frac{n}{10}} = \sum_{n=1}^{\infty} \left(e^{\frac{1}{10}}\right)^n$ is a geometric series with $r = e^{\frac{1}{10}}$. Since for any $x > 0$ $e^x > 1$, we have $r = e^{\frac{1}{10}} > 1$. Therefor, the series $\sum_{n=1}^{\infty} e^{\frac{n}{10}}$ is divergent.

Answer divergent.

Example 6 (Decimal representation). Express the number as a ratio of integers: $1.\overline{027} = 1.027027027\ldots$

Solution $1.\overline{027} = 1 + \frac{27}{1000} + \frac{27}{1000^2} + \frac{27}{1000^3} + \ldots =$

$$= 1 + \sum_{n=1}^{\infty} \frac{27}{1000^n}$$

geometric series with $a = 27$, $r = \frac{1}{1000}$

$$\boxed{\sum_{n=1}^{\infty} ar^n = \frac{ar}{1-r}}.$$ Thus

$$1.\overline{027} = 1 + \frac{27 \cdot \frac{1}{1000}}{1 - \frac{1}{1000}} = 1 + \frac{27}{999} = 1\frac{1}{37} = \frac{38}{37}$$

Answer: $\frac{38}{37}$

Example 7 (Telescoping sum) Show that the series $\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)$ is divergent using the telescoping sum.

Solution The idea of telescoping sum is representing the terms of the series as differences such that after summation most of the new terms cancel.

We have: $a_n = \ln\left(1 + \frac{1}{n}\right) = \ln \frac{n+1}{n} = \ln(n+1) - \ln n$.

Thus, $\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right) = \sum_{n=1}^{\infty} \left(\ln(n+1) - \ln n\right)$. Notice that in the last expression we can use the difference law because both series are divergent ($\sum_{n=1}^{\infty} \ln(n+1)$, $\sum_{n=1}^{\infty} \ln n$). But we obtain:

$$\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right) = (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + (\ln 4 - \ln 3) + \ldots$$

We see that terms cancel, but there are infinitely many of them, to be accurate we need to use partial sums:

$$S_n = \sum_{k=1}^{n} a_k = (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \ldots + (\ln(n+1) - \ln n)$$

$$= \ln(n+1) - \ln 1 = \ln(n+1).$$

Thus, $S_n = \ln(n+1)$ diverges to ∞ when $n \to \infty$.

By definition, $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right)$ is divergent.

<u>Remark</u> This is another example of the divergent series $\sum_{n=1}^{\infty} a_n$ for which $\lim_{n\to\infty} a_n = 0$.

Example 8 (The Divergence Test) Determine whether the series is divergent or convergent. If convergent find the sum.

$\sum_{n=1}^{\infty} \tan \frac{n-1}{n}$.

Solution Let's investigate the terms $a_n = \tan \frac{n-1}{n}$ of the series. Observe that $\frac{n-1}{n} = 1 - \frac{1}{n} \to 1$ when n goes to ∞. Since $\tan$ is continuous on $(-\frac{\pi}{2}, \frac{\pi}{2})$ (in particular, at 1) we obtain

$\lim_{n\to\infty} \tan \frac{n-1}{n} = \tan 1$. Thus, $\lim_{n\to\infty} a_n \neq 0$.

By Divergence Test, $\sum_{n=1}^{\infty} a_n$ is divergent.

Answer: divergent.

Example 9 (Using integral test).

Determine whether $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges.

Remark The series start from $n=2$. For $n=1$ the expression $\frac{1}{n(\ln n)^2} = \frac{1}{0}$ does not make sense.

Solution $\frac{1}{n(\ln n)^2} = f(n)$ where $f(x) = \frac{1}{x(\ln x)^2}$.

$f(x) > 0$ and decreasing for $x > 1$. By the integral test, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges if and only if $\int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx$ converges.

Consider $\int_2^{t} \frac{1}{x(\ln x)^2}\, dx$. use change of variable $u = \ln x$. We have: $du = \frac{1}{x}\, dx$

$$\Rightarrow \int_2^t \frac{1}{x(\ln x)^2}\, dx = \int_{\ln 2}^{\ln t} \frac{1}{u^2}\, du = -\frac{1}{u}\Big|_{\ln 2}^{\ln t} =$$

$$= -\frac{1}{\ln t} + \frac{1}{\ln 2}$$ Since $\frac{1}{\ln t} \to 0$ when $t \to \infty$,

We get: $\int_2^t \frac{1}{x(\ln x)^2}\, dx \to \frac{1}{\ln 2}$ when $t \to \infty$.

Thus, $\int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx = \frac{1}{\ln 2}$ is convergent.

By The Integral Test, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ is convergent.

Example 10 (Comparison Test). Determine whether the following series is convergent. If convergent find an upper bound for their sum:

1) $\frac{1}{2} + \frac{1}{8} + \frac{1}{14} + \frac{1}{20} + \dots$

2) $\sum_{n=1}^{\infty} \frac{3}{2^n+5}$

Solution 1) The general term is $a_n = \frac{1}{6n-4}$.

Observe that $a_n > \frac{1}{6n} > 0$. Since the series $\sum_{n=1}^{\infty} \frac{1}{6n} = \frac{1}{6} \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (the p-series with $p=1$) by Comparison Test we obtain that $\sum_{n=1}^{\infty} a_n$ is also divergent.

2) Observe that $0 < \frac{3}{2^n+5} < \frac{3}{2^n}$ for all n. The series $\sum_{n=1}^{\infty} \frac{3}{2^n}$ is the geometric series with $r = \frac{1}{2}$, $|r| < 1$ Therefore, it is convergent and

$$\sum_{n=1}^{\infty} \frac{3}{2^n} = \frac{3 \cdot \frac{1}{2}}{1 - \frac{1}{2}} = 3.$$

By Comparison Test, $\sum_{n=1}^{\infty} \frac{3}{2^n+5}$ is also convergent and bounded from above by 3:

$$\sum_{n=1}^{\infty} \frac{3}{2^n+5} < 3.$$

Example 11 Find the radius of convergence and the interval of convergence.

1) $\sum_{n=1}^{\infty} \frac{6^n (2x-3)^n}{3n^2-7}$; 2) $\sum_{n=5}^{\infty} n^n \cdot (x+3)^n$; 3) $\sum_{n=1}^{\infty} \frac{(7x-1)^n}{n!+\sin n}$

Solution 1) $\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{\frac{6^{n+1} \cdot (2x-3)^{n+1}}{3(n+1)^2-7}}{\frac{6^n \cdot (2x-3)^n}{3n^2-7}} \right| =$

$= 6|2x-3| \cdot \frac{3n^2-7}{3(n+1)^2-7} = 6 \cdot |2x-3| \cdot \frac{3-\frac{7}{n^2}}{3+\frac{6}{n}-\frac{4}{n^2}} \longrightarrow 6|2x-3|$

when $n \to \infty$. By the ratio test, if $6|2x-3| < 1$ (equivalently, $|x-\frac{3}{2}| < \boxed{\frac{1}{12} = R}$, $x \in \left(\frac{3}{2}-\frac{1}{12}, \frac{3}{2}+\frac{1}{12}\right) = \left(\frac{17}{12}, \frac{19}{12}\right)$) then the series is convergent. If $6|2x-3| > 1$ then the series is divergent. Let's verify whether the series is convergent for $6|2x-3| = 1$ (equivalently, $|x-\frac{3}{2}| = \frac{1}{12}$, $x = \frac{17}{12}$ or $x = \frac{19}{12}$). When $6|2x-3| = 1$ we have: $|a_n| = \left|\frac{(6(2x-3))^n}{3n^2-7}\right| = \frac{1}{|3n^2-7|}$

Set $b_n = \frac{1}{n^2}$. Then $\lim_{n\to\infty} \frac{|a_n|}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{|3n^2-7|}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{|3-\frac{7}{n^2}|} = \frac{1}{3}$. Also, $\sum_{n=1}^{\infty} b_n$ is convergent as the p-series with $p = 2 > 1$. By the limit comparison test, $\sum_{n=1}^{\infty} |a_n|$ is convergent. By the absolute convergence test, $\sum_{n=1}^{\infty} a_n$ is also convergent. Thus, for $x = \frac{17}{12}$ and $\frac{19}{12}$ the series is convergent.

Answer: $R = \frac{1}{12}$, $\left[\frac{17}{12}, \frac{19}{12}\right]$

2) $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{n+1}\cdot|x+3|^{n+1}}{n^n\ |x+3|^n} = \frac{(n+1)^{n+1}}{n^n}\cdot|x+3| >$

$\frac{n^{n+1}}{n^n}\cdot|x+3| = n\cdot|x+3|$. If $x+3\neq 0$ then

$n\cdot|x+3| \longrightarrow \infty$ when $n\to\infty$. Therefore,

$\left|\frac{a_{n+1}}{a_n}\right| \longrightarrow \infty$. By the ratio test,

$\sum_{n=5}^{\infty} n^n\cdot(x+3)^n$ diverges for all x except $x=-3$.

Thus, $R=0$, the interval of convergence is one point $x=-3$.

Answer: $R=0$, $\{-3\}$.

3) $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(7x-1)^{n+1}}{(n+1)!+\sin(n+1)}}{\frac{(7x-1)^n}{n!+\sin n}}\right| = |7x-1|\cdot\frac{n!+\sin n}{(n+1)!+\sin(n+1)} =$

$= |7x-1|\cdot\frac{1+\frac{\sin n}{n!}}{1+\frac{\sin(n+1)}{(n+1)!}}\cdot\frac{1}{n+1} \longrightarrow 0$

1

when $n\to\infty$.

By the Ratio Test, $\sum_{n=7}^{\infty} a_n$ converges for all x.

Answer: $R=\infty$, $(-\infty,\infty)$.

Example 11 Find a power series representation for a function and determine the interval of convergence.

1) $f(x)=\dfrac{7x^2}{3+5x^4}$; 2) $f(x)=\ln(2-x^2)$;

3) $f(x)=\int \frac{x}{1+x^3}\,dx$.

Solution 1) $f(x)=\dfrac{7x^2}{3}\cdot\dfrac{1}{1+\frac{5}{3}x^4}$.

We know: $\dfrac{1}{1-x}=\sum\limits_{n=0}^{\infty}x^n=1+x+x^2+\dots$ for $|x|<1$

plugging $-\frac{5}{3}x^4$ instead of x we get:

$$\frac{1}{1+\frac{5}{3}x^4}=\sum_{n=0}^{\infty}\left(-\frac{5}{3}x^4\right)^n=\sum_{n=0}^{\infty}(-1)^n\cdot\left(\frac{5}{3}\right)^n x^{4n}=$$

$=1-\frac{5}{3}x^4+\left(\frac{5}{3}\right)^2x^8-\left(\frac{5}{3}\right)^3x^{12}+\dots$ for $|\frac{5}{3}x^4|<1$

or equivalently $|x^4|<\frac{3}{5}$, $|x|<\left(\frac{3}{5}\right)^{\frac{1}{4}}$. Thus, the latter expansion has radius of convergence $R=\left(\frac{3}{5}\right)^{\frac{1}{4}}$. We have:

$$f(x)=\frac{7x^2}{3}\cdot\sum_{n=0}^{\infty}(-1)^n\cdot\left(\frac{5}{3}\right)^n\cdot x^{4n}=$$

$=\sum\limits_{n=0}^{\infty}\frac{7}{3}\cdot(-1)^n\cdot\left(\frac{5}{3}\right)^n\cdot x^{4n+2}$. Notice that multiplying by a monomial does not change the radius of convergence $\Rightarrow$ for $f(x)$ $R=\left(\frac{3}{5}\right)^{\frac{1}{4}}$ too. Check the boundary points: $x=\pm\left(\frac{3}{5}\right)^{\frac{1}{4}}$,

$x^4=\frac{3}{5}$, then $f(x)=\sum\limits_{n=0}^{\infty}\left(\frac{7}{3}\right)(-1)^n\cdot\left(\frac{5}{3}\right)^n\cdot\left(\frac{3}{5}\right)^n\cdot\sqrt{\frac{3}{5}}=$

$=\sum\limits_{n=0}^{\infty}\frac{7}{\sqrt{15}}(-1)^n$ divergent by the Divergence Test (lim a_n does not exist, since a_n oscillates between $-\frac{7}{\sqrt{15}}$ and $\frac{7}{\sqrt{15}}$).

__Answer__ $f(x) = \sum_{n=0}^{\infty} \frac{7}{3} (-1)^n \cdot \left(\frac{5}{3}\right)^n x^{4n+2}$, $x \in \left(-\left(\frac{3}{5}\right)^{\frac{1}{4}}, \left(\frac{3}{5}\right)^{\frac{1}{4}}\right)$.

2) $\ln(2-x^2) = \ln\left(2 \cdot \left(1 - \frac{x^2}{2}\right)\right) = \ln 2 + \ln\left(1 - \frac{x^2}{2}\right)$.

We know: $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

plugging $-\frac{x^2}{2}$ for x we get:

$$\ln\left(1 - \frac{x^2}{2}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot \left(-\frac{x^2}{2}\right)^n}{n} = \sum_{n=1}^{\infty} \frac{-x^{2n}}{n \cdot 2^n} =$$

$$= -\frac{x^2}{1 \cdot 2} - \frac{x^4}{2 \cdot 4} - \frac{x^6}{3 \cdot 8} - \frac{x^8}{4 \cdot 16} - \ldots \quad \text{when } \left|\frac{x^2}{2}\right| < 1,$$

~~That~~ equivalently, $|x| < \sqrt{2}$. Thus,

$$f(x) = \ln(2-x^2) = \ln 2 - \sum_{n=1}^{\infty} \frac{x^{2n}}{n \cdot 2^n}$$ When

$x = \pm\sqrt{2}$, $x^2 = 2$, so $\sum_{n=1}^{\infty} \frac{x^{2n}}{n \cdot 2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (p-series with $p=1$, also known as harmonic series). Thus, the interval of convergence is $(-\sqrt{2}, \sqrt{2})$

__Answer__ $f(x) = \ln 2 - \sum_{n=1}^{\infty} \frac{x^{2n}}{n \cdot 2^n}$, $x \in (-\sqrt{2}, \sqrt{2})$.

3) $\cancel{f(x)}\ \frac{x}{1+x^3} = x \cdot \frac{1}{1+x^3} = x \cdot \sum_{n=1}^{\infty} (-x^3)^n =$

$= \sum_{n=0}^{\infty} (-1)^n x^{3n+1} = x - x^4 + x^7 - x^{10} + \ldots$, when $|x^3|<1$, $|x|<1$

By Theorem on Integration and Differentiation of series, $\int \sum_{n=0}^{\infty} (-1)^n x^{3n+1}\, dx = \sum_{n=0}^{\infty} \int (-1)^n x^{3n+1}\, dx =$

$= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+2}}{3n+2} = C + \frac{x^2}{2} - \frac{x^5}{5} + \frac{x^8}{8} - \frac{x^{11}}{11} + \ldots$

Radius of convergence is the same as for the integrand: $R=1$. Check the boundary values of x for convergence: $x = \pm 1$.

a) $x=-1$: $f(-1) = C + \frac{1}{2} + \frac{1}{5} + \frac{1}{8} + \frac{1}{11} = C + \sum_{n=0}^{\infty} \frac{1}{3n+2}$.

Set $a_n = \frac{1}{3n+2}$, $b_n = \frac{1}{n}$. We have: $a_n > 0$, $b_n > 0$,

$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{3} > 0$, $\sum_{n=1}^{\infty} b_n$ is divergent

(harmonic series) $\Rightarrow$ by Limit Comparison Test, $\sum_{n=0}^{\infty} a_n$ is divergent.

b) $x=1$ $f(1) = C + \frac{1}{2} - \frac{1}{5} + \frac{1}{8} - \frac{1}{11} + \ldots = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{3n+2}$.

Since $a_n = \frac{1}{3n+2}$ is decreasing and convergent to 0, by Alternating Series Test the series is convergent.

Answer: $C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+2}}{3n+2}$, $x \in (-1, 1]$.

Example 12 Show that $(1+x)^{-\frac{1}{2}}$ is equal to its Mclaurin series for $|x| < \frac{1}{2}$ using an estimation of $R_n(x)$.

Solution We will use the Taylor inequality. Let $f(x) = (1+x)^{-\frac{1}{2}}$. Then $f'(x) = -\frac{1}{2}(x+1)^{-\frac{3}{2}}$, $f''(x) = (-\frac{1}{2})\cdot(-\frac{3}{2})(x+1)^{-\frac{5}{2}}$, etc. $f^{(n)}(x) = (-\frac{1}{2})\cdot(-\frac{3}{2})\cdot\ldots\cdot(-\frac{2n-1}{2})\cdot(x+1)^{-\frac{2n+1}{2}}$. Let $d = -\frac{1}{2}$. Then on $[-d, d] = [-\frac{1}{2}, \frac{1}{2}]$ we have $|x+1| \geq \frac{1}{2}$, therefore $|x+1|^{-\frac{2n+3}{2}} \leq \left(\frac{1}{2}\right)^{-\frac{2n+3}{2}} = 2^{\frac{2n+3}{2}}$, and thus

$$|f^{(n+1)}(x)| = \frac{1}{2}\cdot\frac{3}{2}\cdot\ldots\cdot\frac{2n+1}{2}\cdot|x+1|^{-\frac{2n+3}{2}} \leq \underbrace{\frac{1}{2}\cdot\frac{3}{2}\cdot\ldots\cdot\frac{(2n+1)}{2}}_{n+1 \text{ term}}\cdot 2^{\frac{2n+3}{2}} = 1\cdot 3\cdot 5\ldots(2n+1)\cdot\sqrt{2} = M.$$

By Taylor's inequality, on $[-\frac{1}{2}, \frac{1}{2}]$ we have:

$$|R_n(x)| \leq \frac{M}{(n+1)!}\cdot|x|^{n+1} = \frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n+1)\cdot\sqrt{2}}{1\cdot 2\cdot\ldots\cdot(n+1)}\cdot|x|^{n+1}.$$

Now, if $|x| < \frac{1}{2}$ then

$$|R_n(x)| = \frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n+1)\sqrt{2}}{1\cdot 2\cdot 3\ldots\cdot(n+1)\cdot 2^{n+1}}\cdot|2x|^{n+1} = \frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n+1)\sqrt{2}}{2\cdot 4\cdot 6\cdot\ldots\cdot(2n+2)}\cdot|2x|^{n+1} \leq \sqrt{2}\cdot|2x|^{n+1} \to 0,$$

since $|2x| < 1$. Therefore, since $R_n(x) \to 0$ we obtain that $(1+x)^{-\frac{1}{2}}$ is equal to its Taylor series at 0, that is to its Mclaurin series.

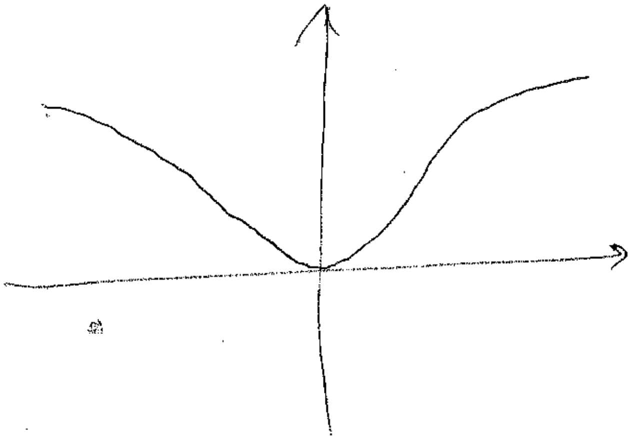
Example (N7.1.13)

Match the differential equations with the solution curves.

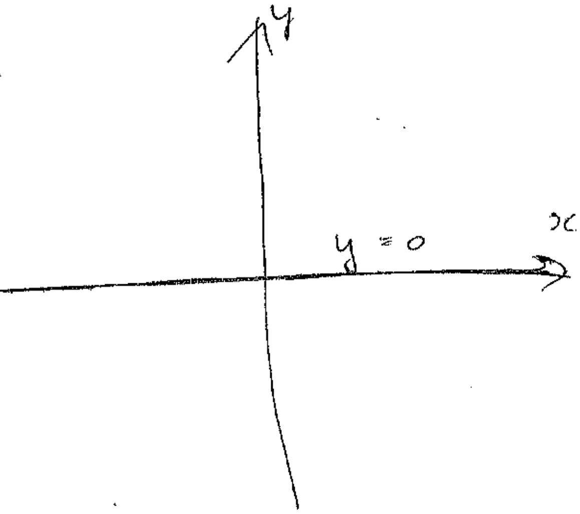
(a) $y' = 1 + x^2 + y^2$; (b) $y' = x e^{-x^2 - y^2}$

(c) $y' = \frac{1}{1 + e^{x^2 + y^2}}$; (d) $y' = \sin(xy)\cos(xy)$.

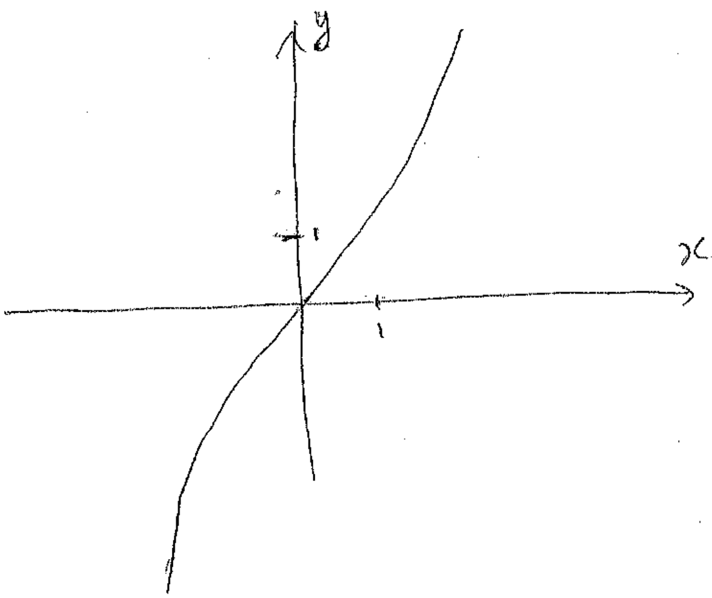
I



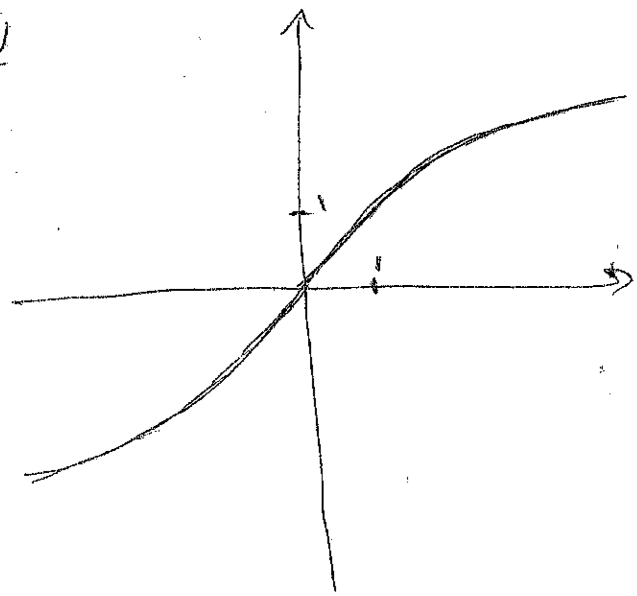
II



III



IV



Solution The simplest solution curve is II: $y=0$, let's check which of the differential equation has $y=0$ as a solution.

(a) $0 = 1+x^2$ No

(b) $0 = x e^{-x^2}$ No

(c) $0 = \frac{1}{1+e^{x^2}}$ No

(d) $0 = \sin 0 \cdot \cos 0 = 0$ Yes, true!

Thus, II can correspond only to (d).

Further, check the rest of the equation for increasing/decreasing solutions.

(a) $y' = 1+x^2+y^2 > 0 \quad \forall x, y \Rightarrow$ increasing

(b) $y' = x e^{-x^2-y^2} > 0$ for $x > 0$ and < 0 for $x < 0 \Rightarrow$ solutions are decreasing for $x < 0$ and increasing for $x > 0$

(c) $y' = \frac{1}{1+e^{x^2+y^2}} > 0$ for all $x, y \Rightarrow$ increasing.

Thus, only I can match only (b).

Further, in (a) $y' = 1+x^2+y^2 \geq 1$ for all x, y, in (c) $y' = \frac{1}{1+e^{x^2+y^2}} < 1$ for all x, y.

Since in III the slope > 1 and in IV the slope < 1, III corresponds to (a) and IV corresponds to (c)

Answer: (a) III, (b) I, (c) IV, (d) II

Problem 1. Show that the function $y(x) = \sqrt{x}e^x$ is a solution of the equation

$$2xy' = (2x+1)y$$

on the infinite segment $(0, +\infty)$.

Solution. Using the product rule for differentiation, we obtain

$$y' = (\sqrt{x}e^x)' = \tfrac{1}{2\sqrt{x}}e^x + \sqrt{x}e^x.$$

Thus,

$$2xy' = \sqrt{x}e^x + 2x\sqrt{x}e^x = (1+2x)\sqrt{x}e^x = (2x+1)y.$$

Problem 2. Solve the initial value problem

$$\begin{aligned} \tfrac{dx}{dt} &= 2t^2 + \cos t - 1, \\ x(0) &= 1. \end{aligned}$$

Solution. Integrating, we obtain:

$$x = \int (2t^2 + \cos t - 1)dt = \tfrac{2}{3}t^3 + \sin t - t + C.$$

Substituting $t = 0$, we find: $1 = x(0) = C$. Thus,

$$x(t) = \tfrac{2}{3}t^3 + \sin t - t + 1.$$

Problem 3. Solve the initial value problem:

$$y' = 2xy^2 + y^2 + 2x + 1, \quad y(0) = 1.$$

Solution. The right-hand side of the equation can be rewritten as $(2x+1)(y^2+1)$. Thus, the equation is separable. Separating the variables and integrating, we obtain:

$$\begin{aligned} \tfrac{dy}{y^2+1} &= (2x+1)dx, \\ \int \tfrac{dy}{y^2+1} &= \int (2x+1)dx, \\ \arctan y &= x^2+x+C. \end{aligned}$$

Substituting $x=0$ we find that

$$C = \arctan(y(0)) = \arctan(1) = \tfrac{\pi}{4}.$$

Thus, the answer is

$$y(x) = \tan(x^2+x+\tfrac{\pi}{4}).$$

Problem 4. Find the general solution of the first-order differential equation:

$$t^2 \tfrac{dx}{dt} + x = 1.$$

Solution. The equation can be rewritten as follows:

$$\tfrac{dx}{x-1} = -\tfrac{dt}{t^2}$$

(assuming $x \neq 1$). Integrating, we obtain:

$$\ln|x-1| = \tfrac{1}{t}+C, \ \ (x-1) = \pm\exp(C)\exp(-\tfrac{1}{t}) = A\exp(-\tfrac{1}{t}),$$

where $A = \pm\exp(C)$ can be any number except 0. Now, since we divided by $x-1$, we need to check $x=1$. Substituting $x=1$ into the original equation we see that $x=1$ is a solution. This solution corresponds to the case $A = 0$. Therefore, the general solution is

$$x(t) = 1 + A\exp(\tfrac{1}{t}),$$

where A is any number.

Problem 5. Solve the differential equation

$$y' = (x+y)^{\frac{1}{3}} - 1. \tag{1}$$

Solution. Substitute $v = x + y$. We have: $\frac{dv}{dx} = \frac{dy}{dx} + 1$. Therefore,

$$\tfrac{dv}{dx} = v^{\frac{1}{3}} - 1 + 1 = v^{\frac{1}{3}}.$$

This is a separable equation. Assuming $v \neq 0$, we obtain:

$$v^{-\frac{1}{3}}dv = dx, \quad \int v^{-\frac{1}{3}}dv = \int dx,$$

$$\tfrac{3}{2}v^{\frac{2}{3}} = x + C \Rightarrow v = \pm(\tfrac{2}{3}x + \tfrac{2}{3}C)^{\frac{3}{2}}.$$

Thus,

$$y = \pm(\tfrac{2}{3}x + \tfrac{2}{3}C)^{\frac{3}{2}} - x.$$

Since we assumed $v \neq 0$, we need to check the case $v = 0$. If $v = 0$ then $y = -x$. Clearly, this is a solution. Therefore, $y = -x$ is a particular solution of (1). Thus, the solutions of (1) are

$$y = \pm(\tfrac{2}{3}x + C)^{\frac{3}{2}} - x, \text{ where } C \text{ is a constant; } y = -x. \tag{2}$$

Remark. Notice that the formula $y = \pm(\frac{2}{3}x + C)^{\frac{3}{2}} - x$ is *a general solution* since it gives a family of solutions depending on a parameter C, but not *the general solution*, since there is a solution $y = -x$ not included in the formula.

Problem 11. Solve the differential equation

$$y' = (2x - y)^2 + 3.$$

Solution. On the right hand side of the equation we see a noticeable expression $2x - y$, therefore it is natural to try the substitution $v = 2x - y$. We have: $v' = 2 - y'$. Thus,

$$v' = 2 - ((2x - y)^2 + 3) = -1 - v^2.$$

This is a separable equation. Separating variables and integrating, we get

$$\tfrac{dv}{v^2+1} = -dx, \quad \tan^{-1}(v) = -x + C, \quad v = \tan(C - x).$$

Thus, we obtain:

$$y = 2x - v = 2x - \tan(C - x) = 2x + \tan(x - C).$$

Since C is arbitrary constant, we can replace C with $-C$.
Answer: $y(x) = 2x + \tan(x + C)$, where C is an arbitrary constant.

Problem 7 (Logistic equation). Solve the initial value problem

$$\tfrac{dx}{dt} = 3x(5 - x), \quad x(0) = 8.$$

Solution. Observe that $x = 0$ and $x = 5$ are particular solutions of the equation $\frac{dx}{dt} = 3x(5 - x)$. If $x \neq 0$ and $x \neq 5$, we have:

$$\tfrac{dx}{3x(5-x)} = dt.$$

Use partial fractions:

$$\frac{1}{x(5-x)} = \frac{A}{x} + \frac{B}{5-x} = \frac{A(5-x)+Bx}{x(5-x)}.$$

Then $5A + (B-A)x = 1$. Therefore, $A = \frac{1}{5}$ and $B = A$. Thus,

$$\int \frac{dx}{3x(5-x)} = \frac{1}{15}\int (\frac{1}{x} + \frac{1}{5-x})dx = \frac{1}{15}(\ln|x| - \ln|5-x|) = \frac{1}{15}\ln|\frac{x}{5-x}|.$$

Thus, $\frac{1}{15}\ln|\frac{x}{5-x}| = t + C$, $\ln|\frac{x}{5-x}| = 15(t+C)$. Exponentiating, we obtain:

$$\frac{x}{5-x} = \pm e^{15C}e^{15t} = Ke^{15t}, \quad \text{where} \quad K = const.$$

It is convenient to plug the initial condidion $x(0) = 8$ in this formula. We obtain:

$$-\frac{8}{3} = K.$$

Thus,

$$x(t) = \frac{5Ke^{15t}}{1+Ke^{15t}} = \frac{40e^{15t}}{8e^{15t}-3}.$$

Problem 8. Suppose that a body moves through a resisting medium with resistance proportional to the square of its velocity v, so that

$$\frac{dv}{dt} = -kv^2 \quad \text{for some constant} \quad k.$$

The body starts moving with the velocity $10m/s$. After 10 seconds its velocity decreases to $5m/s$. Find the velocity of the body in $1.5min$ $(90s)$ after it started moving.

Solution. To find a formula for the velocity let us solve the given differential equation. Separating the variables, we obtain:

$$\frac{dv}{v^2} = -kdt, \quad -\frac{1}{v} = -kt + C, \quad v(t) = \frac{1}{kt-C}.$$

Substituting $t = 0$ and using the initial condition $v(0) = 10$ we get:

$$\tfrac{1}{-C} = 10 \;\Rightarrow\; C = -0.1,\; v(t) = \tfrac{1}{kt+0.1}.$$

Substituting $t = 10$ and using the condition $v(10) = 5$ we get:

$$\tfrac{1}{10k+0.1} = 5 \;\Rightarrow\; k = 0.01,\; v(t) = \tfrac{1}{0.01t+0.1}.$$

Thus, the velocity after 90 seconds is $v(90) = \frac{1}{0.9+0.1} = 1$.

Problem 9. A tank contains 1000 liters (L) of a solution consisting of 100 kg of salt dissolved in water. Pure water is pumped into the tank at the rate of $5L/s$, and the mixture – kept uniform by stirring – is pumped out at the same rate. How long will it be until only 10 kg of salt remains in the tank?

Solution. Observe that no new salt is coming into the tank, but some salt is leaving the tank with the solution. Thus, the amount of salt in the tank changes with time. Let $x(t)$ be the amount of salt in the tank at time t. Then the ratio of the salt in the solution is $x/1000$ (kg/L). Therefore, salt leaves the tank with the speed

$$5 \cdot x/1000 = x/200 \; (kg/s).$$

We obtain that

$$\tfrac{dx}{dt} = -x/200.$$

Solving this separable equation we find

$$\tfrac{dx}{x} = -\tfrac{dt}{200}, \quad \ln|x| = C - \tfrac{t}{200},$$
$$x = Ae^{-\frac{t}{200}}, \quad \text{where} \quad A = \text{const}.$$

Observe that $x(0) = 100$ kg (the amount of salt at the beginning). Therefore, $A = 100$. Thus, $x(t) = 100e^{-\frac{t}{200}}$. Solving the equation $100e^{-\frac{t}{200}} = 10$ we find

$$e^{-\frac{t}{200}} = 0.1, \quad t = -200 \ln \tfrac{1}{10},$$

and thus 10 kg of salt remains after $t = -200 \ln \frac{1}{10} \approx 461$ seconds.

Problem 10 (Population model). The time rate of change of a rabbit population P is proportional to the square root of P. At time $t = 0$ (months) the population numbers 100 rabbits and is increasing at the rate of 20 rabbits per month. How many rabbits will there be one year later?

Solution. This rabbit population $P(t)$ satisfies the differential equation

$$\tfrac{dP}{dt} = k\sqrt{P}, \tag{3}$$

where k is some constant. According to the conditions of the problem, we have: $P(0) = 100$, $P'(0) = 20$. Thus, plugging $t = 0$ into the equation (3) we obtain:

$$20 = k\sqrt{100} \Rightarrow k = 2.$$

Solving the differential equation $\frac{dP}{dt} = 2\sqrt{P}$ we find:

$$\tfrac{dP}{2\sqrt{P}} = dt, \quad \sqrt{P} = t + C.$$

For $t = 0$ we obtain: $\sqrt{100} = 0 + C \Rightarrow C = 10$. Thus,

$$P(t) = (10 + t)^2.$$

So, after 1 year (12 months) there will be $P(12) = (10+12)^2 = 484$ rabbits.

Problem 11. Apply Euler's method with step size $h = 0.25$ to find approximate value of the solution of the initial value problem

$$\frac{dy}{dx} = \frac{y}{x^2+1}, \quad y(2) = 1$$

at point $x = 3$.

Solution. Since the initial condition is given at $x_0 = a = 2$ and we are asked to find a value at $b = 3$, we will use the Euler's method for the segment $[a, b] = [2, 3]$. We have: $x_k = x_0 + kh = 2+0.25k$. Thus, $3 = x_4$. Inductively, define approximations y_k of $y(x_k)$ by the formulas $y_0 = 1,\ y_{k+1} = y_k + hf(x_k, y_k) = y_k + \frac{0.25y_k}{x_k^2+1}$. We obtain the following values:

$$y_1 = 1.05,\ y_2 \approx 1.0933,\ y_3 \approx 1.1310,\ y_4 \approx 1.1640.$$

Thus, $y(3) \approx 1.1640$.

Problem 12. Given the initial value problem

$$\frac{dx}{dt} = \frac{t}{x}, \quad x(0) = 1$$

apply Euler's method with step size $h = 0.1$ to find an approximation of the value $x(0.2)$.

Solution. Since the initial condition is given at $t = 0$ we set $t_0 = 0$. Construct according to the Euler's method points $t_k = t_0 + kh:\ t_1 = 0.1,\ t_2 = 0.2$. Given $x_0 = x(0) = 1$ we define inductively approximations x_k of $x(t_k)$ by the formula

$$x_{k+1} = x_k + hf(t_k, x_k) = x_k + \frac{0.1t_k}{x_k}.$$

We have:

$$x_1 = x_0 + \frac{0.1t_0}{x_0} = 1, \quad x_2 = x_1 + \frac{0.1t_1}{x_1} = 1 + \frac{0.1 \cdot 0.1}{1} = 1.01.$$

Thus, $x(0.2)$ is approximately equal to $x_2 = 1.01$.