MAT 126: Calculus B
Spring 2014

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General Information

Course description:
MAT 126 is a continuation of MAT 125, covering integral calculus: the fundamental theorem, symbolic and numeric methods of integration, area under a curve, volume, applications such as work and probability.

Prerequisite:
C or higher in MAT 125 or 131 or 141 or AMS 151 or level 6 on the mathematics placement examination

Textbook: Single Variable Calculus (Stony Brook University 4th Edition) by James Stewar, Chapters 5 and 6.

Course coordinator: Artem Dudko, artem.dudko@stonybrook.edu

Time, location and instructor's name for each lecture and recitation can be found **here**

Your instructor's **office hours** can be found **here** (starting the second week of the semester)

Tests:

Midterm Test I: Tuesday, February 25, 8:45pm-10:15pm.
Midterm Test II: Tuesday, April 8, 8:45-10:15pm.
Final Exam: Wednesday, May 14, 11:15am-1:45pm.
Last day of classes: Monday, May 12.

Midterm II info:

Time: Tuesday, April 8, 8:45pm-10:15pm.

Location: same as for midterm I. Depends on your lecture section and the first letter of your last name (except L03).
L01, A-K: Javits 110.
L01, L-Z: Frey 100.
L02, A-K: Harriman 137.
L02, L-Z: Frey 102.
L03, all students in Javits 100.

The test covers sections 5.5-5.7 and 5.10. There will be 6 questions, some divided into 2 parts. The problems will be similar to those which you had in homeworks. There will be no questions requiring complicated numerical calculations. **No calculators allowed. No table of integrals or derivatives** will be provided during the test.

Here are links to midterms and practice midterms from previous years:
midterm 2 of spring 2000 (**solutions**)
practice midterm 2 of spring 2007 (**solutions**)
midterm 2 of spring 2010 (**solutions**)

A set of problems **to practice for Midterm 2** is posted on WebAssign. However, it is highly recommended that you not just submit your answers while practicing, but also try writing down solutions on paper. Also, practice not to use calculator unless it is completely necessary. These problems are for practice purposes only and will not affect your grades.

There will be a **review** provided by MLC for midterm 2 on Friday, April 4th from 7pm till 9pm in Harriman 137 (here are **handwritten solutions** of the problems from the review).

Midterm I info:

Time: Tuesday, February 25, 8:45pm-10:15pm.

Location depends on your lecture section and the first letter of your last name (except L03).
L01, A-K: Javits 110.
L01, L-Z: Frey 100.
L02, A-K: Harriman 137.
L02, L-Z: Frey 102.
L03, all students in Javits 100.

The test covers sections 5.1-5.4. There will be 5 questions, some divided into 2 parts. The problems will be similar to those which you had in homeworks. There will be no questions requiring complicated numerical calculations. **No calculators allowed. No table of integrals or derivatives** will be provided during the test. You should memorize the **table of basic indefinite integrals**.

Here are links to midterms and practice midterms from previous years:
practice midterm 1 of spring 2006 (**solutions**)
midterm 1 of fall 2007 (**solutions**)

A set of problems **to practice for Midterm 1** is posted on WebAssign. However, it is highly recommended that you not just submit your answers while practicing, but also try writing down solutions on paper. Also, practice not to use calculator unless it is completely necessary. These problems are for practice purposes only and will not affect your grades.

There will be **no makeup midterms**. However, if you miss a midterm because of an emergency and provide a documentation your final grade will be rescaled so that you don't lose points because of the missed midterm.

There will be a **review** provided by MLC for midterm 1 on Monday, Feb. 24th from 7pm till 8:30pm in Javits 100.

Information for students with disabilities

If you have a physical, psychological, medical, or learning disability that may impact your course work, please contact Disability Support Services at (631) 632-6748 or http://studentaffairs.stonybrook.edu/dss/. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential.

Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to the following website:
http://www.sunysb.edu/ehs/fire/disabilities.shtml

Academic integrity

Each student must pursue his or her academic goals honestly and be personally accountable for all submitted work. Representing another person's work as your own is always wrong. Faculty are required to report any suspected instances of academic dishonesty to the Academic Judiciary. Faculty in the Health Sciences Center (School of Health Technology & Management, Nursing, Social Welfare, Dental Medicine) and School of Medicine are required to follow their school-specific procedures. For more comprehensive information on academic integrity, including categories of academic dishonesty, please refer to the academic judiciary website
http://www.stonybrook.edu/uaa/academicjudiciary/

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Syllabus

The following is a tentative schedule for MAT 126.

Week of	Section	Notes
Jan 27	5.1 Areas and distances	
Feb 3	5.2 The definite integral	
Feb 10	4.8 Brief review of antiderivatives 5.3 Evaluating definite integrals	
Feb 17	5.4 The fundamental theorem of calculus	**Review**
Feb 24	5.5 The substitution rule (beginning)	**Midterm I on Tuesday, Feb 25, 8:45pm**
March 3	5.5 The substitution rule (continuation) 5.6 Integration by parts	
March 10	5.7 Additional techniques of integration	
March 17		**Springbreak**
March 24	5.9 Approximate integration	
March 31	5.10 Improper integrals	**Review**
April 7	6.1 More on areas	**Midterm II on Tuesday, April 8, 8:45pm**
April 14	6.2 Volumes 6.3 Cylindrical shells	
April 21	6.4 Arc length 6.5 Average Value of a Function	
April 28	6.6 Applications to physics and engineering 6.7 Applications to Economics and Biology	
May 5	6.8 Probability	**Review** **Classes end May 9** **Final Exam: Wednesday, May 14, 11:15AM-1:45PM**

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Homeworks

There will be weekly homeworks, typically consisting of two parts: WebAssign and paper homeworks.

Paper homeworks

Click **here** to access paper homeworks. The assignments in the file are in the newest to oldest order (so the latest assignment starts on the first page). Make sure you scroll down to see all the problems from the current assignment.

These homeworks are to be written on a piece of paper (with your name, recitation section and ID number) and handed to your recitation instructor in the specified week. You must write **not only the answers but also your reasoning and explanation**, which must be written clearly enough so that the grader should be able to follow your reasoning. Answers without explanation will get only parital credit.

Solutions of paper homework 10

WebAssign

You should access WebAssign through **Blackboard**. From within Blackboard, click on the link for your MAT 126 lecture. Then go to the left panel and click on "Tools". Finally, click on "Access WebAssign".

At the beginning of the semester there is a 2 week "grace period" during which you may access WebAssign without an access code. But within the first 2 weeks you are required to purchase a WebAssign access code (either bundled with a textbook, or as a stand-alone access code, whichever you prefer). Without a WebAssign access code, you will not be able to continue accessing WebAssign. That means you will not be able to complete the WebAssign assignments.

After they are assigned, the online problems may be completed anytime before the assigned deadline. You can look at problems online, print them out, work on them as long as you like, and then answer them in a later Internet session (before the deadline). The online problems are automatically graded with instant feedback.

Notice that different students are assigned different variants of the same problem. So do not try to compare your answers to another student's answers.

The WebAssign assignments due dates may be on different days of the week. You can always see the due date next to the assignment on WebAssign. Late submissions will not be accepted except in cases of emergency.

If you got the answer wrong in a numerical question, you can retry it two times, so you have trhee attempts in total. For a multiple choice question you have only one attempt.

Doing the Assignment

WebAssign has a variety of different question types, ranging from multiple choice to fill-in-the-blank to symbolic questions. Here are some things to keep in mind as you work through your assignments:

- WebAssign does not work well in Internet Explorer. So if you have troubles with WebAssign while using Internet Explorer try switching to a different browser.
- Some questions require entering symbolic notation. Answer symbolic questions by using calculator notation. You must use the exact variables specified in the questions. The order is not important as long as it is mathematically correct.
- When you click on some WebAssign math questions an input palette will open. This palette, called mathPad, will help you enter your answer in proper notation.
- You can save your work without grading by selecting the Save Work button at the end of the question. After you save your work, it will be available to you the next time you click the assignment.
- Please note that WebAssign will not automatically submit your answers for scoring if you only Save your work. Please be sure to Submit prior to the due date and time.
- You can submit answers by question part or for the entire assignment. To submit an individual question answer for grading, click the "Submit New Answers to Question" button at the bottom of each question. To submit the entire assignment for grading, click the "Submit All New Answers" button at the end of the assignment.
- Some WebAssign questions check the number of significant figures in your answer. If you enter the correct value with the wrong number of significant figures, you will not receive credit, but you will receive a hint that your number does not have the correct number of significant figures.
- While different students may get slightly different versions of the assignment, your questions will be the same every time you return. This means you can print out the assignment, work the problems, and then come back later and put in your answers. Since you get multiple attempts to get the question correct, be sure to leave yourself enough time to rework the problems that you did wrong.
- Each assignment worth 10 points in total. Your final WebAssign grade will be equal to the average of the grades for individual assignments.
- If you have **issues with WebAssign**, please, e-mail Yury Sobolev yury@math.sunysb.edu or contact WebAssign support http://www.webassign.net/manual/student_guide/c_a_support_documents.htm

Stony Brook
University

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Final exam and final grade Information

Final Exam info:

Time: Wednesday, May 14, 11:15am-1:45pm.

Location: Depends on your lecture section and the first letter of your last name (except L03).
L01, A-K: Javits 110.
L01, L-Z: Stonybrook Union Auditorium, Room 123.
L02, A-K: Javits 100.
L02, L-Z: Javits 102.
L03, all students in Javits 100.

There will be **10 questions,** first four on sections 5.1-5.7 and 5.10, the rest on sections 6.1-6.8. From the sections 6.6-6.8 you must know how to find 1) work, 2) moments and center of mass of a flat plate, 3) probability that a random variable lies between given bounds and average value in examples similar to considered in the course book. The problems will be similar to those which you had in homeworks. There will be no questions requiring complicated numerical calculations. **No calculators allowed. No table of integrals, derivatives or other formulas** will be provided during the test.

Important notice: if your final exam grade will be better than the average of the midterm 1 and 2 grades it will replace your midterm 1 and 2 grades (so your total grade will improve). Thus, even if you did poorly on the midterms, you have a chance to get a good grade.

Here are links to two **practice finals** from previous years:
spring 2007 (**solutions**)
fall 2009 (**solutions**)

A set of problems **to practice for Final exam** is posted on WebAssign.

There will be a **review** provided by MLC on Monday, May 12. The review will be given in two identical sessions: 5:30-7:30pm in Frey Hall 100 and 7-9pm in Harriman 137.

Course grade is computed by the following scheme:
Paper Homeworks: 10%
WebAssign: 10%
Midterm Test I: 20%
Midterm Test II: 20%
Final Exam: 40%

Letter grade cutoffs:
85-100 A
80-85 A-
75-80 B+
65-75 B
60-65 B-
55-60 C+
45-55 C
35-45 D
0-35 F

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This section contains the list of updates on the course webpage including brief description of the changes

Feb 16, 2014

- Assignment 3 is posted in the Homeworks section http://www.math.sunysb.edu/~artem/mat126-spring14/HW.pdf (due on the week of Feb 24; scroll down to see all the problems and older assignments)

- Info on WebAssign due dates is updated (because of the class cancellations the deadlines for WebAssign homeworks 2 and 3 were shifted to Sundays; in future the due dates may be on different days of the week; you can always see the due day next to the assignment on WebAssign)

Feb 17, 2014

- Information about Midterm 1 added in the "General information" section.

Feb 18, 2014

- A set of problems to practice for Midterm 1 is posted on WebAssign.

Feb 20, 2014

- **WebAssign support** e-mail changed. Starting from this day, please, contact Yury Sobolev yury@math.sunysb.edu regarding issues with WebAssign homeworks.

March 31, 2014

- Information about Midterm 2 added in the "General information" section.

April 2, 2014

- Locations for Midterm 2 are posted (same as for midterm 1).

April 5, 2014

- **Handwritten solutions** of the problems from the MLC review 2 are posted

May 2, 2014

- Information about Final exam added in the "General information" section.

May 5, 2014

- For convenience, information about Final exam and final grade is moved to a separate section (and updated). Topic "consumer surplus" is removed from the list of topics for the final exam. Letter grade cutoffs are posted.

My NAME is:

Problem	1	2	3	4	Total
Score					

MAT 126
Calculus B
Midterm 2

April 5, 2000

Show all your work on these pages! Total score = 100

1. (30 points) Evaluate each of the following definite integrals.

   (a) $\int_0^1 \sin(\pi x)\ dx$

   (b) $\int_0^1 x\sqrt{1-x^2}\ dx$

   (c) $\int_{-1}^1 xe^x\ dx$

2. (30 points) Find the following antiderivatives.

(a) $\int x^2 \sin x \; dx$

(b) $\int \frac{x-1}{x^2+1} \; dx$

(c) $\int e^{-x} \sin(2x) \; dx$

3. (20 points) The function f is given by the table of values below.

x	0	0.5	1	1.5	2	2.5	3
$f(x)$	1	0.96	0.84	0.66	0.45	0.24	0.05

Approximate $\int_0^3 f(x)\,dx$ by using

(a) the left sum with 3 subintervals

(b) the trapezoid rule with 3 subintervals

(c) Simpson's rule with 6 subintervals.

4. (20 points)

Evaluate the following indefinite integrals.

(a) $\int \frac{2\ dx}{(2+x)(3-x)}$

(b) $\int \sin^3(x)\cos^4(x)\ dx$

My NAME is:

Problem	1	2	3	4	Total
Score					

MAT 126
Calculus B
Midterm 2, Solutions
April 5, 2000

Show all your work on these pages! Total score = 100

1. (30 points) Evaluate each of the following definite integrals.

(a) $\int_0^1 \sin(\pi x)\ dx$

Solution: Let $u = \pi x$. Then $du = \pi\,dx$. When $x = 0, u = 0$, and when $x = 1$, $u = \pi$. Thus

$$\int_0^1 \sin(\pi x)\ dx = \int_0^\pi \frac{\sin(u)}{\pi}\,du = -\frac{\cos(u)}{\pi}\Bigg|_0^\pi = \frac{-\cos(\pi)}{\pi} - \frac{-\cos(0)}{\pi} = \frac{1}{\pi} + \frac{1}{\pi} = \frac{2}{\pi}$$

(b) $\int_0^1 x\sqrt{1-x^2}\ dx$

Solution: Let $u = 1 - x^2$. Then $du = -2x\,dx$. When $x = 0$, $u = 1$ and when $x = 1$, $u = 0$. Thus,

$$\int_0^1 x\sqrt{1-x^2}\ dx = -\frac{1}{2}\int_1^0 \sqrt{u}\,du = -\frac{1}{2}\cdot\frac{2}{3}x^{\frac{3}{2}}\Bigg|_1^0 = -\frac{1}{3}\Big(0-1\Big) = \frac{1}{3}.$$

(c) $\int_{-1}^1 xe^x\ dx$

Solution: We integrate by parts. Take $u = x$ and $dv = e^x\,dx$. Then $du = dx$ and $v = e^x$. So we have

$$\int_{-1}^1 xe^x\ dx = xe^x\Bigg|_0^1 - \int_{-1}^1 e^x\,dx = \Big(xe^x - e^x\Big)\Bigg|_{-1}^1 = (e^1 - e^1) - (-e^{-1} - e^{-1}) = 2e^{-1} = \frac{2}{e}.$$

2. (30 points) Find the following antiderivatives.

(a) $\int x^2 \sin x \; dx$

Solution: We integrate by parts twice. First, $u = x^2$ and $dv = \sin x \, dx$, giving $du = 2x \, dx$ and $v = -\cos x$. So, we have

$$\int x^2 \sin x \; dx = -x^2 \cos x + 2 \int x \cos x \, dx.$$

To do the second integral, we take $u = x$ and $dv = \cos x \, dx$. Then $du = dx$ and $v = \sin x$. So, we have

$$\int x^2 \sin x \; dx = -x^2 \cos x + 2 \left(x \sin x - \int \sin x \, dx \right) = -x^2 \cos x + 2x \sin x + 2 \cos x + C.$$

(b) $\int \frac{x-1}{x^2+1} \; dx$

Solution:

$$\int \frac{x-1}{x^2+1} \; dx = \int \frac{x}{x^2+1} \; dx - \int \frac{1}{x^2+1} \; dx$$

To do the first integral, let $u = x^2 + 1$ so $du = 2x \, dx$. The second is immediate. We have

$$\int \frac{x-1}{x^2+1} \; dx = \frac{1}{2} \ln \left| x^2 + 1 \right| + \arctan(x) + C$$

(c) $\int e^{-x} \sin(2x) \; dx$

Solution: We integrate by parts twice, and solve.

First, take $u = \sin(2x)$ and $dv = e^{-x} \, dx$, giving $du = 2\cos(2x)$ and $v = -e^{-x}$. This gives

$$\int e^{-x} \sin(2x) \; dx = -e^{-x} \sin(2x) + 2 \int e^{-x} \cos(2x) \, dx$$

Now take $u = \cos(2x)$ and $dv = e^{-x} \, dx$, giving $du = -2\sin(2x)$ and $v = -e^{-x}$.

$$\int e^{-x} \sin(2x) \; dx = -e^{-x} \sin(2x) + 2 \left(-e^{-x} \cos(2x) - 2 \int e^{-x} \sin(2x) \; dx \right).$$

That is, we've shown

$$\int e^{-x} \sin(2x) \; dx = -e^{-x} \sin(2x) - 2e^{-x} \cos(2x) - 4 \int e^{-x} \sin(2x) \; dx,$$

or, solving for the integral, we have

$$5 \int e^{-x} \sin(2x) \; dx = -e^{-x} \sin(2x) - 2e^{-x} \cos(2x) + C.$$

That is,

$$\int e^{-x} \sin(2x) \; dx = -\frac{1}{5} e^{-x} \sin(2x) - \frac{2}{5} e^{-x} \cos(2x) + C$$

3. (20 points) The function f is given by the table of values below.

x	0	0.5	1	1.5	2	2.5	3
$f(x)$	1	0.96	0.84	0.66	0.45	0.24	0.05

Approximate $\int_0^3 f(x)\,dx$ by using

(a) the left sum with 3 subintervals

Solution: Using 3 intervals, we have $\Delta\, x = 1$, so our approximation is

$$L_3 = \Big(f(0) + f(1) + f(2)\Big) = \Big(1 + 0.84 + 0.45\Big) = 2.29.$$

(b) the trapezoid rule with 3 subintervals

Solution:

$$T_3 = \frac{1}{2}\Big(f(0) + 2f(1) + 2f(2) + f(3)\Big) = \frac{1}{2}\Big(1 + 1.68 + 0.9 + 0.1\Big) = \frac{3.68}{2} = 1.84.$$

(c) Simpson's rule with 6 subintervals.

Solution: With 6 intervals, $\Delta\, x = 0.5$, so

$$\begin{aligned} S_6 &= \frac{0.5}{3}\Big(f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + f(3)\Big) \\ &= \frac{1}{6}\Big(1 + 3.84 + 1.68 + 2.64 + 0.9 + 0.96 + 0.05\Big) \\ &= \frac{11.07}{6} = 0.9225. \end{aligned}$$

4. (20 points)

Evaluate the following indefinite integrals.

(a) $\displaystyle\int \frac{2\,dx}{(2+x)(3-x)}$

Solution: We use partial fractions to separate the integrand. Let

$$\frac{2}{(2+x)(3-x)} = \frac{A}{2+x} + \frac{B}{3-x}$$

so

$$2 = A(3-x) + B(2+x).$$

Thus

$$\begin{aligned} 2 &= 3A + 2B \\ 0 &= -A + B \end{aligned}$$

So $A = B$ and so $5A = 2$. That is, $A = B = 2/5$. So we have

$$\int \frac{2\,dx}{(2+x)(3-x)} = \int \frac{2/5}{2+x} + \frac{2/5}{3-x}\,dx = \frac{2}{5}\ln|2+x| - \frac{2}{5}\ln|3-x| + C.$$

(We get the negative sign on the second term by letting $u = 3 - x$ so $du = -dx$.)

(b) $\int \sin^3(x)\cos^4(x)\ dx$

Solution: Use the identity $\sin^2 x = 1 - \cos^2 x$ to rewrite the integral as follows

$$\int \sin^3(x)\cos^4(x)\ dx = \int \sin(x)(1-\cos^2(x))\cos^4(x)\ dx = -\int (1-u^2)u^4\ du$$

where we took $u = \cos(x)$ so $du = -\sin(x)\,dx$. Now

$$-\int (1-u^2)u^4\ du = \int u^6 - u^4\ dx = \frac{u^7}{7} - \frac{u^5}{5} + C$$

Thus, we have

$$\int \sin^3(x)\cos^4(x)\ dx = \frac{\cos^7(x)}{7} - \frac{\cos^5(x)}{5} + C.$$

MAT 126 Calculus B Spring 2007
Practice Midterm II

Answer each question in the space provided and on the reverse side of the sheets. Show your work whenever possible. Unless otherwise indicated, **answers without justification will get little or no partial credit!** Cross out anything that grader should ignore and circle or box the final answer. The actual exam will contain 5 problems. This practice test contains more problems to give you more practice.

1. Evaluate the following definite integrals

(a)

$$\int_0^{13} \frac{2}{(2x+1)^{\frac{2}{3}}} dx$$

(b)

$$\int_0^{\frac{\pi}{2}} e^{\sin x} \cos x \, dx$$

(c)

$$\int_0^1 x^4(1+x^5)^{20} dx$$

(d)

$$\int_0^1 \tan^{-1} x \, dx$$

(e)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin t| \, dt.$$

(f)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+t^2)^2 \sin^5 t \, dt.$$

2. Evaluate the following indefinite integrals

(a)

$$\int x^3 e^{x^4} dx$$

(b)

$$\int te^t dt$$

(c)

$$\int x^2 \cos x \, dx$$

(d)

$$\int \cos(\sqrt{x})dx$$

3. Evaluate the following indefinite integrals

(a)

$$\int \frac{1}{x^2} \ln x \, dx$$

(b)

$$\int \frac{1}{x} (\ln x)^2 \, dx$$

(c)

$$\int x^7 \ln x \, dx$$

4. Evaluate the following indefinite integrals

(a)

$$\int \frac{2x^2}{x^2+1} dx$$

(b)

$$\int \frac{2x}{x^2+1} dx$$

5. (a) Write a formula for $\tan x$ in terms of $\sin x$ and $\cos x$.

(b) Evaluate

$$\int \tan x \, dx$$

6. Evaluate

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x}{\sin^2 x} \, dx$$

7. (a) Set

$$f(x) = \int_1^{x^2} \sin t^3 dt + x^3$$

Find $f(1)$ and $f'(x)$.

(b) Set

$$f(x) = \int_{\sqrt{x}}^{x-2} \tan^2 t \, dt$$

Find $f(4)$ and $f'(x)$.

MAT 126 Calculus B Spring 2007
Practice Midterm II Solutions

Answer each question in the space provided and on the reverse side of the sheets. Show your work whenever possible. Unless otherwise indicated, **answers without justification will get little or no partial credit!** Cross out anything that grader should ignore and circle or box the final answer. The actual exam will contain 5 problems. This practice test contains more problems to give you more practice.

1. Evaluate the following definite integrals

(a)

$$\int_0^{13} \frac{2}{(2x+1)^{\frac{2}{3}}} dx$$

Solution. Substitution $u = 2x + 1$ gives $du = 2dx$ and $x = 0$ corresponds to $u = 1$ and $x = 13$ — to $u = 27$. Thus by the substitution rule,

$$\int_0^{13} \frac{2}{(2x+1)^{\frac{2}{3}}} dx = \int_1^{27} \frac{1}{u^{\frac{2}{3}}} du = 3u^{\frac{1}{3}}\Big|_1^{27}$$
$$= 3(27)^{\frac{1}{3}} - 3 = 3 \cdot 3 - 3 = 6.$$

(b)

$$\int_0^{\frac{\pi}{2}} e^{\sin x} \cos x \, dx$$

Solution. Substitution $u = \sin x$ gives $du = \cos x dx$ and $x = 0$ corresponds to $u = 0$ and $x = \pi/2$ — to $u = 1$. Thus by the substitution rule,

$$\int_0^{\frac{\pi}{2}} e^{\sin x} \cos x \, dx = \int_0^1 e^u du = e^u\big|_0^1 = e - 1.$$

(c)

$$\int_0^1 x^4(1 + x^5)^{20} dx$$

Solution. Substitution $u = 1 + x^5$ gives $du = 5x^4 dx$ and $x = 0$ corresponds to $u = 1$ and $x = 1$ — to $u = 2$. Thus by the substitution rule,

$$\int_0^1 x^4(1+x^5)^{20} dx = \frac{1}{5}\int_1^2 u^{20} du = \frac{u^{21}}{5 \cdot 21}\Big|_1^2 = \frac{2^{21} - 1}{105}.$$

(d)

$$\int_0^1 \tan^{-1} x\,dx$$

Solution. We use integration by parts with $u = \tan^{-1} x$ and $dv = dx$. We have $du = \dfrac{dx}{1+x^2}$ and $v = x$, so that using $\tan^{-1}(1) = \dfrac{\pi}{4}$, we get

$$\begin{aligned}\int_0^1 \tan^{-1} x\,dx &= \int_0^1 udv = uv|_0^1 - \int_0^1 vdu \\ &= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}dx.\end{aligned}$$

To evaluate the remaining integral, we use the substitution $u = 1 + x^2$, so that $du = 2xdx$ and $x = 0$ corresponds to $u = 1$ and $x = 1$ — to $u = 2$. Thus

$$\int_0^1 \frac{x}{1+x^2}dx = \frac{1}{2}\int_1^2 \frac{du}{u} = \left.\frac{\ln u}{2}\right|_1^2 = \frac{\ln 2}{2}.$$

Therefore, we get

$$\int_0^1 \tan^{-1} x\,dx = \frac{\pi}{4} - \frac{\ln 2}{2}.$$

(e)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin t|\,dt.$$

Solution. This is the integral over the symmetric interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ of the even function

$$f(t) = |\sin t|.$$

Since $\sin t$ is non-negative on $[0, \frac{\pi}{2}]$, we get

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin t|\,dt = 2\int_0^{\frac{\pi}{2}} |\sin t|\,dt = 2\int_0^{\frac{\pi}{2}} \sin t\,dt \\ = -2\,\cos t|_0^{\frac{\pi}{2}} = -2(\cos\tfrac{\pi}{2} - \cos 0) = 2.\end{aligned}$$

(f)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+t^2)^2 \sin^5 t\,dt.$$

Solution. It is an integral over the symmetric interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ of the odd function

$$f(t) = (1+t^2)^2 \sin^5 t.$$

(Verify that $f(-t) = -f(t)$!). Therefore,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+t^2)^2 \sin^5 t\, dt = 0.$$

2. Evaluate the following indefinite integrals

(a)

$$\int x^3 e^{x^4} dx$$

Solution. Setting $u = x^4$, we get $du = 4x^3 dx$, so by the substitution rule,

$$\int x^3 e^{x^4} dx = \frac{1}{4} \int e^u du = \frac{1}{4} e^u + C = \frac{1}{4} e^{x^4} + C.$$

(b)

$$\int te^t dt$$

Solution. We use integration by parts with $u = t$ and $dv = e^t dt$. We have $du = dt$ and $v = e^t$, so that

$$\int te^t dt = \int udv = uv - \int vdu = te^t - \int e^t dt = te^t - e^t + C.$$

(c)

$$\int x^2 \cos x\, dx$$

Solution. We use integration by parts with $u = x^2$ and and $dv = \cos x dx$. We have $du = 2xdx$ and $v = \sin x$, so that

$$\int x^2 \cos x\, dx = \int udv = uv - \int vdu = x^2 \sin x - 2 \int x \sin x\, dx.$$

For the remaining integral we again use integration by parts with $u = 2x$ and $dv = \sin x dx$, so that $du = 2dx$ and $v = -\cos x$. We have

$$\begin{aligned} 2\int x \sin x\, dx &= \int udv = uv - \int vdu = -2x \cos x + 2 \int \cos x\, dx \\ &= -2x \cos x + 2 \sin x + C, \end{aligned}$$

so that

$$\int x^2 \cos x\, dx = x^2 \sin x + 2x \cos x - 2 \sin x + C.$$

(Double-check the answer by differentiating!)

(d)

$$\int \cos(\sqrt{x})dx$$

Solution. First, we use the substitution rule with $t = \sqrt{x}$, so that $dt = \frac{1}{2\sqrt{x}}\,dx$, or $dx = 2\sqrt{x}\,dt = 2t\,dt$. We get

$$\int \cos(\sqrt{x})dx = 2\int t\cos t dt.$$

To evaluate this integral, we use integration by parts with $u = 2t$ and $dv = \cos t dt$. We have $du = 2dt$ and $v = \sin t$, so that

$$\begin{aligned} 2\int t\cos t dt &= \int u dv = uv - \int v du \\ &= 2t\sin t - 2\int \sin t dt = 2t\sin t + 2\cos t + C. \end{aligned}$$

Finally, remembering that $t = \sqrt{x}$, we get

$$\int \cos(\sqrt{x})dx = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C.$$

3. Evaluate the following indefinite integrals

(a)

$$\int \frac{1}{x^2}\ln x dx$$

Solution. Here we use integration by parts with $u = \ln x$ and $dv = \frac{1}{x^2}dx$, so that

$$du = \frac{1}{x}dx \quad \text{and} \quad v = -\frac{1}{x}.$$

(Note that substitution rule with $u = \ln x$ does not simplify the integral since in the denominator we have x^2; if it was x, then the substitution rule would work.) Thus we have

$$\begin{aligned} \int \frac{1}{x^2}\ln x dx &= \int u dv = uv - \int v du \\ &= -\frac{\ln x}{x} + \int \frac{1}{x^2}dx = -\frac{\ln x}{x} - \frac{1}{x} + C. \end{aligned}$$

(b)

$$\int \frac{1}{x}(\ln x)^2 dx$$

Solution. Here we use the substitution rule with $u = \ln x$ and $du = \frac{1}{x}dx$ (since we have x in the denominator). Therefore,

$$\int \frac{1}{x}(\ln x)^2 dx = \int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}(\ln x)^3 + C.$$

(c)

$$\int x^7 \ln x \, dx$$

Solution. Here we use integration by parts with $u = \ln x$ and $dv = x^7 dx$, so that

$$du = \frac{1}{x}dx \quad \text{and} \quad v = \frac{1}{8}x^8.$$

Thus we have

$$\begin{aligned}\int x^7 \ln x \, dx &= \int u dv = uv - \int v du \\ &= \frac{1}{8}x^8 \ln x - \frac{1}{8}\int x^8 \frac{dx}{x} = \frac{1}{8}x^8 \ln x - \frac{1}{64}x^8 + C.\end{aligned}$$

4. Evaluate the following indefinite integrals

(a)

$$\int \frac{2x^2}{x^2+1}dx$$

Solution. We have

$$\frac{2x^2}{x^2+1} = 2 - \frac{2}{x^2+1}$$

(either by doing the long division, or by writing $2x^2 = 2x^2 + 2 - 2 = 2(x^2+1) - 2$, and dividing both terms by x^2+1). Thus

$$\int \frac{2x^2}{x^2+1}dx = \int \left(2 - \frac{2}{x^2+1}\right) dx = 2x - 2\tan^{-1} x + C.$$

(b)

$$\int \frac{2x}{x^2+1}dx$$

Solution. To evaluate this integral, we use the substitution $u = 1 + x^2$, so that $du = 2xdx$ (compare with the last integral in problem **2** (d)). We have

$$\int \frac{2x}{x^2+1}dx = \int \frac{du}{u} = \ln|u| + C = \ln(1+x^2) + C.$$

5. (a) Write a formula for $\tan x$ in terms of $\sin x$ and $\cos x$.
Solution.

$$\tan x = \frac{\sin x}{\cos x}.$$

(b) Evaluate

$$\int \tan x \, dx$$

Solution. Using part (a) we have

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx,$$

which suggests the substitution $u = \cos x$. We have $du = -\sin x dx$, so that

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{du}{u} = -\ln |u| + C = -\ln |\cos x| + C.$$

6. Evaluate

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x}{\sin^2 x} \, dx$$

Solution. We use integration by parts with $= x$ and

$$dv = \frac{1}{\sin^2 x} \, dx,$$

so that $du = dx$ and $v = -\cot x$. We have, using that antiderivative of $\cot x$ is $\ln |\sin x|$ (compare with part (a)),

$$\begin{aligned}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x}{\sin^2 x} \, dx &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} udv = uv|_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} vdu \\ &= -x \cot x|_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x \, dx \\ &= \frac{\pi}{4} + \ln |\sin x| \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\pi}{4} + \frac{1}{2} \ln 2,\end{aligned}$$

where we have used that $\cot \frac{\pi}{4} = 1$, $\cot \frac{\pi}{2} = 0$ and $\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $\sin \frac{\pi}{2} = 1$.

7. (a) Set

$$f(x) = \int_1^{x^2} \sin t^3 dt + x^3$$

Find $f(1)$ and $f'(x)$.
Solution. First, $f(1) = 1^3 = 1$, since in this case the interval of integration shrinks to a point and the integral

is zero. Second, we get by the FTC and the chain rule, setting $u = x^2$,

$$f'(x) = \frac{du}{dx}\ \frac{d}{du}\left(\int_1^u \sin t^3 dt\right)\Big|_{u=x^2} + 3x^2$$
$$= 2x\,(\sin u^3)\big|_{u=x^2} + 3x^2 = 2x\sin x^6 + 3x^2.$$

(b) Set

$$f(x) = \int_{\sqrt{x}}^{x-2} \tan^2 t\, dt$$

Find $f(4)$ and $f'(x)$

Solution. We have

$$f(4) = \int_{\sqrt{4}}^{4-2} \tan^2 t\, dt = \int_2^2 \tan^2 t\, dt = 0.$$

To find $f'(x)$, we write

$$f(x) = \int_{\sqrt{x}}^{x-2} \tan^2 t\, dt = \int_{\sqrt{x}}^{0} \tan^2 t\, dt + \int_0^{x-2} \tan^2 t\, dt$$
$$= -\int_0^{\sqrt{x}} \tan^2 t\, dt + \int_0^{x-2} \tan^2 t\, dt,$$

and apply the FTC and the chain rule (for the first integral we use $= \sqrt{x}$, and for the second integral we use $u = x-2$). We get

$$f'(x) = -\frac{du}{dx}\ \frac{d}{du}\left(\int_0^u \tan^2 t\, dt\right)\Big|_{u=\sqrt{x}} + \frac{du}{dx}\ \frac{d}{du}\left(\int_0^u \tan^2 t\, dt\right)\Big|_{u=x-2}$$
$$= -\frac{1}{2\sqrt{x}} \tan^2(\sqrt{x}) + \tan^2(x-2).$$

MATH 126 Second Midterm

Thursday March 25, 2010

Name: ______________________ **ID:** __________________ **Rec:** ____

Question:	1	2	3	4	5	6	Total
Points:	30	45	45	45	45	40	250
Score:							

There are 6 problems in this exam, printed on 6 pages (not including this cover sheet). Make sure that you have them all.

Do all of your work in this exam booklet, and cross out any work that the grader should ignore. You may use the backs of pages, but indicate **clearly** what is where if you expect someone to look at it. **Books, calculators, extra papers, and discussions with friends are not permitted.** Leave all answers in exact form (that is, do *not* approximate π, square roots, and so on.)

If you wish to use your psychic abilities to read the proctor's mind for the answers, you may do so. However, remember that he may be deliberately thinking of the *wrong* answers during the test.

You must give a correct justification of all answers to receive credit.

You have 90 minutes to complete this exam.

Name: ________________________ **Id:** ________________________

1. Determine these EASY antiderivatives. You should be able to do these **very well**. In these problems, no justification is needed. Remember the '+C'.

6 pts (a) $\displaystyle\int \frac{2}{x}\,dx$

6 pts (b) $\displaystyle\int 2\sin(x)\,dx$

6 pts (c) $\displaystyle\int e^{4x}\,dx$

6 pts (d) $\displaystyle\int \frac{2}{t^2+1}\,dt$

6 pts (e) $\displaystyle\int \frac{1}{\sqrt{1-u^2}}\,du$

2. In this question we tell you which method we suggest you use. Use the back of the previous page if you need more space.

15 pts

(a) Suggested method: substitution $\displaystyle\int \frac{y}{1+y^2}\,dy$

15 pts

(b) Suggested method: substitution $\displaystyle\int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}\,dx$

15 pts

(c) Suggested method: substitution $\displaystyle\int \frac{\ln(z)}{z}dz$

3. In this question we tell you which method we suggest you use. Use the back of the previous page if you need more space.

15 pts (a) Suggested method: integration by parts $\int x^6 \ln(x)\, dx$

15 pts (b) Suggested method: integration by parts $\int xe^{2x}\, dx$

15 pts (c) Suggested method: integration by parts $\int \sin(x)e^{3x}\, dx$

4. Determine the following antiderivatives. Use the back of the previous page if you need more space.

15 pts (a) $\displaystyle\int \sin^3(x)\,dx$

15 pts (b) $\displaystyle\int \frac{1}{\sec(2x)}\,dx$

15 pts (c) $\displaystyle\int \frac{1}{x^2\sqrt{x^2-1}}\,dx$

5. Evaluate these definite integrals. Use the back of the previous page if you need more space.

15 pts (a) $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{1-x^2}\,dx$

15 pts (b) $\displaystyle\int_{-100}^{100} \frac{\sin^{21}(x)}{1+e^{x^2}}dx$

15 pts (c) $\displaystyle\int_{0}^{1} \frac{x}{\sqrt{4-x^2}}\,dx$

Name: ____________________ **Id:** ____________________

6. Since $\int_0^1 \frac{1}{1+x^2}\,dx = \arctan(1) = \frac{\pi}{4}$, evaluating the integral $\int_0^1 \frac{4}{1+x^2}\,dx$ gives π.

$\boxed{\text{20 pts}}$ (a) Use Simpson's rule with 2 intervals to estimate $\int_0^1 \frac{4}{1+x^2}\,dx.$

$\boxed{\text{20 pts}}$ (b) How many intervals are needed to estimate $\int_0^1 \frac{4}{1+x^2}\,dx = \pi$ within $.0001$ using the trapezoid rule?[1]

---

[1] Use the following estimate for E_T using n intervals: If $|f''(x)| \leq K$ then $E_T \leq K\,\frac{(b-a)^3}{12n^2}$.
If $f(x) = \frac{1}{1+x^2}$, then $f''(x) = \frac{6x^2-2}{(1+x^2)^3}$

MATH 126 Solutions to Midterm 2 (electric)

1. Determine these EASY antiderivatives. You should be able to do these **very well**. In these problems, no justification is needed. Remember the '+C'.

6 pts (a) $\displaystyle\int \frac{2}{x}\,dx$

Solution:

$$\int \frac{2}{x}\,dx = \boxed{2\ln|x| + C}$$

6 pts (b) $\displaystyle\int 2\sin(x)\,dx$

Solution:

$$\int 2\sin(x)\,dx = \boxed{-2\cos(x) + C}$$

6 pts (c) $\displaystyle\int e^{4x}\,dx$

Solution:

$$\int e^{4x}\,dx = \boxed{\frac{1}{4}e^{4x} + C}$$

6 pts (d) $\displaystyle\int \frac{2}{t^2+1}\,dt$

Solution:

$$\int \frac{2}{t^2+1}\,dt = \boxed{2\arctan(t) + C}$$

6 pts (e) $\displaystyle\int \frac{1}{\sqrt{1-u^2}}\,du$

Solution:

$$\int \frac{1}{\sqrt{1-u^2}}\,du = \boxed{\arcsin(u) + C}$$

2. In this question we tell you which method we suggest you use. Use the back of the previous page if you need more space.

15 pts (a) Suggested method: substitution $\displaystyle\int \frac{y}{1+y^2}\,dy$

Solution: Make the substitution $u = 1 + y^2$, so that $du = 2dy$, or $\frac{1}{2}du = dy$. Then

$$\int \frac{y}{1+y^2}\,dy = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \boxed{\frac{1}{2}\ln|1+y^2| + C}$$

Note that since $1 + y^2 > 0$ for all y, the absolute value is not necessary; the answer $\dfrac{\ln(1+y^2)}{2} + C$ is fine, too.

15 pts (b) Suggested method: substitution $\displaystyle\int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}\,dx$

Solution: Make the substitution $u = \sqrt{x+1}$. Then

$$du = \frac{1}{2\sqrt{x+1}}\,dx \qquad \text{or} \quad 2du = \frac{dx}{\sqrt{x+1}}.$$

Thus,

$$\int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}\,dx = 2\int e^u\,du = 2e^u + C = \boxed{2e^{\sqrt{x+1}} + C}$$

15 pts (c) Suggested method: substitution $\displaystyle\int \frac{\ln(z)}{z}dz$

Solution: Here, we let $u = \ln(z)$ and so $du = \dfrac{dz}{z}$. This means we have

$$\int \frac{\ln(z)}{z}dz = \int u\,du = \frac{u^2}{2} + C = \boxed{\frac{\left(\ln(z)\right)^2}{2} + C}$$

3. In this question we tell you which method we suggest you use. Use the back of the previous page if you need more space.

15 pts (a) Suggested method: integration by parts $\displaystyle\int x^6 \ln(x)\,dx$

Solution: Take $u = \ln(x)$ and $dv = x^6\,dx$. Then $du = \frac{1}{x}\,dx$ and $v = \frac{x^7}{7}$. So:

$$\int x^6 \ln(x)\,dx = \frac{x^7\ln(x)}{7} - \frac{1}{7}\int x^7 \cdot \frac{1}{x}\,dx = \frac{x^7\ln(x)}{7} - \frac{1}{7}\int x^6\,dx = \boxed{\frac{x^7\ln(x)}{7} - \frac{x^7}{49} + C}$$

15 pts (b) Suggested method: integration by parts $\displaystyle\int xe^{2x}\,dx$

Solution: Take $u = x$ and $dv = e^{2x}\,dx$. Then $du = dx$ and $v = \frac{1}{2}e^{2x}$, and so we have

$$\int xe^{2x}\,dx = \frac{x}{2}e^{2x} - \frac{1}{2}\int e^{2x}\,dx = \boxed{\frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C}$$

15 pts (c) Suggested method: integration by parts $\displaystyle\int \sin(x)e^{3x}\,dx$

Solution: Take $u = e^{3x}$ and $dv = \sin(x)\,dx$. Then $du = 3e^{3x}\,dx$ and $v = -\cos(x)$. So we have

$$\int \sin(x)e^{3x}\,dx = -\cos(x)e^{3x} + 3\int \cos(x)e^{3x}\,dx$$

(we have a $+$ before the integral because we were subtracting a negative). To do the second integral, we take $u = e^{3x}$ and $dv = \cos x\,dx$. Then $du = 3e^{3x}\,dx$ and $v = \sin x$. This gives us

$$\int \sin(x)e^{3x}\,dx = -\cos(x)e^{3x} + 3\Big(\sin(x)e^{3x} - 3\int \sin(x)e^{3x}\,dx\Big)$$

Multiplying out gives

$$\int \sin(x)e^{3x}\,dx = -\cos(x)e^{3x} + 3\sin(x)e^{3x} - 9\int \sin(x)e^{3x}\,dx$$

or, equivalently,

$$10\int \sin(x)e^{3x}\,dx = -\cos(x)e^{3x} + 3\sin(x)e^{3x} + C$$

Thus, we have

$$\int \sin(x)e^{3x}\,dx = \boxed{\frac{-\cos(x)e^{3x} + 3\sin(x)e^{3x}}{10} + C}$$

4. Determine the following antiderivatives. Use the back of the previous page if you need more space.

15 pts (a) $\int \sin^3(x)\,dx$

Solution: We use the identity $\sin^2(x) = 1 - \cos^2(x)$ to get

$$\int \sin^3(x)\,dx = \int \big(1 - \cos^2(x)\big)\sin(x)\,dx.$$

Now take $u = \cos(x)$ and $du = -\sin(x)\,dx$, giving

$$\int \sin^3(x)\,dx = -\int (1-u^2)\,du = -u + \frac{u^3}{3} + C = \boxed{\frac{\cos^3(x)}{3} - \cos(x) + C}$$

15 pts (b) $\int \frac{1}{\sec(2x)}\,dx$

Solution:

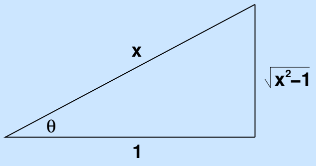
$$\int \frac{1}{\sec(2x)}\,dx = \int \cos(2x)\,dx = \boxed{\frac{1}{2}\sin(2x) + C}$$

15 pts (c) $\int \frac{1}{x^2\sqrt{x^2-1}}\,dx$

Solution: Take $x = \sec\theta$ so $dx = \sec\theta\tan\theta\,d\theta$. Then we have

$$\int \frac{1}{x^2\sqrt{x^2-1}}\,dx = \int \frac{\sec\theta\tan\theta\,d\theta}{\sec^2\theta\sqrt{\sec^2\theta - 1}} = \int \frac{\tan\theta\,d\theta}{\sec\theta\sqrt{\tan^2\theta}} = \int \frac{d\theta}{\sec\theta} = \int \cos\theta d\theta$$

This means we have $\sin\theta + C$ as our answer, but of course we need the answer in terms of x. Recall that we took $x = \sec\theta$, and so $\sin\theta = \frac{\sqrt{x^2-1}}{x}$ (see figure). Thus, we have shown



$$\int \frac{1}{x^2\sqrt{x^2-1}}\,dx = \boxed{\frac{\sqrt{x^2-1}}{x} + C}$$

5. Evaluate these definite integrals. Use the back of the previous page if you need more space.

15 pts (a) $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{1-x^2}\, dx$

Solution: We use partial fractions:

$$\frac{1}{1-x^2} = \frac{A}{1+x} + \frac{B}{1-x}$$

so $\quad 1 = A(1-x) + B(1+x).\quad$ Thus

$$A + B = 1 \qquad -A + B = 0 \qquad \text{hence} \quad A = \frac{1}{2},\, B = \frac{1}{2}$$

$$\begin{aligned}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{1-x^2}\, dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1/2}{1+x} + \frac{1/2}{1-x}\, dx &= \frac{1}{2}\ln|1+x| - \frac{1}{2}\ln|1-x|\Big|_{-1/2}^{1/2} \\ &= \frac{1}{2}\left[\ln\left(\frac{3}{2}\right) - \ln\left(\frac{1}{2}\right) - \ln\left(\frac{1}{2}\right) + \ln\left(\frac{3}{2}\right)\right] \\ &= \ln\left(\frac{3}{2}\right) - \ln\left(\frac{1}{2}\right) = \boxed{\ln(3)}\end{aligned}$$

15 pts (b) $\displaystyle\int_{-100}^{100} \frac{\sin^{21}(x)}{1+e^{x^2}} dx$

Solution: Since $\frac{\sin^{21}(x)}{1+e^{x^2}}$ is an odd function and the bounds are symmetric with respect to 0, the value of the integral is $\boxed{0}$.

15 pts (c) $\displaystyle\int_0^1 \frac{x}{\sqrt{4-x^2}}\, dx$

Solution: Let $u = 4 - x^2$ so that $du = -2x\, dx$. When $x = 0$, $u = 4$ and when $x = 1$, $u = 3$. Thus we have

$$\int_0^1 \frac{x}{\sqrt{4-x^2}}\, dx = -\int_4^3 \frac{du}{2\sqrt{u}} = -\sqrt{u}\Big|_4^3 = -\sqrt{3} + \sqrt{4} = \boxed{2-\sqrt{3}}.$$

6. Since $\int_0^1 \frac{1}{1+x^2}\,dx = \arctan(1) = \frac{\pi}{4}$, evaluating the integral $\int_0^1 \frac{4}{1+x^2}\,dx$ gives π.

20 pts

(a) Use Simpson's rule with 2 intervals to estimate $\int_0^1 \frac{4}{1+x^2}\,dx$.

Solution: Since there are two intervals, the width of each is $1/2$. Thus, Simpson's rule gives:

$$\frac{1}{3}\cdot\frac{1}{2}\Big(f(0)+4f(1/2)+f(1)\Big) = \frac{1}{6}\Big(4+4\left(\frac{4}{1+1/4}\right)+2\Big) = \boxed{\frac{94}{30}} \approx 3.13333$$

20 pts

(b) How many intervals are needed to estimate $\int_0^1 \frac{4}{1+x^2}\,dx = \pi$ within $.0001$ using the trapezoid rule?[1]

Solution: We use the information in the footnote. We need to determine n so that

$$\frac{1}{12n^2}K \le .0001$$

where K is the maximum of the absolute value of the second derivative of $4/(1+x^2)$ for x between 0 and 1. Since $\left|\frac{4(6x^2-2)}{(1+x^2)^3}\right|$ is a decreasing function on this interval, the maximum occurs at $x = 0$, so we take $K = |-8/1| = 8$.

To solve $\frac{8}{12n^2} \le .0001$, we multiply both sides by $10000n^2$ to get

$$\frac{80000}{12} \le n^2,$$

so n is the smallest integer bigger than $\sqrt{20000/3} \approx 81.6$.

Thus, $\boxed{n = 82}$.

---

[1] Use the following estimate for E_T using n intervals: If $|f''(x)| \le K$ then $E_T \le K\,\frac{(b-a)^3}{12n^2}$.

If $f(x) = \frac{1}{1+x^2}$, then $f''(x) = \frac{6x^2-2}{(1+x^2)^3}$

MAT 126 04/04/2014

Midterm 2 Review

5.5.24, 50, 64
5.6.10, 26
5.7.2, 6, 16, 22, 32
5.10.6, 16, 34, 30

Silvia

5.5

(24) $\int \frac{\sin(\ln x)}{x}\,dx =$ $u = \ln x$, $du = \frac{dx}{x}$

$$= \int \sin(u)\,du = -\cos(u) + C = \boxed{-\cos(\ln x) + C}$$

(50)

$$\int_{-\pi/2}^{\pi/2} \frac{x^2 \sin(x)}{1+x^6}\,dx$$

$f(x) = \frac{x^2 \sin(x)}{1+x^6}$. If $f(x)$ is odd, since the integral is done on a symmetric interval we get that the answer is 0.

$$f(-x) = \frac{(-x)^2 \sin(-x)}{1+(-x)^6} = \frac{-x^2 \sin(x)}{1+x^6} = -f(x)$$ ODD

$$\Rightarrow \int_{-\pi/2}^{\pi/2} \frac{x^2 \sin(x)}{1+x^6}\,dx = \boxed{0}$$

(64) A bacteria population starts with 400 bacteria and grows at a rate of $r(t) = (450.268)e^{1.12567t}$ bacteria per hour. How many bacteria will there be after three hours?

The answer is gonna be $\int_0^3 r(t)\,dt + 400$

$(P'(t) = r(t)$, $P(0) = 400 \rightarrow P(3) = \int_0^3 r(t)\,dt + 400.)$

$$\int 450.268\, e^{1.12567t}\, dt = 450.268 \int e^{1.12567t}\, dt =$$

$u = 1.12567t$
$du = 1.12567\,dt$
$dt = 0.88\,du$
$\left(0.88 = \frac{1}{1.12567}\right)$

$$= 450.268 \int e^{u}\,(0.88\,du) =$$
$$= 400 \int e^{u}\,du = 400e^{u} + C$$
$$= 400\,e^{1.12567t} + C$$

$$\Rightarrow \int_0^3 450.268\, e^{1.12567t}\, dt = 400\, e^{1.12567t} \Big|_0^3 =$$
$$= 400\, e^{1.12567 \cdot 3} - 400\, e^{0} =$$
$$= 400\, e^{3.377} - 400$$

$$\Rightarrow P(3) = 400\, e^{3.377} - \cancel{400} + \cancel{400} = \boxed{400\, e^{3.377}}$$

$$\int u\,dv = uv - \int v\,du$$

5.6 ⑩ $\int p^5 \ln p \, dp$

$u = \ln p \rightarrow du = \frac{1}{p} dp$
$dv = p^5 dp \rightarrow v = p^6/6$

$$\int p^5 \ln p \, dp = p^6/6 \ln p - \int p^{\overset{5}{\cancel{6}}}/6 \cdot \frac{1}{\cancel{p}} dp = p^6/6 \ln(p) - 1/6 \int p^5 dp =$$

$$= p^6/6 \ln(p) - 1/6 \, p^6/6 + C =$$

$$= \boxed{p^6/6 \left(\ln(p) - 1/6\right) + C.}$$

㉖ $\int t^3 e^{-t^2} dt =$ $\quad x = t^2$
$dx = 2t\,dt$

$$= \int \frac{t^2}{2} e^{-t^2} 2t\,dt = \int \frac{x}{2} e^{-x} dx = \frac{1}{2} \int x e^{-x} dx =$$

$u = x$
$dv = e^{-x} dx$
$\downarrow$
$du = dx$
$v = -e^{-x}$

$$= \frac{1}{2}\left(-x e^{-x} - \int (-e^{-x}) dx\right) =$$

$$= -\frac{1}{2} x e^{-x} + \frac{1}{2} \int e^{-x} dx = -\frac{1}{2} x e^{-x} - \frac{1}{2} e^{-x} + C =$$

$$= \boxed{-\frac{1}{2} t^2 e^{-t^2} - \frac{1}{2} e^{-t^2} + C}$$

5.7 ②

$$\int_0^{\pi/2} \cos^5 x \, dx = \int_0^{\pi/2} \cos^4 x \cos x \, dx = \int_0^{\pi/2} (\cos^2 x)^2 \cos x \, dx =$$

$$= \int_0^{\pi/2} (1 - \sin^2 x)^2 \cos x \, dx = \int_0^{\pi/2} (1 + \sin^4 x - 2 \sin^2 x) \cos x \, dx =$$

$$= \int_0^{\pi/2} (\cos x + \sin^4 x \cos x - 2 \sin^2 x \cos x) \, dx =$$

$$= \int_0^{\pi/2} \cos x \, dx + \int_0^{\pi/2} \sin^4 x \cos x \, dx - 2 \int_0^{\pi/2} \sin^2 x \cos x \, dx =$$

$\downarrow$ ① $\quad \downarrow$ ② $\quad \downarrow$ ③

as we noticed during the review we can do the substitution from here! ($u = \sin x$)
Thank you guys that noticed that!
My bad! SORRY

① $\int_0^{\pi/2} \cos x \, dx = \sin x \Big|_0^{\pi/2} = \sin(\pi/2) - \sin(0) = 1$

② $\int_0^{\pi/2} \sin^4 x \cos x \, dx$

$u = \sin x$
$du = \cos x \, dx$

$\int \sin^4 x \cos x \, dx = \int u^4 \, du = \frac{u^5}{5} + C = \frac{(\sin x)^5}{5} + C$

$\Rightarrow \int_0^{\pi/2} \sin^4 x \cos x \, dx = \frac{(\sin x)^5}{5} \Big|_0^{\pi/2} = \frac{1}{5} - 0 = \frac{1}{5}$

~~change the bounds?~~
(if you know what I mean...)

③ $-2 \int_0^{\pi/2} \sin^2 x \cos x \, dx$

$u = \sin x$
$du = \cos x \, dx$

$\int \sin^2 x \cos x \, dx = \int u^2 \, du = u^3/3 + C = \frac{(\sin x)^3}{3} + C$

$\Rightarrow -2 \int_0^{\pi/2} \sin^2 x \cos x \, dx = -2 \left(\frac{\sin x)^{3.}}{3} \right|_0^{\pi/2} = -2 \left(\frac{1}{3} - 0 \right) = -2/3$

$\int_0^{\pi/2} \cos^5 x \, dx = ① + ② + ③ = 1 + 1/5 - 2/3 = \frac{15 + 3 - 10}{15} = \boxed{\frac{8}{15}}$

⑥ $\int_0^{\pi/2} \sin^2 x \cos^2 x\,dx = \int_0^{\pi/2} (1-\cos^2 x)\cos^2 x\,dx =$

$= \int_0^{\pi/2} \cos^2 x\,dx - \int_0^{\pi/2} \cos^4 x\,dx$

① ②

$$\boxed{\begin{aligned} \cos^2 x &= \frac{1}{2}(1+\cos 2x) \\ \cos^2 2x) &= \frac{1}{2}(1+\cos 4x) \end{aligned}}$$

① $\int_0^{\pi/2} \cos^2 x\,dx = \int_0^{\pi/2} \frac{1}{2}(1+\cos 2x)\,dx =$

$= \frac{1}{2}\int_0^{\pi/2} dx + \frac{1}{2}\int_0^{\pi/2} \cos(2x)\,dx = \frac{1}{2}x\Big|_0^{\pi/2} + \frac{1}{2}\frac{\sin(2x)}{2}\Big|_0^{\pi/2} =$

$= \frac{1}{2}(\pi/2 - 0) + \frac{1}{2}(0-0) = \pi/4$

② $\int_0^{\pi/2} \cos^4 x\,dx = \int_0^{\pi/2} (\cos^2 x)^2\,dx = \int_0^{\pi/2} \left(\frac{1}{2}(1+\cos(2x))\right)^2 dx =$

$= \int_0^{\pi/2} 1/4\,(1+\cos^2(2x)+2\cos(2x))\,dx = \frac{1}{4}\int_0^{\pi/2} dx + \frac{1}{4}\int_0^{\pi/2} \cos^2(2x)\,dx + \frac{1}{4}.2\int_0^{\pi/2} \cos(2x)\,d$

$= \frac{1}{4}\int_0^{\pi/2} dx + \frac{1}{4}\int_0^{\pi/2} \left(\frac{1}{2}(1+\cos 4x)\right)dx + \frac{1}{2}\int_0^{\pi/2} \cos(2x)\,dx =$

$= \frac{1}{4}\int_0^{\pi/2} dx + \frac{1}{8}\int_0^{\pi/2} dx + \frac{1}{8}\int_0^{\pi/2} \cos(4x)\,dx + \frac{1}{2}\int_0^{\pi/2} \cos(2x)\,dx =$

$= \frac{1}{4}x\Big|_0^{\pi/2} + \frac{1}{8}x\Big|_0^{\pi/2} + \frac{1}{8}\frac{\sin(4x)}{4}\Big|_0^{\pi/2} + \frac{1}{2}\frac{\sin(2x)}{2}\Big|_0^{\pi/2} =$

$= \frac{1}{4}\left(\frac{\pi}{2}-0\right) + \frac{1}{8}(\pi/2 - 0) + \frac{1}{8}(0-0) + \frac{1}{2}(0-0) = \frac{\pi}{8} + \frac{\pi}{16} = \frac{3\pi}{16}$

$\Rightarrow \int_0^{\pi/2} \sin^2 x \cos^2 x\,dx =$ ① - ② $= \frac{\pi}{4} - \frac{3\pi}{16} = \frac{4\pi - 3\pi}{16} = \boxed{\frac{\pi}{16}}$

⑯

$$\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\,dx$$

$$\frac{x^3}{\sqrt{16-x^2}} = \frac{x^3}{\sqrt{\frac{16}{16}\cdot(16-x^2)}} = \frac{x^3}{4\sqrt{1-\frac{x^2}{16}}} = \frac{x^3}{4\sqrt{1-(\frac{x}{4})^2}}$$

$$\int \frac{x^3}{\sqrt{16-x^2}}\,dx = \int x^3\cdot\frac{1}{4}\cdot\frac{1}{\sqrt{1-(x/4)^2}}\,dx =$$

apparently, a guy suggested a better way but.. well, that's what I've got.

$x/4 = \sin t \qquad t = \arcsin(x/4)$

$x = 4\sin t \qquad dt = \frac{1}{\sqrt{1-(x/4)^2}}\cdot\frac{1}{4}\,dx$

$$= \int (4\sin t)^3\,dt = 64\int \sin^3 t\,dt = 64\int \sin^2 t\,\sin t\,dt = 64\int (1-\cos^2 t)\sin t\,dt =$$

$$=64\int \sin t\,dt - 64\int \cos^2 t\,\sin t\,dt = -64\cos t + 64\int u^2\,du = -64\cos t + 64\,\frac{u^3}{3} =$$

$u = \cos t$

$du = -\sin t\,dt$

$$= -64\cos t + 64\,\frac{(\cos t)^3}{3} = -64\cos(\arcsin(x/4)) + 64\cdot\frac{\cos^3(\arcsin(x/4))}{3}$$

$$\Rightarrow \int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\,dx = 64\left[\frac{\cos^3(\arcsin(x/4))}{3} - \cos(\arcsin(x/4))\right]\Bigg|_0^{2\sqrt{3}} =$$

$$=64\left[\left(\frac{\cos^3(\arcsin(\sqrt{3}/2))}{3} - \cos(\arcsin(\sqrt{3}/2))\right) - \left(\frac{\cos^3(\arcsin(0))}{3} - \cos(\arcsin(0))\right) =\right.$$

$$= 64\left(\frac{\cos^3(\pi/3)}{3} - \cos(\pi/3) - \frac{\cos^3(0)}{3} + \cos(0)\right) =$$

$$= 64\left(\frac{(1/2)^3}{3} - 1/2 - 1/3 + 1\right) = 64\left(\frac{1}{24} - \frac{1}{2} - \frac{1}{3} + 1\right) = 64\,\frac{(1-12-8+24)}{24} = \frac{\overset{8}{\cancel{64}}\cdot 5}{\cancel{24}_3} =$$

$$= \boxed{40/3}$$

㉒ $\int_0^1 \frac{x-4}{x^2-5x+6}\,dx$

- $x^2-5x+6=0$ $\qquad \frac{5\pm\sqrt{25-24}}{2} = \frac{5\pm 1}{2} \begin{matrix} \nearrow 3 \\ \searrow 2 \end{matrix}$

$\Rightarrow \quad x^2-5x+6=(x-2)(x-3)$

$$\frac{x-4}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)} = \frac{Ax-3A+Bx-2B}{(x-2)(x-3)}$$

$$\Rightarrow \begin{cases} A+B=1 \\ -3A-2B=-4 \end{cases} \qquad \begin{matrix} A=1-B \\ -3+3B-2B=-4 \\ B=-1 \\ A=2 \end{matrix}$$

$$\frac{x-4}{(x-2)(x-3)} = \frac{2}{x-2} - \frac{1}{x-3}$$

$$\int_0^1 \frac{x-4}{x^2-5x+6}\,dx = 2\int_0^1 \frac{1}{x-2}\,dx - \int_0^1 \frac{1}{x-3}\,dx =$$

$$= 2\ln|x-2|\Big|_0^1 - \ln|x-3|\Big|_0^1 =$$

$$= 2\left(\ln|-1| - \ln|-2|\right) - \left(\ln|-2| - \ln|-3|\right) =$$

$$= 2\left(0-\ln(2)\right) - \left(\ln(2)-\ln(3)\right) =$$

$$= -2\ln(2) - \ln(2) + \ln(3) = \ln(3) - 3\ln(2) =$$

$$= \ln(3) - \ln(2^3) = \ln(3) - \ln(8) = \boxed{\ln(3/8)}$$

(32) $\displaystyle\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx$

$$\begin{array}{l|l} x^3 \quad -4x-10 & x^2-x-6 \\ \hline -x^3+x^2+6x & x+1 \\ \hline // \quad x^2+2x-10 & \\ \quad -x^2+x+6 & \\ \hline // \quad 3x-4 & \end{array}$$

$\Rightarrow x^3-4x-10=(x^2-x-6)(x+1)+(3x-4)$

$$\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx=\int_0^1\left(\frac{\cancel{(x^2-x-6)}(x+1)}{\cancel{x^2-x-6}}+\frac{3x-4}{x^2-x-6}\right)dx=$$

$$=\int_0^1 (x+1)\,dx+\int_0^1 \frac{3x-4}{x^2-x-6}\,dx$$

$x^2-x-6=0 \qquad \dfrac{1\pm\sqrt{1+24}}{2}=\dfrac{1\pm 5}{2} \begin{array}{l} \nearrow 3 \\ \searrow -2 \end{array}$

$x^2-x-6=(x-3)(x+2)$

$$\frac{3x-4}{(x-3)(x+2)}=\frac{A}{x-3}+\frac{B}{x+2}=\frac{A(x+2)+B(x-3)}{(x-3)(x+2)}=\frac{Ax+2A+Bx-3B}{(x-3)(x+2)}$$

$\begin{cases} A+B=3 \\ 2A-3B=-4 \end{cases} \qquad B=3-A$

$2A-3(3-A)=-4$

$2A-9+3A=-4$

$5A=5$

$A=1 \longrightarrow B=3-1=2$

$$\int_0^1 \frac{3x-4}{x^2-x-6}\,dx=\int_0^1 \frac{1}{x-3}\,dx+2\int_0^1 \frac{1}{x+2}\,dx$$

↳

$$\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx = \int_0^1 (x+1)\,dx + \int_0^1 \frac{1}{x-3}\,dx + 2\int_0^1 \frac{1}{x+2}\,dx =$$

$$= \frac{x^2}{2}+x\,\Big|_0^1 + \ln|x-3|\,\Big|_0^1 + 2\ln|x+2|\,\Big|_0^1 =$$

$$= \left(\frac{1}{2}+1-0\right) + \left(\ln|-2| - \ln|-3|\right) + 2\left(\ln|3| - \ln|2|\right) =$$

$$= 3/2 + \ln|2| - \ln|3| + 2\ln|3| - 2\ln|2| = 3/2 + \ln|3| - \ln|2| =$$

$$= \boxed{\frac{3}{2} + \ln(3/2)}$$

$\boxed{5.10}$ ⑥ $\int_0^\infty \frac{1}{\sqrt[4]{1+x}}\,dx$ convergent or divergent?

if convergent, evaluate it

$$\int_0^\infty \frac{1}{\sqrt[4]{1+x}}\,dx = \lim_{t\to+\infty}\int_0^t \frac{1}{\sqrt[4]{1+x}}\,dx = \lim_{t\to+\infty}\int_0^t (1+x)^{-1/4}\,dx =$$

$$= \lim_{t\to+\infty} \frac{(1+x)^{-1/4+1}}{-1/4+1}\Big|_0^t = \lim_{t\to+\infty} \frac{4}{3}(1+x)^{3/4} = +\infty$$

$\boxed{\text{DIVERGENT}}$

⑯ $\int_{-\infty}^{+\infty} \cos(\pi t)\,dt$ convergent or divergent?

if convergent, evaluate it

$$\int_{-\infty}^{+\infty} \cos(\pi t)\,dt = \underbrace{\int_{-\infty}^{0} \cos(\pi t)\,dt}_{①} + \underbrace{\int_{0}^{+\infty} \cos(\pi t)\,dt}_{②}$$

① $\int_{-\infty}^{0} \cos(\pi t)\,dt = \lim_{r\to-\infty}\int_r^0 \cos(\pi t)\,dt = \lim_{r\to-\infty} \frac{\sin(\pi t)}{\pi}\Big|_r^0 =$

$$= \lim_{r\to-\infty}\left(0 - \frac{\sin(\pi r)}{\pi}\right)$$

② $\lim_{r\to+\infty}\int_0^r \cos(\pi t)\,dt = \lim_{r\to+\infty}\frac{\sin(\pi t)}{\pi}\Big|_0^r = \lim_{r\to+\infty}\frac{\sin(\pi r)}{\pi}$

$\int_{-\infty}^{+\infty}\cos(\pi t)\,dt = \lim_{r\to-\infty} -\frac{\sin(\pi r)}{\pi} + \lim_{r\to+\infty}\frac{\sin(\pi r)}{\pi} =$

$= \lim_{r\to+\infty}\frac{\sin(\pi r)}{\pi} + \lim_{r\to+\infty}\frac{\sin(\pi r)}{\pi} =$ DOES NOT EXIST

$= \frac{2}{\pi}\lim_{r\to+\infty}\sin(\pi r)$

$\boxed{\text{NOT CONVERGENT}}$

convergent or divergent?
if convergent, evaluate

(30) $\int_0^1 \frac{1}{4y-1}\,dy$ $\quad$ $4y-1=0$

$y = 1/4 \quad 1/4 \in [0,1]$

$$\int_0^1 \frac{1}{4y-1}\,dy = \overset{①}{\int_0^{1/4} \frac{1}{4y-1}\,dy} + \overset{②}{\int_{1/4}^1 \frac{1}{4y-1}\,dy}$$

① $$\int_0^{1/4} \frac{1}{4y-1}\,dy = \lim_{t\to 1/4^-} \int_0^t \frac{1}{4y-1}\,dy = \lim_{t\to 1/4^-} \left.\frac{\ln|4y-1|}{4}\right|_0^t =$$

$$= \lim_{t\to 1/4^-} \left(\frac{\ln|4t-1|}{4} - \frac{\ln|1|}{4}\right) = \lim_{t\to 1/4^-} \frac{\ln|4t-1|}{4} = -\infty$$

② $$\int_{1/4}^1 \frac{1}{4y-1}\,dy = \lim_{t\to 1/4^+} \int_t^1 \frac{1}{4y-1}\,dy = \lim_{t\to 1/4^+} \left.\frac{\ln|4y-1|}{4}\right|_t^1 =$$

$$= \lim_{t\to 1/4^+} \left(\frac{\ln|3|}{4} - \frac{\ln|4t-1|}{4}\right) =$$

$$= \frac{\ln|3|}{4} - \lim_{t\to 1/4^+} \ln\frac{|4t-1|}{4}$$

you can avoid ②

the fact that ① diverges it is enough.

NOT CONVERGENT

㉞ $\int_0^1 \frac{\ln x}{\sqrt{x}}dx$ convergent or divergent?
if convergent, evaluate

$$\int_0^1 \frac{\ln x}{\sqrt{x}}dx = \lim_{t\to 0}\int_t^1 \frac{\ln x}{\sqrt{x}}dx = \qquad u = \ln x \quad du = \frac{1}{x}dx \qquad dv = \frac{dx}{\sqrt{x}} \quad v = 2\sqrt{x}$$

$$= \lim_{t\to 0}\left(2\sqrt{x}\ln(x)\Big|_t^1 - \int_t^1 \frac{2\cancel{\sqrt{x}}}{x\sqrt{x}}dx\right) =$$

$$= \lim_{t\to 0}\left(2\sqrt{x}\ln(x)\Big|_t^1 - 2\int_t^1 x^{-1/2}dx\right) = \lim_{t\to 0}\left(2\sqrt{x}\ln(x)\Big|_t^1 - 4x^{1/2}\Big|_t^1\right) =$$

$$= \lim_{t\to 0}\left((0 - 2\sqrt{t}\ln(t)) - (4 - 4t^{1/2})\right) =$$

$$= \lim_{t\to 0}\left(-2\sqrt{t}\ln(t) + 4t^{1/2} - 4\right) = -2\lim_{t\to 0}\sqrt{t}\ln(t) + 4\lim_{t\to 0}t^{1/2} - 4 =$$

$$= -2\lim_{t\to 0}\sqrt{t}\ln(t) - 4$$

$$\lim_{t\to 0}\sqrt{t}\ln(t) = \lim_{t\to 0}\frac{\ln(t)}{\frac{1}{\sqrt{t}}} \overset{\text{de l'Hôpital}}{=} \lim_{t\to 0}\frac{1/t}{-\frac{1}{2}\frac{1}{t^{3/2}}} = \lim_{t\to 0} -2\frac{t^{3/2}}{t} =$$

$$= -2\lim_{t\to 0}t^{1/2} = 0$$

$$\boxed{\text{CONVERGENT}}$$

$$\Rightarrow \int_0^1 \frac{\ln x}{\sqrt{x}}dx = \boxed{-4}$$

IF THERE ARE MISTAKES (or even just stupid things) I'M VERY SORRY (probably there are some...)

Good luck with your midterm! Cya Silvia

1 Table of Indefinite Integrals

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx \qquad \int cf(x)\,dx = c\int f(x)\,dx$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

MAT 126 Calculus B Spring 2006
Practice Midterm I

Name:____

I.D.:__________________________ **Section number:** _________

Answer each question in the space provided and on the reverse side of the sheets. Show your work whenever possible. Unless otherwise indicated, **answers without justification will get little or no partial credit!** Cross out anything that grader should ignore and circle or box the final answer.

1. (a) (10 points) Estimate the area under the graph of $f(x) = 16 - x^2$ from $x = 0$ to $x = 4$ using four rectangles and right endpoints. Sketch the graph and rectangles. Is your estimate and underestimate or an overestimate?

(b) (10 points) Repeat part (a) using left endpoints.

2. (a) (10 points) Evaluate integral by interpreting it as area

$$\int_{-5}^{5} \sqrt{25 - x^2}dx$$

(b) (5 points) Determine a region whose area is equal to

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan \frac{i\pi}{4n}$$

Do not evaluate the limit.

3. Given two functions $f(x)$ and $g(x)$ which satisfy

$$\int_0^3 f(x)dx = 5, \quad \int_0^5 f(x)dx = 7,$$
$$\int_3^5 g(x)dx = 1, \quad \int_0^5 g(x)dx = 9,$$

find

(a) (5 points)

$$\int_3^5 (3f(x) - g(x))dx$$

(b) (5 points)

$$\int_0^3 (f(x) + 2g(x))dx$$

4. (5 points) Express the limit as a definite integral on the given interval $[0,4]$:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{e^{x_i}}{1+x_i}\Delta x$$

5. Evaluate the following indefinite integrals

(a) (5 points)

$$\int(3\cos x-4\sin x)dx$$

(b) (10 points)

$$\int\frac{\cos x}{1-\cos^2 x}dx$$

6. Evaluate the following definite integrals

(a) (5 points)

$$\int_1^2 x^{-2}dx$$

(b)

$$\int_1^8\frac{x-1}{\sqrt[3]{x^2}}dx$$

(c) (5 points)

$$\int_1^{27}\frac{1}{9t}dt$$

(d) (5 points)

$$\int_{\ln 3}^{\ln 6}5e^x dx$$

(e) (10 points)

$$\int_{\pi/3}^{\pi/2}\csc x\cot x dx$$

MAT 126 Calculus B Spring 2006
Practice Midterm I — Solutions

Name:____

I.D.:______________________ **Section number:** ________

Answer each question in the space provided and on the reverse side of the sheets. Show your work whenever possible. Unless otherwise indicated, **answers without justification will get little or no partial credit!** Cross out anything that grader should ignore and circle or box the final answer.

1. (a) (10 points) Estimate the area under the graph of $f(x) = 16 - x^2$ from $x = 0$ to $x = 4$ using four rectangles and right endpoints. Sketch the graph and rectangles. Is your estimate and underestimate or an overestimate?
Solution.

$$R_4 = f(1) + f(2) + f(3) + f(4) = 15 + 12 + 7 = 34$$

The function $f(x)$ is concave downward, so that $R_4 < A$, i.e., it is an underestimate (sketch the graph!)

(b) (10 points) Repeat part (a) using left endpoints.
Solution.

$$L_4 = f(0) + f(1) + f(2) + f(3) = 50$$

It is an overestimate, $L_4 > A$ (sketch the graph!)

2. (a) (10 points) Evaluate integral by interpreting it as area

$$\int_{-5}^{5} \sqrt{25 - x^2} dx$$

Solution. It is the upper half of the circle of radius 5 centered at the origin, so

$$\int_{-5}^{5} \sqrt{25 - x^2} dx = \frac{1}{2}\pi\, 5^2 = 12.5\pi$$

(b) (5 points) Determine a region whose area is equal to

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan\frac{i\pi}{4n}$$

Do not evaluate the limit.
Solution. We get $\Delta x = \pi/4n$, so that $b - a = \pi/4$. Since $x_i = i\Delta x$, we get $a = x_0 = 0$, so that $b = \pi/4$. The sum is

the right endpoint sum R_n for $f(x) = \tan x$ on the interval $\left[0, \frac{\pi}{4}\right]$, and we get

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan \frac{i\pi}{4n} = \int_0^{\pi/4} \tan x dx,$$

which is the area of the region under the graph of the function $f(x) = \tan x$ from $x = 0$ to $x = \pi/4$.

3. Given two functions $f(x)$ and $g(x)$ which satisfy

$$\int_0^3 f(x)dx = 5, \quad \int_0^5 f(x)dx = 7,$$
$$\int_3^5 g(x)dx = 1, \quad \int_0^5 g(x)dx = 9,$$

find

(a) (5 points)

$$\int_3^5 (3f(x) - g(x))dx$$

Solution. We have

$$\int_3^5 f(x)dx = \int_0^5 f(x)dx - \int_0^3 f(x)dx = 7 - 5 = 2$$

so that

$$\int_3^5 (3f(x) - g(x))dx = 3\int_3^5 f(x)dx - \int_3^5 g(x)dx = 6 - 1 = 5$$

(b) (5 points)

$$\int_0^3 (f(x) + 2g(x))dx$$

Solution. Similarly,

$$\int_0^3 g(x)dx = 9 - 1 = 8$$

and

$$\int_0^3 (f(x) + 2g(x))dx = 5 + 2 \times 8 = 21$$

4. (5 points) Express the limit as a definite integral on the given interval $[0, 4]$:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{e^{x_i}}{1 + x_i} \Delta x$$

Solution.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{e^{x_i}}{1+x_i}\Delta x = \int_0^4 \frac{e^x}{1+x}dx$$

5. Evaluate the following indefinite integrals

(a) (5 points)

$$\int (3\cos x - 4\sin x)dx$$

Solution.

$$\int (3\cos x - 4\sin x)dx = 3\sin x + 4\cos x + C$$

(b) (10 points)

$$\int \frac{\cos x}{1-\cos^2 x}dx$$

Solution. Using the fundamental trigonometric identity,

$$\int \frac{\cos x}{1-\cos^2 x}dx = \int \frac{\cos x}{\sin^2 x}dx = -\frac{1}{\sin x} + C$$

— and integral from the table of indefinite integrals.

6. Evaluate the following definite integrals

(a) (5 points)

$$\int_1^2 x^{-2}dx$$

Solution.

$$\int_1^2 x^{-2}dx = \left.\frac{x^{-1}}{-1}\right|_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}$$

(b)

$$\int_1^8 \frac{x-1}{\sqrt[3]{x^2}}dx$$

Solution.

$$\begin{aligned}\int_1^8 \frac{x-1}{\sqrt[3]{x^2}}dx &= \int_1^8 (x^{1/3} - x^{-2/3})dx \\ &= \left.\left(\frac{3}{4}x^{4/3} - 3x^{1/3}\right)\right|_1^8 = \left(\frac{3}{4}\times 8^{4/3} - 3\times 8^{1/3}\right) - \left(\frac{3}{4} - 3\right) = 8.25\end{aligned}$$

where we have used that $8 = 2^3$.

(c) (5 points)

$$\int_1^{27} \frac{1}{9t} dt$$

Solution.

$$\int_1^{27} \frac{1}{9t} dt = \frac{1}{9} \ln t \bigg|_1^{27} = \frac{1}{9} \ln 27 = \frac{1}{3} \ln 3,$$

where we have used that $\ln 1 = 0$ and $27 = 3^3$.

(d) (5 points)

$$\int_{\ln 3}^{\ln 6} 5e^x dx$$

Solution.

$$\int_{\ln 3}^{\ln 6} 5e^x dx = 5e^x|_{\ln 3}^{\ln 6} = 5(e^{\ln 6} - e^{\ln 3}) = 5(6-3) = 15$$

(e) (10 points)

$$\int_{\pi/3}^{\pi/2} \csc x \cot x dx$$

Solution.

$$\begin{aligned}\int_{\pi/3}^{\pi/2} \csc x \cot x dx &= \int_{\pi/3}^{\pi/2} \frac{\cos x}{\sin^2 x} dx \\ &= \left(-\frac{1}{\sin x}\right)\bigg|_{\pi/3}^{\pi/2} = -1 + \frac{2}{\sqrt{3}}.\end{aligned}$$

Math 126 **First Midterm, Fall 2007**

Instructions: Do all your work in the provided blue book. Make sure you write your name and recitation number on the blue book.

Write all your answers exactly. Do not use approximations for numbers like π or $\sqrt{2}$. NO CALCULATORS!

Good luck!

1. A liquid leaked from a tank at a rate $r(t)$, where r is in liters per hour, and t is in hours. The rate decreased as time passed. This rate was measured every 2 hours, and the result is given below. Find an upper estimate for the total amount of oil that has leaked out after 6 hours.

t	0	2	4	6
r(t)	9	7.5	7	6

2.

$$\int_2^3 f(x)dx = 1/3, \qquad \int_3^5 f(x)dx = 6, \qquad \int_2^5 g(x)dx = 4/5.$$

Find

$$\int_2^5 (g(x) + 4f(x) + 5)dx.$$

3. Express the integral as the limit of Riemann sums. Do not evaluate the limit. Do not use Δx or x_i^* in your final answer; instead, plug in the formulas for these.

$$\int_2^{12} \frac{\sqrt[3]{x}dx}{2+3x}$$

4. Let

$$f(w) = \frac{w^4 - 2w^2\sqrt{w}}{w}.$$

(a) Find $f'(w)$.

(b) Find $\int f(w)dw$.

5. Simplify.

(a)

$$\frac{d}{dx}\left(\int_{\pi}^{e^x} 3\cos t dt\right).$$

(b)

$$\int (2x^{-1} + 7\sin x)dx$$

6. The velocity of a particle at time t is given by $v(t) = 3t^2 - 1$. The position of the particle at time $t = 0$ is 1. Find the position of the partice at time $t = 3$.

7. Consider the following integral

$$\int_1^2 3 + 2x\ dx$$

(a) Express the integral as the limit of Riemann sums. Do not evaluate the limit. Do not use Δx or x_i^* in your final answer; instead, plug in the formulas for these.

(b) Use the formula $\displaystyle\sum_{i=1}^{n} i = \frac{n^2+n}{2}$ to evaluate the limit in the previous part. DO NOT CALCULATE THE INTEGRAL DIRECTLY, or you will get no credit.

1. A liquid leaked from a tank at a rate $r(t)$, where r is in liters per hour, and t is in hours. The rate decreased as time passed. This rate was measured every 2 hours, and the result is given below. Find an upper estimate for the total amount of oil that has leaked out after 6 hours.

t	0	2	4	6
r(t)	9	7.5	7	6

Solution: We are being asked to calculate a Riemann sum. Since the rate of leakage was decreasing, the left sum will give us an upper estimate. There are 6 hours involved, measured every 2 hours, we will have a sum with 3 rectangles, each of width 2.

Our estimate is

$$2r(0) + 2r(2) + 2r(4) = 2 \cdot 9 + 2 \cdot 7.5 + 2 \cdot 7 = 47$$

There should be no more than 47 liters of oil that leaked out during the 6 hours. (Note that we do NOT want to use $r(6)$ in our estimate, since that would tell us about what happened AFTER the 6th hour ended.

(Although it wasn't asked, we could do a right sum to get a lower bound on the amount of oil lost. In this case, we would get $2r(2) + 2r(4) + 2r(6) = 41$, so the total amount of oil that leaked in the given period was between 41 and 47 liters.)

2.

$$\int_2^3 f(x)dx = 1/3, \qquad \int_3^5 f(x)dx = 6, \qquad \int_2^5 g(x)dx = 4/5.$$

Find

$$\int_2^5 \big(g(x) + 4f(x) + 5\big)dx.$$

Solution: We rewrite this in terms of what we are given. We have

$$\int_2^5 (g(x) + 4f(x) + 5)dx = \int_2^5 g(x)\ dx + 4\int_2^5 f(x)\ dx + \int_2^5 5\ dx.$$

The first integral is given as 4/5, and the last is the area of a 5×3 rectangle, so it is 15. To do the middle integral, we use the fact that $\int_2^5 f(x)\ dx = \int_2^3 f(x)\ dx + \int_3^5 f(x)\ dx = 1/3 + 6$. This means we have

$$\int_2^5 \big(g(x) + 4f(x) + 5\big)dx = \frac{4}{5} + 4\left(6 + \frac{1}{3}\right) + 15 = \frac{617}{15}$$

Sorry about the fractions...

3. Express the integral as the limit of Riemann sums. Do not evaluate the limit. Do not use Δx or x_i^* in your final answer; instead, plug in the formulas for these.

$$\int_2^{12} \frac{\sqrt[3]{x}dx}{2+3x}$$

Solution: Let's write this as a right sum with n equal rectangles.

The width Δx of each rectangle will be $\dfrac{12-2}{n} = \dfrac{10}{n}$.

Since we are doing right sums, we divide the interval $[2, 10]$ into n pieces, and choose our x_i^* on the right (larger) side of each piece. Thus, we have $x_1^* = 2 + \frac{10}{n}$, $x_2^* = 2 + \frac{20}{n}$, $x_3^* = 2 + \frac{30}{n}$, and so on. More compactly, we can write this as

$$x_i^* = 2 + \frac{10i}{n} \qquad i \leq i \leq n.$$

This means that the area of the i^{th} rectangle will be

$$(\Delta x) f\left(x_i^*\right) = \left(\frac{10}{n}\right) f\left(2 + \frac{10i}{n}\right) = \left(\frac{10}{n}\right) \frac{\sqrt[3]{2+\frac{10i}{n}}}{2+3\left(2+\frac{10i}{n}\right)} = \frac{10\sqrt[3]{2+\frac{10i}{n}}}{8n+30i},$$

and so we have the area with n rectangles as $R_n = \sum_{i=1}^{n} \frac{10\sqrt[3]{2+\frac{10i}{n}}}{8n+30i}$.

Since the exact value of the integral is the limit as the number of rectangles goes to infinity, we have

$$\int_2^{12} \frac{\sqrt[3]{x}dx}{2+3x} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{10\sqrt[3]{2+\frac{10i}{n}}}{8n+30i}$$

4. Let $f(w) = \dfrac{w^4 - 2w^2\sqrt{w}}{w}$.

(a) Find $f'(w)$.

Solution: Observe that $f(w) = w^3 - 2w^{3/2}$. So $f'(w) = 3w^2 - 3w^{1/2}$.

(b) Find $\int f(w)dw$.

Solution:

$$\int f(w)dw = \int w^3 - 2w^{3/2}dw = \frac{1}{4}w^4 - 2\cdot\frac{2}{5}w^{5/2} + C = \frac{w^4}{4} - \frac{4w^{5/2}}{5} + C$$

5. Simplify.

(a) $\dfrac{d}{dx}\left(\displaystyle\int_{\pi}^{e^x} 3\cos t dt\right)$.

Solution: We use the fundamental theorem of calculus, and remember that we must use the chain rule (since the upper bound of integration is not x):

$$\frac{d}{dx}\left(\int_{\pi}^{e^x} 3\cos t dt\right) = 3\cos\left(e^x\right)\cdot\left(\frac{d}{dx}e^x\right) = 3e^x\cos(e^x)$$

(b) $\displaystyle\int \left(2x^{-1} + 7\sin x\right)\, dx$

Solution:

$$\int \left(2x^{-1} + 7\sin x\right)\, dx = 2\ln|x| - 7\cos(x) + C$$

6. The velocity of a particle at time t is given by $v(t) = 3t^2 - 1$. The position of the particle at time $t = 0$ is 1. Find the position of the particle at time $t = 3$.

Solution: We remember that if $s(t)$ is the position at time t, the velocity is $v(t) = s'(t)$. We have to find an the antiderivative $s(t) = \int v(t)\, dt$ for which $s(0) = 1$.

$$s(t) = \int 3t^2 - 1\, dx = t^3 - t + C$$

Since $s(0) = 1$, we must have $C = 1$. Thus

$$s(t) = t^3 - t + 1.$$

This means the position at $t = 3$ is $s(3) = 27 - 3 + 1 = 25$.

7. Consider the following integral

$$\int_1^2 3 + 2x\ dx$$

(a) Express the integral as the limit of Riemann sums. Do not evaluate the limit. Do not use Δx or x_i^* in your final answer; instead, plug in the formulas for these.

Solution: First, we write the right approximation with n rectangles (right, left, midpoint, . . . any will do). Since we have $\Delta x = (2-1)/n = 1/n$, we also have $x_i^* = 1+i/n$. Thus,

$$R_n = \sum_{i=0}^{n} \left(\frac{1}{n}\right)\left(3 + 2\left(1 + \frac{i}{n}\right)\right) = \sum_{i=0}^{n}\left(\frac{5}{n} + \frac{2i}{n^2}\right)$$

Taking the limit as $n \longrightarrow \infty$ gives us the integral:

$$\int_1^2 3 + 2x\ dx \ = \ \lim_{n\to\infty} \sum_{i=0}^{n}\left(\frac{5}{n} + \frac{2i}{n^2}\right)$$

(b) Use the formula $\sum_{i=1}^{n} i = \frac{n^2+n}{2}$ to evaluate the limit in the previous part. DO NOT CALCULATE THE INTEGRAL DIRECTLY, or you will get no credit.

Solution: In order to compute the limit, it will probably help to rearrange the above answer a bit first.
Observe that $\sum_1^n 1 = 1 + 1 + \ldots + 1 = n$. We have

$$\sum_{i=0}^{n}\left(\frac{5}{n} + \frac{2i}{n^2}\right) = \frac{5}{n}\sum_{i=0}^{n} 1 + \frac{2}{n^2}\sum_{i=0}^{n} i = \frac{5}{n}\cdot n + \frac{2}{n^2}\left(\frac{n^2+n}{2}\right) = 5 + \frac{n^2+n}{n^2}$$

Thus, we have

$$\int_1^2 3 + 2x\ dx = \lim_{n\to\infty}\left(5 + \frac{n^2+n}{n^2}\right) = 5 + 1 = 6.$$

Just to make sure we didn't do something stupid, we can check our answer by calculating the definite integral

$$\int_1^2 3 + 2x\ dx = 3x + x^2\Big|_1^2 = (6+4) - (3+1) = 6.$$

But doing that is worth no credit, just peace of mind.

Paper homeworks.

Assignment 10. For practice purpose only (not for grading).

Problem 1. Jim and John are doing bench press in a gym. Jim lifted a 100 kg weight 10 times by 36 cm, John lifted 120 kg weight 8 times by 40 cm. Who did more work?

Problem 2. A particle is moving along the x-axis by a force that measures $2e^{-3x}$ pounds at a point x feet from the origin. Find the work done in moving the particle from the origin to a distance 5 feet.

Problem 3. A spring is hanging from a ceiling. When a 1 kg weight was attached to the end of the spring it stretched by 24.5 cm. Find the work done by the gravity.

Remark. There are two forces affecting the weight: the force of gravity and the elastic force of the spring. The resulting force is the difference of these two forces.

Problem 4. Find center of mass of a lamina with uniform density bounded by *a*) $x = 0, y = 0$ and $y = 1 - x$, *b*) $x = 1$, $x = 2$ and $y = \frac{1}{x}$.

Problem 5. The demand function for a product, in dollars, is *a*) $p(x) = 50 - x + 0.005x^2$, *b*) $p = 100e^{-0.01x} - 50$. Sketch the demand curve. Find the consumer surplus when the sales level is 50.

Problem 6. Assume that the probability density function for the time (in minutes) required to solve certain WebAssign problem is

$$f(x) = \begin{cases} 0.06x - 0.006x^2, & \text{if } 0 \leqslant x \leqslant 10, \\ 0, & \text{if } x < 0 \text{ or } x > 10. \end{cases}$$

What is more likely: to spend 3 to 7 minutes for solving this problem, or to spend less than 5 minutes? Find the average time to solve this problem.

Problem 7. The probability density function for the waiting time (in minutes) in a queue at the cashier of some store is

$$f(x) = \begin{cases} 0.1e^{-0.1x}, & \text{if } x \geqslant 0, \\ 0, & \text{if } x < 0. \end{cases}$$

Find the probability that a customer has to wait in the queue less then 10 minutes. Find the average waiting time.

Assignment 9. Due on the week of May 5.

Problem 1. Find the length of the curve:

$$x(t) = \tfrac{4}{3}t^{\frac{3}{2}},\ y(t) = \tfrac{1}{2}t^2 - t,\ 0 \leqslant t \leqslant 1.$$

Problem 2. Find the length of the curve:

$$y = \tfrac{1}{4}x^2 - \tfrac{1}{2}\ln x, 1 \leqslant x \leqslant 2.$$

Problem 3. Chip and Dale start running from point A at $(0, 0)$ to point B at $(1, 2)$ at the same time and with the same constant speed. Chip runs over the curve $y = 2x^2$ and Dale runs over the curve $y = 2\sqrt{x}$. Who will finish first?

Remark: use CAS to calculate the integrals you obtain. For instance, in maple to calculate a numerical value for the integral of $f(x)$ between a and b you can use the command

$$evalf(int(f(x), x = a..b));$$

Problem 4. Find the length of the astroid

$$x(t) = \cos^3 t, y(t) = \sin^3 t, 0 \leqslant t \leqslant 2\pi.$$

Problem 5. Find the average value of a function on the given interval:

$$f(x) = \sin(2x) - \sqrt{x},\ x \in [0, \pi].$$

Problem 6. A population in a small country changed from 1980 till 1990 according to the rule:

$$P(t) = 5 + \tfrac{1}{2}e^{\frac{t}{10}} \text{ millions people,}$$

where $0 \leqslant t \leqslant 10$ is the number of years since 1980. Find the average size of the population during this time period. Using SAC approximate the time t at which the size of the population was equal to the average.

Remark: for instance, in maple to solve an equation $f(t) = a$ you can use the command

$$solve(f(t) = a, t);$$

Problem 7. In a certain city the temperature t hours after 9 am was modeled by the function $T(t) = 40 + \frac{1}{4}t(12 - t)$. Find the average temperature during the period from 9 am to 9 pm. At what times the temperature was equal to the average temperature?

Assignment 8. Due on the week of April 28.

Problem 1. A football has a shape of a solid obtained by rotating the region between $y = \frac{4}{3}x - \frac{4}{9}x^2$ and the x-axis around the x-axis. Find the volume of this football.

In problems 2 and 3 using the washer method find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region, the solid, and a typical washer.

Problem 2. $y = \frac{2}{x}$, $y = 1$ and $x = 1$; about $x = 2$.

Problem 3. $y = 2x + 1$, $y = 3x + 1$, $x = 3$; about the x-axis.

Problem 4. Each integral below represents the volume of a solid. Describe the solid.

$$a)\ \pi \int\limits_1^2 (x^2 e^{2x} - 1)\mathrm{d}x, \qquad b)\ \pi \int\limits_1^5 ((\ln x + 1)^2 - \tfrac{1}{x^2})\mathrm{d}x.$$

Problem 5. A solid lies between the planes $x = 0$ and $x = 1$. It is known that it's cross-section in a plane P_x is a rectangle with sides of length $x + 1$ and $2 - x$. Find the volume of the solid.

Problem 6. A parabolic antenna has a shape of a solid obtained by rotating the region between the curves $y = \frac{1}{8}x^2$, $y = \frac{1}{8}x^2 + \frac{1}{50}$, $x = 0$ and $x = 1$ around the y-axis. Find the volume of the antenna.

Problem 7. Use the method of cylindrical shells to find the volume of the solid from problem 2. Sketch the region and a typical shell.

Problem 8. A mathematician has a swimming pool. The top view of the pool has a shape of a ring with the inner radius $\frac{\pi}{2}$ meters and the outer radius $\frac{3\pi}{2}$ meters. The bottom of the pool has a shape generated by rotating the curve $y = \cos x$, $x \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ about the y-axis. Find the volume of the pool.

Assignment 6. Due on the week of March 31.

In the problems below round your answers to 6 decimal places.

Problem 1. Use (a) Midpoint Rule, (b) Trapezoidal Rule with $n = 5$ to estimate $\int_1^2 xe^x \mathrm{d}x$.

Problem 2. Compute $\int_1^2 xe^x \mathrm{d}x$ using integration by parts. Compare with the results of Problem 1. Find the error terms E_M and E_T.

Problem 3. What n should one take to guarantee that the Trapezoidal Rule for $\int_1^2 xe^x \mathrm{d}x$ gives an approximation accurate to within 10^{-4}?

Problem 4. Use Simpson's Rule with $n = 6$ to estimate

$$\int_{-1}^{0} \frac{\mathrm{d}t}{t^2 + t + 1}.$$

Problem 5. Find the exact value of

$$\int_{-1}^{0} \frac{\mathrm{d}t}{t^2 + t + 1}.$$

Compare with the result of Problem 4. Find the error term E_S.

Assignment 5. Due on the week of March 24.

Problem 1. Evaluate the integral using integration by parts with the indicated choices of u and $\mathrm{d}v$:

$$\int \tfrac{2\ln y}{y^2}\mathrm{d}y, \quad u = \ln y, \ \mathrm{d}v = \tfrac{2}{y^2}\mathrm{d}y.$$

Problem 2. Evaluate the integral:

$$\int t\cos(5t)\mathrm{d}t.$$

Problem 3. A particle moving along a straight line has velocity $V(t) = te^{-t/10}$ m/s after t seconds. Find the distance traveled by the particle after 20 seconds.

Problem 4. (a) Use the substitution $x = 1 - t^2$ to evaluate the integral

$$\int \frac{t}{\sqrt{1-t^2}}\mathrm{d}t.$$

(b) Use the result of part (a) to find the integral

$$\int \sin^{-1} t\mathrm{d}t.$$

Problem 5. Given that $f(x)$ is a differentiable function on $[0,3]$ such that $\int\limits_0^3 xf'(x)\mathrm{d}x = 10$ and $f(3) = 4$. Find $\int\limits_0^3 f(x)\mathrm{d}x$.

Problem 6. Evaluate the integrals

$$(a) \int \sin^3 x\mathrm{d}x, \quad (b) \int \sin^4 x\mathrm{d}x$$

Problem 7. Use long division to evaluate the integral:

$$\int\limits_1^5 \frac{t^2+1}{t+1}\mathrm{d}t.$$

Problem 8. A population of a small country is growing with the rate $r(t) = \frac{1}{(t+5)(t+10)}$ million people per year. At the present moment ($t = 0$

years) the population is one million people. Find the population in 10 years.

Problem 9. Evaluate the integral $\int \frac{1}{x^2\sqrt{1+x^2}}\mathrm{d}x$ by using the substitution $x = \tan t$.

Assignment 4. Due on the week of March 10.

Problem 1. Evaluate the integral by making the given substitution:

$$\int \frac{2e^x}{e^x - 3}\mathrm{d}x, \quad u = e^x - 3.$$

Problem 2. Evaluate the integral by making the given substitution:

$$\int x\cos(x^2)\mathrm{d}x, \quad t = x^2.$$

Problem 3. Evaluate the integral:

$$\int \frac{\tan^{-1} t}{t^2 + 1}\mathrm{d}t.$$

Problem 4. Evaluate the definite integral:

$$\int\limits_1^2 \frac{1}{(2x+1)^2}\mathrm{d}x.$$

Problem 5. Evaluate the definite integral:

$$\int\limits_1^{e^4} \frac{3\sqrt{\ln x}}{x}\mathrm{d}x.$$

Problem 6. An oil storage tank ruptures at time $t = 0$ and oil leaks from the tank at a rate of $r(t) = 20e^{-0.02t}$ liters per minute. How much oil leaks during the first 10 minutes?

Problem 7. Show that $\frac{\sin t}{t^4+1}$ is an odd function. Use this and properties of definite integrals to show that

$$0 \leq \int\limits_{-1}^{2} \frac{\sin t}{t^4 + 1}\mathrm{d}t \leq \frac{1}{2}.$$

Assignment 3. Due on the week of Feb 24. However, it is highly recommended that you do the assignment before Midterm 1 (for instance, on the weekend of Feb 22-23, after we cover all the material).

Problem 1. Evaluate the integral:

$$\int_{1}^{4} \left(\sqrt{\tfrac{1}{x}} - 2\right) \mathrm{d}x.$$

Problem 2. Evaluate the integral:

$$\int_{-\frac{1}{2}}^{\frac{\sqrt{2}}{2}} \left(\frac{1}{2\sqrt{1-t^2}} + 4t^3\right) \mathrm{d}t.$$

Problem 3. The velocity function (in meters per second) for a particle moving along a line is given:

$$v(t) = 10t^{\frac{1}{2}} - 20, \quad 1 \leqslant t \leqslant 9.$$

Find (a) the displacement and (b) the distance traveled by the particle during the given time interval.
Notice: if $v(t)$ changes sign it means that the particle starts moving in an opposite direction.

Problem 4. Verify by differentiation that the formula is correct:

$$\int e^t \sin t \mathrm{d}t = \tfrac{1}{2}(e^t \sin t - e^t \cos t) + C.$$

Problem 5. Evaluate the limit by first recognizing the sum as a Riemann sum of a function defined on $[0, \frac{\pi}{2}]$:

$$\lim_{n\to\infty} \tfrac{\pi}{2n} \sum_{i=1}^{n} \sin \tfrac{i\pi}{2n}.$$

Problem 6. Use the Fundamental Theorem of Calculus to find the derivative of a function:

$$g(x) = \int_{-1}^{x} \frac{1}{e^t + 1} \mathrm{d}t.$$

Problem 7. Use the Fundamental Theorem of Calculus to find the derivative of a function:

$$g(x) = \int\limits_{x^4}^{1} 2\sin(t^2)\mathrm{d}t.$$

Assignment 2. Reminder: you need to show you work.

Problem 1. Express $\int\limits_{-5}^{-1} x \sin x \mathrm{d}x$ as a limit using left end-points.

Problem 2. Write $\lim\limits_{n\to\infty} \frac{2}{n} \sum\limits_{i=1}^{n} 3^{1+i/n}$ as an integral.

Problem 3. Find $\int\limits_{0}^{3} (1 - \frac{u}{2}) \mathrm{d}u$ using geometry.

Problem 4. Approximate $\int\limits_{1}^{4} \frac{\ln t}{t} \mathrm{d}t$ using the midpoint rule with $n = 6$.

Problem 5. Find $\int\limits_{-2}^{1} f(x) \mathrm{d}x$ where

$$f(x) = \begin{cases} \sqrt{1 - x^2}, & \text{if } 0 \leqslant x \leqslant 1, \\ -1 + x, & \text{if } -2 \leqslant x < 0. \end{cases}$$

Problem 6. Show that $\int\limits_{1}^{5} \ln x \geqslant \frac{\ln 5}{2}$.

Assignment 1. Due on your recitation on the week of February 3.

Problem 1. Estimate the area under the graph of $f(x) = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$ using 6 approximating rectangles and left end-points. Sketch the graph and the rectangles. Is your estimate an underestimate or an overestimate?

Problem 2. Determine a region whose area is equal to the given limit:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \tfrac{1}{n} \cos\left(\pi + \tfrac{2i}{n}\right).$$

Do not evaluate the limit.

Problem 3. Express the area under the curve $y = \frac{1}{x}$ from 2 to 4 as a limit with sample points chosen to be the midpoints of the corresponding segments. Do not evaluate the limit.

Problem 4. Speedometer readings $v(t)$ ft/s of a car at sample times are:

$$v(0) = 30,\ v(4) = 32,\ v(10) = 35,\ v(13) = 31,\ v(17) = 28,\ v(20) = 25.$$

Estimate the distance traveled by the car in 20 seconds approximating the speed on each time interval by a) the initial speed on this interval, b) the ending speed.

Problem 5. Exercise 22 for section 5.1 of the course book.

Assignment 10 Solutions

1. $F = mg$ gravitational force

$W = Fd = mgd.$ $36\text{ cm} = 0.36\text{ m}$, $40\text{ cm} = 0.4\text{ m}$

Work done by Jim:

$10 \cdot 100 \cdot 9.8 \cdot 0.36 = 3528\text{ J}$

Work done by John:

$8 \cdot 120 \cdot 9.8 \cdot 0.4 = 3763.20\text{ J}$

Thus, John's done more work

2. $W = \int_a^b F(x)\,dx$

$$W = \int_0^5 2e^{-3x}\,dx = \frac{2e^{-3x}}{-3}\Big]_0^5 = -\frac{2}{3}e^{-15} + \frac{2}{3} \approx$$

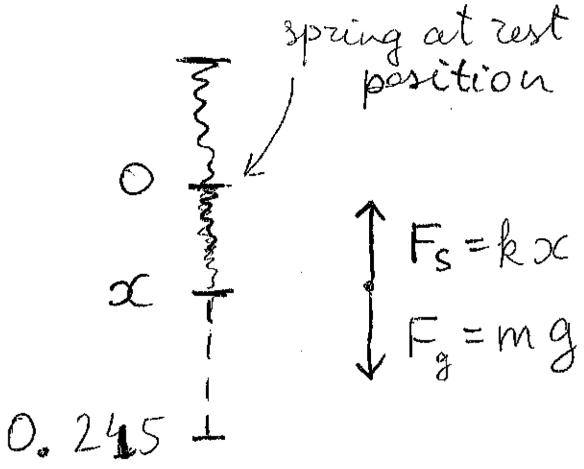
$0.6666646 \approx \frac{2}{3}$ ft-lb

3. $24.5\text{ cm} = 0.245\text{ m}$

When stretched to 0.245 m gravity and spring elasticity are equal:

$0.245\,k = 1 \cdot 9.8$

$k = \frac{9.8}{0.245} = 40\text{ N/m}$



The resulting force acting on the weight at the point when the spring is stretched x meters below its rest :

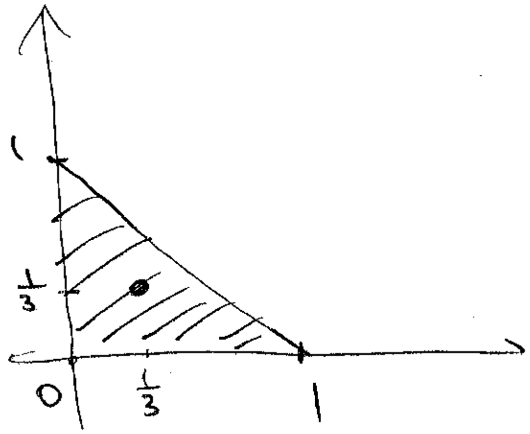
$$F = F_g - F_s = mg - kx = 9.8 - 40x.$$

Thus, $$W = \int_0^{0.245} (9.8 - 40x)\,dx =$$

$$= (9.8x - 20x^2)\Big]_0^{0.245} = 1.2005 \text{ J}$$

4. a) $f(x) = 1 - x$,

$a = 0$, $b = 1$



$$M_y = \rho \int_0^1 x(1-x)\,dx =$$

$$\rho\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big]_0^1 = \frac{\rho}{6}$$

$$M_x = \rho \int_0^1 \frac{1}{2}(1-x)^2\,dx = -\frac{\rho}{2} \cdot \frac{(1-x)^3}{3}\Big]_0^1 =$$

$$m = \rho \cdot A$$

$$= \frac{\rho}{6}$$

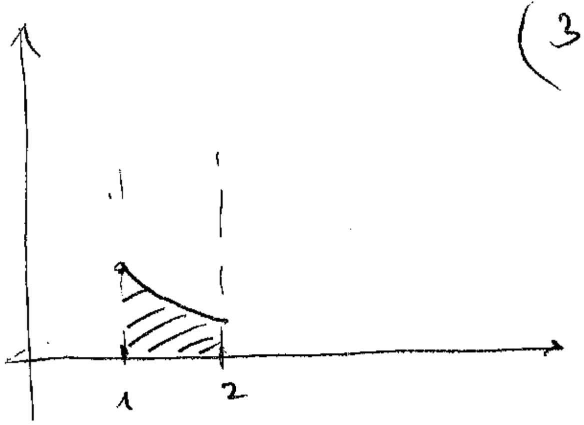
$$A = \int_0^1 (1-x)\,dx = \left(x - \frac{x^2}{2}\right)\Big]_0^1 = \frac{1}{2}.$$

(Or area of the triangle formula: $A = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$)

$$\bar{x} = \frac{M_y}{m} = \frac{\frac{\rho}{6}}{\frac{1}{2}} = \frac{1}{3}, \quad \bar{y} = \frac{M_x}{m} = \frac{1}{3}$$

answer $\left(\frac{1}{3}, \frac{1}{3}\right)$

b) $f(x) = \frac{1}{x}$



$M_y = \rho \int_1^2 x \cdot \frac{1}{x}\, dx =$

$= \rho \int_1^2 dx = \rho$

$M_x = \rho \int_1^2 \frac{1}{2} \cdot \left(\frac{1}{x}\right)^2 dx =$

$= -\frac{\rho}{2x}\Big]_1^2 = -\frac{\rho}{4} + \frac{\rho}{2} = \frac{\rho}{4}.$

$m = \rho A \quad , A = \int_1^2 \frac{1}{x}\, dx = \ln 2 - \ln 1 \approx 0.693$

$\bar{x} = \frac{M_y}{m} = \frac{1}{\ln 2} \approx 1.4427$

$\bar{y} = \frac{M_x}{m} = \frac{\frac{1}{4}}{\ln 2} \approx 0.3607$

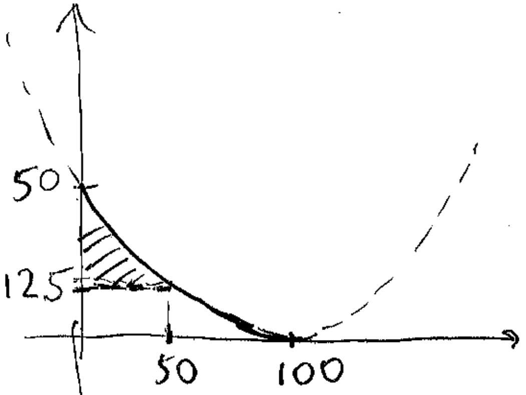
answer $\left(\frac{1}{\ln 2}, \frac{1}{4\ln 2}\right) \approx (1.4427, 0.3607)$

5. a) $50 - x + 0.005x^2 = 0.005(x^2 - 200) + 50 =$

$= 0.005\left((x-100)^2 - 100^2\right) + 50 = 0.005(x-100)^2$

a parabola with the vertex $(100, 0)$

$X = 50$, $P = p(X) = 12.5$



$CS = \int_0^X (p(x) - P)\, dx =$

$\int_0^{50} \left(0.005(x-100)^2 - 12.5\right) dx =$

$$0.005\,\frac{(x-100)^3}{3}\Big]_0^{50} - 50\cdot 12.5 \approx 833.33$$

(4

Answer $CS \simeq 833.3$

b) To graph $100e^{-0.01x}-50$ take the graph of e^{-x}, stretch by factor 100 in y direction and by factor $\frac{1}{0.01}=100$ in x direction, shift by -50 in y direction

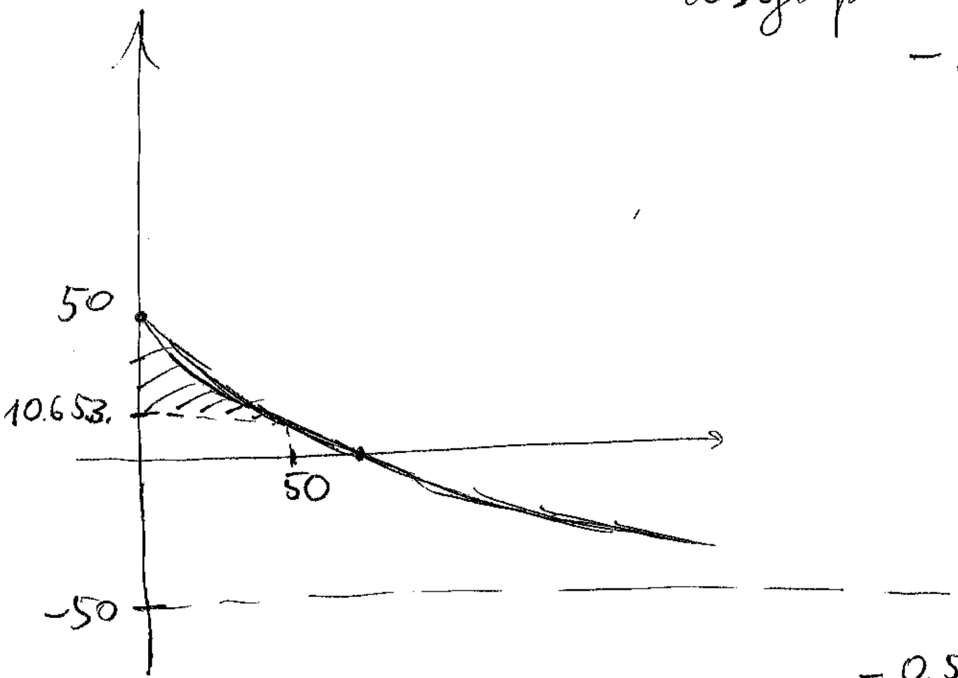
asymptote when $x\to\infty$: -50.

x-intercept:

$e^{-0.01x}=\frac{1}{2}$,

$x=-\frac{\ln\frac{1}{2}}{0.01}=\frac{\ln 2}{0.01}\simeq$

69.31



$$P=p(X)=p(50)=100e^{-0.5}-50\simeq 10.653$$

$$CS=\int_0^X (p(x)-P)\,dx=\int_0^{50}(100e^{-0.01x}-50-10.653)\,dx=$$

$$=-\frac{100}{0.01}e^{-0.01x}\Big]_0^{50}-60.653\cdot 50=$$

$$-10^4e^{-0.5}+10^4-3032.65\simeq 902.04.$$

Answer $CS \simeq 902.04$

(5

6. $P(3 \le X \le 7) = \int_3^7 f(x)\,dx =$

$$\int_3^7 (0.06x - 0.006x^2)\,dx = \left(0.03x^2 - 0.02x^3\right)\Big]_3^7 =$$

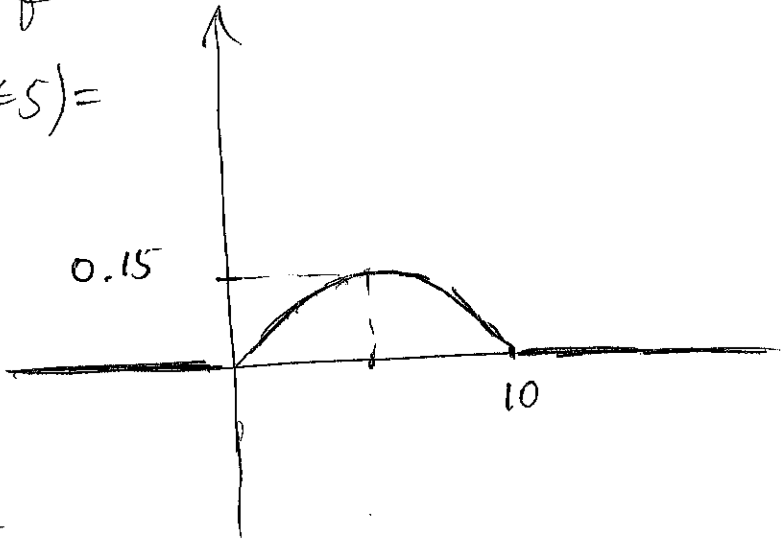
$$= 0.03\cdot 49 - 0.02\cdot 343 - (0.03\cdot 9 - 0.02\cdot 27) = 0.568$$

$$P(X \le 5) \overset{\substack{\text{since } f(x)=0 \text{ for } x<0 \\ \downarrow}}{=} P(0 \le X \le 5) =$$

$$\int_0^5 (0.06x - 0.006x^2)\,dx =$$

$$\left(0.03x^2 - 0.02x^3\right)\Big]_0^5 =$$

$$0.03\cdot 5^2 - 0.02\cdot 5^3 = 0.5$$



$P(3 \le X \le 7) > P(X \le 5)$. Thus, it is more likely to spend 3 to 7 minutes.

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{10} x\cdot(0.06x - 0.006x^2)\,dx =$$

$$= \left(0.02x^3 - 0.0015x^4\right)\Big]_0^{10} = 20 - 15 = 5.$$

So, the average time is 5 minutes.

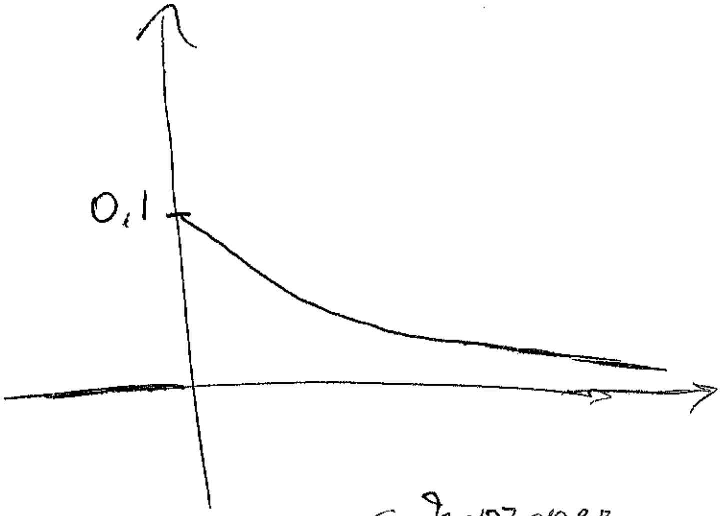
since f(x)=0 for x<0

7. $P(X \le 10) \overset{\downarrow}{=} P(0 \le X \le 10) =$

$$\int_0^{10} 0.1 e^{-0.1x}\,dx =$$

$$= \frac{0.1}{-0.1} e^{-0.1x}\Big]_0^{10} =$$

$$= -e^{-1} + 1 \approx \boxed{0.632}$$



The average time: ← Improper

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} 0.1 \cdot x \cdot e^{-0.1x}\,dx =$$

$$\lim_{t\to\infty} \int_0^t 0.1 x e^{-0.1x}\,dx =$$

By parts: $u = x$
$dv = 0.1 e^{-0.1x}dx$
$v = -e^{-0.1x}dx$

$$\lim_{t\to\infty} \left(-xe^{-0.1x}\Big]_0^t + \int_0^t e^{-0.1x}\,dx \right) =$$

$$= -\lim_{t\to\infty} \frac{t}{e^{0.1t}} - \lim_{t\to\infty} \frac{1}{0.1} e^{-0.1x}\Big]_0 =$$

$$= \underbrace{-\lim_{t\to\infty} \frac{t}{e^{0.1t}}}_{\text{L'Hospitals rule}} + 10 - \boxed{\lim_{t\to\infty} 10e^{-0.1t}} \;\; \bigcirc\!\!\!\!= \;\; 10.$$

$= 0$

$$\lim_{t\to\infty} \frac{t}{e^{0.1t}} = \lim_{t\to\infty} \frac{1}{0.1 e^{0.1t}} = \frac{1}{\infty} = 0.$$

The average waiting time is 10.

MAT 126 Calculus Spring 2007
Practice Final Exam

Answer each question in the space provided and on the reverse side of the sheets. Show your work whenever possible. Unless otherwise indicated, **answers without justification will get little or no partial credit!** Cross out anything that grader should ignore and circle or box the final answer. You **do not need** to simplify numerical answers or write their approximate values. This practice exam contains more problems then the actual test to give you more practice.

1. Evaluate the following definite integrals:

(a)

$$\int_1^9 \ln\sqrt{x}\,dx$$

(b)

$$\int_0^2 \frac{x}{1+2x^2}\,dx$$

(c)

$$\int_1^e \frac{(\ln x)^3}{x}\,dx$$

(d)

$$\int_{-1}^1 x^2\sin(x^5)\,dx$$

(e)

$$\int_0^{1/2} \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$$

(f)

$$\int_1^4 \sqrt{t}\ln t\,dt$$

(g)

$$\int_0^{13} \frac{dx}{\sqrt[3]{(1+2x)^2}}$$

(h)

$$\int_0^{\frac{\pi}{2}} \sin^3 x\,dx$$

2. Evaluate the following indefinite integrals:

(a)

$$\int x^2e^x\,dx$$

(b)

$$\int \frac{2x^3+1}{x^2+1}\,dx$$

(c)

$$\int \frac{\tan^{-1} x}{1+x^2}\,dx$$

(d)

$$\int \sin^{-1} x\,dx$$

(e)

$$\int \frac{x-1}{x^2+3x+2}\,dx$$

(f)

$$\int t^2\cos(1-t^3)\,dt$$

(g)

$$\int e^x\sqrt[3]{1+e^x}\,dx$$

(h)

$$\int \cos^5 x\,dx$$

3. (a) Write a formula for $\cos^2 x$ in terms of $\sin^2 x$.

(b) Evaluate

$$\int \cos^3 x\sin^2 x\,dx$$

4. Let

$$f(x) = \int_2^{\sqrt{x}} \frac{\sin t}{t} dt + x^2$$

(a) Find $f'(x)$.

(b) Evaluate $f(4)$.

5. Find a function f and a number a such that for x,

$$1 + \int_a^x tf(t)\,dt = x^3$$

6. (a) Let

$$I = \int_0^4 e^{x^2}dx$$

For any value of n list the numbers L_n, R_n, M_n, T_n and I in increasing order.

(b) Repeat part (a) for

$$I = \int_0^{\sqrt{2}/2} e^{-x^2} dx$$

7. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

(a)

$$\int_0^\infty e^{-x}\, dx$$

(b)

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx$$

(c)

$$\int_0^3 \frac{1}{x\sqrt{x}}\, dx$$

(d)

$$\int_{-\infty}^\infty xe^{-x^2}\, dx$$

(e)

$$\int_0^1 \frac{1}{4y-1}\, dy$$

8. Find the area of the region bounded by the curves:

(a) $y = x^2$ and $y = x^4$.

(b) $x + y^2 = 2$ and $x + y = 0$.

9. (a) Find the volume of the solid of revolution obtained by rotating the region bounded by the curves $y = x^2$ and $y^2 = x$ about the x-axis.

(b) Find the volume of the solid of revolution obtained by rotating the region bounded by $y = \sec x$, $y = 1$, $x = -1$ and $x = 1$ about the x-axis.

10. Find the length of the following curves:

(a) $y = x^{3/2}$, $0 \leq x \leq 2$.

(b)

$$y = \frac{x^2}{4} - \frac{\ln x}{2}, \quad 1 \leq x \leq 2$$

11. Find the average value f_{ave} of f on the given interval.

(a) $f(x) = x\sin(x^2)$ on $[0, \sqrt{\pi}]$.

(b) $f(x) = 4 - x^2$ on $[0, 3]$.

(c) For f as in part (b) find the number c in $[0, 3]$ such that $f(c) = f_{ave}$.

MAT 126 Calculus B Spring 2007
Practice Final Exam — Solutions

Answer each question in the space provided and on the reverse side of the sheets. Show your work whenever possible. Unless otherwise indicated, **answers without justification will get little or no partial credit!** Cross out anything that grader should ignore and circle or box the final answer. You **do not need** to simplify numerical answers or write their approximate values. This practice exam contains more problems then the actual test to give you more practice.

1. Evaluate the following definite integrals:

(a)

$$\int_1^9 \ln\sqrt{x}\,dx$$

$$\int_1^9 \ln\sqrt{x}\,dx = \frac{1}{2}\int_1^9 \ln x\,dx = \frac{1}{2}\left(x\ln x - x\right)\Big|_1^9$$
$$= \frac{1}{2}(9\ln 9 - 9 - (\ln 1 - 1)) = 9\ln 3 - 4,$$

where we have used that the antiderivative of $\ln x$ is $x\ln x - x$ (see Section 5.6, Example 2).

(b)

$$\int_0^2 \frac{x}{1+2x^2}\,dx$$

Using the substitution $u = 1 + 2x^2$ we get $du = 4xdx$, so that $xdx = \frac{1}{4}du$, and the limits of integration $x = 0$ and $x = 2$ correspond to $u = 1$ and $u = 9$. We get

$$\int_0^2 \frac{x}{1+2x^2}\,dx = \frac{1}{4}\int_1^9 \frac{du}{u}$$
$$= \frac{1}{4}\ln u\Big|_1^9 = \frac{1}{2}\ln 3.$$

(c)

$$\int_1^e \frac{(\ln x)^3}{x}\,dx$$

The substitution $u = \ln x$ gives $du = \frac{dx}{x}$ and limits of integration $x = 1$ and $x = e$ correspond to $u = 0$ and $u = 1$. We have

$$\int_1^e \frac{(\ln x)^3}{x}\,dx = \int_0^1 u^3 du = \frac{1}{4}\,u^4\Big|_0^1 = \frac{1}{4}.$$

(d)

$$\int_{-1}^{1} x^2 \sin(x^5)\, dx$$

The function $f(x) = x^2 \sin(x^5)$ is odd, $f(-x) = -f(x)$, so using the property of symmetric functions (see Section 5.5), we get

$$\int_{-1}^{1} x^2 \sin(x^5)\, dx = 0.$$

(e)

$$\int_{0}^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx$$

Using the substitution $u = \sin^{-1} x$, we get $du = \dfrac{dx}{\sqrt{1-x^2}}$, and the limits of integration $x = 0$ and $x = 1/2$ correspond to $u = 0$ and $u = \pi/6$. We get

$$\int_{0}^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx = \int_{0}^{\pi/6} u\, du = \frac{1}{2}\, u^2\Big|_{0}^{\pi/6} = \frac{\pi^2}{72}.$$

(f)

$$\int_{1}^{4} \sqrt{t} \ln t\, dt$$

Here we use integration by parts with $u = \ln t$ and $dv = \sqrt{t}$. We have $du = \frac{dt}{t}$ and $v = \frac{2}{3} t^{3/2}$, so that

$$\begin{aligned}\int_{1}^{4} \sqrt{t} \ln t\, dt &= \int_{1}^{4} u\, dv = uv|_{1}^{4} - \int_{1}^{4} vdu \\ &= \frac{2}{3}(16 \ln 2) - \frac{2}{3} \int_{1}^{4} \sqrt{t}\, dt \\ &= \frac{32}{3} \ln 2 - \frac{4}{9}\, t^{3/2}\Big|_{1}^{4} = \frac{32}{3} \ln 2 - \frac{28}{9}.\end{aligned}$$

(g)

$$\int_{0}^{13} \frac{dx}{\sqrt[3]{(1+2x)^2}}$$

We use the substitution $u = 1 + 2x$, so that $du = 2dx$ and the limits of integration $x = 0$ and $x = 13$ correspond to $u = 1$ and $u = 27$. We get

$$\int_{0}^{13} \frac{dx}{\sqrt[3]{(1+2x)^2}} = \frac{1}{2} \int_{1}^{27} u^{-\frac{2}{3}}\, du = \frac{1}{2}\, 3u^{\frac{1}{3}}\Big|_{1}^{27} = 3.$$

(h)

$$\int_0^{\frac{\pi}{2}} \sin^3 x\, dx$$

This is a trigonometric integral. Writing

$$\sin^3 x = \sin^2 x \sin x = (1 - \cos^2 x) \sin x,$$

we recognize the substitution $u = \cos x$. We have $du = -\sin x dx$, and the limits of integration $x = 0$ and $x = \frac{\pi}{2}$ correspond to $u = 1$ and $u = 0$. We get

$$\int_0^{\frac{\pi}{2}} \sin^3 x\, dx = -\int_1^0 (1 - u^2) du = \int_0^1 (1 - u^2) du = \left(u - \frac{u^3}{3} \right)\Big|_0^1 = \frac{2}{3}.$$

2. Evaluate the following indefinite integrals:

(a)

$$\int x^2 e^x\, dx$$

This is Example 3 in Section 5.6.

(b)

$$\int \frac{2x^3 + 1}{x^2 + 1}\, dx$$

Doing long division, or simplifying as follows:

$$\frac{2x^3 + 1}{x^2 + 1} = \frac{(2x^3 + 2x) + 1 - 2x}{x^2 + 1} = 2x + \frac{1 - 2x}{x^2 + 1},$$

we get

$$\begin{aligned} \int \frac{2x^3 + 1}{x^2 + 1}\, dx &= \int \left(2x + \frac{1 - 2x}{x^2 + 1} \right) dx \\ &= x^2 + \int \frac{1}{x^2 + 1}\, dx - 2 \int \frac{x}{x^2 + 1}\, dx \\ &= x^2 + \tan^{-1} x - \ln(x^2 + 1) + C, \end{aligned}$$

where in the last integral we have used the substitution $u = x^2 + 1$.

(c)

$$\int \frac{\tan^{-1} x}{1 + x^2}\, dx$$

Using the substitution $u = \tan^{-1} x$, we get $du = \dfrac{dx}{1 + x^2}$, so that

$$\int \frac{\tan^{-1} x}{1 + x^2}\, dx = \int u\, du = \frac{1}{2} u^2 + C = \frac{1}{2} (\tan^{-1} x)^2 + C.$$

(d)

$$\int \sin^{-1} x\, dx$$

Using integration by parts with $u = \sin^{-1} x$ and $dv = dx$ we get

$$du = \frac{dx}{\sqrt{1-x^2}} \quad \text{and} \quad v = x,$$

so that

$$\begin{aligned}\int \sin^{-1} x\, dx = \int u\, dv = uv - \int v\, du \\ = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx.\end{aligned}$$

To compute the last integral, use the substitution $u = 1 - x^2$, so that $du = -2xdx$ and

$$\int \frac{x}{\sqrt{1-x^2}} dx = -\frac{1}{2} \int \frac{du}{\sqrt{u}} = -\sqrt{u} + C = -\sqrt{1-x^2} + C.$$

Thus, finally,

$$\int \sin^{-1} x\, dx = x \sin^{-1} x + \sqrt{1-x^2} + C.$$

(e)

$$\int \frac{x-1}{x^2+3x+2}\, dx$$

Using partial fractions,

$$\frac{x-1}{x^2+3x+2} = \frac{x-1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2},$$

where A and B are such that

$$x - 1 = A(x+2) + B(x+1)$$

holds for all x. Setting $x = -1$, we get $A = -2$, and setting $x = -2$, we get $B = 3$. Thus

$$\begin{aligned}\int \frac{x-1}{x^2+3x+2}\, dx = \int \left(-\frac{2}{x+1} + \frac{3}{x+2} \right) dx \\ = -\log(x+1)^2 + \log|x+2|^3 + C.\end{aligned}$$

(f)

$$\int t^2 \cos(1-t^3)\, dt$$

Using the substitution $u = 1 - t^3$ we get $du = -3t^2 dt$ and

$$\int t^2 \cos(1-t^3)\, dt = -\frac{1}{3}\int \cos u\, du = -\frac{1}{3}\sin u + C = -\frac{1}{3}\sin(1-t^3) + C.$$

(g)

$$\int e^x \sqrt[3]{1+e^x}\, dx$$

Using the substitution $u = 1 + e^x$ we get $du = e^x dx$ and

$$\int e^x \sqrt[3]{1+e^x}\, dx = \int u^{\frac{1}{3}}\, du = \frac{3}{4}u^{\frac{4}{3}} + C = \frac{3}{4}(1+e^x)^{\frac{4}{3}} + C.$$

(h)

$$\int \cos^5 x\, dx$$

This is a trigonometric integral. Writing

$$\cos^5 x = \cos^4 \cos x = (1 - \sin^2 x)^2 \cos x,$$

we recognize the substitution $u = \sin x$. We have $du = \cos x dx$ and

$$\int \cos^5 x\, dx = \int (1 - \sin^2 x)^2 \cos x\, dx = \int (1-u^2)^2\, du = \int (1 - 2u^2 + u^4) du$$
$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.$$

3. (a) Write a formula for $\cos^2 x$ in terms of $\sin^2 x$.

$$\cos^2 x = 1 - \sin^2 x.$$

(b) Evaluate

$$\int \cos^3 x \sin^2 x\, dx$$

$$\int \cos^3 x \sin^2 x\, dx = \int (1 - \sin^2 x)\sin^2 x \cos x\, dx,$$

and using the substitution $u = \sin x$, we get $du = \cos x\, dx$ and

$$\int \cos^3 x \sin^2 x\, dx = \int u^2(1-u^2) du = \int (u^2 - u^4) du$$
$$= \frac{u^3}{3} - \frac{u^5}{5} + C$$
$$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.$$

4. Let

$$f(x) = \int_2^{\sqrt{x}} \frac{\sin t}{t} dt + x^2$$

(a) Find $f'(x)$.
Using the chain rule with $u = \sqrt{x}$ and the Fundamental Theorem of Calculus, we get

$$\begin{aligned} \frac{df}{dx}(x) &= \frac{d}{du}\left(\int_2^u \frac{\sin t}{t}\, dt\right) \frac{du}{dx}\bigg|_{u=\sqrt{x}} + 2x \\ &= \frac{\sin\sqrt{x}}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} + 2x = \frac{\sin\sqrt{x}}{2x} + 2x. \end{aligned}$$

(b) Evaluate $f(4)$.

$$f(4) = \int_2^{\sqrt{4}} \frac{\sin t}{t} dt + 4^2 = \int_2^2 \frac{\sin t}{t}\, dt + 16 = 16.$$

5. Find a function f and a number a such that for x,

$$1 + \int_a^x t f(t)\, dt = x^3$$

Setting in the equation $x = a$, we get $1 = a^3$, so that $a = 1$. Differentiating both sides of the equation with respect to x and using the Fundamental Theorem of Calculus, we get

$$x f(x) = 3x^2,$$

so that $f(x) = 3x$.

6. (a) Let

$$I = \int_0^4 e^{x^2} dx$$

For any value of n list the numbers L_n, R_n, M_n, T_n and I in increasing order.
The function $f(x) = e^{x^2}$ is increasing and concave upward on the real line (check it using the second derivative test). From the graph (sketch it!) we get

$$L_n < M_n < I < T_n < R_n.$$

(Here we used the analog of Fig. 5 on p. 419 (sketch it!) to get the relation $M_n < I < T_n$).

(b) Repeat part (a) for

$$I = \int_0^{\sqrt{2}/2} e^{-x^2} dx$$

The function $f(x) = e^{-x^2}$ is decreasing and concave downward when $0 \le x \le \sqrt{2}/2$ (check it using the second derivative test) . From the graph and analog of Fig. 5 (sketch them!) we get

$$R_n < T_n < I < M_n < L_n.$$

7. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

(a)

$$\int_0^\infty e^{-x}\,dx$$

The integral $\int_0^t e^{-x}\,dx$ exists for every number $t \ge 0$ and

$$\begin{aligned}\lim_{t\to\infty}\int_0^t e^{-x}\,dx &= \lim_{t\to\infty}\left(-e^{-x}\right)\Big|_0^t \\ &= \lim_{t\to\infty}(1 - e^{-t}) = 1.\end{aligned}$$

The improper integral of Type 1 is convergent and

$$\int_0^\infty e^{-x}\,dx = 1.$$

(b)

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

The integral $\int_t^1 \frac{1}{\sqrt{x}}\,dx$ exists for every number $t > 0$ and

$$\begin{aligned}\lim_{t\to 0}\int_t^1 \frac{1}{\sqrt{x}}\,dx &= \lim_{t\to 0}\left(2\sqrt{x}\right)\Big|_t^1 \\ &= 2\lim_{t\to 0}(1 - \sqrt{t}) = 2.\end{aligned}$$

The improper integral of Type 2 is convergent and

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2.$$

(c)

$$\int_0^3 \frac{1}{x\sqrt{x}}\,dx$$

The integral $\int_t^3 \frac{1}{x\sqrt{x}}\,dx$ exists for every number $t > 0$ and

$$\lim_{t\to 0}\int_t^3 \frac{1}{x\sqrt{x}}\,dx = \lim_{t\to 0}\left(-2x^{-1/2}\right)\Big|_t^3 = -\frac{2}{\sqrt{3}} + 2\lim_{t\to 0}\frac{1}{\sqrt{t}}.$$

The last limit is ∞ — does not exist as a finite number, so that improper integral of Type 2 is divergent.

(d)

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx$$

Since

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \int_{-\infty}^{0} xe^{-x^2}\,dx + \int_{0}^{\infty} xe^{-x^2}\,dx,$$

we must evaluate both integrals separately. Using the substitution $u = x^2$ (one could also use $u = -x^2$), we get $du = 2xdx$ and

$$\int_0^{\infty} xe^{-x^2}\,dx = \lim_{t\to\infty}\int_0^t xe^{-x^2}\,dx = \frac{1}{2}\lim_{t\to\infty}\int_0^{t^2} e^{-u}du = \frac{1}{2}\lim_{t\to\infty}\left(-e^{-u}\right)\Big|_0^{t^2} = \frac{1}{2}\lim_{t\to\infty}(1 - e^{-t^2}) = \frac{1}{2}.$$

Using this result and the fact that xe^{-x^2} is an odd function we get

$$\int_{-\infty}^{0} xe^{-x^2}\,dx = \lim_{t\to-\infty}\int_t^0 xe^{-x^2}\,dx = -\lim_{-t\to\infty}\int_0^{-t} xe^{-x^2}\,dx = -\frac{1}{2}.$$

Thus the improper integral of Type 1 is convergent and

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \frac{1}{2} + \left(-\frac{1}{2}\right) = 0.$$

(e)

$$\int_0^1 \frac{1}{4y-1}\,dy$$

The integrand is $f(y) = \frac{1}{4y-1}$, and it is discontinuous (blows up) at $y = \frac{1}{4}$. Thus

$$\int_0^1 \frac{1}{4y-1}\,dy = \int_0^{\frac{1}{4}} \frac{1}{4y-1}\,dy + \int_{\frac{1}{4}}^1 \frac{1}{4y-1}\,dy,$$

and we need to investigate both improper integrals of Type 2. For the first integral we have, using the substitution $u = 4y - 1, du = 4dy$,

$$\begin{aligned}\int_0^{\frac{1}{4}} \frac{1}{4y-1}\,dy = \lim_{t\to\frac{1}{4}} \int_0^t \frac{1}{4y-1}\,dy &= \frac{1}{4} \lim_{t\to\frac{1}{4}} \int_{-1}^{4t-1} \frac{1}{u}\,du \\ \frac{1}{4} \lim_{t\to\frac{1}{4}} \ln|u|\Big|_{-1}^{4t-1} &= \frac{1}{4} \lim_{t\to\frac{1}{4}} (\ln|4t-1| - \ln|-1|) \\ &= \frac{1}{4} \lim_{t\to\frac{1}{4}} \ln|4t-1| = -\infty,\end{aligned}$$

since $\ln 0 = -\infty$. Thus the first improper integral is divergent, so that the integral in question is also divergent.

8. Find the area of the region bounded by the curves:

(a) $y = x^2$ and $y = x^4$.

The curves intersect at the points $x = -1, 0, 1$ and the top and bottom boundaries of the enclosed region are $y = x^2$ and $y = x^4$ (sketch the graph!). We have

$$\begin{aligned}A &= \int_{-1}^1 (x^2 - x^4)dx = 2\int_0^1 (x^2 - x^4)dx \\ &= 2\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Bigg|_0^1 = \frac{4}{15}.\end{aligned}$$

(b) $x + y^2 = 2$ and $x + y = 0$.

The curves intersect at the points with coordinates $(-2, 2)$ and $(1, -1)$, and the top and bott om boundaries of the enclosed region are $x = 2 - y^2$ and $x = -y$, where we are using y as an independent variable (sketch the graph!). We have

$$\begin{aligned}A &= \int_{-1}^2 (2 - y^2 - (-y))dy = \int_{-1}^2 (2 - y^2 + y)dy \\ &= \left(2y - \frac{y^3}{3} + \frac{y^2}{2}\right)\Bigg|_{-1}^2 = 4\frac{1}{2}.\end{aligned}$$

9. (a) Find the volume of the solid of revolution obtained by rotating the region bounded by the curves $y = x^2$ and $y^2 = x$ about the x-axis.

The curves $y = x^2$ and $y = \sqrt{x}$ (we solved the second equation for y, which is assumed to be positive) intersect at $x = 0$ and $x = 1$. The region has the curve $y = \sqrt{x}$ as the top boundary and the curve $y = x^2$ as the bottom boundary (sketch the graph!). A cross-section is a washer with the inner radius x^2 and the outer radius $\sqrt{x}$. The cross-sectional area is $A(x) = \pi(x - x^4)$, and the volume of the solid of revolution is

$$\begin{aligned} V &= \int_0^1 A(x)dx = \pi \int_0^1 (x - x^4)dx \\ &= \pi \left(\frac{x^2}{2} - \frac{x^5}{5} \right)\Big|_0^1 = \frac{3\pi}{10}. \end{aligned}$$

(b) Find the volume of the solid of revolution obtained by rotating the region bounded by $y = \sec x$, $y = 1$, $x = -1$ and $x = 1$ about the x-axis.

The region has the curve $y = \sec x$ as the top boundary, the horizontal line $y = 1$ as the bottom boundary, and the lines $x = -1$ and $x = 1$ as the vertical boundaries (sketch the graph!). A cross-section is a washer with the inner radius 1 and the outer radius $\sec x$. The cross-sectional area is $A(x) = \pi(\sec^2 x - 1)$, and the volume of the solid of revolution is

$$\begin{aligned} V &= \int_{-1}^1 A(x)dx = \pi \int_{-1}^1 (\sec^2 x - 1)dx = 2\pi \int_0^1 (\sec^2 x - 1)dx \\ &= 2\pi \left(\tan x - x\right)\big|_0^1 = 2\pi(\tan 1 - 1). \end{aligned}$$

10. Find the length of the following curves:

(a) $y = x^{3/2}$, $0 \le x \le 2$.

$$L = \int_0^2 \sqrt{1 + (y')^2}\, dx = \int_0^2 \sqrt{1 + \tfrac{9}{4}x}\, dx.$$

Using the substitution $u = 1 + \frac{9}{4}x$, we get

$$\begin{aligned} L &= \frac{4}{9} \int_1^{11/2} \sqrt{u}\, du \\ &= \frac{4}{9} \cdot \frac{2}{3}\, u^{3/2}\big|_1^{11/2} = \frac{8}{27} \left(\left(\frac{11}{2} \right)^{3/2} - 1 \right). \end{aligned}$$

(b)

$$y = \frac{x^2}{4} - \frac{\ln x}{2}, \quad 1 \le x \le 2$$

$$L = L = \int_1^2 \sqrt{1+(y')^2}\,dx = \int_1^2 \sqrt{1+\frac{1}{4}\left(x-\frac{1}{x}\right)^2}\,dx.$$

We have, by simple algebra,

$$\begin{aligned} 1+\frac{1}{4}\left(x-\frac{1}{x}\right)^2 &= 1+\frac{1}{4}(x^2-2+x^{-2}) = \frac{1}{4}(x^2+x^{-2})+\frac{1}{2} \\ &= \frac{1}{4}(x^2+2+x^{-2}) = \frac{1}{4}(x+x^{-1})^2. \end{aligned}$$

Thus

$$L = \int_1^2 \sqrt{\frac{1}{4}(x+x^{-1})^2}dx = \frac{1}{2}\int_1^2 (x+x^{-1})dx = \frac{1}{2}\left(\frac{x^2}{2}+\ln x\right)\Bigg|_1^2 = \frac{3}{4}+\frac{\ln 2}{2}.$$

11. Find the average value f_{ave} of f on the given interval.

(a) $f(x) = x\sin(x^2)$ on $[0, \sqrt{\pi}]$.

We get, using the substitution $u = x^2$, $du = 2xdx$,

$$\begin{aligned} f_{ave} &= \frac{1}{\sqrt{\pi}}\int_0^{\sqrt{\pi}} x\sin(x^2)dx = \frac{1}{2\sqrt{\pi}}\int_0^{\pi} \sin u\,du \\ &= \frac{1}{2\sqrt{\pi}}\left(-\cos u\right)\Big|_0^{\pi} = \frac{1}{2\sqrt{\pi}}(-\cos\pi+\cos 0) = \frac{1}{\sqrt{\pi}}. \end{aligned}$$

(b) $f(x) = 4 - x^2$ on $[0, 3]$.

$$\begin{aligned} f_{ave} &= \frac{1}{3}\int_0^3 (4-x^2)\,dx = \frac{1}{3}\left(4x-\frac{x^3}{3}\right)\Bigg|_0^3 \\ &= \frac{1}{3}(12-9) = 1. \end{aligned}$$

(c) For f as in part (b) find the number c in $[0, 3]$ such that $f(c) = f_{ave}$.

Solving $4-c^2 = 1$ we get $c = \sqrt{3}$ as the only solution which belongs to the interval $[0, 3]$.

MAT126 Fall 2009

Practice Final

The actual Final exam will consist of twelve problems

Problem 1

1. Evaluate $\int_{\pi/3}^{\pi/2} \sin^3(x)\cos^2(x)dx$
2. Evaluate $\int \sin^2(x)dx$

Problem 2 1. Estimate the integral

$$\int_7^8 \frac{dx}{\ln(x)}$$

using three rectangles and

(a) right endpoints

(b) left endpoints

(c) Are your answers in 1a and 1b over- or under-estimates of the actual integral?

2. Do the same for the under-integral function $f(t) = e^{t^3}$

Problem 3 Integrate

1.

$$\int \cos(\ln(x))dx$$

2.

$$\int e^{\sqrt{x}}dx$$

Problem 4 Find the following indefinite integrals

1. $$\int \frac{dx}{(x+1)\sqrt{x}}$$

2. $$\int \frac{(e^x+1)dx}{e^x(e^x+2)}$$

Problem 5 Find the following indefinite integrals

1. $$\int \frac{x^2+1}{x^2-3x+2}$$

2. $$\int \frac{x^3+1}{x^2-4x+3}$$

Problem 6 1. Use trapezoidal approximation with $n = 4$ intervals of subdivision (each of equal length) to estimate the following integral

$$\int_0^{\pi} \sin^2(x)dx.$$

Estimate the error of approximation.

2. Use Simpson's rule with $n = 4$ to estimate $\int_0^1 x^5 dx$. Find the precise error of approximation.

Problem 7 1. Does the following integral converge? If yes, evaluate it:

$$\int_0^{+\infty} \exp(-t)\cos^2(t)dt$$

2. Does the following integral converge? If yes, evaluate it:

$$\int_1^{+\infty} \frac{dt}{t(t+1)}$$

3. Does the following integral converge? If yes, evaluate it:

$$\int_1^{\infty} \frac{dx}{x[\ln x]^2}$$

Problem 8

1. Find the volumes of the bodies obtained from the region enclosed by

$$y = \sin(x), 0 \leq x \leq \pi$$

   and $y = 0$ by revolving about a) the x-axis, b) the line $x = -2$

2. Find the volumes of the bodies obtained from the region enclosed by

$$y = \frac{1}{x^5}, 1 \leq x < \infty,$$

   $y = 0$, $x = 1$ by revolving about a) the line $y = -2$, b) the y-axis.

Problem 9 Find the length of the curve given by the graph of the function

$$y = \ln(\cos(x))$$

between points $(0, 0)$ and $(a, \ln(\cos(a)))$, $0 < a < \frac{\pi}{2}$.

Problem 10

1. A cable that weights 2lb/ft is used to lift 800lb of coal from a mineshaft 500 ft deep. Find the work done.
2. A 10 ft chain weights 25 lb and hangs from the ceiling. Find the work done in lifting the middle of the chain to the ceiling so that it is level with the upper end.**Final**

Problem 11 A granary has the shape of a half cylinder lying on its rectangular side (the cut). The cylinder's height is 10m, and the radius of the base is 2m. If the granary is full of barley, with density $600kg/m^3$, how much work is done in removing all the grain via an opening at the top of the granary?

Problem 12 If $f(x)$ is an increasing function on [0, 1], rank the following in order from least to greatest:

- f(0)
- f(1)
- The left endpoint approximation to $\int_0^1 f(x)dx$ with $n = 5$ rectangles.
- The right endpoint approximation to $\int_0^1 f(x)dx$ with $n = 5$ rectangles.
- The average value of f on $[0, 1]$.

Problem 13 For each of the following, determine if the improper integral converges or diverges. If it converges, evaluate the integral.

1. $\int_{-\infty}^{\infty} 2xe^{-x^2}dx$
2. $\int_{-\infty}^{\infty} \frac{1}{2x}e^{-x^2}dx$

Problem 14 A particle starts at the origin at time $t = 0$, and traces out a path given by

$$\begin{aligned} x(t) &= t \\ y(t) &= 2t^2 \end{aligned}$$

for each $t \geq 0$.

1. Express the length $l(T)$ of the path traced out by the particle from $t = 0$ to a time $t = T$ as an integral, but do not evaluate it.
2. Find $l'(T)$, the speed of the particle at time T.
3. Evaluate $l'(2)$.

MAT126 Fall 2009

Practice Final

The actual Final exam will consist of twelve problems

Problem 1 1. Evaluate $\int_{\pi/3}^{\pi/2} \sin^3(x)\cos^2(x)dx$

2. Evaluate $\int \sin^2(x)dx$

Solution:

1. First, recognize that the integrand consists of $\sin(x)$ raised to an odd power, multiplied by $\cos(x)$ raised to an even power. Such a function is a prime candidate for the substitution $u = \cos(x)$ (so that one of the $\sin(x)$ can be used for the change of variable, and the rest may be expressed in terms of $\cos(x)$ via the identity $\sin^2(x) = 1 - \cos^2(x)$).

Therefore we substitute $u = \cos(x)$, whereby $du = -\sin(x)dx$, and we can write (noting that $u = \cos(\pi/3) = 1/2$ when $x = \pi/3$, and $u = \cos(\pi/2) = 0$ when $x = \pi/2$)

$$\begin{aligned}
\int_{x=\pi/3}^{x=\pi/2} \sin^3(x)\cos^2(x)dx &= \int_{x=\pi/3}^{x=\pi/2} (-\sin^2(x)\cos^2(x))(-\sin(x)dx) \\
&= \int_{x=\pi/3}^{x=\pi/2} (-1+\cos^2(x))(\cos^2(x))(-\sin(x)dx) \\
&= \int_{u=1/2}^{u=0} (u^2-1)(u^2)du \\
&= \int_{u=1/2}^{u=0} (u^4-u^2)du \\
&= \frac{u^5}{5} - \frac{u^3}{3}\Bigg|_{u=1/2}^{u=0} \\
&= \left(\frac{(0)^5}{5} - \frac{(0)^3}{3}\right) - \left(\frac{(1/2)^5}{5} - \frac{(1/2)^3}{3}\right) \\
&= -\frac{2^{-5}}{5} + \frac{2^{-3}}{3}
\end{aligned}$$

2. There are two ways to approach this problem. The first is to use the identity

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x)$$

to get

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$

This implies that

$$\begin{aligned}\int \sin^2(x)dx &= \int \frac{1}{2}(1-\cos(2x))dx \\ &= \int \frac{1}{2}dx - \int \frac{1}{2}\cos(2x)dx \\ &= \frac{1}{2}x - \frac{1}{2}\int \cos(2x)dx \\ &= \frac{1}{2}x - \frac{1}{2}\frac{\sin(2x)}{2} + C \\ &= \frac{1}{2}x - \frac{1}{4}\sin(2x) + C\end{aligned}$$

The second approach is to use an integration by parts, setting $u = \sin x$ and $dv = \sin x dx$; this means that $du = \cos x dx$ and $v = -\cos x$. Therefore, integration by parts gives

$$\int \sin^2(x)dx = (\sin x)(-\cos x) - \int (-\cos x)(\cos x dx) = -\sin x \cos x + \int \cos^2(x)dx$$

Now we plug in the identity $\cos^2(x) = 1 - \sin^2(x)$ to get

$$\int \sin^2(x)dx = -\sin x \cos x + \int (1-\sin^2(x))dx = -\sin x \cos x + \int dx - \int \sin^2(x)dx$$

Adding $\int \sin^2(x)dx$ to both sides of the equation gives

$$2\int \sin^2(x)dx = -\sin(x)\cos(x) + \int dx = -\sin(x)\cos(x) + x + C$$

so that

$$\int \sin^2(x)dx = \frac{1}{2}x - \frac{1}{2}\sin(x)\cos(x) + C$$

Notice that both approaches give the same answer, because of the identity $\sin(2x) = 2\sin(x)\cos(x)$.

Problem 2 1. Estimate the integral

$$\int_7^8 \frac{dx}{\ln(x)}$$

using three rectangles and

(a) right endpoints

(b) left endpoints

(c) Are your answers in 1a and 1b over- or under-estimates of the actual integral?

2. Do the same for the under-integral function $f(t) = e^{t^3}$

Solution:

1. The problem asks for three rectangles, so we divide the interval $[7, 8]$ into three equal subintervals:

$$\left[7, 7\frac{1}{3}\right] \qquad \left[7\frac{1}{3}, 7\frac{2}{3}\right] \qquad \left[7\frac{2}{3}, 8\right]$$

Each subinterval has length $\frac{1}{3}$.

(a) For right endpoints, evaluate $f(x) = \frac{1}{\ln(x)}$ at the right endpoint of each subinterval, and multiply the sum by $\frac{1}{3}$, the length of the subintervals:

$$\frac{1}{3}\left[f\left(7\frac{1}{3}\right) + f\left(7\frac{2}{3}\right) + f(8)\right] = \frac{1}{3}\left(\frac{1}{\ln\left(7\frac{1}{3}\right)} + \frac{1}{\ln\left(7\frac{2}{3}\right)} + \frac{1}{\ln(8)}\right)$$

(b) For left endpoints, evaluate $f(x) = \frac{1}{\ln(x)}$ at the left endpoint of each subinterval, and multiply the sum by $\frac{1}{3}$, the length of the subintervals:

$$\frac{1}{3}\left[f(7) + f\left(7\frac{1}{3}\right) + f\left(7\frac{2}{3}\right)\right] = \frac{1}{3}\left(\frac{1}{\ln(7)} + \frac{1}{\ln\left(7\frac{1}{3}\right)} + \frac{1}{\ln\left(7\frac{2}{3}\right)}\right)$$

(c) First, notice that the function $\ln(x)$ is an increasing function of x, so that $f(x) = \frac{1}{\ln(x)}$ is a *decreasing* function of x. Since $f(x)$ is decreasing, the right endpoint $f(x_i)$ is always less than the other values of $f(x)$ in the subinterval $[x_{i-1}, x_i]$, and the left endpoint $f(x_{i-1})$ is always greater than the other values in the subinterval. Thus the right endpoint approximation from part (a) is an under-estimate, and the left endpoint approximation from part (b) is an over-estimate.

2. As in (1), we divide the interval $[7, 8]$ into three equal subintervals:

$$\left[7, 7\frac{1}{3}\right] \qquad \left[7\frac{1}{3}, 7\frac{2}{3}\right] \qquad \left[7\frac{2}{3}, 8\right]$$

of length $\frac{1}{3}$.

(a) For right endpoints, evaluate $f(x) = e^{t^3}$ at the right endpoint of each subinterval, and multiply the sum by $\frac{1}{3}$, the length of the subintervals:

$$\frac{1}{3}\left[f\left(7\frac{1}{3}\right) + f\left(7\frac{2}{3}\right) + f(8)\right] \quad = \quad \frac{1}{3}\left(e^{\left(7\frac{1}{3}\right)^3} + e^{\left(7\frac{2}{3}\right)^3} + e^{8^3}\right)$$

(b) For left endpoints, evaluate $f(x) = \frac{1}{\ln(x)}$ at the left endpoint of each subinterval, and multiply the sum by $\frac{1}{3}$, the length of the subintervals:

$$\frac{1}{3}\left[f(7) + f\left(7\frac{1}{3}\right) + f\left(7\frac{2}{3}\right)\right] \quad = \quad \frac{1}{3}\left(e^{7^3} + e^{\left(7\frac{1}{3}\right)^3} + e^{\left(7\frac{2}{3}\right)^3}\right)$$

(c) First, notice that the function x^3 is an increasing function of x, and e^x is also an increasing function of x, so that $f(x) = e^{x^3}$ is an increasing function of x. Since $f(x)$ is increasing, the right endpoint $f(x_i)$ is always greater than the other values of $f(x)$ in the subinterval $[x_{i-1}, x_i]$, and the left endpoint $f(x_{i-1})$ is always less than the other values in the subinterval. Thus the right endpoint approximation from part (a) is an over-estimate, and the left endpoint approximation from part (b) is an under-estimate.

Problem 3 Integrate

1. $$\int \cos(\ln(x))dx$$

2. $$\int e^{\sqrt{x}}dx$$

Solution:

1. The first thing to recognize is the presence of a composite function, $\cos(\ln(x))$, in the integrand. This immediately suggests a substitution $u = \ln(x)$, or $x = e^u$. This implies that $dx = e^u du$, and so we have

$$\int \cos(\ln(x))dx = \int \cos(u)dx = \int \cos(u) \cdot e^u du$$

Observe that the integrand is the product of two functions, e^u and $\cos(u)$, whose antiderivatives are not too exotic— therefore this integral is a candidate for integration by parts. Setting $w = \cos(u)$ and $dv = e^u du$, we get that $dw = -\sin(u)du$ and $v = e^u$, so integration by parts gives

$$\int \cos(u)e^u du = \cos(u)e^u - \int e^u(-\sin(u)du) = \cos(u)e^u + \int e^u \sin(u)du \tag{1}$$

The integral on the right is again a candidate for integration by parts, this time setting $w = \sin(u)$ and $dv = e^u du$, whereby $dw = \cos(u)du$ and $v = e^u$, so that

$$\int e^u \sin(u)du = e^u \sin(u) - \int e^u \cos(u)du \tag{2}$$

Plugging (2) into (1) gives

$$\int \cos(u)e^u du = \cos(u)e^u + \int e^u \sin(u)du = \cos(u)e^u + e^u \sin(u) - \int e^u \cos(u)du$$

Adding $\int e^u \cos(u)du$ to both sides gives

$$2\int \cos(u)e^u du = \cos(u)e^u + e^u \sin(u) + C$$

and we have

$$\int \cos(u)e^u du = \frac{1}{2}e^u(\cos(u) + \sin(u)) + C$$

Plugging back in our substitution $u = \ln(x)$, we have

$$\int \cos(\ln(x))dx = \frac{1}{2}e^{\ln(x)}(\cos(\ln(x)) + \sin(\ln(x))) + C = \frac{1}{2}x(\cos(\ln(x)) + \sin(\ln(x))) + C$$

2. As with part (1), notice first that the composite function $e^{\sqrt{x}}$ suggests that the substitution $u = \sqrt{x}$ may be useful. This substitution gives $x = u^2$, so that $dx = 2udu$. Therefore

$$\int e^{\sqrt{x}}dx = \int e^u dx = \int e^u \cdot 2udu$$

Since the derivative of $2u$ is simpler than $2u$ (a polynomial of lower degree; in this case, a constant), while the antiderivative of e^u is not more complicated than e^u (in this case, the anti-derivative of e^u is itself), this integral is a good candidate for integration by parts. Since we want to differentiate $2u$, take $w = 2u$ and $dv = e^u du$. This gives $dw = 2du$ and $v = e^u$, so integration by parts yields

$$\int e^u 2udu = e^u 2u - \int 2e^u du = 2ue^u - 2e^u + C = 2(u-1)e^u + C$$

Plugging back our substitution $u = \sqrt{x}$, this gives us

$$\int e^{\sqrt{x}}dx = 2(\sqrt{x} - 1)e^{\sqrt{x}} + C$$

Problem 4 Find the following indefinite integrals

1. $$\int \frac{dx}{(x+1)\sqrt{x}}$$

2. $$\int \frac{(e^x+1)dx}{e^x(e^x+2)}$$

Solution:

1. Here the substitution is more subtle; however, one can recognize that $\sqrt{x}$ and x are both easily expressed in terms of the variable $u = \sqrt{x}$, and that the presence of $\sqrt{x}$ in the denominator makes this a good candidate for substitution. Since

$$du = \frac{1}{2}x^{-1/2}dx = \frac{1}{2}\frac{1}{\sqrt{x}}dx$$

we can write

$$\begin{aligned}\int \frac{dx}{(x+1)\sqrt{x}} &= \int \frac{2}{x+1}\frac{1}{2}x^{-1/2}dx \\ &= \int \frac{2}{u^2+1}du\end{aligned}$$

Now, notice that the resulting integral is one that we recognize: the anti-derivative of $\frac{1}{u^2+1}$ is arctan(u), so we get

$$\begin{aligned}\int \frac{dx}{(x+1)\sqrt{x}} &= \int \frac{2}{u^2+1}du \\ &= 2\arctan(u) + C \\ &= 2\arctan(\sqrt{x}) + C\end{aligned}$$

2. The simplest solution to this problem is to write

$$\int \frac{e^x+1}{e^x(e^x+2)}dx = \int \frac{e^x+1}{e^x+2}e^{-x}dx$$

and then substitute $u = e^{-x}$, with $du = -e^{-x}dx$. This gives

$$\begin{aligned}\int \frac{e^x+1}{e^x+2}(e^{-x}dx) &= \int \frac{u^{-1}+1}{u^{-1}+2}(-du) \\ &= -\int \frac{1+u}{1+2u}du\end{aligned}$$

by multiplying the top and bottom of the integrand by u (since $u = e^{-x} > 0$ for all x, this is legal). Using long division we find that

$$1 + u = \frac{1}{2}(1 + 2u) + \frac{1}{2}$$

so that

$$\begin{aligned}
\int \frac{e^x + 1}{e^x + 2}(e^{-x}dx) &= -\int \frac{1 + u}{1 + 2u} du \\
&= -\int \left(\frac{\frac{1}{2}(1 + 2u)}{1 + 2u} + \frac{\frac{1}{2}}{1 + 2u} \right) du \\
&= -\int \frac{1}{2} du - \int \frac{1}{2 + 4u} du \\
&= -\frac{1}{2}u - \frac{1}{4} \ln|2 + 4u| + C
\end{aligned}$$

using the substitution $w = 2+4u$, with $du = \frac{1}{4}dw$, in the rightmost integral (the absolute value sign is actually irrelevant, since $2 + 4u = 2 + 4e^{-x} > 0$ is positive for all x). We then substitute back $u = e^{-x}$ and get

$$\int \frac{e^x + 1}{e^x + 2}(e^{-x}dx) = -\frac{1}{2}e^{-x} - \frac{1}{4} \ln(2 + 4e^{-x}) + C$$

Note: It is perhaps more natural, at first glance, to try to substitute $u = e^x$ instead of e^{-x}. This means that $x = \ln(u)$, so that $dx = \frac{1}{u}du$, and this substitution gives

$$\begin{aligned}
\int \frac{e^x + 1}{e^x(e^x + 2)} dx &= \int \frac{u + 1}{u(u + 2)} dx \\
&= \int \frac{u + 1}{u(u + 2)} \frac{1}{u} du \\
&= \int \frac{u + 1}{u^2(u + 2)} du
\end{aligned}$$

This integral could be computed using partial fractions, but since the denominator has a double root (one of the factors is u^2), we won't go down this path.

Problem 5 Find the following indefinite integrals

1.
$$\int \frac{x^2+1}{x^2-3x+2}dx$$

2.
$$\int \frac{x^3+1}{x^2-4x+3}dx$$

Solution: Both of these integrals require the method of partial fractions. Remember that this technique requires that the numerator have degree *lower* than the degree of the denominator; this is not the case for either integral, so we must first use long division to decompose the numerator appropriately.

1. The first step, as mentioned above, is to use long division to expand the numerator. Since the highest (x^2) term has the same coefficient in the numerator and denominator, we know that

$$x^2+1 = 1 \cdot (x^2-3x+2) + (\text{degree 1 polynomial})$$

Comparing terms shows that $x^2+1-(x^2-3x+2) = 3x-1$, so that

$$x^2+1 = (x^2-3x+2)+(3x-1)$$

(it's a good idea to check this to verify the arithmetic!)

Therefore we can write

$$\frac{x^2+1}{x^2-3x+2} = \frac{(x^2-3x+2)+(3x-1)}{x^2-3x+2} = 1+\frac{3x-1}{x^2-3x+2}$$

which tells us that

$$\int \frac{x^2+1}{x^2-3x+2}dx = \int dx + \int \frac{3x-1}{x^2-3x+2}dx = x + \int \frac{3x-1}{x^2-3x+2}dx \tag{3}$$

Now the problem is reduced to solving the integral on the right, in which the numerator has degree 1— lower than the denominator's 2— so partial fractions can be applied. Write

$$\frac{3x-1}{x^2-3x+2} = \frac{3x-1}{(x-2)(x-1)} = \frac{A}{x-2}+\frac{B}{x-1}$$

Putting the right hand side over common denominator $x^2 - 3x + 2$ and comparing the two numerators, we see that

$$3x - 1 = A(x - 1) + B(x - 2) = (A + B)x + (-A - 2B)$$

implying the system of equations

$$\begin{aligned} A + B &= 3 \\ -A - 2B &= -1 \end{aligned}$$

The solution of this system is $A = 5$, $B = -2$. Therefore we have our partial fractions decomposition

$$\frac{3x - 1}{x^2 - 3x + 2} = \frac{5}{x - 2} + \frac{-2}{x - 1}$$

and the integral

$$\int \frac{3x - 1}{x^2 - 3x + 2} dx = \int \left(\frac{5}{x - 2} + \frac{-2}{x - 1} \right) dx = 5 \ln |x - 2| - 2 \ln |x - 1| + C$$

Plugging this into (3) finally yields

$$\int \frac{x^2 + 1}{x^2 - 3x + 2} dx = x + (5 \ln |x - 2| - 2 \ln |x - 1|) + C$$

2. Proceed as in part (1) above, but note that the numerator now has degree 3, so there are two steps to the long division. First note that the leading term of the numerator (x^3) is x times the leading term of the denominator (x^2), so that

$$x^3 + 1 = x \cdot (x^2 - 4x + 3) + (\text{degree 2 polynomial})$$

and we subtract $x^3 + 1 - x(x^2 - 4x + 3)$ to find the degree 2 polynomial on the right

$$x^3 + 1 = x \cdot (x^2 - 4x + 3) + 4x^2 - 3x + 1$$

Now the remainder term on the right is still degree 2 (not degree lower than the denominator yet), so we apply long division to $4x^2 - 3x + 1$ to obtain

$$4x^2 - 3x + 1 = 4 \cdot (x^2 - 4x + 3) + 13x - 11$$

All together, we have the partial fraction decomposition

$$\frac{x^3 + 1}{x^2 - 4x + 3} = \frac{x(x^2 - 4x + 3) + 4(x^2 - 4x + 3) + 13x - 11}{x^2 - 4x + 3} = x + 4 + \frac{13x - 11}{x^2 - 4x + 3}$$

so that

$$\begin{aligned} \int \frac{x^3+1}{x^2-4x+3}dx &= \int \left(x+4+\frac{13x-11}{x^2-4x+3} \right) dx \\ &= \frac{x^2}{2}+4x+\int \frac{13x-11}{x^2-4x+3}dx \qquad (4) \end{aligned}$$

Next, we apply partial fractions to the integral on the right. Write

$$\frac{13x-11}{x^2-4x+3} = \frac{13x-11}{(x-3)(x-1)} = \frac{A}{x-3}+\frac{B}{x-1}$$

Again, putting the right side over a common denominator x^2-4x+3 and comparing numerators, we get

$$13x-11 = A(x-1)+B(x-3) = (A+B)x+(-A-3B)$$

and the system of equations

$$\begin{aligned} A+B &= 13 \\ -A-3B &= -11 \end{aligned}$$

which has solution $A = 14$, $B = -1$. Therefore

$$\int \frac{13x-11}{x^2-4x+3}dx = \int \left(\frac{14}{x-3}+\frac{-1}{x-1} \right) dx = 14\ln|x-3| - \ln|x-1| + C$$

Plugging this into (4) gives the final answer

$$\int \frac{x^3+1}{x^2-4x+3}dx = \frac{x^2}{2}+4x+14\ln|x-3|-\ln|x-1|+C$$

Problem 6 1. Use trapezoidal approximation with $n = 4$ intervals of subdivision (each of equal length) to estimate the following integral

$$\int_0^\pi \sin^2(x)dx.$$

Estimate the error of approximation.

2. Use Simpson's rule with $n = 4$ to estimate $\int_0^1 x^5 dx$. Find the precise error of approximation.

Solution:

1. The trapezoidal approximation is given by

$$T_n = \frac{1}{2n}(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n))$$

In our case, $n = 4$, and therefore the interval $[0, \pi]$ is divided into 4 equal subintervals, with sample points $x_0 = 0$, $x_1 = \pi/4$, $x_2 = \pi/2$, $x_3 = 3\pi/4$, and $x_4 = \pi$. We evaluate $f(x) = \sin^2(x)$ at each of these sample points, and get

$$\begin{aligned}
T_4 &= \frac{1}{8}(f(0) + 2f(\pi/4) + 2f(\pi/2) + 2f(3\pi/4) + f(\pi)) \\
&= \frac{1}{8}(\sin^2(0) + 2\sin^2(\pi/4) + 2\sin^2(\pi/2) + 2\sin^2(3\pi/4) + \sin^2(\pi)) \\
&= \frac{1}{8}\left(0 + 2\left(\frac{1}{\sqrt{2}}\right)^2 + 2(1)^2 + 2\left(\frac{1}{\sqrt{2}}\right)^2 + 0\right) \\
&= \frac{1}{8}\left(2 \cdot \frac{1}{2} + 2 + 2 \cdot \frac{1}{2}\right) \\
&= \frac{1}{8}(4) = \frac{1}{2}
\end{aligned}$$

To estimate the error in this approximation, we need to estimate the second derivative $f''(x)$ for all x in the interval $[0, \pi]$. First calculate (using the Chain Rule):

$$\begin{aligned}
f(x) &= \sin^2(x) \\
f'(x) &= 2\sin(x)\cos(x) = \sin(2x) \\
f''(x) &= \cos(2x) \cdot 2 = 2\cos(2x)
\end{aligned}$$

Since $|\cos(2x)| \le 1$ for any x (in particular, any x in our interval $[0, \pi]$), we have

$$|f''(x)| = |2\cos(2x)| = 2|\cos(2x)| \le 2$$

for all x in our interval $[0, \pi]$, we can take $K = 2$ in the trapezoid error approximation formula

$$|E_T| \le \frac{K(b-a)^3}{12n^2} = \frac{2(\pi-0)^3}{12 \cdot 4^2} = \frac{\pi^3}{96}$$

2. The Simpson's rule approximation is given by

$$S_n = \frac{1}{3n}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n))$$

In our case, $n = 4$, and therefore the interval $[0, 1]$ is divided into 4 equal subintervals, with sample points $x_0 = 0$, $x_1 = 1/4$, $x_2 = 1/2$, $x_3 = 3/4$, and $x_4 = 1$. We evaluate $f(x) = x^5$ at each of these sample points, and get

$$\begin{aligned} S_4 &= \frac{1}{12}(f(0) + 4f(1/4) + 2f(1/2) + 4f(3/4) + f(1)) \\ &= \frac{1}{12}((0)^5 + 4(0.25)^5 + 2(0.5)^5 + 4(0.75)^5 + (1)^5) \\ &= \frac{1}{12}(0 + .00390625 + .0625 + .94921875 + 1) \\ &= \frac{1}{12}(2.015625) \approx 0.168 \end{aligned}$$

To estimate the error in this approximation, we need to estimate the fourth derivative $f^{(4)}(x)$ for all x in the interval $[0, 1]$. It is straightforward to see that the fourth derivative of $f(x) = x^5$ is $f^{(4)}(x) = 5{\cdot}4{\cdot}3{\cdot}2x = 120x$. On the interval $[0, 1]$ this function is *increasing and non-negative*, and so achieves its maximum absolute value at the right endpoint of interval— i.e. at $x = 1$— where it takes the value $f^{(4)}(1) = 120 \cdot 1 = 120$.

Therefore we have

$$|f^{(4)}(x)| \le |f^{(4)}(1)| = 120$$

for all x in our interval $[0, 1]$, and we can take $K = 120$ in the Simpson's rule error approximation formula

$$|E_S| \le \frac{K(b-a)^5}{180n^4} = \frac{120(1-0)^5}{180 \cdot 4^4} = \frac{120}{180 \cdot 256} \approx 0.003$$

The actual value of the integral is easy to compute:

$$\int_0^1 x^5 dx = \left.\frac{x^6}{6}\right|_{x=0}^{x=1} = \frac{1}{6} - 0 = \frac{1}{6} \approx 0.167$$

Note that this differs by only 0.001 from our approximation; this error is within our estimate of 0.003.

Problem 7 1. Does the following integral converge? If yes, evaluate it:

$$\int_0^{+\infty} e^{-t}\cos^2(t)dt$$

2. Does the following integral converge? If yes, evaluate it:

$$\int_1^{+\infty} \frac{dt}{t(t+1)}$$

3. Does the following integral converge? If yes, evaluate it:

$$\int_1^{\infty} \frac{dx}{x[\ln x]^2}$$

Solution:

1. The integrand is everywhere continuous, so the problem is Type I, and we only have to worry about the convergence at the upper limit.

Notice that the integrand is non-negative, and since $|\cos(t)| \leq 1$, we have

$$0 \leq e^{-t}\cos^2(t) \leq e^{-t}$$

for all t. Therefore by comparison, since $\int_0^{+\infty} e^{-t}dt$ converges, we know that our integral $\int_0^{+\infty} e^{-t}\cos^2(t)dt$ also converges.

In order to compute what this integral converges to, it is best to first use the identity

$$\cos^2(t) = \frac{1}{2} + \frac{1}{2}\cos(2t)$$

so that we have

$$\int_0^{\infty} e^{-t}\cos^2(t) = \int_0^{\infty} \frac{1}{2}e^{-t}dt + \int_0^{\infty} \frac{1}{2}\cos(2t)e^{-t}dt$$

The first integral on the right hand side is relatively easy to evaluate:

$$\begin{aligned}
\frac{1}{2}\int_0^{\infty} e^{-t}dt &= \frac{1}{2}\lim_{R\to\infty}\int_0^{R} e^{-t}dt \\
&= \frac{1}{2}\lim_{R\to\infty}\left(-e^{-t}\right)\Big|_{t=0}^{t=R} \\
&= \frac{1}{2}\lim_{R\to\infty}(-e^{-R} - (-e^{-0})) \\
&= \frac{1}{2}\left(-\lim_{R\to\infty} e^{-R} + e^{0}\right) \\
&= \frac{1}{2}(-0+1) = \frac{1}{2}
\end{aligned}$$

The second integral requires an integration by parts; setting $u = \cos(2t)$ and $dv = e^{-t}dt$, we get $du = -2\sin(2t)dt$ and $v = -e^{-t}$, so integration by parts gives

$$\int e^{-t}\cos(2t)dt = -e^{-t}\cos(2t) - \int(-e^{-t})(-2\sin(2t)dt) = -e^{-t}\cos(2t) - 2\int e^{-t}\sin(2t)dt$$

Apply another integration by parts to the integral on the right with $u = \sin(2t)$ and $dv = e^{-t}dt$ (so that $du = 2\cos(2t)dt$ and $v = -e^{-t}$) to get

$$\begin{aligned}\int e^{-t}\cos(2t)dt &= -e^{-t}\cos(2t) - 2\left[(-e^{-t})(\sin(2t)) - \int(-e^{-t})(2\cos(2t)dt)\right] \\ &= -e^{-t}\cos(2t) + 2e^{-t}\sin(2t) - 4\int e^{-t}\cos(2t)dt\end{aligned}$$

which, after some algebraic rearranging, gives

$$\int e^{-t}\cos(2t)dt = \frac{1}{5}\left(-e^{-t}\cos(2t) + 2e^{-t}\sin(2t)\right) + C$$

Therefore,

$$\begin{aligned}\int_0^\infty \frac{1}{2}\cos(2t)e^{-t}dt &= \frac{1}{2}\lim_{R\to\infty}\int_0^R \cos(2t)e^{-t}dt \\ &= \frac{1}{2}\lim_{R\to\infty}\frac{1}{5}\left(-e^{-t}\cos(2t) + 2e^{-t}\sin(2t)\right)\Big|_{t=0}^{t=R} \\ &= \frac{1}{10}\left(\lim_{R\to\infty}(-e^{-R}\cos(2R) + 2e^{-R}\sin(2R)) - (-e^{0}\cos(0) + 2e^{0}\sin(0))\right) \\ &= \frac{1}{10}[(0) - (-1\cdot 1 + 2\cdot 1\cdot 0)] = \frac{1}{10}[1] = \frac{1}{10}\end{aligned}$$

Putting together the two pieces of the original integral yields

$$\int_0^\infty e^{-t}\cos^2(t) = \frac{1}{2} + \frac{1}{10} = 0.5 + 0.1 = 0.6$$

2. Once again the integrand is continuous on the interval of integration (it has discontinuities at $t = 0$ and $t = -1$, but neither of these is in our interval of integration $[0, \infty)$) and non-negative, and we notice furthermore that

$$\frac{1}{t(t+1)} = \frac{1}{t^2 + t} < \frac{1}{t^2}$$

for all $1 \leq t < \infty$. Therefore, since $\int_1^\infty \frac{1}{t^2}dt$ converges, we know by comparison that our integral $\int_1^\infty \frac{1}{t(t+1)}dt$ converges.

In order to calculate this integral, we need to use a partial fractions decomposition. The numerator 1 is already of lower degree than the denominator $(t^2 + t)$, which—conveniently— is already factored. Therefore set

$$\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1}$$

and put the right hand side over the common denominator $t(t + 1)$, and then compare numerators of left and right hand sides to get

$$1 = A(t + 1) + Bt = (A + B)t + A$$

which yields the system of equations

$$\begin{aligned} A + B &= 0 \\ A &= 1 \end{aligned}$$

with solution $A = 1$, $B = -1$. Therefore

$$\int \frac{1}{t(t+1)} dt = \int \left(\frac{1}{t} - \frac{1}{t+1}\right) dt = \ln|t| - \ln|t+1| + C = \ln\left|\frac{t}{t+1}\right|$$

Returning to our problem, calculate

$$\begin{aligned} \int_1^{+\infty} \frac{dt}{t(t+1)} &= \lim_{R\to\infty} \int_1^R \frac{dt}{t(t+1)} \\ &= \lim_{R\to\infty} \left(\ln\left|\frac{t}{t+1}\right|\right)\Bigg|_{t=1}^{t=R} \\ &= \lim_{R\to\infty} \left(\ln\frac{R}{R+1} - \ln\frac{1}{1+1}\right) \\ &= \ln\left(\lim_{R\to\infty} \frac{R}{R+1}\right) - \ln\left(\frac{1}{2}\right) \\ &= \ln(1) - (-\ln(2)) = 0 + \ln(2) = \ln(2) \end{aligned}$$

3. It is not easy to test the convergence of this integral without calculating the limit directly, so let's attack the integral itself. The integrand is continuous at all points in the interval $[1, \infty)$ except for $x = 1$, where $\ln^2(1) = 0$ and the denominator goes to zero. Therefore, *if* (and only if) the integral converges, it is defined by the limit

$$\begin{aligned} \int_1^{\infty} \frac{dx}{x\ln^2(x)} &= \int_1^2 \frac{dx}{x\ln^2(x)} + \int_2^{\infty} \frac{dx}{x\ln^2(x)} \\ &= \lim_{R\to 1} \int_R^2 \frac{dx}{x\ln^2(x)} + \lim_{R\to\infty} \int_2^{\infty} \frac{dx}{x\ln^2(x)} \end{aligned}$$

In either case, we can integrate $\frac{1}{x\ln^2(x)}$ by the substitution $u = \ln(x)$, with $du = \frac{1}{x}dx$, giving

$$\begin{aligned}
\int \frac{dx}{x\ln^2(x)} &= \int \frac{1}{\ln^2(x)} \frac{1}{x} dx \\
&= \int u^{-2} du \\
&= -u^{-1} + C \\
&= -(\ln(x))^{-1} + C
\end{aligned}$$

Let's first check if the integral converges near $x = 1$. Recall we have

$$\begin{aligned}
\int_1^2 \frac{dx}{x\ln^2(x)} &= \lim_{R\to 1} \int_R^2 \frac{dx}{x\ln^2(x)} \\
&= \lim_{R\to 1} [(-\ln(2))^{-1} - (-\ln(R))^{-1})] \\
&= (-\ln(2))^{-1} + \lim_{R\to 1} (\ln(R))^{-1}
\end{aligned}$$

Now as $R \to 1$, we have $\ln(R) \to 0$, and so $(\ln(R))^{-1} \to \infty$ does not converge as $R \to 1$. Therefore, the integral $\int_1^2 \frac{dx}{x\ln^2(x)}$ does *not* converge. This means that the full integral $\int_1^\infty \frac{dx}{x\ln^2(x)}$ does *not* converge— regardless of whether the second part $\int_2^\infty \frac{dx}{x\ln^2(x)}$ converges (which it does)!

So, the answer is NO, the integral does not converge.

Problem 8 1. Find the volumes of the bodies obtained from the region enclosed by

$$y = \sin(x), 0 \leq x \leq \pi$$

and $y = 0$ by revolving about a) the x-axis, b) the line $x = -2$

2. Find the volumes of the bodies obtained from the region enclosed by

$$y = \frac{1}{x^5}, 1 \leq x < \infty,$$

$y = 0$, $x = 1$ by revolving about a) the line $y = -2$, b) the y-axis.

Solution:

1. First step: DRAW A PICTURE!! (see accompanying diagrams)

Since $y = \sin(x)$ is not a pleasant function to invert, it's a good idea to integrate over x using vertical strips (notice, for example, that horizontal strips go from one $\arcsin(y)$ to another $\arcsin(y)$... not something that we want to start having to compute).

(a) Each value of x between 0 and π gives a vertical strip, of thickness dx, that generates a disk when rotated about the x-axis. Thus we use the formula for volume of rotation via the disk method,

$$V = \int_0^\pi \pi R^2 dx$$

In our case, the radius of the disk is given by the distance from $(x, \sin x)$ to $(x, 0)$ for each x, and is therefore given by $R = \sin x$. Hence

$$V = \int_0^\pi \pi (\sin x)^2 dx$$

We can compute this integral by applying the identity $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$ (see Problem 1, part (2)):

$$\begin{aligned} V &= \pi \int_0^\pi \sin^2(x) dx = \pi \left(\frac{x}{2} - \frac{\sin(2x)}{4} \right) \Bigg|_{x=0}^{x=\pi} \\ &= \pi \left(\frac{\pi}{2} - \frac{\sin(2\pi)}{4} - \frac{0}{2} + \frac{\sin(0)}{4} \right) \\ &= \pi \left(\frac{\pi}{2} - 0 - 0 + 0 \right) = \frac{\pi^2}{2} \end{aligned}$$

(b) Since we are now rotating about the (vertical) line $x = -2$, our vertical strips no longer generate disks, but instead generate cylindrical shells. The formula for the volume of rotation via the cylindrical shell method is

$$V = \int_0^\pi 2\pi \cdot R \cdot h dx$$

Here R is the radius of rotation, given by the distance from x to the line of rotation at -2; this distance is $R = x + 2$. The height h is the height of the vertical strip, equal to $\sin x$ as before in part (a). Thus

$$\begin{aligned} V &= \int_0^\pi 2\pi(x+2)\sin(x)dx \\ &= 2\pi \int_0^\pi x\sin(x)dx + 2\pi \int_0^\pi 2\sin(x)dx \\ &= 2\pi \int_0^\pi x\sin(x)dx - 4\pi\cos(x)|_{x=0}^{x=\pi} \\ &= 2\pi \int_0^\pi x\sin(x)dx - (4\pi\cdot(-1) - 4\pi\cdot(1)) \\ &= 2\pi \int_0^\pi x\sin(x)dx + 8\pi \end{aligned}$$

The integral $\int_0^\pi x\sin(x)dx$ can be evaluated by an integration by parts, setting $u = x$ and $dv = \sin(x)dx$, whereby $du = dx$ and $v = -\cos(x)$. This gives

$$\begin{aligned} \int_0^\pi x\sin(x)dx &= -x\cos(x)|_{x=0}^{x=\pi} - \int_0^\pi(-\cos(x))dx \\ &= -\pi\cos(\pi) - 0 + \int_0^\pi \cos(x)dx \\ &= -\pi\cdot(-1) + \sin(x)|_{x=0}^{x=\pi} \\ &= \pi + (0-0) = \pi \end{aligned}$$

Putting all this together, we have

$$V = 2\pi \int_0^\pi x\sin(x)dx + 8\pi = 2\pi(\pi) + 8\pi = 2\pi^2 + 8\pi$$

2. As in part (1), we will use the disk/washer method for part (a) and the cylindrical shell method for part (b); although the function $y = x^{-5}$ is more easily inverted than the function in part (a), and we could exchange disk/washer and cylindrical shell methods by considering $x = y^{-1/5}$, it seems easier to integrate over x since we are given the information about the region in terms of y as a function of x.

(a) We have as above vertical strips stretching from (x, x^{-5}) to $(x, 0)$ for each x in $[0, \infty)$. However, unlike part (a), the axis of rotation is now $y = -2$, not the x-axis; therefore the region under consideration does NOT border the axis of rotation, and we have to use the washer method

$$V = \int_1^\infty (\pi R^2 - \pi r^2) dx$$

Here R is the outer radius of rotation, given by the distance of the graph of the function $y = x^{-5}$ from the line $y = -2$; so $R = x^{-5} + 2$. The inner radius r is the distance from the inner curve $y = 0$ to the line $y = -2$, which is $r = 2$ for all x. Therefore

$$\begin{aligned} V &= \pi \int_1^\infty ((x^{-5} + 2)^2 - 2^2) dx \\ &= \pi \int_1^\infty (x^{-10} + 2x^{-5} + 4 - 4) dx \\ &= \pi \int_1^\infty (x^{-10} + 2x^{-5}) dx \\ &= \pi \left(\frac{x^{-9}}{-9} + 2\frac{x^{-4}}{-4} \right) \Bigg|_{x=1}^{x=\infty} \\ &= \pi(0 + 0 - (-1/9 - 1/4)) = \pi(1/9 + 1/4) = \frac{13\pi}{36} \end{aligned}$$

(b) Once again we use the cylindrical shells formula

$$V = \int_1^\infty 2\pi \cdot R \cdot h dx$$

where R is the radius of rotation, given by the distance from x to the y-axis; therefore $R = x$. The height is given by the height of the vertical strip which, as in part (a), is given by $h = x^{-5}$.

Thus

$$\begin{aligned} V &= 2\pi \int_1^\infty x \cdot x^{-5} dx = 2\pi \int_1^\infty x^{-4} dx \\ &= 2\pi \left(\frac{x^{-3}}{-3} \Bigg|_{x=1}^{x=\infty} \right) \\ &= 2\pi \left(0 - \frac{1}{-3} \right) = \frac{2\pi}{3} \end{aligned}$$

Problem 9 Find the length of the curve given by the graph of the function

$$y = \ln(\cos(x))$$

between points $(0,0)$ and $(a, \ln(\cos(a))), 0 < a < \frac{\pi}{2}$.

Solution: Observe that the starting point $(0,0)$ of the curve corresponds to $x = 0$ (and $y = \ln(\cos(0)) = \ln(1) = 0$), and the terminal point $(a, \ln(\cos(a)))$ corresponds to $x = a$. Since the curve is parametrized by x, we have the arc length formula

$$L = \int_{x=0}^{x=a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

So the next step is to differentiate

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(\ln(\cos(x))) \\
&= \frac{1}{\cos(x)} \cdot \frac{d}{dx}(\cos(x)) \\
&= \frac{1}{\cos(x)} \cdot (-\sin(x)) \\
&= \frac{-\sin(x)}{\cos(x)} = -\tan(x)
\end{aligned}$$

using the Chain Rule.

Plugging this into the arc length formula, and recalling that

$$1 + \tan^2(x) = \frac{\cos^2(x)}{\cos^2(x)} + \frac{\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$

we get that the arc length is

$$\begin{aligned}
L &= \int_0^a \sqrt{1 + (-\tan(x))^2} dx \\
&= \int_0^a \sqrt{1 + \tan^2(x)} dx \\
&= \int_0^a \sqrt{\sec^2(x)} dx \\
&= \int_0^a |\sec(x)| dx \\
&= \int_0^a \sec(x) dx
\end{aligned}$$

since $0 \leq x \leq a < \pi/2$ implies that $\sec(x) \geq 0$ on this interval.

Therefore

$$\begin{aligned}
L &= \int_0^a \sec(x)dx \\
&= (\ln|\sec(x) + \tan(x)|)\Big|_{x=0}^{x=a} \\
&= \ln(\sec(a) + \tan(a)) - \ln(\sec(0) + \tan(0)) \\
&= \ln(\sec(a) + \tan(a)) - \ln(1 + 0) = \ln(\sec(a) + \tan(a)) - \ln(1) \\
&= \ln(\sec(a) + \tan(a))
\end{aligned}$$

Problem 10 1. A cable that weights 2lb/ft is used to lift 800lb of coal from a mineshaft 500 ft deep. Find the work done.

2. A 10 ft chain weights 25 lb and hangs from the ceiling. Find the work done in lifting the middle of the chain to the ceiling so that it is level with the upper end.

Solution:

1. There are two parts to the work done: the work done in lifting the coal out of the mine, and the work done in lifting the chain itself out of the mine.

For the first part, the problem states that 800 lbs. of coal are lifted 500 ft., and so the total work done to lift the coal up out of the mine is

$$W_1 = (800 \text{ lbs. }) \cdot (500 \text{ ft. }) = 400,000 \text{ ft.-lbs.}$$

For the second part, we have to do an integral. Let x be the depth at which each piece of the chain is located— this means $0 \leq x \leq 500$. Each piece of chain of length dx has weight (2 lb/ft.)(dx ft.) $= 2dx$ lbs., and is lifted a distance x to the top of the mine. Therefore the work done in lifting the chain out of the mine is given by the integral

$$W_2 = \int_0^{500} 2dx \cdot x = \int_0^{500} 2xdx = x^2|_{x=0}^{x=500} = 250,000 \text{ ft.-lbs.}$$

So the total work done is

$$W_1 + W_2 = 400,000 \text{ ft.-lbs. } + 250,000 \text{ ft.-lbs. } = 650,000 \text{ ft.-lbs.}$$

2. Here again, there are multiple parts to the work done. If the middle of the chain is lifted to be even with the top, then the top $1/4$ of the chain— the first 2.5 ft.— is not moved at all. On the other hand, the bottom half (5 ft.) of the chain is lifted a total distance of 5 ft.; or, more precisely, each piece of the bottom half of the chain is lifted to a point 5 ft. above its original position. Therefore, since the bottom half of the chain weighs $\frac{1}{2} \cdot 25$ lbs. $= 12.5$ lbs. , the work done in moving the bottom half of the chain 5 ft. upward is

$$W_1 = (12.5 \text{ lbs. }) \cdot (5 \text{ ft. }) = 62.5 \text{ ft.-lbs.}$$

It remains to examine the second $\frac{1}{4}$ of the chain. Here, each piece of the chain is lifted a distance equal to twice its original distance from the point P located $\frac{1}{4}$ of the way down the chain (eg., the point P does not move at all, but the midpoint of the chain starts 2.5 ft. below P, and ends up 2.5 ft. above P, which is $2 \cdot 2.5$ ft. $=$ 5 ft. above its original position). Therefore, if we set x to be the starting distance below P, then we let x range over $0 \leq x \leq 2.5$ ft., and each piece of thickness dx is moved upward a distance $2x$. The weight density of the chain is given by

$$\frac{25 \text{ lbs.}}{10 \text{ ft.}} = 2.5 \text{ lbs./ft.}$$

This means that a piece of chain of length dx will have weight $2.5dx$ lbs., and since it is moved a distance $2x$, the work done is given by the integral

$$\begin{aligned} W_2 &= \int_0^{2.5} (2.5dx)(2x) = \int_0^{2.5} 5xdx \\ &= 2.5x^2|_{x=0}^{x=2.5} = 2.5 \cdot (2.5)^2 - 0 \\ &= 15.625 \text{ ft.-lbs.} \end{aligned}$$

Putting the two parts of the work together, the total work done is

$$W_1 + W_2 = 62.5 \text{ ft.-lbs. } + 15.625 \text{ ft.-lbs. } = 78.125 \text{ ft.-lbs.}$$

Problem 11 A granary has the shape of a half cylinder lying on its rectangular side (the cut). The cylinder's height is 10m, and the radius of the base is 2m. If the granary is full of barley, with density $600 kg/m^3$, how much work is done in removing all the grain via an opening at the top of the granary?

Solution: Step 1— DRAW A PICTURE!! (see accompanying diagrams).

We divide the granary into horizontal slices, each at a height z above the base of the granary (so that z runs from 0 to 2m). For each z, the slice at height z is a rectangle of length 10m (the "height" of the sideways cylinder), and thickness dz, whose width stretches from one side of the semi-circular cross-section to the opposite side. A simple application of the Pythagorean Theorem shows that this width is $2\sqrt{4-z^2}$. Thus the volume of the slice at height z is given by

$$dV = (10 \text{ m})(2\sqrt{4-z^2} \text{ m})(dz \text{ m}) = 20\sqrt{4-z^2}dz \text{ m}^3$$

Since the density of the barley is 600 kg/m^3, the mass of barley in this slice is

$$dm = (600 \text{ kg/m}^3)(dV) = 600 \cdot 20\sqrt{4-z^2}dz \text{ kg} = 12,000\sqrt{4-z^2}dz \text{ kg}$$

From the law $F = mg = m \cdot 9.8$, we have a force

$$dF = dm \cdot 9.8 = (9.8)(12,000\sqrt{4-z^2}dz) = 117,600\sqrt{4-z^2}dz \text{ N}$$

acting on the slice at height z.

Next, observe that the slice at height z is moved a distance $2-z$ upward to get out the top of the granary. Therefore,

$$W = \int_0^2 (117,600\sqrt{4-z^2}dz) \cdot (2-z) = 117,600 \int_0^2 (2-z)\sqrt{4-z^2}dz$$

This integral can be split into two parts. The first part is

$$\begin{aligned} W_1 &= 117,600 \int_0^2 2\sqrt{4-z^2}dz \\ &= 235,200 \int_0^2 \sqrt{2^2-z^2}dz \end{aligned}$$

Observe that this integral is a candidate for the trigonometric substitution $z = 2\sin(u)$, with $dz = 2\cos(u)du$, so that

$$\int_{z=0}^{z=2} \sqrt{2^2 - z^2}dx = \int_{u=0}^{u=\pi/2} \sqrt{2^2 - 2^2\sin^2(u)}(2\cos(u)du) = \int_0^{\pi/2} 4\cos^2(u)du$$

since

$$\sqrt{2^2(1-\sin^2(u))} = 2\sqrt{\cos^2(u)} = \cos(u)$$

for $0 \le u \le \pi/2$, since $\cos(u) \ge 0$ on this interval.

Therefore, using the identity $\cos^2(u) = \frac{1}{2} + \frac{1}{2}\cos(2u)$, this work becomes

$$\begin{aligned} W_1 &= 235,200\int_{u=0}^{u=\pi/2} 4\cos^2(u)du \\ &= 235,200\cdot 4\left[\int_0^{\pi/2}\left(\frac{1}{2}+\frac{1}{2}\cos(2u)\right)du\right] \\ &= 235,200\cdot 4\left(\frac{u}{2}+\frac{\sin(2u)}{4}\right)\Bigg|_0^{\pi/2} \\ &= 235,200\cdot 4\left(\frac{\pi/2}{2}+\frac{\sin(\pi)}{4}-\frac{0}{2}-\frac{\sin(0)}{4}\right) \\ &= 235,200\cdot 4\cdot\frac{\pi}{4} = 235,200\pi \end{aligned}$$

The second part can be computed via the substitution $u = 4 - z^2$, with $du = -2zdz$:

$$\begin{aligned} W_2 &= 117,600\int_{z=0}^{z=2}(-z)\sqrt{4-z^2}dz \\ &= 58,800\int_{z=0}^{z=2}\sqrt{4-z^2}(-2zdz) \\ &= 58,800\int_{u=4}^{u=0}\sqrt{u}du \\ &= 58,800\left.\frac{u^{3/2}}{3/2}\right|_{u=4}^{u=0} \\ &= 58,800\left(0-\frac{2}{3}4^{3/2}\right) \\ &= -58,800\cdot\frac{2}{3}\cdot 8 = -313,600 \end{aligned}$$

Thus the total work done is

$$W_1 + W_2 = 235,200\pi - 313,600$$

Problem 12 If $f(x)$ is an increasing function on $[0, 1]$, rank the following in order from least to greatest:

- f(0)
- f(1)
- The left endpoint approximation to $\int_0^1 f(x)dx$ with $n = 5$ rectangles.
- The right endpoint approximation to $\int_0^1 f(x)dx$ with $n = 5$ rectangles.
- The average value of f on $[0, 1]$.

Solution: The first step is to reinterpret some of the quantities listed here. Notice that, since the interval of integration has length 1, the average value of f on $[0, 1]$ is given by

$$f_{\text{avg}} = \frac{1}{1-0}\int_0^1 f(t)dt = \int_0^1 f(t)dt$$

We know that, whenever f is an increasing function, the left endpoint of an interval gives a value lower than the other values in the interval, and the right endpoint gives a value greater than the other values in the interval; thus the left endpoint approximation will be an underestimate for the integral, and the right endpoint will be an overestimate. Therefore, we have

left endpoint approximation $<$ average value of f $<$ right endpoint approximation

It remains to see where $f(0)$ and $f(1)$ fit in to the heirarchy. Notice that the left endpoint approximation is given by

$$L_5 = \frac{1}{5}(f(0) + f(1) + f(2) + f(3) + f(4))$$

which is the average of the values $f(0)$, $f(1)$, $f(2)$, $f(3)$, and $f(4)$. Since each of these is at least as big as $f(0)$, and the latter 4 are greater than $f(0)$ (since f is increasing), this left endpoint approximation is greater than $f(0)$. Similarly, the right endpoint approximation is the average of $f(1)$, $f(2)$, $f(3)$, $f(4)$, and $f(5)$— all of which are no greater than $f(1)$, and the first 4 of which are strictly less than $f(1)$. Thus the right endpoint approximation is less than $f(1)$.

Summarizing, we have the following order from least to greatest:

- f(0)
- The left endpoint approximation to $\int_0^1 f(x)dx$ with $n = 5$ rectangles.
- The average value of f on $[0, 1]$.
- The right endpoint approximation to $\int_0^1 f(x)dx$ with $n = 5$ rectangles.
- f(1)

It may be useful to interpret these quantities as given by the areas of rectangles over the x-axis (see attached figure). The value $f(0)$ is equal to the area of a rectangle streching from 0 to 1 of height $f(0)$, which lies entirely underneath all points on the graph of f from 0 to 1. The left endpoint approximation consists of rectangles each lying underneath the graph *in each respective subinterval*, though all but the leftmost stretch above $f(0)$. Similarly on the other extreme, the right endpoint approximation is given by the areas of rectangles, each stretching above the graph in each subinterval, though all but the rightmost lie underneath the value $f(1)$. The value $f(1)$ corresponds to a rectangle of length 1 and height $f(1)$, which is entirely above all points in the graph of f (and rectangles of the right endpoint approximation).

Problem 13 For each of the following, determine if the improper integral converges or diverges. If it converges, evaluate the integral.

1. $\int_{-\infty}^{\infty} 2xe^{-x^2}dx$
2. $\int_{-\infty}^{\infty} \frac{1}{2x}e^{-x^2}dx$

Solution:

1. First, observe that the integrand is everywhere continuous, so we do not have to worry about any "Type II" convergence issues; the only thing to check is whether or not the integral converges out to $\pm\infty$. For this, split the integral into two parts, one for each (infinite) limiting endpoint:

$$\begin{aligned} \int_{-\infty}^{\infty} 2xe^{-x^2}dx &= \int_{-\infty}^{0} 2xe^{-x^2}dx + \int_{0}^{\infty} 2xe^{-x^2}dx \\ &= \lim_{R\to\infty}\int_{-R}^{0} 2xe^{-x^2}dx + \lim_{R\to\infty}\int_{0}^{R} 2xe^{-x^2}dx \end{aligned}$$

if both of these limits exist.

Next, notice that we can compute these (finite) integrals with the substitution $u = -x^2$, since this gives $du = -2xdx$, and we have

$$\begin{aligned} \int 2xe^{-x^2}dx &= -\int e^{-x^2}(-2xdx) \\ &= -\int e^{u}du = -e^{u} + C \\ &= -e^{-x^2} + C \end{aligned}$$

Therefore we have

$$\begin{aligned} \lim_{R\to\infty}\int_{-R}^{0} 2xe^{-x^2}dx &= \lim_{R\to\infty} -e^{-x^2}\Big|_{x=-R}^{x=0} \\ &= \lim_{R\to\infty}\left(-e^{-0^2} + e^{-(-R)^2}\right) \\ &= -e^{0} + \lim_{R\to\infty} -e^{-R^2} \\ &= -1 + 0 = -1 \end{aligned}$$

Similarly,

$$\begin{aligned}\lim_{R\to\infty}\int_0^R 2xe^{-x^2}dx &= \lim_{R\to\infty} -e^{-x^2}\Big|_{x=0}^{x=R} \\ &= \lim_{R\to\infty} -e^{-R^2}+e^{-0^2} \\ &= 0+1=1\end{aligned}$$

This means that both halves converge, and so our original integral from $-\infty$ to ∞ converges, and its value is equal to

$$\int_{-\infty}^{\infty} 2xe^{-x^2}dx = \lim_{R\to\infty}\int_{-R}^{0} 2xe^{-x^2}dx + \lim_{R\to\infty}\int_{0}^{R} 2xe^{-x^2}dx = -1+1=0$$

Thus the integral converges to 0.

2. Here there are two issues to worry about: not only is the interval of integration infinite, but the integrand is also discontinuous at 0, so we have both "Type I" and "Type II" convergence issues. Break up the integral into 4 pieces, one for each potential convergence problem:

$$\int_{-\infty}^{\infty}\frac{1}{2x}e^{-x^2}dx = \int_{-\infty}^{-1}\frac{1}{2x}e^{-x^2}dx + \int_{-1}^{0}\frac{1}{2x}e^{-x^2}dx + \int_{0}^{1}\frac{1}{2x}e^{-x^2}dx + \int_{1}^{\infty}\frac{1}{2x}e^{-x^2}dx$$

This integral is not easily evaluated; however, let's try to see if we can determine whether or not it converges. Near $\pm\infty$, the factor e^{-x^2} is very very small, and multiplying it by $\frac{1}{2x}$ only makes it smaller— this suggests that the first and last pieces should converge (a rigorous argument would use the Comparison Test, eg. comparing with the function e^x and e^{-x}, for x negative and x positive, respectively). On the other hand, near 0, the factor e^{-x^2} is close to a constant, while $\frac{1}{2x}$ gets very large, and the whole integral behaves like $\int \frac{\text{constant}}{x}dx$, which we know diverges.

So let's apply the Comparison Test to show that the integral from 0 to 1 diverges. For $0 \le x \le 1$, the function e^{-x^2} satisfies

$$e^{-x^2} \ge e^{-1} \ge \frac{1}{3}$$

Therefore

$$\int_0^1 \frac{1}{2x}e^{-x^2}dx \ge \int_0^1 \frac{1}{2x}\cdot\frac{1}{3}dx = \frac{1}{6}\int_0^1\frac{dx}{x}$$

But we know that this last integral diverges, since

$$\begin{aligned}\lim_{R\to 0}\int_R^1 \frac{dx}{x} &= \lim_{R\to 0}\ln(x)|_{x=R}^{x=1}\\ &= \ln(1) - \lim_{R\to 0}\ln(R) = -\lim_{R\to 0}\ln(R)\end{aligned}$$

and $\ln(R) \to -\infty$ (as $R \to 0$) does not converge.

Therefore, our original integral $\int_{-\infty}^{\infty} \frac{1}{2x} e^{-x^2} dx$ diverges.

Problem 14 A particle starts at the origin at time $t = 0$, and traces out a path given by

$$\begin{aligned} x(t) &= t \\ y(t) &= 2t^2 \end{aligned}$$

for each $t \geq 0$.

1. Express the length $l(T)$ of the path traced out by the particle from $t = 0$ to a time $t = T$ as an integral, but do not evaluate it.
2. Find $l'(T)$. (Side note: this is the speed of the particle at time T.)
3. Evaluate $l'(2)$.

Solution:

1. Since we are given the parametrization of the curve, this is simply given by the arc length formula

$$l(T) = \int_{t=0}^{t=T} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

So the next step is to evaluate the derivatives:

$$\begin{aligned} \frac{dx}{dt} &= \frac{d}{dt}(t) = 1 \\ \frac{dy}{dt} &= \frac{d}{dt}(2t^2) = 4t \end{aligned}$$

Therefore

$$l(T) = \int_{t=0}^{t=T} \sqrt{1 + (4t)^2} dt = l(T) = \int_{t=0}^{t=T} \sqrt{1 + 16t^2} dt$$

2. Since $l(T)$ is defined as the integral of $f(t) = \sqrt{1 + 16t^2}$ from 0 to T, the Fundamental Theorem of Calculus implies that

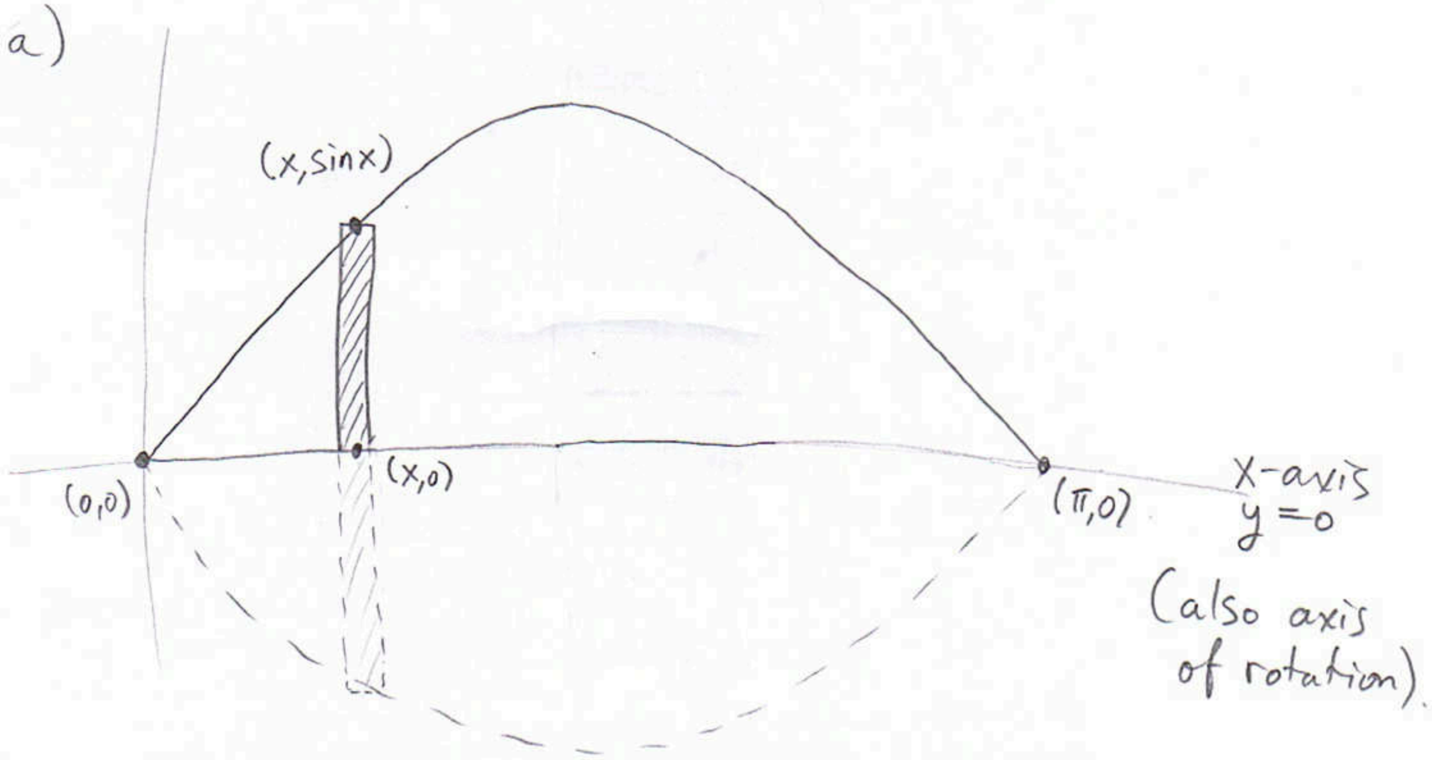
$$l'(T) = f(T) = \sqrt{1 + 16T^2}$$

3. Since part (2) gave us the function $l'(T) = \sqrt{1 + 16T^2}$, we simply evaluate this function at the point $T = 2$, giving

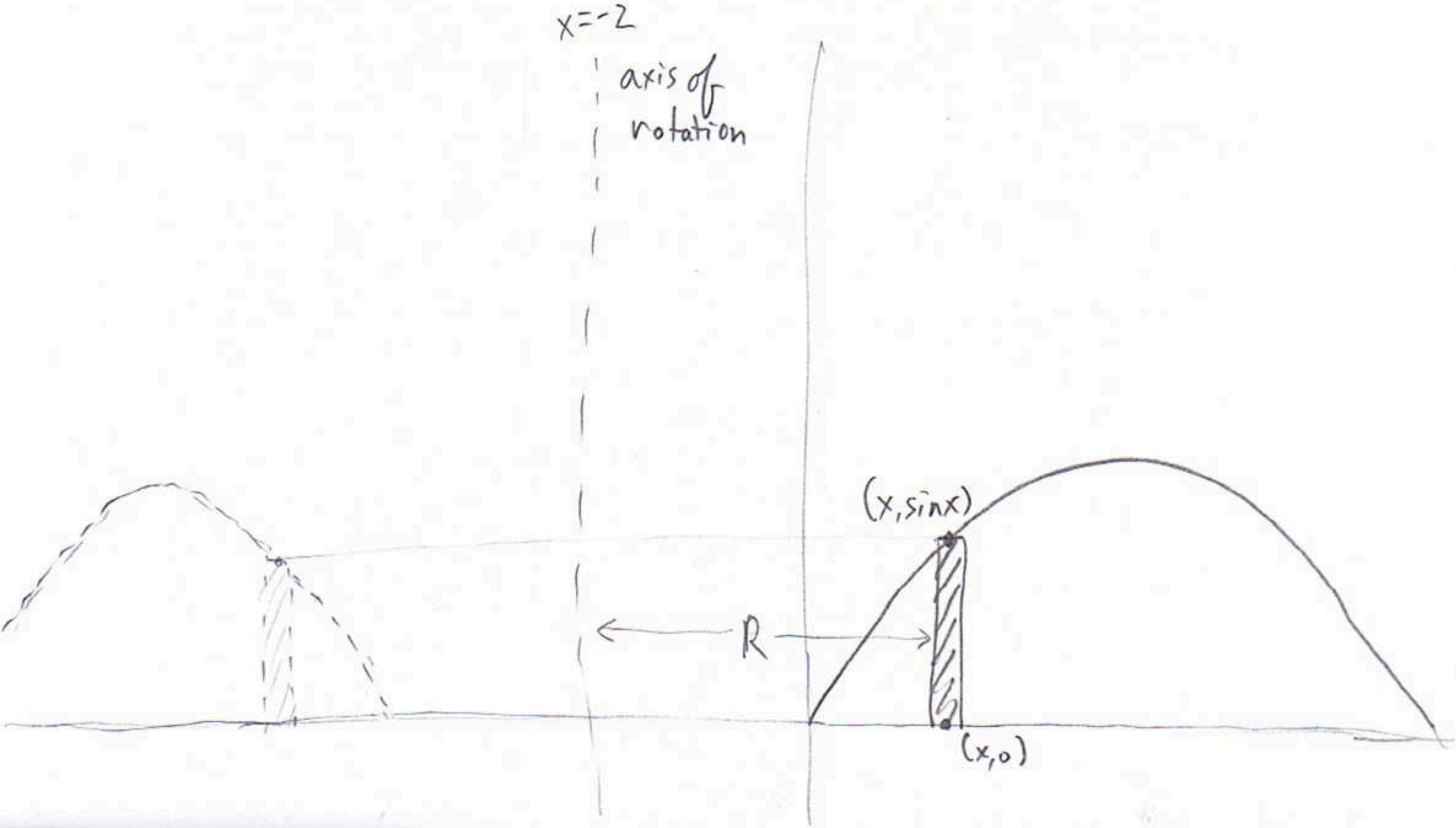
$$l'(2) = \sqrt{1 + 16(2)^2} = \sqrt{1 + 16 \cdot 4} = \sqrt{65}$$

Problem 8

1. a)

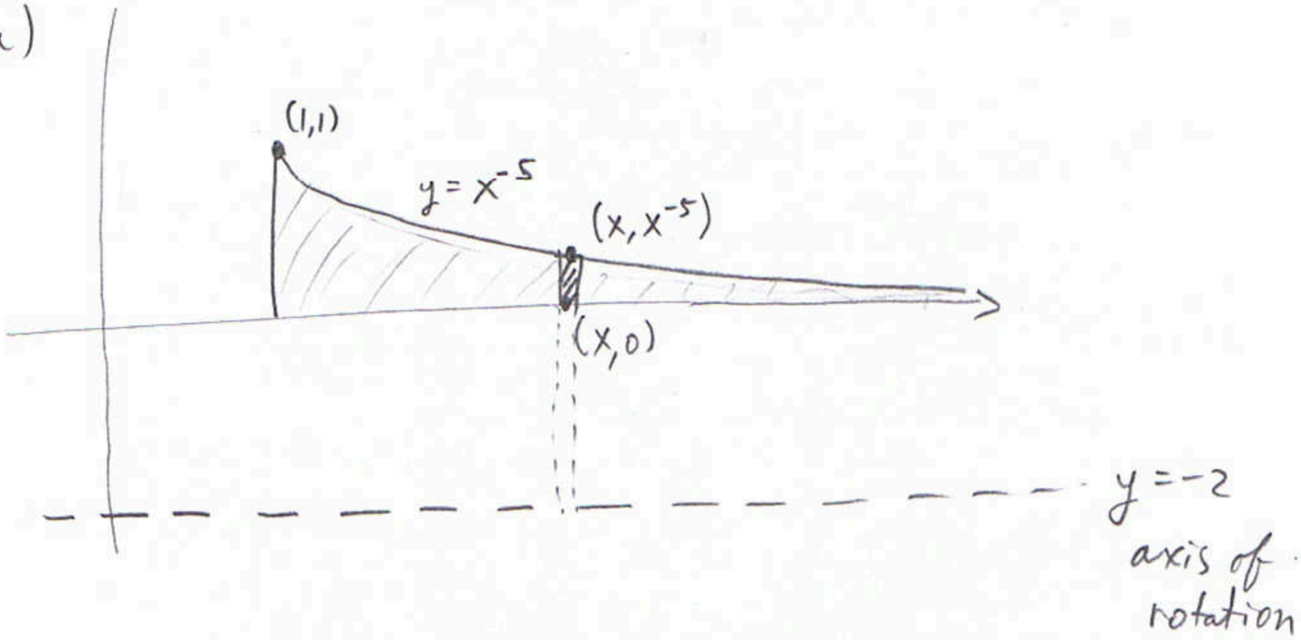


1. b)

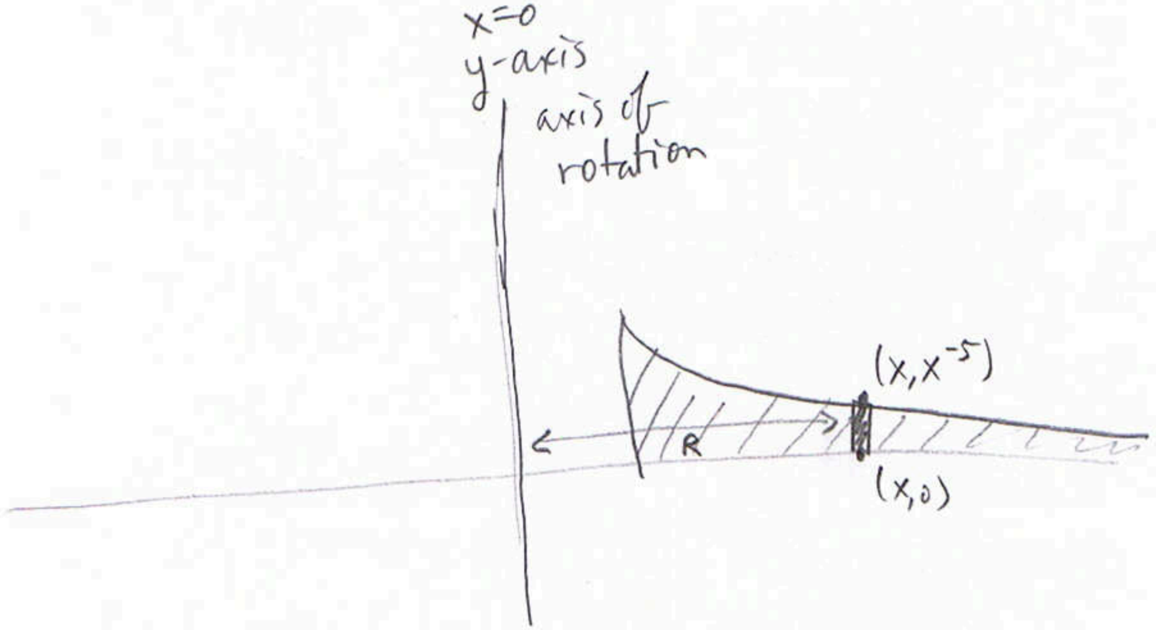


Problem 8

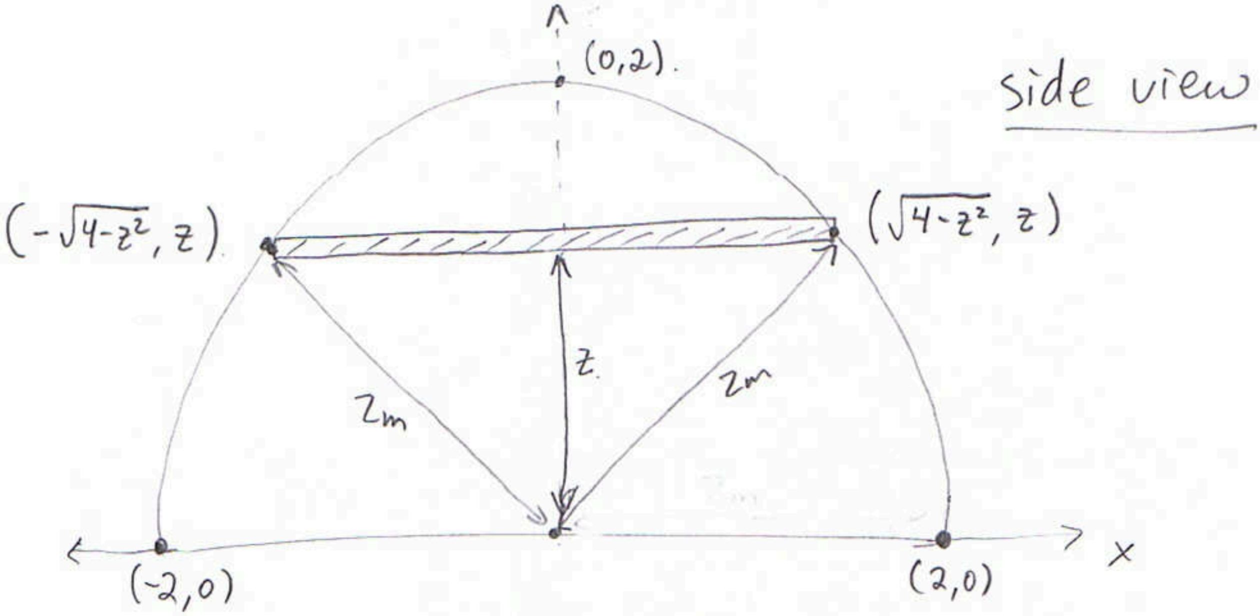
2. (a)



(b)



Problem 11



top view of each slice: