MAT 122: Overview of Calculus

Summer Session 2, 2014

Home Syllabus Schedule and Homework Quizzes and Exams

Home

Welcome to the course website for MAT 122: Overview of Calculus with Applications. Please check this website frequently, especially the announcements below and the course schedule availble on the schedule and homework page. These sections will be updated frequently throughout the course. Use the navigation bar above to access other important information about the course.

Announcements

All course announcements will be posted here.

- **August 14**: Information about final grades has been posted to the Quizzes and Exams section. I will be available in my office on Friday, Aug 15, in the afternoon if you'd like to turn in the extra credit, pick up any old homework, look over your final exam (I will keep all final exams to file with the Math department), or if you have any questions about grades or the course in general.
- **August 8**: The sample final exam is posted, and available on the Quizzes and Exams page. An optional extra credit assignment has also been posted to the Schedule and Homework section.
- **July 18**: Review matericals for the upcoming midterm exam have been posted to the Quizzes and Exams section. They include a list of topics for the midterm, a sample midterm, and solutions to the sample midterm.
- **July 9**: Solutions to Quiz 1 have been posted. For an interesting discussion of the discount rate and present value of a future payment (section 1.7), see the story and podcast here.
- **June 18**: Course website available

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Syllabus

The course syllabus, with schedule, is also available as a PDF here. Note that the schedule in this document will not be updated as the course progresses. The most up-to-date schedule is availble on the Schedule and Homework page.

General Information

Course: MAT 122, Overview of Calculus with Applications
Instructor: Joseph Thurman
Time and Location: MW 1:30–4:55 pm, Physics P-112
Instructor email: jthurman AT math.sunysb.edu
Instructor's Office: 2-107 in the Math building
Textbook: *Applied Calculus*, by Hughes-Hallett, Gleason, *et al.*, 4th Edition

Email and this course website will be the main avenue for communication outside of class. Please check this website frequently so that you are always aware of upcoming assignments and exams. Urgent announcements (e.g., class cancellations) will be emailed to enrolled students using Blackboard. Make sure to check the email address listed in your contact information on Blackboard (most likely "firstname.lastname@stonybrook.edu").

Course Description

The aim of this course is to introduce students to the basic ideas and techniques of differential and integral calculus. The focus is on computation and problem solving, while avoiding technical details. This course is appropriate for students who intend to take only a single semester of calculus.

From the course catalog:

> The basics of calculus in a self-contained, one-semester course. Properties and applications of polynomial, exponential, and logarithmic functions. Derivatives: slopes, rates of change, optimization, integrals, area, cumulative change, and average. The fundamental theorem of calculus. Emphasis on modeling examples from economics.

Office Hours

The instructor will hold office hours on Tuesdays from 1-3 pm and on Thursdays from 1-2 pm. All scheduled office hours will take place in the Math Learning Center (MLC), in the basement of the Mathematics building. If you are unable to attend normal office hours, please email me to schedule a meeting.

In addition to the instructor's office hours, the MLC is also open during the summer sessions. The MLC is a drop-in help center available to all students, staffed by graduate students in the Mathematics department. During the second summer session, the MLC will be open from 11 am - 6 pm on Mondays - Thursdays.

Grading Policy

Grades will be assigned based on student performance on quizzes, homework, a midterm exam, and a final exam.

Quizzes: Starting with the second class meeting, each class will begin with a short (10-15 minute) quiz covering the material from the previous lecture. There will be no quizzes on exam days. The lowest quiz score will be dropped. Quiz performance will count for **10%** of the overall grade.

Homework: Homework will be assigned at the end of each class meeting, to be turned in at the start of the next class. Most homework will consist of problem sets of exercises selected from the textbook. Homework assignments will be posted to the Schedules and Homework page. **Late homework will not be accepted**. Homework performance will count for **20%** of the overall grade.

In addition to the required problems, recommended homework problems will also be assigned to give students more opportunity for practice. These problems will not be graded and do not count for any credit.

Midterm Exam: There will be an in-class midterm during the first half of class on Wednesday, July 23. Information about the format and content of the exam will be posted to the Quizzes and Exams section of the website in the week before the exam. The midterm will account for **30%** of the overall grade.

Final Exam: There will be an in-class final exam during the final class period on Wednesday, August 13. The exam will be cumulative. Information about the format and content of the exam will be posted to the Quizzes and Exams section of the website in the week before the exam. The final exam will account for **40%** of the overall grade.

Calculator Use

A scientific calculator with basic arithmetic and trigonometric functions may be necessary in this course to complete some homework assignments. A graphing calculator will never be necessary. No calculators of any kind will be allowed during in-class quizzes or exams.

Academic Integrity

Each student must pursue his or her academic goals honestly and be personally accountable for all submitted work. Representing another person's work as your own is always wrong. Faculty are required to report any suspected instance of academic dishonesty to the Academic Judiciary. For more comprehensive information on academic integrity, including categories of academic dishonesty, please refer to the academic judiciary website at stonybrook.edu/uaa/academicjudiciary.

For this course, students may collaborate on homework assignments. However, all written work must be the student's own. Students who collaborate on homework must write their homework separately, and in their own words. Collaboation is not allowed on in-class quizzes or exams.

Disability Support Services

If you have a physical, psychological, medical, or learning disability that may impact your course work, please contact Disability Support Services at (631) 632-6748 or studentaffairs.stonybrook.edu/dss/. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential.

Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to stonybrook.edu/ehs/fire/disabilities.shtml

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Schedule and Homework

Please check this page often, as it will be frequently updated with homework assignments. The schedule is tentative, and will be adjusted as the class progresses.

Extra Credit- An optional extra credit assignment, due 8/15, is available here.

Class	Topic	Required HW	Suggested HW	Note
7/7	Overview of Course and Syllabus 1.1 — Functions 1.2 — Linear functions 1.3 — Rates of change	1.1 — 6,12,20 1.2 — 2,10,12,22 1.3 — 10,12,24	1.1 — 1,3,15,23 1.2 — 3,7,13,17 1.3 — 9,17,27	Solutions
7/9	1.5, 1.7 — Exponential Functions 1.6 — Logarithms	1.5 — 6,8,22 1.6 — 10,16,22 1.7 — Read Section, 4	1.5 — 7,15,25 1.6 — 13,17,19,39 1.7 — 9,13	Solutions
7/14	1.8 — New Functions from Old 1.9 — Polynomial Functions	See PDF here	See PDF here	Solutions
7/16	2.1 — Instantaneous Rates of Change 2.2 — The Derivative Function 2.3 — Interpreting the derivative	2.1 — 2,10,16,18,20 2.2 — 10,14,16,30 2.3 — 2	2.1 — 3,5,13,15,19 2.2 — 11,13,15,25	Solutions
7/21	2.3 — Interpreting the derivative, continued 2.4 — The Second Derivative Review for Midterm	None - Study for Midterm	Sample Midterm 2.3 — 15,25,27 2.4 — 5,7,9,17,19,21	No Homework Due
7/23	Midterm Exam 3.1 — Derivatives of Polynomial Functions	3.1 — 12,18,34,46,50a	3.1 — 1-35, odds	Solutions
7/28	3.2 — Derivatives of Exponential and Log functions 3.4 — The Product and Quotient Rules 3.3 — The Chain Rule	3.2 — 10,12,28,34 3.4 — 10,16,24,34,36	3.2 — 1-27, odds 3.4 — 1,3,7,9,11,23,25,27, 29,31	Solutions
7/30	3.3 — The Chain Rule, continued 4.1 — Local Extrema 4.2 — Inflection Points	See PDF here.	See PDF here. Extra Credit assignment on page 2	Solutions

8/4	4.3 — Global Extrema 5.2 — The Definite Integral 5.3 — The Definite Integral as Area 5.4 — Interpretations of the Definite Integral	Chp 3, "Focus on Practice", pg 168 — 34, 42, 58 4.3 — 6, 20, 34,36 5.1 — 10, 12b	Chp 3, "Focus on Practice" — All odds without sine or cosine 4.3 — 19,37,39 5.1 — 9,15,17	Solutions
8/6	Sections 5.3 and 5.4, continued 7.1 — Antiderivatives 5.5, 7.3 — The Fundamental Theorem of Calculus	5.2 — 4,16,18 7.1 — 38,40,46,50 7.3 — 4,6,16	5.2 — 3,7,19 7.1 — 1-23, 33-51 odds 7.3 — 1-15, odds	Solutions
8/11	7.2 — Integration by Substitution Applications of the Definite Integral Review for Final	None - Study for Final	5.3 — 18,19 6.1 — 11,12a 7.2 — 1-13 odd 7.3 — 23,25,33	No Homework Due
8/13	Final Exam	None	Optional extra credit due at 5 pm on 8/15	—

The schedule was last modified on 10/10/48321 03:21:13

MAT 122: Overview of Calculus

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Quizzes and Exams

Quizzes

- Solutions to Quiz 1
- Solutions to Quiz 2. This quiz was not collected for a grade.
- Solutions to Quiz 3
- Solutions to Quiz 4
- Solutions to Quiz 5
- Solutions to Quiz 6
- A blank copy of Quiz 7, and solutions. We didn't take this quiz in class due to time constraints. However, please take a look at this quiz - a question with the same format will be on the final exam.
- Solutions to Quiz 8
- Solutions to Quiz 9

Midterm Exam

A blank copy of the midterm is available here, while a copy of the midterm with solutions is here. For reference, the sample midterm and solutions to the sample midterm are still available.

Grades for the midterm are available on Blackboard. Also posted on Blackboard is an "Estimated Grade." This is meant to give you a rough idea of your current standing in the class. This is no guarantee of final grades - for example, if you have approximately a B now, this does not mean you will get a B as your final grade, especially if you do not perform well on the final exam. This score was computed using all graded work up to and including the midterm. In particular, it includes your scores on quizzes 1, 3, and 4, even though the lowest quiz grade will be dropped when computing the final grades. The approximate ranges for the letter grades are as follows -

Letter Grade	Range
A	83 — 100 %
B	65 — 82 %
C	50 — 64 %
D	40 — 49 %
F	39 % or less

Please note that this table and the "Estimated Grade" column in Blackboard will not be updated further.

Final Exam

A blank copy of the final exam is available here. Solutions to the exam are here. For reference, the sample final and sample final solutions are still available.

Overall grades have been calculated for the course and posted to the Blackboard grade center in the "Overall %" column. This grade is an averaging of your quiz, homework, midterm, and final exam percentages, weighted 10%, 20%, 30%, and 40%, respectively, as described in the course syllabus. The lowest quiz grade was dropped when computing the quiz percentage. The table below shows how these percentages correspond to letter grades.

Letter Grade	Range
A	85 — 100 %
A -	81 — 84 %
B+	77 — 80 %
B	67 — 76 %
B-	63 — 66 %
C+	59 — 62 %
C	50 — 58 %
C-	46 — 49 %
D	37 — 45 %
F	36 % or less

If you do not complete the extra credit assignment, this will be your final grade. If you do complete the extra credit assignment, your grade will be updated sometime on Saturday, Aug 16, after the extra credit has been graded. Final grades will be posted to SOLAR over the weekend.

MAT 122 - Extra Credit

Grading Information— This extra credit assignment is optional, and due on Friday, 8/15, at 5:00pm. It can be turned in to my office (2-107 in the Math building) anytime - if I'm not there, simply slide it under the door. It will be worth 50 homework points, as this extra credit is about the same length as a normal homework assignment.

Do not think about this extra credit as an opportunity to improve your overall grade in the course. If you're interested in improving your grade, you'd be better off spending your time studying for the final exam, which is worth 40% of your overall grade. This is also why this extra credit is due the Friday after the exam - you can spend the days before the final focusing on studying for the final, and still have time to attempt the extra credit after the exam if you so wish.

Calculus in Business and Economics— Calculus applications are very important in business and economics, providing ways to optimize business strategies, to understand how economies are changing, and to give a theoretical underpinning to economics thinking. Even the basic techniques of calculus that we've learn (simple differentiation and integration) can be very powerful in these contexts. The difficulty in understanding these applications is usually not in the calculus itself, but rather in understanding how economics concepts like cost, revenue, supply, demand, etc, can be modeled by functions. In this extra credit assignment, you'll learn a what some of these concepts are, how they can be modeled mathematically, and how those mathematical models can be better understood using calculus.

This extra credit assignment has three parts -

1. Before using calculus, we need to understand some business terms. For this, you should
   (a) Read section 1.4, focusing on cost, revenue, profit, and marginal cost/revenue/profit.
   (b) Do problems 2, 10, and 14 from section 1.4.
2. As we have seen in class, a main application for differential calculus is finding maximum or minimum values of a function. This is useful in business, as it gives a way for businesses to maximize profit, for example, or minimize costs. To understand this application,
   (a) Read sections 2.5 and 4.4.
   (b) Complete problem 12 and 14 from section 2.5 and problems 2 and 24 from section 4.4.
3. We learned one way to calculate the present and future value of a payment in section 1.7. A more sophisticated way of computing present and future value is to use the definite integral. To understand this application,
   (a) Read section 6.3.
   (b) Complete problems 8 and 10 from this section.

Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to `http://www.stonybrook.edu/ehs/fire/disabilities.shtml`.

Tentative Course Schedule

Below is a tentative schedule for the course. For the most up-to-date schedule, which will include homework assignments, see the course web page. The numbers refer to sections of the textbook.

Date	**Topics Covered**
7/7	• Overview of Course and Syllabus • 1.1 — Functions • 1.2 — Linear Functions • 1.3 — Rates of Change • 1.4 — Applications to Economics
7/9	• 1.9 — Polynomial Functions • 1.5, 1.7 — Exponential Functions • 1.6 — Logarithms
7/14	• 1.8 — New Functions from Old • 2.1 — Instantaneous Rates of Change
7/16	• 2.2 — The Derivative Function • 2.3 — Interpreting the Derivative • 2.5 — Marginal Cost and Revenue
7/21	• 2.4 — The Second Derivative • 3.1 — Derivatives of Polynomial Functions • Review for Midterm Exam
7/23	• Midterm Exam • 3.2 — Derivatives of Exponential and Logarithmic Functions • 3.4 — The Product and Quotient Rules
7/28	• 3.3 — The Chain Rule • 4.1, 4.3 — Local and Global Extrema
7/30	• 4.2 — Inflection Points • 4.4 — Profit, Cost, and Revenue • 5.1 — Distance and Accumulated Change
8/4	• 5.2 — The Definite Integral • 5.3 — The Definite Integral as Area • 5.4 — Interpretations of the Definite Integral
8/6	• 7.1 — Antiderivatives • 5.5, 7.3 — The Fundamental Theorem of Calculus
8/11	• 7.2 — Integration by Substitution • Applications of the Definite Integral • Review for Final
8/13	Final Exam

MAT 122 Homework 1 Solutions

Section 1.1, Problem 6 —

Part a: When $p = 0$, the graph shows that $r = 8$. (This is the vertical intercept of the graph). When $p = 3$, we have $r = 7$.

Part b: $f(2)$ represent the value of r when $p = 2$. Using the graph, $f(2) = 10$.

Section 1.1, Problem 12 — Let $f(x) = x^2 + 2$.

Part a: The value when $x = 0$ is $f(0) = 0^2 + 2 = 2$. (This is the value of the vertical intercept).

Part b: $f(3) = 3^2 + 2 = 9 + 2 = 11$.

Part c: We need to solve the equation $11 = x^2 + 2$. Subtracting 2 from each side gives that $x^2 = 9$, so $x = 3$ and $x = -3$.

Part d: There are no values of x that give y value of 1. If there were such a value, we could solve the equation $1 = x^2 + 2$. This is the same as $x^2 = -1$, but the square of a real number is always positive. Therefore there are no solutions, so there cannot be any value of x that gives y a value of 1.

Section 1.1, Problem 20 —

Part a: $f(30) = 10$ means that after 30 minutes, the temperature of the object is $10°C$. The units (minutes, $°C$) are necessary in the answer.

Part b: The vertical intercept is the value of the function when $t = 0$. On the graph, we see that the value of the vertical intercept is a. This means that the object had a temperature of $a°C$ when placed outside. A horizontal intercept is a value for t where the graph of the function crosses the horizontal axis, that is, a value for t such that $f(t) = 0$. From the graph, we see that there is a horizontal intercept at b. This means that after b minutes, the temperature of the object is $0°C$.

Section 1.2, Problem 2 —

We need to rewrite the equation in slope-intercept form, $y = mx + b$. We have

$$\begin{aligned} 3x + 2y = 8 &\quad \Rightarrow \\ 2y = -3x + 8 &\quad \Rightarrow \\ y = \frac{-3}{2}x + 4 & \end{aligned}$$

Therefore the slope of the line is $m = \frac{-3}{2}$ and the y-intercept is $b = 4$.

Section 1.2, Problem 10 —

Part a: Lines l_2 and l_3 have the same slope. Notice that this means they are parallel - they never meet. Line l_2 has the larger y-intercept, because it crosses the y-axis at a larger value than the line l_3.

Part b: Lines l_1 and l_3 have the same y-intercept, because they cross the y-axis at the same point. Line l_1 has the larger slope, as it is increasing faster than line l_3.

Section 1.2, Problem 12 —

The cost is $C(m) = 25 + (.05)m$.

Section 1.2, Problem 22 —

Part a: We are told this is a linear function. Using the point-slope equation for a line, we know that the formula must be given by $P - P_0 = m(t - t_0)$. Rearranging this expression to isolate P on the left, we have that the function must look like

$$P(t) = m(t - t_0) + P_0$$

were P represents production in million tons of grain, t represents years since 1975, m represents the slope, and (t_0, P_0) is a known point on the line.

From the problem statement, we have that when $t = 0$ (that is, for the year 1975) we have that P $= 1241$, and when $t = 30$ (the year 2005), $P = 2048$. Therefore our line passes through the points $(0, 1241)$ and $(30, 2048)$. Using these two points to compute the slope, we have

$$m = \frac{2048 - 1241}{30 - 0} = \frac{807}{30} = 26.9$$

Putting this value into the equation above, along with the point $(0, 1241)$ - you could also use the point $(30, 2048)$ - gives the function

$$P(t) = 26.9t + 1241$$

Part b: The slope, with units, is $26.9 \frac{\text{million tons}}{\text{years}}$. This means that each year the amount of grain produced increases by 26.9 million tons.

Part c: The vertical intercept is the value of the function when $t = 0$, and is therefore 1241 million tons. This means that worldwide grain production in 1975 was 1241 million tons.

Part d: The year 2015 is 40 years after 1975, so we are looking for the value of P when $t = 40$. This is

$$P(40) = 26.9 \cdot 40 + 1241 = 2317$$

so the world grain production in 2015 is predicted to be 2317 million tons.

Part e: We are trying to find t such that $P(t) = 2500$. Solving this equation, we have

$$\begin{aligned} 26.9t + 1241 &= 2500 \quad \Rightarrow \\ 26.9t &= 1259 \quad \Rightarrow \\ t &= \frac{1259}{26.9} \approx 46.8 \end{aligned}$$

Therefore grain production will reach 2500 million tons approximately 46.8 years after 1975, or about in 2022.

Section 1.3, Problem 10 —
Part a: From D to E and from H to I.
Part b: From A to B and from E to F.
Part c: From C to D and from G to H.
Part d: From B to C and from F to G.

Section 1.3, Problem 12 — Using the formula, the average rate of change of the function $f(x) = 2x^2$ from $x = 1$ to $x = 3$ is

$$avgROC = \frac{f(3) - f(1)}{3 - 1} = \frac{2 \cdot 3^2 - 2 \cdot 1^2}{3 - 1} = \frac{18 - 2}{2} = \frac{16}{2} = 8$$

Section 1.3, Problem 24 —

Part a: Using the formula, the average rate of change from 1996 to 2003 is

$$avgROC = \frac{831 - 1517}{2003 - 1996} = \frac{-686}{7} = -98$$

This quantity has units of millions of pounds per year. This means that on average, the amount of tobacco produced is decreasing by 98 million pounds every year during the time period from 1996 to 2003.

Part b: A *positive* average rate of change will occur when the amount of tobacco produced increases over an interval. The only interval when this occurs is between 1996 and 1997. For any other pair of years, the amount of tobacco produced the later year is less than the amount produced the earlier year.

MAT 122 Homework 2 Solutions

Section 1.5, Problem 6 — Let C represent the cost of the item, and let t denote the number of days after today.

Part a: Because the cost is decreasing by \$4 every day, this is a linear relationship. The initial cost is \$80, which means that when $t = 0$, $C = 80$. Thus 80 is the vertical intercept. Decreasing by \$4 a day means that the slope of the line is -4. Putting these facts together, the cost is $C(t) = -4t + 80$. Notice that this model wouldn't make good economic sense - on the 20th day the item would be free, and after that the merchant would have to pay people to take the item.

Part b: Because the cost is decreasing by 5% every day, this is an exponential relationship, and "decreasing" means that this is an exponential decay. The initial value is 80 and the rate is $r = -.05$ (negative because the price is decreasing). Using the formula for exponential functions, the cost is

$$C(t) = 80(1 - .05)^t = 80(.95)^t$$

Notice that this model makes better sense. Although the price is always decreasing, it will never be 0 or negative.

Section 1.5, Problem 8 —

Part a: The usual formula for exponential growth is $P(t) = P_0(1 + r)^t$, where r represents the annual growth rate. Writing the function $P(t) = 6.4(1.0126)^t$ in this form gives $P(t) = 6.4(1 + .0126)^t$. Therefore the current growth rate is $r = .0126$, or 1.26%.

Part b: The year 2004 corresponds to $t = 0$. Recalling that P_0 represents the vertical intercept (initial value), the population in 2004 if 6.4 billion. The year 2010 corresponds to $t = 6$. Therefore the population in 2010 is predicted to be $P(6) = 6.4(1.0126)^6 \approx 6.9$ billion people.

Part c: Using the formula for the average rate of change,

$$avgROC = \frac{6.9 - 6.4}{6 - 0} \approx .083$$

so the word population is increasing by 83 million people a year, on average.

Section 1.5, Problem 22 —

If we write P to denote the number of mussels in the bay and t to denote the number of years since 2003, then the information in the problem statement says that $P(0) = 2700$ and $P(1) = 3186$.

Part a: The slope of a line going through the points $(0, 2700)$ and $(1, 3186)$ is

$$m = \frac{3186 - 2700}{1 - 0} = 486$$

The units are mussels per year, and this means that the population is growing by adding 486 mussels every year.

Part b: Recall that exponential functions exhibit a constant relative growth rate. The relative growth rate for one year is

$$r = \frac{P(1) - P(0)}{P(0)} = \frac{3186 - 2700}{2700} = \frac{486}{2700} = .18$$

So the growth rate of the mussels is 18% per year.

Section 1.6, Problem 10 —

$$\begin{aligned}
10 &= 6e^{.5t} & \Rightarrow\\
\frac{10}{6} &= e^{.5t} & \Rightarrow\\
\ln\left(\frac{10}{6}\right) &= \ln\left(e^{.5t}\right) & \Rightarrow\\
\ln\left(\frac{10}{6}\right) &= .5t & \Rightarrow\\
t = 2\ln\left(\frac{10}{6}\right) &\approx & 1.022
\end{aligned}$$

Here we used the logarithm rule $\ln(e^x) = x$.

Section 1.6, Problem 16 —

$$\begin{aligned}
7\cdot 3^t &= 5\cdot 2^t & \Rightarrow\\
\ln(7\cdot 3^t) &= \ln(5\cdot 2^t) & \Rightarrow\\
\ln(7)+\ln(3^t) &= \ln(5)+\ln(2^t) & \Rightarrow\\
\ln(7)+t\ln(3) &= \ln(5)+t\ln(2) & \Rightarrow\\
t\left(\ln(3)-\ln(2)\right) &= \ln(5)-\ln(7) & \Rightarrow\\
t = \frac{\ln(5)-\ln(7)}{\ln(3)-\ln(2)} &= \frac{\ln\left(\frac{5}{7}\right)}{\ln\left(\frac{3}{2}\right)} \approx & -.8298
\end{aligned}$$

Here we used the logarithm rules $\ln(AB) = \ln(A) + \ln(B)$ and $\ln(A^p) = p\ln(A)$.

Section 1.6, Problem 22 —

Part a: For Question (i), we use the formula $P(t) = P_0(1+r)^t$ because the rate is an annual rate. P_0 represents the initial population, so we use $P_0 = 1000$. r represents the rate, so we use $r = .05$. Therefore the formula is $P(t) = 1000(1.05)^t$.

For Question (ii), we use the formula for a continuous rate, $P(t) = P_0e^{kt}$, where P_0 again represents the initial amount ($P_0 = 1000$) and k represents the continuous rate, again $k = .05$. The formula is then $P(t) = 1000e^{.05t}$.

Part b: Using the formula from question (i) above (annual rate), the estimate is

$$P(10) = 1000(1.05)^{10} = 1628.89$$

so the population will be approximately 1629 people.

Using the formula from question (ii) above (continuous rate), the estimate is

$$P(10) = 1000e^{.05\cdot 10} \approx 1648.72$$

so the population will be approximately 1649 people. Observe that a continuous growth rate of 5% gives more growth than a growth rate of 5% per year.

Section 1.7, Problem 4 —

We are asked to compute the present value of a payment of \$20,000 to be made in 6 years, with 10% interest compounded continuously. The formula for the present value P of a future payment B with continuous interest is

$$P = \frac{B}{e^{rt}}$$

where r is the interest rate and t is the time. In our case $B = 20000$, $r = .1$, and $t = 6$, so plugging these values into the formula gives.

$$P = \frac{20000}{e^{.1 \cdot 6}} \approx 10976.23$$

Therefore you need to save only \$10,976.23 now to to be able to pay \$20,000 in 6 years' time.

MAT 122 Homework 3 - Due 7/16

Required
Complete and turn in the following problems from the textbook -

Section 1.8— 6, 14, 22 [only find values for $f(g(x))$], 30
Section 1.9— 14

In addition to these problems from the textbook, please solve the following equations.

$$2x^2 + 6x = 8$$
$$2x^2 + 7x + 1 = 0$$

You may use any method - factoring or the quadratic formula.

Suggested
For extra practice, try

Section 1.8— 7, 13, 31
Section 1.9— 15

MAT 122 Homework 3 Solutions

Section 1.8, Problem 6 — Given that $f(x) = \frac{1}{x}$ and $g(x) = 3x + 4$, we have

Part a: We observe that $g(1)$ is $g(1) = 3 \cdot 1 + 4 = 7$, while $f(7) = \frac{1}{7}$. Therefore $f(g(1)) = \frac{1}{7}$.

Part b: We observe that $f(1)$ is $f(1) = \frac{1}{1} = 1$, which we already saw that $g(1) = 7$. Therefore $g(f(1)) = 7$.

Part c:

$$f(g(x)) = f(3x+4) = \frac{1}{3x+4}$$

Notice that if we plug 1 in for x in this expression we get $\frac{1}{7}$, our answer in part (a).

Part d:

$$g(f(x)) = g\left(\frac{1}{x}\right) = 3 \cdot \frac{1}{x} + 4 = \frac{3}{x} + 4$$

Notice that if we plug 1 in for x in this expression we get 7, our answer in part (b).

Part e: This is multiplication of functions, and we need to replace all of the x's with t's. This gives

$$f(t)g(t) = \left(\frac{1}{t}\right)(3t+4) = \frac{3t+4}{t} = 3 + \frac{4}{t}$$

Section 1.8, Problem 14 — There are a number of correct answers to this question, as there are many ways to write a given function as a composition of two different functions.

Part a: For the function $y = (5t^2 - 2)^6$, we could choose the inside function to be $u = 5t^2 - 2$. Then the outside function is $y = u^6$.

Part b: For the function $P = 12e^{-0.6t}$, we could choose the inside function to be $u = -0.6t$. Then the outside function is $P = 12e^u$.

Part c: For the function $C = 12\ln(q^3 + 1)$, we could choose the inside function to be $u = q^3 + 1$. Then the outside function is $C = 12\ln(u)$.

Section 1.8, Problem 22 — Using the table, we see that

$$f(g(-3)) = f(3) = 0$$

Computing the other values similarly gives the table

x	-3	-2	-1	0	1	2	3
$f(g(x))$	0	1	1	3	1	1	0

Section 1.8, Problem 30 — To make the functions easier for me to graph with a computer, I've assumed that the function is $f(x) = 2^x$. The graphs are shown in Figure 1 on the last page. The original function and each new function are shown. In the graphs showing two functions, the original function is shown in blue, while the answer to the question is shown in yellow.

The transformations, as described in words, are

(a) Shifted left 2 units
(b) Shifted right 1 unit
(c) Shifted down 4 units
(d) Shifted left 1 unit and up 3 units
(e) Stretched vertically, 3x

(f) Reflected over the x-axis, then shifted up 1 unit

Section 1.9, Problem 14 —

$$E = k \cdot v^3$$

Extra Problems— *Part a:* This equation can be solved by factoring.

$$\begin{aligned} 2x^2 + 6x = 8 &\ \Rightarrow \\ 2x^2 + 6x - 8 = 0 &\ \Rightarrow \\ 2(x^2 + 3x - 4) = 0 &\ \Rightarrow \\ 2(x-1)(x+4) = 0 & \end{aligned}$$

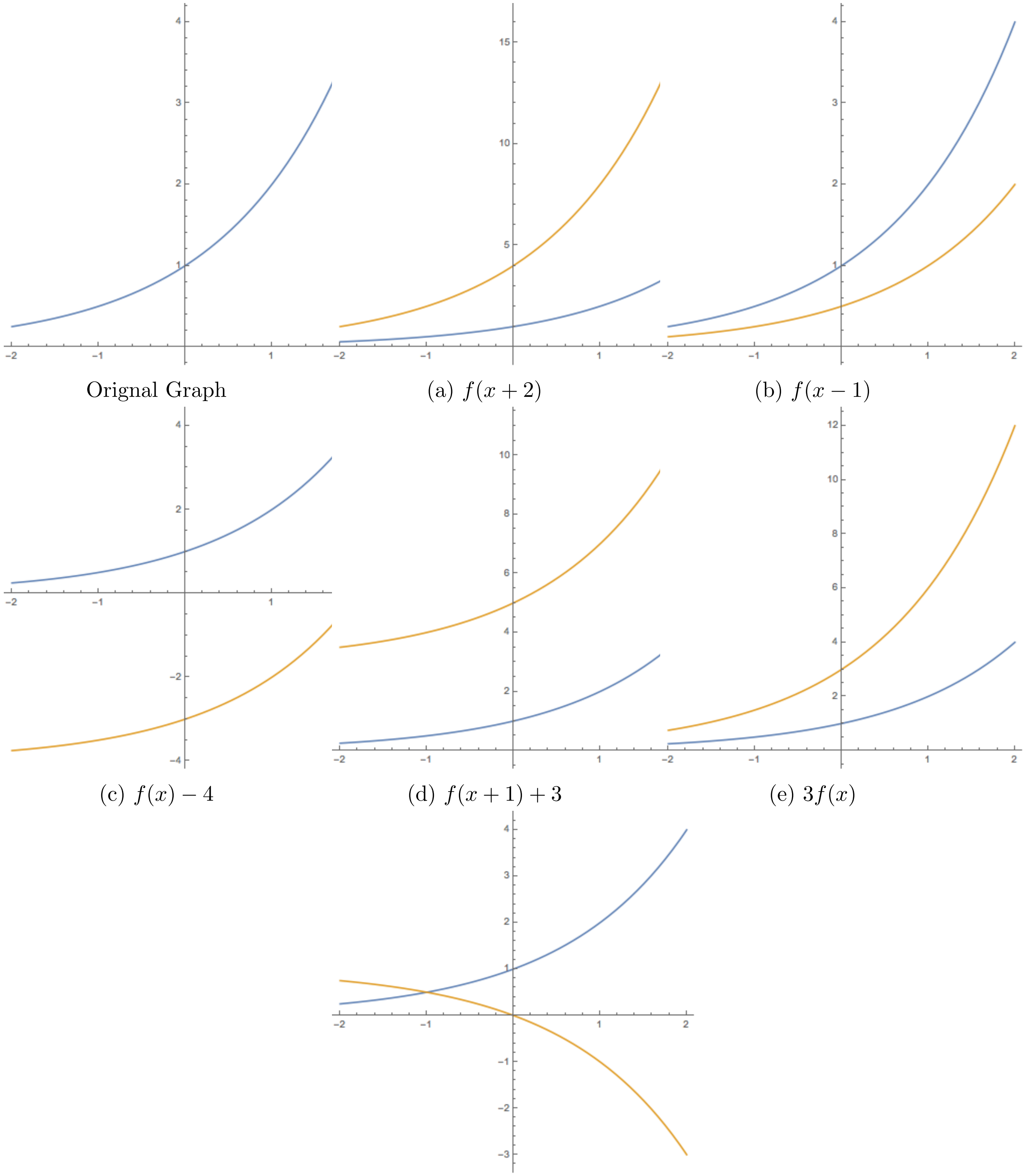
so the solutions are $x = 1$ and $x = -4$.

Part b: This equation does not factor, so we use the quadratic formula.

$$x = \frac{-7 \pm \sqrt{7^2 - 4 \cdot 2 \cdot 1}}{2 \cdot 2} = \frac{-7 \pm \sqrt{41}}{4}$$

We cannot simplify $\sqrt{41}$, so this is the final answer. The approximate values are

$$\begin{aligned} \frac{-7 + \sqrt{41}}{4} &\approx -0.149219 \\ \frac{-7 - \sqrt{41}}{4} &\approx -3.35078 \end{aligned}$$



Orignal Graph

(a) $f(x+2)$

(b) $f(x-1)$

(c) $f(x)-4$

(d) $f(x+1)+3$

(e) $3f(x)$

(f) $-f(x)+1$

Figure 1: Graphs for Problem 1.8.30

MAT 122 Homework 4 Solutions

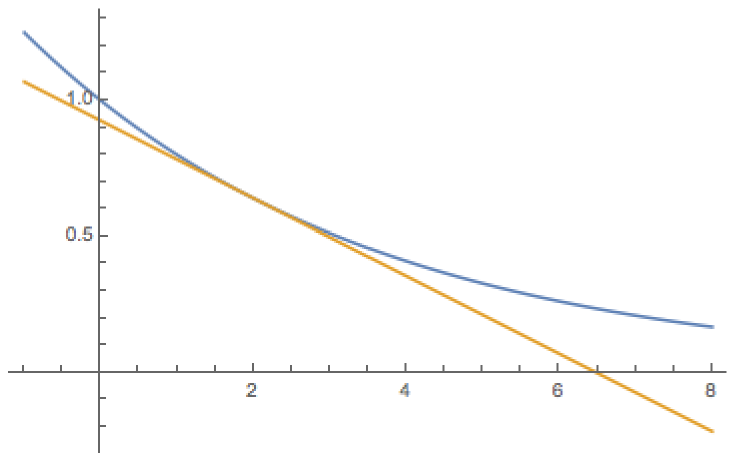
Section 2.1, Problem 2 —

Part a: The value of $f'(1950)$ is negative. Observe that the tangent line for the graph at that point would appear to be a decreasing linear function, hence have negative slope. This means that the number of farms is decreasing.

Part b: The tangent line at $t = 1960$ would appear to be decreasing more quickly than the tangent line at $t = 1980$. This means that the slope of the tangent line at $t = 1960$ is more negative, so $f'(1960)$ is more negative that $f'(1980)$.

Section 2.1, Problem 10 —

Part a: A graph of $g(t) = (0.8)^t$ is shown below in blue, while the tangent line at $t = 2$ is graphed in yellow. The slope of this tangent line is the value $g'(2)$. Since this line has negative slope, the value of $g'(2)$ is negative.



Part b: We estimate $g'(2)$ using the small interval $[2, 2.01]$. The average value of g over that interval is

$$avgROC = \frac{g(2.01) - g(2)}{2.01 - 2} \approx -0.142653$$

We can therefore estimate that $g'(2) = -0.142653$. Other answers on this question are possible.

We will later show how this derivative can be computed exactly. This will give that

$$g'(2) = \frac{-16}{25} \ln\left(\frac{5}{4}\right) \approx -0.142812$$

so we can see that our estimate is relatively accurate.

Section 2.1, Problem 16 —

Part a: The function appears to be increasing on the entire interval $[4, 8]$, so we can guess that $f'(6)$ should be positive. This indicates that the percentage of household with cable TV is growing in the year 1996.

Part b: To estimate the value of a derivative from a table, we take the average rate of change on the smallest interval possible. Here, these intervals have length 2. To estimate the value of the derivative at $t = 2$, we compute the average rate of change on the intervals $[0, 2]$ and $[2, 4]$. These are

$$\begin{aligned}
avgROC \text{ for } [0,2] &= \frac{61.5-59.0}{2-0} = 1.25 \\
avgROC \text{ for } [2,4] &= \frac{63.4-61.5}{4-2} = 0.95
\end{aligned}$$

Either of these could be an estimate for the value of $f'(2)$, but we can get another estimate by averaging them, to obtain the estimate

$$f'(2) \approx \frac{1.25+.95}{2} = 1.1$$

We repeat the same process for $f'(10)$ —

$$\begin{aligned}
avgROC \text{ for } [8,10] &= \frac{67.8-67.4}{10-8} = 0.2 \\
avgROC \text{ for } [10,12] &= \frac{68.9-67.8}{12-10} = 0.55
\end{aligned}$$

Again either of these would be an acceptable estimate, but their average yields another estimate,

$$f'(10) \approx \frac{0.2+0.55}{2} = 0.375$$

Any of these estimates has units of % of households/year. Each says that the percentage of people with cable television is growing each year, although we can observe that the growth is slower in the year 2000 ($t = 10$) than in the year 1992 ($t = 2$).

Section 2.1, Problem 18 —

Point A is a point on the graph of our function, with x-coordinate 4. Since we have that $f(4) = 25$, the coordinates of point A are $(4, 25)$.

Points B and C are points on the line tangent to the graph at $x = 4$. We know that this line passes through the point $(4, 25)$, and we also know that the slope of this line is 1.5, since $f'(4) = 1.5$. Using point-slope form, the equation of this line is

$$y - 25 = 1.5(x-4) \Leftrightarrow y = 1.5x + 19$$

Therefore to find the y-coordinates of each point, we need only plug the x-coordinate of each into the equation of this line. Since point C has an x-coordinate of 3.9, it has y-coordinate $(1.5)(3.9) + 19 = 24.85$. A similar computation shows that the y-coordinate of B is $(1.5)(4.2) + 19 = 25.3$. Thus the coordinates of B and C are $(4.2, 25.3)$ and $(3.9, 24.85)$ respectively.

Section 2.1, Problem 20 — An approximate reproduction of the graph is shown below, which we'll use to answer the questions.

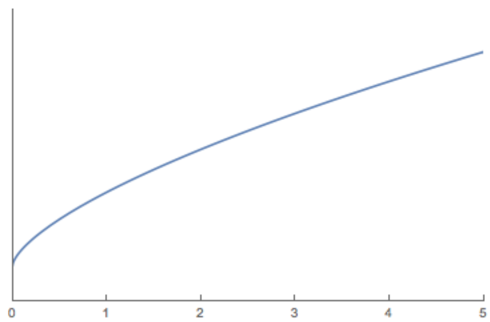


Figure 1: Graph for Problem 2.1.20

Part a: The value of $f(4)$ is the y-value of the graph of $f(x)$ at $x = 4$. The yellow dotted line below is the constant function with value equal to $f(4)$.

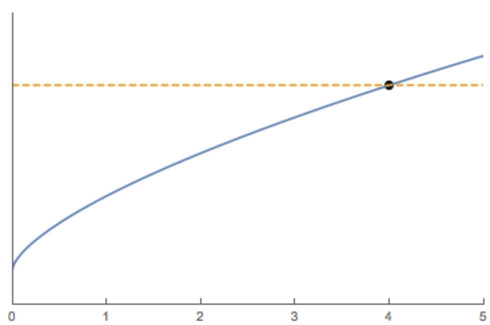


Figure 2: Graph for Part A

Part b: The difference $f(4) - f(2)$ is the difference between the y-values of the points on the graph at $x = 2$ and $x = 4$. It is represented by the distance between the two dotted lines on the graph below.

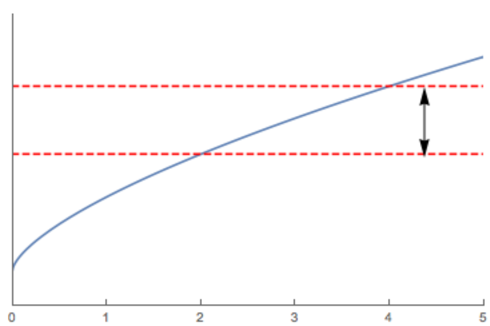


Figure 3: Graph for Part B

Part c: The value $\dfrac{f(5) - f(2)}{5 - 2}$ is the slope of the secant line that passes through the graph at $x = 2$ and $x = 5$, shown below on the graph in yellow.

Part d: The value of $f'(3)$ is the slope of the line tangent to the graph at the point $x = 3$, shown below on the graph in yellow.

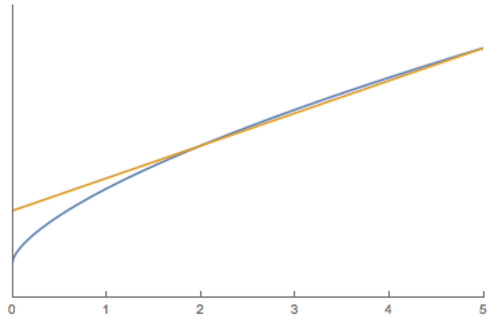


Figure 4: Graph for Part C

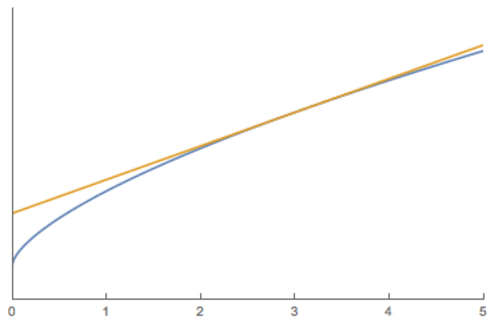


Figure 5: Graph for Part D

Section 2.2, Problem 10 —

Part a: $f(x_3)$ is the largest, as this is the place where the function's value is the largest.

Part b: $f(x_4)$ is the smallest, as this is the place where the function's value is least.

Part c: $f'(x_5)$ is the largest, as this is the place where the slope of the tangent line to the graph is the greatest.

Part d: $f'(x_3)$ is the smallest, as this is the place where the slope of the tangent line to the graph is the least. Notice that this is the only one of the given points where the slope of the tangent line is negative.

Section 2.2, Problem 14 — One possible graph is shown below. Notice that the original graph is increasing on the interval $(-\infty, 1)$ and decreasing on the interval $(1, \infty)$, with a "peak" at $x = 1$ that has a horizontal tangent line. Therefore the derivative function should be positive on the interval $(-\infty, 1)$, have a horizontal intercept at $x = 1$, and be negative on the interval $(1, \infty)$.

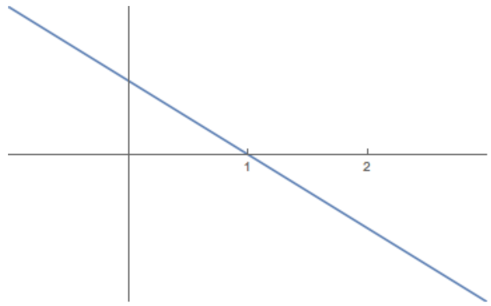


Figure 6: Graph for Problem 14

Section 2.2, Problem 16 — One possible graph is shown below. Notice that the original graph in always increasing, so the derivative should be always ≥ 0. The original graph also has a horizontal tangent line at $x = 0$, so the derivative function should have a value of 0 at $x = 0$.

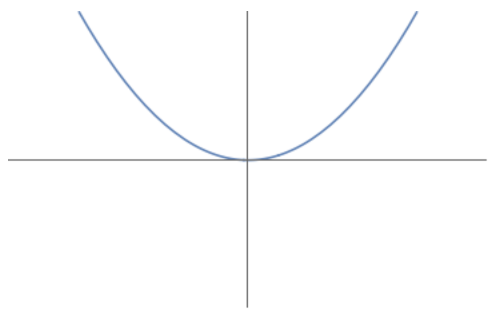


Figure 7: Graph for Problem 16

Section 2.2, Problem 30 —

Part a: That $f'(1) > 0$ means that the function should have a tangent line at $x = 1$ with positive slope. We see that the two options for this are graphs II and III. In graph III we observe that $f'(x)$ is decreasing. As we move right along the graph, the slopes of the tangent lines get smaller, with $f'(2) = 0$ and negative values for $f'(x)$ when $x > 2$. Therefore the correct graph is Graph III.

Part b: Again we have that the graph must be either Graph II or III. Since Graph III was the answer to the previous question, we know that the correct graph is Graph II.

Part c: By process of elimination, we must have that this is either graph I or IV. Note that this matches with the fact that $f'(0) < 0$, as both graphs I and IV have tangent lines at $x = 1$ with negative slope. We observe that as we move right along the graph in Graph IV, the slopes of the tangent lines are decreasing (becoming more negative), so the correct graph is Graph IV.

Part d: Process of elimination gives that the correct graph is Graph I. We can also observe, though, that Graph I matches the given description.

Section 2.3, Problem 2 —

The derivative can be written as $\dfrac{dD}{dt}$, and has units $feet/minute$.

MAT 122 Sample Midterm

1 Conceptual Questions.

This section contains some general questions to help you test your understanding of some of the key concepts we've learned so far. These questions are not necessarily representative of the kind of questions that will appear on the midterm - the true sample midterm is in the next section. To understand the last question, read the section on the second derivative (to be covered in class on Monday, 7/21).

1. Why do linear functions have the same average rate of change on every interval?
2. If $f(x)$ is an increasing function, explain why $f'(x) \geq 0$.
3. Can the graph of a function have more than one vertical intercept? Can the graph of a function have more than one horizontal intercept? Explain your answers.
4. Give an example of a function that is both concave up and increasing. Given an example of a function that is concave up and decreasing. Give an example of a function that is concave down and increasing. Give an example of a function that is concave down and decreasing.
5. What information do you need to know in order to write the equation of a line? What information do you need to know in order to write the equation of an exponential function?
6. A savings account at Bank A pays 5% interest, compounded continuously. Savings accounts at Bank B pay 5% interest, compounded annually. Which bank account pays more interest?
7. Draw the graph of a function $f(x)$ such that $f(0) = 0$ and $f'(0) = 0$.
8. A hiker starts climbing a mountain at 8:00 am, reaches the summit at 1:00 pm, and then climbs back down to arrive back at her starting point at 4:00 pm. Let A represent the altitude of the hiker, in feet, and let t represent the time after 8:00 am, in hours.
   (i) We can consider A as a function of t, but cannot consider t as a function of A. Explain why.
   (ii) Thinking of $A(t)$ as a function, consider the intervals $[0, 5]$ and $[5, 8]$ for the t-values. Over which of these intervals is the average rate of change the greatest? Over which of these intervals is the *absolute value* of the average rate of change the greatest?
9. Give as many interpretations for the derivative of a function at a point as you can.
10. What are the differences between exponential functions and power functions?
11. Let $f(x) = -e^{-x}$. Are there any values a for which $f''(a) \geq 0$?

2 Sample Midterm

This sample midterm is meant to have the same style of questions and the same length and difficulty as the midterm. However, topics covered on this sample midterm may not be on the actual midterm, and there may be topics on the midterm that do not appear on the sample midterm. You are responsible for knowing all of the concepts on the topics sheet for the midterm. There will be no "cheat sheet" or list of formulas included with the midterm.

You may not be able to answer every part of the last two questions without reading more of sections 2.3 and section 2.5 in the textbook. These sections will be covered on Monday.

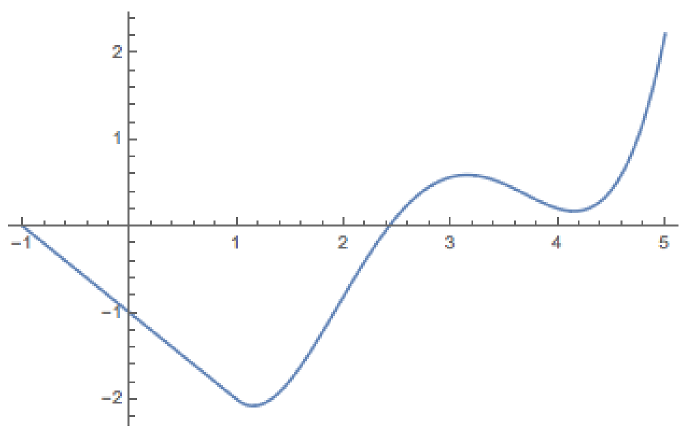
Question 1: Consider the function $f(x) = x^2 + x - 6$.

(a) Compute the average rate of change of the function f on the intervals $[0, 1]$, $[0, 1/2]$, and $[0, 1/4]$.

(b) Using your answer to part (a), estimate the value of $f'(0)$.

(c) Let $F(x)$ be a function such that $F'(x) = f(x)$. Find all numbers a such that the graph of $F(x)$ has a horizontal tangent line at $x = a$.

Question 2: Consider the graph of $f(x)$ below. Note that the graph is a straight line on the interval $[-1, 1]$.



(a) On what intervals is the function increasing and decreasing?

(b) Sketch a graph of the derivative $f'(x)$. Label all relevant points, including the vertical and horizontal intercepts. Explain how your graph agrees with your answer in part (a).

(c) Sketch a graph of $-2f'(x)$. Label the vertical and horizontal intercepts.

Question 3:

(a) Write an equation for a line that passes through the points $(2, 5)$ and $(6, 13)$.

(b) Assume that $f(x)$ is an exponential function with $f(0) = 75$ that is decaying at a continuous rate of 25%. Write an equation for $f(x)$.

(c) Let $f(x)$ be the function you obtained in part (b). Find the value of a such that $f(a) = e^2$. Simplify your answer as much as possible, and use logarithm rules to write it using only one logarithm function.

Question 4: Let $f(x) = 3x^2$, $g(x) = x - 1$.

(a) Write a formula for $f(g(x))$.

(b) Graph the function $f(g(x))$.

(c) Write a formula for $g(f(x))$.

(d) Graph the function $g(f(x))$.

Question 5: A deposit of \$10,000 is made into a bank account. After 1 year the account has a balance of \$10,750. Let B denote that balance of the bank account and t denote the number of years since the initial deposit.

(a) What is the relative (percent) change in the balance of the account after 1 year?

(b) Assuming that the bank continues to pay the same interest rate, compounded annually, write a formula for the balance in the account as a function of time.

(c) A table showing the value of the function B for different values of t is shown below. Use this to estimate the value of $B'(3)$. What units does this quantity have?

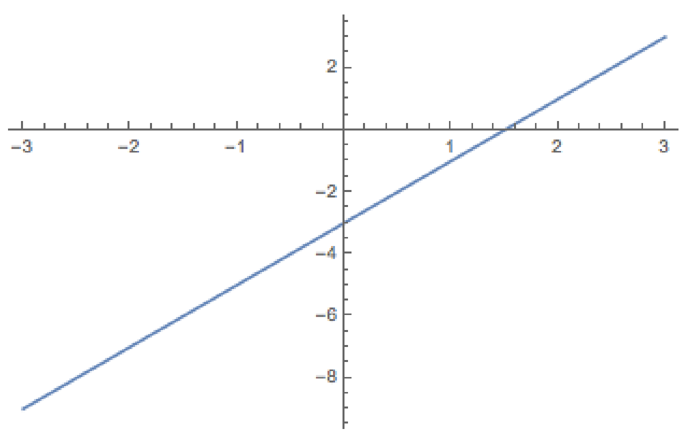
t	0	1	2	3	4	5
B(t)	10,000	10,750	11,556	12,423	13,355	14,356

Question 6: Let $f(x)$ be some function with $f(2) = 3$, $f'(2) = -2$, and $f''(2) = 1$.

(a) Use the local linear approximation at $x = 2$ to estimate the value of $f(2.5)$.

(b) Assume that $f''(x) > 0$ when $x > 2$. Is your estimate in part (a) larger or smaller than the actual value of $f(2.5)$?

(c) Assume that the graph below is a graph of $f''(x)$. What can you say about the concavity of the original function?



Question 7: A man stands on the edge of a 100 m building and throws a ball straight upward with an initial velocity of 20 m/s. The ball first flies upward, then falls back down, falling past the top of the building and down to the ground. Let h denote the height of the ball above the ground, in meters, and let t denote the time that has passed since the ball was thrown, in seconds. Ignoring wind resistance, physics tells us that the force of gravity causes the ball to accelerate toward the ground at rate of 9.8 m/sec^2. The ball hits the ground after about 7 seconds.

(a) Interpret $h'(t)$ in terms of the motion of the object. Include units in your answer.

(b) Would you expect that $h'(1)$ is positive or negative? Would you expect that $h'(6)$ is positive or negative? Explain your answer.

(c) What is the value of $\left.\dfrac{d^2h}{dt^2}\right|_{t=1}$? What is the value of $\left.\dfrac{d^2h}{dt^2}\right|_{t=6}$? Include units in your answer.

(d) The the ball speeding up or slowing down at time $t = 1$? Is the ball speeding up or slowing down at time $t = 6$? Explain your answer in terms of the signs of the first and second derivatives of the function h.

MAT 122 Homework 5 Solutions

Section 3.1, Problem 12 —

$$\begin{aligned}
\frac{d}{dx}\left(6x^3+4x^2-2x\right) &= \frac{d}{dx}\left(6x^3\right)+\frac{d}{dx}\left(4x^2\right)-\frac{d}{dx}\left(2x\right)\\
&= 6\cdot\frac{d}{dx}\left(x^3\right)+4\cdot\frac{d}{dx}\left(x^2\right)-2\cdot\frac{d}{dx}\left(x\right)\\
&= 6\cdot 3x^2+4\cdot 2x-2\cdot 1\\
&= 18x^2+8x-2
\end{aligned}$$

Section 3.1, Problem 18 —

Note that $f(z)=\dfrac{-1}{z^{6.1}}=-z^{-6.1}$, so we have

$$\begin{aligned}
f'(z) &= \frac{d}{dz}\left(-z^{-6.1}\right)\\
&= -1\cdot\frac{d}{dz}\left(z^{-6.1}\right)\\
&= -1\cdot(-6.1)z^{-6.1-1}\\
&= 6.1z^{-7.1}
\end{aligned}$$

Section 3.1, Problem 34 —

$$\begin{aligned}
\frac{dV}{dr} &= \frac{d}{dr}\left(\frac{4}{3}\pi r^2 b\right)\\
&= \frac{4}{3}\pi b\cdot\frac{d}{dr}\left(r^2\right)\\
&= \frac{4}{3}\pi b\cdot 2r\\
&= \frac{8}{3}\pi br
\end{aligned}$$

Section 3.1, Problem 46 —

$$\begin{aligned}
f(t) &= t^4 - 3t^2 + 5t \\
f'(t) &= \frac{d}{dt}\left(t^4 - 3t^2 + 5t\right) \\
&= \frac{d}{dt}(t^4) - \frac{d}{dt}(3t^2) + \frac{d}{dt}(5t) \\
&= \frac{d}{dt}(t^4) - 3 \cdot \frac{d}{dt}(t^2) + 5 \cdot \frac{d}{dt}(t) \\
&= 4t^3 - 3 \cdot 2t + 5 \cdot 1 \\
&= 4t^3 - 6t + 5 \\
f''(t) &= \frac{d}{dt}\left(4t^3 - 6t + 5\right) \\
&= \frac{d}{dt}(4t^3) - \frac{d}{dt}(6t) + \frac{d}{dt}(5) \\
&= 4 \cdot \frac{d}{dt}(t^3) - 6 \cdot \frac{d}{dt}(t) + \frac{d}{dt}(5) \\
&= 4 \cdot 3t^2 - 6 \cdot 1 + 0 \\
&= 12t^2 - 6
\end{aligned}$$

Section 3.1, Problem 50a —

The tangent line will pass through the point $(2, f(2)) = (2, 8)$ as $f(x) = x^3$. The derivative is given by $f'(x) = 3x^2$ using the power rule. Therefore the slope of the tangent line at $x = 1$ is $f'(2) = 3 \cdot 2^2 = 12$. The equation of a line through the point $(2, 8)$ with slope 12 is $y - 8 = 12(x - 2)$.

MAT 122 Homework 6 Solutions

Section 3.2, Problem 10 —

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(5\cdot 2^x - 5x + 4\right) \\
&= \frac{d}{dx}(5\cdot 2^x) - \frac{d}{dx}(5x) + \frac{d}{dx}(4) \\
&= 5\frac{d}{dx}(2^x) - 5\frac{d}{dx}(x) + 0 \\
&= 5\ln(2)\cdot 2^x - 5
\end{aligned}$$

Section 3.2, Problem 12 —

$$\frac{dy}{dt} = \frac{d}{dt}\left(e^{0.7t}\right) = 0.7\cdot e^{0.7t}$$

Section 3.2, Problem 28 —

$$\begin{aligned}
f'(t) &= \frac{d}{dt}\left(Ae^t + B\ln t\right) \\
&= \frac{d}{dt}(Ae^t) + \frac{d}{dt}(B\ln t) \\
&= A\cdot\frac{d}{dt}(e^t) + B\cdot\frac{d}{dt}(\ln t) \\
&= Ae^t + B\cdot\frac{1}{t} \\
&= Ae^t + \frac{B}{t}
\end{aligned}$$

Section 3.2, Problem 34 — The function $f(t) = 6.8e^{0.012t}$ describes the world population, in billions, as a function of t, the time in years since 2009.

Part a: $f(0) = 6.8e^{0.012\cdot 0} = 6.8\cdot 1 = 6.8$. This means that the world population in the year 2009 is 6.8 billion.

Part b: We first find the derivative, which is

$$\begin{aligned}
f'(t) &= \frac{d}{dt}\left(6.8e^{0.012t}\right) \\
&= \frac{d}{dt}\left(6.8e^{0.012t}\right) \\
&= 6.8\cdot\frac{d}{dt}\left(e^{0.012t}\right) \\
&= 6.8\cdot 0.012\cdot e^{0.012t} \\
&= 0.0816\cdot e^{0.012t}
\end{aligned}$$

Next, we evaluate the derivative at $t = 0$ to find

$$f'(0) = 0.0816\cdot e^{0.012\cdot 0} = 0.0816$$

This means that the world population is growing at an (instantaneous) rate of 0.0816 billion people per year in the year 2009 (or 81.6 million people per year).

Part c: $f(10) = 6.8e^{0.012 \cdot 10} = 6.8e^{.12} = 7.66698$, so the world population in 2019 is estimated to be 7.7 billion people.

Part d: Using the expression we found earlier for the derivative, we have that

$$f'(10) = 0.0816 \cdot e^{0.012 \cdot 10} = 0.0816 \cdot e^{0.12} = 0.0920$$

This means that the world population is (estimated to be) growing at an (instantaneous) rate of 0.092 billion people per year in the year 2019 (or 92 million people per year).

Section 3.4, Problem 10 —

$$\begin{aligned}
\frac{dy}{dt} &= \frac{d}{dt}\left[(t^3 - 7t^2 + 1)e^t\right] \\
&= \frac{d}{dt}(t^3 - 7t^2 + 1) \cdot e^t + (t^3 - 7t^2 + 1) \cdot \frac{d}{dt}(e^t) \\
&= (3t^2 - 14t)e^t + (t^3 - 7t^2 + 1)e^t \\
&= (3t^2 - 14t + t^3 - 7t^2 + 1)e^t \\
&= (t^3 - 4t^2 - 14t + 1)e^t
\end{aligned}$$

Section 3.4, Problem 16 —

$$\begin{aligned}
f'(z) &= \frac{d}{dz}\left(\sqrt{z}e^{-z}\right) = \frac{d}{dz}\left(z^{1/2}e^{-z}\right) \\
&= \frac{d}{dz}(z^{1/2}) \cdot e^{-z} + z^{1/2} \cdot \frac{d}{dz}(e^{-z}) \\
&= \frac{1}{2}z^{-1/2} \cdot e^{-z} + z^{1/2} \cdot -e^{-z} \\
&= \frac{1}{2\sqrt{z}e^z} - \sqrt{z}e^{-z}
\end{aligned}$$

Section 3.4, Problem 24 —

$$\begin{aligned}
\frac{dw}{dz} &= \frac{d}{dz}\left(\frac{3z}{1+2z}\right) \\
&= \frac{(1+2z)\frac{d}{dz}(3z) - (3z)\frac{d}{dz}(1+2z)}{(1+2z)^2} \\
&= \frac{(1+2z) \cdot 3 - (3z) \cdot 2}{(1+2z)^2} \\
&= \frac{3 + 6z - 6z}{(1+2z)^2} \\
&= \frac{3}{(1+2z)^2}
\end{aligned}$$

Section 3.4, Problem 34 —

$$\begin{aligned}
f(x) &= (3x+8)(2x-5) \\
f'(x) &= \frac{d}{dx}\left[(3x+8)(2x-5)\right] \\
&= \frac{d}{dx}(3x+8)\cdot(2x-5)+(3x+8)\cdot\frac{d}{dx}(2x-5) \\
&= 3\cdot(2x-5)+(3x+8)\cdot 2 \\
&= 6x-15+6x+16 \\
&= 12x+1 \\
f''(x) &= \frac{d}{dx}(12x+1) \\
&= 12
\end{aligned}$$

Section 3.4, Problem 36 — To find the equation of the tangent line at $x = 0$, we need to find the point the line will pass through and its slope.

The point will have coordinates $(0, f(0))$. We have $f(x) = \dfrac{2x-5}{x+1}$, so $f(0) = \dfrac{-5}{1} = -5$. Note that this gives that the y-intercept of the line is at -5.

The slope is given by the value of the derivative at 0, that is, $f'(0)$. We first find $f'(x)$. Using the quotient rule,

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left(\frac{2x-5}{x+1}\right) \\
&= \frac{(x+1)\frac{d}{dx}(2x-5)-(2x-5)\frac{d}{dx}(x+1)}{(x+1)^2} \\
&= \frac{(x+1)\cdot 2-(2x-5)\cdot 1}{(x+1)^2} \\
&= \frac{2x+2-2x+5}{(x+1)^2} \\
&= \frac{7}{(x+1)^2}
\end{aligned}$$

Evaluating at $x = 0$, we have that $f'(0) = \frac{7}{(0+1)^2} = 7$. Therefore the tangent line has slope 7.

The equation of a line with slope 7 and y-intercept -5 is $y = 7x - 5$.

MAT 122 Homework 7 - Due 8/4

Required

Complete and turn in the following problems from the textbook -

Section 3.3— 4, 10, 24, 44
Section 4.1— 8, 16
Section 4.2— 24

Extra Question Consider the functions.

$$f(x) = \frac{x^3}{6} + \frac{x^2}{4} - x + 2 \qquad\qquad g(x) = x^4 - 4x^3 + 10$$

For each function, identify -

(i) The vertical intercept.
(ii) The critical points of the function.
(iii) The local maxima and minima of the function, if any.
(iv) The intervals on which the function is increasing and decreasing.
(v) The intervals on which the function is concave up or down.
(vi) The inflection points, if any.

Use this information to draw a graph of the function. The vertical intercept, local max and min, and inflection points should all be labeled on the graph, and the graph should have the proper increasing/decreasing intervals and concavity.

Suggested

You may want to attempt the following problems for extra practice.

Section 3.3— 1-27 odd, 45
Section 4.1— 9,17
Section 4.1— For questions 11, 17, and 19, answer the same questions as in the extra question above. Also try 23, 25, 29

Extra Credit

The second derivative test says the following - if $f(x)$ is a function and $x = a$ is a critical point of that function (that is, $f'(a) = 0$), then we can decide if $x = a$ is a local maximum or minimum via the following -

- If $f''(a) < 0$, then $x = a$ is a local maximum.
- If $f''(a) > 0$, then $x = a$ is a local minimum.
- If $f''(a) = 0$, the second derivative test is inconclusive.

Here "inconclusive" means that anything can happen - the point $x = a$ could be a local maximum, it could be a local minimum, or it could be neither. The goal of this extra credit assignment is to find examples of functions that give each of these cases. Namely, find each of the following.

1. Find a function $f(x)$ such that $f'(0) = 0$, $f''(0) = 0$, and the point $x = 0$ is a local maximum.
2. Find a function $f(x)$ such that $f'(0) = 0$, $f''(0) = 0$, and the point $x = 0$ is a local minimum.
3. Find a function $f(x)$ such that $f'(0) = 0$, $f''(0) = 0$, and the point $x = 0$ is a neither a local maximum nor a local minimum.

This extra credit assignment is worth 10 quiz points, is not required, and is due with the homework at the start of class on 8/4.

MAT 122 Homework 7 Solutions

Section 3.3, Problem 4 —

For the function $w = (t^2+1)^{100}$, we take the inside function to be $z = t^2 + 1$ and the outside function to be z^{100}. The derivative of the inside function is $z' = 2t$, while the derivative of the outside function is $100z^{99}$. The chain rule therefore give that the derivative is

$$\frac{dw}{dt} = 100(t^2+1)^{99} \cdot (2t) = 200t(t^2+1)^{99}$$

Section 3.3, Problem 10 —

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(6e^{5x} + e^{-x^2}\right) \\ &= 6\frac{d}{dx}(e^{5x}) + \frac{d}{dx}(e^{-x^2}) \end{aligned}$$

We take the derivative of each term separately, using the chain rule for each one. For the first term, we take the inside function to be $z = 5x$ and the outside function to be e^z. The derivative of the inside function is 5, and the derivative of the outside function is e^z. We therefore have that

$$\frac{d}{dx}(e^{5x}) = 5e^{5x}$$

To deal with the second term, we take the inside function to be $z = -x^2$ and the outside function to be e^z. The derivative of the inside function is $-2x$, and the derivative of the outside function is e^z. The chain rule then gives

$$\frac{d}{dx}(e^{-x^2}) = e^{-x^2} \cdot (-2x)$$

The derivative of $f(x)$ is therefore

$$f'(x) = 30e^{5x} - 2xe^{-x^2}$$

Section 3.3, Problem 24 —

For the function $y = (5+e^x)^2$, we take the inside function to be $z = 5 + e^x$ and the outside function to be z^2. The derivative of the inside function is e^x, and the derivative of the outside function is $2z$. The chain rule give that

$$\frac{dy}{dx} = 2(5+e^x) \cdot e^x = 2e^x(5+e^x)$$

Section 3.3, Problem 44 —

Using the chain rule,

$$\left.\frac{d}{dx}f(g(x))\right|_{x=30} = f'(g(30)) \cdot g'(30)$$

Looking at the graph of g, we observe that $g(30) = 55$ and $g'(30) = 1/2$. (Notice that on the interval $[0, 40]$, the graph of g is a line going through the points $(0, 40)$ and $(40, 60)$. Such a line has slope $1/2$ giving the derivative, and $30 * 1/2 + 40 = 55$.)

To finish we must find $f'(g(30)) = f'(55)$. We notice that on the interval $[40, 80]$, the graph of f is a line with slope 1 (this line passes through the points $(40, 0)$ and $(80, 40)$.) Therefore $f'(55) = 1$.

Combining these observations, we have that

$$\left.\frac{d}{dx} f(g(x))\right|_{x=30} = 1 \cdot \frac{1}{2} = \frac{1}{2}$$

Section 4.1, Problem 8 —

The graph of f' shown in the book is a parabola with two zeros. Each zero (horizontal intercept) is a critical point of the function f.

The first horizontal intercept (on the left) is a place where the graph of f' switches from positive to negative. Therefore it corresponds to a place where the original graph switched from increasing to decreasing, a local maximum.

The second horizontal intercept (on the right) is a place where the graph of f' switched from negative to positive. Therefore it corresponds to a place where the original graph switched from increasing to decreasing, a local minimum.

(In both of these answers we are using the first derivative test to determine if a critical point is a maximum or a minimum.)

Section 4.1, Problem 16 —

The graph of f' shown in the book has three horizontal intercepts, at $x = 1, x = 3$, and $x = 5$. These are critical points for the original graph.

The first derivative test gives that there is a local maximum at $x = 1$, as this is a place where the first derivative switches signs from positive to negative.

The first derivative test gives that there is a local minimum at $x = 3$, as this is a place where the first derivative switches signs from negative to positive.

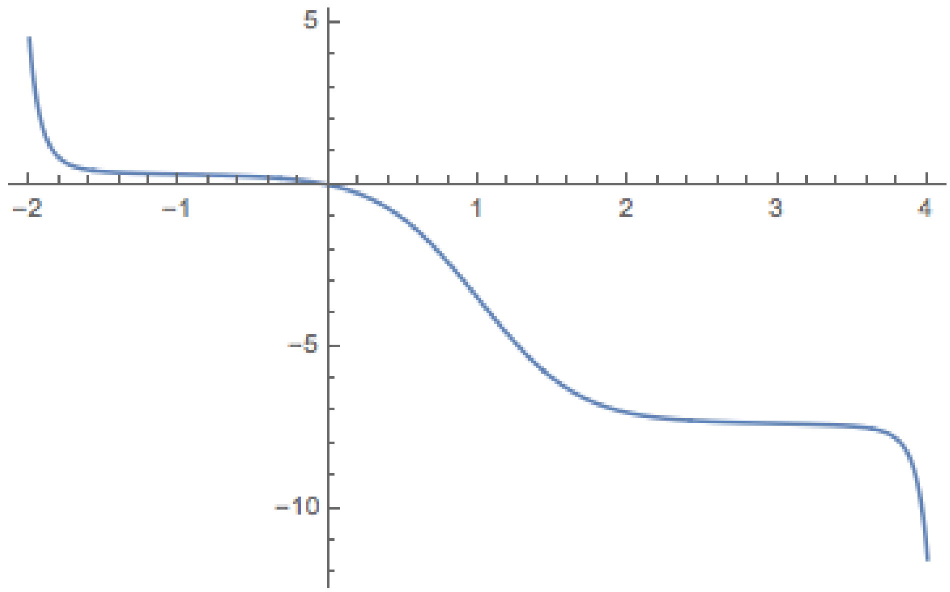
The first derivative test gives that $x = 5$ is neither a maximum nor a minimum. On either side of 5 we can observe that the derivative is positive. (The point $x = 5$ will be an inflection point, as it is a place where the derivative switches from decreasing to increasing. Said another way, that means that the second derivative f'' switches signs from negative to positive at $x = 5$.)

Section 4.2, Problem 24 —

The information given says that $f' < 0$ for all values. This means that the function will be always decreasing.

However, the second derivative is zero at three points, x_1, x_2, and x_3. We can also see that the second derivative switches signs at each point - from positive to negative at x_1, from negative to positive at x_2, and from positive to negative at x_3. This means that each of these points will be an inflection point on the graph.

So we need to draw a graph that is always decreasing (in particular, as no horizontal tangent lines) that begins concave up and then switches concavity 3 times. There are many possible answers to this question - one is shown below.



The figure shown has inflection points at $x = -1$, $x = 1$, and $x = 3$, which we take to be x_1, x_2, and x_3, respectively. Notice that the graph does not have any horizontal tangent lines, so that $f' < 0$ always.

Extra Question

Part a: We first answer the various questions for $f(x) = \frac{x^3}{6} + \frac{x^2}{4} - x + 2$.

(i) The vertical intercept.

The just means plugging 0 into the function, where we observe that $f(0) = 2$.

(ii) The critical points of the function.

Critical points are where the derivative function has a zero. The first derivative is

$$f'(x) = \frac{x^2}{2} + \frac{x}{2} - 1$$

We can solve this equation either by factoring or using the quadratic formula. The zeros are $x = 1$ and $x = -2$, hence these are the critical points.

(iii) The local maxima and minima of the function, if any.

We'll use the first derivative test. By checking the sign of the first derivative on various intervals, we find

The first derivative test then gives that the function has a local maximum at $x = -2$ and a local minimum at $x = 1$.

(iv) The intervals on which the function is increasing and decreasing.

From the signs of the first derivative we found above, the function is increasing on the intervals $(-\infty, -2)$ and $(1, \infty)$, while it is decreasing on the interval $(-2, 1)$.

(v) The intervals on which the function is concave up or down.

For this we use the second derivative. The second derivative of the function is

$$f''(x) = x + \frac{1}{2}$$

Notice that the second derivative is equal to 0 at $x = \frac{-1}{2}$, so we might guess that this is an inflection point. We need to decide if $f''(x)$ switches signs at $x = \frac{-1}{2}$. Since this is a linear function, it is not hard to see that for values bigger than $\frac{-1}{2}$ the second derivative will be positive, while for values smaller than $\frac{-1}{2}$ the second derivative will be negative.

Therefore the function $f(x)$ is concave up on the interval $(-1/2, \infty)$ and concave down on the interval $(-\infty, -1/2)$.

(vi) The inflection points, if any.

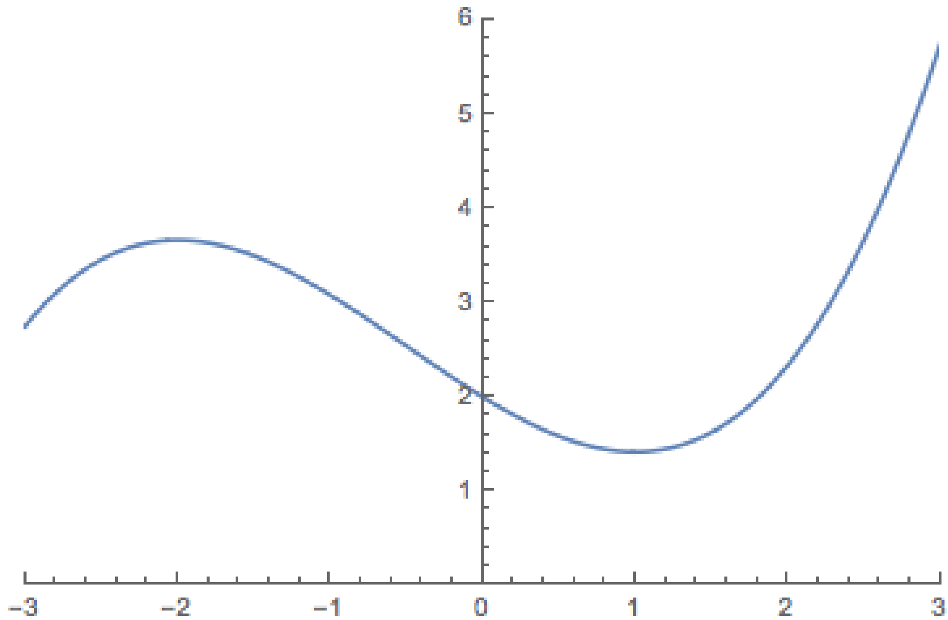
In the previous section we saw that the second derivative changes sign at $x = -1/2$, which means that there is an inflection point at $x = -1/2$.

(vii) Graph the function.

To graph, we first find the values of the function at the various local extrema and inflection points we calculated. We observe

$$f(-2) = \frac{11}{3} \qquad f(1) = \frac{17}{12} \qquad f(-1/2) = \frac{61}{24}$$

Plotting these points on a graph, along with the vertical intercept, and then plotting a graph with the appropriate intervals of increasing/decreasing/concavity gives the plot



Part b: We now consider the function $g(x) = x^4 - 4x^3 + 10$.

(i) The vertical intercept.

$g(0) = 10$ is the vertical intercept.

(ii) The critical points of the function.

The first derivative is

$$\begin{aligned} g'(x) &= 4x^3 - 12x^2 \\ &= 4x^2(x-3) \end{aligned}$$

Once factored, we can see that $g'(x) = 0$ at $x = 0$ and $x = 3$, so these are the critical points.

(iii) The local maxima and minima of the function, if any.

Investigating the sign of $g'(x)$ at various points gives the following number line.

Using the first derivative test, we see that $x = 0$ is neither a local max nor local min, since the derivative does not switch signs at this point. We also see that $x = 3$ is a local minimum, as this is a place where the derivative switched from negative to positive.

(iv) The intervals on which the function is increasing and decreasing.

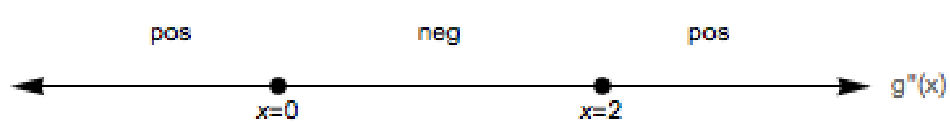
From our answer to the previous question, we can see that the function is increasing on the interval $(3, \infty)$, while the function is decreasing on the intervals $(-\infty, 0)$ and $(0, 3)$.

(v) The intervals on which the function is concave up or down.

Again we need the second derivative, which is

$$\begin{aligned} g''(x) &= 12x^2 - 24x \\ &= 12x(x-2) \end{aligned}$$

We observe that this function has zeros at $x = 0$ and $x = 2$, so these are possible inflection points. To find the intervals of concavity we find where this second derivative is positive and negative, which we can display as a number line as



This says that the function $f(x)$ is concave up on the intervals $(-\infty, 0)$ and $(2, \infty)$, while it is concave down on the intervals $(0, 2)$.

(vi) The inflection points, if any.

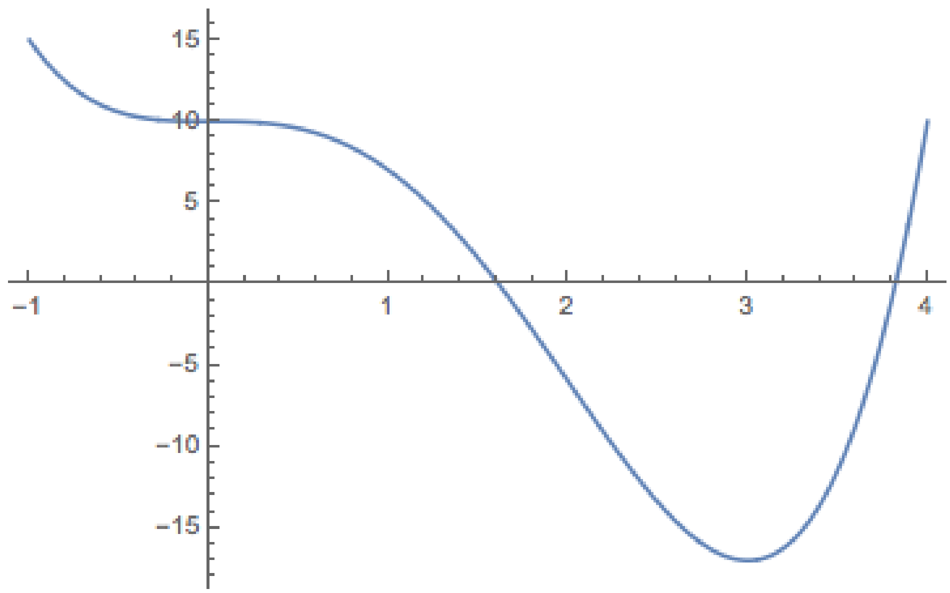
We can see from our previous answer that the second derivative switches sign at both $x = 0$ and $x = 2$, so these are inflection points.

(vii) Graph

To graph, we first find the values of the function at the various local extrema and inflection points we calculated. We observe

$$g(0) = 10 \qquad g(2) = -6 \qquad g(3) = -17$$

Plotting these points on a graph and then drawing a graph with the appropriate intervals of increasing/decreasing/concavity gives the graph



Extra Credit Problem —

1. Find a function $f(x)$ such that $f'(0) = 0$, $f''(0) = 0$, and the point $x = 0$ is a local maximum.

   One example of such a function is $f(x) = -x^4$.

2. Find a function $f(x)$ such that $f'(0) = 0$, $f''(0) = 0$, and the point $x = 0$ is a local minimum.

   One example of such a function is $f(x) = x^4$.

3. Find a function $f(x)$ such that $f'(0) = 0$, $f''(0) = 0$, and the point $x = 0$ is a neither a local maximum nor a local minimum.

   One example of such a function is $f(x) = x^3$.

You should compute the various derivatives of these functions for yourself to see why $f'(0) = 0$ and $f'(0) = 0$ for all of these functions, and then use the first derivative test to see why each critical point is either a local maximum, local minimum, or neither.

MAT 122 Homework 8 Solutions

Focus on Practice, Problem 34 — For the function

$$y = \left(\frac{x^2+2}{3}\right)^2$$

we start with the chain rule. The inner function is $z = \dfrac{x^2+2}{3}$, and the outside function is z^2. Using the chain rule

$$\begin{aligned}\frac{dy}{dx} &= 2z \cdot \frac{d}{dx}\left(\frac{x^2+2}{3}\right) \\ &= 2\left(\frac{x^2+2}{3}\right) \cdot \frac{1}{3}\frac{d}{dx}(x^2+2) \\ &= \frac{2}{9}(x^2+2)\cdot(2x) \\ &= \frac{4x}{9}(x^2+2)\end{aligned}$$

Focus on Practice, Problem 42 — For the function

$$h(w) = w^3 \ln(10w)$$

we begin with the product rule:

$$\begin{aligned}h'(w) &= \frac{d}{dw}\left(w^3 \ln(10w)\right) \\ &= \frac{d}{dw}(w^3)\cdot \ln(10w) + w^3 \frac{d}{dw}\ln(10w) \\ &= 3w^2 \ln(10w) + w^3 \frac{d}{dw}\left(\ln(10w)\right)\end{aligned}$$

To take the remaining derivative, we use the chain rule with $z = 10w$ as the inside function and $\ln z$ as the outside function. So the chain rule gives

$$\frac{d}{dw}\left(\ln(10w)\right) = \frac{1}{z}\cdot 10 = \frac{1}{10w}\cdot 10 = \frac{1}{w}$$

Therefore the final derivative is

$$\begin{aligned}h'(w) &= 3w^2 \ln(10w) + w^3 \cdot \frac{1}{w} \\ &= 3w^2 \ln(10w) + w^2 \\ &= w^2\left(3\ln(10w)+1\right)\end{aligned}$$

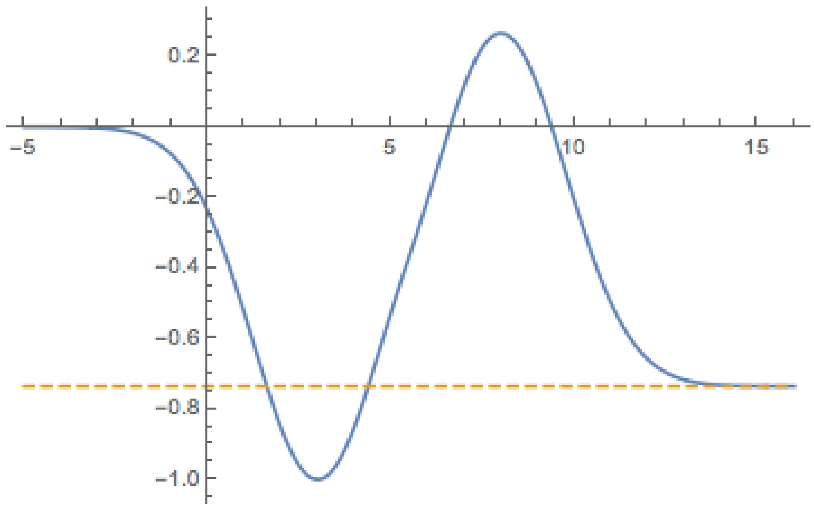
Focus on Practice, Problem 58 — For the function

$$g(x) = \frac{x^2+\sqrt{x}+1}{x^{3/2}} = \frac{x^2+x^{1/2}+1}{x^{3/2}}$$

we use the quotient rule. We have

$$\begin{aligned} g'(x) &= \frac{x^{3/2} \cdot \frac{d}{dx}\left(x^2 + x^{1/2} + 1\right) - \left(x^2 + x^{1/2} + 1\right) \cdot \frac{d}{dx}(x^{3/2})}{(x^{3/2})^2} \\ &= \frac{x^{3/2} \cdot \left(2x + \frac{1}{2}x^{-1/2}\right) - \left(x^2 + x^{1/2} + 1\right) \cdot \frac{3}{2}x^{1/2}}{x^3} \end{aligned}$$

Section 4.3, Problem 6 — A number of graphs are possible. Below is one possible example.



Notice that the graph has a local min at $x = 3$ and a local max at $x = 8$. The graph as shown also has a horizontal asymptote at $y = 0$, not shown, and at the dotted yellow line, so that we can be sure the function is never smaller than the value at $x = 3$ or larger than the value at $x = 8$. Thus these local extrema are also global extrema.

Section 4.3, Problem 20 — The function is $f(x) = x^3 - 3x^2 - 9x + 15$, on the interval $[-5, 4]$.

Part a: The function is a polynomial, so we take its derivatives using a combination of the constant multiple, sum/diffference, and power rules for taking derivatives, to obtain

$$\begin{aligned} f'(x) &= 3x^2 - 6x - 9 \\ f''(x) &= 6x - 6 \end{aligned}$$

Part b: Critical points are where the first derivative is equal to 0. So we solve the equation

$$3x^2 - 6x - 9 = 0 \Leftrightarrow 3(x+1)(x-3) = 0$$

(Instead of factoring you could also use the quadratic formula). This gives that the critical points are at $x = -1$ and $x = 3$. Both of these critical points are in the interval $[-5, 4]$.

Part c: Inflection points, if they occur, happen where the second derivative is zero. Solving the equation $6x - 6 = 0$, we have that $x = 1$ may be an inflection point. To check that it is an inflection point, we need to check that the second derivative switches sign at this point.

By plugging in numbers less than 1 and greater than 1 into the second derivative, we can see that the second derivative changes sign from negative to positive at $x = 1$, so that $x = 1$ is indeed an inflection point.

Part d: To find global maxima and minima, we evaluate $f(x)$ at the critical points and at the endpoints of the interval, getting the numbers

$$f(-5) = -140 \qquad f(-1) = 20 \qquad f(3) = -12 \qquad f(4) = -5$$

Thus the global maximum value is 20, occurring at $x = -1$, and the global minimum value is -140, occurring at $x = -5$.

Both of these are local max/min, respectively, as a global extremum is always a local extremum. To find other local extrema, we need to decide if the remaining critical point $x = 3$ is a local max or local min. In this case, we use the second derivative test, and observe that

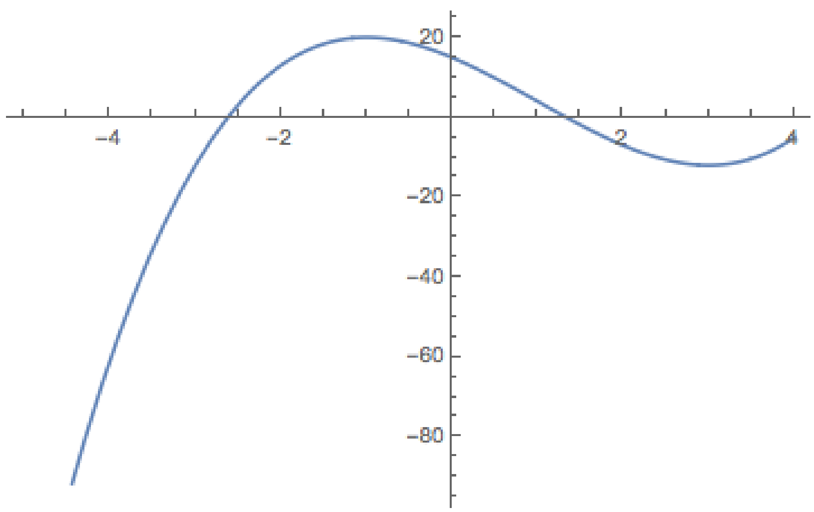
$$f''(3) = 6 \cdot 3 - 6 = 12 > 0$$

so the second derivative test says that a local minimum occurs at $x = 3$. (You could also use the first derivative test here, if you prefer.

Part e: To graph the function, we can first plot the points we know, which are $(-5, -140)$ and $(4, -5)$ — the endpoints — $(-1, 20)$ and $(3, -12)$ — a global max and a local min — and the inflection point $(1, 4)$.

We then draw a curve through those points with the proper increasing/decreasing behavior and concavity. We can check the sign of the first derivative to see that the function is increasing on $(-5, -1)$ and on $(3, 4)$, while the function is decreasing on $(-1, 3)$. The function is concave down to the left of $x = 1$ and concave up to the right of $x = 1$.

Putting all of this together gives the graph



Section 4.3, Problem 34 —

Part a: The volume of a box is given by the product of the length, width, and height. Let x denote the length of the sides of the square base of the box, and let h denote the height of the box. The volume is therefore

$$V = x^2 h$$

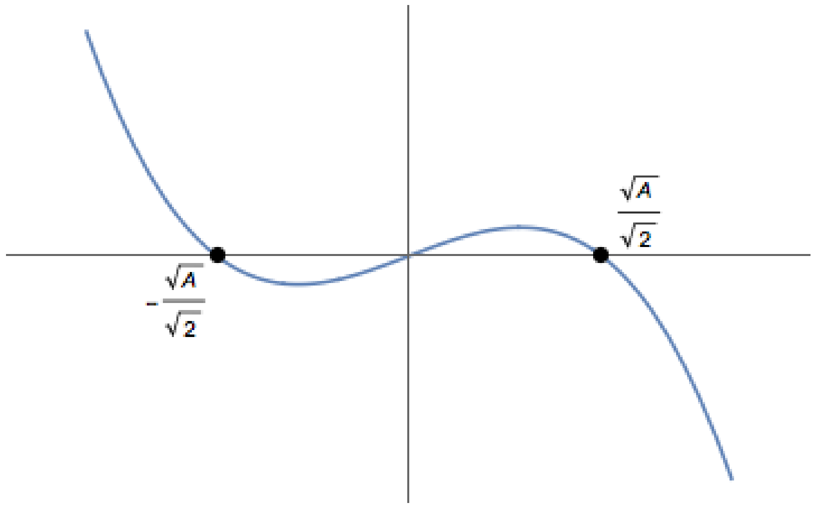
This is not a function, however, as it depends on two variables, x and h. However, we are also told in the problem statement that the box has a fixed surface area A, which we can treat as a constant. Using the formula for the surface area of a box,

$$A = 2x^2 + 4xh \Leftrightarrow h = \frac{A - 2x^2}{4x}$$

Substituting this into our earlier expression for V gives that V can be considered as a function of x by

$$V(x) = x^2 \cdot \frac{A - 2x^2}{4x} = \frac{x(A - 2x^2)}{4} = \frac{A}{4}x - \frac{1}{2}x^3$$

Part b: The graph is a cubic function, with zeros at $x = 0$ and $x = \pm\sqrt{A/2}$. We can also observe that for very large values of x, the value of V is negative. This says that the graph must have the following shape-



Observe from the graph that the function will have a maximum somewhere between 0 and $\sqrt{A/2}$, so this is what we will try to find in the next part.

Part c: We need to find the global maximum value of the function $V(x)$. First, we consider whether there are any physical constraints that restrict the domain of our function. We observe that we must have $x \geq 0$, as it does not make sense for the side of a box to have negative length. We also need to have that $x \leq \sqrt{A/2}$. If $x > \sqrt{A/2}$, then the area of the top and bottom of the box will be greater than A, which we cannot have due to the constraints of the problem. We could also notice from our graph above that these constraints come from requiring that both x is positive and that the volume V is positive as well.

Thus, to maximize the volume we need to find the global maximum value of the function $V(x) = \frac{A}{4}x - \frac{1}{2}x^3$ on the interval $[0, \sqrt{A/2}]$.

Following the steps for finding the global extrema of a function on an interval, we begin by finding critical points. Taking the derivative of V gives

$$V'(x) = \frac{A}{4} - \frac{3}{2}x^2$$

Setting this expression equal to 0 and solving for x gives $x = \pm\sqrt{A/6}$. We can ignore the negative value here, as it is not within the interval $[0, \sqrt{A/2}]$.

We then evaluate the function V at this critical point $x = \sqrt{A/6}$, and also at the endpoints $0, \sqrt{A/2}$ of out interval. We have

$$\begin{aligned} V(0) &= 0 \\ V(\sqrt{A/2}) &= 0 \\ V(\sqrt{A/6}) &= \frac{1}{6\sqrt{6}}A^{3/2} \end{aligned}$$

The last number on this list is clearly the largest value, hence is the global maximum on the interval, and is therefore the largest possible value of the volume of the box.

We can notice that the maximum value of the volume occurs when $x = \sqrt{A/6}$. Notice that this means the third side of the box, h, has a length of

$$h = \frac{A - 2(\sqrt{A/6})^2}{4\sqrt{A/6}} = \sqrt{A/6}$$

so the box that has the most volume for a fixed surface area is actually a cube (that is, all sides have the same length).

Section 4.3, Problem 36 —

Consider the equation

$$I = 192 \ln\left(\frac{S}{192}\right) - S + 763$$

First we notice that this equation must have $S > 0$, both so that the logarithm is defined but also because S represents a number of children, which cannot be negative. Since there can only be a whole number of children, we can assume that $S \geq 1$. We are also told in the problem that there are 763 students in the boarding school. Since S represents the number of students in the boarding school that are susceptible to infection, we have that $S \leq 763$. Therefore we are trying to find the maximum value of the function I on the interval $[1, 763]$.

We begin as usual by finding critical points. Taking the derivative, we have

$$\begin{aligned} \frac{dI}{dt} &= 192 \cdot \frac{1}{\frac{S}{192}} \cdot \frac{1}{192} - 1 \\ &= \frac{192}{S} - 1 \end{aligned}$$

Notice that we used the chain rule to take the derivative. Solving for when this equation is 0 gives $S = 192$ as the critical point, which is within our interval under consideration.

Next, we evaluate I at the endpoints $S = 1$, $S = 763$, and at the critical point $S = 192$. This gives

$$\begin{aligned} I(1) &\approx -512.102 \\ I(763) &\approx 0.251803 \\ I(192) &\approx 306.337 \end{aligned}$$

The largest number here is about 306, so the maximum possible number of infected children is about 306.

Section 5.1, Problem 10 —

Part a: We will estimate the change in population by computing a left Riemann sum, a right Riemann sum, and averaging the two.

Notice that the total time period is $200 - 1950 = 50$ years, and that the table divides this period into $n = 5$ subintervals, with $\Delta t = 10$ years.

For the left Riemann sum, we use the values of the function at the left endpoint of each subinterval. This gives

$$L_5 = 37 \cdot 10 + 41 \cdot 10 + 78 \cdot 10 + 77 \cdot 10 + 86 \cdot 10 = 3190$$

For the right Riemann sum, we use the values of the function at the right endpoint of each subinterval, which gives

$$R_5 = 41 \cdot 10 + 78 \cdot 10 + 77 \cdot 10 + 86 \cdot 10 + 79 \cdot 10 = 3610$$

the average of these estimates is

$$\frac{3190 + 3610}{2} = 3400$$

Therefore, we estimate that the world's population increased by 3,400 million people from the year 1950 to the year 2000.

Part b: If the world population in 1950 was 2555 million and the world population in 2000 was 6085 million people, then the actual change in population was

$$6085 - 2555 = 3530 \text{ million}$$

Thus our estimate of 3400 million is reasonably close to the actual value.

Section 5.1, Problem 12b —

We will compute left and right Riemann sums to give two estimates for the distance traveled. The larger number will be an upper estimate, and the smaller number will be a lower estimate.

The time interval of $t = 0$ minutes to $t = 90$ minutes is divided into $n = 6$ subintervals, each with a length of $(90 - 0)/6 = 15$ minutes, so our $\Delta t = 15$ min. Note that the speeds are given in miles per hour, however, while the time as given is in minutes. We therefore need to switch all times into units of hours. We have that 15 minutes is .25 hour, so we use $\Delta t = 0.25$ in our Riemann sum.

The left Riemann sum is therefore

$$L_6 = 12 \cdot 0.25 + 11 \cdot 0.25 + 10 \cdot 0.25 + 10 \cdot 0.25 + 8 \cdot 0.25 + 7 \cdot 0.25 = 14.5,$$

while the right Riemann sum is

$$R_6 = 11 \cdot 0.25 + 10 \cdot 0.25 + 10 \cdot 0.25 + 8 \cdot 0.25 + 7 \cdot 0.25 + 0 \cdot 0.25 = 11.5$$

Thus a lower estimate for the distance that Roger ran is 11.5 miles, while an upper estimate for the distance that Roger ran is 14.5 miles.

MAT 122 Homework 9 Solutions

Section 5.2, Problem 4 — For this problem, $a = 3, b = 4, n = 5$, and $\Delta t = 0.2$. Notice

$$\frac{b-a}{n} = \frac{4-3}{5} = \frac{1}{5} = 0.2 = \Delta t$$

Using a left Riemann sum, we have

$$\begin{aligned} L_5 &= 25 \cdot 0.2 + 23 \cdot 0.2 + 20 \cdot 0.2 + 15 \cdot 0.2 + 9 \cdot 0.2 \\ &= 18.4 \end{aligned}$$

The right Riemann sum is

$$\begin{aligned} R_5 &= 23 \cdot 0.2 + 20 \cdot 0.2 + 15 \cdot 0.2 + 9 \cdot 0.2 + 2 \cdot 0.2 \\ &= 13.8 \end{aligned}$$

We estimate the value of the integral by averaging these two Riemann sums. We therefore have

$$\int_3^4 W(t)\, dt = \frac{18.4 + 13.8}{2} = 16.1$$

Section 5.2, Problem 16 — Note that the function $f(t) = t^2 + 1$ is increasing, so the left Riemann sum will always be an underestimate and the right Riemann sum is an overestimate.

Part a: With $\Delta t = 4$, we have $n = 2$ subintervals, namely the subintervals $[0, 4]$ and $[4, 8]$. The left and right Riemann sums are therefore

$$\begin{aligned} L_2 &= f(0) \cdot \Delta t + f(4) \cdot \Delta t \\ &= 1 \cdot 4 + 17 \cdot 4 \\ &= 72 \\ R_2 &= f(4) \cdot \Delta t + f(8) \cdot \Delta t \\ &= 17 \cdot 4 + 65 \cdot 4 \\ &= 328. \end{aligned}$$

(Averaging these would give an estimate of 200 for the value of the integral.)

Part b: With $\Delta t = 2$, we have $n = 4$ subintervals, namely the subintervals $[0, 2], [2, 4], [4, 6]$, and $[6, 8]$. The left and right Riemann sums are therefore

$$\begin{aligned} L_4 &= f(0) \cdot \Delta t + f(2) \cdot \Delta t + f(4) \cdot \Delta t + f(6) \cdot \Delta t \\ &= 1 \cdot 2 + 5 \cdot 2 + 17 \cdot 2 + 37 \cdot 2 \\ &= 120 \\ R_4 &= f(2) \cdot \Delta t + f(4) \cdot \Delta t + f(6) \cdot \Delta t + f(8) \cdot \Delta t \\ &= 5 \cdot 2 + 17 \cdot 2 + 37 \cdot 2 + 65 \cdot 2 \\ &= 248. \end{aligned}$$

(Averaging these would give an estimate of 184 for the value of the integral.)

Part c: With $\Delta t = 1$, we have $n = 8$ subintervals, namely the subintervals $[0,1]$, $[1,2]$, $[2,3]$, $[3,4]$, $[4,5]$, $[5,6]$, $[6,7]$, and $[7,8]$. The left and right Riemann sums are therefore

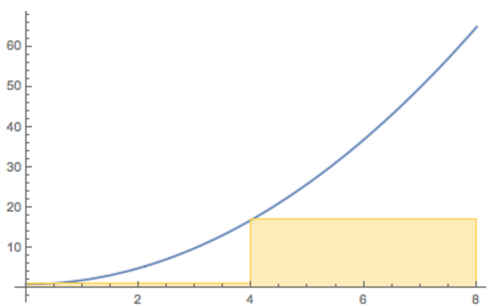
$$\begin{aligned}
L_8 &= f(0)\cdot\Delta t + f(1)\cdot\Delta t + f(2)\cdot\Delta t + f(3)\cdot\Delta t + f(4)\cdot\Delta t + f(5)\cdot\Delta t + f(6)\cdot\Delta t + f(7)\cdot\Delta t \\
&= 1+2+5+10+17+26+37+50 \\
&= 148 \\
R_8 &= f(1)\cdot\Delta t + f(2)\cdot\Delta t + f(3)\cdot\Delta t + f(4)\cdot\Delta t + f(5)\cdot\Delta t + f(6)\cdot\Delta t + f(7)\cdot\Delta t + f(8)\cdot\Delta t \\
&= 2+5+10+17+26+37+50+65 \\
&= 212.
\end{aligned}$$

(Averaging these would give an estimate of 180 for the value of the integral.)

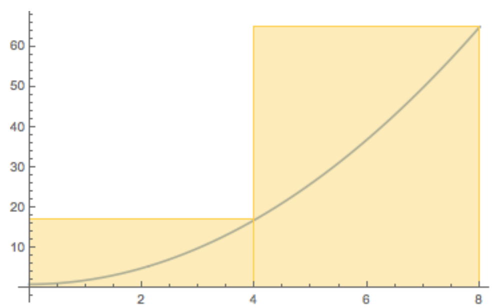
Notice that as Δt gets smaller (as n gets bigger), the left and right Riemann sums get closer together. You should also compare all of these numbers to the actual value of the integral, which you can compute (using the fundamental theorem of calculus) to find is

$$\int_0^8 (t^2+1)\,dt = \frac{560}{3}$$

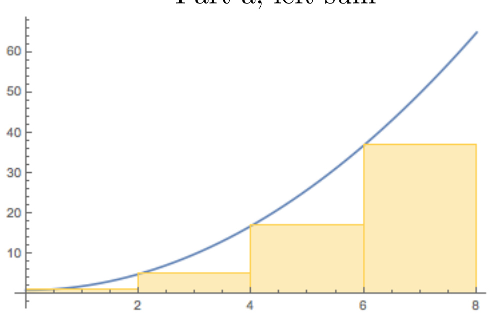
Finally, each Riemann sum is illustrated below.



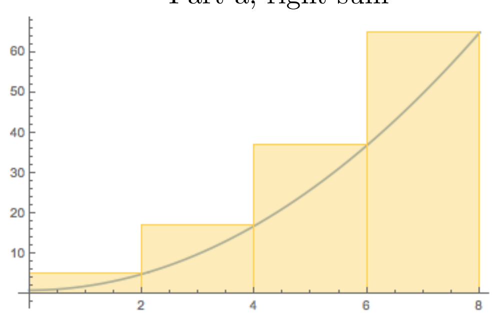
Part a, left sum



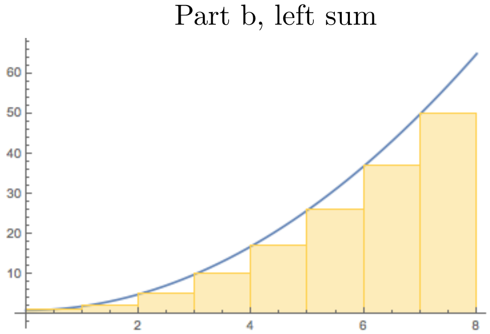
Part a, right sum



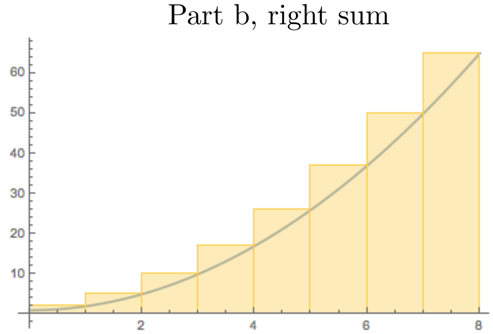
Part b, left sum



Part b, right sum



Part c, left sum



Part c, right sum

Section 5.2, Problem 18 — Recall that in general $\int_a^b f(x)\ dx$ gives the area between the graph of $f(x)$ and the x-axis between the values of $x = a$ and $x = b$.

So for this problem we simply find the area under the provided graph. Notice that a grid is drawn on the graph, with each square on the graph having an area of one square unit. Counting the squares under the graph gives that

$$\int_1^6 f(x)\ dx = 8.5$$

Section 7.1, Problem 38 —

$$\begin{aligned}
\int \left(x^2 + \frac{1}{x^2}\right)\ dx &= \int x^2\ dx + \int x^{-2}\ dx \\
&= \frac{1}{3}x^3 - x^{-1} + C \\
&= \frac{x^3}{3} - \frac{1}{x} + C
\end{aligned}$$

Section 7.1, Problem 40 —

$$\int 5e^z\ dz = 5\int e^z\ dz = 5e^z + C$$

Section 7.1, Problem 46 —

$$\int e^{2t}\ dt = \frac{1}{2}e^{2t} + C$$

Section 7.1, Problem 50 —

$$\begin{aligned}
\int \left(x^2 + \frac{1}{x}\right)\ dx &= \int x^2\ dx + \int \frac{1}{x}\ dx \\
&= \frac{x^3}{3} + \ln|x| + C
\end{aligned}$$

Section 7.3, Problem 4 — We begin by finding the antiderivative of the integrand -

$$\begin{aligned}
\int \left(3t^2 + 4t + 3\right)\ dt &= \int 3t^2\ dt + \int 4t\ dt + \int 3\ dt \\
&= 3\int t^2\ dt + 4\int t\ dt + 3\int\ dt \\
&= 3\cdot\frac{1}{3}t^3 + 4\cdot\frac{1}{2}t^2 + 3t + C \\
&= t^3 + 2t^2 + 3t + C
\end{aligned}$$

Using this antiderivative and the fundamental theorem of calculus,

$$\begin{aligned}
\int_0^2 \left(3t^2 + 4t + 3\right)\ dt &= \left[t^3 + 2t^2 + 3t\right]_0^2 \\
&= \left(2^3 + 2\cdot 2^2 + 3\cdot 2\right) - \left(0^3 + 2\cdot 0^2 + 3\cdot 0\right) \\
&= 22
\end{aligned}$$

Section 7.3, Problem 6 —

$$\begin{aligned}
\int_1^4 \frac{1}{\sqrt{x}}\,dx &= \int_1^4 x^{-1/2}\,dx \\
&= \left[\frac{1}{1/2}x^{1/2}\right]_1^4 \\
&= \left[2\sqrt{x}\right]_1^4 \\
&= 2\sqrt{4} - 2\sqrt{1} \\
&= 2\cdot 2 - 2 = 2
\end{aligned}$$

Section 7.3, Problem 16 —

$$\begin{aligned}
\int_0^1 2e^x\,dx &= [2e^x]_0^1 \\
&= (2e^1) - (2e^0) \\
&= 2e - 2
\end{aligned}$$

Recall that e^x is its own antiderivative.

MAT 122 Quiz 1 Solutions

Question 1: A line passes through the points $(-2, -1)$ and $(1, 5)$.

(a) Write the equation of the line using point-slope form with the point $(1, 5)$.

The slope of the line is given by

$$m = \frac{5 - (-1)}{1 - (-2)} = \frac{5 + 1}{1 + 2} = \frac{6}{3} = 2$$

Using the point-slope form of the equation of a line with slope point $(1, 5)$ and slope $m = 2$ gives the equation

$$y - 5 = 2(x - 1)$$

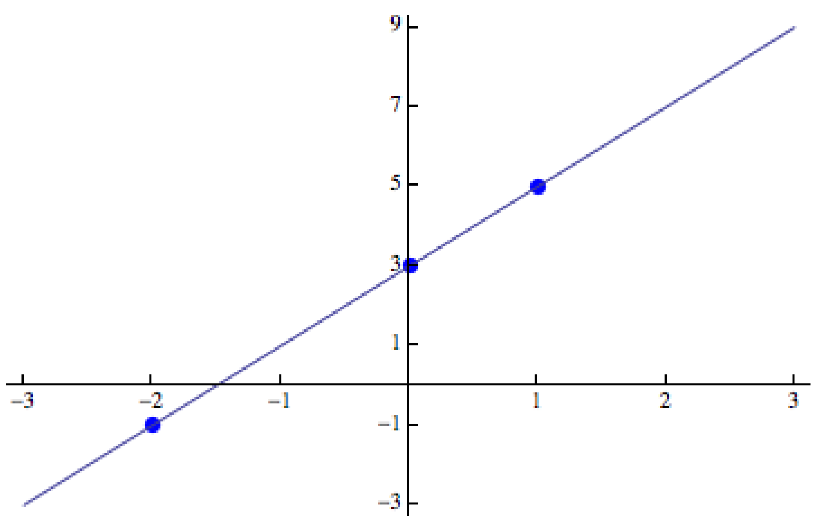
(b) Write the equation of the line in slope-intercept form.

We can rearrange the equation we found above.

$$\begin{aligned} y - 5 = 2(x - 1) &\quad \Rightarrow \\ y - 5 = 2x - 2 &\quad \Rightarrow \\ y = 2x - 2 + 5 &\quad \Rightarrow \\ y = 2x + 3 & \end{aligned}$$

Therefore the y-intercept of the line is at $y = 3$.

(c) Graph the line below. Label at least 2 points.



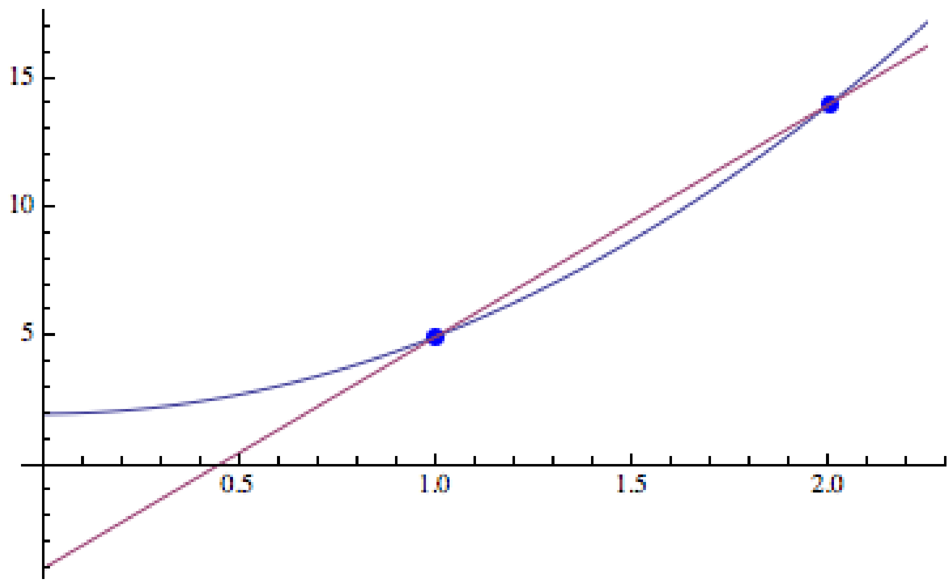
Question 2: Consider the function $f(x) = 3x^2 + 2$.

(a) Compute the average rate of change of the function on the interval $[1, 2]$.

Using the formula for the average rate of change,

$$\text{Average rate of change } = \frac{f(2) - f(1)}{2-1} = \frac{14-5}{2-1} = \frac{9}{1} = 9$$

(b) The graph of $f(x)$ is shown below. Draw the secant line through the points at $x = 1$ and $x = 2$. How is the secant line related to the average rate of change?



The function $f(x)$ is graphed in blue, and the secant line is graphed in red. The slope of the secant line is the same as the average rate of change for that interval.

MAT 122 Quiz 2 Solutions

Question 1: A biologist is studying a population of rabbits in the wild. Each year she counts the total number of rabbits. Her data is given in the table below.

Year	0	1	2	3	4
Population	200	220	242	270	297

(a) What is the average rate of change in population from year 0 to year 1? What is the average rate in change from year 3 to year 4? Include units in your answer.

Using the formula for average rate of change, the rate for the interval $[0, 1]$ is

$$avgROC = \frac{220 - 200}{1 - 0} = 20$$

while the average rate of change for $[3, 4]$ is

$$avgROC = \frac{297 - 270}{4 - 3} = 27.$$

Both of these answers have units of rabbits per year.

(b) What is the relative change in population between years 0 to year 1? What is the relative population change from years 3 to year 4?

Using the formula for relative change, the relative change for the first period is

$$\text{relative change} = \frac{220 - 200}{200} = \frac{20}{200} = .1 = 10\%$$

and the relative change for the second period is

$$\text{relative change} = \frac{297 - 270}{270} = \frac{27}{270} = .1 = 10\%$$

(c) Based on your answers to parts (a) and (b), would you assume the population is undergoing linear or exponential growth? Explain your answer.

The population appears to growing exponentially, because the relative rate of change per year appears to be constant. In fact, computing the relative rate of change for all other periods above would give numbers close to 10%. (The growth cannot be linear because the average rate of change is not constant).

(d) Using your answer on part (c), write a function that gives the population of rabbits P as a function of the year t.

The formula for exponential growth is $P(t) = P_0(1 + r)^t$, where P_0 represents the initial population and r represents the annual growth rate. Using $P_0 = 200$ and $r = .1$, we have that the function is

$$P(t) = 100(1.1)^t$$

Question 2: Using logarithm rules, write the expression below using only a single logarithm.

$$3\ln(x) + \ln(5) - \ln(b)$$

We first use the rule $\ln(A^p) = p\ln(A)$ to rewrite the first term, giving

$$\ln(x^3) + \ln(5) - \ln(b)$$

Next we use the rule $\ln(AB) = \ln(A) + \ln(B)$ to combine the first two terms, giving

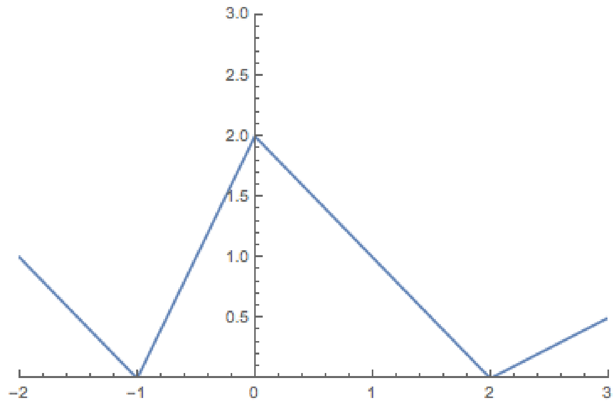
$$\ln(5x^3) - \ln(b)$$

Finally, we combine these terms using the rule $\ln(A/B) = \ln(A) - \ln(B)$, for a final answer of

$$\ln\left(\frac{5x^3}{b}\right)$$

MAT 122 Quiz 3 Solutions

Question 1: The graph of some function $f(x)$ is shown below.



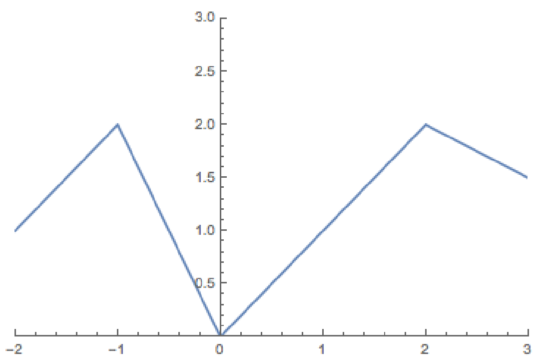
Match the functions listed below to their graphs. (Pay attention to the labels on the axes)

(i) $g(x) = f(2x)$ Graph C

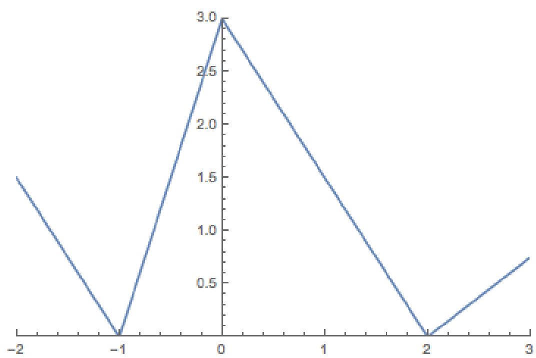
(ii) $h(x) = -f(x) + 2$ Graph A

(iii) $j(x) = f(x-1)$ Graph D

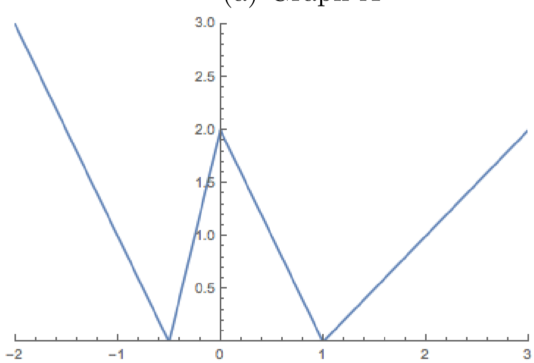
(iv) $k(x) = \frac{3}{2}f(x)$ Graph B



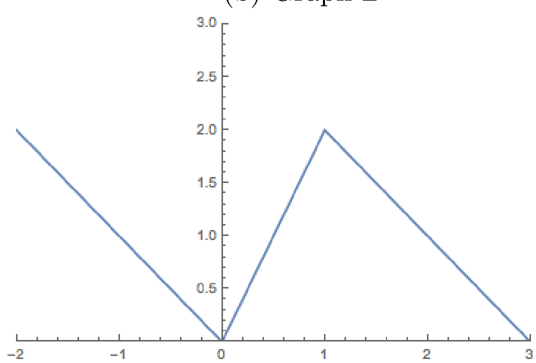
(a) Graph A



(b) Graph B



(c) Graph C



(d) Graph D

The graph of $g(x)$ is obtained from a horizontal shrink of the original graph, which gives Graph C. The graph of $h(x)$ is obtained by reflecting the graph of $f(x)$ over the x-axis and shifting up 2 units, which gives graph A. The graph of $j(x)$ is the graph of $f(x)$ shifted right 1 unit, giving graph D. Finally the graph of $k(x)$ is obtained by stretching the graph of $f(x)$ vertically, which yields graph B.

Question 2: Consider the functions

$$f(x) = \sqrt{x+2} \qquad g(x) = \frac{1}{3x-2} \qquad h(x) = 1 + e^x$$

Find each of the following -

(a) $f(h(0))$

We have that $h(0) = 1 + e^0 = 1 + 1 = 2$. Therefore $f(h(0)) = f(2) = \sqrt{2+2} = 2$

(b) $\dfrac{f(x)}{g(x)}$

Dividing these functions gives

$$\frac{f(x)}{g(x)} = \frac{\sqrt{x+2}}{\frac{1}{3x-2}} = (3x-2)\sqrt{x+2}$$

which cannot be further simplified.

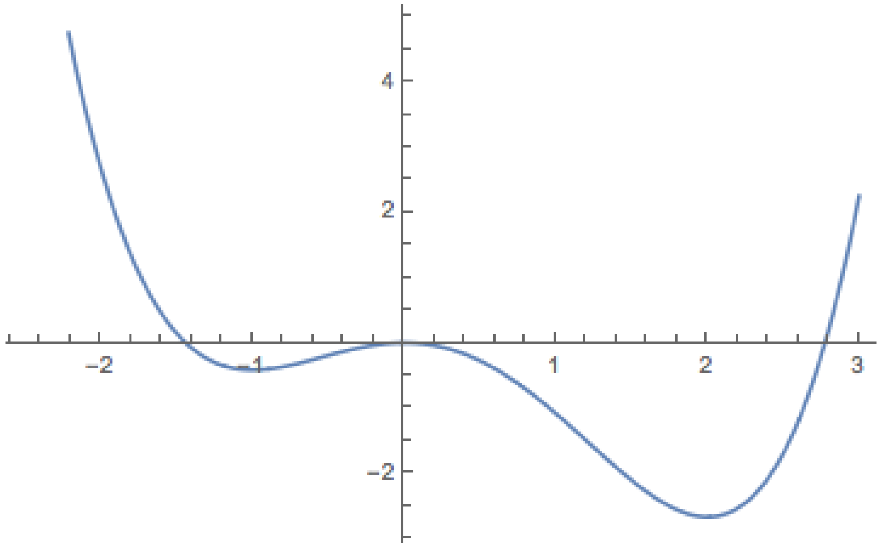
(c) $f(g(h(x)))$

We use function composition multiple times.

$$\begin{aligned} f(g(h(x)) &= f\left(g\left(1+e^x\right)\right) \\ &= f\left(\frac{1}{3(1+e^x)-2}\right) \\ &= f\left(\frac{1}{1+3e^x}\right) \\ &= \sqrt{\frac{1}{1+3e^x}+2} \end{aligned}$$

MAT 122 Quiz 4 Solutions

Question 1: The graph of some function $f(x)$ is shown below.



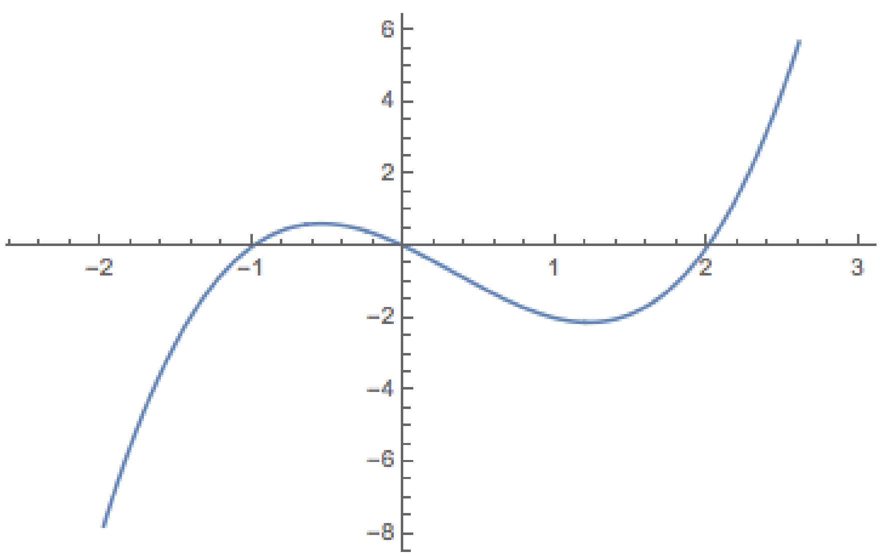
(a) On what intervals is the function increasing? Decreasing?

The function is increasing on the intervals $(-1, 0)$ and $(2, \infty)$. It is decreasing on the intervals $(-\infty, -1)$ and $(0, 2)$.

(b) Where does the graph have horizontal tangent lines?

The graph has horizontal tangent lines when $x = -1$, $x = 0$, and $x = 2$.

(c) On the axes below, sketch a graph of the derivative function $f'(x)$. Label all relevant points. Explain why your graph agrees with your answers in parts (a) and (b).



The value of $f'(x)$ is positive on the same intervals that $f(x)$ is increasing, and negative on the same intervals the original function is decreasing. The derivative function has zeros (x-intercepts) at the same x-values that the original function has horizontal tangent lines.

Question 2: Consider the function $g(x) = x^3 - 2x$. The derivative of this function is $g'(x) = 3x^2 - 2$ (we'll learn in a later chapter how to find this derivative). Find the equation of the tangent line through the graph of $g(x)$ at the point $x = 0$.

The value of the original function at $x = 0$ is $g(0) = 0^3 - 2 \cdot 0 = 0$. Therefore the graph of $g(x)$ passes through the point $(0, 0)$, so the tangent line at $x = 0$ passes through that same point as well.

The value of the derivative at a point is the same as the slope of the tangent line at that point. Therefore the slope of the tangent line at $x = 0$ is $g'(0) = 3 \cdot 0^2 - 2 = -2$.

The tangent line therefore passes through the point $(0, 0)$ and has slope $m = -2$. Using either the point-slope or slope-intercept form of the equation of a line gives that the equation of the tangent line is $y = -2x$.

MAT 122 Quiz 5 Solutions

Question 1: Let $f(x) = 5x^2 + 3x - \frac{2}{x}$.

(a) Find the derivative $f'(x)$.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left(5x^2 + 3x - \frac{2}{x}\right) \\
&= \frac{d}{dx}(5x^2) + \frac{d}{dx}(3x) - \frac{d}{dx}\left(\frac{2}{x}\right) \\
&= 5 \cdot \frac{d}{dx}(x^2) + 3 \cdot \frac{d}{dx}(x) - 2 \cdot \frac{d}{dx}\left(\frac{1}{x}\right) \\
&= 5 \cdot \frac{d}{dx}(x^2) + 3 \cdot \frac{d}{dx}(x) - 2 \cdot \frac{d}{dx}\left(x^{-1}\right) \\
&= 5 \cdot 2x + 3 \cdot 1 - 2 \cdot (-1)x^{-2} \\
&= 10x + 3 + 2x^{-2} = 10x + 3 + \frac{2}{x^2}
\end{aligned}$$

(b) Find the second derivative $f''(x)$.

$$\begin{aligned}
f''(x) &= \frac{d}{dx}\left(f'(x)\right) \\
&= \frac{d}{dx}\left(10x + 3 + 2x^{-2}\right) \\
&= \frac{d}{dx}(10x) + \frac{d}{dx}(3) + \frac{d}{dx}(2x^{-2}) \\
&= 10 \cdot \frac{d}{dx}(x) + \frac{d}{dx}(3) + 2 \cdot \frac{d}{dx}(x^{-2}) \\
&= 10 \cdot 1 + 0 + 2 \cdot (-2)x^{-3} \\
&= 10 - 4x^{-3} = 10 - \frac{4}{x^3}
\end{aligned}$$

Question 2: At what values of t does the graph of $g(t) = 2\sqrt{t} - \dfrac{t}{2}$ have horizontal tangent lines?

The derivative gives the value of the slope of the tangent line, so to find where the function g has horizontal tangent lines (that is, slope zero tangent lines), we need to solve the equation $g'(t) = 0$.

For this, we first find $g'(t)$. We have

$$\begin{aligned}
g'(t) &= \frac{d}{dt}\left(2\sqrt{t} - \frac{t}{2}\right) \\
&= \frac{d}{dt}\left(2t^{1/2} - \frac{t}{2}\right) \\
&= \frac{d}{dt}(2t^{1/2}) - \frac{d}{dt}\left(\frac{t}{2}\right) \\
&= 2 \cdot \frac{d}{dt}(t^{1/2}) - \frac{1}{2} \cdot \frac{d}{dt}(t) \\
&= 2 \cdot \frac{1}{2}t^{-1/2} - \frac{1}{2} \cdot 1 \\
&= t^{-1/2} - \frac{1}{2}
\end{aligned}$$

We are therefore trying to solve the equation $t^{-1/2} - \frac{1}{2} = 0$. Solving this equation gives

$$\begin{aligned}
t^{-1/2} - \frac{1}{2} &= 0 \\
\frac{1}{\sqrt{t}} &= \frac{1}{2} \\
\sqrt{t} &= 2 \\
t &= 4
\end{aligned}$$

Therefore the graph of $g(t)$ has a horizontal tangent line only at $t = 4$.

MAT 122 Quiz 6 Solutions

Question 1: Find the derivative of the following functions. You do not need to simplify your answers.

(a) $f(t) = (t^3 - 3t^2 + 1)e^t$

Using the product rule,

$$\begin{aligned} f'(t) &= \frac{d}{dt}\left[(t^3 - 3t^2 + 1)e^t\right] \\ &= \frac{d}{dt}(t^3 - 3t^2 + 1) \cdot e^t + (t^3 - 3t^2 + 1)\frac{d}{dt}(e^t) \\ &= (3t^2 - 6t)e^t + (t^3 - 3t^2 + 1)e^t \end{aligned}$$

This expression can be simplified to

$$f'(t) = (t^3 - 6t + 1)e^t$$

(b) $g(x) = \dfrac{4x - 2}{x^2 + x + 1}$

Using the quotient rule,

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(\frac{4x - 2}{x^2 + x + 1}\right) \\ &= \frac{(x^2 + x + 1)\frac{d}{dx}(4x - 2) - (4x - 2)\frac{d}{dx}(x^2 + x + 1)}{(x^2 + x + 1)^2} \\ &= \frac{(x^2 + x + 1) \cdot 4 - (4x - 2)(2x + 1)}{(x^2 + x + 1)^2} \end{aligned}$$

This expression can be simplified to

$$g'(x) = \frac{-4x^2 + 4x + 6}{(x^2 + x + 1)^2}$$

(c) $h(x) = 3^x + x^3 + 3\ln(x)$

$$\begin{aligned} h'(x) &= \frac{d}{dx}(3^x + x^3 + 3\ln x) \\ &= \frac{d}{dx}(3^x) + \frac{d}{dx}(x^3) + \frac{d}{dx}(3\ln x) \\ &= \frac{d}{dx}(3^x) + \frac{d}{dx}(x^3) + 3\frac{d}{dx}(\ln x) \\ &= (\ln 3)3^x + 3x^2 + 3\frac{1}{x} = (\ln 3)3^x + 3x^2 + \frac{3}{x} \end{aligned}$$

Name: ____________________

MAT 122 Quiz 7 - Aug 4

Question 1: Consider the function $h(x) = (x^2 - 4)^2$.

(a) Find the first derivative $h'(x)$ using the chain rule.

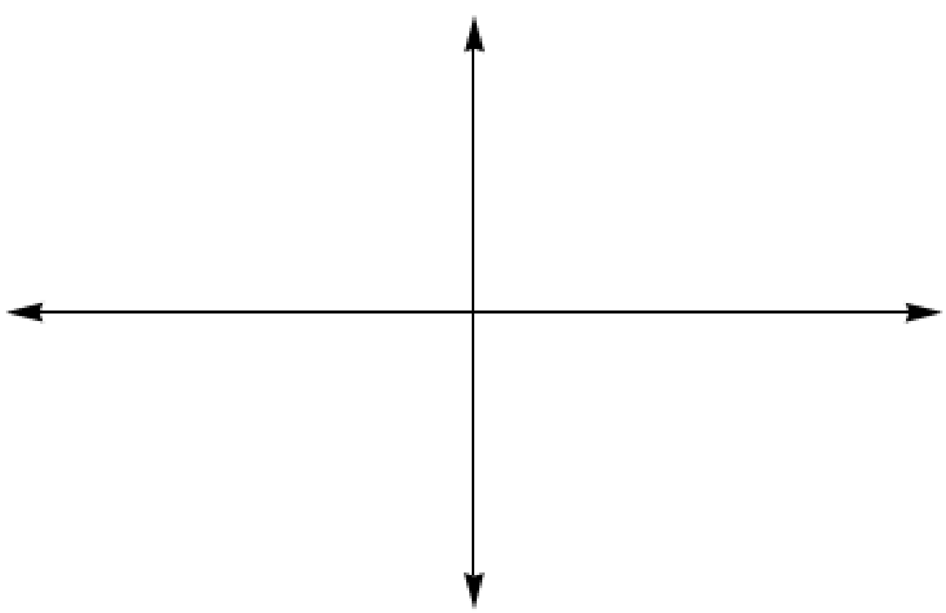
(b) Find the critical points of $h(x)$.

(c) Where does $h(x)$ have local maxima? Where does $h(x)$ have local minima?

(d) On what intervals is $h(x)$ increasing? On what intervals is $h(x)$ decreasing?

(e) Find the inflection points of $h(x)$. On what intervals is $h(x)$ concave up? On what intervals is $h(x)$ concave down?

(f) Sketch a graph of $h(x)$. Label on your graph the horizontal intercepts, the vertical intercepts, all local extrema, and all inflection points.



MAT 122 Quiz 7 Solutions

Question 1: Consider the function $h(x) = (x^2 - 4)^2$.

(a) Find the first derivative $h'(x)$ using the chain rule.

The inside function is $g(x) = x^2 - 4$ and the outside function is $f(z) = z^2$. We have that $g'(x) = 2x$ and $f'(z) = 2z$. Therefore the derivative is

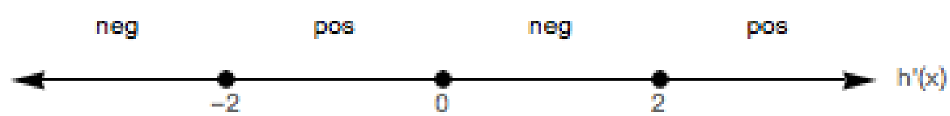
$$\begin{aligned} h'(x) &= f'(g(x)) \cdot g'(x) \\ &= f'(x^2 - 4) \cdot 2x \\ &= 2(x^2 - 4) \cdot 2x \\ &= 4x(x^2 - 4) \end{aligned}$$

(b) Find the critical points of $h(x)$.

The critical points are where $h'(x) = 0$. The derivative we obtained above factors as $4x(x - 2)(x + 2)$, and so we have that the critical points are $x = 0, x = 2$, and $x = -2$.

(c) Where does $h(x)$ have local maxima? Where does $h(x)$ have local minima?

We'll use the first derivative test. We need to check the sign of the derivative on the intervals $(-\infty, -2)$, $(-2, 0)$, $(0, 2)$, and $(2, \infty)$. Using a number line to display our answer gives



Using the first derivative test, we have local minima at $x = -2$ and $x = 2$ and a local maximum at $x = 0$.

(d) On what intervals is $h(x)$ increasing? On what intervals is $h(x)$ decreasing?

Using the signs of the derivative we found in the previous part, the function is increasing on the intervals $(-2, 0)$ and $(2, \infty)$ and decreasing on the intervals $(-\infty, -2)$ and $(0, 2)$.

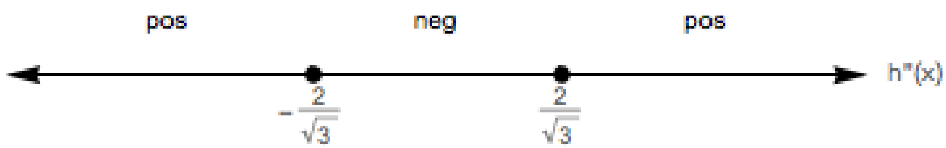
(e) Find the inflection points of $h(x)$. On what intervals is $h(x)$ concave up? On what intervals is $h(x)$ concave down?

To find inflection points, we first need to find the second derivative. We observe

$$\begin{aligned} h'(x) &= 4x(x^2 - 4) \\ &= 4x^3 - 16x \\ h''(x) &= 12x^2 - 16 \end{aligned}$$

(We could also have used the product rule to take the derivative). Solving for where $h''(x) = 0$, we have that the zeros of the second derivative occur at $x = \pm\sqrt{\frac{16}{12}} = \pm\frac{2}{\sqrt{3}}$.

We then check the sign of the second derivative, and displaying our answers on a number line gives



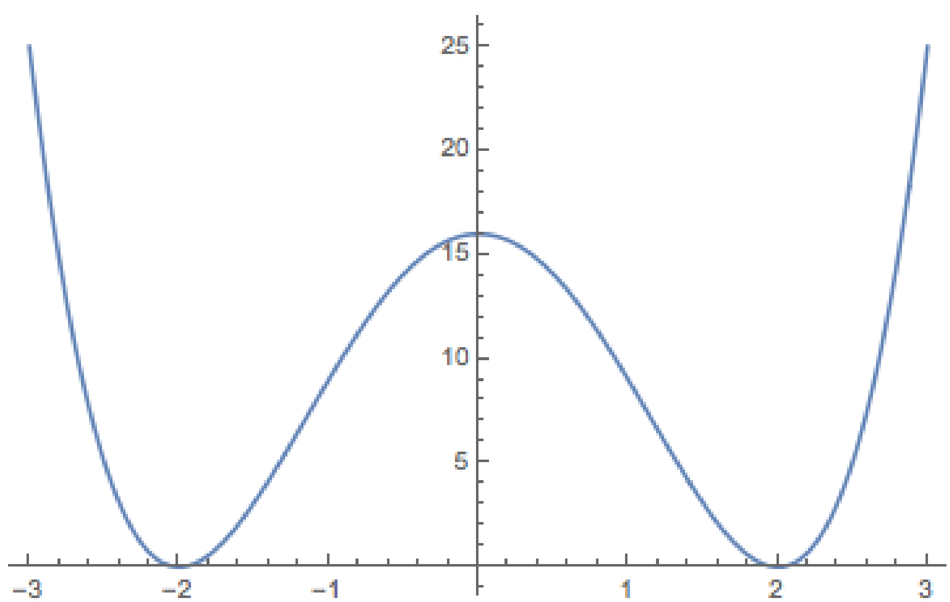
Therefore the function has inflection points at $x = \pm\frac{2}{\sqrt{3}}$. The function is concave up on the intervals $\left(-\infty, \frac{-2}{\sqrt{3}}\right)$ and $\left(\frac{2}{\sqrt{3}}, \infty\right)$ and is concave down on the interval $\left(\frac{-2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$

(f) Sketch a graph of $h(x)$. Label on your graph the horizontal intercepts, the vertical intercepts, all local extrema, and all inflection points.

The horizontal intercepts are at $x = \pm 2$. Notice that the function $h(x)$ is already factored, which makes finding the horizontal intercepts easier. The vertical intercept is at 16. We find that the y-values associated with the various extrema and inflection points are

$$\begin{aligned} h(-2) &= 0 \\ h(0) &= 16 \\ h(2) &= 0 \\ h\left(\frac{-2}{\sqrt{3}}\right) &= 64/9 \approx 7.11 \\ h\left(\frac{2}{\sqrt{3}}\right) &= 64/9 \approx 7.11 \end{aligned}$$

Plotting all of these points and the drawing the graph gives



MAT 122 Quiz 8 Solutions

Question 1: Find the global maximum and minimum values of the function

$$f(x) = e^{\frac{-x^3}{3}+\frac{x}{4}}$$

on the interval $[0,2]$. (Remember the following fact about exponential functions - if a, b are any two real numbers with $a < b$, then $e^a < e^b$.)

To find the global maximum and minimum values of a function on a closed interval, we must first find the critical points, where the derivative is equal to zero. To compute the derivative we use the chain rule. The inside function is $z = \frac{-x^3}{3} + \frac{x}{4}$, and the outside function is e^z. Therefore the derivative is

$$\begin{aligned} f'(x) &= e^{\frac{-x^3}{3}+\frac{x}{4}} \cdot \frac{d}{dx}\left(\frac{-x^3}{3}+\frac{x}{4}\right) \\ &= e^{\frac{-x^3}{3}+\frac{x}{4}} \cdot \left(-x^2+\frac{1}{4}\right) \\ &= e^{\frac{-x^3}{3}+\frac{x}{4}} \left(\frac{1}{2}-x\right)\left(\frac{1}{2}+x\right) \end{aligned}$$

Notice that we have factored above, so that we can see that the derivative is zero when $x = \pm\frac{1}{2}$. (Exponential functions have no zeros).

Thus the critical points are $x = -1/2$ and $x = 1/2$. Notice that $-1/2$ is not in the interval $[0,2]$ that we are considering, so we ignore it - we are only looking for the global max and min on the interval $[0,2]$.

We then evaluate the function at the critical point $x = 1/2$ and the endpoints $x = 0, x = 2$. This gives

$$\begin{aligned} f(0) &= e^0 = 1 \\ f\left(\frac{1}{2}\right) &= e^{\frac{1}{12}} \\ f(2) &= e^{\frac{-13}{6}} \end{aligned}$$

The largest number on this list is $e^{\frac{1}{12}}$, so that is therefore the global maximum. The smallest number on this list is $e^{\frac{-13}{6}}$, so that is the global minimum.

Question: A sprinter runs around a track. While she is running, her coach measures her speed at different times. Her speed v, in meters per second, after t seconds of running is given below.

t	0	5	10	15	20	25	30
v(t)	0	2	3	4	5	5	0

(a) Using a left Riemann sum with $n = 3$ subintervals, estimate how far the sprinter runs on the time interval $[0,30]$.

Using $n = 3$, each subinterval will have a width of

$$\Delta t = \frac{30-0}{3} = 10$$

Dividing the interval $[0, 30]$ into $n = 3$ subintervals gives

$$t_0 = 0 \qquad t_1 = 10 \qquad t_2 = 20 \qquad t_3 = 30$$

The left Riemann sum approximation with $n = 3$ is then

$$\begin{aligned} L_3 &= v(t_0) \cdot \Delta t + v(t_1) \cdot \Delta t + v(t_2) \cdot \Delta t \\ &= v(0) \cdot \Delta t + v(10) \cdot \Delta t + v(20) \cdot \Delta t \\ &= 0 \cdot 10 + 3 \cdot 10 + 5 \cdot 10 \\ &= 80 \end{aligned}$$

so this estimate gives that the sprinter runs approximately 80 m.

(b) Using a right Riemann sum with $n = 6$ subintervals, estimate how far the sprinter runs on the time interval $[0, 30]$.

Using $n = 6$, each subinterval will have a width of

$$\Delta t = \frac{30 - 0}{6} = 5$$

Dividing the interval $[0, 30]$ into $n = 6$ subintervals gives

$$t_0 = 0 \quad t_1 = 5 \quad t_2 = 10 \quad t_3 = 15 \quad t_4 = 20 \quad t_5 = 25 \quad t_6 = 30$$

The right Riemann sum approximation with $n = 6$ is then

$$\begin{aligned} R_6 &= v(t_1) \cdot \Delta t + v(t_2) \cdot \Delta t + v(t_3) \cdot \Delta t + v(t_4) \cdot \Delta t + v(t_5) \cdot \Delta t + v(t_6) \cdot \Delta t \\ &= v(5) \cdot \Delta t + v(10) \cdot \Delta t + v(15) \cdot \Delta t + v(20) \cdot \Delta t + v(25) \cdot \Delta t + v(30) \cdot \Delta t \\ &= 2 \cdot 5 + 3 \cdot 5 + 4 \cdot 5 + 5 \cdot 5 + 5 \cdot 5 + 0 \cdot 5 \\ &= 10 + 15 + 20 + 25 + 25 + 0 \\ &= 95 \end{aligned}$$

so this estimate gives that the sprinter runs approximately 95 m.

(c) Which estimate do you think is more accurate? Why?

The second estimate is more likely to be accurate, as Riemann sums become more accurate when using more subintervals, that is, for larger values of n.

MAT 122 Quiz 9 Solutions

Question 1: Compute the following integrals. Note that some of these are indefinite integrals and some are definite integrals.

(a) $\int (2x^4 - x - 1)\,dx$

$$\begin{aligned}
\int (2x^4 - x - 1)\,dx &= \int 2x^4\,dx - \int x\,dx - \int dx \\
&= 2\int x^4\,dx - \int x\,dx - \int dx \\
&= 2\cdot\frac{1}{5}x^5 - \frac{1}{2}x^2 - x + C \\
&= \frac{2}{5}x^5 - \frac{1}{2}x^2 - x + C
\end{aligned}$$

(b) $\int \left(\frac{1}{z} + \sqrt{z} + e^z\right)\,dz$

$$\begin{aligned}
\int \left(\frac{1}{z} + \sqrt{z} + e^z\right)\,dz &= \int \frac{1}{z}\,dz + \int z^{1/2}\,dz + \int e^z\,dz \\
&= \ln|z| + \frac{1}{1/2+1}z^{1/2+1} + e^z + C \\
&= \ln|z| + \frac{2}{3}z^{3/2} + e^z + C
\end{aligned}$$

(c) $\int_0^2 (2^w + w)\,dw$

To compute a definite integral, we use the fundamental theorem of calculus. We first need to find an antiderivative for the integrand $2^w + w$. We have

$$\begin{aligned}
\int (2^w + w)\,dw &= \int 2^w\,dw + \int w\,dw \\
&= \frac{1}{\ln 2}2^w + \frac{1}{2}w^2 + C
\end{aligned}$$

Recall that all antiderivatives of $2^w + w$ have this form, so when we pick any antiderivative to use with the fundamental theorem of calculus. We will pick the constant C to be $C = 0$. Therefore we have

$$\begin{aligned}
\int_0^2 (2^w + w)\,dw &= \left[\frac{1}{\ln 2}2^w + \frac{1}{2}w^2\right]_0^2 \\
&= \left(\frac{1}{\ln 2}\cdot 2^2 + \frac{1}{2}\cdot 2^2\right) - \left(\frac{1}{\ln 2}\cdot 2^0 + \frac{1}{2}\cdot 0^2\right) \\
&= \frac{4}{\ln 2} + 2 - \frac{1}{\ln 2} - 0 \\
&= \frac{3}{\ln 2} + 2
\end{aligned}$$

MAT 122 Midterm Exam

Name: ____

- Please read all instructions before beginning, and do not open the exam until you are told to do so.
- Put away all notes, books, calculators, etc. before beginning the exam, and place them under your desk. The only items on your desk should be this exam, pencils/pens, and an eraser.
- This exam has 7 questions, each with multiple parts. The point value for each question is shown below.
- The final page of the exam is left blank for scratch paper. You may tear this page from the exam booklet, but do not remove any other pages.
- You have 90 minutes to complete the exam.

Question	Points	Score
1	15	
2	10	
3	15	
4	10	
5	15	
6	20	
7	15	
Total	100	

Question 1: Let $f(x)$ be a function such that $f(0) = 20$ and $f(1) = 10$.

(a) Assume that $f(x)$ is a linear function. Write a formula for $f(x)$.

(b) Still assuming that $f(x)$ is a linear function, what is the average rate of change of this function on the interval $[0, 2]$?

(c) Now assume that $f(x)$ is an exponential function. Write a formula for $f(x)$. (Hint: you may need to compute the relative (percent) change.)

(d) Still assuming that $f(x)$ is exponential, and using your formula from part (b), find the average rate of change of the function on the interval $[0, 2]$.

Question 2:

(a) Solve the following equation for w. Simplify your answer as much as possible.

$$3 \cdot 4^w = e^2$$

(b) Solve the following equation for w. Simplify your answer as much as possible.

$$3w^2 + 5w - 3 = -2$$

Question 3: The derivative of the function $f(x) = \ln(x)$ is $f'(x) = \dfrac{1}{x}$. Recall that the domain of $\ln(x)$ is $(0, \infty)$.

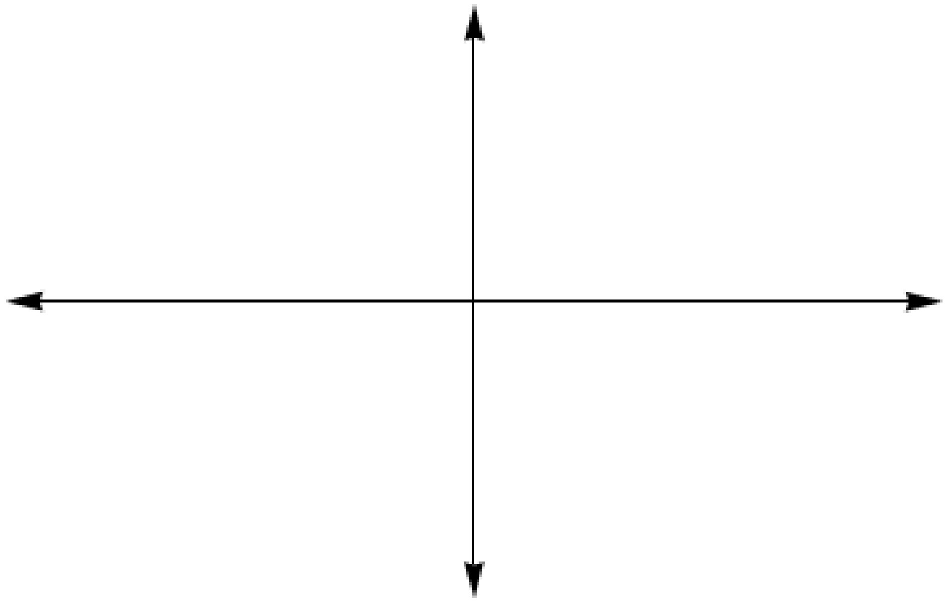
(a) Use the local linear approximation at $x = 1$ to estimate the value of $\ln(1.1)$.

(b) The derivative of the function $\dfrac{1}{x}$ with respect to x is $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = \dfrac{-1}{x^2}$. Using this information, is the graph of $\ln(x)$ concave up or concave down? Explain your answer.

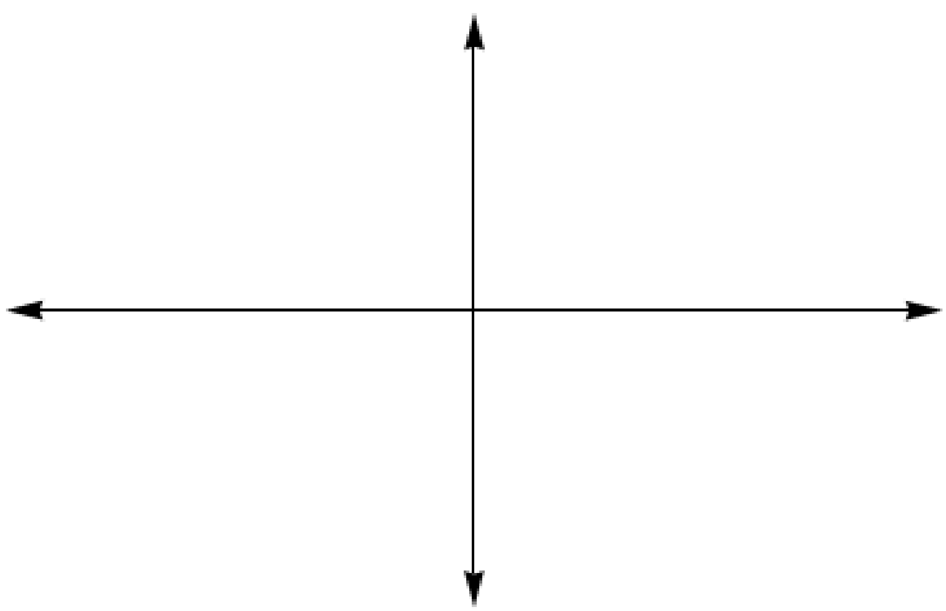
(c) Is the approximate value you found in part (a) larger or smaller than the actual value of $\ln(1.1)$? Use part (b) to justify your answer.

Question 4: Graph the following functions. Find the values of the vertical and horizontal intercepts on each graph.

(a) $f(x) = \frac{1}{2}(x+1)^2$



(b) $g(x) = -(e^{-x})$



Question 5: Let $f(x) = \dfrac{1}{x^2+1}$ and $g(x) = 4^x$. Let $h(x) = f(g(x))$.

(a) Write a formula for $h(x)$. Use that formula to compute $h(0)$.

(b) Estimate the value of $h'(0)$ using the interval $\left[0, \frac{1}{2}\right]$. (Remember that $x^{1/2} = \sqrt{x}$.)

Question 6: Let $F(t)$ be a function such that $F'(t)$ has the graph shown below.

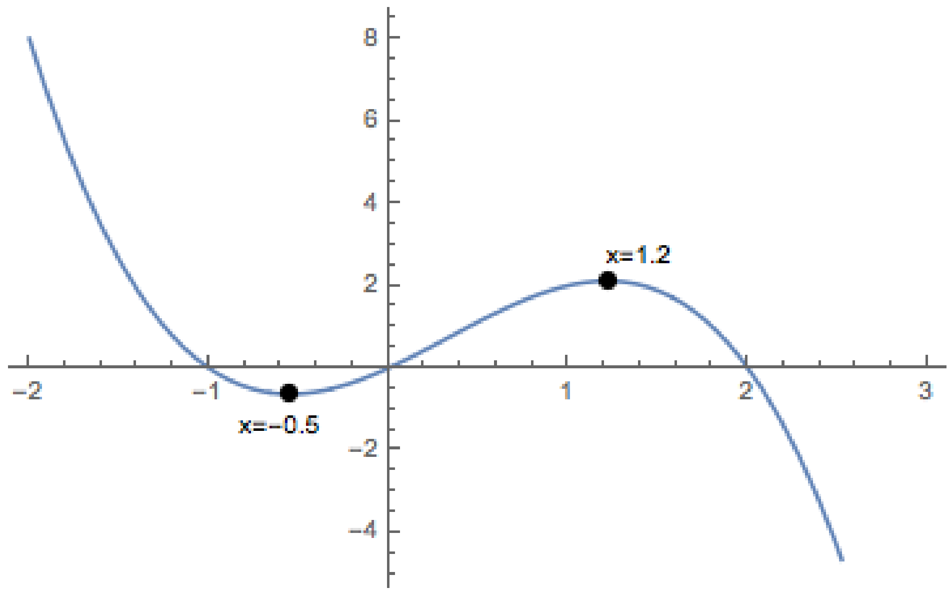
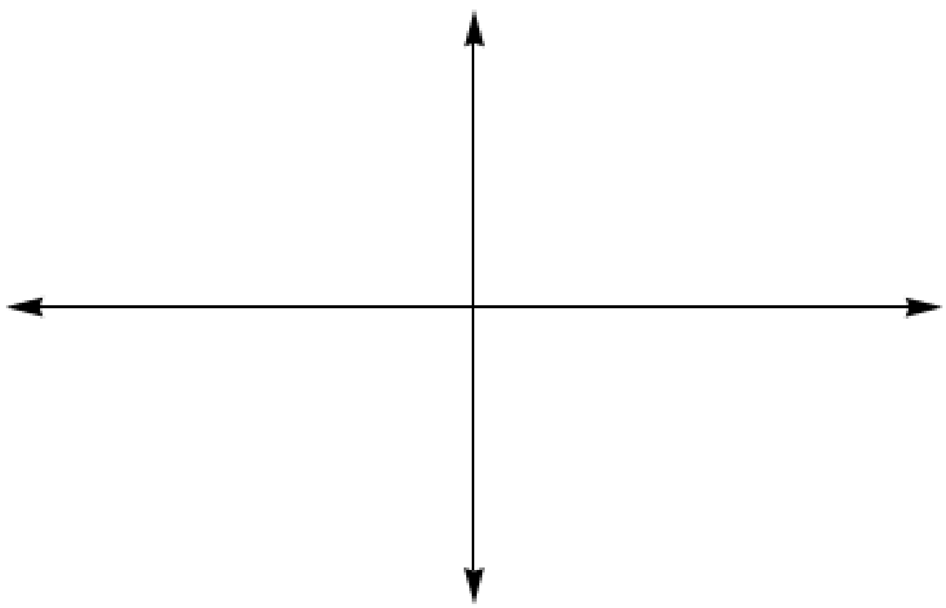


Figure 1: Graph of $F'(t)$

(a) On what intervals is the function $F(t)$ increasing? On what intervals is the function $F(t)$ decreasing? Where does the graph of $F(t)$ have horizontal tangent lines? Explain your answer. (WARNING: Remember that the graph shown is a graph of the derivative $F'(t)$.)

(b) On what intervals is $F'(t)$ increasing and decreasing? Where does the graph of $F'(t)$ have horizontal tangent lines?

(c) Use your answers to part (b) to sketch a graph of the second derivative $F''(t)$. Label the horizontal intercepts. Explain how your graph agrees with your answers in part (b).



(d) On what intervals is the original function $F(t)$ concave up? On what intervals is the original function $F(t)$ concave down? Explain your answer.

Question 7: Bill gets into his car and begins driving east. Let s represent the distance from Bill to his starting point, in kilometers, and let t denote the time he has been driving, in hours. (We take s to be positive when Bill is east of his starting point and negative when Bill is west of his starting point.)

(a) What does $\dfrac{ds}{dt}$ represent physically? Include units in your answer.

(b) What does $\dfrac{d^2s}{dt^2}$ represent physically? Include units in your answer.

(c) Assume that $\left.\frac{ds}{dt}\right|_{t=2} = -50$ and $\left.\frac{d^2s}{dt^2}\right|_{t=2} = 500$. In what direction is Bill traveling after 2 hours, east or west? Is his car speeding up or slowing down? Explain your answers.

(d) The following table gives different values of s and t. Compute the average rate of change in position on the intervals $[4, 4.5]$ and $[4, 4.25]$ (this corresponds to time periods of 30 and 15 minutes, respectively). Using these values, estimate the value of $\left.\frac{ds}{dt}\right|_{t=4}$.

t	4	4.25	4.5
s	100	121	145

MAT 122 Midterm Exam Solutions

Name: KEY

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Question	Points	Score
1	15	
2	10	
3	15	
4	10	
5	15	
6	20	
7	15	
Total	100	

Question 1: Let $f(x)$ be a function such that $f(0) = 20$ and $f(1) = 10$.

(a) Assume that $f(x)$ is a linear function. Write a formula for $f(x)$.

The slope of such a line is

$$m = \frac{10 - 20}{1 - 0} = -10$$

and $f(0) = 20$ means the y-intercept is 20. Using the slope-intercept formula for a line gives

$$f(x) = -10x + 20$$

(b) Still assuming that $f(x)$ is a linear function, what is the average rate of change of this function on the interval $[0, 2]$?

The average rate of change of a linear function is the same as the slope, so it is -10. You could also use the formula to compute

$$avgROC = \frac{f(2) - f(0)}{2 - 0} = \frac{0 - 20}{2} = \frac{-20}{2} = -10$$

(c) Now assume that $f(x)$ is an exponential function. Write a formula for $f(x)$. (Hint: you may need to compute the relative (percent) change.)

The relative change is

$$\frac{10 - 20}{20} = \frac{-10}{20} = \frac{-1}{2} = -.5 = -50\%$$

Note that this is a relative rate of change per unit time. Using the equation for an exponential equation with relative rate per unit time $r = -0.5$ and initial value 20 gives the formula

$$f(x) = 20(1 - 0.5)^x = 20(0.5)^x = 20\left(\frac{1}{2}\right)^x$$

(d) Still assuming that $f(x)$ is exponential, and using your formula from part (b), find the average rate of change of the function on the interval $[0, 2]$.

We'll use the formula for average rate of change. Note that $f(0) = 0$, while

$$f(2) = 20\left(\frac{1}{2}\right)^2 = \frac{20}{4} = 5$$

so applying the average rate of change formula gives

$$avgROC = \frac{f(2) - f(0)}{2 - 0} = \frac{5 - 20}{2 - 0} = \frac{-15}{2}.$$

Question 2:

(a) Solve the following equation for w. Simplify your answer as much as possible.

$$3 \cdot 4^w = e^2$$

We take the logarithm of both sides to isolate the variable w.

$$\begin{aligned}
3 \cdot 4^w &= e^2 \\
\ln\left(3 \cdot 4^w\right) &= \ln\left(e^2\right) \\
\ln(3) + \ln\left(4^w\right) &= \ln\left(e^2\right) \\
\ln(3) + w\ln(4) &= 2 \\
w\ln(4) &= 2 - \ln(3) \\
w &= \frac{2 - \ln(3)}{\ln(4)}
\end{aligned}$$

(b) Solve the following equation for w. Simplify your answer as much as possible.

$$3w^2 + 5w - 3 = 2$$

We put the equation in standard form, and then apply the quadratic formula.

$$\begin{aligned}
3w^2 + 5w - 3 &= -2 \\
3w^2 + 5w - 1 &= 0 \\
x &= \frac{-5 \pm \sqrt{5^2 - 4 \cdot 3 \cdot (-1)}}{2 \cdot 3} \\
x &= \frac{-5 \pm \sqrt{37}}{6}
\end{aligned}$$

Question 3: The derivative of the function $f(x) = \ln(x)$ is $f'(x) = \dfrac{1}{x}$. Recall that the domain of $\ln(x)$ is $(0, \infty)$.

(a) Use the local linear approximation at $x = 1$ to estimate the value of $\ln(1.1)$.

The local linear approximation is simply the equation of the tangent line at $x = 1$. Note that this line passes through the point $(1, \ln(1)) = (1, 0)$. The slope of the tangent line is given by the derivative, so the slope of the tangent line at $x = 1$ is $f'(1) = \frac{1}{1} = 1$. Using point-slope form, the formula for tangent line is therefore $L(x) = 1(x - 1) + 0 = x - 1$.

The local linear approximation says that $\ln(x) \approx L(x)$ near $x = 1$, so we have that $\ln(1.1) \approx L(1.1) = 1.1 - 1 = 0.1$, so our estimate is $\ln(1.1) \approx 0.1$.

(b) The derivative of the function $\dfrac{1}{x}$ with respect to x is $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = \dfrac{-1}{x^2}$. Using this information, is the graph of $\ln(x)$ concave up or concave down? Explain your answer.

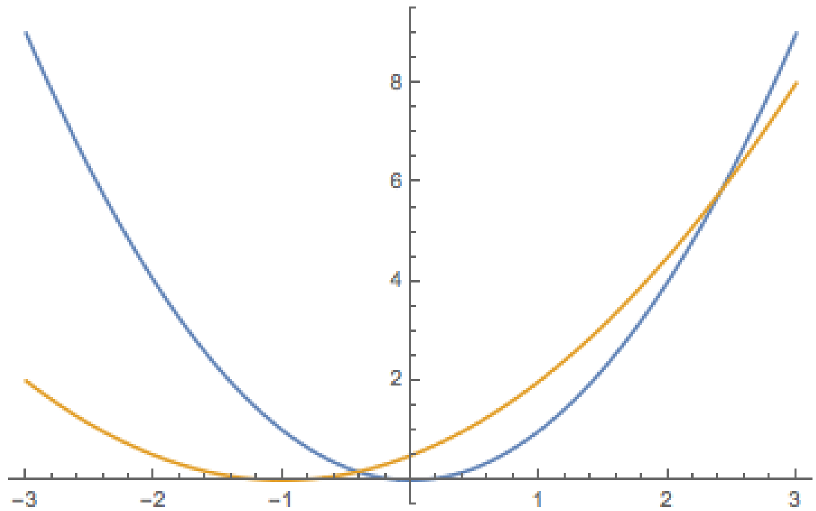
The graph is concave down, as the second derivative is always negative, and the sign of the second derivative determines the concavity.

(c) Is the approximate value you found in part (a) larger or smaller than the actual value of $\ln(1.1)$? Use part (b) to justify your answer.

Because $\ln(x)$ is increasing and concave down (opposite signs), the estimate will be an overestimate. We could see this by sketching a graph - if we draw a function that is increasing and concave down at a point, the tangent line at that point will be above the function itself. (For reference, the actual value is $\ln(1.1) = 0.0953102\ldots$)

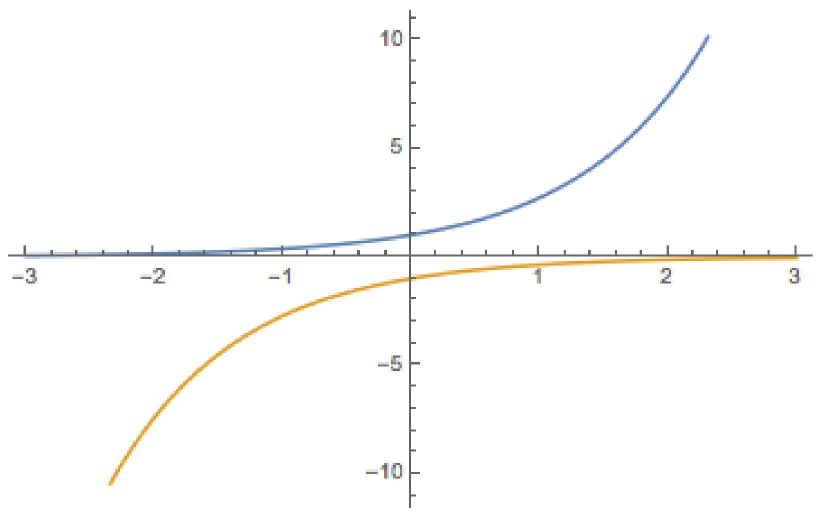
Question 4: Graph the following functions. Find the values of the vertical and horizontal intercepts on each graph.

(a) $f(x) = \frac{1}{2}(x+1)^2$



The graph is obtained by taking the known graph of x^2, shifting it left 1 unit, and shrinking vertically by half. The desired graph is shown in gold above - the graph of x^2 is shown in blue for reference. The vertical intercept of the new graph is $\frac{1}{2}$. The horizontal intercept is at -1.

(b) $g(x) = -(e^{-x})$



The graph is obtained by taking the known graph of e^x and reflecting over both the x- and y-axes. The desired graph is shown in gold above - the graph of e^x is shown in blue for reference. The vertical intercept of the new graph is -1. The new graph does not have a horizontal intercept.

Question 5: Let $f(x) = \dfrac{1}{x^2+1}$ and $g(x) = 4^x$. Let $h(x) = f(g(x))$.

(a) Write a formula for $h(x)$. Use that formula to compute $h(0)$.

$$h(x) = f(g(x)) = f(4^x) = \frac{1}{(4^x)^2+1} = \frac{1}{4^{2x}+1}$$

Plugging in $x = 0$ into the above formula gives that

$$h(0) = \frac{1}{4^0+1} = \frac{1}{1+1} = \frac{1}{2}.$$

Alternatively, we could compute

$$h(0) = f(g(0)) = f\left(4^0\right) = f(1) = \frac{1}{1^2+1} = \frac{1}{2}$$

(b) Estimate the value of $h'(0)$ using the interval $\left[0, \frac{1}{2}\right]$.

We can estimate the value of the derivative at 0 by computing the average rate of change on the interval $[0, 1/2]$. To compute this, we first find $h(1/2)$, which is

$$h\left(\frac{1}{2}\right) = f\left(g\left(\frac{1}{2}\right)\right) = f\left(4^{1/2}\right) = f(2) = \frac{1}{2^2+1} = \frac{1}{5}$$

We could also compute this using the formula found above. We therefore have that

$$h'(0) \approx avgROC \text{ on } \left[0, \frac{1}{2}\right] = \frac{h(1/2)-h(0)}{1/2-0} = \frac{\frac{1}{5}-\frac{1}{2}}{\frac{1}{2}} = 2\left(\frac{2}{10}-\frac{5}{10}\right) = \frac{-3}{5}$$

Question 6: Let $F(t)$ be a function such that $F'(t)$ has the graph shown below.

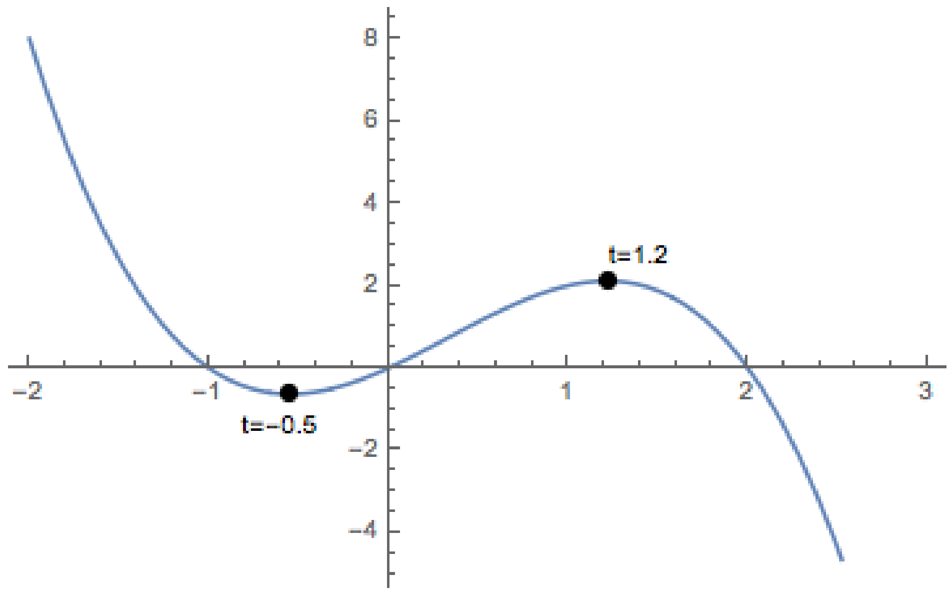


Figure 1: Graph of $F'(t)$

(a) On what intervals is the function $F(t)$ increasing? On what intervals is the function $F(t)$ decreasing? Where does the graph of $F(t)$ have horizontal tangent lines? Explain your answer. (WARNING: Remember that the graph shown is a graph of the derivative $F'(t)$.)

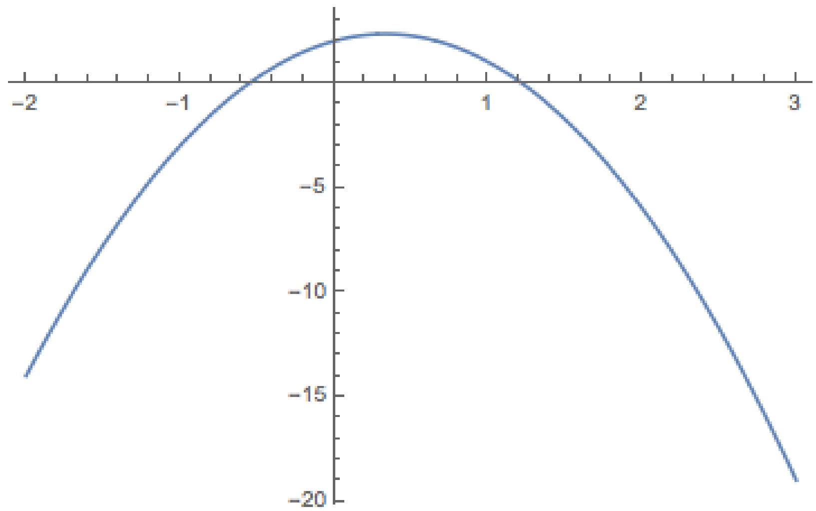
$F(t)$ is increasing where the derivative is positive, and decreasing where the derivative is negative. Therefore $F(t)$ is increasing on the intervals $(-\infty, -1)$ and $(0, 2)$, while it is decreasing on the intervals $(-1, 0)$ and $(2, \infty)$.

The graph of $F(t)$ will have horizontal tangent lines at the places that the derivative has a horizontal intercept, as horizontal lines have slope 0. Therefore the graph of $F(t)$ will have horizontal tangent lines at $t = -1, 0, 2$.

(b) On what intervals is $F'(t)$ increasing and decreasing? Where does the graph of $F'(t)$ have horizontal tangent lines?

$F'(t)$ is increasing on the interval $(-0.5, 1.2)$ and decreasing on the intervals $(-\infty, -0.5)$ and $(1.2, \infty)$. The graph of $F'(t)$ has horizontal tangent lines when $t = -0.5, 1.2$.

(c) Use your answers to part (b) to sketch a graph of the second derivative $F''(t)$. Label the horizontal intercepts. Explain how your graph agrees with your answers in part (b).



A graph of the second derivative of F is the same as the graph of the derivative of F'. So the graph above is the derivative of the original graph. Notice that this graph of the second derivative is positive on the same intervals where $F'(t)$ is increasing, negative on the same intervals where $F'(t)$ is decreasing, and has horizontal intercepts at the same points that $F'(t)$ has horizontal tangent lines.

(d) On what intervals is the original function $F(t)$ concave up? On what intervals is the original function $F(t)$ concave down? Explain your answer.

The function $F(t)$ will be concave up on the intervals where $F''(t)$ is positive, and it will be concave down on the intervals where $F''(t)$ is negative. Equivalently, $F(t)$ will be concave up on the intervals where $F'(t)$ is increasing, and concave down on the intervals where $F'(t)$ is decreasing.

Using either of these methods to determine concavity, we observe that the original function will be concave up on the interval $(-0.5, 1.2)$, and concave down on the intervals $(-\infty, -0.5)$ and $(1.2, \infty)$.

Question 7: Bill gets into his car and begins driving east. Let s represent the distance from Bill to his starting point, in kilometers, and let t denote the time he has been driving, in hours. (We take s to be positive when Bill is east of his starting point and negative when Bill is west of his starting point.)

(a) What does $\frac{ds}{dt}$ represent physically? Include units in your answer.

This represents velocity, with units of kilometers per hour.

(b) What does $\frac{d^2s}{dt^2}$ represent physically? Include units in your answer.

This represents acceleration, with units of kilometers per hour^2.

(c) Assume that $\left.\frac{ds}{dt}\right|_{t=2} = -50$ and $\left.\frac{d^2s}{dt^2}\right|_{t=2} = 500$. In what direction is Bill traveling after 2 hours, east or west? Is his car speeding up or slowing down? Explain your answers.

Bill is traveling west, as the problem states that the positive direction is east, and Bill's velocity is negative. His car is slowing down, because the velocity and acceleration have opposite signs.

(d) The following table gives different values of s and t. Compute the average rate of change in position on the intervals $[4, 4.5]$ and $[4, 4.25]$ (this corresponds to time periods of 30 and 15 minutes, respectively). Using these values, estimate the value of $\left.\frac{ds}{dt}\right|_{t=4}$.

t	4	4.25	4.5
s	100	121	145

Using the formula for average rate of change,

$$\begin{aligned} avgROC \text{ on } [4, 4.5] &= \frac{145 - 100}{4.5 - 4} = \frac{45}{.5} = 90 \\ avgROC \text{ on } [4, 4.25] &= \frac{121 - 100}{4.25 - 4} = \frac{21}{.25} = 84. \end{aligned}$$

From these numbers, we can guess that the average rate of change is approaching 80 as the intervals get smaller and smaller, and we therefore estimate that

$$\left.\frac{ds}{dt}\right|_{t=4} = 80$$

Other answers are possible.

MAT 122 Sample Midterm Solutions

1 Conceptual Questions.

1. Why do linear functions have the same average rate of change on every interval?

   The average rate of change of a function on an interval is essentially the formula for the slope. Lines are defined by the fact that they have a constant slope.

2. If $f(x)$ is an increasing function, explain why $f'(x) \geq 0$.

   The value of $f'(a)$ at any point $x = a$ is the "limit" of values of the average rate of change on small intervals, $\frac{f(b)-f(a)}{b-a}$ where $[a, b]$ is a small interval. If f is increasing, then $f(b) - f(a)$ is positive, so the average rates of change on all of these small intervals will be positive. This means that $f'(x)$ should be positive.

3. Can the graph of a function have more than one vertical intercept? Can the graph of a function have more than one horizontal intercept? Explain your answers.

   A function cannot have more than one vertical intercept, as the vertical intercept is simply the value of the function at 0, and functions have a unique output for a given input. A graph can have many horizontal intercepts, or none, as horizontal intercepts are places where the function's value is 0. Consider the graphs of the functions $x^2, x^2 - 1$, and $x^2 + 1$.

4. Give an example of a function that is both concave up and increasing. Given an example of a function that is concave up and decreasing. Give an example of a function that is concave down and increasing. Give an example of a function that is concave down and decreasing.

   $f(x) = e^x$ is concave up and increasing. $f(x) = e^{-x}$ is concave up and decreasing. $f(x) = -e^{-x}$ is concave down and increasing. $f(x) = -e^x$ is concave down and decreasing.

5. What information do you need to know in order to write the equation of a line? What information do you need to know in order to write the equation of an exponential function?

   The slope and a point on this line, or two points on the line. The initial value, the rate of change, and whether that rate of change is per unit time or continuous.

6. A savings account at Bank A pays 5% interest, compounded continuously. Savings accounts at Bank B pay 5% interest, compounded annually. Which bank account pays more interest?

   The account compounded continually pays more interest.

7. Draw the graph of a function $f(x)$ such that $f(0) = 0$ and $f'(0) = 0$.

   The graph of $f(x) = x^2$ is such a function.

8. A hiker starts climbing a mountain at 8:00 am, reaches the summit at 1:00 pm, and then climbs back down to arrive back at her starting point at 4:00 pm. Let A represent the altitude of the hiker, in feet, and let t represent the time after 8:00 am, in hours.

   (i) We can consider A as a function of t, but cannot consider t as a function of A. Explain why.

(ii) Thinking of $A(t)$ as a function, consider the intervals $[0, 5]$ and $[5, 8]$ for the t-values. Over which of these intervals is the average rate of change the greatest? Over which of these intervals is the *absolute value* of the average rate of change the greatest?

A is a function of t because for any time there is a unique corresponding height. However, to any height we can associate two different times (the time the height is reached while the hiker climbs up and the time when the hiker climbs down). Since a function must have a unique output for every input, t cannot be a function of A. The average rate of change on $[0, 5]$ is greater than the average rate of change on $[5, 8]$, as the first is positive and the second is negative. However, the absolute value of the second is larger, because the climber is climbing down more quickly. She descends the mountain in only 3 hours, while it takes her 5 hours to climb. Note that all average rates of change here have units of feet per hour.

9. Give as many interpretations for the derivative of a function at a point as you can.

The derivative at a point is the instantaneous rate of change of the function at that point. It is the limit of the average rate of change of the function computed on smaller and smaller intervals containing the given point. It is the slope of the line tangent to the graph at the given point.

10. What are the differences between exponential functions and power functions?

In a power function, the variable is being raised to a fixed power. In an exponential function, a fixed number is being raised to the power of the variable. For example, $f(x) = x^3$ is a power function and $f(x) = 3^x$ is an exponential function.

11. Let $f(x) = -e^{-x}$. Are there any values a for which $f''(a) \geq 0$?

We can use graph transformations to draw the graph of this function from the basic graph of e^x. The graph of $f(x)$ is the graph of e^x reflected over both the x- and y-axes. We know that e^x is always concave up, and we can see that these transformations produce a function that is always concave down. If a function is always concave down, then its second derivative is always negative. Therefore there are no such values.

2 Sample Midterm

Question 1: Consider the function $f(x) = x^2 + x - 6$.

(a) Compute the average rate of change of the function f on the intervals $[0, 1]$, $[0, 1/2]$, and $[0, 1/4]$.

Using the formula for the average rate of change

$$\begin{aligned} avgROC \text{ on } [0,1] &= \frac{f(1)-f(0)}{1-0} = -4-(-6) = 2 \\ avgROC \text{ on } [0,1/2] &= \frac{f(1/2)-f(0)}{1/2-0} = \frac{\frac{1}{4}+\frac{1}{2}-6-(-6)}{\frac{1}{2}} = \frac{3}{2} \\ avgROC \text{ on } [0,1/4] &= \frac{f(1/4)-f(0)}{1/4-0} = \frac{\frac{1}{16}+\frac{1}{4}-6-(-6)}{\frac{1}{4}} = \frac{5}{4} \end{aligned}$$

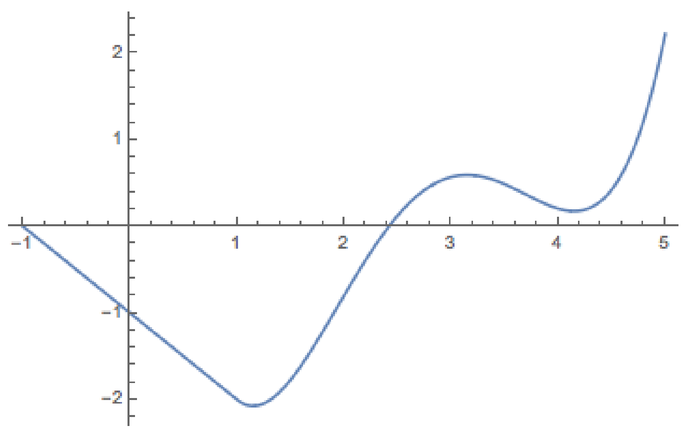
(b) Using your answer to part (a), estimate the value of $f'(0)$.

The values of the average rates of change appear to be approaching 1 as we make the intervals smaller. Therefore we can estimate that $f'(0) = 1$.

(c) Let $F(x)$ be a function such that $F'(x) = f(x)$. Find all numbers a such that the graph of $F(x)$ has a horizontal tangent line at $x = a$.

$F(x)$ will have a horizontal tangent line at points where its derivative has a value of 0. Therefore we are trying to solve the equation $F'(x) = 0$. Since $F'(x) = f(x)$, this is the same as solving the equation $x^2 + x - 6 = 0$. We can solve this using either factoring or the quadratic formula, which gives that the desired values are $a = 2$ or $a = -3$.

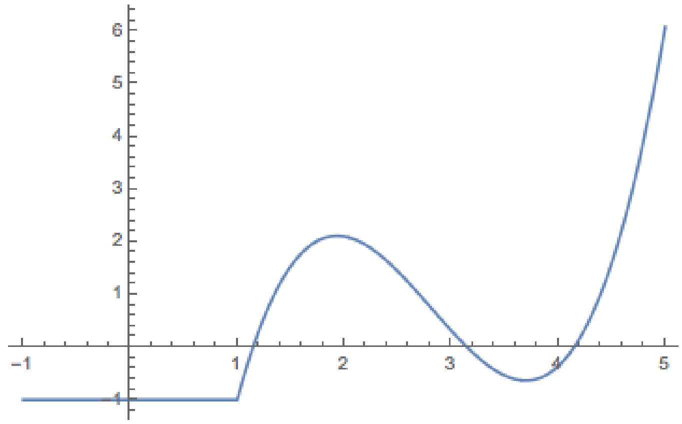
Question 2: Consider the graph of $f(x)$ below. Note that the graph is a straight line on the interval $[-1, 1]$.



(a) On what intervals is the function increasing and decreasing?

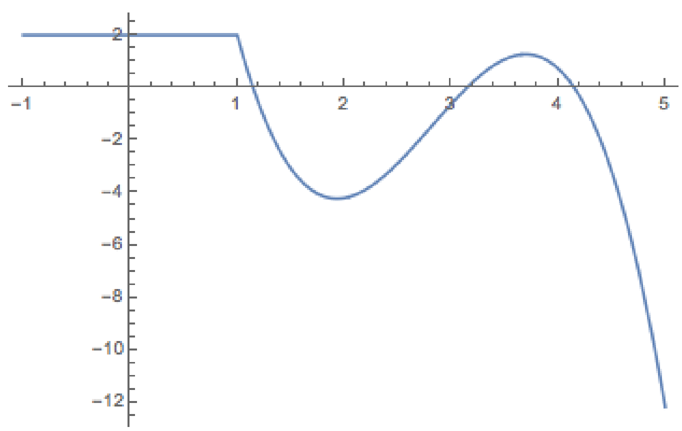
The function is increasing on the intervals $(1.2, 3.2)$ and $(4.2, \infty)$. It is decreasing on the intervals $(-\infty, 1.2)$ and $(3.2, 4.2)$.

(b) Sketch a graph of the derivative $f'(x)$. Label all relevant points, including the vertical and horizontal intercepts. Explain how your graph agrees with your answer in part (a).



Recall that the derivative of a linear function is exactly its slope, so that the derivative function graphed here is constant on the region where the original function is a line, equal to the value of the slope. The vertical intercept is at -1, while horizontal intercepts occur at $x = 1.2, 3.2$, and 4.2, the locations of horizontal tangent lines on the original graph. The derivative function is positive on the same intervals the original function is increasing, and negative on the same intervals the original function is decreasing.

(c) Sketch a graph of $-2f'(x)$. Label the vertical and horizontal intercepts.



This graph is stretched vertically 2 units and reflected over the x-axis. This does not change the horizontal intercepts. The new vertical intercept is 2.

Question 3:

(a) Write an equation for a line that passes through the points $(2, 5)$ and $(6, 13)$.

Such a line has slope

$$m = \frac{13-5}{6-2} = \frac{8}{4} = 2,$$

so using point-slope form for the equation of a line with the point $(2, 5)$ gives the equation

$$y - 5 = 2(x - 2)$$

(b) Assume that $f(x)$ is an exponential function with $f(0) = 75$ that is decaying at a continuous rate of 25%. Write an equation for $f(x)$.

Using the formula for such an exponential function,

$$f(x) = 75e^{-0.25t}$$

(c) Let $f(x)$ be the function you obtained in part (b). Find the value of a such that $f(a) = e^2$. Simplify your answer as much as possible, and use logarithm rules to write it using only one logarithm function.

Solving the equation $75e^{-0.25t} = e^2$ using logarithms gives

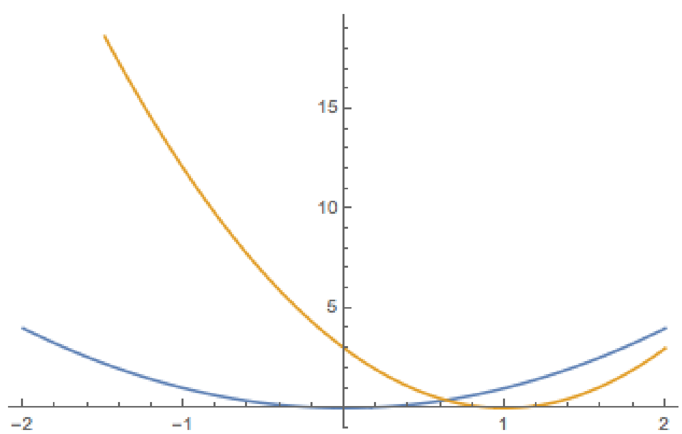
$$\begin{aligned}
75e^{-0.25t} &= e^2 \\
\ln\left(75e^{-0.25t}\right) &= \ln\left(e^2\right) \\
\ln(75) + \ln\left(e^{-0.25t}\right) &= \ln\left(e^2\right) \\
\ln(75) - 0.25t &= 2 \\
t &= -4(2 - \ln(75)) = -8 + 4\ln(75)
\end{aligned}$$

Question 4: Let $f(x) = 3x^2$, $g(x) = x - 1$.

(a) Write a formula for $f(g(x))$.

$$f(g(x)) = 3(x-1)^2$$

(b) Graph the function $f(g(x))$.

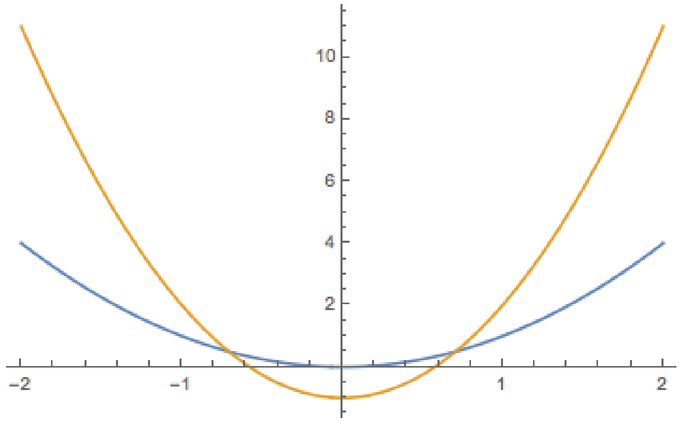


The graph is obtained by taking the known graph of x^2, shifting it right 1 unit, and stretching vertically by 3. On the graph above, the graph of x^2 is shown in blue and the desired graph is shown in gold.

(c) Write a formula for $g(f(x))$.

$$g(f(x)) = 3x^2 - 1$$

(d) Graph the function $g(f(x))$.



The graph is obtained by taking the known graph of x^2, stretching it vertically by 3, then shifting down 1 unit. On the graph above, the graph of x^2 is shown in blue and the desired graph is shown in gold.

Question 5: A deposit of \$10,000 is made into a bank account. After 1 year the account has a balance of \$10,750. Let B denote that balance of the bank account and t denote the number of years since the initial deposit.

(a) What is the relative (percent) change in the balance of the account after 1 year?

$$\frac{10,750 - 10,000}{10,000} = \frac{750}{10000} = .075 = 7.5\%$$

(b) Assuming that the bank continues to pay the same interest rate, compounded annually, write a formula for the balance in the account as a function of time.

Using the formula for exponential growth with initial value 10,000 and rate $r = .075$, the formula is

$$B(t) = 10,000(1.075)^t$$

(c) A table showing the value of the function B for different values of t is shown below. Use this to estimate the value of $B'(3)$. What units does this quantity have?

t	0	1	2	3	4	5
B(t)	10,000	10,750	11,556	12,423	13,355	14,356

$$B'(3) \approx avgROC \text{ on } [3,4] = \frac{13355 - 12,423}{4-3} = 932 \; \frac{\text{dollars}}{\text{year}}$$

Note that other estimates are possible. For example, you could also approximate the value of the derivative using the average rate of change on the interval $[2,3]$. This would give

$$B'(3) \approx avgROC \text{ on } [2,3] = \frac{12,423 - 11556}{3-2} = 867 \; \frac{\text{dollars}}{\text{year}}$$

Question 6: Let $f(x)$ be some function with $f(2) = 3$, $f'(2) = -2$, and $f''(2) = 1$.

(a) Use the local linear approximation at $x = 2$ to estimate the value of $f(2.5)$.

The local linear approximation at $x = 2$ is simply the equation of the tangent line at $x = 2$. This line passes through the point $(2, 3)$ and has slope -2, hence has the equation

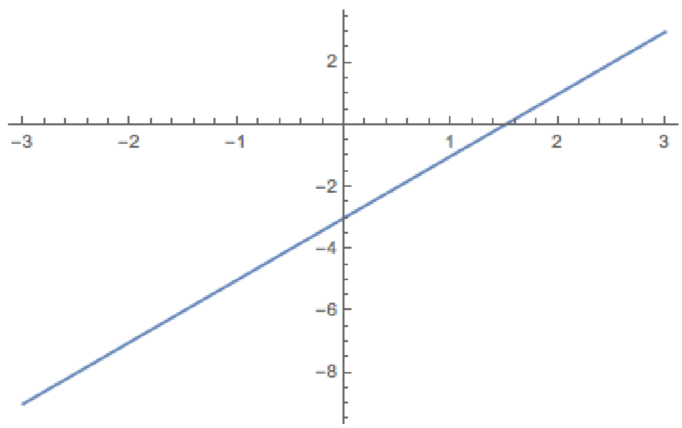
$$L(x) = -2(x - 2) + 3$$

Evaluating $L(x)$ at $x = 2.5$ gives $L(2.5) = 2$. Therefore we can estimate that $f(2.5) \approx 2$.

(b) Assume that $f''(x) > 0$ when $x > 2$. Is your estimate in part (a) larger or smaller than the actual value of $f(2.5)$?

As $f''(2) = 1$, and $f''(x)$ remains positive for $x > 2$, we have that the graph is concave up on $(2, \infty)$. Therefore f' is increasing as we move to the right, and therefore our estimate is an underestimate.

(c) Assume that the graph below is a graph of $f''(x)$. What can you say about the concavity of the original function?



The original function is concave down on the interval $(-\infty, 3/2)$ and concave up on the interval $(3/2, \infty)$.

Question 7: A man stands on the edge of a 100 m building and throws a ball straight upward with an initial velocity of 20 m/s. The ball first flies upward, then falls back down, falling past the top of the building and down to the ground. Let h denote the height of the ball above the ground, in meters, and let t denote the time that has passed since the ball was thrown, in seconds. Ignoring wind resistance, physics tells us that the force of gravity causes the ball to accelerate toward the ground at rate of 9.8 m/sec^2. The ball hits the ground after about 7 seconds.

(a) Interpret $h'(t)$ in terms of the motion of the object. Include units in your answer.

This function gives the velocity of the object at time t, in units of meters/second.

(b) Would you expect that $h'(1)$ is positive or negative? Would you expect that $h'(6)$ is positive or negative? Explain your answer.

$h'(1)$ is likely positive and $h'(6)$ is negative. The ball is still traveling upward after 1 second, hence has positive velocity, while it is traveling toward the ground after 6 seconds, hence has negative velocity.

(c) What is the value of $\left.\frac{d^2h}{dt^2}\right|_{t=1}$? What is the value of $\left.\frac{d^2h}{dt^2}\right|_{t=6}$? Include units in your answer.

Both values are -9.8 meters/second^2. The second derivative represents acceleration, and we are told that the acceleration has a constant value of 9.8. The sign is negative because the acceleration is toward the ground.

(d) The the ball speeding up or slowing down at time $t = 1$? Is the ball speeding up or slowing down at time $t = 6$? Explain your answer in terms of the signs of the first and second derivatives of the function h.

The ball is slowing down at time $t = 1$, because the velocity and the acceleration have opposite signs. The ball is speeding up at time $t = 6$, because the velocity and the acceleration ahve the same sign.

MAT 122 Final Exam

Name: ____

- Please read all instructions before beginning, and do not open the exam until you are told to do so.
- Put away all notes, books, calculators, etc. before beginning the exam, and place them under your desk. The only items on your desk should be this exam, pencils/pens, and an eraser.
- This exam has 12 questions, each with multiple parts. The point value for each question is shown below. A score of 100 on the first part (80%) will guarantee a passing overall grade in the course, regardless of previous performance in the course.
- The final page of the exam is left blank for scratch paper. You may tear this page from the exam booklet, but do not remove any other pages.
- You have 3 hours and 25 minutes to complete the exam.

Question	**Points**	**Score**
1	20	
2	20	
3	25	
4	15	
5	15	
6	15	
7	15	
Part I Total	125	
8	15	
9	15	
10	15	
11	15	
12	15	
Total	200	

Part I

Question 1: Find the derivative of the following functions.

(a) $f(x) = 5x^3 + 2x^2 + x + 1 + \frac{1}{x}$

(b) $g(x) = \ln(3x^2 + x)$

(c) $h(t) = (1 + \sqrt{t})e^t$

(d) $m(z) = \dfrac{3z+1}{3z-1}$

Question 2: Compute the following integrals. Notice that some are indefinite and some are definite.

(a) $\displaystyle\int_0^2 (5t^4 + t^3 - 2)\ dt$

(b) $\displaystyle\int (e^z + 3z + \frac{1}{z})\ dz$

(c) $\int_0^1 x(2x^2+1)^2 \, dx$

(d) $\int (2x^2+1)^2 \, dx$

Question 3: Consider the function $f(x) = 3x^4 + 8x^3 + 6x^2 + 1$.

(a) Find all critical points of $f(x)$.

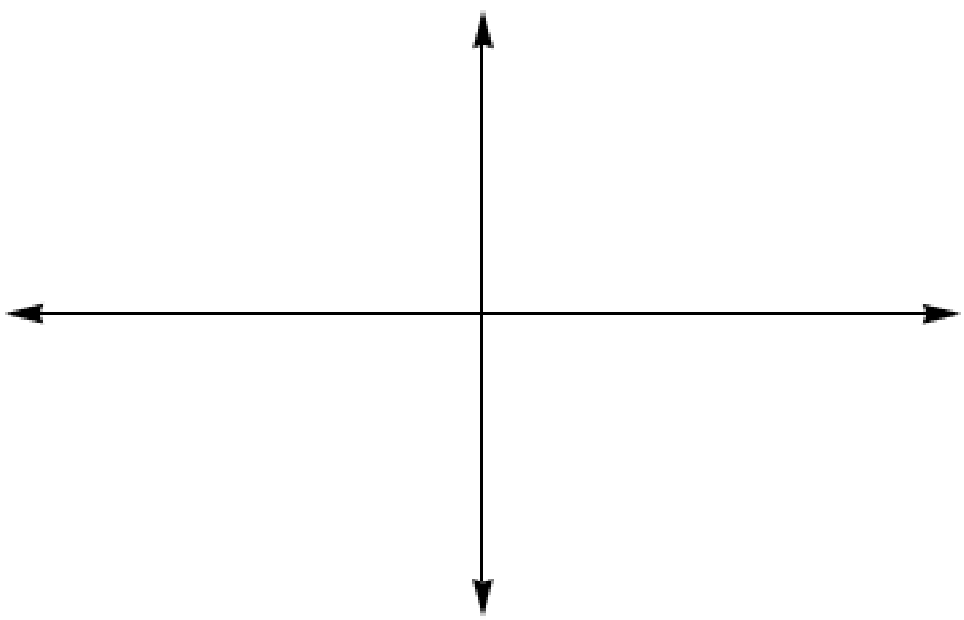
(b) Find the intervals on which $f(x)$ is increasing and decreasing.

(c) Find all local extrema of $f(x)$.

(d) Find the intervals on which $f(x)$ is concave up and concave down.

(e) Find all inflection points of $f(x)$.

(f) Draw a graph of $f(x)$. Label the vertical intercept, critical points, local extrema, and inflection points on your graph.



Question 4: Jason leaves his house to go for a bike ride. He travels either due north or due south, and we take north to be the positive direction. His velocity, in miles per hour, after t hours of riding is given by

$$v(t) = 12t^3 - 27t^2 - \frac{13}{2}t + 15$$

How far from home is he after 2 hours of riding? Is he north or south of home?

Question 5: Estimate the value of

$$\int_{-3}^{5} (2x^2 - 1)\, dx$$

by computing the average of the left and right Riemann sums with $n = 4$

Question 6: Find the global maximum and global minimum of the function

$$g(x) = -x^4 + \frac{16}{3}x^3 - 6x^2 + 1$$

on the interval $\left[-1, \frac{3}{2}\right]$.

Question 7: Use local linear approximation to estimate the value of $\sqrt{66}$. Explain whether your answer is an overestimate or an underestimate.

Part II

Question 8:

(a) Let $k(x) = f(g(h(x)))$. The tables below gives values of f, g, h, and their derivatives at certain x-values. Use these tables to compute $k'(2)$.

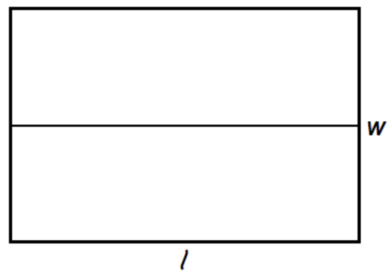
x	-2	-1	0	1	2
$f(x)$	2	0	-2	5	4
$g(x)$	-2	2	-1	7	6
$h(x)$	1	3	-1	2	0

x	-2	-1	0	1	2
$f'(x)$	3	-2	1	1	2
$g'(x)$	-2	3	4	1	0
$h'(x)$	4	-3	0	2	-1

(b) Compute the following definite integral

$$\int_4^9 \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$$

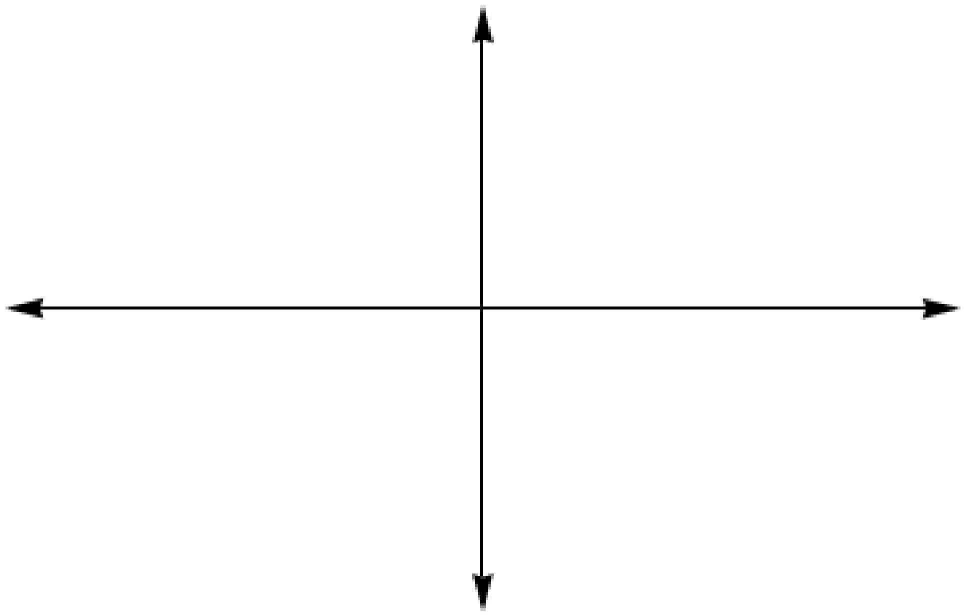
Question 9: A petting zoo has both goats and sheep. The petting zoo is building a new pen to hold both animals, but must build two pens in order to keep them goats and sheep separate. They decide to make two pens by building a large rectangular pen, then dividing the pen in half with a fence down the middle, as illustrated below.



The petting zoo only has 240 ft of fencing. What is the largest area that they can enclose?

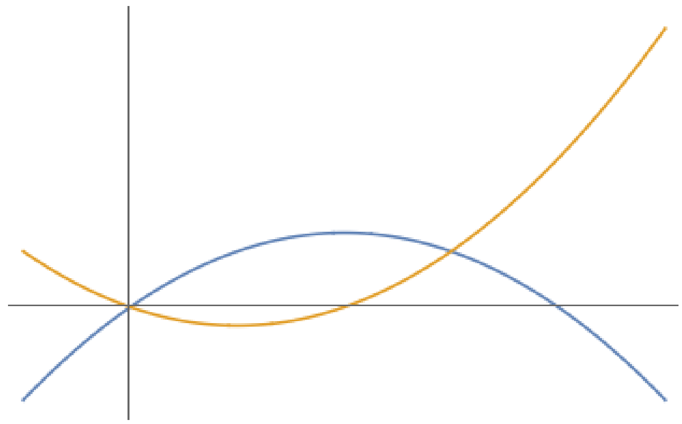
<u>Question 10:</u> Consider the function $f(x) = 2(x-1)^2 - 2$.

(a) Plot the function $f(x)$ on the graph below.



(b) Find the area between the graph of $f(x)$ and the x-axis on the interval $[0, 3]$.

Question 11: Consider the functions $f(x) = x^2 - 2x$ and $g(x) = -x^2 + 4x$. A plot of these functions is shown below - note that the scale is not shown on the plot, and the functions are unlabeled.



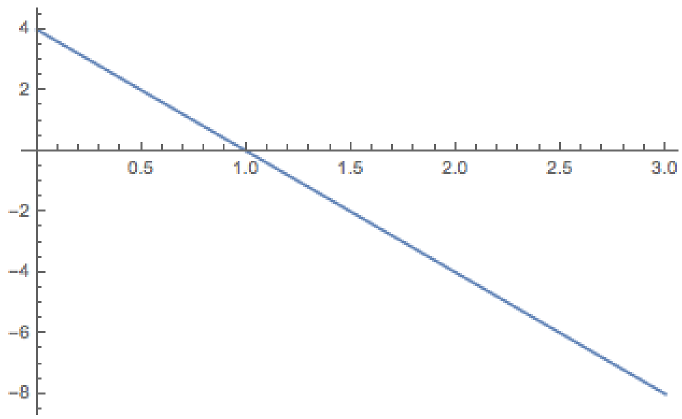
(a) Find the x-coordinates of the intersection points of the graphs.

(b) Find the area enclosed by the two curves.

Question 12: We have learned one part of the fundamental theorem of calculus, which gives a method for evaluating definite integrals. The second part of the fundamental theorem of calculus says:

Theorem (Fundamental Theorem of Calculus, Part 2)**.** *If* $F(x) = \int_0^x f(t)\,dt$*, then the derivative of* $F(x)$ *is* $F'(x) = f(x)$*.*

Consider the function $F(x) = \int_0^x g(t)\,dt$, where the graph of $g(t)$ is shown below.



(a) Compute $F(3)$ and $F'(3)$.

(b) On what intervals if F increasing and decreasing? On what intervals is F concave up and concave down?

MAT 122 Final Exam

Name: KEY

- Please read all instructions before beginning, and do not open the exam until you are told to do so.
- Put away all notes, books, calculators, etc. before beginning the exam, and place them under your desk. The only items on your desk should be this exam, pencils/pens, and an eraser.
- This exam has 12 questions, each with multiple parts. The point value for each question is shown below. A score of 100 on the first part (80%) will guarantee a passing overall grade in the course, regardless of previous performance in the course.
- The final page of the exam is left blank for scratch paper. You may tear this page from the exam booklet, but do not remove any other pages.
- You have 3 hours and 25 minutes to complete the exam.

Question	**Points**	**Score**
1	20	
2	20	
3	25	
4	15	
5	15	
6	15	
7	15	
Part I Total	125	
8	15	
9	15	
10	15	
11	15	
12	15	
Total	200	

Part I

Question 1: Find the derivative of the following functions.

(a) $f(x) = 5x^3 + 2x^2 + x + 1 + \frac{1}{x}$

Recall that $\frac{1}{x} = x^{-1}$.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left(5x^3 + 2x^2 + x + 1 + \frac{1}{x}\right) \\
&= \frac{d}{dx}(5x^3) + \frac{d}{dx}(2x^2) + \frac{d}{dx}(x) + \frac{d}{dx}(x^{-1}) \\
&= 5 \cdot \frac{d}{dx}(x^3) + 2 \cdot \frac{d}{dx}(x^2) + \frac{d}{dx}(x) + \frac{d}{dx}(x^{-1}) \\
&= 5 \cdot 3x^2 + 2 \cdot 2x + 1 + (-1)x^{-2} \\
&= 15x^2 + 4x + 1 - x^{-2}
\end{aligned}$$

(b) $g(x) = \ln(3x^2 + x)$

We use the chain rule. The inside function is $z = 3x^2 + x$, which has derivative $6x + 1$. The outside function is $\ln(z)$, which has derivative $\frac{1}{z}$. We therefore have

$$g'(x) = \frac{1}{3x^2 + x} \cdot (6x + 1)$$

(c) $h(t) = (1 + \sqrt{t})e^t$ Using the product rule,

$$\begin{aligned}
h'(t) &= \frac{d}{dt}(1 + \sqrt{t})e^t + (1 + \sqrt{t})\frac{d}{dt}(e^t) \\
&= \frac{1}{2}t^{-1/2}e^t + (1 + \sqrt{t})e^t
\end{aligned}$$

(d) $m(z) = \dfrac{3z + 1}{3z - 1}$ Using the quotient rule,

$$\begin{aligned}
m'(z) &= \frac{(3z - 1)\frac{d}{dz}(3z + 1) - (3z + 1)\frac{d}{dz}(3z - 1)}{(3z - 1)^2} \\
&= \frac{3(3z - 1) - 3(3z + 1)}{(3z - 1)^2} \\
&= \frac{-6}{(3z - 1)^2}
\end{aligned}$$

Question 2: Compute the following integrals. Notice that some are indefinite and some are definite.

(a) $\int_0^2 (5t^4 + t^3 - 2)\, dt$

We use the fundamental theorem of calculus. First, we find the antiderivative.

$$\begin{aligned}
\int (5t^4 + t^3 - 2)\, dt &= \int 5t^4\, dt + \int t^3\, dt - \int 2\, dt \\
&= 5\int t^4\, dt + \int t^3\, dt - 2\int dt \\
&= 5 \cdot \frac{1}{5}t^5 + \frac{1}{4}t^4 - 2t + C \\
&= t^5 + \frac{1}{4}t^4 - 2t + C
\end{aligned}$$

Then the fundamental theorem of calculus gives

$$\begin{aligned}
\int_0^2 (5t^4 + t^3 - 2)\, dt &= \left[t^5 + \frac{1}{4}t^4 - 2t\right]_0^2 \\
&= \left(2^5 + \frac{1}{4}2^4 - 2 \cdot 2\right) - \left(0^5 + \frac{1}{4}0^4 - 2 \cdot 0\right) \\
&= 32 + 4 - 4 = 32
\end{aligned}$$

(b) $\int (e^z + 3z + \frac{1}{z})\, dz$

$$\begin{aligned}
\int (e^z + 3z + \frac{1}{z})\, dz &= \int e^z\, dx + 3\int z\, dz + \int \frac{1}{z}\, dz \\
&= e^z + \frac{3}{2}z^2 + \ln|z| + C
\end{aligned}$$

(c) $\int_0^1 x(2x^2+1)^2\, dx$

We will use a substitution. We choose the inside function to be $u = 2x^2 + 1$. Then $du = 4x\, dx$. We can rewrite this as $\frac{1}{4}\, du = x\, dx$. We also change the limits. When $x = 0$, we have that $u = 2 \cdot 0^2 + 1 = 1$, while if $x = 1$ we have $u = 2 \cdot 1^2 + 1 = 3$. So our integral is

$$\begin{aligned}
\int_0^1 x(2x^2+1)^2\, dx &= \int_0^1 (2x^2+1)^2 \cdot x\, dx \\
&= \int_1^3 u^2 \frac{1}{4}\, du \\
&= \frac{1}{4}\int_1^3 u^2\, du \\
&= \frac{1}{4}\left[\frac{1}{3}u^3\right]_1^3 \\
&= \frac{1}{4}\left(\frac{1}{3}3^3 - \frac{1}{3}1^3\right) \\
&= \frac{1}{4}\left(9 - \frac{1}{3}\right) = \frac{1}{4} \cdot \frac{26}{3} = \frac{13}{6}
\end{aligned}$$

(d) $\displaystyle\int (2x^2+1)^2\,dx$

We cannot make a substitution as in the previous problem. Instead, we have to multiply out the integrand. We have

$$\begin{aligned}
\int (2x^2+1)^2\,dx &= \int \left(4x^4+4x^2+1\right)\,dx \\
&= \int 4x^4\,dx + \int 4x^2\,dx + \int 1\,dx \\
&= 4\int x^4\,dx + 4\int x^2\,dx + \int 1\,dx \\
&= 4\cdot\frac{1}{5}x^5 + 4\cdot\frac{1}{3}x^3 + x + C \\
&= \frac{4}{5}x^5 + \frac{4}{3}x^3 + x + C
\end{aligned}$$

Question 3: Consider the function $f(x) = 3x^4 + 8x^3 + 6x^2 + 1$.

(a) Find all critical points of $f(x)$.

Critical points are where $f'(x) = 0$. The derivative is

$$\begin{aligned}
f'(x) &= 12x^3 + 24x^2 + 12x \\
&= 12x(x^2+2x+1) \\
&= 12x(x+1)^2
\end{aligned}$$

So the critical points are $x = 0$ and $x = -1$.

(b) Find the intervals on which $f(x)$ is increasing and decreasing.

We divide a number line up into three intervals using the critical points These intervals are $(-\infty,-1),(-1,0)$, and $(0,\infty)$. We then check the sign of the first derivative on each of these integrals. We can observe that the sign will be negative for the first interval, negative for the second interval, and positive for the final interval.

Thus, $f(x)$ is increasing on the interval $(0,\infty)$, while $f(x)$ is decreasing on the intervals $(-\infty,-1)$ and $(-1,0)$.

(c) Find all local extrema of $f(x)$.

The first derivative test and the information in the previous part gives that there is a local minimum at $x = 0$, as this is a place where the first derivative switches sign from negative to positive. The first derivative test also gives that the point $x = -1$ is neither a local max nor a local min, since the first derivative does not change sign at that point.

(d) Find the intervals on which $f(x)$ is concave up and concave down.

This comes from finding the sign of the second derivative. The second derivative is

$$\begin{aligned}
f''(x) &= 36x^2 + 48x + 12 \\
&= 12(3x^2+4x+1) \\
&= 12(3x+1)(x+1)
\end{aligned}$$

Notice that the second derivative is therefore equal to zero at $x = -1/3$ and $x = -1$. So we divide up the number line into the intervals $(-\infty, -1,), (-1, -1/3)$, and $(-1/3, \infty)$, and test the sign of the second derivative on each interval. We can observe that on these intervals, the second derivative is positive, negative, and positive, respectively.

Thus, the original function is concave up on the intervals $(-\infty, -1)$ and $(-1/3, \infty)$, while it is concave down on the interval $(-1, -1/3)$.

(e) Find all inflection points of $f(x)$.

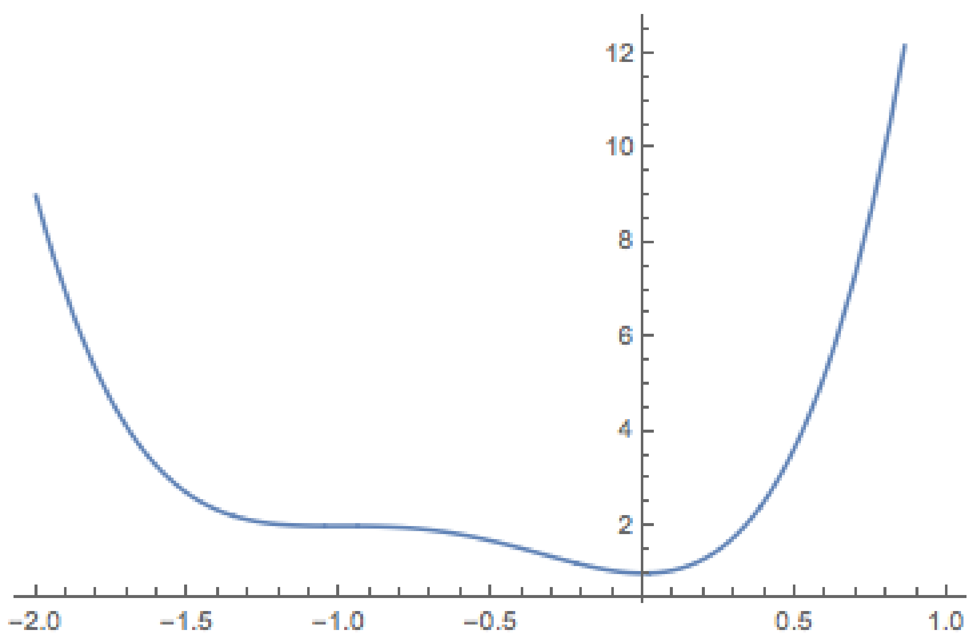
Since inflection points occur where the function switches concavity, that is, where the second derivative switches sign, we have from our answer in the previous section that there are inflection points at $x = -1$ and $x = -1/3$.

(f) Draw a graph of $f(x)$. Label the vertical intercept, critical points, local extrema, and inflection points on your graph.

We should plot the following points on our graph-

$$f(0) = 1 \qquad f(-1) = 2 \qquad f(1/3) = 38/27$$

as there are the vertical intercept, local min, and 2 inflection points. The graph is then



Question 4: Jason leaves his house to go for a bike ride. He travels either due north or due south, and we take north to be the positive direction. His velocity, in miles per hour, after t hours of riding is given by

$$v(t) = 12t^3 - 27t^2 - \frac{13}{2}t + 15$$

How far from home is he after 2 hours of riding? Is he north or south of home?

The change in position is given by integrating velocity. We have

$$\begin{aligned}
\int_0^2 v(t)\,dt &= \int_0^2 \left(12t^3 - 27t^2 - \frac{13}{2}t + 15\right)\,dt \\
&= \left[12 \cdot \frac{1}{4}t^4 - 27 \cdot \frac{1}{3}t^3 - \frac{13}{2} \cdot \frac{1}{2}t^2 + 15t\right]_0^2 \\
&= 3 \cdot 2^4 - 9 \cdot 2^3 - \frac{13}{4} \cdot 2^2 + 15 \cdot 2 \\
&= 48 - 72 - 13 + 30 \\
&= -7
\end{aligned}$$

so the total change in position over the time interval $[0, 2]$ is -7 miles. Using the sign convention given in the problem, this means that Jason is 7 miles south of his house.

Question 5: Estimate the value of

$$\int_{-3}^{5} (2x^2 - 1)\,dx$$

by computing the average of the left and right Riemann sums with $n = 4$

We first divide the interval $[-3, 5]$ into $n = 4$ subintervals. Each subinterval will have a width of

$$\Delta x = \frac{5 - (-3)}{4} = \frac{8}{4} = 2$$

Therefore the subintervals being considered are $[-3, -1], [-1, 1], [1, 3]$, and $[3, 5]$. Thus the left Riemann sum is

$$\begin{aligned}
L_4 &= f(-3) \cdot \Delta x + f(-1) \cdot \Delta x + f(1) \cdot \Delta x + f(3) \cdot \Delta x \\
&= 17 \cdot 2 + 1 \cdot 2 + 1 \cdot 2 + 17 \cdot 2 \\
&= 72
\end{aligned}$$

while the right Riemann sum is

$$\begin{aligned}
R_4 &= f(-1) \cdot \Delta x + f(1) \cdot \Delta x + f(3) \cdot \Delta x + f(5) \cdot \Delta x \\
&= 1 \cdot 2 + 1 \cdot 2 + 17 \cdot 2 + 49 \cdot 2 \\
&= 136
\end{aligned}$$

We therefore estimate

$$\int_{-3}^{5} (2x^2 - 1)\,dx \approx \frac{72 + 96}{2} = 104$$

(Note that we can compute the actual value using the fundamental theorem of calculus. The actual value is $93\frac{1}{3}$.)

Question 6: Find the global maximum and global minimum of the function

$$g(x) = -x^4 + \frac{16}{3}x^3 - 6x^2 + 1$$

on the interval $\left[-1, \frac{3}{2}\right]$.

We begin by finding the critical points. The derivative is

$$\begin{aligned} g'(x) &= -4x^3 + 16x^2 - 12x \\ &= -4x(x^2 - 4x + 3) \\ &= -4x(x-3)(x-1) \end{aligned}$$

so the critical points are at $x = 0, x = 1$, and $x = 3$. Because $x = 3$ is outside the interval we are considering, we throw it out.

We then compute the value of the function at the critical points $x = 0$ and $x = 1$, along with the endpoints $x = -1$ and $x = \frac{3}{2}$. This gives

$$g(-1) = \frac{-34}{3} \qquad g(0) = 1 \qquad g(1) = \frac{-2}{3} \qquad g(3/2) = \frac{7}{16}$$

Thus the global maximum of this function is 1, occurring at $x = 0$, and the global minimum is $\frac{-34}{3}$, occurring at $x = -1$.

Question 7: Use local linear approximation to estimate the value of $\sqrt{66}$. Explain whether your answer is an overestimate or an underestimate.

We consider the function $f(x) = \sqrt{x} = x^{1/2}$. We are trying to compute $f(66)$. We will use the local linear approximation at $x = 64$. Notice that we can easily compute the value of our function at $x = 64$, as $f(64) = 8$.

The local linear approximation at $x = 64$ is the equation of the tangent line at $x = 64$. We have already seen that this tangent line passes through the point $(64, 8)$. The slope of this tangent line is $f'(64)$.

We begin by finding the derivative, which is

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

using the power rule. Therefore the slope is

$$f'(64) = \frac{1}{2\sqrt{64}} = \frac{1}{16}$$

The equation of a line through the point $(64, 8)$ with slope $\frac{1}{16}$ is

$$L(x) = \frac{1}{16}(x - 64) + 8,$$

using the point-slope form of a line.

Therefore we have that

$$\sqrt{66} = f(66) \approx L(66) = \frac{1}{16}(66 - 64) + 8 = \frac{1}{8} + 8 = 8.125$$

This estimate will be an overestimate, because the funciton $f(x) = \sqrt{x}$ is increasing and concave down at $x = 64$. (The actual value is $\sqrt{66} = 8.12404\ldots$).

Part II

Question 8:

(a) Let $k(x) = f(g(h(x)))$. The tables below give values of f, g, h, and their derivatives at certain x-values. Use these tables to compute $k'(2)$.

x	-2	-1	0	1	2
$f(x)$	2	0	-2	5	4
$g(x)$	-2	2	-1	7	6
$h(x)$	1	3	-1	2	0

x	-2	-1	0	1	2
$f'(x)$	3	-2	1	1	2
$g'(x)$	-2	3	4	1	0
$h'(x)$	4	-3	0	2	-1

We use the chain rule repeatedly. First, we apply the chain rule with $f(z)$ as the outside function and $z = g(h(x))$ as the inside function. This gives

$$k'(x) = f'(g(h(x))) \cdot \frac{d}{dx}\left(g(h(x))\right)$$

To compute the remaining derivative, we again use the chain rule, this time with the inside function $z = h(x)$ and outside function $g(z)$. This gives

$$k'(x) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$$

We then evaluate, using the table.

$$\begin{aligned} k'(2) &= f'(g(h(2))) \cdot g'(h(2)) \cdot h'(2) \\ &= f'(g(0)) \cdot g'(0) \cdot (-1) \\ &= f'(-1) \cdot 4 \cdot (-1) \\ &= (-2) \cdot 4 \cdot (-1) = 8 \end{aligned}$$

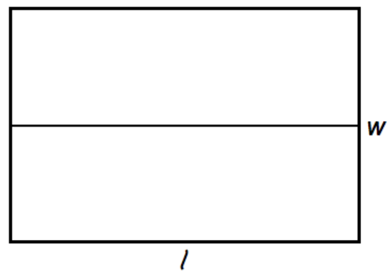
(b) Compute the following definite integral

$$\int_4^9 \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$$

We make the substitution $u = \sqrt{x}$, so that $du = \frac{1}{2}x^{-1/2}\, dx$. Rearranging slightly, this gives $2\, du = \frac{1}{\sqrt{x}}\, dx$. With this substitution, the limits of integration change from 4 and 9 to 2 and 3, respectively. This gives

$$\begin{aligned} \int_4^9 \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx &= \int_4^9 e^{\sqrt{x}} \frac{1}{\sqrt{x}}\, dx \\ &= \int_2^3 e^u \cdot 2\, du \\ &= 2\int_2^3 e^u\, du \\ &= 2\left[e^u\right]_2^3 \\ &= 2\left(e^3 - e^2\right) \end{aligned}$$

Question 9: A petting zoo has both goats and sheep. The petting zoo is building a new pen to hold both animals, but must build two pens in order to keep them goats and sheep separate. They decide to make two pens by building a large rectangular pen, then dividing the pen in half with a fence down the middle, as illustrated below.



The petting zoo only has 240 ft of fencing. What is the largest area that they can enclose?

We are trying to maximize the area, so we begin by writing a function for the area. The area of the pen is $A = \ell w$. This is not a function, however, so we use extra information from the problem to eliminate one of the variables.

We observe that building the pen takes a total of $3\ell + 2w$ feet of fencing. Since the zoo only has 240 feet of fence, this means

$$3\ell + 2w = 240 \Leftrightarrow \ell = \frac{240 - 2w}{3} = 80 - \frac{2}{3}w$$

Thus we can write the area as a function of w as

$$\begin{aligned} A(w) &= \left(80 - \frac{2}{3}w\right) w \\ &= 80w - \frac{2}{3}w^2 \end{aligned}$$

There are also some physical constraints. Clearly $w \geq 0$, as w represents a length. We also must have that $w \leq 120$. The zoo will use at least $2w$ feet of fencing, which must be less than the total of 240 ft that they have, meaning $2w \leq 240$, or $w \leq 120$.

We are therefore trying to find the global maximum of the function $A(w) = 80w - \frac{2}{3}w^2$ on the interval $[0, 120]$. We begin by locating critical points. The derivative is

$$A'(w) = 80 - \frac{4}{3}w$$

which is equal to 0 when $w = \frac{3}{4} \cdot 80 = 60$.

Evaluating the function A at the critical point $w = 60$ and at the endpoints $w = 0, w = 120$ gives

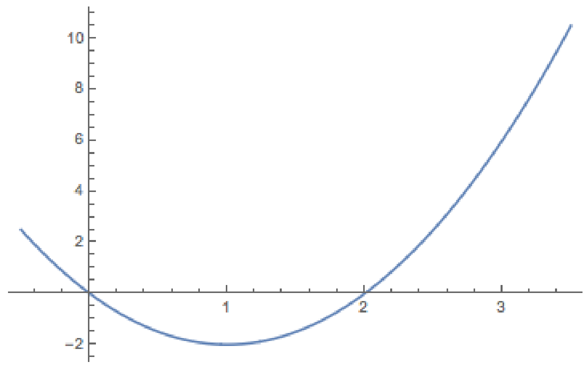
$$A(0) = 0 \qquad A(120) = 0 \qquad A(60) = 2400$$

Therefore the maximum possible area is 2400 ft^2.

Question 10: Consider the function $f(x) = 2(x-1)^2 - 2$.

(a) Plot the function $f(x)$ on the graph below.

Using graph transformations and the known graph of x^2, the graph is



(b) Find the area between the graph of $f(x)$ and the x-axis on the interval $[0, 3]$.

Observe that the graph is negative on the interval $[0, 2]$ and positive on the interval $[2, 3]$. The area below the x-axis is given by

$$\begin{aligned}
\int_0^2 \left|2(x-1)^2 - 2\right| \, dx &= \int_0^2 \left(-2(x-1)^2 + 2\right) \, dx \\
&= \int_0^2 \left(-2(x-1)^2 + 2\right) \, dx \\
&= \int_0^2 \left(-2x^2 + 4x - 2 + 2\right) \, dx \\
&= \int_0^2 \left(-2x^2 + 4x\right) \, dx \\
&= \left[\frac{-2}{3}x^3 + 2x^2\right]_0^2 \\
&= \frac{-2}{3} \cdot 2^3 + 2 \cdot 2^2 \\
&= \frac{-16}{3} + 8 = \frac{8}{3}
\end{aligned}$$

The area above the x-axis is given by

$$\begin{aligned}
\int_2^3 \left|2(x-1)^2 - 2\right|\ dx &= \int_2^3 \left(2(x-1)^2 - 2\right)\ dx \\
&= \int_2^3 \left(2x^2 - 4x\right)\ dx \\
&= \left[\frac{2}{3}x^3 - 2x^2\right]_2^3 \\
&= \left(\frac{2}{3}3^3 - 2\cdot 3^2\right) - \left(\frac{2}{3}\cdot 2^3 - 2\cdot 2^2\right) \\
&= (18-18) + \frac{8}{3} \\
&= \frac{8}{3}
\end{aligned}$$

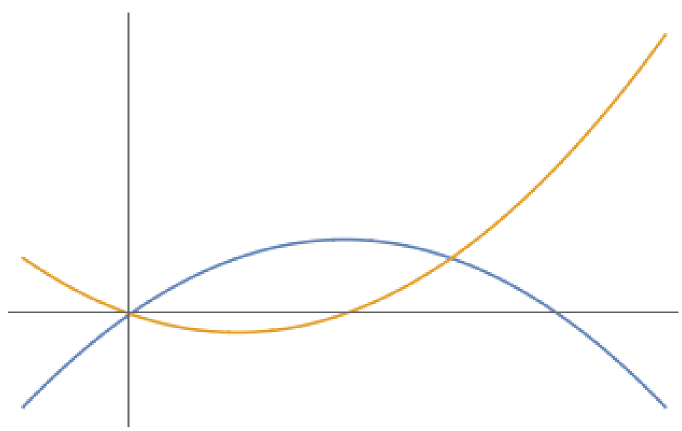
Therefore the total area is

$$Area = \frac{8}{3} + \frac{8}{3} = \frac{16}{3}$$

Note that this is different from

$$\int_0^3 2(x-1)^2 - 2\ dx = 0$$

Question 11: Consider the functions $f(x) = x^2 - 2x$ and $g(x) = -x^2 + 4x$. A plot of these functions is shown below - note that the scale is not shown on the plot, and the functions are unlabeled.



(a) Find the x-coordinates of the intersection points of the graphs.

To find the intersection points, we set the functions equal to one another. We have

$$\begin{aligned}
x^2 - 2x &= -x^2 + 4x \\
2x^2 - 6x &= 0 \\
2x(x-3) &= 0
\end{aligned}$$

so the curves intersect at $x = 0$ and $x = 3$. (We can observe the intersection at $x = 0$ on the provided graph.)

(b) Find the area enclosed by the two curves.

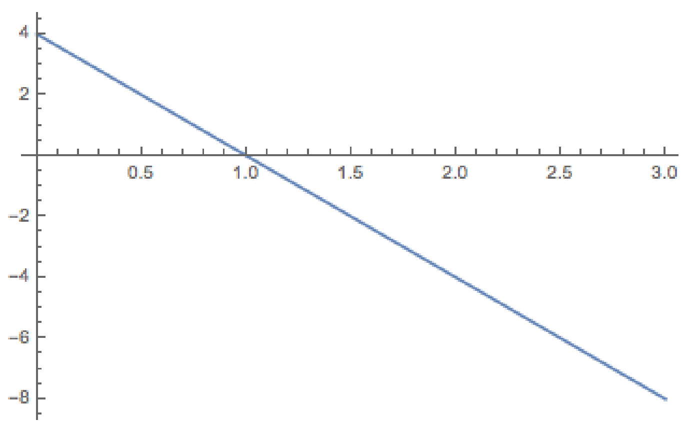
On the interval $[0, 3]$, we can observe that $g(x)$ is the larger function. (For example, $f(1) = -1$ and $g(1) = 3$). Therefore, using the formula for the area between two curves, we have

$$\begin{aligned} Area &= \int_0^3 \left[(-x^2+4x)-(x^2-2x)\right]\ dx \\ &= \int_0^3 \left(-2x^2+6x\right)\ dx \\ &= \left[\frac{-2}{3}x^3+3x^2\right]_0^3 \\ &= \frac{-2}{3}\cdot 3^3 + 3\cdot 3^2 \\ &= -18+27 = 9 \end{aligned}$$

Question 12: We have learned one part of the fundamental theorem of calculus, which gives a method for evaluating definite integrals. The second part of the fundamental theorem of calculus says:

Theorem (Fundamental Theorem of Calculus, Part 2)**.** *If $F(x) = \int_0^x f(t)\ dt$, then the derivative of $F(x)$ is $F'(x) = f(x)$.*

Consider the function $F(x) = \int_0^x g(t)\ dt$, where the graph of $g(t)$ is shown below.



(a) Compute $F(3)$ and $F'(3)$.

To compute $F(3) = \int_0^3 g(t)\ dt$ we can either compute using area, or find an equation for $g(t)$ and evaluate the integral using the (first part) of the fundamental theorem of calculus. Using area, we notice that on $[0, 3]$ the graph of $g(t)$ is divided into two parts, a triangle above the x-axis, and one below. The first has an area of 2, while the second has an area of 8. Therefore we have $F(3) = 2 - 8 = -6$.

To compute $F'(3)$ we use the second part of the fundamental theorem of calculus, explained above, which gives

$$F'(3) = g(3) = -8$$

(b) On what intervals if F increasing and decreasing? On what intervals is F concave up and concave down?

From the second part of the fundamental theorem of calculus, $F'(x) = g(x)$, so F is increasing on the same interval that g is positive, and decreasing where g is negative. Thus F is increasing on the interval $(-\infty, 1)$ and decreasing on the interval $(1, \infty)$.

To find concavity we compute the second derivative, which gives

$$F''(x) = \frac{d}{dx}F'(x) = \frac{d}{dx}g(x) = g'(x)$$

We observe that g is a linear function with slope -4, so that $g'(x) = -4$. Thus $F''(x) = -4$, so F is always concave down, that is, is concave down on the interval $(-\infty, \infty)$.

MAT 122 Sample Final

1 Conceptual Questions

This section contains some general questions to help you test your understanding of the key concepts we've learned in the course. These questions are not necessarily representative of the kind of questions that will appear on the final - the true sample final is in the next section. You might not be able to answer the last three questions until after class on Monday, 8/11.

1. What is the definition of the derivative of a function at a point?
2. What is the definition of $\int_a^b v(t)dt$?
3. What is the Fundamental Theorem of Calculus?
4. Let h be a function of p. Write the Newton notation for the second derivative of h. Write the Leibniz notation for the second derivative of h.
5. Give as many interpretations for the derivative as you can.
6. Give as many interpretations for the definite integral as you can.
7. What are the differences between $\int_a^b f(x)dx$ and $\int f(x)dx$? What does each expression represent?
8. What are the differences between local extrema and global extrema?
9. For each condition below, either give an example of a function with domain all real numbers satisfying that condition, or explain why such a function cannot exist.
   (a) The function has no local max nor local min.
   (b) The function has a local max but no global max.
   (c) The function has a global max but no local max.
   (d) The function has two local maxima, 1 global minimum, and no global maximum.
10. What condition guarantees the existence of a global maximum and a global minimum for a function? What is the procedure for finding these global extrema in that case?
11. What are the first and second derivative tests? What are they used for? Explain why they work.
12. Assume that $0 \leq f(x) \leq g(x)$ for any value of x. What can you say about the relationship between the values $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$?
13. If $f'(x)$ is always positive, what can you say about the sign of $f(x)$? If $f(x)$ is always negative, what can you say about $\int_a^b f(x)$?
14. Give an example of a function $f(x)$ such that
    (a) $f(-1) = 1$ and $f(1) = -1$
    (b) $\int_{-1}^1 f(x)dx = 0$
    (c) $f''(x)$ is not always 0

15. Draw a function for which the quantity $\int_0^1 f(x)dx$ is positive but the quantity $\int_0^2 f(x)dx$ is negative.

2 Sample Final

As discussed in class, the final exam will be divided into two sections. The first section will be comprehensive, and meant to test you on the basics of all of the material covered in the course. A score of 80% or better will be enough to guarantee a passing overall grade in this course, regardless of previous performance in the course. The second section of the exam will include more difficult questions, with a focus on the material learned since the midterm exam.

You may not be able to answer questions 2a, 7 and 10 until after class on Monday.

Part 1

Question 1: Find the derivative of the following functions.

(a) $f(x) = 5e^{x^2+1}$

(b) $w(p) = \dfrac{2^p - p^2}{p+1}$

(c) $Z(t) = t^{10} - 7t^2 + \dfrac{1}{\sqrt[3]{t}}$

(d) $h(y) = y^2 \ln(y)$

Question 2: Compute the following integrals. Notice that some are indefinite and some are definite.

(a) $\displaystyle\int_1^3 x^2\sqrt{x^3+1}\, dx$

(b) $\displaystyle\int (4^x + 4x^4 - 1)\, dx$

(c) $\displaystyle\int \left(\frac{1}{z} + \frac{1}{\sqrt{z}}\right)\, dz$

(d) $\displaystyle\int_{-1}^2 (w^4 - \frac{1}{3}w^2 - 2w)\, dw$

Question 3: Consider the function $f(x) = -2x^3 + 3x^2 + 12x - 7$

(a) Find all critical points of $f(x)$.

(b) Find the intervals on which $f(x)$ is increasing and decreasing.

(c) Find all local extrema of $f(x)$.

(d) Find the intervals on which $f(x)$ is concave up and concave down.

(e) Find all inflection points of $f(x)$.

(f) Draw a graph of $f(x)$. Label the vertical intercept, critical points, local extrema, and inflection points on your graph.

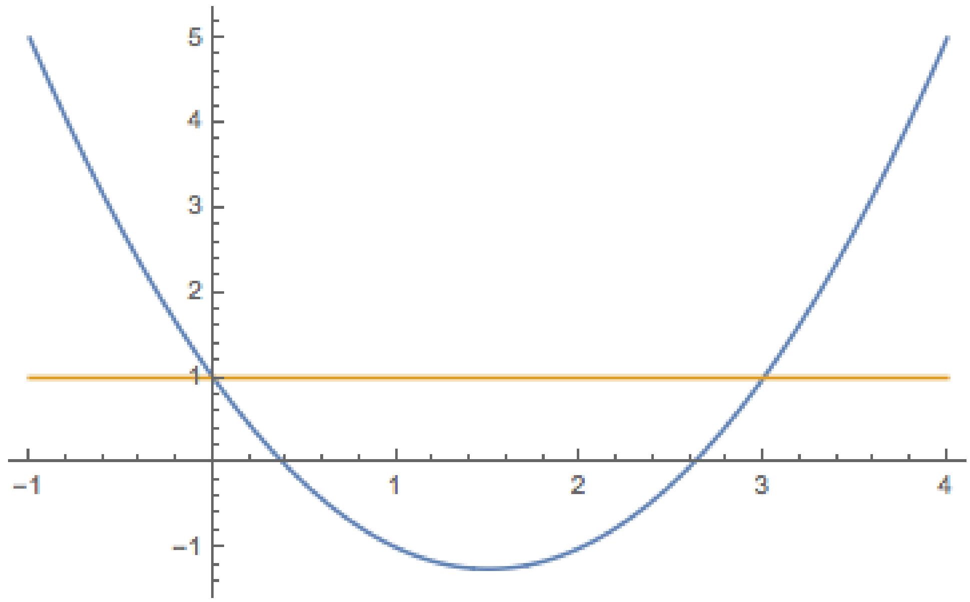
Question 4: A table of values of the function $f(t)$ is below. Use this table to estimate the value of $\int_8^{10} f(t)dt$ by computing the average of left and right Riemann sums.

t	8	8.5	9	9.5	10
f(t)	3	7	12	11	-1

Question 5: Find the global maximum and global minimum of the function $g(x) = x^4 - \frac{8}{3}x^3 - 16x^2 + 2$ on the interval $[-3, 3]$.

Question 6: Use local linear approximation to estimate the value of $(1.1)^{7.2}$.

Question 7: Calculate the area enclosed by the curves $y = 1$ and $y = x^2 - 3x + 1$. A graph of these curves is shown below.



Part II

Question 8: A zoo is building a new enclosure for its monkeys. The enclosure will be a rectangular pen. Three of the walls to the pen will be made of stone, while one of the walls of the pen will be made of glass. The stone wall costs \$20 per foot to build, and the glass wall costs \$30 per foot to build. The zoo can only spend \$3500 to build the enclosure. What is the largest area that the pen can enclose?

Question 9:

(a) Compute $f'(x)$ for the following function. You do not need to simplify your answer.

$$f(x) = 2^{\left(3x^2 \cdot \ln(4x+z^2)\right)}$$

(b) Consider the function $g(t)$ given in the following table.

t	0	1	2	3	4
g(t)	3	1	5	-2	6

Use the table to calculate

$$\int_0^4 \frac{g'(\sqrt{t})}{\sqrt{t}}\,dt$$

Question 10: A particle travels in a straight line along a number line. We take the positive direction to be toward the right. As the particle is traveling, its velocity is given by the function

$$v(t) = \frac{t^3}{2} - 3t^2 + 4t$$

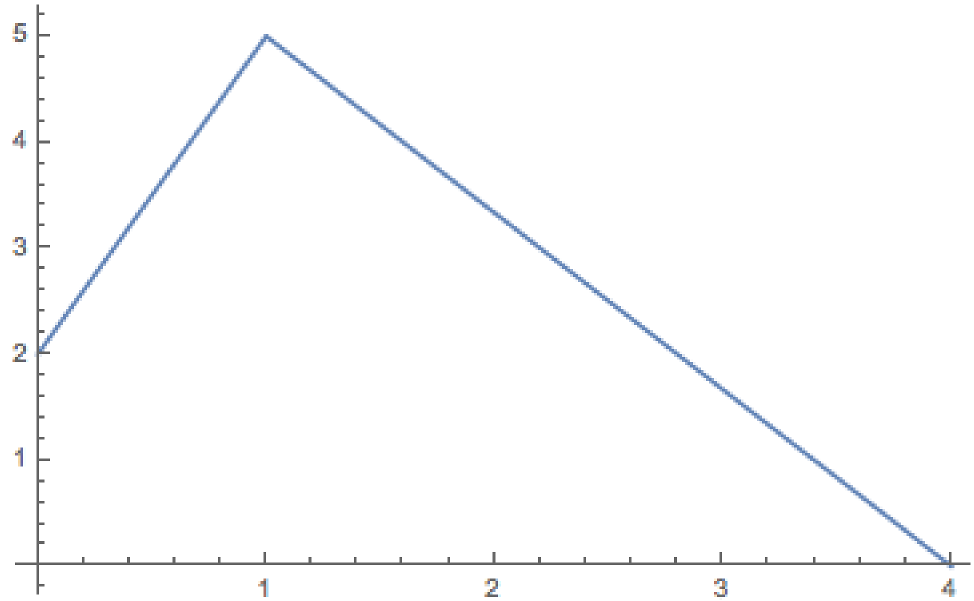
(a) What is the total change in position of the point from time $t = 0$ to time $t = 4$?

(b) What is the total distance that the particle travels during this time?

(c) What is the average velocity of the particle during this time?

(d) What is the average speed of the particle during this time?

Question 11: Consider the function $F(x) = \int_0^x f(t)dt$, where the function $f(t)$ is graphed below:



(a) What is $F(2)$?

(b) What is $f(2)$? What is $f'(2)$?

(c) Is $F'(2)$ positive, negative, or zero?

Question 12: Instead of using the local linear approximation at a point, which only uses the first derivative, to approximate a function, we can use a "local quadratic approximation" at a point, which uses both the first and second derivative. Given a function $f(x)$ and a point $x = a$, the "local quadratic approximation at $x = a$" is given by

$$Q(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$$

Similar to the case of the local linear approximation, the local quadratic approximation is a good estimate of the function near the value $x = a$.

(a) Find the local linear approximation for the function $f(x) = e^{2x}$ at $x = 0$.

(b) Find the local quadratic approximation for the function $f(x) = e^{2x}$ at $x = 0$.

(c) Using your function in part (b), estimate the value of $e^{0.2} = f(0.1)$.

MAT 122 Sample Final - Solutions

1 Conceptual Questions

1. What is the definition of the derivative of a function at a point?

   The "limit" of the average rate of change of the function on smaller and smaller intervals containing the point.

2. What is the definition of $\int_a^b v(t)\,dt$?

   The "limit" of Rieman sums of $v(t)$ on the interval $[a,b]$ as the number of subintervals gets larger and larger.

3. What is the Fundamental Theorem of Calculus?

   If $F(x)$ is an antiderivative for the function $f(x)$ (i.e., $F'(x) = f(x)$), then $\int_a^b f(x)\,dx = F(b) - F(a)$.

4. Let h be a function of p. Write the Newton notation for the second derivative of h. Write the Leibniz notation for the second derivative of h.

   They are $h''(p)$ and $\frac{d^2h}{dp^2}$, respectively.

5. Give as many interpretations for the derivative as you can.

   Limit of the average rate of change. Slope of the tangent line at a point. Instantaneous rate of change.

6. Give as many interpretations for the definite integral as you can.

   Limit of Riemann sum. Area under a curve. Total change or change in position when the integrand represents a rate of change or velocity.

7. What are the differences between $\int_a^b f(x)\,dx$ and $\int f(x)\,dx$? What does each expression represent?

   The first is a definite integral, and represents a single number, which we can interpret as above. The second is an indefinite integral, and represents the family of antiderivatives of the function $f(x)$, each antiderivative differing only by the addition of a constant.

8. What are the differences between local extrema and global extrema?

   The point $x = a$ is a local extremum if the value of the function is larger or smaller at a than at all x-values near a. The point $x = a$ is a global extremum if the value of the function at a is larger or smaller than the value of the function at all other points in the entire domain of the function. A global extremum is always a local extremum. A local extremum does not need to be a global extremum. Local extrema are found by finding the critical points, and testing each critical point using the first or second derivative test. Global extrema (on an interval) are found by evaluating the original function at critical points and at the endpoints of the interval.

9. For each condition below, either give an example of a function with domain all real numbers satisfying that condition, or explain why such a function cannot exist.

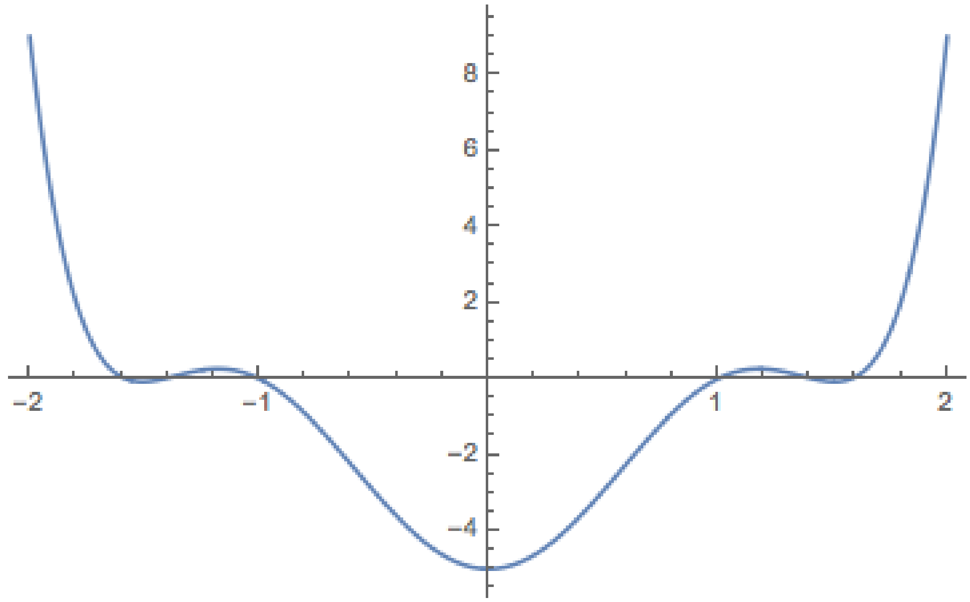
   (a) The function has no local max nor local min.

(b) The function has a local max but no global max.

(c) The function has a global max but no local max.

(d) The function has two local maxima, 1 global minimum, and no global maximum.

Any linear function is an example of (a), as is the function $f(x) = x^3$. The function $f(x) = x^3 - x$ is an example of (b), as it has a local but not global max at $x = -1/\sqrt{3}$. (It also has a local but not global min at $x = 1/\sqrt{3}$). A function described in part (c) cannot exist, as a global maximum is always a local maximum. The graph shown below is one of many possible examples for part (d).



10. What condition guarantees the existence of a global maximum and a global minimum for a function? What is the procedure for finding these global extrema in that case?

    Restricting the domain of the function to a closed interval $[a, b]$ guarantees that the function will have a global maximum and a global minimum. To find the global extrema, find the critical points of the function, then evaluate the original function at these critical points and at the endpoints of the interval. The largest number is the maximum value, while the smallest is the minimum value.

11. What are the first and second derivative tests? What are they used for? Explain why they work.

    The first and second derivative tests are used to decide if critical points of a function are in fact local max or min points. The first derivative test says that if $x = a$ is a critical point and the derivative f' switches from positive to negative at a, then a is a local max, while if the derivative switches from negative to positive at a then a is a local min. If the derivative does not switch sign then a is neither a local max nor a local min. This works because a local max is a place where the function switches from increasing to decreasing, and similarly for a local min.

    The second derivative test says that a critical point $x = a$ is a local max is $f''(a) < 0$, while it is a local min if $f''(a) > 0$. The second derivative test is inconclusive if $f''(0) = 0$. To see why it works, again consider that a local max is a place where the function switched from increasing to decreasing, that is, f' is decreasing, so $f'' < 0$, and similarly for a local min.

12. Assume that $0 \leq f(x) \leq g(x)$ for any value of x. What can you say about the relationship between the values $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$?

    We'll have $\int_a^b f(x)\ dx \leq \int_a^b g(x)\ dx$. Consider the interpretation of the definite integral as an area. Because $f(x)$ is closer to the x-axis, it encloses less area that $g(x)$.

13. If $f'(x)$ is always positive, what can you say about the sign of $f(x)$? If $f(x)$ is always negative, what can you say about $\int_a^b f(x)$?

    Nothing can be said in the first case. We could have that $f(x)$ is always positive (consider e^x) or always negative (consider $-e^{-x}$) or that it changes sign (consider $1/3x^3 + x$). However, if $f(x)$ is always negative, then $\int_a^b f(x) \leq 0$. We can interpret this in terms of area. The area between $f(x)$ and the x-axis will always be entirely below the x-axis, which we interpret at negative area.

14. Give an example of a function $f(x)$ such that

    (a) $f(-1) = 1$ and $f(1) = -1$

    (b) $\int_{-1}^1 f(x)dx = 0$

    (c) $f''(x)$ is not always 0

    The function $f(x) = -x^3$ is a possible answer. Its second derivative is $6x$, which is not always 0, satisfying (c). We can evaluate the function to see that is satisfies (a), and integrate it to see that it satisfies (b). The important consideration to find such a function is that it must enclose the same area above the x-axis as it does below the x-axis. Notice that $-x^3$ is symmetric about the origin, which automatically gives this property.

15. Draw a function for which the quantity $\int_0^1 f(x)dx$ is positive but the quantity $\int_0^2 f(x)dx$ is negative.

    The graph of $f(x) = -(x - 1/2)^3 + 1/8$ is such a function. You can graph this function using graph transformations, and observe that the area above the x-axis on the interval $[0, 1]$ is less than the area below the x-axis on $[1, 2]$, so that $\int_0^1 f(x)dx$ will be positive but $\int_0^2 f(x)dx$ will be negative.

2 Sample Final

Part 1

Question 1: Find the derivative of the following functions.

(a) $f(x) = 5e^{x^2+1}$

Use the chain rule with inside function $z = x^2+1$, which has derivative $2x$, and outside function $5e^z$, which has derivative $5e^z$. The derivative is therefore

$$f'(x) = 5e^{x^2+1}(2x)$$

(b) $w(p) = \dfrac{2^p - p^2}{p+1}$

Using the quotient rule,

$$\begin{aligned} w'(p) &= \frac{(p+1)\frac{d}{dp}(2^p + p^2) - (2^p - p^2)\frac{d}{dp}(p+1)}{(p+1)^2} \\ &= \frac{(p+1)\left(\ln(2)2^p + 2p\right) - (2^p - p^2)}{(p+1)^2} \end{aligned}$$

(c) $Z(t) = t^{10} - 7t^2 + \dfrac{1}{\sqrt[3]{t}}$

We can rewrite the last term as $t^{-1/3}$, and the use the power rule on each term, which gives

$$Z'(t) = 10t^9 - 14t - \frac{1}{3}t^{-4/3}$$

(d) $h(y) = y^2 \ln(y)$

Using the product rule,

$$\begin{aligned} h'(y) &= \frac{d}{dy}(y^2)\ln y + y^2\frac{d}{dy}\ln y \\ &= 2y\ln y + y^2 \cdot \frac{1}{y} \end{aligned}$$

Question 2: Compute the following integrals. Notice that some are indefinite and some are definite.

(a) $\displaystyle\int_1^3 x^2\sqrt{x^3+1}\,dx$

We use a substitution, with $u = x^3 + 1$. Then $du = 3x^2dx$, so that $x^2\ dx = \frac{1}{3}\ du$. The limit $x = 1$ becomes $u = 2$, and the limit $x = 3$ becomes the limit $u = 28$.

$$\begin{aligned}\int_1^3 x^2\sqrt{x^3+1}\,dx &= \int_1^3 \sqrt{x^3+1}x^2\,dx\\ &= \int_2^{28}\sqrt{u}\frac{1}{3}\,du\\ &= \frac{1}{3}\left[\frac{2}{3}u^{3/2}\right]_2^{28}\\ &= \frac{2}{9}\left(28^{3/2}-2^{3/2}\right)\end{aligned}$$

(b) $\int (4^x + 4x^4 - 1)\,dx$

$$\begin{aligned}\int (4^x+4x^4-1)\,dx &= \int 4^x\,dx + \int 4x^4\,dx - \int\,dx\\ &= \int 4^x\,dx + 4\int x^4\,dx - \int\,dx\\ &= \frac{1}{\ln 4}4^x + 4\cdot\frac{1}{5}x^5 \;-x+C\end{aligned}$$

(c) $\int \left(\frac{1}{z}+\frac{1}{\sqrt{z}}\right)\,dz$

$$\begin{aligned}\int \left(\frac{1}{z}+\frac{1}{\sqrt{z}}\right)\,dz &= \int \frac{1}{z}\,dz + \int z^{-1/2}\,dz\\ &= \ln|z| + 2\sqrt{z}+C\end{aligned}$$

(d) $\int_{-1}^{2}(w^4 - \frac{1}{3}w^2 - 2w)\,dw$

$$\begin{aligned}\int_{-1}^{2}(w^4-\frac{1}{3}w^2-2w)\,dw &= \left[\frac{1}{5}w^5-\frac{1}{9}w^3-w^2\right]_{-1}^{2}\\ &= \left(\frac{2^5}{5}-\frac{8}{9}-4\right)-\left(-\frac{1}{5}+\frac{1}{9}-1\right)\\ &= \frac{13}{5}\end{aligned}$$

Question 3: Consider the function $f(x) = -2x^3+3x^2+12x-7$

(a) Find all critical points of $f(x)$.

Critical points are where the first derivative is zero. The first derivative is

$$\begin{aligned}f'(x) &= -6x^2+6x+12\\ &= -6(x^2-x-2)\\ &= -6(x-2)(x+1)\end{aligned}$$

So the critical points are at $x=-1$ and $x=2$.

(b) Find the intervals on which $f(x)$ is increasing and decreasing.

The critical points divide the number line into three intervals, $(-\infty, -1), (-1, 2)$, and $(2, \infty)$. We can observe that $f'(-2) = -24$, so that $f' < 0$ on $(-\infty, -1)$. We also have that $f'(0) = 12$, so $f' > 0$ on $(-1, 2)$, while $f'(3) = -24$, so that $f' < 0$ on $(2, \infty)$.

Therefore $f(x)$ is increasing on the interval $(-1, 2)$, while it is decreasing on the intervals $(-\infty, -1)$ and $(2, \infty)$.

(c) Find all local extrema of $f(x)$.

We need to decide if the critical points $x = -1$ and $x = 2$ are local max, local min, or neither. We could use either the first or the second derivative test. We observe that at $x = -1$ the derivative switches from negative to positive, so $x = -1$ is a local min according to the first derivative test. Similarly the first derivative test gives that $x = 2$ is a local max.

(d) Find the intervals on which $f(x)$ is concave up and concave down.

Concavity is given by the sign of the second derivative. The second derivative is

$$f''(x) = -12x + 6.$$

We observe that the second derivative is zero when $x = 1/2$, and can also verify that the second derivative is positive for values of x that are less that $1/2$ and negative for values greater than $1/2$.

Therefore $f(x)$ is concave up on the interval $(-\infty, 1/2)$ and concave down on the interval $(1/2, \infty)$.

(e) Find all inflection points of $f(x)$.

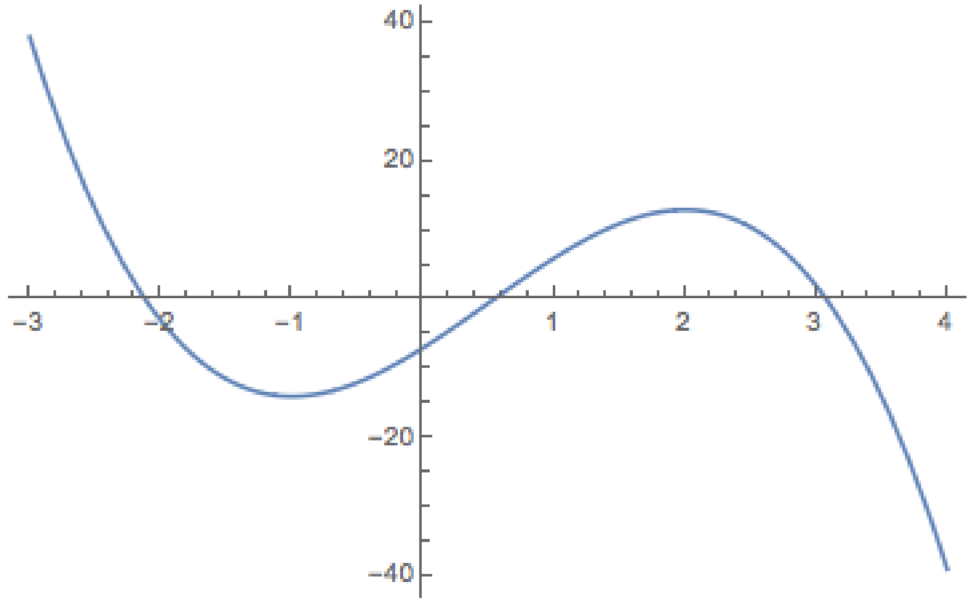
Inflection points are places where the concavity switches sign. From our answer to the previous part we can see that $x = 1/2$ is an inflection points.

(f) Draw a graph of $f(x)$. Label the vertical intercept, critical points, local extrema, and inflection points on your graph.

We need to plot the points

$$f(0) = -7 \qquad f(-1) = -14 \qquad f(2) = 13 \qquad f(1/2) = -1/2$$

on the graph, as these are the vertical intercept, local min, local max, and inflection point, respectively. The graph is



Question 4: A table of values of the function $f(t)$ is below. Use this table to estimate the value of $\int_8^{10} f(t)dt$ by computing the average of left and right Riemann sums.

t	8	8.5	9	9.5	10
f(t)	3	7	12	11	-1

Observe from the table that $\Delta t = 0.5$, which makes $n = 4$. The subintervals are $[8, 8.5]$, $[8.5, 9]$, $[9, 9.5]$, and $[9.5, 10]$. Therefore the left Riemann sum approximation is

$$\begin{aligned} L_4 &= f(8)\Delta t + f(8.5)\Delta t + f(9)\Delta t + f(9.5)\Delta t \\ &= 3 \cdot 0.5 + 7 \cdot 0.5 + 12 \cdot 0.5 + 11 \cdot 0.5 \\ &= \frac{33}{2} = 16.5 \end{aligned}$$

The right Riemann sum is

$$\begin{aligned} L_4 &= f(8.5)\Delta t + f(9)\Delta t + f(9.5)\Delta t + f(10)\Delta t \\ &= 7 \cdot 0.5 + 12 \cdot 0.5 + 11 \cdot 0.5 + (-1) \cdot 0.5 \\ &= \frac{29}{2} = 14.5 \end{aligned}$$

Averaging these gives the approximation

$$\int_8^{10} f(t)dt = \frac{16.5 + 14.5}{2} = \frac{31}{2} = 15.5$$

Question 5: Find the global maximum and global minimum of the function $g(x) = x^4 - \frac{8}{3}x^3 - 16x^2 + 2$ on the interval $[-3, 3]$.

The first derivative is

$$\begin{aligned} g'(x) &= 4x^3 - 8x^2 - 32x \\ &= 4x(x^2 - 2x - 8) \\ &= 4x(x - 4)(x + 2) \end{aligned}$$

Therefore the critical points are $x = 0, 4$, and -2. Notice that $x = 4$ is not in the interval $[-3, 3]$, so we throw it out. We then evaluate the function g at the critical points and at the endpoints. This gives

$$g(-3) = 11 \qquad g(-2) = \frac{-74}{3} \qquad g(0) = 2 \qquad g(3) = -133$$

The global maximum value is therefore 11, which occurs at $x = -3$, and the global minimum value is -133, which occurs at $x = 3$.

Question 6: Use local linear approximation to estimate the value of $(1.1)^{7.2}$.

We use the function $f(x) = x^{7.2}$, and make the approximation at $x = 1$. Recall that the local linear approximation is simply the equation of the line tangent to $f(x)$ at $x = 1$. This line passes through the point

$$(1, f(1)) = (1, 1^{7.2}) = (1, 1)$$

and has slope equal to $f'(1)$. The derivative is $f'(x) = 7.2x^{6.2}$, using the power rule, so we have $f'(1) = 7.2$.

Using the point-slope form for the equation of a line, we have

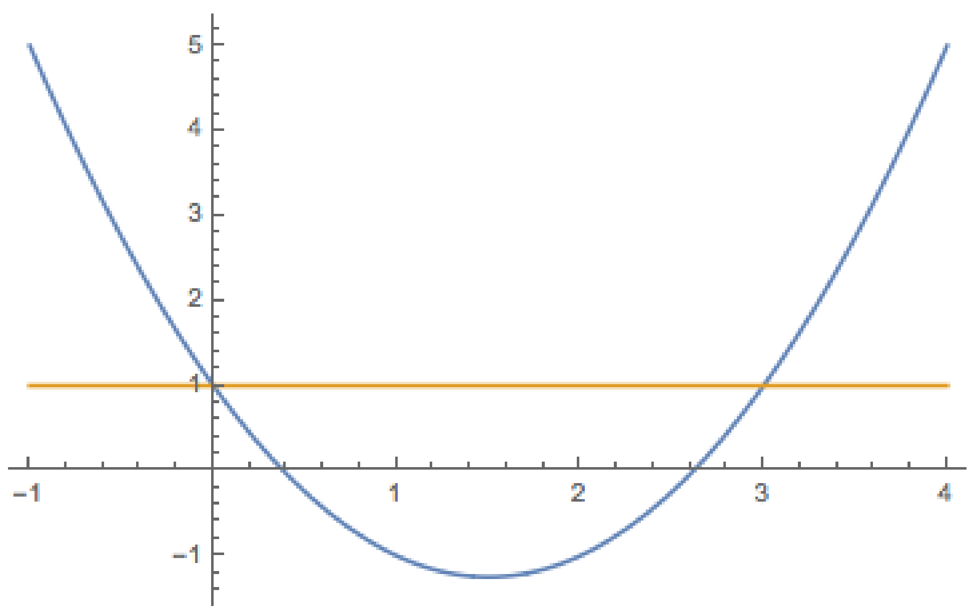
$$L(x) = 7.2(x - 1) + 1$$

Therefore our approximation is

$$(1.1)^{7.2} = f(1.1) \approx L(1.1) = 7.2(1.1 - 1) + 1 = 1.72$$

(The actual value is $1.1^{7.2} \approx 1.98622$.)

Question 7: Calculate the area enclosed by the curves $y = 1$ and $y = x^2 - 3x + 1$. A graph of these curves is shown below.



We first find where the two curves intersect by setting the two equations equal to one another-

$$x^2 - 3x + 1 = 1 \Leftrightarrow x^2 - 3x = 0 \Leftrightarrow x(x - 3) = 0$$

so the two curves intersect at $x = 0$ and $x = 3$. These will be the limits of the integral. To find the integrand, we subtract the smaller function on the interval $[0, 3]$, which is $x^2 - 3x + 2$, from the larger, which is 1. We therefore have that

$$\begin{aligned}
Area &= \int_0^3 \left(1 - (x^2 - 3x + 1)\right)\, dx \\
&= \int_0^3 \left(-x^2 + 3x\right)\, dx \\
&= \left[-\frac{x^3}{3} + \frac{3x^2}{2}\right]_0^3 \\
&= -\frac{3^3}{3} + \frac{3 \cdot 3^2}{2} \\
&= \frac{9}{2}
\end{aligned}$$

Part II

Question 8: A zoo is building a new enclosure for its monkeys. The enclosure will be a rectangular pen. Three of the walls to the pen will be made of stone, while one of the walls of the pen will be made of glass. The stone wall costs \$20 per foot to build, and the glass wall costs \$30 per foot to build. The zoo can only spend \$3500 to build the enclosure. What is the largest area that the pen can enclose?

Let ℓ denote the length of the glass wall in the pen, which we will call the length of the rectangle, and let w denote the width of the rectangle. The area is then

$$A = \ell w$$

The area is the function we are trying to maximize, so we need to be able to write it as a function of a single variable. For this we use the cost. The cost to build a pen with a glass wall of length ℓ is

$$30\ell + (2w + \ell)20$$

As this cost must be \$3500, we have that

$$50\ell + 40w = 3500 \Leftrightarrow \ell = \frac{3500 - 40w}{50}$$

Therefore the area, written as a function of w, is

$$A(w) = \frac{3500 - 40w}{50}w$$

We can restrict the domain of this function. Note that we need $w \geq 0$ to make physical sense, and also $w \leq \frac{175}{2}$. We arrive at this second inequality by noticing that we must buy at least $2w$ feet of stone wall, which costs $40w$, and this quantity must be less that 3500, that is, $w \leq 3500/40 = 175/2$. Therefore, we are trying to compute the maximum value of A on the interval $\left[0, \frac{175}{2}\right]$.

For this we need to find the critical points. Taking the derivative, we have

$$A'(w) = 70 - \frac{8}{5}w$$

so that a critical point occurs at $w = \frac{175}{4} = 43.75$. Notice that this value is in the interval under consideration.

Finally, we evaluate the area function at the endpoints and at this critical point, which gives

$$A(0) = 0 \qquad A\left(\frac{175}{2}\right) = 0 \qquad A\left(\frac{175}{4}\right) = 1531.25$$

Therefore the maximum area that can be enclosed in the pen is 1531.25 ft^2.

Question 9:

(a) Compute $f'(x)$ for the following function. You do not need to simplify your answer.

$$f(x) = 2^{\left(3x^2 \cdot \ln(4x+z^2)\right)}$$

We begin with the chain rule. The inside functions is $w = 3x^2 \cdot \ln(4x + z^2)$, and the outside function is 2^w. So we have

$$f'(x) = \ln(2)2^{\left(3x^2 \cdot \ln(4x+z^2)\right)} \cdot \frac{d}{dx}\left(3x^2 \cdot \ln(4x + z^2)\right)$$

Next we apply the product rule, which gives

$$\begin{aligned} f'(x) &= \ln(2)2^{\left(3x^2 \cdot \ln(4x+z^2)\right)} \cdot \left(\frac{d}{dx}(3x^2) \cdot \ln(4x + z^2) + 3x^2 \cdot \frac{d}{dx}\ln(4x + z^2)\right) \\ &= \ln(2)2^{\left(3x^2 \cdot \ln(4x+z^2)\right)} \cdot \left(6x\ln(4x + z^2) + 3x^2 \cdot \frac{d}{dx}\ln(4x + z^2)\right) \end{aligned}$$

For the remaining derivative we again use the chain rule. The inside function is $t = 4x + z^2$, which has derivative 4 (z^2 is a constant). The outside function is $\ln t$, which has derivative $1/t$. We therefore have that the overall derivative is

$$f'(x) = \ln(2)2^{\left(3x^2 \cdot \ln(4x+z^2)\right)} \cdot \left(6x\ln(4x + z^2) + 3x^2 \cdot \frac{4}{4x + z^2}\right)$$

(b) Consider the function $g(t)$ given in the following table.

t	0	1	2	3	4
g(t)	3	1	5	-2	6

Use the table to calculate

$$\int_0^4 \frac{g'(\sqrt{t})}{\sqrt{t}}\, dt$$

We make the substitution $u = \sqrt{t}$, so that $du = \frac{1}{2}t^{-1/2}\, dt$. Rearranging slightly, this gives $2\, du = \frac{1}{\sqrt{t}}\, dt$. With this substitution, the limits of integration change from 0 and 4 to 0 and 2, respectively.

This gives

$$\begin{aligned} \int_0^4 \frac{g'(\sqrt{t})}{\sqrt{t}}\, dt &= \int_0^4 g'(\sqrt{t}) \cdot \frac{1}{\sqrt{t}}\, dt \\ &= \int_0^2 g'(u) \cdot 2\, du \\ &= 2\int_0^2 g'(u)\, du \\ &= 2\left[g(u)\right]_0^2 \\ &= 2\left(g(2) - g(0)\right) \\ &= 2\,(5 - 3) \\ &= 4 \end{aligned}$$

Question 10: A particle travels in a straight line along a number line. We take the positive direction to be toward the right. As the particle is traveling, its velocity is given by the function

$$v(t) = \frac{t^3}{2} - 3t^2 + 4t$$

(a) What is the total change in position of the point from time $t = 0$ to time $t = 4$?

Recalling that one of our interpretations of the definite integral is that integrating a velocity function gives change in position, we have

$$\begin{aligned}
\text{Change in position} &= \int_0^4 \left(\frac{t^3}{2} - 3t^2 + 4t\right)\, dt \\
&= \left[\frac{t^4}{8} - t^3 + 2t^2\right]_0^4 \\
&= \frac{4^4}{8} - 4^3 + 2 \cdot 4^2 \\
&= 0
\end{aligned}$$

Thus at times $t = 0$ and $t = 4$ the particle is at the same location.

(b) What is the total distance that the particle travels during this time?

The total distance is

$$\text{Total distance} = \int_0^4 |v(t)|\, dt$$

To compute this, we need to find the intervals on which f is positive and negative. We observe that $v(t)$ factors as

$$v(t) = \frac{t}{2}(t-2)(t-4)$$

so we need to check the sign of $v(t)$ on the intervals $[0, 2]$ and $[2, 4]$. By plugging in values we find that v is positive on the first interval and negative on the second. Therefore $|v(t)| = v(t)$ on $[0, 2]$ and $|v(t)| = -v(t)$ on $[2, 4]$. We therefore have

$$\begin{aligned}
\int_0^4 |v(t)|\, dt &= \int_0^2 |v(t)|\, dt + \int_2^4 |v(t)|\, dt \\
&= \int_0^2 v(t)\, dt - \int_2^4 v(t)\, dt \\
&= \left[\frac{t^4}{8} - t^3 + 2t^2\right]_0^2 - \left[\frac{t^4}{8} - t^3 + 2t^2\right]_2^4 \\
&= 2\left(\frac{2^4}{8} - 2^3 + 2 \cdot 2^2\right) \\
&= 4
\end{aligned}$$

The particle therefore travels a total of 4 units.

(c) What is the average velocity of the particle during this time?

Using the definition of the average value of a function, we have that

$$v_{avg} = \frac{1}{4-0}\int_0^4 v(t)\, dt = \frac{1}{4} \cdot 0 = 0$$

We could also observe this directly from out answer in part (a) - the the positions at $t = 0$ and $t = 4$ are the same, the average rate of change in position (i.e., the average velocity) on the interval $[0, 4]$ is zero.

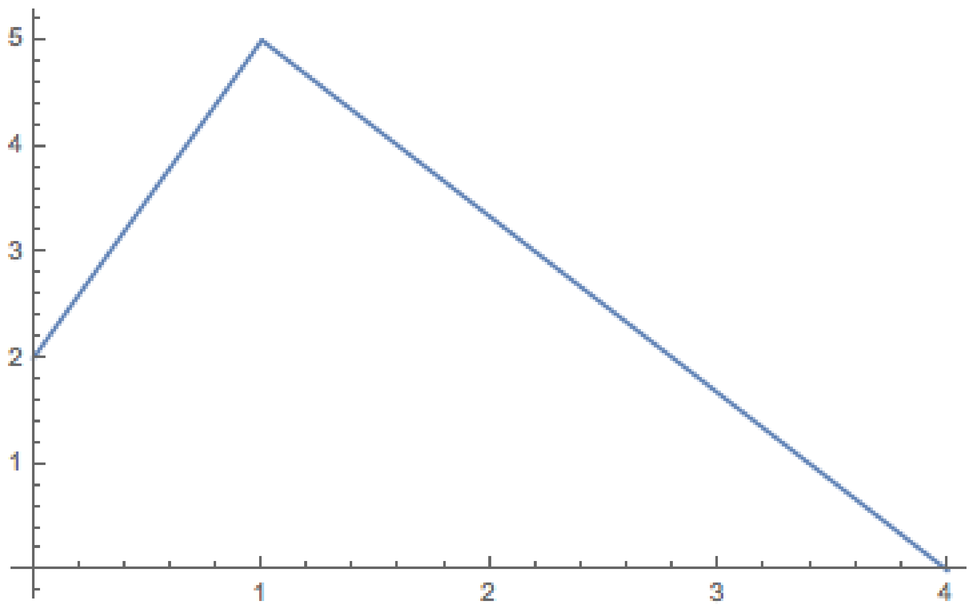
(d) What is the average speed of the particle during this time?

Using the definition of the average value of a function, and recalling that speed is equal to $|v(t)|$, we have

$$\begin{aligned}\text{Average speed} &= \frac{1}{4-0}\int_0^4 |v(t)|\,dt \\ &= \frac{1}{4}\cdot 4 = 1\end{aligned}$$

so that the particle moves with an average speed of 1 unit distance per unit time.

Question 11: Consider the function $F(x) = \int_0^x f(t)dt$, where the function $f(t)$ is graphed below:



(a) What is $F(2)$?

$$\begin{aligned}F(2) &= \int_0^2 f(t)\,dt \\ &= \frac{1}{2}(2+5)\cdot 1 + \frac{1}{2}\cdot(5+\frac{-5}{3})\cdot 1 \\ &= \frac{7}{2}+\frac{5}{3} = \frac{31}{6}\end{aligned}$$

We compute the integral by interpreting it as an area. The two terms in the second line above correspond the areas of the trapezoid formed by the graph on the interval $[0,1]$ and the trapezoid formed by the graph on the interval $[1,2]$.

(b) What is $f(2)$? What is $f'(2)$?

The graph of f on the interval $[1,4]$ is a straight line, starting at $(1,5)$ and ending at $(4,0)$. Such a line has slope $\frac{0-5}{4-1} = \frac{-5}{3}$. Therefore

$$f(2) = 5 + 1\cdot\frac{-5}{3} = 10/3$$

Since the derivative of a linear function is simply the slope, we have that $f'(2) = \frac{-5}{3}$.

(c) Is $F'(2)$ positive, negative, or zero?

Another way to ask this question is to ask whether $F(x)$ is increasing or decreasing at $x = 2$. Consider the difference between $F(2)$ and $F(2.1)$. The difference between these two values is the area under the curve $f(t)$ on the interval $[2, 2.1]$, which we can see is positive. More generally, for any value a slightly larger than 2 we will have that $F(a) > F(2)$, so the function $F(x)$ is increasing. Thus $F'(2) > 0$.

Question 12: Instead of using the local linear approximation at a point, which only uses the first derivative, to approximate a function, we can use a "local quadratic approximation" at a point, which uses both the first and second derivative. Given a function $f(x)$ and a point $x = a$, the "local quadratic approximation at $x = a$" is given by

$$Q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

Similar to the case of the local linear approximation, the local quadratic approximation is a good estimate of the function near the value $x = a$.

(a) Find the local linear approximation for the function $f(x) = e^{2x}$ at $x = 0$.

The local linear approximation passes through the point $(0, f(0)) = (0, 1)$. The slope of the tangent line for $f(x)$ at $x = 0$ is $f'(0)$. Computing this, we have

$$f'(x) = 2e^{2x} \Rightarrow f'(0) = 2$$

Therefore the local linear approximation is

$$L(x) = 2(x-0) + 1 = 2x + 1$$

(b) Find the local quadratic approximation for the function $f(x) = e^{2x}$ at $x = 0$.

We have already found that $f(0) = 1$ and $f'(0) = 2$. To write the local quadratic approximation, we only need to figure out $f''(0)$. Computing this, we have

$$f''(x) = 4e^{2x} \Rightarrow f''(0) = 4$$

Therefore the local quadratic approximation is

$$\begin{aligned} Q(x) &= 1 + 2(x-0) + \frac{4}{2}(x-0)^2 \\ &= 2x^2 + 2x + 1 \end{aligned}$$

(c) Using your function in part (b), estimate the value of $e^{0.2} = f(0.1)$.

$$e^{0.2} = f(0.1) \approx Q(0.1) = 2(0.1)^2 + 2(0.1) + 1 = 1.22$$