Superstring scattering amplitudes and modular forms

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Versions given in Hannover and Berlin in 2010

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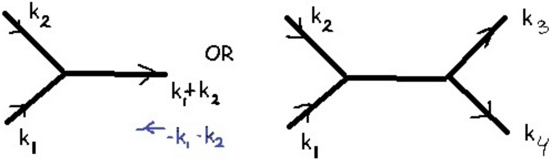

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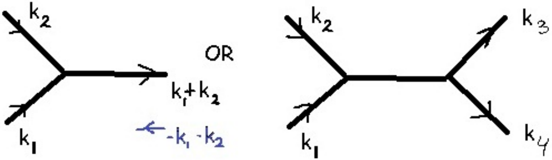

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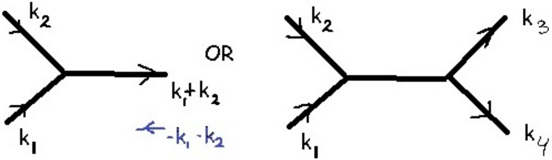

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- *Probability amplitude*:
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- *Problem:*
the integral for many (> 4) interactions diverges

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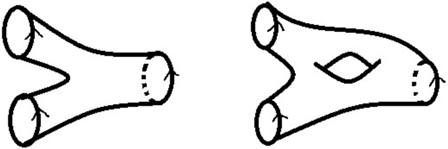

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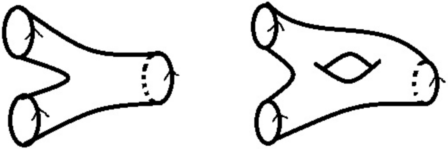

- *Probability amplitude*:
  integral over all possible trajectories, i.e. over all possible surfaces with given end-circles.
- *Question*:
  how to assign a weight to a given surface, i.e. what is the probability distribution on the set of all trajectories?

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Simplest case: no incoming or outgoing particles.
Free energy = vacuum-to-vacuum probability.

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(*This is a physical argument: reducing an infinite-dimensional integral over all worldsheets to a finite-dimensional one*)

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Genus 0: $\mathcal{M}_0 =$one point, there is nothing to integrate.

(*An interesting expression for the 4-point function* $A_0(k_1, k_2, k_3, k_4)$)

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In genus 1 we have

$$A_1^{bos} = \int_{\mathcal{M}_1 = \mathbb{H}/SL(2,\mathbb{Z})} \frac{1}{(\mathrm{Im}\,\tau)^{14}} \left| \frac{d\tau}{\prod \theta_m^8(\tau)} \right|^2$$

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Problem:

The bosonic measure is not integrable: it blows up near $\partial\overline{\mathcal{M}_g}$ as $\frac{dz}{z^2}$, and thus $\int_{\mathcal{M}_g} ||d\mu^2_{bos}|| = \infty$

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$$-\psi_-^\mu \partial_z \psi_-^\nu - \frac{1}{2}\chi_{\bar z}^+ \chi_z^- \psi_+^\mu \psi_-^\nu + \chi_{\bar z}^+ \psi_+^\mu \partial_z x^\nu + \chi_z^- \psi_-^\mu \partial_{\bar z} x^\nu\Big)$$

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⇒everything should depend only on the superRiemann surface, so

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`D'Hoker-Phong`: successfully dealt with this for $g = 2$.

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Ansatz (G.)

$$d\mu_{ss}[m](\tau) = d\mu_{bos} \sum_{i=0}^{g} (-1)^i 2^{i(i-1)/2} \sum_{V \subset (\mathbb{Z}/2)^{2g}; \dim V = i} \prod_{n \in V} \theta_{n+m}^{2^{4-i}}$$

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The action of $\mathrm{Sp}(2g, \mathbb{Z})$ on $\mathcal{H}_g$ is given by

$$\gamma \circ \tau := (C\tau + D)^{-1}(A\tau + B)$$

for an element $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{Z})$

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Definition

A modular form of weight k with respect to $\Gamma \subset \mathrm{Sp}(2g,\mathbb{Z})$ is a function $F : \mathcal{H}_g \to \mathbb{C}$ such that

$$F(\gamma \circ \tau) = \det(C\tau + D)^k F(\tau) \qquad \forall\gamma \in \Gamma, \forall\tau \in \mathcal{H}_g$$

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If m is odd, i.e. $a \cdot b = 1 \in \mathbb{Z}/2$, then $\theta_m(\tau) \equiv 0$.

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- $\mathcal{A}_g$: moduli space of principally polarized abelian varieties.

Moduli of abelian varieties

- Abelian variety A: a projective variety with a group structure $\Longleftrightarrow A_\tau = \mathbb{C}^g/\mathbb{Z}^g + \mathbb{Z}^g\tau$.
- Principal polarization Θ on A: an ample divisor on A with $h^0(A,\Theta)=1$. $\Longleftrightarrow \Theta_\tau = \{z \in A_\tau | \theta(\tau,z)=0\}$.
  (*Theta function is a section of a line bundle on* A_τ*, i.e.* $\forall a,b \in \mathbb{Z}^g$ *we have* $\theta(\tau, z+\tau a+b) = \exp(\cdot)\theta(\tau,z)$)
- $\mathcal{A}_g$: moduli space of principally polarized abelian varieties.
- Then $\mathcal{H}_g \to \mathcal{A}_g$ (sending τ to (A_τ,Θ_τ)) is the universal cover, and $\mathcal{A}_g = \mathcal{H}_g/\operatorname{Sp}(2g,\mathbb{Z})$.

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Thus to construct $d\mu_{bos}$ could try to write a modular form of weight 13 on $\mathcal{A}_g$, and restrict it to $\mathcal{M}_g$.

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Mathematics approach: construct a finite cover $\mathcal{S}_g \to \mathcal{M}_g$ such that $s\mathcal{M}_g \to \mathcal{S}_g$

Spin curves

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Depending on whether $h^0(X, \eta)$ is even or odd (generically 0 or 1), have two irreducible components $\mathcal{S}_g = \mathcal{S}_g^+ \sqcup \mathcal{S}_g^-$.

For supersymmetry reasons, the measure on $\mathcal{S}_g^-$ (where η has a non-trivial section) is identically zero.

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where $\nu_1,\ldots,\nu_6$ are the 6 odd spin structures such that $m=\nu_1+\nu_2+\nu_3$.

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This is very hard, uses heavily the hyperelliptic representation of genus 2 surfaces. So how one can generalize to higher genus?

D'Hoker-Phong program: factorization constraint

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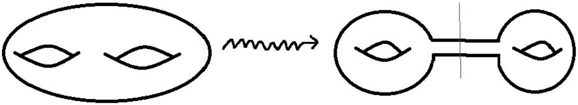
Study the degeneration of $d\mu_{ss}$ and $d\mu_{bos}$ under degeneration



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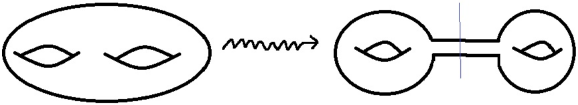

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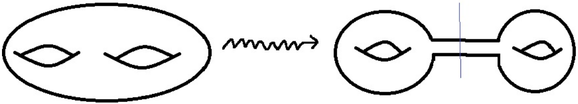

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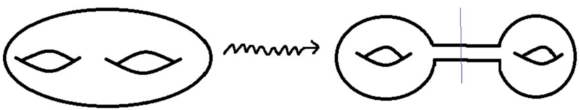

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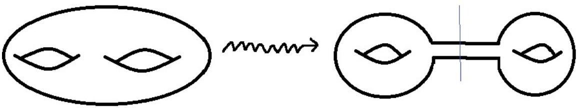

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(*Degeneration to* $\mathcal{M}_{g-1,2}\subset\overline{\mathcal{M}_g}$ *is not clear — off-shell amplitudes*)

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(Provided the roots can be chosen consistently)

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Theorem (Dalla Piazza-van Geemen)

$\Xi^{(4)}[m]$ is the unique modular form satisfying factorization.

Theorem (Oura-Poor-Salvati Manni-Yuen)

$\Xi^{(5)}[m]$ extends holomorphically to all of $\mathcal{A}_5$.

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Theorem (G.-Salvati Manni)

For $g \leq 5$ the cosmological constant $\Xi^{(g)}(\tau)$ is proportional to the Schottky-Igusa form

$$F_g(\tau) := 2^g \sum_{m \in (\mathbb{Z}/2)^{2g}} \theta_m^{16}(\tau) - \left(\sum_{m \in (\mathbb{Z}/2)^{2g}} \theta_m^{8}(\tau) \right)^2$$

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This conjecture is false for any $g \geq 5$.
(in fact the zero locus of F_5 on $\mathcal{M}_5$ is the divisor of trigonal curves)

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Thus $\Xi^{(5)}(\tau) = \mathrm{const} \cdot F_5(\tau) \not\equiv 0$ on $\mathcal{J}_5$, i.e. the cosmological constant for the proposed ansatz does not vanish for genus 5.

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What happens for higher genera?

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- Apply these modular forms to approach the Schottky problem in genus 5 and higher.