MAT 319

Practice Midterm II. 28 October, 2013

This is a closed notes/ closed book/ electronics off exam.

You are allowed and encouraged to motivate your reasoning, but at the end your proofs should be formal logical derivations, whether proving that something holds for all, or proving that your example works.

You can use any theorem or statement proven in the book; please refer to it in an identifiable way, eg. "by the completeness axiom", "by the definition of the limit", etc.

You should attempt Problem 1 and three of the remaining four questions. If you attempt all four questions, your total score will be made up of the score for Problem 1 and your best three scores on the remaining questions.

Please write legibly and cross out anything that you do not want us to read.

Each problem is worth 25 points.

Name:

Problem	1	2	3	4	5	Total
Grade						

Problem 1.

a) Define what it means to say that the series
$$\sum_{n=1}^{\infty} a_n$$
diverges.

b) Show that the series
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$$
diverges. (Hint: show $\frac{1}{\sqrt{n+1}+\sqrt{n}} = \sqrt{n+1}-\sqrt{n}$).

c) Define what it means for a function $f\colon \mathrm{dom}(f) \to \mathbb{R}$ to be *sequentially continuous* at a point $x_0 \in \mathrm{dom}(f)$.

d) Suppose that for all $\epsilon > 0$ there exists $\delta > 0$ such that
$$|x - x_0| < \delta \text{ and } x \in \mathrm{dom}(f) \implies |f(x) - f(x_0)| < \epsilon$$
(that is, suppose f is continuous at x_0). Show that f is sequentially continuous at x_0.

Problem 2. In this question, you may use standard results about the convergence of series of the form $\sum \frac{1}{n^k}$ and of geometric series. You may also any other results concerning the convergence of series, as long as you state when you are using them. Prove the convergence or divergence of the following.

a)
$$\sum_{n=0}^{\infty} \frac{n^2}{n^3+1}.$$

b)
$$\sum_{n=1}^{\infty} \frac{n^{2013}}{2^n}$$

c)
$$\sum_{n=2}^{\infty} \frac{1}{\log n}$$

d)
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$$

Problem 3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 2x & \text{if } x \in \mathbb{Q} \\ x+1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Show that f is continuous at $x = 1$ not continuous at any x in $\mathbb{R} \setminus \{1\}$.

Problem 4.

a) Let $g\colon \mathbb{R} \to \mathbb{R}$ be continuous and suppose that $g(x) = 0$ for all $x \in \mathbb{Q}$. Show that $g(x) = 0$ for all $x \in \mathbb{R}$.

b) Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be continuous functions. Show that if $f(x) = g(x)$ for all $x \in \mathbb{Q}$ then $f(x) = g(x)$ for all $x \in \mathbb{R}$.

c) Suppose that g is continuous at 0 with $g(0) = 0$ and that $|f(x)| \leq |g(x)|$ for all x. Show that f is continuous at 0.

Problem 5.

a) State the *Extreme Value Theorem.*
b) Give an example of a function $f\colon [0,1] \to \mathbb{R}$ which is not bounded.
c) Give an example of a continuous function $f\colon (0,1) \to \mathbb{R}$ which is not bounded.
d) Let $f\colon [0,4] \cap \mathbb{Q} \to \mathbb{R}$ be given by $f(x) = |x^2 - 2|$. Is f bounded? Does there exist $x_0,\ y_0 \in [0,4] \cap \mathbb{Q}$ such that $f(x_0) \leq f(x) \leq f(y_0)$ for all $x \in [0,4] \cap \mathbb{Q}$?