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On Basic Concepts of Tropical Geometry

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Abstract—We introduce a binary operation over complex numbers that is a tropical analog of addition. This operation, together with the ordinary multiplication of complex numbers, satisfies axioms that generalize the standard field axioms. The algebraic geometry over a complex tropical hyperfield thus defined occupies an intermediate position between the classical complex algebraic geometry and tropical geometry. A deformation similar to the Litvinov–Maslov dequantization of real numbers leads to the degeneration of complex algebraic varieties into complex tropical varieties, whereas the amoeba of a complex tropical variety turns out to be the corresponding tropical variety. Similar tropical modifications with multivalued additions are constructed for other fields as well: for real numbers, p-adic numbers, and quaternions.

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1. INTRODUCTION

1.1. Tropical geometry. When one wishes to describe tropical geometry by a single phrase, one says that it is algebraic geometry over the semifield $\mathbb{T} = \mathbb{R}_{\max,+} \cup \{-\infty\}$. The elements of the semifield $\mathbb{T}$ are real numbers augmented with $-\infty$; the role of addition is played by the operation of taking the maximum of two numbers: $(a, b) \mapsto \max(a, b)$; and the role of multiplication is played by the ordinary addition of numbers. The role of zero is played by $-\infty$, while the role of unity, by $0 \in \mathbb{R}$. The standard properties of addition and multiplication of elements of a field are valid with the following exception: addition is completely noninvertible; i.e., for any $a \in \mathbb{R}$, there does not exist an $x \in \mathbb{R}$ such that $\max(a, x) = -\infty$. This implies the absence of subtraction. Instead, addition possesses the property of idempotency, $\max(a, a) = a$.

With this definition of tropical geometry, it may seem that the subject is exotic and remote from the central fields of mathematics. But this is a wrong impression. Tropical geometry is used for solving difficult classical problems of algebraic geometry over the fields of complex and real numbers. In fact, it stemmed from the solution of such problems. Tropical varieties appeared under different names in various mathematical contexts: Bergman's logarithmic limit sets [1], the Bieri–Groves sets [3], and Kapranov's non-Archimedean amoebas [12]. Tropical curves are closely related to combinatorial patchworking, a powerful method developed by the author [28, 11] for constructing real algebraic curves with controlled topology. Tropical curves are a key element of a powerful method developed by Mikhalkin [19] for calculating plane Gromov–Witten invariants.

Although some of the above-mentioned germs of tropical geometry arose long time ago (some of them can be traced back to Newton), as an independent subject it was realized only about nine years ago; the very term tropical geometry appeared about 2002. In spite of its early age, tropical geometry is well presented in the literature. Here are some surveys that give a rather comprehensive picture of various aspects of tropical geometry at different stages of its development: [25, 26, 9, 13, 20–22, 7]. Closely related to this subject are numerous aspects of algebraic geometry that

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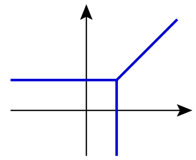

Fig. 1. Tropical line corresponding to the tropical linear form $\max(x, y, 1)$.

group around the concept of Newton polytope (see the monograph by Gelfand, Kapranov, and Zelevinsky [10]), Berkovich's studies on p-adic analytic spaces (see [2]), and studies on homological mirror symmetry in the style of Kontsevich and Soibelman [14].

1.2. Tropical varieties. The main objects of study in tropical geometry are quite simple. To give an idea of these objects, I will restrict myself to tropical hypersurfaces of the space $\mathbb{R}^n_{\max,+}$. Just as in the classical algebraic geometry, any affine hypersurface is defined by a single polynomial equation; however, the polynomial should naturally be tropical, i.e., over $\mathbb{R}_{\max,+}$.

A tropical polynomial is a tropical sum of several tropical monomials, i.e., the maximum of several tropical monomials. A tropical monomial is a tropical product (i.e., sum) of the coefficient and tropical powers of the variables. Raising to the kth power in the tropical sense means just multiplying by k. Thus, a tropical polynomial in n variables in terms of ordinary arithmetic operations is

$$p(x_1, \dots, x_n) = \max_{k=(k_1,\dots,k_n)} (a_k + k_1 x_1 + \dots + k_n x_n),$$

i.e., a convex piecewise linear function.

One could expect that a tropical hypersurface defined by the polynomial p is described by the equation $p(x_1, \dots, x_n) = -\infty$, because $-\infty$ plays the role of zero in the tropical semifield $\mathbb{T}$; however, this equation has no solutions, and one introduces the tropical hypersurface defined by the polynomial p as the set in which this polynomial is nondifferentiable as a function. In other words, a point belongs to the tropical hypersurface defined by the tropical polynomial $p(x_1, \dots, x_n) = \max_{k=(k_1,\dots,k_n)}(a_k + k_1 x_1 + \dots + k_n x_n)$ if the value of the polynomial at this point is equal to the values of at least two linear functions $a_k + k_1 x_1 + \dots + k_n x_n$ (i.e., the maximum is attained on more than one linear function).

For example, the tropical line defined by the tropical linear form $\max(x, y, 1)$ (which corresponds to the classical linear form $x + y + 1$) is shown in Fig. 1.

1.3. Dequantization. Relations between tropical geometry and other parts of mathematics are diverse; however, many of them are based on the same phenomenon. There is a continuous deformation that transforms the semifield $\mathbb{R}_{\geq 0}$ of nonnegative real numbers with ordinary operations of addition and multiplication into the tropical semifield $\mathbb{T}$ (see [15, 29]). This deformation is called the *Litvinov–Maslov dequantization* of real numbers.

Formally speaking, the Litvinov–Maslov dequantization is a family of semifields $\{T_h\}_{h\in[0,\infty)}$. As a set, T_h is $\mathbb{R}$ for any h. Binary operations $\oplus_h$ and $\odot_h$ in the semiring T_h are defined as follows:

$$a \oplus_h b = \begin{cases} h\ln(e^{a/h} + e^{b/h}) & \text{for } h > 0, \\ \max\{a, b\} & \text{for } h = 0, \end{cases} \tag{1}$$

$$a \odot_h b = a + b. \tag{2}$$

These operations depend *continuously* on h. For $h > 0$,

$$D_h\colon \mathbb{R}_{>0} \to T_h\colon x \mapsto h\ln x$$

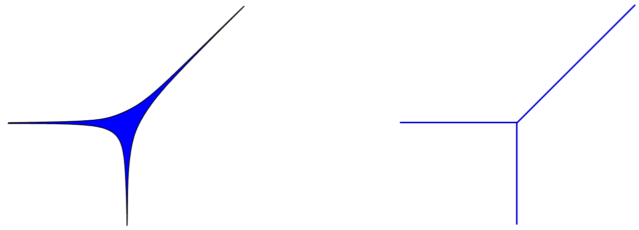

Fig. 2. The amoeba of a straight line (left) is contracted into a tropical line (right).

is an isomorphism of the semiring $\{\mathbb{R}_{>0}, +, \cdot\}$ on the semiring $\{T_h, \oplus_h, \odot_h\}$. Thus, for $h > 0$, the semiring T_h is a copy of the semiring $\mathbb{R}_{>0}$ with ordinary operations. The semiring T_0 is the tropical semiring $\mathbb{R}_{\max,+}$.

Any one-parameter family of objects in which all objects except one are mutually isomorphic, while this special object is in a sense degenerate, is considered as a kind of quantization of this degenerate object. In the case of the family T_h, this is even more justified because T_h was discovered in connection with quantum mechanics (see [15]). From the mathematical point of view, T_h is a continuous degeneration of the semiring $\mathbb{R}_{>0}$ into $\mathbb{R}_{\max,+}$. From the quantum point of view, T_0 is a *classical* object (the idempotent semiring $\mathbb{R}_{\max,+}$, which is not so classical in mathematics), whereas T_h for $h \neq 0$ are *quantum* objects (although very classical in mathematics), and the entire family T_h is a quantization of the semiring $\mathbb{R}_{\max,+}$. The Litvinov–Maslov dequantization continuously deforms the graph of a polynomial over T_h into the graph of the same polynomial over the tropical semiring.

The combination of the relative simplicity of tropical varieties with the possibility of their subsequent transformation, via the Litvinov–Maslov quantization, into complex and real algebraic varieties with preservation of many geometric properties allows one to prove the existence of algebraic varieties with interesting properties by means of tropical geometry.

However, the Litvinov–Maslov dequantization is applied to algebraic varieties over the field $\mathbb{C}$ somewhat indirectly. What is deformed and then degenerated into a tropical variety is not the complex variety itself but its *amoeba*, i.e., the image of a variety $V \subset (\mathbb{C} \setminus 0)^n$ under the map

$$\mathrm{Log}\colon\ (\mathbb{C} \setminus 0)^n \to \mathbb{R}^n\colon\ (z_1, \dots, z_n) \mapsto (\log|z_1|, \dots, \log|z_n|).$$

For example, the amoeba of a straight line is contracted into a tropical line (Fig. 2).

It is well known to the experts that in many cases an appropriate deformation can also be applied to the variety itself. Moreover, the limit objects have been analyzed in detail. In particular, to solve problems of enumerative geometry by tropical techniques, Mikhalkin [19] considered plane complex tropical curves as images of curves over the field of Puiseux series and as the limits of holomorphic curves under appropriate degeneration of the complex structure (see [19, Sect. 6]). Mikhalkin also considered complex tropical hypersurfaces (see [18, Sect. 6.3]). However, these hypersurfaces appeared as auxiliary objects and were not regarded as algebraic varieties over a field.

1.4. Complex tropical geometry. In this paper, we construct objects that fill the above-mentioned gap between the classical algebraic geometry over $\mathbb{C}$ and tropical geometry. We construct tropical degeneration of the field $\mathbb{C}$. This degeneration turns out to be somewhat exotic: the operation of addition in this field is multivalued. Nevertheless, it is largely similar to ordinary fields, and, in particular, there is algebraic geometry over it.

Nonsingular hypersurfaces in this geometry are topological manifolds; they can be obtained from complex algebraic hypersurfaces homeomorphic to them by degeneration similar to the Litvinov–Maslov dequantization, while their amoebas are tropical varieties.

1.5. Hyperfields. Tropical degeneration of the field $\mathbb{C}$ satisfies a system of axioms that is maximally close to the system of field axioms. All the differences are associated exclusively with the fact that addition is multivalued. Similar degenerations can be applied to addition in many other algebraic systems, for example, in the fields of real and p-adic numbers and in the skew field of quaternions. Here we touch not a single example but an unstudied phenomenon of quite general nature whose analysis, not to mention evaluation, goes beyond the scope of the present paper.

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2. ALGEBRA OF MULTIVALUED OPERATIONS

2.1. Multivalued maps. The symbol 2^X stands for the set of all subsets of a set X. A *multivalued map* of a set X into a set Y is nothing but a map $X \to 2^Y$ that is treated for some reason as a map $X \to Y$ that does not satisfy the requirement of being single-valued (according to which each element of the set X is mapped to *exactly one* element of the set Y).

Such a deviation from the standards of the language of modern mathematics is usually motivated by the desire to emphasize the analogy with the situations where the relevant map is single-valued. For example, in the present study we deal with a generalization of addition when a sum may be multivalued. The use of the modern set-theoretical terminology would obscure the analogy with ordinary addition beyond recognition; this fact forces us to employ the nontraditional terminology of multivalued maps.

A multivalued map f of a set X to a set Y is denoted by $f\colon X \multimap Y$.

Just as other deviations from set-theoretical standards, this implies a whole lot of changes in the conventional definitions and notations. Some changes are straightforward and do not lead to confusion. For example, $f(a)$ stands for a subset of Y that is the image of an element $a \in X$ under the corresponding map $X \to 2^Y$, whereas $f(A)$ for a subset $A \subset X$ denotes the subset $\bigcup_{x\in A} f(x)$ of the set Y rather than the subset $\{f(x)\colon x \in A\}$ of the set 2^Y.

In the same spirit, the composition of multivalued maps $f\colon X \multimap Y$ and $g\colon Y \multimap Z$ is the multivalued map $g \circ f\colon X \multimap Z$ that sends $a \in X$ to $g(f(a)) = \bigcup_{y\in f(a)} g(y)$.

Other changes are less obvious. For example, what is the preimage of a set $B \subset Y$ under a multivalued map $f\colon X \multimap Y$? Is this $\{a \in X\colon f(a) \subset B\}$ or $\{a \in X\colon f(a) \cap B \neq \varnothing\}$? Hence, the concept of preimage splits when passing from single-valued to multivalued maps. In the cases of such splitting, it is necessary to introduce new terms. The set $\{a \in X\colon f(a) \subset B\}$ is called the *upper preimage* of a set B under f and is denoted by $f^+(B)$, whereas the set $\{a \in X\colon f(a) \cap B \neq \varnothing\}$ is called the *lower preimage* of B under f and is denoted by $f^-(B)$. These terms are somewhat strange because $f^+(B) \subset f^-(B)$; i.e., the upper preimage is less than the lower preimage.

When we wish to take refuge in the standard set-theoretical terminology, we will pass from a multivalued map $f\colon X \multimap Y$ to the corresponding single-valued map $X \to 2^Y$. The latter will be denoted by $f^{\uparrow}$.

2.2. Multivalued binary operations. A *multivalued binary operation* in a set X is a multivalued map $X \times X \multimap X$ with nonempty values, i.e., any map $X \times X \to 2^X \setminus \{\varnothing\}$.

A binary multivalued operation $f\colon X \times X \multimap X$ is said to be *commutative* if $f(a,b) = f(b,a)$ for all $a, b \in X$.

A binary multivalued operation $f\colon X \times X \multimap X$ is said to be *associative* if $f(f(a,b),c) = f(a, f(b,c))$ for all $a, b, c \in X$. Naturally, by f in the last formula one should mean the natural extension of the operation f to all subsets of the set X, i.e.,

$$2^X \times 2^X \to 2^X\colon\ (A,B) \mapsto \bigcup_{a\in A,\ b\in B} f(a,b).$$

Let $Y \subset X$, and let $f\colon X \times X \multimap X$ be a multivalued binary operation. A multivalued binary operation $g\colon Y \times Y \multimap Y$ is said to be *induced* by the operation f if $g(a,b) = f(a,b) \cap Y$ for all $a, b \in Y$. The induced operation is completely determined by the original operation. It exists if and only if $f(a,b) \cap Y \neq \varnothing$ for any $a, b \in Y$ (recall that the definition of a multivalued binary operation prohibits the set $g(a,b)$ from being empty).

2.3. Multivalued groups. A set X with a *multivalued* operation $(a,b) \mapsto a \top b$ is called a (*commutative*) *hypergroup* if

(1) the operation $\top$ is associative and commutative;

(2) there is an element 0 in X such that $0 \top a = a$ for any $a \in X$;

(3) for each $a \in X$, there exists a unique element $-a \in X$ such that $0 \in a \top (-a)$.

This is a direct generalization of the notion of abelian group: a hypergroup in which the operation is single-valued (i.e., $a \top b$ consists of a single element for any a and b) is an abelian group.

Of course, nothing prevents one from considering noncommutative hypergroups as well; however, these hypergroups are not needed in the present study.

Here we use the symbol $\top$ (rather than $+$) because we will usually have to consider the operation $(a,b) \mapsto a \top b$ together with ordinary addition $(a,b) \mapsto a + b$.

2.4. The smallest hypergroup. In the set $\{0,1\}$, define an operation $\top$ by the formulas $0 \top 0 = 0$, $0 \top 1 = 1 = 1 \top 0$, and $1 \top 1 = \{0,1\}$. It is easily checked that this is a hypergroup. Following Marshall [16], we denote it by Q_1. This is a unique hypergroup consisting of two elements that is not a group.

2.5. Remarks on the history of the notion of hypergroup. Hypergroups appeared repeatedly in different contexts and sometimes under different names (such as *multigroup* or *polygroup*). The earliest publications [17, 30] on hypergroups that I could find date back to 1934. Some authors who defined them apparently did not know of their predecessors. I am grateful to A.M. Vershik, who helped me out of such confusion.

Often the terms multigroup and hypergroup denoted objects of wider classes. For example, Dresher and Ore [6] used the term multigroup in a much wider sense, while in their terminology what we call a hypergroup would be a regular multigroup reversible in itself with an absolute unit.

The definition given above seems to be the narrowest multivalued generalization of the notion of abelian group. In relatively recent publications, the same notion was considered by Comer [5] (he used the term *polygroup*) and Marshall [16].

There is another version of the notion of hypergroup in which the operation takes a fixed number of values (some of them may coincide with each other). Thus, the operation takes values in the nth symmetric power of the set rather than in the set of all of its subsets. This version of multigroups was considered by Wall [31] and, in our days, by Buchstaber and Rees [4].

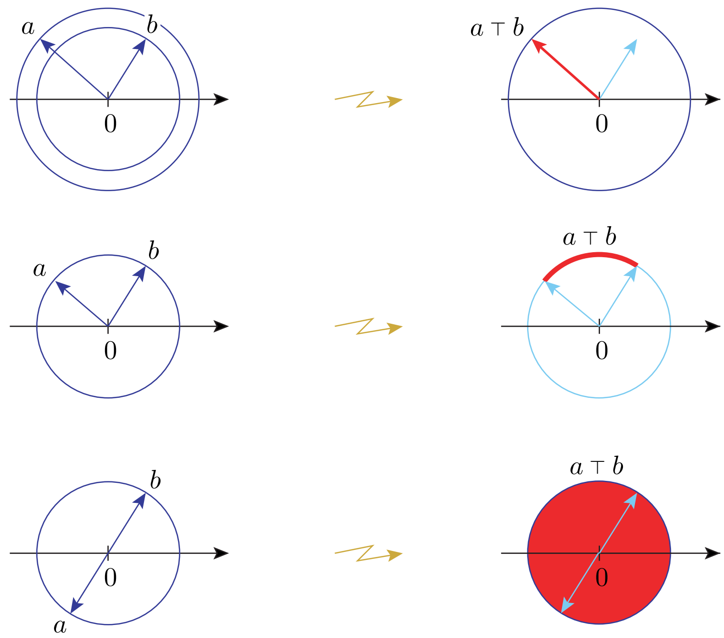


Fig. 3. Tropical addition of complex numbers.

2.6. Tropical addition of complex numbers. Let a and b be arbitrary complex numbers. Set (see Fig. 3)

$$a \top b = \begin{cases} \{a\} & \text{if } \ |a| > |b|, \\ \{b\} & \text{if } \ |a| < |b|, \\ \{|a|e^{\varphi i} \colon \varphi \in [\alpha, \beta]\} & \text{if } \ a = |a|e^{\alpha i}, \quad b = |a|e^{\beta i}, \quad \beta - \alpha < \pi, \\ \{c \in \mathbb{C} \colon |c| \le |a|\} & \text{if } \ a + b = 0. \end{cases}$$

We will call the set $a \top b$ the *tropical sum* of the numbers a and b.

Theorem 2.A. *The set of all complex numbers equipped with the tropical addition is a hypergroup.*

Proof. The commutativity of the tropical addition is obvious. Obviously, the neutral element is given by 0. For any complex number a, the only complex number b such that $0 \in a \top b$ is $-a$. Among all the axioms of the hypergroup, only the associativity requires real verification. Let us formulate it as a separate lemma. $\square$

Lemma 2.B. *Tropical addition of complex numbers is associative.*

Proof. Let us prove that $(a \top b) \top c = a \top (b \top c)$ for any complex numbers a, b, and c. The following list exhausts all types of triples of complex numbers:

(1) the absolute value of one of the numbers, say, a, is greater than the absolute values of the other two numbers: $|a| > |b|, |c|$;

(2) $|a| = |c| > |b|$;

(3) $|a| = |b| > |c|$ with

  (a) $a \neq -b$;

  (b) $a = -b$;

(4) $|a| = |b| = |c|$ with

  (a) $a + b \neq 0 \neq b + c$;

(b) either $a+b=0$ or $b+c=0$, but not both equalities simultaneously;

(c) $a+b=0=b+c$, but $a\neq 0$;

(d) $a=b=c=0$.

Let us prove associativity in each of these cases. Within the proof, it is convenient to introduce additional notations for the sets that arise as tropical sums. When the tropical sum of complex numbers a and b is an arc (i.e., $|a|=|b|$ and $a+b\neq 0$), we will denote this arc by $\frown(ab)$.

(1) In the first case (i.e., when $|a|>|b|,|c|$), the tropical sum is equal to a, the term with the maximum absolute value. Irrespective of the order of operations, this term dominates the other terms and turns out to be the final result. $\square$

(2) If $|a|>|b|$ and $|b|<|c|$, then $a\top b=a$ and $b\top c=c$. Therefore, $(a\top b)\top c=a\top c$ and $a\top(b\top c)=a\top c$. $\square$

(3a) Since $|a|=|b|$ and $a\neq -b$ in this case, we have $a\top b=\frown(ab)$; and since $|c|<|a|$, we have $c\top x=x$ for any x with $|x|=|a|$. Therefore, $(a\top b)\top c=(\frown(ab))\top c=\frown(ab)$. On the other hand, $a\top(b\top c)=a\top b=\frown(ab)$. $\square$

(3b) We have

$$
\begin{aligned}
(a\top -a)\top c &= \{x\colon |x|\le|a|\}\top c\\
&= \begin{pmatrix}\{x\colon |c|<|x|\le|a|\}\cup\\ \{x\colon |x|=|c|,\, x\neq -c\}\cup\\ \{-c\}\cup\\ \{x\colon |x|<|c|\}\end{pmatrix}\top c=\begin{pmatrix}\{x\colon |c|<|x|\le|a|\}\cup\\ \{x\colon |x|=|c|,\, x\neq -c\}\cup\\ \{x\colon |x|\le|c|\}\cup\\ \{c\}\end{pmatrix}=\{x\colon |x|\le|a|\}.
\end{aligned}
$$

On the other hand, $a\top(-a\top c)=a\top(-a)=\{x\colon |x|\le|a|\}$. $\square$

(4a) We have

$$
(a\top b)\top c=(\frown(ab))\top c=\begin{cases}\{x\colon |x|\le|a|\} & \text{if } -c\in(\frown(ab)),\\ (\frown(ac))\cup(\frown(bc)) & \text{if } -c\notin(\frown(ab)).\end{cases}
$$

On the other hand,

$$
a\top(b\top c)=a\top(\frown(bc))=\begin{cases}\{x\colon |x|\le|a|\} & \text{if } -a\in(\frown(bc)),\\ (\frown(ab))\cup(\frown(ac)) & \text{if } -a\notin(\frown(bc)).\end{cases}
$$

The conditions $-c\in(\frown(ab))$ and $-a\in(\frown(bc))$ are equivalent. Indeed, it is easily seen that each of these conditions is equivalent to the fact that the convex hull of the three-point set $\{a,b,c\}$ contains 0. If this condition is not satisfied, then $\{a,b,c\}$ is contained in a half of the circle $\{x\colon |x|=|a|\}$, and then $(\frown(ac))\cup(\frown(bc))=(\frown(ab))\cup(\frown(ac))$ is the shortest arc of the circle that contains a, b, and c, i.e., a kind of convex hull of the set $\{a,b,c\}$ in the semicircle. $\square$

(4b) If $|a|=|b|=|c|$ and $a+b=0$ but $b+c\neq 0$, then $(a\top b)\top c=\{x\colon |x|\le|a|\}\top c=(\{-c\}\cup\{x\colon x\neq -c,\, |x|\le|a|\})\top c=\{x\colon |x|\le|a|\}$. On the other hand, we have $a\top(-a\top c)=a\top(\frown(-a,c))=\{x\colon |x|\le|a|\}$. $\square$

(4c) If $|a|=|b|=|c|\neq 0$ and $a+b=0=b+c$, then $(a\top b)\top c=(a\top -a)\top a=a\top(-a\top a)=a\top(b\top c)$. $\square$

(4d) Does not need a proof. $\square$

Theorem 2.C. *Let $a_1,\dots,a_n$ be complex numbers with absolute values equal to r. Then*

- *either $a_1\top\dots\top a_n$ is the closed disk of radius r centered at 0 and is obtained as a sum of at most three terms among $a_1,\dots,a_n$ and $0\in\operatorname{Conv}(a_1,\dots,a_n)$,*

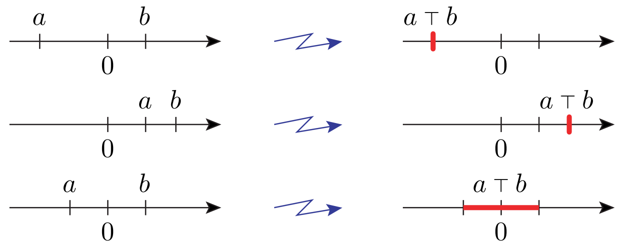


Fig. 4. Tropical addition of real numbers.

- *or* $a_1 \top \ldots \top a_n$ *is an arc which is contained in a semicircle of radius* r *centered at* 0 *and is a tropical sum of at most two terms among* $a_1, \ldots, a_n$.

Proof. For $n = 2$, this assertion is an immediate corollary to the definition of the tropical sum. Suppose that the assertion of the lemma is proved for any $n < k$ and prove it for $n = k$.

By the assumption, the tropical sum of the first $k-1$ terms is either the entire closed disk, and then $0 \in \operatorname{Conv}(a_1, \ldots, a_{k-1})$, or an arc of the circle that fits in a semicircle. In the first case, the tropical sum of all k terms is the same disk because $-a_k \in a_1 \top \ldots \top a_{k-1}$, and $0 \in \operatorname{Conv}(a_1, \ldots, a_k)$ because $0 \in \operatorname{Conv}(a_1, \ldots, a_{k-1})$.

In the second case, two mutually exclusive situations are possible: either $-a_k$ belongs to the arc $a_1 \top \ldots \top a_{k-1}$, and then $a_1 \top \ldots \top a_k$ is the disk, or $-a_k$ does not belong to the arc $a_1 \top \ldots \top a_{k-1}$.

In the first situation, the diameter of the disk connecting the points a_k and $-a_k$ divides the arc $a_1 \top \ldots \top a_{k-1}$ and, hence, also divides the chord connecting the endpoints of this arc. The center of the disk lies on the segment of this diameter between a_k and the chord subtending the arc $a_1 \top \ldots \top a_{k-1}$. By the induction hypothesis, the endpoints of this arc are some of the first $k-1$ terms. Hence, $0 \in \operatorname{Conv}(a_1, \ldots, a_k)$.

In the second situation, either a_k lies on the arc $a_1 \top \ldots \top a_{k-1}$, and then $a_1 \top \ldots \top a_k = a_1 \top \ldots \top a_{k-1}$ so that the second alternative holds, or a_k does not lie on the arc $a_1 \top \ldots \top a_{k-1}$, and then this arc lies on one side of the diameter connecting a_k with $-a_k$. In this case, the whole sum $a_1 \top \ldots \top a_k$ is an arc one of whose endpoints is a_k and the other is one of the endpoints of the arc $a_1 \top \ldots \top a_{k-1}$. □

Corollary 2.D. *The tropical sum of any finite set of complex numbers is equal to the tropical sum of a subset consisting of at most three terms. If the tropical sum does not contain zero, then the number of terms can be reduced to two.* □

Corollary 2.E. *The tropical sum of a finite number of complex numbers contains zero if and only if zero is contained in the convex hull of those terms that have the maximum absolute value.* □

2.7. Tropical addition of real numbers. Tropical addition of complex numbers induces a binary multivalued operation $a \top_{\mathbb{R}} b = (a \top b) \cap \mathbb{R}$ on $\mathbb{R}$. More explicitly, the set $a \top_{\mathbb{R}} b$ is described as follows (see also Fig. 4):

$$a \top_{\mathbb{R}} b = \begin{cases} \{a\} & \text{if } |a| > |b|, \\ \{b\} & \text{if } |a| < |b|, \\ \{a\} & \text{if } a = b, \\ [-|a|, |a|] & \text{if } a = -b. \end{cases}$$

The operation $(a, b) \mapsto a \top_{\mathbb{R}} b$ is called a *real tropical sum*, or simply a *tropical sum* if it is clear from the context that the real version is meant.

It is easy to check that the set $\mathbb{R}$ of real numbers equipped with tropical addition is a hypergroup.

2.8. Homomorphisms. Let X and Y be hypergroups. A map $f\colon X \to Y$ is called a (*hypergroup*) *homomorphism* if $f(a \top b) \subset f(a) \top f(b)$ for any $a, b \in X$.

Example. A non-Archimedean norm $K \to \mathbb{R}$ satisfies the ultrametric triangle inequality $|a+b| \le \max(|a|,|b|)$ for any $a,b \in K$. This means that it is a homomorphism of the additive group of K into the hypergroup $(\mathbb{R}, \top_{\mathbb{R}})$ defined in Subsection 2.7.

2.9. Rings and fields with multivalued addition. A set X equipped with a binary multivalued operation $\top$ and (single-valued) multiplication is called a *hyperring* if it is a commutative hypergroup with respect to $\top$ and the multiplication is associative and commutative, as well as distributive with respect to $\top$.

A multivalued ring X is called a *hyperfield* if $X \setminus 0$ is a group with respect to multiplication.

Examples of hyperfields. The hypergroup Q_1 introduced above in Subsection 2.4 turns into a hyperfield in a unique way. Recall that $Q_1 = \{0,1\}$ with the operation $\top$ defined by the formulas $0 \top 0 = 0$, $0 \top 1 = 1 = 1 \top 0$, and $1 \top 1 = \{0,1\}$. Multiplication in Q_1 is defined by the formulas $0 \cdot 0 = 0 \cdot 1 = 0$ and $1 \cdot 1 = 1$.

The hypergroups $(\mathbb{R}, \top_{\mathbb{R}})$ and $(\mathbb{C}, \top)$ with ordinary multiplication are hyperfields. When speaking of hyperfields $\mathbb{R}$ and $\mathbb{C}$, we will always bear in mind $\mathbb{R}$ and $\mathbb{C}$ with ordinary multiplication and with tropical addition $\top_{\mathbb{R}}$ and $\top$, respectively. Other similar examples are given in Appendix 1.

Complex conjugation $\mathbb{C} \to \mathbb{C}\colon z \mapsto \bar{z}$ is an automorphism of the hyperfield $\mathbb{C}$.

Triangle hyperfield. In the set $\mathbb{R}_{\ge 0}$ of nonnegative real numbers, define a multivalued addition $\triangledown$ by the formula

$$a \triangledown b = \{c \in \mathbb{R}_{\ge 0}\colon |a-b| \le c \le a+b\}.$$

In other words, $a \triangledown b$ is the set of all real numbers c such that there exists a triangle with side lengths a, b, and c.

Theorem 2.F. *The set $\mathbb{R}_{\ge 0}$ with multivalued addition $\triangledown$ and ordinary multiplication is a hyperfield.*

Proof. It is obvious that this addition is commutative. To show that it is associative, notice that both $(a \triangledown b) \triangledown c$ and $a \triangledown (b \triangledown c)$ are sets of real numbers x such that there exists a quadrangle with side lengths a, b, c, and x.

Ordinary multiplication is distributive with respect to $\triangledown$. The role of zero is played by 0. Finally, for any $a \in \mathbb{R}_{\ge 0}$, the only real number x such that $0 \in a \triangledown x$ is the number a itself. $\square$

We will call this hyperfield a *triangle hyperfield* and denote it by ∇.

Linear order hyperfield. Let X be a multiplicative group with a linear order $\prec$ such that if $a \prec b$, then $ac \prec bc$ for any $a,b,c \in X$. Let $Y = X \cup \{0\}$. We extend the order $\prec$ from X to Y, setting $0 \prec x$ for any $x \in X$. In Y, we define a multivalued addition $\curlyvee$,

$$(a,b) \mapsto a \curlyvee b = \begin{cases} \max(a,b) & \text{if } a \ne b, \\ \{x \in X\colon x \preceq a\} & \text{if } a = b. \end{cases}$$

It is easily seen that X with $\curlyvee$ is a hypergroup in which $-a = a$ for any $a \in X$. Let us extend multiplication from X to Y by setting $x0 = 0$ for any $x \in Y$. It is easily seen that Y with multivalued addition $\curlyvee$ and with this multiplication is a hyperfield.

In the case of the trivial group $X = \{1\}$, this construction yields Q_1.

Ultrametric triangle hyperfield. This construction, applied to the multiplicative group of positive real numbers equipped with ordinary order $<$, defines the structure of a hyperfield in $\mathbb{R}_{\ge 0}$. Recall that addition in this construction is defined by the formula

$$(a,b) \mapsto a \curlyvee b = \begin{cases} \max(a,b) & \text{if } a \ne b, \\ \{x \in \mathbb{R}_{\ge 0}\colon x \le a\} & \text{if } a = b, \end{cases}$$

and multiplication is the ordinary multiplication of real numbers.

This hyperfield can also be defined in a different way. To this end, we should replace the triangle inequalities in the construction of the triangle hyperfield by the non-Archimedean (i.e., ultrametric) triangle inequalities $|c| \le \max(|a|,|b|)$. We will call this hyperfield an *ultrametric triangle hyperfield* and denote it by $U\Delta$.

Tropical hyperfield. The map $\log\colon \mathbb{R}_{>0} \to \mathbb{R}$ extends naturally to a map $\mathbb{R}_{\ge 0} \to \mathbb{R} \cup \{-\infty\}$, which sends 0 to $-\infty$. Denote this map by the same symbol log. This is a bijection, and the structure of the ultrametric triangle hyperfield $U\Delta$ is translated by log to the set $\mathbb{R} \cup \{-\infty\}$. Denote the hyperfield obtained by $\mathbb{Y}$ and call it the *tropical hyperfield*.

The tropical hyperfield can also be obtained by the construction of a linear order hyperfield applied to the additive group of all real numbers with ordinary order $<$. The addition in this hyperfield differs from the addition $(a,b) \mapsto \max(a,b)$ in the semifield $\mathbb{T}$ only on the diagonal: $\max(a,a) = a \ne a \curlyvee a = \{x \in \mathbb{T}\colon x \le a\}$; note that $\max(a,a) \in a \curlyvee a$.

Example of a hyperring: *integer numbers.* In the set $\mathbb{Z}$ of integers, the tropical addition induced by the tropical addition in $\mathbb{C}$ turns $\mathbb{Z}$ into a hypergroup; combined with ordinary multiplication, it turns $\mathbb{Z}$ into a hyperring, but not into a hyperfield.

The notion of hyperfield is a direct generalization of the notion of field: a field is nothing but a hyperfield with single-valued addition. I could find only one publication [16] in which the notion of hyperfield was addressed.[1] Below, this notion arises naturally as the degeneration of a field under tropical deformation.

In the set $\mathbb{R}_{\ge 0}$ of nonnegative real numbers, tropical addition $\top$ induces the ordinary operation max of taking the maximum of two numbers. Note that $a \top_{\mathbb{R}} b \subset \mathbb{R}_{\ge 0}$ for any $a, b \in \mathbb{R}_{\ge 0}$. Thus, the semifield $\mathbb{R}_{\ge 0,\max,\times}$ arises as a subset of the hyperfield $\mathbb{R}$ that is closed with respect to both binary operations, and its binary operations are identical to the operations of the hyperfield $\mathbb{R}$. In particular, the inclusion $\mathbb{R}_{\ge 0,\max,\times} \to \mathbb{R}_{\top_{\mathbb{R}},\times}$ is a homomorphism.

Caution. There is a natural map in the opposite direction, $\mathbb{R} \to \mathbb{R}_{\ge 0}\colon x \mapsto |x|$. It is a right inverse of inclusion. However, it is not a homomorphism for $\top$. Indeed, $x \top (-x) = [-|x|, |x|]$ for any $x \in \mathbb{R}$; next, the map $x \mapsto |x|$ sends $[-|x|,|x|]$ to $[0,|x|]$, but $|x| \top |-x| = |x|$, which is different from $[-|x|,|x|]$ for $x \ne 0$.

The map $\mathbb{R} \to \mathbb{R}_{\ge 0}\colon x \mapsto |x|$ is a homomorphism of the hyperfield of real tropical numbers to the ultrametric triangle hyperfield. Moreover, the same map is a homomorphism of the field of real numbers to the triangle hyperfield.

The map $\mathbb{C} \to \mathbb{R}_{\ge 0}\colon x \mapsto |x|$ is a homomorphism of the hyperfield of complex tropical numbers to the ultrametric triangle hyperfield. Moreover, the same map is a homomorphism of the ordinary field of complex numbers to the triangle hyperfield.

2.10. Tropical complex numbers and polynomials. A map w that is defined and discussed in this and the next subsections was actually introduced by Mikhalkin [19] for his definition of complex tropical curves. However, the algebraic properties were left out of sight because the tropical addition was not discussed.

Let $p(X) \in \mathbb{C}[X]$ be a polynomial in one variable X with complex coefficients, $p(X) = \sum_{k=0}^{n} a_k X^k$, where $a_k \in \mathbb{C}$, $a_n \ne 0$. Set $w(p) = \frac{a_n}{|a_n|} e^n$. This defines a map $\mathbb{C}[X] \to \mathbb{C}\colon p \mapsto w(p)$.

Theorem 2.G. *The map w is a homomorphism of the polynomial ring $\mathbb{C}[X]$ to the hyperfield of tropical complex numbers $\mathbb{C}_{\top,\times}$; i.e., $w(p+q) \in w(p) \top w(q)$ and $w(pq) = w(p)w(q)$ for any $p, q \in \mathbb{C}[X]$.*

Proof. The fact that w is a multiplicative homomorphism is obvious. Indeed, the value of w on a polynomial is equal to its value on the leading term of this polynomial; the leading term of

[1] Hyperfields were introduced by Marc Krasner [32] in 1956.

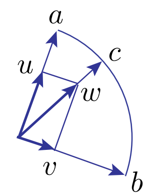


Fig. 5. Construction of $u, v, w \in \mathbb{C}$ such that $A = uX^r$, $B = vX^r$, and $C = wX^r$ with $r = \log|a|$ and $w(A) = a$, $w(B) = b$, and $w(C) = c$, $A + B = C$, where $a, b, c \in \mathbb{C}$ are given numbers.

the product of polynomials is the product of the leading terms of the factors; and the value of the map w on a monomial $p(X) = aX^n$ is calculated by the formula $\frac{p(e)}{|p(1)|}$, which obviously defines a multiplicative homomorphism.

Let us prove that $w(p+q) \in w(p) \top w(q)$ for any $p, q \in \mathbb{C}[X]$. If the degrees of the leading terms of the polynomials p and q are different, then the leading term of the polynomial $p+q$ is equal to the leading term of one of the polynomials p and q whose exponent is greater; this immediately implies that $w(p+q) = w(p) \top w(q)$ in this case.

If the degrees of the leading terms of the polynomials p and q are equal and these terms have nonopposite coefficients and therefore do not cancel each other out under addition, then the leading term of the polynomial $p+q$ is the sum of the leading terms of the polynomials p and q. Its exponent is equal to the exponents of the leading terms of the polynomials p and q, the coefficient is equal to the sum of the leading coefficients of the polynomials p and q, but the argument of its coefficient is not defined by the arguments of the leading coefficients of the polynomials p and q because the argument of the sum of two complex numbers is not defined by the arguments of the summands. It can take any value in the open interval between the arguments of the summands. In particular, it takes values in the set of arguments of numbers from $w(p) \top w(q)$.

If the degrees of the leading terms of the polynomials p and q are equal and these terms have opposite coefficients and therefore may cancel each other out under addition, then the leading term of the polynomial $p+q$ either is the sum of the leading terms of the polynomials p and q, or is obtained from lower degree terms and cannot be calculated by the leading terms. The only thing that can be said about this term based only on $w(p)$ and $w(q)$ (i.e., based on the arguments of the leading coefficients and their degrees) is that the degree of this term is not greater than the degrees of the summands; however, in the present case this implies that $w(p+q) \in w(p) \top w(q)$. $\square$

2.11. Tropical complex numbers and complex polynomials with real exponents. The image of the homomorphism w consists only of those complex numbers whose moduli are powers of the number e. However, a similar construction can also provide a map to the whole $\mathbb{C}$. To this end, it suffices to replace ordinary polynomials by polynomials with arbitrary real exponents, i.e., to start with the group algebra $\mathbb{C}[\mathbb{R}]$ of the additive group of real numbers rather than with $\mathbb{C}[X]$. An element of this algebra can be represented as $\sum_k a_k X^{r_k}$, where $a_k \in \mathbb{C}$ and $r_k \in \mathbb{R}$. The formal variable X indicates a transition from the additive notation for addition in $\mathbb{R}$ to the multiplicative notation in $\mathbb{C}[\mathbb{R}]$, where additive notation is reserved for a formal sum.

The elements of the algebra $\mathbb{C}[\mathbb{R}]$ can be interpreted as functions $\mathbb{C} \to \mathbb{C}$. To this end, replacing X by e^T, we transform $\sum_k a_k X^{r_k}$ into a sum of exponential functions $\sum_k a_k e^{r_k T}$.

The map $w\colon \mathbb{C}[X] \to \mathbb{C}$ extends to $\mathbb{C}[\mathbb{R}]$ in an obvious way: from the sum $\sum_k a_k X^{r_k}$, we take the term with the greatest exponent, say, $a_n X^{r_n}$, and apply the same formula $\frac{a_n}{|a_n|} e^{r_n}$ to it. The map obtained is an epimorphism of the ring $\mathbb{C}[\mathbb{R}]$ onto the hyperfield of tropical complex numbers $\mathbb{C}_{\top,\times}$. The proof of the fact that this is a homomorphism repeats word for word the proof of Theorem 2.G.

This construction illustrates how tropical addition of complex numbers is obtained from the ordinary addition of polynomials. Here it is quite clear why it should be multivalued. For complex numbers a and b with $|a| = |b|$, $a \neq -b$, and any c from the open arc $(a \top b) \setminus \{a, b\}$, there exist $A, B, C \in \mathbb{C}[\mathbb{R}]$ such that $w(A) = a$, $w(B) = b$, and $w(C) = c$ (see Fig. 5). Opposite complex numbers (i.e., numbers $a, b \in \mathbb{C}$ such that $a + b = 0$) are represented as the images under w of polynomials $A, B \in \mathbb{C}[\mathbb{R}]$ whose leading terms are mutually opposite and therefore cancel each other out under the addition of these polynomials. The leading term of the sum $A + B$ is by no means controlled by the leading terms of the summands A and B, except that its order is not greater than the order of the leading terms of the summands.

3. TROPICAL COMPLEX NUMBERS AS A RESULT OF DEQUANTIZATION

3.1. Subtropical deformation of the field of complex numbers. For any positive real h, let $S_h\colon \mathbb{C} \to \mathbb{C}$ be a map defined by the formula

$$z \mapsto \begin{cases} |z|^{\frac{1}{h}} \dfrac{z}{|z|} & \text{for } z \neq 0, \\ 0 & \text{for } z = 0. \end{cases}$$

This is an invertible map. The inverse map is defined by the formula

$$S_h^{-1}\colon\ z \mapsto \begin{cases} |z|^{h} \dfrac{z}{|z|} & \text{for } z \neq 0, \\ 0 & \text{for } z = 0. \end{cases}$$

Obviously, the map S_h is an isomorphism with respect to multiplication: $S_h(ab) = S_h(a)S_h(b)$. However, it does not commute with addition. To make the map S_h into an isomorphism with respect to addition as well, we redefine addition on the domain of definition of this map, i.e., induce a binary operation on the set of complex numbers:

$$a \oplus_h b = S_h^{-1}(S_h(a) + S_h(b)).$$

Thus, we obtain a field $\mathbb{C}_h = \mathbb{C}_{\oplus_h, \times}$ (which is nothing but a copy of the field $\mathbb{C}$) and an isomorphism $S_h\colon \mathbb{C}_h \to \mathbb{C}$.

Remark. A similar deformation of the complex torus $(\mathbb{C} \setminus 0)^n$ (rather than of the field $\mathbb{C}$) was used by Mikhalkin, in particular, to establish relations between tropical and complex algebraic geometries (see [19, Sect. 6]).

3.2. Limit of addition under subtropical deformation. One can easily verify that, as h tends to zero, $a \oplus_h b$ tends to a certain limit. Namely,

- if $|a| > |b|$, then $\lim_{h\to 0}(a \oplus_h b) = a$;
- if $|a| = |b|$ and $a + b \neq 0$, then $\lim_{h\to 0}(a \oplus_h b) = |a| \frac{a+b}{|a+b|}$;
- if $a + b = 0$, then $\lim_{h\to 0}(a \oplus_h b) = 0$.

Denote $\lim_{h\to 0}(a \oplus_h b)$ by $a \oplus_0 b$ (see Fig. 6).

The operation $(a, b) \mapsto a \oplus_0 b$ possesses a number of nice properties. It is commutative and distributive with respect to the ordinary multiplication of complex numbers, and zero behaves properly: $a \oplus_0 0 = a$ for any $a \in \mathbb{C}$. Finally, for any $a \in \mathbb{C}$, there exists a unique complex number b such that $a \oplus_0 b = 0$, and this b is nothing else than $-a$.

However, the operation $(a, b) \mapsto a \oplus_0 b$ is far from perfect: first, as a function of the variables a and b, it is not continuous, and, second, it is not associative. To verify the latter, let us compare $(-1 \oplus_0 i) \oplus_0 1$ and $-1 \oplus_0 (i \oplus_0 1)$:

$$(-1 \oplus_0 i) \oplus_0 1 = \Bigl(\exp(\pi i) \oplus_0 \exp\Bigl(\frac{\pi i}{2}\Bigr)\Bigr) \oplus_0 1 = \exp\Bigl(\frac{3\pi i}{4}\Bigr) \oplus_0 \exp(0) = \exp\Bigl(\frac{3\pi i}{8}\Bigr).$$

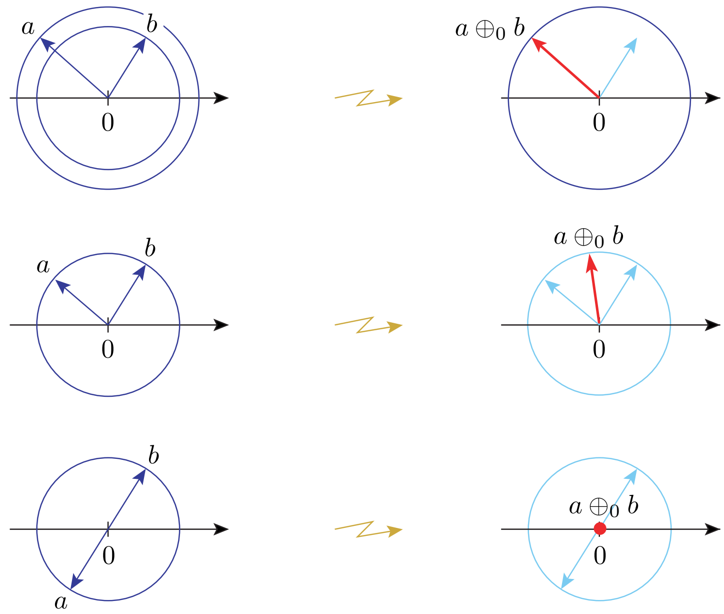


Fig. 6. Limit $a \oplus_0 b$ of sums $a \oplus_h b$ as $h \to 0$.

On the other hand,

$$-1 \oplus_0 (i \oplus_0 1) = \exp(\pi i) \oplus_0 \Big(\exp\Big(\frac{\pi i}{2}\Big) \oplus_0 \exp(0)\Big) = \exp(\pi i) \oplus_0 \exp\Big(\frac{\pi i}{4}\Big) = \exp\Big(\frac{5\pi i}{8}\Big).$$

The tropical addition introduced in Subsection 2.6 above is free of this drawback. It is associative (see Lemma 2.B). But it is multivalued. Among its advantages is a kind of continuity. Just as $\oplus_0$, it is a limit of the operations $\oplus_h$. However, it should be explained in what sense it is a limit and in what sense it is continuous, because the relevant definitions and theorems in general topology are not commonly known (see Section 4).

3.3. Subtropical deformation and asymptotic behavior. Under subtropical deformation, a point z moves along a straight line with exponentially varying speed:

$$z \mapsto \frac{z}{|z|}|z|^{\frac{1}{h}},$$

where h can be interpreted as time. This law of motion can be rewritten as

$$T \mapsto ae^{rT},$$

where $a = \frac{z}{|z|}$ and $e^{rT} = |z|^{\frac{1}{h}}$; i.e., $rT = \frac{1}{h}\log|z|$, $T = \frac{1}{h}$, and $r = \log|z|$. The motion is defined by two parameters: the complex number a, which defines the direction of motion, and the real parameter r, which defines the speed. The number z is recovered by these parameters as follows: $z = ae^r$.

From this viewpoint, the operation $\oplus_h$ looks as follows: starting with the numbers $z = ae^r$ and $w = be^s$, we construct ae^{rT} and be^{sT}, add them up, and seek c and t such that $ae^{rT} + be^{sT} = ce^{tT}$. Then $z \oplus_{\frac{1}{T}} w = ce^t$.

Since the parameters c and t define the motion and we consider this motion for large T, the matter concerns the asymptotic behavior of the curve $T \mapsto ae^{rT} + be^{sT}$ from the viewpoint of the comparison scale $T \mapsto ae^{rT}$. An answer to the question about the asymptotics of the sum is given by the operation $\oplus_0$. However, this operation is not continuous; hence, the answer is unstable, and it makes sense to consider what happens with the asymptotics of the sum under perturbations of the summands.

As a perturbation of the function $T \mapsto ae^{rT}$, we can consider a curve defined by an arbitrary exponential polynomial

$$T \mapsto \sum_k a_k e^{r_k T} \tag{3}$$

with arbitrary complex coefficients a_k and real r_k. In the comparison scale $T \mapsto ae^{rT}$, the asymptotic behavior of function (3) is determined by its leading term, i.e., by the term of sum (3) with the maximum r_k. Moreover, the role of the absolute value of the coefficient of the leading term is negligible if we do not try to improve our comparison scale. However, the asymptotic behavior of the sum is not always determined by the asymptotic behavior of the summands, and the set of asymptotics that arise under addition of functions with given asymptotics defines a binary operation on the set of asymptotics. Thus, we return to the material of Subsection 2.11 and see that, to speak of the asymptotics of the sum, we should turn to the homomorphism w.

4. TOPOLOGY OF MULTIVALUED MAPS

It is impossible to obtain a multivalued function as the limit of a family of single-valued functions $f_h\colon X \to Y$ while remaining in any space of *single-valued functions* $X \to Y$. However, this can be done by replacing functions by their graphs considered as subsets of the space $X \times Y$. To speak of the limits of graphs of a family of functions, one needs a topological structure in the space of subsets of the space $X \times Y$.

In the set of all subsets of a topological space, there exist different natural topological structures; however, there is no structure among them that would be satisfactory in all respects. The most classical among them are three topological structures introduced by Vietoris [27] in 1922. In our situation, none of these structures is suitable for passing to the limit, but a modification proposed by Fell [8] works and gives the graph of tropical addition. Tropical addition turns out to be continuous with respect to the upper topologies of Vietoris and Fell, and this provides important properties of tropical polynomial functions.

4.1. Vietoris topologies. The *upper Vietoris topology* in the set 2^X of all subsets of a topological space X is the topology generated by sets of the form $2^U \subset 2^X$, where U is open in X. A neighborhood of a set $A \subset X$ in the upper Vietoris topology should contain all subsets of some set U that is open in X and contains the set A.

This is a somewhat unusual topology. For example, it is far from being Hausdorff: in this topology, intersecting sets cannot have disjoint neighborhoods. Therefore, the limits in the upper Vietoris topology are, as a rule, nonunique. In particular, increasing the limit (i.e., adding new points to it), we obtain other limits. This fact is likely to be responsible for the word *upper* in the name of the topology.

There is also a *lower* Vietoris topology. The *lower Vietoris topology* in the set 2^X of all subsets of a topological space X is the topology generated by sets of the form $2^X \setminus 2^C$, where C is a closed subset of the space X. In other words, the lower Vietoris topology is generated by sets of the form $\{Y \subset X\colon Y \cap U \neq \varnothing\}$, where U is an open set of the space X. In the lower Vietoris topology, closed sets are formed from the closed sets of the original space most naturally; a closed set $C \subset X$ gives a set $2^C \subset 2^X$, which is closed in the lower Vietoris topology. Recall that open sets in the upper Vietoris topology are generated in exactly the same way by open sets of the original space. A neighborhood of a set $A \in 2^X$ in the lower Vietoris topology should contain all sets that intersect several open sets $U_1, \dots, U_n \subset X$ intersecting A. The limit in the lower Vietoris topology is not unique in general either, but for the opposite reason: it remains a limit under decrease (i.e., under the removal of points).

The topology induced by the upper and lower Vietoris topologies is called simply the *Vietoris topology*.

4.2. What is not good in the Vietoris topologies. Take the plane $\mathbb{R}^2$ as X. Consider a subset P of the space 2^X whose elements are straight lines parallel to the abscissa axis: $P = \{L_a \subset \mathbb{R}^2 \colon L_a = \{(x,y) \in \mathbb{R}^2 \colon y = a\}\}$. It is natural to expect that the space P with any reasonable topological structure is homeomorphic to a straight line.

However, the upper Vietoris topology of the space 2^X induces a discrete topology in P. Indeed, the straight line L_a on the plane has a neighborhood $\{(x,y) \colon |x(y-a)| < 1\}$, which contains only one horizontal line. The corresponding neighborhood of the straight line L_a in 2^X intersects P only at the point L_a. Hence, each point of the set P is open in the induced topology.

Since the Vietoris topology contains the upper Vietoris topology, it also induces a discrete topology in P. In fact, the graphs of the operations $\oplus_h$ do not converge to the graph of tropical addition in the upper Vietoris topology of the space $\mathbb{C}^3$ for the same reason: there is a too large supply of open sets. The graph of tropical addition of complex numbers can be represented as the intersection of all limits of the graphs of operations $\oplus_h$ as $h \to 0$ in the upper Vietoris topology. Another natural approach is to relax the topology. An appropriate natural relaxation, the upper Fell topology, is discussed in Appendix 2 of the present paper. However, the most important means to deal with the objects of complex tropical geometry are provided by the upper Vietoris topology.

It is easily seen that the lower Vietoris topology induces in P the desired topology of a straight line. However, it has a drawback (described in Theorem 4.A below) that is much more significant from the viewpoint of the present study.

4.3. Semicontinuity of tropical addition. A multivalued map $X \multimap Y$ is said to be

- *upper semicontinuous* if the corresponding map $f^{\uparrow} \colon X \to 2^Y$ is continuous with respect to the upper Vietoris topology in 2^Y;
- *lower semicontinuous* if $f^{\uparrow} \colon X \to 2^Y$ is continuous with respect to the lower Vietoris topology in 2^Y;
- *continuous* if $f^{\uparrow} \colon X \to 2^Y$ is continuous with respect to the Vietoris topology in 2^Y (i.e., $X \multimap Y$ is both upper and lower semicontinuous).

Recall that the set $\{a \in X \colon f(a) \subset B\}$ is called the *upper preimage* of a set B under f, and the set $\{a \in X \colon f(a) \cap B \neq \varnothing\}$ is called the *lower preimage* of a set B under f.

It is easily seen that $f \colon X \multimap Y$ is upper (respectively, lower) semicontinuous if and only if the upper (respectively, lower) preimage of any open set in Y is open in X.

Theorem 4.A. *Tropical addition* $\mathbb{C} \times \mathbb{C} \multimap \mathbb{C} \colon (a,b) \mapsto a \top b$ *is not lower semicontinuous* (*i.e., the corresponding map* $\mathbb{C} \times \mathbb{C} \to 2^{\mathbb{C}}$ *is not a continuous map with respect to the classical topology in* $\mathbb{C}^2$ *and the lower Vietoris topology in* $2^{\mathbb{C}}$).

Proof. To prove this theorem, it suffices to produce a set that is open in the lower Vietoris topology and whose preimage is not open in the classical topology of the space $\mathbb{C}^2$. Take, for example, the set H consisting of sets A that intersect the open disk of radius 1 centered at 0. Its preimage under our map is the set of pairs of complex numbers whose tropical sum intersects this disk. It is easily seen that this preimage consists of those pairs of complex numbers (a,b) that satisfy one of the two following conditions: either $|a| < 1$ and $|b| < 1$, or $a = -b$. It is clear that this set is not open. $\square$

Theorem 4.B. *Tropical addition* $\mathbb{C} \times \mathbb{C} \multimap \mathbb{C} \colon (a,b) \mapsto a \top b$ *is upper semicontinuous* (*i.e., the corresponding map* $\mathbb{C} \times \mathbb{C} \to 2^{\mathbb{C}}$ *is continuous with respect to the classical topology in* $\mathbb{C}^2$ *and the upper Vietoris topology in* $2^{\mathbb{C}}$).

Proof. Let us prove the local continuity; i.e., let us prove that for any neighborhood $V \subset 2^{\mathbb{C}}$ of the image $a \top b$ of a point (a,b), there exists a neighborhood $U \subset \mathbb{C}^2$ of the point (a,b) whose image is contained in V. In the upper Vietoris topology, the neighborhood basis of a set A consists of sets of all subsets of open sets $W \supset A$, so that indeed it suffices to find, for an arbitrarily close neighborhood $W \supset a \top b$, a neighborhood U of the point (a,b) in $\mathbb{C}^2$ such that $x \top y \subset W$ for any point $(x,y) \in U$.

Consider separately each of the three types of images of the point (a,b) under tropical addition.

If $|a| > |b|$, then $a \top b = a$. Any neighborhood of the set a contains an open disk with center at the point a. Let us reduce this disk, if necessary, so that its radius r becomes less than $\frac{1}{2}(|a|-|b|)$. Take, as W, an open disk $B_r(a)$ of radius r centered at the point a. Then, as U, we can take the neighborhood $B_r(a) \times B_r(b)$ of the point (a,b). It is clear that $B_r(a) \top B_r(b) \subset B_r(a)$.

If $|a| = |b|$ and $a+b \neq 0$, then $a \top b$ is the shortest arc C connecting a with b in the circle centered at zero. Let r be a positive real number so small that the disks $B_r(a)$ and $B_r(b)$ do not contain points symmetric to each other with respect to zero. Any neighborhood of the arc C in $\mathbb{C}$ contains $W = B_\rho(a) \top B_\rho(b)$ with some $\rho \in (0,r)$. Take the neighborhood $B_\rho(a) \times B_\rho(b)$ of the point (a,b) as U.

If $|a| = |b|$ and $a+b = 0$, then $a \top b$ is the closed disk of radius $|a|$ centered at zero. Any neighborhood of this disk in $\mathbb{C}$ contains a concentric open disk of some radius $r > |a|$. Take this disk as U. Its image under tropical addition is obviously equal to the disk itself. □

4.4. Properties of upper semicontinuous multivalued maps. According to Theorem 4.B, tropical addition is upper semicontinuous. Since we have to deal with this addition, we will need a few simple and well-known properties of upper semicontinuous maps.

First of all, note that for single-valued maps the concept of upper semicontinuity is equivalent to ordinary continuity.

Moreover, it is obvious that a composition of upper semicontinuous maps is upper semicontinuous.

These two assertions immediately imply that a multivalued function defined by a complex tropical polynomial (i.e., by a tropical sum of monomials with complex coefficients) is upper semicontinuous.

Theorem 4.C. *Let X and Y be topological spaces, $f\colon X \multimap Y$ be an upper semicontinuous multivalued map, and $C \subset Y$ be a closed set. Then the set $\{a \in X\colon f(a) \cap C \neq \varnothing\}$ is closed.*

Proof. The set $\{B \in 2^Y\colon B \subset Y \setminus C\}$ is open in the upper Vietoris topology of the space 2^Y. Since the multivalued map f is upper semicontinuous, the preimage $\{a \in X\colon f(a) \subset Y \setminus C\}$ of this set under the single-valued version $X \to 2^Y$ of the multivalued map f is open. Hence, the set $\{a \in X\colon f(a) \cap C \neq \varnothing\} = X \setminus \{a \in X\colon f(a) \subset X \setminus C\}$ is closed. □

Corollary 4.D. *For any complex or real tropical polynomial $\top_{k=(k_1,\dots,k_n)} a_k x_1^{k_1} \dots x_{k_1}^{k_n}$, the set defined by the condition $0 \in \top_{k=(k_1,\dots,k_n)} a_k x_1^{k_1} \dots x_{k_1}^{k_n}$ in $\mathbb{C}^n$ or $\mathbb{R}^n$, respectively, is closed.* □

Finally, I present two well-known theorems on upper semicontinuous maps without proof.

Theorem 4.E. *The image of a connected set under an upper semicontinuous map is connected if the images of points under this map are connected.* □

Theorem 4.F. *The image of a compact set under an upper semicontinuous map is compact if the images of points under this map are compact.* □

Corollary 4.G. *For any tropical polynomial $\top_{k=(k_1,\dots,k_n)} a_k x_1^{k_1} \dots x_{k_1}^{k_n}$, the multivalued map defined by it is upper semicontinuous. It sends connected sets into connected sets and compacts sets into compact sets. In particular, the graphs of tropical polynomials are connected.* □

Corollary 4.G applies to both complex and real tropical polynomials and to the multivalued maps $\mathbb{C}^n \multimap \mathbb{C}$ and $\mathbb{R}^n \multimap \mathbb{R}$ defined by them.

5. COMPLEX TROPICAL EQUATIONS AND VARIETIES

5.1. Irreducible similar terms. Since addition in a hyperfield is multivalued, habitual reflexes acquired at primary school may be misleading. We know that

$$x + 1 + (-1) = x.$$

Is this so in a hyperfield? Does the equality

$$x \top 1 \top (-1) = x$$

hold? The answer is "it depends on x!" In any case,

$$x \in x \top 1 \top (-1),$$

but if $|x| \leq 1$, then

$$x \top 1 \top (-1) = 1 \top (-1) = \{z \in \mathbb{C}\colon |z| \leq 1\}.$$

Compare the real graphs of the functions $y = x$ (Fig. 7a) and $y = x \top 1 \top (-1)$ (Fig. 7b).

Thus, the addition of similar terms, which is habitual from childhood, requires care and may generally cause errors in tropical complex and real algebras.

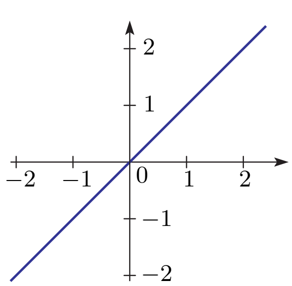
Now, consider an example showing that sometimes cancellation of similar terms is nevertheless possible. Is it true that

$$x^2 \top -1 = (x \top 1)(x \top -1)?$$

At first glance, not, because, opening the brackets on the right-hand side of this equality, we obtain $x^2 \top x \top (-x) \top (-1)$, and, to obtain the left-hand side, we would like to cancel the similar terms x and $-x$, which, as we know, is risky. However, the equality nevertheless holds for any x. Indeed, x^2 is a dominant term for $|x| \geq 1$, and then the terms x and $-x$ make no contribution to the sum, whereas for $|x| \leq 1$ the dominant term is -1, so the terms x and $-x$ for no values of x influence the sum and can be removed (Fig. 8).

A tropical polynomial is said to be *clean* if it has no monomials with equal exponents. A tropical polynomial is said to be *cleanable* if some of its monomials can be removed without changing the multivalued function defined by the polynomial so that the resulting polynomial is clean. For example, as we have seen, the polynomial $x^2 \top x \top (-x) \top (-1)$ is cleanable, and the result of cleaning is $x^2 \top (-1)$.

5.2. Equations over a hyperfield. Expressions involving multivalued addition define, as a rule, multivalued functions. How should one understand, say, the equation $f(x, y) = g(x, y)$ both sides of which contain such expressions? To solve such an equation means to find the values of



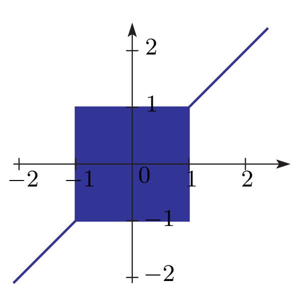


Fig. 7. Real graphs of the functions (a) $y = x$ and (b) $y = x \top 1 \top (-1)$.

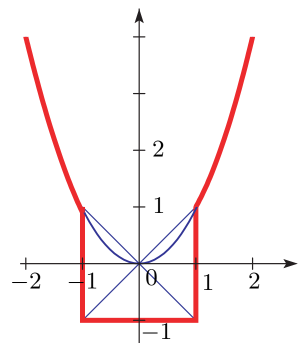


Fig. 8. Graph of the function $x^2 \top x \top (-x) \top (-1) = x^2 \top (-1)$.

unknowns for which the values of the multivalued functions $f(x,y)$ and $g(x,y)$ intersect, while the sets of all values do not necessarily coincide.

In particular, to solve the equation $f(x,y)=0$ with a multivalued function $f(x,y)$ on the left-hand side and zero on the right-hand side means to find values of unknowns for which $0 \in f(x,y)$.

Due to the last hypergroup axiom, the equations $f(x,y)=g(x,y)$ and $f(x,y) \top (-g(x,y))=0$ are equivalent, which agrees with experience in dealing with ordinary equations. However, as we have seen in the previous subsection, in the case of multivalued functions it is not legitimate to transform equations in a more general way by adding the same expression to both sides. Indeed, the equation $x=0$ has a unique solution, whereas the equation $x \top 1 = 1$ has many solutions: each x with $|x| \le 1$ is a solution. Equivalent are the equations $x \top 1 = 1$ and $x \top 1 \top (-1) = 0$.

As a rule, we will consider only equations of the form $f(x,y,\dots)=0$. As already mentioned, such an equation should be understood as $0 \in f(x,y,\dots)$. In view of Theorem 4.C, the set of solutions of such an equation with upper semicontinuous multivalued function f is closed.

5.3. Monomials. A tropical polynomial is a tropical sum of ordinary monomials. A monomial has no roots in the complex torus $(\mathbb{C}\setminus 0)^n$ and represents a single-valued function $(\mathbb{C}\setminus 0)^n \to \mathbb{C}\colon z \mapsto m(z) = az_1^{k_1}\dots z_n^{k_n}$. It is convenient to consider a monomial together with the function $\log|m(z)|$, which is factored through the map

$$\mathrm{Log}\colon\ (\mathbb{C}\setminus 0)^n \to \mathbb{R}^n\colon\ (z_1,\dots,z_n) \mapsto \bigl(\log|z_1|,\dots,\log|z_n|\bigr)$$

and yields a linear function

$$\mathrm{Log}\, m\colon\ \mathbb{R}^n \to \mathbb{R}\colon\ (x_1,\dots,x_n) \mapsto \log|a| + \sum_{i=1}^n k_i x_i.$$

On a fiber of the map Log, which is naturally identified with the torus $(S^1)^n$, a monomial defines a character $(z_1,\dots,z_n) \mapsto z_1^{k_1}\dots z_n^{k_n}$ multiplied by a constant (by the coefficient a).

5.4. Binomial equations. As was mentioned in Subsection 5.2 above, the binomial equation

$$m_1(z) \top m_2(z) = 0,$$

where $m_1(z)$ and $m_2(z)$ are monomials, is equivalent to the equation $m_1(z) = -m_2(z)$. In this case, tropical addition disappears and we deal with an ordinary equation over the field of complex numbers.

The equality $m_1(z) = -m_2(z)$ may only hold for those values of the variable z for which $|m_1(z)| = |m_2(z)|$. Hence, the tropical hypersurface defined by the equation $az_1^{p_1}\dots z_n^{p_n} \top$

$bz_1^{q_1} \dots z_n^{q_n} = 0$ is contained in the preimage under Log of the hyperplane

$$\log|a| + \sum_{i=1}^{n} p_i x_i = \log|b| + \sum_{i=1}^{n} q_i x_i.$$

Let us consider what happens in the fiber of the fibration $\mathrm{Log}\colon (\mathbb{C}\setminus 0)^n \to \mathbb{R}^n$ over a point of this hyperplane. The points of the fiber $\mathrm{Log}^{-1}(x)$ are represented as

$$z = (z_1, \dots, z_n) = (e^{x_1 + i\varphi_1}, \dots, e^{x_n + i\varphi_n}).$$

The original equation turns into a linear relation between two characters

$$az_1^{p_1} \dots z_n^{p_n} = -bz_1^{q_1} \dots z_n^{q_n}.$$

In the phase coordinates φ_k, this equation is linear:

$$\alpha + (p_1 - q_1)\varphi_1 + \dots + (p_n - q_n)\varphi_n = 0 \mod 2\pi, \tag{4}$$

where $-\frac{a/b}{|a|/|b|} = e^{i\alpha}$.

The difference $p - q$ of the vectors $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$ composed of the exponents in the monomials is a normal vector to the hyperplane defined by equation (4). If the vector $p - q$ is primitive, i.e., if its coordinates are relatively prime, then equation (4) defines a hypertorus (i.e., a subtorus of codimension 1) in the torus $\mathrm{Log}^{-1}(x_1, \dots, x_n)$. If $p - q = mv$, where m is a positive integer and v is a primitive integer vector, then equation (4) defines m parallel hypertori of the torus $\mathrm{Log}^{-1}(x_1, \dots, x_n)$.

The hypertori arising in this manner in different fibers are identified with each other by the natural trivialization of the fibration Log via the phase coordinates. Indeed, the coordinates of the point x are not involved in equation (4).

5.5. Principal part of a tropical polynomial at a point. Consider a clean tropical polynomial $p(z_1, \dots, z_n) = \top_{k=(k_1,\dots,k_n)\in I} a_k z_1^{k_1} \dots z_n^{k_n}$ with complex coefficients, and let $w = (w_1, \dots, w_n) \in \mathbb{C}^n$. Denote by $p_{(w)}$ the tropical polynomial composed of those monomials of the tropical polynomial p whose absolute values at the point w are maximal among the absolute values of all monomials of the tropical polynomial p. In formulas,

$$p_{(w)}(z) = \mathop{\top}_{k=(k_1,\dots,k_n)\in I_{(w)}} a_k z_1^{k_1} \dots z_n^{k_n},$$

where $I_{(w)} = \{k \in I\colon |a_k w_1^{k_1} \dots w_n^{k_n}| = \max_{j\in I} |a_j w_1^{j_1} \dots w_n^{j_n}|\}$.

It is obvious that $p(w) = p_{(w)}(w)$. Moreover, there exists a neighborhood U of w such that the multivalued functions $z \mapsto p(z)$ and $z \mapsto p_{(w)}(z)$ coincide on U. Therefore, in a local analysis of the hypersurface defined by the equation $p(z) = 0$, one can replace $p(z) = 0$ by a simpler equation $p_{(w)}(z) = 0$.

We call the number of monomials in $p_{(w)}$ the *size* of the tropical polynomial p at w, and call the Newton polytope of the tropical polynomial $p_{(w)}$ the *Newton polytope of p at w*. Denote the size of the tropical polynomial p at w by $r_p(w)$. It is clear that if $0 \in p(w)$, then $r_p(w) \geq 2$.

Note that the set of monomials appearing in $p_{(w)}$ depends only on the absolute values of the coordinates of the point w and therefore remains invariant within a fiber of the fibration Log.

5.6. Amoebas of complex tropical varieties. Recall that the *amoeba* of a variety $V \subset (\mathbb{C}\setminus 0)^n$ is the image of the variety V under the map

$$\mathrm{Log}\colon\ (\mathbb{C}\setminus 0)^n \to \mathbb{R}^n\colon\ (z_1,\dots,z_n)\mapsto (\log|z_1|,\dots,\log|z_n|).$$

It is clear that this concept applies not only to classical complex varieties, but also to sets defined by complex tropical equations.

By the *amoeba* of a clean complex tropical polynomial

$$p(z_1,\dots,z_n) = \mathop{\top}_{k=(k_1,\dots,k_n)\in I} a_k z_1^{k_1}\dots z_n^{k_n}$$

with complex coefficients we mean the $\mathbb{R}_{\max,+}$-polynomial

$$\max\{\log|a_k| + k_1x_1 + \ldots + k_nx_n\colon k\in I\}.$$

Denote this $\mathbb{R}_{\max,+}$-polynomial by $\mathrm{Log}\,p$. (In the particular case of monomials, this notation was introduced in Subsection 5.3 above.)

Recall that a tropical hypersurface defined by an $\mathbb{R}_{\max,+}$-polynomial $\max\{b_k+k_1x_1+\ldots+k_nx_n\colon k\in I\}$ is the set of points at which at least two of the linear functions $b_k + k_1x_1 + \ldots + k_nx_n$ are equal to $\max\{b_k + k_1x_1 + \ldots + k_nx_n\colon k\in I\}$.

Denote by V_p the complex tropical hypersurface $\{z\in(\mathbb{C}\setminus 0)^n\colon p(z)=0\}$ defined by a complex tropical polynomial p.

Theorem 5.A. *Let*

$$p(z) = \mathop{\top}_{k=(k_1,\dots,k_n)\in I} a_k z_1^{k_1}\dots z_n^{k_n}$$

be a clean tropical polynomial with complex coefficients. Then the amoeba $\mathrm{Log}(V_p)$ *of the complex tropical hypersurface* V_p *is a tropical hypersurface defined by the amoeba* $\mathrm{Log}\,p$ *of the tropical complex polynomial* p.

Proof. It is clear that $\mathrm{Log}(V_p)$ is contained in the tropical hypersurface defined by the amoeba $\mathrm{Log}\,p$ of the tropical complex polynomial p. Indeed, at the points of the space $(\mathbb{C}\setminus 0)^n$ that are not mapped by Log onto the tropical hypersurface T defined by the $\mathbb{R}_{\max,+}$-polynomial $\mathrm{Log}\,p$, the size of the tropical polynomial p is 1 (the absolute value of one of the monomials in p is greater than the absolute values of other monomials); therefore, the tropical sum of all monomials cannot contain zero.

Let us prove that each point of the tropical hypersurface T belongs to $\mathrm{Log}(V_p)$. Let $x\in T$ and $S=\mathrm{Log}^{-1}(x)\subset(\mathbb{C}\setminus 0)^n$. Take two monomials in the principal part of p at the points of the fiber S. This is possible because the size of the tropical polynomial p on $\mathrm{Log}^{-1}(T)$ is greater than 1. As we have seen in Subsection 5.2, there is a point in S at which the sum of these monomials vanishes. Then the tropical sum of all monomials contains zero. □

5.7. Roots of complex tropical polynomials in one variable. Let $p(z) = \top_{k=0}^{n} a_k z^k$ be a clean complex tropical polynomial in one variable, and let w be a complex number at which the size of the tropical polynomial p is greater than 1. Let us see what may be the set of solutions to the equation $p(z)=0$ in the circle $C=\mathrm{Log}^{-1}\mathrm{Log}(w)$.

Without loss of generality, we may assume that $p=p_w$. If the number of monomials in this tropical polynomial is two, then the answer to the question immediately follows from the results of Subsection 5.4: the number of roots of the equation $p(z)=0$ in C is equal to the difference of the exponents of the monomials, and the roots themselves are uniformly distributed on the circle C.

If the number of monomials is greater than two, then roots always exist. Indeed, the tropical sum of two monomials with the same absolute values on the fiber C of the Log-fibration necessarily

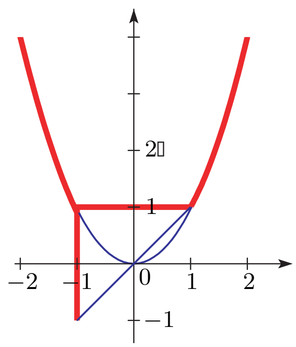


Fig. 9. Graph of the function $1 \top x \top x^2$.

has a root (cf. the proof of Theorem 5.A), and if the tropical sum of two monomials contains zero, then the tropical addition of new monomials with the same absolute value to this sum does not change the sum.

However, here we observe a new phenomenon: as a rule, the sum of more than two terms has infinitely many roots on such a fiber, and the set of roots has a nonempty interior in C. I know only one (up to simple transformations) exception to this rule: the tropical polynomial $1 \top z \top -z^2$ on the circle $|z| = 1$ has only two roots ($z = \pm 1$).

Infinite sets of roots with nonempty interior in C arise as follows. If three monomials have equal absolute values on C at some point $z_0 \in C$ and the interior of the convex hull of their values contains zero, then the tropical sum of these monomials also contains zero (see Corollary 2.E) and, in some neighborhood of the point z_0 on C, in view of the continuity of the monomials, zero still belongs to the convex hull of the values of the monomials and to the tropical sum of these three monomials.

For example, the tropical polynomial $1 \top z \top z^2$ on the circle $|z| = 1$ has zero value at all points with $\operatorname{Re} z \le 0$. These are, however, all the roots of this complex tropical polynomial. Figure 9 shows the real graph of this polynomial. It is connected (as it should be according to Theorem 4.G), intersects the abscissa axis at a single point (the other roots are imaginary), but is anomalous in another sense: it is not a one-dimensional manifold.

Thus, the structure of roots of a complex tropical polynomial in one variable may radically differ from that of ordinary complex polynomials. However, complex tropical polynomials in one variable that exhibit this phenomenon, i.e., those with the size at some point greater than 2, are quite degenerate. In a nondegenerate situation, the size of a complex tropical polynomial in one variable at a point is not greater than 2, and then the number of its roots is equal to the degree of the polynomial, just as in the classical case.

Note also that real tropical polynomials in one variable have a finite number of roots because the real analog of the circle $|z| = \text{const}$ consists of two points.

5.8. Complex tropical line. Consider the trinomial equation

$$z \top w \top 1 = 0. \qquad (5)$$

According to Theorem 5.A, the amoeba of the set defined by equation (5) is a tropical line (see Fig. 1).

Over the internal points of the rays, the principal part of the tropical polynomial $z \top w \top 1$ consists of two terms: over the points of the horizontal ray, these are $w \top 1$; over the points of the vertical ray, these are $z \top 1$; and over the points of the oblique ray, these are $z \top w$. The intersection of the torus $\operatorname{Log}^{-1}(x, y)$ with the complex tropical curve over the points of these rays is described by the equations $\psi = \pi$, $\varphi = \pi$, and $\varphi = \psi \pm \pi$, respectively, in the coordinates $\varphi = -i \log \frac{z}{|z|}$ and $\psi = -i \log \frac{w}{|w|}$. Thus, over every point of the horizontal ray, we have the parallel $\psi = \pi$ of the

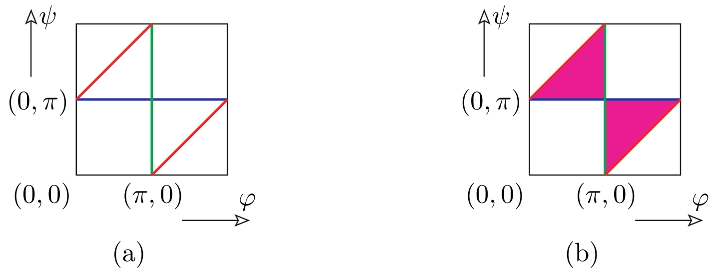


Fig. 10. Sets of points corresponding to equation (5) in the square parameterizing the torus.

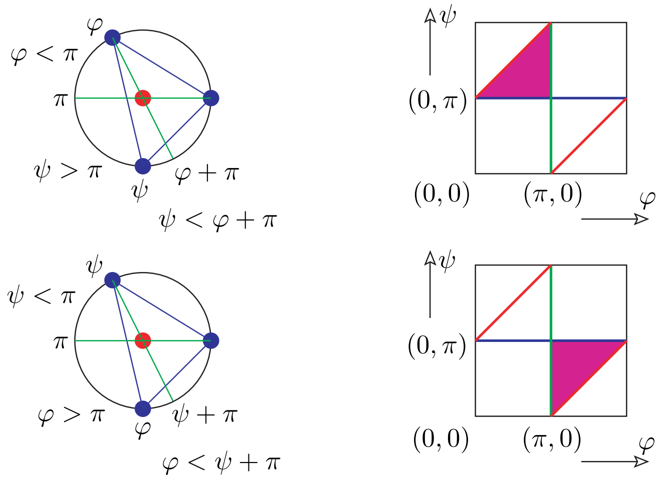


Fig. 11. Illustration of how Fig. 10b is obtained.

torus; over every point of the vertical ray, we have the meridian $\varphi = \pi$; and over every point of the oblique ray, we have the diagonal circle $\varphi = \psi \pm \pi$. In the square parameterizing the torus, these circles look as shown in Fig. 10a. Over the common initial point of all the three rays, equation (5) defines a domain consisting of two triangles (Fig. 10b).

Figure 11 provides a proof of the fact that Fig. 10b indeed represents a part of our tropical line situated over the initial point of the three rays.

It can easily be verified that the union of the three cylinders projected onto the rays with these two triangles makes up a pair of pants (i.e., a sphere with three holes).

We have studied the part of the complex tropical line $z \top w \top 1 = 0$ available for analysis by means of its amoeba. This is the part contained in the complex torus $(\mathbb{C} \setminus 0)^2$, on which the map Log is defined. However, the equation $z \top w \top 1 = 0$ makes sense on the entire complex plane $\mathbb{C}^2$. On the coordinate axes $z = 0$ and $w = 0$, it defines one point on each: on the axis $z = 0$, this is the point $(0, 1)$, and on the axis $w = 0$, this is $(1, 0)$. Adding these two points to the pants transforms the latter into a plane (the points fill two holes, i.e., sew up the pants legs).

We can proceed further and consider a complex tropical projective line. To this end, just as in classical geometry, we should pass to projective homogeneous coordinates $(z_0 : z_1 : z_2)$ and to the homogeneous version $z_1 \top z_2 \top z_0 = 0$ of the equation $z \top w \top 1 = 0$. This procedure adds the last point to our line, thus transforming it into a sphere.

Note that the topology of all three types of complex tropical lines—torus line in $(\mathbb{C} \setminus 0)^2$, affine in $\mathbb{C}^2$, and projective in $\mathbb{C}\mathrm{P}^2$—does not differ at all from the topology of the respective types of the classical complex straight line. However, the geometry is different: the tropical line is nonsmooth; it has corners.

5.9. Classical spaces in complex tropical geometry. Up to the end of the previous subsection, we considered the zero sets of complex tropical polynomials either in the affine space $\mathbb{C}^n$ or in the complex torus $(\mathbb{C}\setminus 0)^n$. However, at the end of the previous subsection, the complex projective tropical line appeared.

Tropical polynomials may be homogeneous, i.e., composed of monomials of the same degree. When all variables are multiplied by the same number, the value of the tropical sum of monomials of the same degree is multiplied by a power of this number, and if the value of the tropical sum contains zero before multiplication, then it will also contain zero after multiplication. Thus, the equation $p(z_0,\dots,z_n)=0$, where p is a tropical homogeneous polynomial, defines a set in $\mathbb{C}\mathrm{P}^n$. This set is naturally called a *tropical complex projective hypersurface*. The complex tropical projective line on the plane that appeared in the previous subsection is one of the simplest examples.

Complex projective spaces belong to a wider class of complex algebraic varieties, namely, the class of complex toric varieties, on which tropical polynomials may yield equations. The point is that a toric variety can be defined by an atlas of local coordinate systems and the definition of transition functions between them does not require the addition operation: these functions are purely monomial. Therefore, a tropical polynomial defined in one of these local coordinate systems naturally remains a tropical polynomial under transition to another system. Each of its constituent monomials is just transformed separately.

Although the sets of points of complex toric varieties remain the same as in the classical complex algebraic geometry, the structure sheaf is replaced by a totally different one, because the role of addition in tropical geometry is played by tropical addition, and the role of polynomials, by tropical polynomials.

The supply of main spaces is also changed under transition from the classical complex to the complex tropical geometry. For example, it includes the complex tropical line considered above.

Complex toric varieties are convenient from the tropical point of view for one more reason: they are covered by complex tori, and in each complex torus the map Log is defined.

5.10. Complex tropical hyperplane. Consider a direct generalization of the complex tropical line, namely, the set H defined by the equation

$$z_0 \top z_1 \top \dots \top z_n = 0 \tag{6}$$

in the complex projective space $\mathbb{C}\mathrm{P}^n$. Our primary interest lies in the local structure and the topology of this set.

Theorem 5.B. *The subspace H of the space $\mathbb{C}\mathrm{P}^n$ is a topological manifold of dimension $2n-2$.*

I am going to publish a detailed proof of this theorem in a separate article. Here we restrict ourselves to a few remarks on the structure of the set H.

At every point $w=(w_0\colon w_1\colon \dots \colon w_n)\in\mathbb{C}\mathrm{P}^n$, the tropical polynomial $p(z)=z_0\top z_1\top\dots\top z_n$ has a principal part $p_{(w)}(z)=z_{k_1}\top z_{k_2}\top\dots\top z_{k_m}$ and size m. Recall that the principal part of a tropical polynomial at a point w is the tropical sum of those of its monomials that have the maximum absolute value at w among all monomials of this tropical polynomial. The size of a tropical polynomial at w is the number of monomials of its principal part at this point.

At every point $w\in H$, the principal part $p_{(w)}$ of the tropical polynomial p vanishes; by Theorem 2.D, this implies that there are at most three monomials in $p_{(w)}$ whose tropical sum is zero. Denote by $s(w)$ the minimum number of such monomials. For every $w\in H$, this number is either 2 or 3. For $w\in H$, set $d(w)=r_p(w)-s(w)$. Denote by H_d the set of points $w\in H$ at which $d(w)=d$.

Lemma 5.C. *The set H_0 is a smooth subvariety of the space $\mathbb{C}\mathrm{P}^n$ of dimension $2n-2$; it is open and dense in H.*

Proof. Let $w \in H_0$. If $s(w) = 2$, then $w_i = -w_j$ for some i and j and $|w_k| < |w_i|$ for any $k \neq i, j$. Then, in some neighborhood of the point w in H, the equation $z_i = -z_j$ defines H, and this neighborhood is contained in H_0. We see that in the neighborhood of such a point, $H = H_0$ is an open set of the complex nontropical hyperplane.

If $s(w) = 3$, then $|w_i| = |w_j| = |w_k|$ and $0 \in \operatorname{Int}\operatorname{Conv}(w_i, w_j, w_k)$ for some i, j, and k and $|w_m| < |w_k|$ for any $m \neq i, j, k$. Then, in some neighborhood of the point w in H, the equations $|z_i| = |z_j| = |z_k|$ define H, and this neighborhood is contained in H_0. We see that in the neighborhood of such a point, $H = H_0$ is an open set in a real smooth algebraic subvariety of real codimension 2.

Now, let w be an arbitrary point of the set H. At this point we have $p(w) = 0$. Hence, according to Theorem 2.D, there are at most three monomials in $p_{(w)}$ whose tropical sum contains zero. Each monomial in p is one of the coordinates. In an arbitrarily small neighborhood of the point w, there exists a point at which $s(w)$ coordinates from $p_{(w)}$ have zero tropical sum (these are the same coordinates as those of w with zero tropical sum), while the other coordinates have slightly smaller absolute values. This point belongs to H_0. This proves that H_0 is dense in H. $\square$

The set H_0 is partitioned into disjoint open sets in each of which the polynomial p has a fixed principal part. We will call these open sets the *principal strata* of the set H. According to Lemma 5.C, the closures of the principal strata cover the whole H.

The closures of principal strata have a very simple structure. These are semialgebraic sets without singularities; more precisely, they are smooth varieties with boundary and corners along the boundary. They are locally diffeomorphic to the product of several copies of the line $\mathbb{R}$ and the half-line $\mathbb{R}_{\geq 0}$. At a boundary point, the set of tangent vectors directed inside the stratum is a cone isomorphic to such a product. This can easily be shown with the use of nearly the same arguments as those in the proof of Lemma 5.C; however, we first make a change of scenery because this will also be useful for the subsequent local analysis of the whole set H.

Note that the principal part of equation (6) at any point looks like the whole equation but contains a smaller number of coordinates. In the neighborhood of a point, the complex tropical hyperplane is defined by the principal part of the equation and is therefore locally homeomorphic to the product of the neighborhood of the point of the complex tropical hyperplane in the space of lower dimension by a complex affine space of appropriate dimension. Therefore, when studying the local structure of the complex tropical hyperplane and its strata, one can restrict oneself to the points at which the size of the equation is maximal, i.e., is $n+1$ in the case of the complex tropical hyperplane of the space $\mathbb{C}\mathrm{P}^n$. The points at which the size of the polynomial p is $n+1$ fill the torus T defined by the equation $|z_0| = |z_1| = \ldots = |z_n|$. (Of course, not all points of the torus T belong to H.)

So, let $w \in H$ and $p_{(w)} = p$. Then definitely $w \notin H_0$, but $w \in \operatorname{Cl} H_0$. The homogeneous coordinates $w_0, w_1, \ldots, w_n$ of the point w have the same absolute values. It will be convenient to interpret them as a configuration of points on a circle. Denote this configuration by W. In terms of this configuration we will describe both the strata of the set H that are adjacent to w and the cones in the tangent space that are composed of vectors directed inside these strata.

First of all, we list the principal strata that are adjacent to w. They are of two types: with $s = 3$ and with $s = 2$.

The principal strata of the set H that are adjacent to w and on which $s = 3$ are in one-to-one correspondence with triples of points w_i, w_j, w_k such that $0 \in \operatorname{Conv}(w_i, w_j, w_k)$. The close (to w) points z of the closure of the principal stratum $H_0^{i,j,k}$ corresponding to the triple w_i, w_j, w_k have homogeneous coordinates that are close to the corresponding homogeneous coordinates of the point w and satisfy the following conditions: $|z_i| = |z_j| = |z_k| = |w_i|$ and $|z_m| \leq |w_i|$ for $m \neq i, j, k$. The tangent vectors directed inside this stratum form a convex cone affinely isomorphic to the product of the space $\mathbb{R}^n$ by a cone over an $(n-3)$-simplex.

The principal strata of the set H that are adjacent to w and on which $s = 2$ are in one-to-one correspondence with pairs of points w_i, w_j such that $w_i = -w_j$. The close (to w) points z of the closure of the principal stratum $H_0^{i,j}$ corresponding to the pair w_i, w_j have homogeneous coordinates that are close to the corresponding homogeneous coordinates of the point w and satisfy the following conditions: $z_i = -z_j$ and $|z_m| \le |z_i|$ for all m. The tangent vectors directed inside this stratum form a convex cone that is affinely isomorphic to the product of the space $\mathbb{R}^{n-1}$ by a cone over an $(n-2)$-simplex.

Note that the strata $H_0^{i,j,k}$ are preserved under a small perturbation of the point w, whereas the strata $H_0^{i,j}$ arise for exceptional w, namely, when the pair of points w_i, w_j consists of antipodes. The strata of positive codimension admit a similar description.

The number of triangles that enclose zero and have vertices in the set W (and, hence, the number of cones that cover H locally) depends on the configuration of the points $w_0, w_1, \dots, w_n$ on the circle. The simplest configuration appears when there is a diameter such that two of these points, say, w_i and w_j, are located on one side of this diameter close to its opposite endpoints, while the other points are located on the other side of the diameter and not very close to its endpoints. Such a configuration arises when a point configuration on the circle whose convex hull does not contain zero is deformed and one of the sides of the convex hull passes through zero.

Then any triangle with vertices in W that encloses zero has two vertices at w_i and w_j, whereas the third vertex can be any of the remaining $n-1$ points. Any two such triangles have a common side $[w_i w_j]$, so the corresponding cones intersect along a face of highest dimension. On the whole, all the cones constitute a structure isomorphic to the structure of faces of an $(n-2)$-simplex. The point under consideration enters the boundaries of $n-1$ strata $H_0^{i,j,k}$. Each of these strata is locally a cone over an $(n-3)$-simplex multiplied by $\mathbb{R}^n$, and their union is a cone over the boundary of an $(n-2)$-simplex multiplied by $\mathbb{R}^n$. Thus, at the points of the type under consideration, the set H is locally Euclidean.

To study the structure of the set at other points, it suffices to analyze how its local stratification changes under the motion of the point w. It is easily seen, for example, that when one of the points w_i passes through $-w_j$, a Pachner transformation [23] is applied to the partition of the tangent cone to H at the point w into principal strata. At the instant of passage, there arises a stratum of type $H_0^{i,j}$, which immediately disappears. However, the topological type of the tangent cone to H at the point w is preserved. Similar arguments concerning deeper degenerations of the configuration W prove Theorem 5.B.

5.11. Nonsingular complex tropical hypersurfaces. A clean tropical complex polynomial in n variables is said to be *nonsingular* if, at every point of the space $(\mathbb{C} \setminus 0)^n$, the Newton polytope of its principal part is a primitive simplex of dimension $\le n$. A simplex with vertices at points with integer coordinates is said to be *primitive* if the vectors connecting one of its vertices with the other vertices form a basis of the intersection of the integer lattice with the minimal affine space that contains this simplex.

Theorem 5.D. *A set X defined in $(\mathbb{C} \setminus 0)^n$ by a nonsingular tropical complex polynomial p is a topological manifold of dimension $2n-2$.*

Reduction to a linear equation. Let $w \in (\mathbb{C} \setminus 0)^n$ be a point of the set X, and let

$$p_{(w)}(z) = \mu_0(z) \top \mu_1(z) \top \ldots \top \mu_m(z)$$

be the principal part of the tropical polynomial p at w. Since p is nonsingular, the ratios of monomials $u_i = \frac{\mu_i(z)}{\mu_0(z)}$ with $i = 1, \dots, m$ define a submersion $(\mathbb{C} \setminus 0)^n \to (\mathbb{C} \setminus 0)^m$. Let us extend this submersion to a diffeomorphism $(\mathbb{C} \setminus 0)^n \to (\mathbb{C} \setminus 0)^n$ by adding some monomials $u_{m+1}, \dots, u_n$ to $u_1, \dots, u_m$. We will consider this diffeomorphism as a local system of coordinates in the neighborhood of the point w. Although the coordinates are defined globally, they are necessary for the local analysis of the set X in the neighborhood of the point w, where $p = p_{(w)}$ and the set X coincides with the set defined by the equation $p_{(w)} = 0$. Denote this set by Y.

In the coordinates $u_1, \dots, u_n$, the set Y is defined by the equation $1 \top u_1 \top \dots \top u_m = 0$, and the absolute values of the first m coordinates of the point w are equal to 1. The values of the last $n-m$ coordinates are inessential because they do not enter the equation and the set Y has a structure of a cylindrical set with $(n-m)$-dimensional generatrices parallel to the last $n-m$ complex coordinate axes.

Thus, the assertion of the theorem reduces to the fact that the complex linear tropical equation $1 \top u_1 \top \dots \top u_m = 0$ defines a topological manifold of (real) dimension $2n-2$ in the space $(\mathbb{C} \setminus 0)^n$. This follows from Theorem 5.B. $\square$

5.12. The Zharkov–Itenberg–Mikhalkin conjecture on the tropical limits of Hodge structures. Complex tropical hypersurfaces, just as more complicated complex tropical varieties left beyond the scope of the present paper, possess a natural amoeba map onto the respective tropical varieties (over $\mathbb{R}_{\max,+}$) (see Subsection 5.6 above). The latter, being polyhedra, possess natural skeletons. The preimages of the skeletons under the amoeba projection constitute a natural filtration of a complex tropical variety. It gives a filtration of the homological groups of this variety.

Let X be a complex tropical variety and X_q be the preimage of the q-dimensional skeleton of the tropical variety under the amoeba projection. Define $H_n^q(X) = \mathrm{Im}(\mathrm{in}_*\colon H_n(X_q) \to H_n(X))$ and set $H_{p,q}(X) = H_{p+q}^q(X)/H_{p+q}^{q-1}(X)$. According to the conjecture formulated under different (and, apparently, more general) assumptions by Itenberg, Zharkov, and Mikhalkin, the group $H_{p,q}(X) \otimes \mathbb{C}$ is isomorphic to the Hodge group $H^{p,q}(X_h)$ of a complex variety that degenerates into X.

Appendix 1. OTHER TROPICAL ADDITIONS

A1.1. Tropical addition of quaternions. Denote by $\mathbb{H}$ the skew field of quaternions $\{x + y\mathbf{i} + z\mathbf{j} + t\mathbf{k}\colon x, y, z, t \in \mathbb{R}\}$. Let $a, b \in \mathbb{H}$. By analogy with the construction of Subsection 2.7, we set

$$a \top b = \begin{cases} \{a\} & \text{if } |a| > |b|, \\ \{b\} & \text{if } |a| < |b|, \\ \text{The set of points of the shortest arc of} & \\ \text{the geodesic connecting the quaternions} & \\ a \text{ and } b \text{ on the sphere } \{c \in \mathbb{H}\colon |c| = |a|\} & \text{if } |a| = |b|,\ a + b \neq 0, \\ \text{The ball } \{c \in \mathbb{H}\colon |c| \le |a|\} & \text{if } a + b = 0. \end{cases}$$

The set $a \top b$ is called the *tropical sum* of quaternions a and b.

Theorem A1.A. *The set $\mathbb{H}$ equipped with tropical addition is a commutative hypergroup.*

The proof repeats the proof of Theorem 2.A almost word for word. $\square$

It is easily seen that multiplication of quaternions is distributive with respect to tropical addition, so we have a *skew hyperfield.*

A1.2. Vector spaces. The construction of tropical addition of quaternions is a particular case of a more general construction. In an arbitrary normed vector space V over $\mathbb{C}$, define the following operation $(a, b) \mapsto a \top b$:

$$a \top b = \begin{cases} \{a\} & \text{if } |a| > |b|, \\ \{b\} & \text{if } |a| < |b|, \\ \mathrm{Cl}\left\{ \dfrac{|a|}{|\alpha a + \beta b|}(\alpha a + \beta b) \in V\colon \alpha, \beta \in \mathbb{R}_{>0} \right\} & \text{if } |a| = |b|,\ a + b \neq 0, \\ \{c \in V\colon |c| \le |a|\} & \text{if } a + b = 0. \end{cases}$$

This operation makes V into a hypergroup and is distributive with respect to the multiplication of vectors by complex numbers. In this case, $av \top bv = (a \top b)v$. In other words, V becomes a vector space over the hyperfield of tropical complex numbers in the sense of the following definition.

Let F be a hyperfield. A set V with a multivalued binary operation $(v,w) \mapsto v \top w$ and with an action $(a,v) \mapsto av$, $a \in F$, $v \in V$, of the multiplicative semigroup of the hyperfield F is called a *vector space* over F if

- the operation $\top$ defines the structure of a commutative hypergroup in V;
- $(ab)v = a(bv)$ for all $a, b \in F$ and $v \in V$;
- $1v = v$ for all $v \in V$;
- $a(v \top w) = av \top aw$ for all $a \in F$ and $v, w \in V$;
- $(a \top b)v = av \top bv$ for all $a, b \in F$ and $v \in V$.

Of course, any hyperfield is a vector space over itself. A copy of this vector space is contained in any vector space over the hyperfield. Indeed, if V is a vector space over a hyperfield F and $w \in V$, then the subset $W = \{aw \colon a \in F\}$ is a vector subspace of the space V in the obvious sense, and the map $F \to V \colon a \mapsto aw$ is an isomorphism of F, considered as a vector space over the hyperfield F, onto W.

Just as in the category of vector spaces over a field, the Cartesian product $V \times W$ of vector spaces V and W over a hyperfield F is naturally equipped with the structure of a vector space over F:

$$(v_1, w_1) \top (v_2, w_2) = \{(v, w) \colon v \in v_1 \top v_2,\ w \in w_1 \top w_2\}, \qquad a(v, w) = (av, aw).$$

Note, however, the following contrast with vector spaces over a field. If a vector space over a hyperfield is generated by a finite number of its elements, then it is not necessarily isomorphic to the Cartesian product of its subspaces each of which is generated by a single element. Indeed, the vector space over the hyperfield of tropical complex numbers constructed as above from a two-dimensional normed vector space over $\mathbb{C}$ is not isomorphic to the product of two copies of this hyperfield.

A1.3. Fields of monomials. The following example has been inspired by the work of Brett Parker [24], which was also motivated by the desire to understand the tropical degeneration of complex structures.

What if we apply the same construction as in Subsection 2.11 but do not ignore the absolute value of the monomial? Consider the set of monomials at^r with complex coefficients $a \neq 0$ and real exponents r. Let us adjoin zero to this set. Formally, this is $(\mathbb{C} \setminus 0) \times \mathbb{R} \cup \{0\}$. Denote it by P and define arithmetic operations on it.

Multiplication is defined as the ordinary multiplication of monomials. The set of nonzero monomials is an abelian group with respect to multiplication, and it is naturally isomorphic to the product of the multiplicative group of nonzero complex numbers by the additive group of all real numbers.

Addition is multivalued and is defined as follows:

$$at^r \top bt^s = \begin{cases} at^r & \text{if } \ r > s, \\ bt^s & \text{if } \ s > r, \\ (a+b)t^r & \text{if } \ s = r,\ a + b \neq 0, \\ \{ct^u \colon u < r,\ c \in \mathbb{C} \setminus \{0\}\} \cup \{0\} & \text{if } \ s = r,\ a + b = 0, \end{cases} \qquad 0 \top x = x.$$

It is clear that this addition is commutative, distributive with respect to multiplication, possesses a neutral element 0, and, for any monomial x, there is a unique y such that $x \top y \ni 0$. Let us check the associativity.

If one of the three terms vanishes, then the associativity holds, and the proof is straightforward: $(x \top 0) \top y = x \top y = x \top (0 \top y)$.

Consider three nonzero monomials at^u, bt^v, and ct^w. The following list exhausts all the possibilities:

(1) the exponent of one of the monomials is greater than the exponents of the other two, say $u > v, w$;

(2) two exponents, say u and v, are equal, while the third is less, and $a + b \neq 0$;

(3) two exponents, say u and v, are equal, while the third is less, and $a + b = 0$;

(4) all three exponents are equal, and, in addition,

  (a) none of the sums $a + b$, $b + c$, or $a + b + c$ vanishes;

  (b) the sum of two coefficients vanishes, say $a + b = 0$ (but $a + b + c \neq 0$);

  (c) $a + b + c = 0$.

Let us prove associativity in each of these cases.

(1) The sum is equal to the term with the greatest exponent irrespective of the order in which the operations are performed: for any order, this term dominates the other terms and turns out to be the final result. $\square$

(2) $(at^u \top bt^u) \top ct^w = (a+b)t^u \top ct^w = (a+b)t^u$; on the other hand, $at^u \top (bt^u \top ct^w) = at^u \top bt^u = (a + b)t^u$. $\square$

(3) We have

$$(at^u \top -at^u) \top ct^w = (\{xt^r : r < u\} \cup \{0\}) \top ct^w$$
$$= \begin{pmatrix} \{xt^r : w < r < u\} \cup \\ \{xt^r : r = w,\, x \neq -c\} \cup \\ \{-ct^w\} \cup \\ \{xt^r : r < w\} \cup \{0\} \end{pmatrix} \top ct^w = \begin{pmatrix} \{xt^r : w < r < u\} \cup \\ \{xt^w : x \neq 0,\, x \neq c\} \cup \\ \{xt^r : r < w\} \cup \{0\} \cup \\ \{ct^w\} \end{pmatrix} = \{xt^r : r < u\} \cup \{0\};$$

on the other hand,

$$at^u \top (-at^u \top ct^w) = at^u \top (-at^u) = \{xt^r : r < u\} \cup \{0\}. \quad \square$$

(4a) $(at^u \top bt^u) \top ct^u = (a + b)t^u \top ct^u = (a + b + c)t^u$ and $at^u \top (bt^u \top ct^u) = at^u \top (b + c)t^u = (a + b + c)t^u$. $\square$

(4b) If $a + b = 0$ and none of the sums $b + c$ or $a + b + c$ vanishes, then $(at^u \top -at^u) \top ct^u = (\{xt^r : r < u\} \cup \{0\}) \top ct^u = ct^u$; on the other hand, $at^u \top (-at^u \top ct^u) = at^u \top (-a + c)t^u = ct^u$. $\square$

(4c) If all three exponents are equal and $a + b + c = 0$, then

$$(at^u \top bt^u) \top ct^u = (a + b)t^u \top ct^u = (-c)t^u \top ct^u = \{xt^r : r < u\} \cup \{0\};$$

on the other hand,

$$at^u \top (bt^u \top ct^u) = at^u \top (b + c)t^u = at^u \top (-a)t^u = \{xt^r : r < u\} \cup \{0\}. \quad \square$$

Remark. There are numerous variants of the construction considered above. For example, in the definition of the tropical addition of monomials, all inequalities can be replaced by the reverse inequalities. Another possibility is to restrict the analysis to monomials whose exponents take only rational or only integer values. More generally, the exponents can be taken from any linearly ordered abelian group.

A1.4. Tropical addition of p-adic numbers. The construction of Subsection 2.11 admits a modification that can be applied to any field with non-Archimedean valuation. In every such field, we can define a tropical addition that, together with the original multiplication, yields the structure of a tropical field. Below, this scheme is implemented only in the case of the field of p-adic numbers. The general case will be considered elsewhere.

Recall that a p-adic number is defined as a series

$$\sum_{n=-v(a)}^{\infty} a_n p^n,$$

where a_n takes values in the set of integers from the interval $[0, p-1]$ and $a_{-v(a)} \neq 0$. Define a tropical sum of p-adic numbers $a = \sum_{n=-v(a)}^{\infty} a_n p^n$ and $b = \sum_{n=-v(b)}^{\infty} b_n p^n$ by the formula

$$a \top b = \begin{cases} a & \text{if } v(a) > v(b), \\ b & \text{if } v(b) > v(a), \\ a+b & \text{if } v(a) = v(b),\ a_{-v(a)} + b_{-v(b)} \neq p, \\ \{x\colon v(x) < v(a)\} & \text{if } v(a) = v(b),\ a_{-v(a)} + b_{-v(b)} = p. \end{cases} \qquad (7)$$

Just as in the previous subsection, one can prove that this operation is associative and, together with ordinary multiplication, yields the structure of a tropical field in the set of p-adic numbers.

Appendix 2. THE UPPER FELL TOPOLOGY

A2.1. The upper Fell topology. It is easy to show that the graph of tropical addition is not the limit of the graphs of operations $(a,b) \mapsto a \oplus_h b$ with respect to the upper Vietoris topology. This drawback, as well as some other drawbacks of the upper Vietoris topology mentioned above, can be overcome by a small change in its definition.

The *upper Fell topology* in the set 2^X of all subsets of a topological space X is the topology induced by sets of the form $2^U \subset 2^X$, where U is the complement of a compact subset of the space X.

The *Fell topology* is defined as the topology induced by the upper Fell topology and the lower Vietoris topology. It inherits drawbacks from the lower Vietoris topology and will not interest us here.

Note that if the space X is compact and Hausdorff, then the upper Fell topology coincides with the upper Vietoris topology. However, in the example in Subsection 4.2, the upper Fell topology induces a topology with the desired properties.

More generally, if X is a smooth manifold and $F_t\colon M \to X$ is a smooth isotopy of embeddings of a manifold M in X, then the curve $t \mapsto F_t(M)$ in the space 2^X with the upper Fell topology is continuous. In particular, the family of graphs $\Gamma_h \subset \mathbb{C}^3$ of binary operations $\oplus_h$ with $h > 0$ represents a continuous map $\mathbb{R}_{>0} \to 2^{\mathbb{C}^3}$.

We want to understand what limit object may appear as a degeneration of a submanifold (for example, the graph Γ_h of addition $\oplus_h$) under a limit transition with respect to the upper Fell topology.

A2.2. Limits in the upper Fell topology. Let X be a topological space and $\mathbb{R}_{>0} \to 2^X\colon h \mapsto F_h$ be an arbitrary map, i.e., an arbitrary family of sets parameterized by positive real numbers. Here we do not assume any continuity of this family with respect to any topology of the space 2^X.

In spite of the absence of any assumptions, F_h has a limit as $h \to 0$; i.e., there exists a set $A \subset X$ such that $F_h \to A$ as $h \to 0$ with respect to the upper Fell topology. For example, $A = X$ is a limit in the upper Fell topology for any family F_h of subsets of the space X. However, we would like to distinguish a more interesting special limit.

Denote by L the set

$$\{a \in X\colon\ \exists h_n \to 0,\ \exists x_n \in F_{h_n},\ x_n \to a\}, \tag{8}$$

i.e., the set of limits of converging sequences of points from F_h as $h \to 0$.

Theorem A2.A. *Let X be a locally compact regular topological space satisfying the first axiom of countability. Then L is the least* (*with respect to inclusion*) *closed limit of the family of sets F_h as $h \to 0$ in the upper Fell topology.*

This theorem follows immediately from Lemmas A2.B, A2.C, and A2.D given below.

Lemma A2.B. *If the space X satisfies the first axiom of countability, then the set L is closed in X.* □

Lemma A2.C. *If the space X satisfies the first axiom of countability, then L is a limit of the sets F_h as $h \to 0$ in the upper Fell topology.* □

Lemma A2.D. *If the space X is locally compact and regular, then L is contained in any closed set that is a limit of the family of sets F_h as $h \to 0$ in the upper Fell topology.* □

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