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REAL ALGEBRAIC PLANE CURVES: CONSTRUCTIONS WITH CONTROLLED TOPOLOGY

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ABSTRACT. This is a survey of the topology of real algebraic plane curves, concentrating on the constructive aspect of this theory, i.e., the problem of constructing curves of a given degree with a prescribed arrangement of its components. A large part of the paper is concerned with introductory material—the formulation of the basic problems and the history of the early development of the subject—so that the exposition is essentially self-contained. We give a detailed presentation of the technique of perturbing singular curves with controlled variation of the topology. We plan to publish the final part of the survey in the next issue.

CONTENTS

The connection between the algebraic properties of an equation in two variables and the geometrical properties of the curve defined by the equation is a subject which has attracted the attention and efforts of mathematicians from the time of Descartes. It is the topological properties of a curve which are the most qualitative, and at the same time the coarsest of its geometrical properties. One is attracted to the study of these properties both by the undeniable importance and foundational aspect of the subject and by the fact that the simplicity of the questions asked makes it possible to encompass a broader class of objects than in the case of the finer properties of curves.

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There are two natural directions in the study of the topology of real algebraic curves: first, the search for restrictions which the algebraic nature of a curve imposes on its topology; and, second, the proof that curves exist which satisfy these restrictions. During the past 18 years there has been an especially intensive development of the topology of real algebraic curves: at the beginning of this period, the basic achievements fell in the first of these two directions, while the past decade has seen significant progress in the second direction. The surveys which are available [2], [8], [12], [23], [25], [26], [43], [47] are primarily devoted to the first direction (although the last two touch upon the second direction as well). The present article is an attempt to explain the basic concepts, methods, and results belonging to the second direction.

The prohibitions and constructions (as we shall call the results in the first and second direction of research, respectively) cannot be considered separately from one another. Rather, they are two complementary ways of answering the same question: What topology is possible for a real algebraic curve of a given class? Hence, in our article considerable attention will also be paid to prohibitions, but as a rule the proofs will not be given, and instead the reader will be referred to the surveys and original articles. However, the author could not resist the temptation of writing *ab ovo* and so the article can probably be read without reference to other treatments of real algebraic geometry. In any case, everything that relates to the central theme—techniques which enable one successively to construct equations according to the topology of the curves they define—is presented in full detail. Experience talking with specialists in other areas of mathematics shows that this procedure is a good idea far beyond the confines of the theory of real algebraic curves.

The main objects in this article are curves. It is only in a context where the dimension is not essential that we shall consider varieties of arbitrary dimension. Space limitations prevent us from discussing phenomena which occur for higher dimensional varieties. Information about such varieties can be found in the surveys by D. A. Gudkov [12] and V. M. Kharlamov [25] and in the author's papers [4], [5], [8].

Chapter 1
Preliminaries

§1. The early topological study of real algebraic plane curves

1.1. Basic definitions and problems. A curve (at least, an algebraic curve) is something more than just the set of points which belong to it. There are many ways to introduce algebraic curves. In the elementary situation of real projective plane curves the simplest and most convenient is the following definition, which at first glance seems to be overly algebraic.

By a *real projective algebraic plane curve of degree* m ([1]) we mean a homogeneous real polynomial of degree m in three variables, considered up to constant factors. If a is such a polynomial, then the equation $a(x_0, x_1, x_2) = 0$ defines the *set of real points of the curve* in the real projective plane $\mathbf{R}P^2$. We let $\mathbf{R}A$ denote the set of real points of the curve A. Following tradition, we shall also call this set a curve, avoiding this terminology only in cases where confusion could result.

([1])Of course, the full designation is used only in formal situations. One normally adopts an abbreviated terminology. We shall say simply a *curve* in contexts where this will not lead to confusion.

A point $(x_0 : x_1 : x_2) \in \mathbf{R}P^2$ is called a (real) *singular point* of the curve A if $(x_0, x_1, x_2) \in \mathbf{R}^3$ is a critical point of the polynomial a which defines the curve. The curve A is said to be (*real*) *nonsingular* if it has no real singular points. The set of real points of a nonsingular real projective plane curve is a smooth closed one-dimensional submanifold of the projective plane.

In the topology of nonsingular real projective algebraic plane curves, as in other similar areas, the first natural questions that arise are classification problems.

1.1.A. The topological classification problem: *Up to homeomorphism, what are the possible sets of real points of a nonsingular real projective algebraic plane curve of degree* m?

1.1.B. The isotopy classification problem: *Up to homeomorphism, what are the possible pairs* $(\mathbf{R}P^2, \mathbf{R}A)$ *where* A *is a nonsingular real projective algebraic plane curve of degree* m?

It is well known that the components of a closed one-dimensional manifold are homeomorphic to a circle, and the topological type of the manifold is determined by the number of components; thus, the first problem reduces to asking about the number of components of a curve of degree m. The answer to this question, which was found by Harnack [35] in 1876, is described in §§1.6 and 1.8 below.

The second problem has a more naive formulation as the question of how a nonsingular curve of degree m can be situated in $\mathbf{R}P^2$. Here we are really talking about the isotopy classification, since any homeomorphism $\mathbf{R}P^2 \to \mathbf{R}P^2$ is isotopic to the identity map. At present the second problem has been solved only for $m \leq 7$. The solution is completely elementary when $m \leq 5$: it was known in the last century, and we shall give the result in this section. But before proceeding to an exposition of these earliest achievements in the study of the topology of real algebraic curves, we shall recall the isotopy classification of closed one-dimensional submanifolds of the projective plane.

1.2. Digression: the topology of closed one-dimensional submanifolds of the projective plane. For brevity, we shall refer to closed one-dimensional submanifolds of the projective plane as *topological plane curves*, or simply *curves* when there is no danger of confusion.

A connected curve can be situated in $\mathbf{R}P^2$ in two topologically distinct ways: *with two sides*, i.e., as the boundary of a disc in $\mathbf{R}P^2$, and *with one side*, i.e., as a projective line. A two-sided connected curve is called an *oval*. The complement of an oval in $\mathbf{R}P^2$ has two components, one of which is homeomorphic to a disc and the other homeomorphic to a Möbius strip. The first is called the *inside* and the second is called the *outside*. The complement of a connected one-sided curve is homeomorphic to a disc.

Any two one-sided connected curves intersect, since each of them is a realization of a nonzero element of the group $H_1(\mathbf{R}P^2; \mathbf{Z}_2)$ having nonzero self-intersection. Hence, a topological plane curve has at most one one-sided component. The existence of such a component can be expressed in terms of homology: it exists if and only if the curve represents a nonzero element of the group $H_1(\mathbf{R}P^2; \mathbf{Z}_2)$. If it exists, then we say that the whole curve is *one-sided*; otherwise, we say that the curve is *two-sided*.

Two disjoint ovals can be situated in two topologically distinct ways: each may lie outside the other one—i.e., each is in the outside component of the

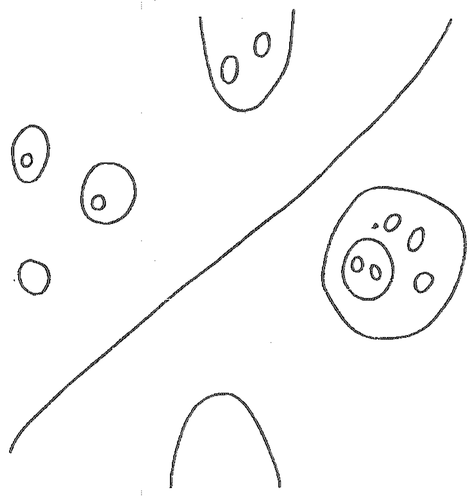

FIGURE 1

complement of the other—or else they may form an *injective pair*, i.e., one of them is in the inside component of the complement of the other—in that case, we say that the first is the *inner* oval of the pair and the second is the *outer* oval. In the latter case we also say that the outer oval of the pair *envelopes* the inner oval.

A set of h ovals of a curve any two of which form an injective pair is called a *nest of depth* h.

The pair $(\mathbf{R}P^2, X)$, where X is a topological plane curve, is determined up to homeomorphism by whether or not X has a one-sided component and by the relative location of each pair of ovals. We shall adopt the following notation to describe this. A curve consisting of a single oval will be denoted by the symbol $\langle 1\rangle$. The empty curve will be denoted by $\langle 0\rangle$. A one-sided connected curve will be denoted by $\langle J\rangle$. If $\langle A\rangle$ is the symbol for a certain two-sided curve, then the curve obtained by adding a new oval which envelopes all of the other ovals will be denoted by $\langle 1\langle A\rangle\rangle$. A curve which is a union of two disjoint curves $\langle A\rangle$ and $\langle B\rangle$ having the property that none of the ovals in one curve is contained in an oval of the other is denoted by $\langle A \amalg B\rangle$. In addition, we use the following abbreviations: if $\langle A\rangle$ denotes a certain curve, and if a part of another curve has the form $A \amalg A \amalg \cdots \amalg A$, where A occurs n times, then we let $n \times A$ denote $A \amalg \cdots \amalg A$. We further write $n \times 1$ simply as n.

When depicting a topological plane curve one usually represents the projective plane either as a disc with opposite points of the boundary identified, or else as the compactification of $\mathbf{R}^2$, i.e., one visualizes the curve as its preimage under either the projection $D^2 \to \mathbf{R}P^2$ or the inclusion $\mathbf{R}^2 \to \mathbf{R}P^2$. In this article we shall use the second method. For example, Figure 1 shows a curve corresponding to the symbol $\langle J \amalg 1 \amalg 2\langle 1\rangle \amalg 1\langle 2\rangle \amalg 1\langle 3 \amalg 1\langle 2\rangle\rangle\rangle$.

1.3. Bézout's prohibitions and the Harnack inequality. The most elementary prohibitions, it seems, are the topological consequences of Bézout's theorem. In any case, these were the first prohibitions to be discovered.

1.3.A. BÉZOUT'S THEOREM (see, for example, [24], [30]). *If A_1 and A_2 are nonsingular curves of degree m_1 and m_2, and if the set $\mathbf{R}A_1 \cap \mathbf{R}A_2$ is finite, then this set contains at most $m_1 m_2$ points. If, in addition, $\mathbf{R}A_1$ and $\mathbf{R}A_2$ are transversal to one another, then the number of points in the intersection $\mathbf{R}A_1 \cap \mathbf{R}A_2$ is congruent to $m_1 m_2$ modulo 2.*

1.3.B. Corollary 1. *A nonsingular plane curve of degree m is one-sided if and only if m is odd. In particular, a curve of odd degree is nonempty.*

In fact, in order for a nonsingular plane curve to be two-sided, i.e., to be homologous to zero mod 2, it is necessary and sufficient that its intersection index with the projective line be zero mod 2. By Bézout's theorem, this is equivalent to the degree being even. ●

1.3.C. Corollary 2. *The number of ovals in the union of two nestings of a nonsingular plane curve of degree m does not exceed $m/2$. In particular, a nesting of a curve of degree m has depth at most $m/2$, and if a curve of degree m has a nesting of depth $[m/2]$, then it does not have any ovals not in the nesting.*

To prove Corollary 2 it suffices to apply Bézout's theorem to the curve and to a line which passes through the insides of the smallest ovals in the nestings. ●

1.3.D. Corollary 3. *There can be no more than m ovals in a set of ovals which is contained in a union of ≤ 5 nestings of a nonsingular plane curve of degree m and which does not contain an oval enveloping all of the other ovals of the set.*

To prove Corollary 3 it suffices to apply Bézout's theorem to the curve and to a conic which passes through the insides of the smallest ovals in the nestings. ●

One can give corollaries whose proofs use curves of higher degree than lines and conics (see §2.5). The most important such result is Harnack's inequality.

1.3.E. Corollary 4 (Harnack's inequality [35]). *The number of components of a nonsingular plane curve of degree m is at most $(m^2 - 3m + 4)/2$.*

For the derivation of Harnack's inequality from Bézout's theorem, see [35], and also [12]. Incidentally, it is possible to prove Harnack's inequality without using Bézout's theorem; see, for example, [12], [47].

1.4. Curves of degree ≤ 5. If $m \leq 5$, then it is easy to see that the prohibitions in the previous subsection are satisfied only by the following isotopy types.

Table 1

m	Isotopy types of nonsingular plane curves of degree m						
1	$\langle J \rangle$						
2	$\langle 0 \rangle$,	$\langle 1 \rangle$					
3	$\langle J \rangle$,	$\langle J \amalg 1 \rangle$					
4	$\langle 0 \rangle$,	$\langle 1 \rangle$, $\langle 2 \rangle$,	$\langle 1\langle 1 \rangle \rangle$,	$\langle 3 \rangle$, $\langle 4 \rangle$			
5	$\langle J \rangle$	$\langle J \amalg 1 \rangle$,	$\langle J \amalg 2 \rangle$,	$\langle J \amalg 1\langle 1 \rangle \rangle$,	$\langle J \amalg 3 \rangle$,	$\langle J \amalg 5 \rangle$,	$\langle J \amalg 6 \rangle$

For $m \leq 3$ the absence of other types follows from 1.3.B and 1.3.C; for $m = 4$ it follows from 1.3.B, 1.3.C and 1.3.D, or else from 1.3.B, 1.3.C and 1.3.E; and for $m = 5$ it follows from 1.3.B, 1.3.C and 1.3.E. It turns out that it is possible to realize all of the types in Table 1; hence, we have the following theorem.

1.4.A. Isotopy classification of nonsingular plane curves of degree ≤ 5. *An isotopy class of topological plane curves contains a nonsingular curve of degree $m \leq 5$ if and only if it occurs in the mth row of Table 1.*

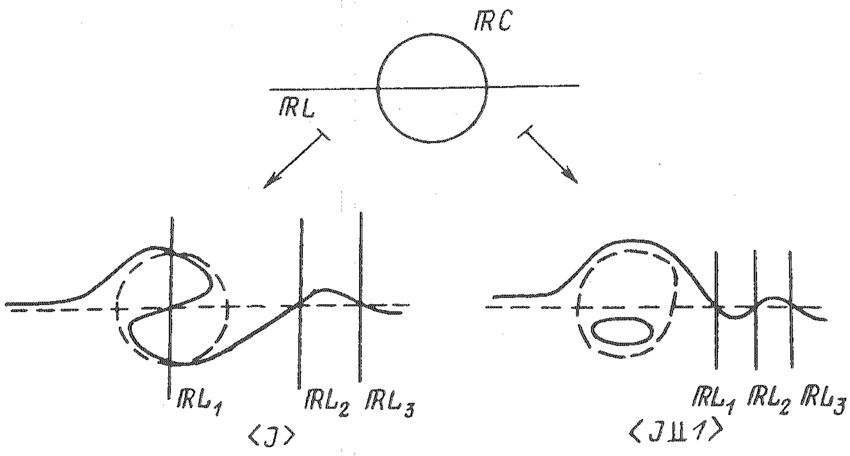

FIGURE 2

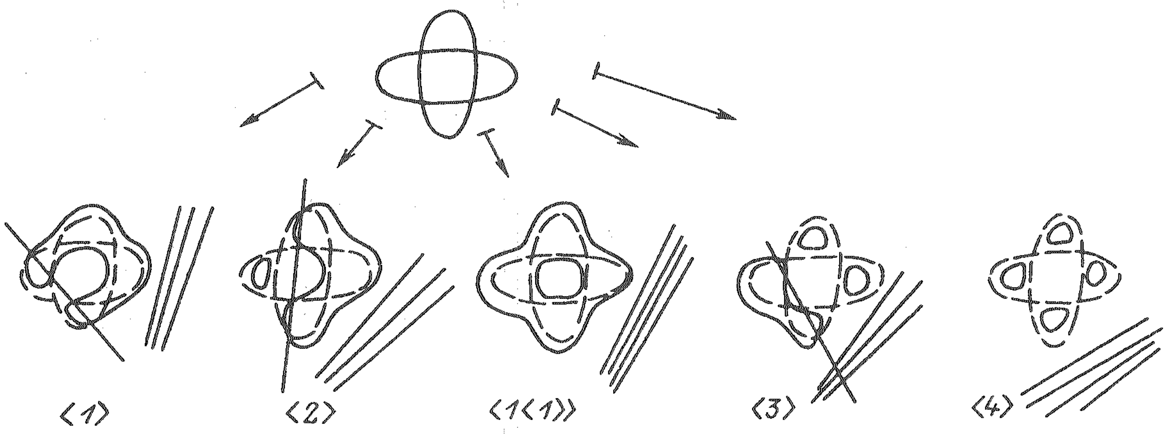

FIGURE 3

The curves of degree ≤ 2 are known to everyone. Both of the isotopy types of nonsingular curves of degree 3 can be realized by small perturbations of the union of a line and a conic which intersect in two real points (Figure 2). One can construct these perturbations by replacing the left side of the equation $cl = 0$ defining the union of the conic C and the line L by the polynomial $cl + \varepsilon l_1 l_2 l_3$, where $l_i = 0$, $i = 1, 2, 3$, are the equations of the lines shown in Figure 2, and ε is a nonzero real number which is sufficiently small in absolute value.

It will be left to the reader to prove that one in fact obtains the curves in Figure 2 as a result; alternatively, the reader can deduce this fact from the theorem in the next subsection.

The isotopy types of nonempty nonsingular curves of degree 4 can be realized in a similar way by small perturbations of a union of two conics which intersect in four real points (Figure 3). The empty curve of degree 4 can be defined, for example, by the equation $x_0^4 + x_1^4 + x_2^4 = 0$.

All of the isotopy types of nonsingular curves of degree 5 can be realized by small perturbations of the union of two conics and a line, shown in Figure 4. ●

For the isotopy classification of nonsingular curves of degree 6 it is no longer sufficient to use this type of construction, or even the prohibitions in the previous subsection. See §§1.13 and 5.1.

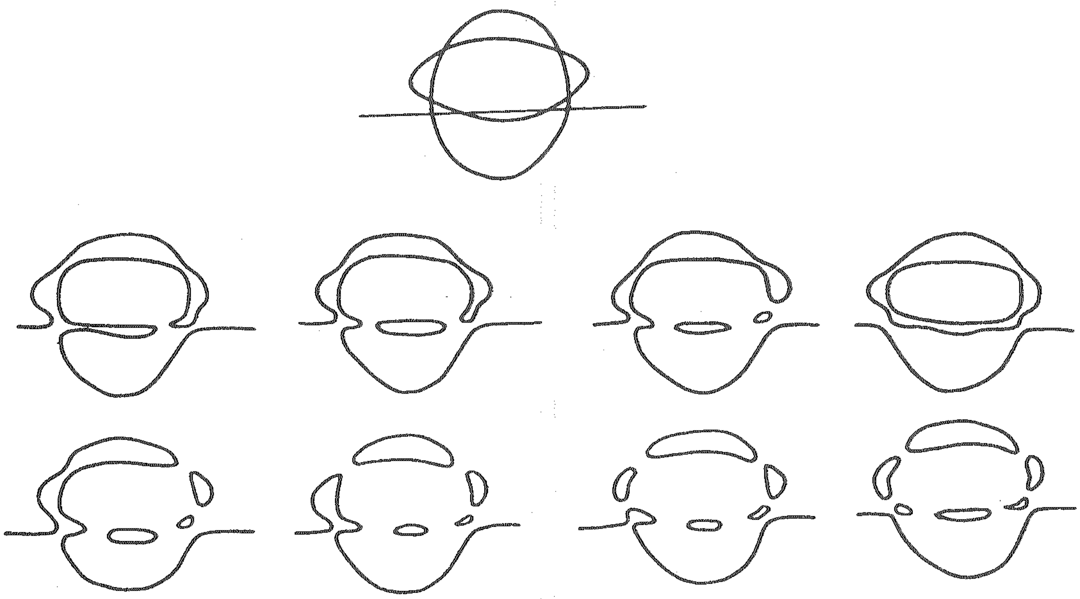

FIGURE 4

1.5. The classical method of constructing nonsingular plane curves. All of the classical constructions of the topology of nonsingular plane curves are based on a single construction, which I will call *classical small perturbation*. Some special cases were given in the previous subsection. Here I will give a detailed description of the conditions under which it can be applied and the results.

We say that a real singular point $\xi = (\xi_0 : \xi_1 : \xi_2)$ of the curve A is an *intersection point of two real transversal branches*, or, more briefly, a *crossing*,([2]) if the polynomial a defining the curve has matrix of second partial derivatives at the point (ξ_0, ξ_1, ξ_2) with both a positive and a negative eigenvalue, or, equivalently, if the point ξ is a nondegenerate critical point of index 1 of the functions $\{x \in \mathbf{R}P^2 | x_i \neq 0\} \to \mathbf{R}: x \mapsto a(x)/x_i \deg a$ for i with $\xi_i \neq 0$. By Morse's lemma, in a neighborhood of such a point the curve looks like a union of two real lines. Conversely, if $\mathbf{R}A_1, \dots, \mathbf{R}A_k$ are nonsingular mutually transverse curves no three of which pass through the same point, then all of the singular points of the union $\mathbf{R}A_1 \cup \dots \cup \mathbf{R}A_k$ (this is precisely the pairwise intersection points) are crossings.

1.5.A. CLASSICAL SMALL PERTURBATION THEOREM (see Figure 5). *Let A be a plane curve of degree m all of whose singular points are crossings, and let B be a plane curve of degree m which does not pass through the singular points of A. Let U be a regular neighborhood of the curve $\mathbf{R}A$ in $\mathbf{R}P^2$, represented as the union of a neighborhood U_0 of the set of singular points of A and a tubular neighborhood U_1 of the submanifold $\mathbf{R}A \setminus U_0$ in $\mathbf{R}P^2 \setminus U_0$.*

Then there exists a nonsingular plane curve X of degree m such that:

(1) $\mathbf{R}X \subset U$.

(2) *For each component V of U_0 there exists a homeomorphism $h: V \to D^1 \times D^1$ such that $h(\mathbf{R}A \cap V) = D^1 \times 0 \cup 0 \times D^1$ and $h(\mathbf{R}X \cap V) = \{(x, y) \in D^1 \times D^1 | xy = 1/2\}$.*

(3) $\mathbf{R}X \setminus U_0$ *is a section of the tubular fibration $U_1 \to \mathbf{R}A \setminus U_0$.*

([2])Sometimes other names are used. For example: a node, a point of type A_1 with two real branches, a nonisolated nondegenerate double point.

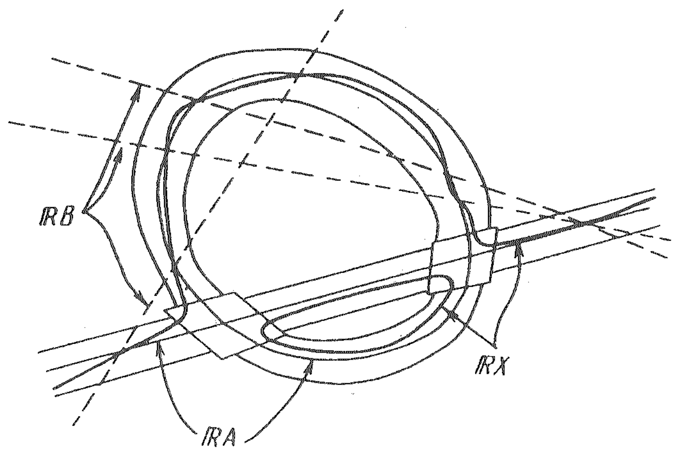

FIGURE 5

(4) $\mathbf{R}X \subset \{(x_0 : x_1 : x_2) \in \mathbf{R}P^2 | a(x_0, x_1, x_2)b(x_0, x_1, x_2) \leq 0\}$, *where* a *and* b *are polynomials defining the curves* A *and* B.

(5) $\mathbf{R}X \cap \mathbf{R}A = \mathbf{R}X \cap \mathbf{R}B = \mathbf{R}A \cap \mathbf{R}B$.

(6) *If* $p \in \mathbf{R}A \cap \mathbf{R}B$ *is a nonsingular point of* B *and* $\mathbf{R}B$ *is transversal to* $\mathbf{R}A$ *at this point, then* $\mathbf{R}X$ *is also transversal to* $\mathbf{R}A$ *at the point.*

There exists $\varepsilon > 0$ *such that for any* $t \in (0, \varepsilon]$ *the curve given by the polynomial* $a + tb$ *satisfies all of the above requirements imposed on* X.

It follows from (1)–(3) that for fixed A the isotopy type of the curve $\mathbf{R}X$ depends on which of two possible ways it behaves in a neighborhood of each of the crossings of the curve A, and this is determined by condition (4). Thus, conditions (1)–(4) characterize the isotopy type of the curve $\mathbf{R}X$. Conditions (4)–(6) characterize its position relative to $\mathbf{R}A$.

We say that the curves defined by the polynomials $a + tb$ with $t \in (0, \varepsilon]$ are obtained by *small perturbations of* A *by means of the curve* B. It should be noted that the curves A and B do not determine the isotopy type of the perturbed curves: since both of the polynomials b and $-b$ determine the curve B, it follows that the polynomials $a - tb$ with small $t > 0$ also give small perturbations of A by means of B. But these curves are not isotopic to the curves given by $a + tb$ (at least not in U), if the curve A actually has singularities.

PROOF OF THEOREM 1.5.A. We set $x_t = a + tb$. It is clear that for any $t \neq 0$ the curve X_t given by the polynomial x_t satisfies conditions (5) and (6), and if $t > 0$ it satisfies (4). For small $|t|$ we obviously have $\mathbf{R}X_t \subset U$. Furthermore, if $|t|$ is small, the curve $\mathbf{R}X_t$ is nonsingular at the points of intersection $\mathbf{R}X_t \cap \mathbf{R}B = \mathbf{R}A \cap \mathbf{R}B$, since the gradient of x_t differs very little from the gradient of a when $|t|$ is small, and the latter gradient is nonzero on $\mathbf{R}A \cap \mathbf{R}B$ (this is because, by assumption, B does not pass through the singular points of A). Outside $\mathbf{R}B$ the curve $\mathbf{R}X_t$ is a level curve of the function a/b. On $\mathbf{R}A \setminus \mathbf{R}B$ this level curve has critical points only at the singular points of $\mathbf{R}A$, and these critical points are nondegenerate. Hence, for small t the behavior of $\mathbf{R}X_t$ outside $\mathbf{R}B$ is described by the implicit function theorem and Morse's lemma (see, for example, [14]); in particular, for small $t \neq 0$ this curve is nonsingular and satisfies conditions (2) and (3). Consequently, there exists $\varepsilon > 0$ such that for any $t \in (0, \varepsilon]$ the curve $\mathbf{R}X_t$ is nonsingular and satisfies (1)–(6).

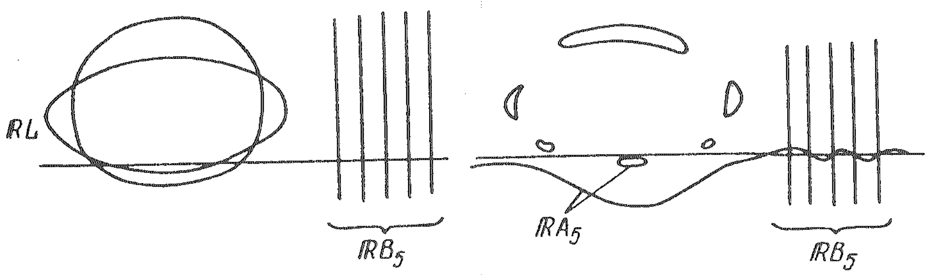

FIGURE 6

1.6. Harnack curves. In 1876, Harnack [35] not only proved the inequality 1.3.E in §1.3, but also completed the topological classification of nonsingular plane curves by proving the following theorem.

1.6.A. HARNACK'S THEOREM. *For any natural number* m *and any integer* c *satisfying the inequalities*

$$\frac{1-(-1)^m}{2} \le c \le \frac{m^2-3m+4}{2}, \qquad (1)$$

there exists a nonsingular plane curve of degree m *consisting of* c *components.*

The inequality on the right in (1) is Harnack's inequality. The inequality on the left is part of Corollary 1 of Bézout's theorem (see §1.3.B). Thus, Harnack's theorem together with §§1.3.B and 1.3.E actually give a complete characterization of the set of topological types of nonsingular plane curves of degree m, i.e., they solve problem 1.1.A.

Curves with the maximum number of components (i.e., with $(m^2-3m+4)/2$ components, where m is the degree) are called M-curves. Curves of degree m which have $(m^2-3m+4)/2-a$ components are called $(M-a)$-curves. We begin the proof of Theorem 1.6.A by establishing that the Harnack inequality 1.3.B is best possible.

1.6.B. *For any natural number* m *there exists an* M*-curve of degree* m.

PROOF. We shall actually construct a sequence of M-curves. At each step of the construction we add a line to the M-curve just constructed, and then give a slight perturbation to the union. We can begin the construction with a line or, as in Harnack's proof in [35], with a circle. However, since we have already treated curves of degree ≤ 5 and constructed M-curves for those degrees (see §1.4), we shall begin by taking the M-curve of degree 5 that was constructed in §1.4, so that we can immediately proceed to curves that we have not encountered before.

Recall that we obtained a degree 5 M-curve by perturbing the union of two conics and a line L. This perturbation can be done using various curves. For what follows it is essential that the auxiliary curve intersect L in five points which are outside the two conics. For example, let the auxiliary curve be a union of five lines which satisfies this condition (Figure 6). We let B_5 denote this union, and we let A_5 denote the M-curve of degree 5 that is obtained using B_5.

We now construct a sequence of auxiliary curves B_m for $m > 5$. We take B_m to be a union of m lines which intersect L in m distinct points lying, for even m, in an arbitrary component of the set $\mathbb{R}L \setminus \mathbb{R}B_{m-1}$ and for odd m in the component of $\mathbb{R}L \setminus \mathbb{R}B_{m-1}$ containing $\mathbb{R}L \cap \mathbb{R}B_{m-2}$.

We construct the M-curve A_m of degree m using small perturbation of the union $A_{m-1} \cup L$ by means of B_m. Suppose that the M-curve A_{m-1} of

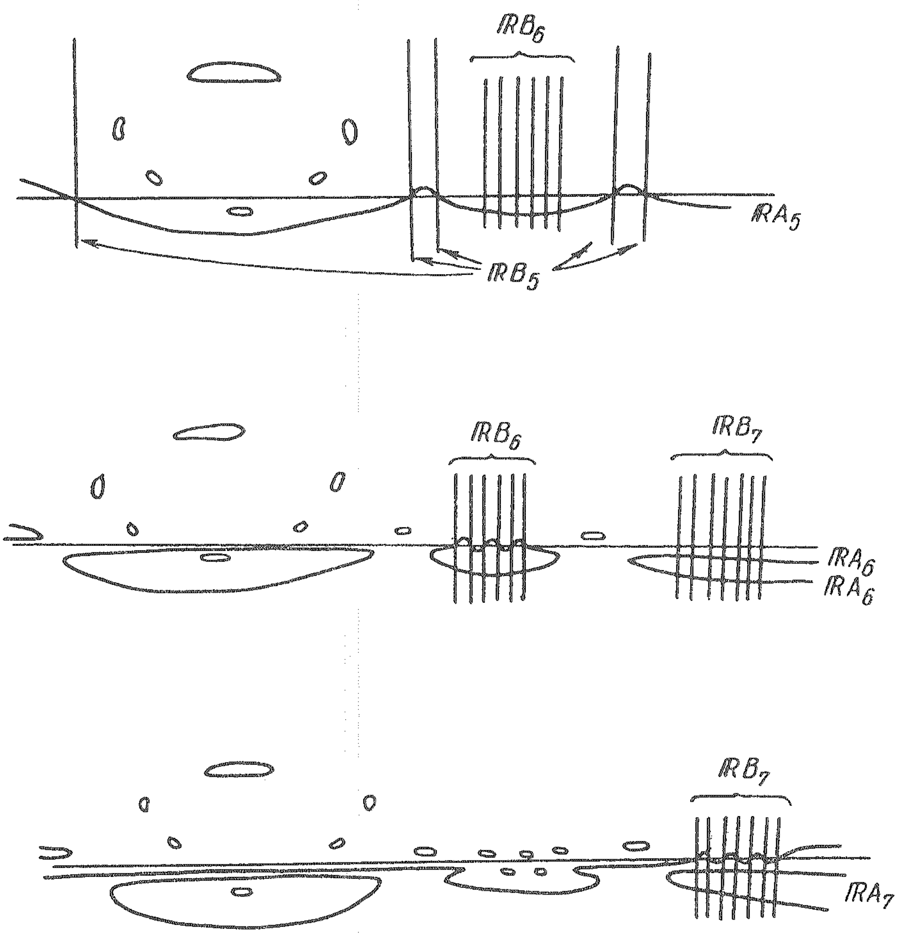

FIGURE 7

degree $m-1$ has already been constructed, and suppose that $\mathbb{R}A_{m-1}$ intersects $\mathbb{R}L$ transversally in the $m-1$ points of the intersection $\mathbb{R}L \cap \mathbb{R}B_{m-1}$ which lie in the same component of the curve $\mathbb{R}A_{m-1}$ and in the same order as on $\mathbb{R}L$. It is not hard to see that, for one of the two possible directions of a small perturbation of $A_{m-1} \cup L$ by means of B_m, the line $\mathbb{R}L$ and the component of $\mathbb{R}A_{m-1}$ that it intersects give $m-1$ components, while the other components of $\mathbb{R}A_{m-1}$, of which, by assumption, there are

$$((m-1)^2 - 3(m-1) + 4)/2 - 1 = (m^2 - 5m + 6)/2,$$

are only slightly deformed—so that the number of components of $\mathbb{R}A_m$ remains equal to $(m^2 - 5m + 6)/2 + m - 1 = (m^2 - 3m + 4)/2$. We have thus obtained an M-curve of degree m. This curve is transversal to $\mathbb{R}L$, it intersects $\mathbb{R}L$ in $\mathbb{R}L \cap \mathbb{R}B_m$ (see 1.5.A), and, since $\mathbb{R}L \cap \mathbb{R}B_m$ is contained in one of the components of the set $\mathbb{R}L \setminus \mathbb{R}B_{m-1}$, it follows that the intersection points of our curve with $\mathbb{R}L$ are all in the same component of the curve and are in the same order as on $\mathbb{R}L$ (Figure 7). ●

The proof that the left inequality in (1) is best possible, i.e., that there is a curve with the minimum number of components, is much simpler. For example, we can take the curve given by the equation $x_0^m + x_1^m + x_2^m = 0$. Its set of real points is obviously empty when m is even, and when m is odd the set of real points is homeomorphic to $\mathbb{R}P^1$ (we can get such a homeomorphism onto $\mathbb{R}P^1$, for example, by projection from the point $(0:0:1)$).

By choosing the auxiliary curves B_m in different ways in the construction of M-curves in the proof of Theorem 1.6.B, we can obtain curves with any intermediate number of components. However, to complete the proof of Theorem 1.6.A in this way would be rather tedious, even though it would not require any new ideas. We shall instead turn to a less explicit, but simpler and more conceptual method of proof, which is based on objects and phenomena not encountered above.

1.7. Digression: the space of real projective plane curves. By the definition of the set of all real projective algebraic plane curves of degree m, they form a real projective space of dimension $m(m+3)/2$. The homogeneous coordinates in this projective space are the coefficients of the polynomials defining the curves. We shall denote this space by the symbol $\mathbf{R}C_m$. Its only difference with the standard space $\mathbf{R}P^{m(m+3)/2}$ is the unusual numbering of the homogeneous coordinates. The point is that the coefficients of a homogeneous polynomial in three variables have a natural double indexing by the exponents of the monomials:

$$a(x_0, x_1, x_2) = \sum_{\substack{i,j\geq 0\\ i+j\leq m}} a_{ij}x_0^{m-i-j}x_1^i x_2^j.$$

We let $\mathbf{R}NC_m$ denote the subset of $\mathbf{R}C_m$ corresponding to the real nonsingular curves. It is obviously open in $\mathbf{R}C_m$. Moreover, any nonsingular curve of degree m has a neighborhood in $\mathbf{R}NC_m$ consisting of isotopic nonsingular curves. Namely, small changes in the coefficients of the polynomial defining the curve lead to polynomials which give smooth sections of a tubular fibration of the original curve. This is an easy consequence of the implicit function theorem; compare with 1.5.A condition (3).

Curves which belong to the same component of the space $\mathbf{R}NC_m$ of nonsingular degree m curves are isotopic—this follows from the fact that nonsingular curves which are close to one another are isotopic. Suppose that we have an isotopy in $\mathbf{R}NC_m$ of the set of real points of a curve in $\mathbf{R}P^2$ which consists of the set of real points of curves of degree m. Such an isotopy is said to be *rigid*. This definition naturally gives rise to the following classification problem, which is every bit as classical as problems 1.1.A and 1.1.B.

1.7.A. Rigid isotopy classification problem: *Classify the nonsingular curves of degree m up to rigid isotopy, i.e., study the partition of the space $\mathbf{R}NC_m$ of nonsingular degree m curves into its components.*

If $m \leq 2$, it is well known that the solution of this problem is identical to that of problem 1.1.B. Isotopy also implies rigid isotopy for curves of degree 3 and 4. This was known in the last century; however, we shall not discuss this further here, since it has little relevance to the theme of the article. At present problem 1.7.A has been solved for $m \leq 6$.

Although this section is devoted to the early stages of the theory, I cannot resist commenting in some detail about a more recent result. In 1978, V. A. Rokhlin [23] discovered that for $m \geq 5$ isotopy of nonsingular curves of degree m no longer implies rigid isotopy. The simplest example is given in Figure 8, which shows two curves of degree 5. They are obtained by slightly perturbing the very same curve in Figure 4 which is made up of two conics and a line. It becomes clear that these curves are not rigid isotopic if we note that the first curve has an oval lying inside a triangle which does not intersect the one-sided component and which has its vertices inside the other three ovals, and

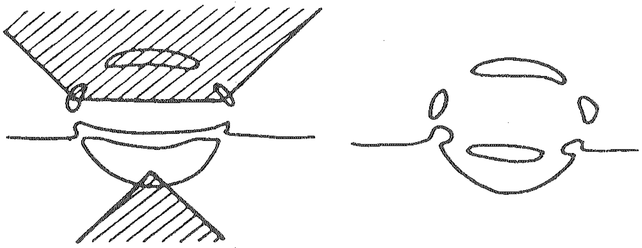

FIGURE 8

the second curve does not have such an oval—but under a rigid isotopy the oval cannot leave the triangle, since that would entail a violation of Bézout's theorem.

We now examine the subset of $\mathbf{R}C_m$ made up of real singular curves.

It is clear that a curve of degree m has a singularity at $(1:0:0)$ if and only if its polynomial has zero coefficients of the monomials x_0^m, $x_0^{m-1}x_1$, $x_0^{m-1}x_2$. Thus, the set of real projective plane curves of degree m having a singularity at a particular point forms a subspace of codimension 3 in $\mathbf{R}C_m$.

We now consider the space S of pairs of the form (p, C), where $p \in \mathbf{R}P^2$, $C \in \mathbf{R}C_m$, and p is a singular point of the curve C. S is clearly an algebraic subvariety of the product $\mathbf{R}P^2 \times \mathbf{R}C_m$. The restriction to S of the projection $\mathbf{R}P^2 \times \mathbf{R}C_m \to \mathbf{R}P^2$ is a locally trivial fibration whose fiber is the space of curves of degree m with a singularity at the corresponding point, i.e., the fiber is a projective space of dimension $m(m+3)/2-3$. Thus, S is a smooth manifold of dimension $m(m+3)/2-1$. The restriction $S \to \mathbf{R}C_m$ of the projection $\mathbf{R}P^2 \times \mathbf{R}C_m \to \mathbf{R}C_m$ has as its image precisely the set of all real singular curves of degree m, i.e., $\mathbf{R}C_m \setminus \mathbf{R}NC_m$. We let $\mathbf{R}SC_m$ denote this image. Since it is the image of a $(m(m+3)/2-1)$-dimensional manifold under smooth map, its dimension is at most $m(m+3)/2-1$. On the other hand, its dimension is at least equal $m(m+3)/2-1$, since otherwise, as a subspace of codimension ≥ 2, it would not partition the space $\mathbf{R}C_m$, and all nonsingular curves of degree m would be isotopic.

Using an argument similar to the proof that $\dim \mathbf{R}SC_m \leq m(m+3)/2-1$, one can show that the set of curves having at least two singular points and the set of curves having a singular point where the matrix of second derivatives of the corresponding polynomial has rank ≤ 1, each has dimension at most $m(m+3)/2-2$. Thus, the set $\mathbf{R}SC_m$ has an open everywhere dense subset consisting of curves with only one singular point, which is a nondegenerate double point (meaning that at this point the matrix of second derivatives of the polynomial defining the curve has rank 2). This subset is called the *principal part* of the set $\mathbf{R}SC_m$. It is a smooth submanifold of codimension 1 in $\mathbf{R}C_m$. In fact, its preimage under the natural map $S \to \mathbf{R}C_m$ is obviously an open everywhere dense subset in the manifold S, and the restriction of this map to the preimage is easily verified to be a one-to-one immersion, and hence a smooth imbedding.

There are two types of nondegenerate real points on a plane curve. We say that a nondegenerate real double point $(\xi_0 : \xi_1 : \xi_2)$ on a curve A is *solitary* if the matrix of second partial derivatives of the polynomial defining A has either two nonnegative or two nonpositive eigenvalues at the point (ξ_0, ξ_1, ξ_2). A solitary nondegenerate double point of A is an isolated point of the set $\mathbf{R}A$.

In general, a singular point of A which is an isolated point of the set $\mathbf{R}A$ will be called a solitary real singular point. The other type of nondegenerate real double point is a crossing; crossings were discussed in §1.5 above. Corresponding to this division of the nondegenerate real double points into solitary points and crossings, we have a partition of the principal part of the set of real singular curves of degree m into two open sets.

If a curve of degree m moves as a point of $\mathbf{R}C_m$ along an arc which has a transversal intersection with the half of the principal part of the set of real singular curves consisting of curves with a solitary singular point, then the set of real points on this curve undergoes a Morse modification of index 0 or 2 (i.e., either the curve acquires a solitary double point, which then becomes a new oval, or else one of the ovals contracts to a point (a solitary nondegenerate double point) and disappears). In the case of a transversal intersection with the other half of the principal part of the set of real singular curves one has a Morse modification of index 1 (i.e., two arcs of the curve approach one another and merge, with a crossing at the point where they come together, and then immediately diverge in their modified form, as happens, for example, with the hyperbola in the family of affine curves of degree 2 given by the equation $xy = t$ at the moment when $t = 0$).

A line in $\mathbf{R}C_m$ is called a (real) pencil of curves of degree m. If a and b are polynomials defining two curves of the pencil, then the other curves of the pencil are given by polynomials of the form $\lambda a + \mu b$ with $\lambda, \mu \in \mathbf{R} \setminus 0$.

By the transversality theorem, the pencils which intersect the set of real singular curves only at points of the principal part and only transversally form an open everywhere dense subset of the set of all real pencils of curves of degree m.

1.8. End of the proof of theorem 1.6.A. In §1.6 it was shown that for any m there exist nonsingular curves of degree m with the minimum number $(1-(-1)^m)/2$ or with the maximum number $(m^2 - 3m + 4)/2$ of components. Nonsingular curves which are isotopic to one another form an open set in the space $\mathbf{R}C_m$ of real projective plane curves of degree m (see §1.7). Hence, there exists a real pencil of curves of degree m which connects a curve with minimum number of components to a curve with maximum number of components and which intersects the set of real singular curves only in its principal part and only transversally. As we move along this pencil from the curve with minimum number of components to the curve with maximum number of components, the curve only undergoes Morse modifications, each of which changes the number of components by at most 1. Consequently, this pencil includes nonsingular curves with an arbitrary intermediate number of components. ●

1.9. Isotopy types of Harnack M-curves. Harnack's construction of M-curves in [35] differs from the construction in the proof of Theorem 1.6.B in that a conic, rather than a curve of degree 5, is used as the original curve. Figure 9 shows that the M-curves of degree ≤ 5 which are used in Harnack's construction [35]. For $m \geq 6$ Harnack's construction gives M-curves with the same isotopy types as in the construction in §1.6.

In these constructions one obtains different isotopy types of M-curves depending on the choice of auxiliary curves (more precisely, depending on the relative location of the intersections $\mathbf{R}B_m \cap \mathbf{R}L$). Recall that in order to obtain M-curves it is necessary for the intersection $\mathbf{R}B_m \cap \mathbf{R}L$ to consist of m points and lie in a single component of the set $\mathbf{R}L \setminus \mathbf{R}B_{m-1}$, where for odd m this

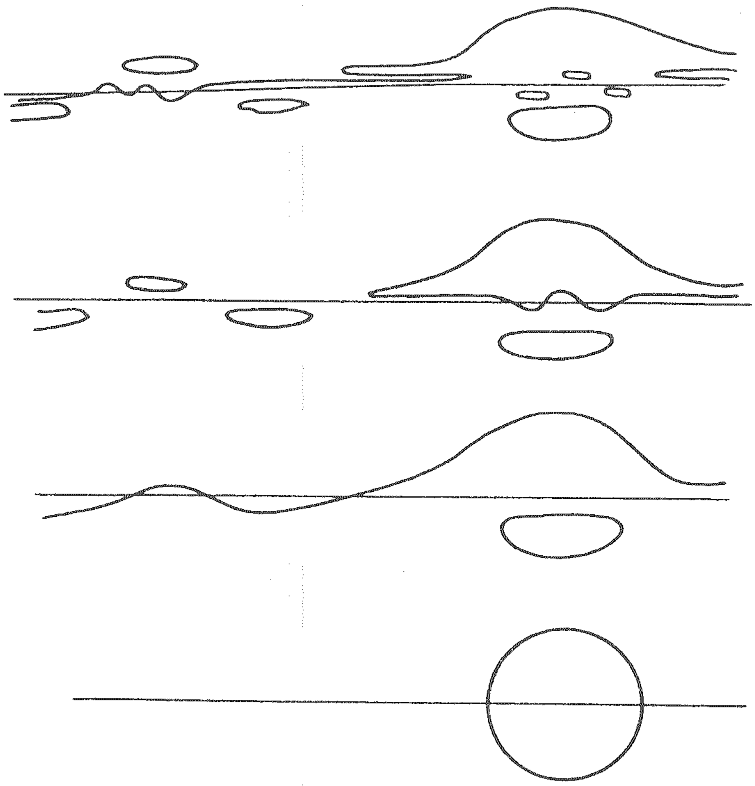

FIGURE 9

component must contain $\mathbf{R}B_{m-2} \cap \mathbf{R}L$. It is easy to see that the isotopy type of the resulting M-curve of degree m depends only on the choice of the components of $\mathbf{R}L \setminus \mathbf{R}B_{r-1}$ for even $r < m$ where the intersections $\mathbf{R}L \cap \mathbf{R}B_r$ are to be found. If we take the components containing $\mathbf{R}L \cap \mathbf{R}B_{r-2}$ for even r as well, then the degree m M-curve obtained from the construction has isotopy type $\langle J \amalg (m^2-3m+2)/2\rangle$ for odd m and $\langle (3m^2-6m)/8 \amalg 1\langle (m^2-6m+8)/8\rangle\rangle$ for even m. In Table 2 we have listed the isotopy types of M-curves of degree ≤ 10 which one obtains from Harnack's construction using all possible B_m.

TABLE 2

m	Isotopy types of Harnack M-curves of degree m			
2	$\langle 1\rangle$			
3	$\langle J \amalg 1\rangle$			
4	$\langle 4\rangle$			
5	$\langle J \amalg 6\rangle$			
6	$\langle 9 \amalg 1\langle 1\rangle\rangle$			
7	$\langle J \amalg 15\rangle$		$\langle J \amalg 13 \amalg 1\langle 1\rangle\rangle$	
8	$\langle 18 \amalg 1\langle 3\rangle\rangle$		$\langle 17 \amalg 1\langle 1\rangle \amalg 1\langle 2\rangle\rangle$	
9	$\langle J \amalg 28\rangle$	$\langle J \amalg 24 \amalg 1\langle 3\rangle\rangle$	$\langle J \amalg 26 \amalg 1\langle 1\rangle\rangle$	$\langle J \amalg 23 \amalg 1\langle 1\rangle \amalg 1\langle 2\rangle\rangle$
10	$\langle 30 \amalg 1\langle 6\rangle\rangle$	$\langle 29 \amalg 2\langle 3\rangle\rangle$	$\langle 29 \amalg 1\langle 1\rangle \amalg 1\langle 5\rangle\rangle$	$\langle 28 \amalg 1\langle 1\rangle \amalg 1\langle 2\rangle \amalg 1\langle 3\rangle\rangle$

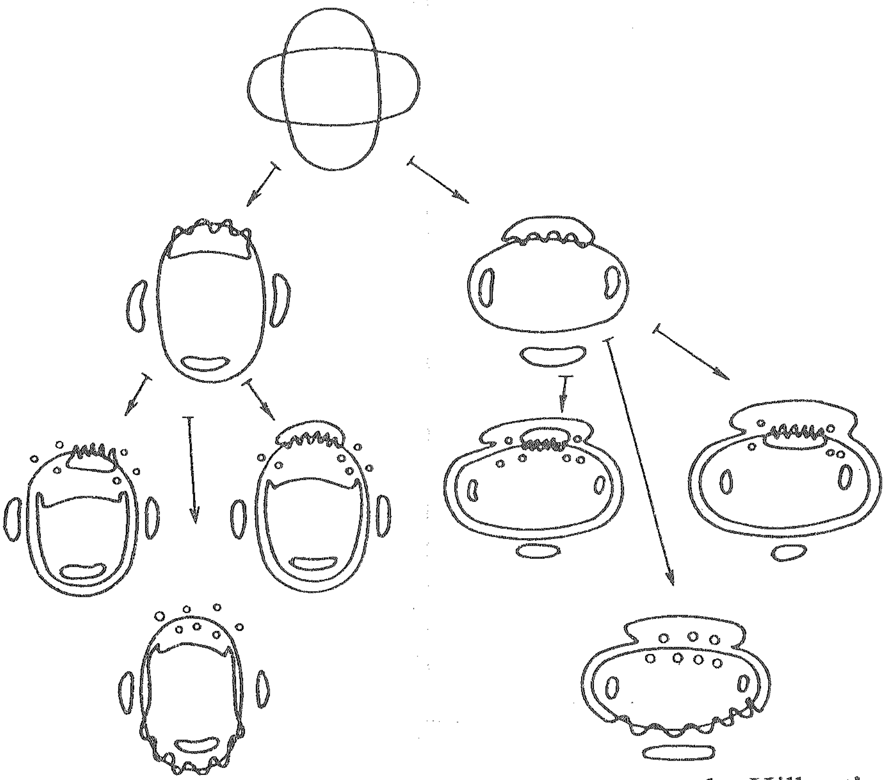

FIGURE 10. Construction of even degree curves by Hilbert's method. Degrees 4 and 6.

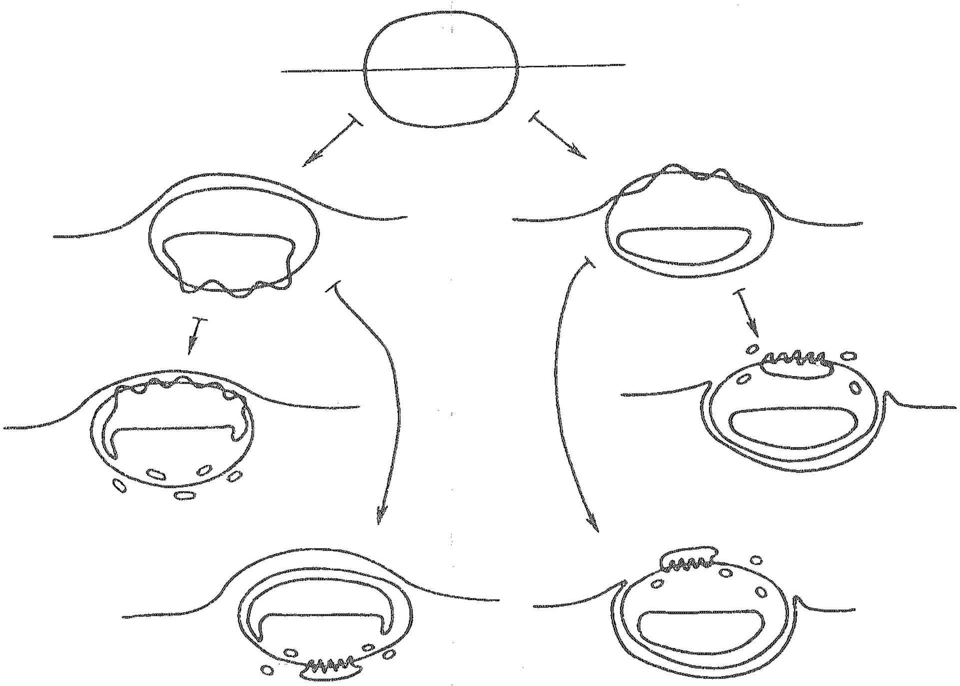

FIGURE 11. Construction of odd degree curves by Hilbert's method. Degrees 3 and 5.

In conclusion, we mention two curious properties of Harnack M-curves, for which the reader can easily furnish a proof.

1.9.A. *The depth of a nesting in a Harnack M-curve is at most* 2.

$\chi(\mathbb{R}P_-) = n - p + 1$. Hence, one should pay special attention to the numbers p and n. (It is amazing that essentially these considerations were stated in a

By analyzing the constructions, Ragsdale [44] made the following observations.

1.11.A. (compare with 1.9A and B). *For any Harnack M-curve of even degree m,*

$$p = (3m^2 - 6m + 8)/8, \qquad n = (m^2 - 6m + 8)/8.$$

1.11.B. *For any Hilbert M-curve of even degree m,*

$$(m^2 - 6m + 16)/8 \le p \le (3m^2 - 6m + 8)/8,$$
$$(m^2 - 6m + 8)/8 \le n \le (3m^2 - 6m)/8.$$

This gave her evidence for the following conjecture.

1.11.C. Ragsdale conjecture. For any curve of even degree m,

$$p \le (3m^2 - 6m + 8)/8, \qquad n \le (3m^2 - 6m)/8.$$

In §5 we shall return to this very first conjecture of a general nature on the topology of real algebraic curves. At this point we shall only mention that several weaker assertions have been proved and examples have been constructed which made it necessary to weaken the second inequality by 1. In the weaker form the Ragsdale conjecture has not yet been either proved or disproved.

The numbers p and n introduced by Ragsdale occur in many of of the prohibitions that were subsequently discovered. While giving full credit to Ragsdale for her insight, we must also say that, if she had looked more carefully at the experimental data available to her, she should have been able to find some of these prohibitions. For example, it is not clear what stopped her from making the conjecture which was made by Gudkov [9] in the late 1960's. Proof of these conjectures marked the beginning of the most recent stage in the development of the topology of real algebraic curves.

1.12. Generalizations of Harnack's and Hilbert's methods. Brusotti. Wiman. Ragsdale's work [44] was partly inspired by the erroneous paper of Halbrat, containing a proof of the false assertion that an M-curve can be obtained by means of a classical small perturbation (see §1.5) from only two M-curves, one of which must have degree ≤ 2. If this had been true, it would have meant that an inductive construction of M-curves by classical small perturbations starting with curves of small degree must essentially be either Harnack's method or Hilbert's method.

In 1910–1917, L. Brusotti showed that this is not the case. He found inductive constructions of M-curves based on classical small perturbation which were different from the methods of Harnack and Hilbert.

Before describing Brusotti's constructions, we need some definitions. A simple arc X in the set of real points of a curve A of degree m is said to be a *base of rank* ρ if there exists a curve of degree ρ which intersects the arc in ρm (distinct) points. A base of rank ρ is clearly also a base of rank any multiple of ρ (for example, one can obtain the intersecting curve of the corresponding degree as the union of several copies of the degree ρ curve, each copy shifted slightly).

An M-curve A is called a *generating curve* if it has disjoint bases X and Y whose ranks divide twice the degree of the curve. An M-curve A_0 of degree m_0 is called an *auxiliary curve* for the generating curve A of degree m with bases X and Y if the following conditions hold:

a) The intersection $\mathbf{R}A \cap \mathbf{R}A_0$ consist of mm_0 distinct points and lies in a single component K of $\mathbf{R}A$ and in a single component K of $\mathbf{R}A_0$.

b) The cyclic orders determined on the intersection $\mathbb{R}A \cap \mathbb{R}A_0$ by how it is situated in K and in K_0 are the same.

c) $X \subset \mathbb{R}A \setminus \mathbb{R}A_0$.

d) If K is a one-sided curve and $m_0 \equiv \bmod 2$, then the base X lies outside the oval K_0.

e) The rank of the base X is a divisor of the numbers $m + m_0$ and $2m$, and the rank of Y is a divisor of $2m + m_0$ and $2m$.

An auxiliary curve can be the empty curve of degree 0. In this case the rank of X must be a divisor of the degree of the generating curve.

Let A be a generating curve of degree m, and let A_0 be a curve of degree m_0 which is an auxiliary curve with respect to A and the bases X and Y. Since the rank of X divides $m + m_0$, we may assume that the rank is equal to $m + m_0$. Let C be a real curve of degree $m + m_0$ which intersects X in $m(m + m_0)$ distinct points. It is not hard to verify that a classical small perturbation of the curve $A \cup A_0$ by means of L will give an M-curve of degree $m + m_0$, and that this M-curve will be an auxiliary curve with respect to A and the bases obtained from Y and X (the bases must change places). We can now repeat this construction, with A_0 replaced by the curve that has just been constructed. Proceeding in this way, we obtain a sequence of M-curves whose degree forms an arithmetic progression: $km + m_0$ with $k = 1, 2, \ldots$. This is called the construction by Brusotti's method, and the sequence of M-curves is called a *Brusotti series*.

Any simple arc of a curve of degree ≤ 2 is a base of rank 1 (and hence of any rank). This is no longer the case for curves of degree ≤ 3. For example, an arc of a curve of degree 3 is a base of rank 1 if and only if it contains a point of inflection. (We note that a base of rank 2 on a curve of degree 3 might not contain a point of inflection: it might be on the oval rather than on the one-sided component where all of the points of inflection obviously lie. A curve of degree 3 with this type of base of rank 2 can be constructed by a classical small perturbation of a union of three lines.)

If the generating curve has degree 1 and the auxiliary curve has degree 2, then the Brusotti construction turns out to be Harnack's construction. The same happens if we take an auxiliary curve of degree 1 or 0. If the generating curve has degree 2 and the auxiliary curve has degree 1 or 2 (or 0), then the Brusotti construction is the same as Hilbert's construction.

In general, not all Harnack and Hilbert constructions are included in Brusotti's scheme; however, the Brusotti construction can easily be extended in such a way as to be a true generalization of the Harnack and Hilbert constructions. This extension involves allowing the use of an arbitrary number of bases of the generating curve. Such an extension is particularly worthwhile when the generating curve has degree ≤ 2, in which case there are arbitrarily many bases.

It can be shown that Brusotti's construction with generating curve of degree 1 and auxiliary curve of degree ≤ 4 gives the same types of M-curves as Harnack's construction. But as soon as one uses auxiliary curves of degree 5, one can obtain new isotopy types from Brusotti's construction. It was only in 1971 that Gudkov [11] found an auxiliary curve of degree 5 that did this. His construction was rather complicated, and so I shall only give some references [11], [12], [33] and present Figure 12, which illustrates the location of the degree 5 curve relative to the generating line. Even with the first stage of Brusotti's

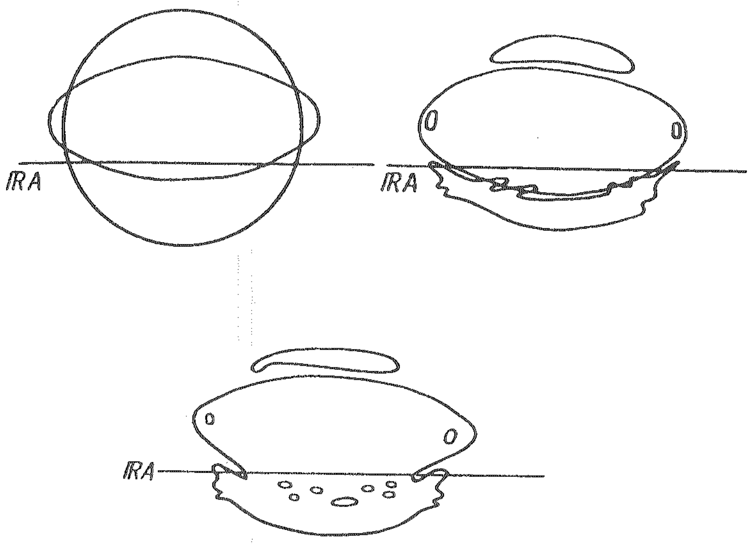

FIGURE 14. In the construction by Hilbert's method, we keep track of the locations relative to a fixed line A. The union of two conics is perturbed by means of a 4-tuple of lines. A curve of degree 4 is obtained. We add one of the original conics to this curve, and then perturb the union.

construction, i.e., the classical small perturbation of the union of the curve and the line, one obtains an M-curve (of degree 6) which has isotopy type $\langle 5 \amalg 1\langle 5\rangle\rangle$, an isotopy type not obtained using the constructions of Harnack and Hilbert. Such an M-curve of degree 6 was first constructed in a much more complicated way by Gudkov [9], [10] in the late 1960's.

In Figures 13 and 14 we show the construction of two curves of degree 6 which are auxiliary curves with respect to a line. In this case the Brusotti construction gives new isotopy types beginning with degree 8.

In the Hilbert construction we keep track of the location relative to a fixed line A. The union of two conics is perturbed by means of a quadruple of lines. One obtains a curve of degree 4. To this curve one then adds one of the original conics, and the union is perturbed.

In numerous papers by Brusotti and his students, many series of Brusotti M-curves were found. Generally, new isotopy types appear in them beginning with degree 9 or 10. In these constructions they paid much attention to combinations of nestings of different depths—a theme which no longer seems to be very interesting. An idea of the nature of the results in these papers can be obtained from Gudkov's survey [12]; for more details, see Brusotti's survey [34] and the papers cited there.

An important variant of the classical constructions of M-curves, of which we shall need to make use in the next section, is not subsumed under Brusotti's scheme even in its extended form. This variant, proposed by Wiman [48], consists in the following. We take an M-curve A of degree k having base X of rank dividing k; near this curve we construct a curve A' transversally intersecting A in k^2 points of X, after which we can subject the union $A \cup A'$ to a classical small perturbation, giving an M-curve of degree $2k$ (for example, a perturbation by means of an empty curve of degree $2k$). The resulting M-curve has the following topological structure: each of the components of the curve A except for one (i.e., except for the component containing X) is doubled, i.e., is replaced by a pair of ovals which are each close to an oval of the original curve, and the component containing X gives a chain of k^2 ovals. This new curve does not necessarily have a base, so that in general one cannot construct a series of M-curves in this way.

1.13. The first prohibitions not obtained from Bézout's theorem. The techniques discussed above are, in essence, completely elementary. As we saw (§1.4), they are sufficient to solve the isotopy classification problem for nonsingular projective curves of degree ≤ 5. However, even in the case of curves of degree 6 one needs subtler considerations. Not all of the failed attempts to construct new isotopy types of M-curves of degree 6 (after Hilbert's 1891 paper [36], there were two that had not been realized: $\langle 9 \amalg 1\langle 1\rangle\rangle$ and $\langle 1 \amalg 1\langle 9\rangle\rangle$) could be explained on the basis of Bézout's theorem. Hilbert undertook an attack on M-curves of degree 6. He was able to grope his way toward a proof that isotopy types cannot be realized by curves of degree 6, but the proof required a very involved investigation of the natural stratification of the space $\mathbf{R}C_6$ of real curves of degree 6. In [45], Rohn, developing Hilbert's approach, proved (while stating without proof several valid technical claims which he needed) that the types $\langle 11\rangle$ and $\langle 1\langle 10\rangle\rangle$ cannot be realized by curves of degree 6. It was not until the 1960's that the potential of this approach was fully developed by Gudkov. By going directly from Rohn's 1913 paper [45] to the work of Gudkov, I would violate the chronological order of my presentation of the history of prohibitions. But in fact I would only be omitting one important episode, to be sure a very remarkable one: the famous work of I. G. Petrovskiĭ [41], [42] in which he proved the first prohibition relating to curves of arbitrary even degree and not a direct consequence of Bézout's theorem.

1.13.A. PETROVSKIĬ'S THEOREM ([41], [42]). *For any nonsingular real projective algebraic plane curve of degree* $m = 2k$

$$-\tfrac{3}{2}k(k-1) \leq p - n \leq \tfrac{3}{2}k(k-1) + 1 .$$

(Recall that p denotes the number of even ovals on the curve (i.e., ovals each of which is enveloped by an even number of other ovals, see §1.11), and n denotes the number of odd ovals.)

Petrovskiĭ's proof was based on a technique that was new in the study of the topology of real curves: the Euler-Jacobi interpolation formula. Petrovskiĭ's theorem was generalized by Petrovskiĭ and Oleĭnik [18] to the case of varieties of arbitrary dimension, and by Oleĭnik [17] to the case of curves on a surface. More about the proof and the influence of Petrovskiĭ's work on the subsequent development of the subject can be found in Kharlamov's survey [26] in Petrovskiĭ's collected works. I will only comment that in application to nonsingular projective plane curves, the full potential of Petrovskiĭ's method, insofar as we are able to judge, was immediately realized by Petrovskiĭ himself.

We now turn to Gudkov's work. In a series of papers in the 1950's and 1960's, he completed the development of the techniques needed to realize Hilbert's approach to the problem of classifying curves of degree 6 (these techniques were referred to as the Hilbert-Rohn method by Gudkov), and he used the techniques to solve this problem (see [9]). The answer turned out to be elegant and stimulating.

1.13.B. Gudkov's theorem [9]. *The 56 isotopy types that follow, and no others, can be realized by nonsingular real projective algebraic plane curves of degree 6.*

$$\langle 9\amalg 1\langle 1\rangle\rangle \qquad \langle 5\amalg 1\langle 5\rangle\rangle \qquad \langle 1\amalg 1\langle 9\rangle\rangle$$

$$\langle 10\rangle \quad \langle 8\amalg 1\langle 1\rangle\rangle \quad \langle 5\amalg 1\langle 4\rangle\rangle \quad \langle 4\amalg 1\langle 5\rangle\rangle \quad \langle 1\amalg 1\langle 8\rangle\rangle \quad \langle 1\langle 9\rangle\rangle$$

$$\langle 9\rangle \quad \langle 7\amalg 1\langle 1\rangle\rangle \quad \langle 6\amalg 1\langle 2\rangle\rangle \quad \langle 5\amalg 1\langle 3\rangle\rangle \quad \langle 4\amalg 1\langle 4\rangle\rangle \quad \langle 3\amalg 1\langle 5\rangle\rangle \quad \langle 2\amalg 1\langle 6\rangle\rangle \quad \langle 1\amalg 1\langle 7\rangle\rangle \quad \langle 1\langle 8\rangle\rangle$$

$$\langle 8\rangle \quad \langle 6\amalg 1\langle 1\rangle\rangle \quad \langle 5\amalg 1\langle 2\rangle\rangle \quad \langle 4\amalg 1\langle 3\rangle\rangle \quad \langle 3\amalg 1\langle 4\rangle\rangle \quad \langle 2\amalg 1\langle 5\rangle\rangle \quad \langle 1\amalg 1\langle 6\rangle\rangle \quad \langle 1\langle 7\rangle\rangle$$

$$\langle 7\rangle \quad \langle 5\amalg 1\langle 1\rangle\rangle \quad \langle 4\amalg 1\langle 2\rangle\rangle \quad \langle 3\amalg 1\langle 3\rangle\rangle \quad \langle 2\amalg 1\langle 4\rangle\rangle \quad \langle 1\amalg 1\langle 5\rangle\rangle \quad \langle 1\langle 6\rangle\rangle$$

$$\langle 6\rangle \quad \langle 4\amalg 1\langle 1\rangle\rangle \quad \langle 3\amalg 1\langle 2\rangle\rangle \quad \langle 2\amalg 1\langle 3\rangle\rangle \quad \langle 1\amalg 1\langle 4\rangle\rangle \quad \langle 1\langle 5\rangle\rangle$$

$$\langle 5\rangle \quad \langle 3\amalg 1\langle 1\rangle\rangle \quad \langle 2\amalg 1\langle 2\rangle\rangle \quad \langle 1\amalg 1\langle 3\rangle\rangle \quad \langle 1\langle 4\rangle\rangle$$

$$\langle 4\rangle \quad \langle 2\amalg 1\langle 1\rangle\rangle \quad \langle 1\amalg 1\langle 2\rangle\rangle \quad \langle 1\langle 3\rangle\rangle$$

$$\langle 3\rangle \quad \langle 1\amalg 1\langle 1\rangle\rangle \quad \langle 1\langle 2\rangle\rangle \qquad\qquad \langle 1\langle 1\langle 1\rangle\rangle\rangle$$

$$\langle 2\rangle \quad \langle 1\langle 1\rangle\rangle$$

$$\langle 1\rangle$$

$$\langle 0\rangle$$

This result, along with the available examples of curves of higher degree, led Gudkov to the following conjectures.

1.13.C. Gudkov's conjectures [9]. (i) *For any M-curve of even degree $m = 2k$*

$$p - n \equiv k^2 \mod 8 .$$

(ii) *For any $(M-1)$-curve of even degree $m = 2k$*

$$p - n \equiv k^2 \pm 1 \mod 8 .$$

While attempting to prove conjecture 1.13.C(i), V. I. Arnol′d [1] discovered some striking connections between the topology of a real algebraic plane curve and the topology of its complexification. Although he was able to prove the conjecture itself only in a weaker form (modulo 4 rather than 8), the new point of view he introduced to the subject opened up a remarkable perspective, and in fact immediately brought fruit: in the same paper [1] Arnol′d proved several new prohibitions (in particular, he strengthened Petrovskiĭ's inequalities 1.13.A). The full conjecture 1.13.C(i) and its high-dimensional generalizations were proved by Rokhlin [21], based on the connections discovered by Arnol′d in [1]. I am recounting this story briefly here only to make the preliminary exposition more complete. For details and information about further work, see Gudkov's survey [12]. To learn about the many results obtained using methods from the modern topology of manifolds and complex algebraic geometry (the use of which was begun by Arnol′d in [1]), the reader is referred to the surveys [47], [23], [2], [25], [26], [8].

§2. Prohibitions

In this section we describe the current state of prohibitions on the topology of real projective algebraic plane curves of a given degree. We do this to convey a general impression rather than anything more serious: although we give careful, and when possible complete statements of results, the proofs are hardly discussed at all. See the above surveys and the papers cited there.

2.1. Real algebraic curves from a complex viewpoint. According to a tradition going back to Hilbert, for a long time the main question concerning the topology of real algebraic curves was considered to be the determination of which isotopy types are realized by nonsingular real projective algebraic plane curves of given degree (i.e., problem 1.1.B above). However, as early as 100 years ago F. Klein [39] posed the question more broadly. He was also interested in how the isotopy type of a curve is connected with the way its set $\mathbf{R}A$ of real points is situated in the set $\mathbf{C}A$ of complex points (i.e., the set of points of the complex projective plane whose homogeneous coordinates satisfy the equation defining the curve A).

The set $\mathbf{C}A$ is an oriented smooth two-dimensional submanifold of the complex projective plane $\mathbf{C}P^2$ which is invariant under the complex conjugation involution $\operatorname{conj}\colon \mathbf{C}P^2 \to \mathbf{C}P^2 : (z_0 : z_1 : z_2) \mapsto (\bar{z}_0 , \bar{z}_1 , \bar{z}_2)$. The curve $\mathbf{R}A$ is the set of fixed points of the restriction of this involution to $\mathbf{C}A$. The real curve may or may not divide $\mathbf{C}A$. In the first case we say that A is a *dividing* curve or a curve of *type* I, and in the second case we say that it is a *nondividing* curve or a curve of *type* II. In the first case $\mathbf{R}A$ divides $\mathbf{C}A$ into two connected pieces. The natural orientations of these two halves determine two opposite orientations on $\mathbf{R}A$ (which is the common boundary); they are called the *complex orientations* on the curve.

The scheme of the relative location of the ovals of a curve is called the *real scheme* of the curve. If along with the real scheme we indicate the type of the curve, and in the case of type I also the complex orientations, then we call this information the *complex scheme* of the curve.

We say that the real scheme of a curve of degree m is *of type* I (*of type* II) if any curve of degree m having this real scheme is a curve of type I (of type II). Otherwise (i.e., if there exist both curves of type I and curves of type II with the given real scheme), we say that the real scheme is of *indeterminate type*.

The division of curves into types is due to Klein [39]. It was Rokhlin [22] who introduced the complex orientations into the study of the topology of real algebraic curves. He also introduced the notion of a complex scheme and its type [23]. These ideas occupy a central place in the contemporary development of the subject. In recent years our point of view concerning the problems in the topology of real algebraic varieties has broadened so that the basic object is no longer the real manifolds themselves, but rather the real manifolds together with their location in their complexification. This viewpoint is also largely due to Rokhlin.

2.2. Flexible curves. A large part of all of the known prohibitions follow from a relatively small number of purely topological properties of algebraic curves. Hence, along with algebraic curves it is useful to consider objects which imitate them in the topological sense.

An oriented smooth closed and connected two-dimensional submanifold M of the complex projective plane $\mathbf{C}P^2$ is called a *flexible curve of degree* m if:

(i) it realizes a class $m[\mathbf{C}P^1] \in H_2(\mathbf{C}P^2)$;

(ii) its genus is equal to $(m-1)(m-2)/2$;

(iii) it is invariant under conj;

(iv) its field of tangent planes on $M \cap \mathbf{R}P^2$ can be deformed in the class of planes invariant under conj into the field of lines in $\mathbf{C}P^2$ which are tangent to $M \cap \mathbf{R}P^2$.

A flexible curve intersects $\mathbf{R}P^2$ in a smooth one-dimensional submanifold, which is called the *real part* of the flexible curve. Obviously, the set of complex points of a nonsingular algebraic curve of degree m is a flexible curve of degree m. Everything said in the last subsection about algebraic curves and their (real and complex) schemes carries over without any changes to the case of flexible curves. We say that a prohibition on the schemes of curves of degree m *comes from topology* if it can be proved for the schemes of flexible curves of degree m. The classification that we have for the schemes of curves of degree ≤ 6 can be obtained from prohibitions that come from topology, i.e., for $m \leq 6$ all prohibitions come from topology.

2.3. Prohibitions on the real schemes of degree m curves which come from topology. In this subsection I will list all such prohibitions that I know of at this point, including the ones already referred to above, but excluding prohibitions which follow from the other prohibitions given here or from the prohibitions on the complex schemes which are given in the next subsection.

2.3.A. *A curve is one-sided if and only if it has odd degree.*

This fact was given before as a corollary of Bézout's theorem (see §1.3). But it also holds for flexible curves. The same can be said about Harnack's inequality, which is undoubtedly the best known and most important prohibition.

2.3.B. HARNACK'S INEQUALITY. *The number of components of the set of real points of a curve of degree m is at most $(m-1)(m-2)/2+1$.*

In prohibitions 2.3.C–2.3.M the degree m of the curve is even: $m = 2k$.

Extremal properties of Harnack's inequality.

2.3.C. THE GUDKOV-ROKHLIN CONGRUENCE. *In the case of an M-curve (i.e., when $p + n = (m-1)(m-2)/2 + 1$),*

$$p - n \equiv k^2 \mod 8.$$

2.3.D. THE GUDKOV-KRAKHNOV-KHARLAMOV CONGRUENCE. *In the case of an $(M-1)$-curve (i.e., when $p + n = (m-1)(m-2)/2$),*

$$p - n \equiv k^2 \pm 1 \mod 8.$$

The Euler characteristic of a component of the complement of a curve in $\mathbf{R}P^2$ is called the *characteristic* of the oval which bounds the component from outside.

2.3.E. FIDLER'S CONGRUENCE. *If the curve is an M-curve, $m \equiv 4 \mod 8$, and every even oval has even characteristic, then*

$$p - n \equiv -4 \mod 16.$$

2.3.F. Nikulin's congruence. *If the curve is an M-curve, $m \equiv 0 \bmod 8$, and every even oval has characteristic divisible by 2^r, then*

$$\text{either } p - n \equiv 0 \mod 2^{r+3},$$
$$\text{or else } p - n = 4^q \chi,$$

where $q \geq 2$ and $\chi \equiv 1 \bmod 2$.

2.3.G. Nikulin's congruence. *If the curve is an M-curve, $m \equiv 2 \bmod 4$, and every odd oval has characteristic divisible by 2^r, then*

$$p - n \equiv 1 \mod 2^{r+3}.$$

We let p^+ denote the number of even ovals with positive characteristic, we let p^0 denote the number of even ovals with zero characteristic, and we let p^- denote the number of even ovals with negative characteristic. We similarly define n^+, n^0 and n^- for the odd ovals; and we let l^+, l^0 and l^- be the corresponding numbers for both even and odd ovals together.

Refined Petrovskiĭ inequalities.

2.3.H. $p - n^- \leq (3k^2 - 3k + 2)/2$.

2.3.I. $n - p^- \leq (3k^2 - 3k)/2$.

Refined Arnol'd inequalities.

2.3.J. $p^- + p^0 \leq (k-1)(k-2)/2 + (1 + (-1)^k)/2$.

2.3.K. $n^- + n^0 \leq (k-1)(k-2)/2$.

Extremal properties of the refined Arnol'd inequalities.

2.3.L. *If k is even and $p^- + p^0 = (k-1)(k-2)/2 + 1$, then $p^- = p^+ = 0$.*

2.3.M. *If k is odd and $n^- + n^0 = (k-1)(k-2)/2$, then $n^- = n^+ = 0$ and there is only one outer oval in all.*

Besides Harnack's inequality, we know only one prohibition coming from topology which extends to real schemes of both even and odd degree.

2.3.N. The Viro-Zvonilov inequality. *If h is a divisor of m which is a power of an odd prime, and if $m \neq 4$, then*

$$l^- + l^0 \leq (m-3)^2/4 + (m^2 - h^2)/4h^2.$$

If m is even, this inequality follows from 2.3.J–L.

2.3.O. Extremal property of the Viro-Zvonilov inequality. *If $l^- + l^0 = (m-3)^2/4 + (m^2 - h^2)/4h^2$, where h is a divisor of m and a power of an odd prime p, then there exist $\alpha_1, \dots, \alpha_r \in \mathbf{Z}_p$ and components $B_1, \dots, B_r$ of the complement $\mathbf{R}P^2 \backslash \mathbf{R}A$ with $\chi(B_1) = \dots = \chi(B_r) = 0$, such that the boundary of the chain $\sum_{i=1}^r \alpha_i [B_i] \in C_2(\mathbf{R}P^2; \mathbf{Z}_p)$ is $[\mathbf{R}A] \in C_1(\mathbf{R}P^2; \mathbf{Z}_p)$.*

2.4. Prohibitions on the complex schemes of degree m curves which come from topology. Recall that l denotes the total number of ovals on the curve.

2.4.A. *If the curve is a partitioning curve, then $l \equiv [m/2] \bmod 2$.*

An injective pair of ovals (i.e., a pair of ovals one of which is enveloped by the other) on an oriented topological plane curve is said to be *positive* if the orientations of the ovals determined by the orientation of the entire curve are induced by an orientation of the annulus bounded by the ovals. Otherwise, the injective pair of ovals is said to be *negative*. It is clear that the division of pairs of ovals into positive and negative pairs does not change if the orientation of the entire curve is reversed; thus, the injective pairs of ovals on a curve of type I are divided into positive and negative pairs relative to the complex orientations of the curve. We let Π^+ denote the number of positive pairs, and Π^- denote the number of negative pairs.

2.4.B. ROKHLIN'S FORMULA. *If the degree* m *is even and the curve is of type* I, *then*

$$2(\Pi^+ - \Pi^-) = l - m^2/4.$$

The ovals of an oriented curve can be divided into positive and negative ovals. Namely, we consider the Möbius strip which is obtained when an inner oval is removed from $\mathbf{R}P^2$. If the integral homology classes which are realized in this strip by the oval and by the doubled one-sided component with orientations determined by the orientation of the entire curve have different signs, then we say that the oval is *positive*; otherwise, we say that it is *negative*. In the case of a two-sided oriented curve, only the nonouter ovals can be divided into positive and negative ovals. Namely, a nonouter oval is said to be *positive* if it forms a positive pair with the outer oval which envelops it; otherwise, it is said to be *negative*. As in the case of pairs, if the orientation of the curve is reversed, the division of ovals into positive and negative ones does not change. We let Λ^+ denote the number of positive ovals on a curve, and we let Λ^- denote the number of negative ones.

2.4.C. THE ROKHLIN-MISHACHEV FORMULA. *If* m *is odd and the curve is of type* I, *then*

$$\Lambda^+ - \Lambda^- + 2(\Pi^+ - \Pi^-) = l - (m^2 - 1)/4.$$

Extremal properties of Harnack's inequality.

2.4.D. *Any* M*-curve is of type* I.

2.4.E. THE KHARLAMOV-MAREN CONGRUENCE. *Any* $(M-2)$*-curve of even degree* $m = 2k$ *with* $p - n \equiv (k^2 + 4) \bmod 8$ *is of type* I.

Extremal properties of the refined Arnol'd inequalities.

2.4.F. *If* $m \equiv 0 \bmod 4$ *and* $p^- + p^0 = (m^2 - 6m + 16)/8$, *then the curve is of type* I.

2.4.G. *If* $m \equiv 0 \bmod 4$ *and* $n^- + n^0 = (m^2 - 6m + 8)/8$, *then the curve is of type* I.

Congruences.

2.4.H. THE NIKULIN-FIDLER CONGRUENCE. *If* $m \equiv 0 \bmod 4$, *the curve is of type* I, *and every even oval has even characteristic, then* $p - n \equiv 0 \bmod 8$.

I will state two more congruences, which are consequences of Rokhlin's formula 2.4.B.

2.4.I. ARNOL'D'S CONGRUENCE. *If* m *is even and the curve is of type* I, *then* $p - n \equiv m^2/4 \bmod 4$.

2.4.J. SLEPIAN'S CONGRUENCE. *If* m *is even, the curve is of type* I, *and every odd oval has even characteristic, then* $p - n \equiv m^2/4 \bmod 8$.

We let π and ν denote the number of even and odd nonempty ovals, respectively, bounding from the outside those components of the complement of the curve which have the property that each of the ovals bounding them from the inside envelops an odd number of other ovals.

Rokhlin's inequalities.

2.4.K. *If the curve is of type* I *and* $m \equiv 0 \bmod 4$, *then*

$$4\nu + p - n \leq (m^2 - 6m + 16)/2.$$

2.4.L. *If the curve is of type* I *and* $m \equiv 2 \bmod 4$, *then*

$$4\pi + n - p \leq (m^2 - 6m + 14)/2.$$

2.5. Prohibitions not proved for flexible curves. As a rule, these prohibitions are hard to visualize, in the sense that it is difficult to state in full generality the results obtained by some particular method. To one extent or another all of them are consequences of Bézout's theorem.

To state the simplest of these prohibitions we introduce the following notation. We let h_r denote the maximum number of ovals occurring in a union of $\leq r$ nestings. We let h'_r denote the maximum number of ovals in a set of ovals contained in a union of $\leq r$ nestings but not containing an oval which envelops all of the other ovals in the set. With this notation Theorems 1.3.C and 1.3.D can be stated as follows:

2.5.A. $h_2 \leq m/2$; *in particular, if* $h_1 = [m/2]$, *then* $l = [m/2]$.

2.5.B. $h'_5 \leq m$; *in particular, if* $h'_4 = m$, *then* $l = m$.

These statements suggest a whole series of similar assertions. We let $c(q)$ denote the greatest number c such that there is a connected curve of degree q passing through any c points of $\mathbf{R}P^2$ in general position. It is known that $c(1) = 2$, $c(2) = 5$, $c(3) = 8$, $c(4) = 13$.

2.5.C (generalization of Theorem 2.5.A). *If* $r \leq c(q)$ *with* q *odd, then*

$$h_r + [c(q) - r/2] \leq qm/2.$$

In particular, if $h_{c(q)-1} = [qm/2]$, *then* $l = [qm/2]$.

2.5.D (generalization of Theorem 2.5.B). *If* $r \leq c(q)$ *with* q *even, then*

$$h'_r + [(c(q) - r)/2] \leq qm/2.$$

In particular, if $h'_{c(q)-1} = qm/2$, then $l = qm/2$.

We have the following two restrictions on complex schemes which are similar to Theorem 2.5.A and B. However, I do not know of analogues of 2.5.C and D.

2.5.E. *If* $h_1 = [m/2]$, *then the curve is of type* I.

2.5.F. *If* $h'_4 = m$, *then the curve is of type* I.

Here I will not discuss the methods of proof for prohibitions which do not come from topology. See [8], [7]. I will only give some statements of results which have been obtained for curves of small degree.

2.5.G. *There is no curve of degree* 7 *with the real scheme* $\langle J \amalg 1\langle 14\rangle\rangle$.

2.5.H. *If an* M*-curve of degree* 8 *has real scheme* $\langle \alpha \amalg 1\langle\beta\rangle \amalg 1\langle\gamma\rangle \amalg 1\langle\delta\rangle\rangle$ *with nonzero* β, γ *and* δ, *then* β, γ *and* δ *are odd.*

2.5.I. *If an* $(M-2)$*-curve of degree* 8 *with* $p - n \equiv 4 \bmod 8$ *has real scheme* $\langle \alpha \amalg 1\langle\beta\rangle \amalg 1\langle\gamma\rangle \amalg 1\langle\delta\rangle\rangle$ *with nonzero* β, γ *and* δ, *then two of the numbers* β, γ, δ *are odd and one is even.*

2.6. Sharpness of the inequalities. The arsenal of constructions in §1 and the supply of curves constructed there, which are very modest from the point of view of classification problems, turn out to be quite rich if we are interested in the problem of sharpness of the inequalities in §2.3.

The Harnack curves of even degree m with scheme

$$\langle (3m^2 - 6m)/8 \amalg 1\langle (m^2 - 6m + 8)/8\rangle\rangle$$

which were constructed in §1.6 (see also §1.9) not only show that Harnack's inequality 2.3.B is best possible, but also show the same for the refined Petrovskiĭ inequality 2.3.H.

One of the simplest variants of Hilbert's construction (see §1.10) leads to the construction of a series of M-curves of degree $m \equiv 2 \bmod 4$ with scheme $\langle (m^2 - 6m + 8)/8 \amalg 1 \langle (3m^2 - 6m)/8 \rangle \rangle$. This proves that the refined Petrovskiĭ inequality 2.3.I for $m \equiv 2 \bmod 4$ is sharp. If $m \equiv 0 \bmod 4$, the methods in §1 do not show that this inequality is best possible. That fact will be proved below in §5.4.

The refined Arnol′d inequality 2.3.J is best possible for any even m. If $m \equiv 2 \bmod 4$, this can be proved using the Wiman M-curves (see the end of §1.12). If $m \equiv 0 \bmod 4$, it follows using curves obtained from a modification of Wiman's construction: the construction proceeds in exactly the same way, except that the opposite perturbation is taken, as a result of which one obtains a curve that can serve as the boundary of a tubular neighborhood of an M-curve of degree $m/2$.

The last construction (doubling), if applied to an M-curve of odd degree, shows that the refined Arnol′d inequality 2.3.K is best possible for $m \equiv 2 \bmod 4$. If $m \equiv 0 \bmod 4$, almost nothing is known about sharpness of the inequality 2.3.K, except that for $m = 8$ the right side can be lowered by 2.

CHAPTER 2
Constructions Using Curves with Complicated Singularities and Their Perturbations

§3. Perturbations of curves with semi-quasihomogeneous singularities

The classical constructions in the topological theory of real algebraic curves (i.e., the constructions considered above) proceed according to the following general scheme. First one constructs two nonsingular curves which are transversal to one another, and then one slightly perturbs their union to remove the singularities. In his classification of curves of degree 6, Gudkov departed from this scheme; however, as before, all of the curves that he perturbed had only nondegenerate double points. There are two circumstances which stand in the way of allowing more complicated singularities when constructing real algebraic curves with prescribed topology. In the first place, if the singularities are not very complicated, they give nothing more than one obtains with nondegenerate double points—to get something new one must go to nondegenerate 5-fold multiple points or to points of tangency of three branches. In the second place, one needs a special technique in order to carry out controlled perturbations of curves with complicated singularities.

In 1980 I proposed a method of constructing perturbations of curves with a semi-quasihomogeneous singularity. From a topological point of view, the perturbation causes a neighborhood of the singular point to be replaced by a model curve fragment prepared in advance. This technique made it possible to enlarge the possible constructions significantly. We could then complete the isotopy classification of nonsingular curves of degree 8 and refine Ragsdale's conjecture.

This section and the one that follows are devoted to developing perturbation techniques for curves with singularities.

3.1. Newton polygons. Let f be a polynomial in two variables over $\mathbf{C}$ or $\mathbf{R}$: $f(x, y) = \sum_{i,j} a_{ij} x^i y^j$. The monomials which occur in f can be depicted in a natural way on the plane: to a monomial $a_{ij} x^i y^j$ we associate the point $(i, j) \in \mathbf{R}^2$. It was Newton who noticed the usefulness of this representation of the monomials: it turns out that the relative position of these points (i, j) has a remarkable connection with the role played by the corresponding monomials for various special values of x and y. A lot of information about f and about the geometry of the curve $f(x, y) = 0$ is contained even in the convex hull of the set $\{(i, j) \in \mathbf{R}^2 | a_{ij} \neq 0\}$, which we denote $\Delta(f)$ and call the *Newton polygon* of f.

We list some obvious connections between the geometry of the curve $f(x, y) = 0$ and the properties of the Newton polygon $\Delta(f)$.

The polygon $\Delta(f)$ does not contain $(0, 0)$ if and only if the curve $f(x, y) = 0$ passes through the point $(0, 0)$.

The polygon $\Delta(f)$ does not contain $(0, 0)$ but does contain $(1, 0)$ or $(0, 1)$ if and only if the origin $(0, 0)$ is a regular point of the curve $f(x, y) = 0$.

More generally, the point $(0, 0)$ is an n-fold singular point of the curve $f(x, y) = 0$ if and only if n is the least number such that the line $x + y = n$ intersects $\Delta(f)$.

These facts are included in the following principle, various manifestations of which we will encounter often: the behavior of the curve $f(x, y) = 0$ near the origin is determined *to a first approximation* by the monomials of f corresponding to the points of the part of the boundary $\Delta(f)$ which faces the origin. This is because those monomials are the leading terms of f as x and y tend to zero.

Not only invariants of the singular points of the curve $f(x, y) = 0$, but also several global invariants can be expressed in terms of the Newton polygon $\Delta(f)$, see [27], [28]. In particular, if the curve $f(x, y) = 0$ has no complex singular points in $(\mathbf{C} \setminus 0) \times (\mathbf{C} \setminus 0)$ (i.e., in the complement of the coordinate axes), then its genus is equal to the number of points with integer coordinates lying inside $\Delta(f)$.

Given a set $\Gamma \subset \mathbf{R}^2$ and a polynomial $f(x, y) = \sum a_{ij} x^i y^j$, we let f^Γ denote the polynomial $\sum_{(i,j)\in\Gamma} a_{ij} x^i y^j$, i.e., the sum of the monomials of f which correspond to points in Γ; we shall call this the Γ-truncation of f.

The definition of the Newton polygon of a polynomial in two variables carries over in the obvious way to any multivariate polynomial (where, of course, we speak of the *Newton polyhedron* rather than the Newton polygon). If our polynomial is a homogeneous polynomial a of degree m in three variables, then it turns out to be a polygon lying inside the triangle defined by the conditions

$$i_0 + i_1 + i_2 = m, \qquad i_0 \geq 0,\ i_1 \geq 0,\ i_2 \geq 0.$$

But in practice it is convenient to replace this polygon by its projection onto the plane $i_0 = 0$, which is the Newton polygon of the polynomial $a(1, x, y)$. That is, we represent the monomials in a in tabular form on the plane, associating a monomial $a_{ij} x_0^{m-i-j} x_1^i x_2^j$ to the point $(i, j) \in \mathbf{R}^2$. This will be our convention: thus, we let $\Delta(a)$ denote the Newton polygon of the polynomial $a(1, x, y)$.

What was said before about the connection between the geometry of an affine curve and the geometry of its Newton polygon has obvious analogues in the projective situation. In particular, the behavior of a degree m curve $a(x_0, x_1, x_2) = 0$ nears the points $(1:0:0)$, $(0:1:0)$, and $(0:0:1)$ is determined to a first approximation by the monomials of a corresponding to the points of the part of the boundary $\Delta(a)$ which faces $(0, 0)$, $(m, 0)$ and $(0, m)$, respectively.

3.2. Singularities of a hypersurface. Much of what we say applies to either real or complex curves. In such cases I will use the following notation to encompass both situations. We let K denote the ground field ($\mathbf{R}$ or $\mathbf{C}$). When we discuss the singular points of algebraic curves, it costs us almost nothing to make another extension of the type of objects under consideration by passing from singularities of *algebraic* curves to singularities of *analytic* curves. Finally, many of the statements carry over without change to the case of isolated singularities on a hypersurface. One could go even further and not limit oneself to hypersurfaces—but this would lead to essential complications. In this subsection we shall consider some general definitions and results on isolated singularities of real or complex analytic hypersurfaces.

Let $G \subset K^n$ be an open set, and let $\varphi: G \to K$ be an analytic function. For $U \subset G$ we let $V_U(\varphi)$ denote the set $\{x \in U / \varphi(x) = 0\}$. By a *singularity* of the hypersurface $V_G(\varphi)$ at the point $x_0 \in U_G(\varphi)$ we mean the class of germs of hypersurfaces which are diffeomorphic to the germ of the hypersurface $V_G(\varphi)$ at x_0. In other words, two hypersurfaces $V_G(\varphi)$ and $V_H(\psi)$ have the same singularity at the points x_0 and y_0, if there exist neighborhoods M and N of x_0 and y_0 such that the pairs $(M, V_M(\varphi))$, $(N, V_N(\psi))$ are diffeomorphic.

When we consider the singularity of a hypersurface at a point x_0, to simplify the formulas we shall suppose that $x_0 = 0$. The *Milnor number* of the hypersurface $V_G(\varphi)$ at 0 is the dimension

$$\dim_K K[[x_1, \dots, x_n]]/(\partial f/\partial x_1, \dots, \partial f/\partial x_n)$$

of the quotient of the formal power series ring by the ideal generated by the partial derivatives $\partial f/\partial x_1, \dots, \partial f/\partial x_n$ of the Taylor series f of the function φ at 0. This number is an invariant of the singularity (see [3]). If it is finite, then we say that the singularity has *finite multiplicity*. In order for the singularity of the hypersurface $V_G(\varphi)$ at zero to have finite multiplicity, it is necessary (and when $K = \mathbf{C}$ it is also sufficient) that it be isolated, i.e., that there exist a neighborhood $U \subset K^n$ of zero which does not contain nonzero singular points of $V_G(\varphi)$. In the case of an isolated singularity, a ball $B \subset K^n$ centered at zero of sufficiently small radius has boundary ∂B which intersects $V_G(\varphi)$ only at nonsingular points and only transversally, and the pair $(B, V_{\partial B}(\varphi))$ is homeomorphic to the cone over its boundary $(\partial B, V_{\partial B}(\varphi))$ (see [15], Theorem 2.10). In this case the pair $(\partial B, V_{\partial B}(\varphi))$ is called the *link* of the singularity of $V_G(\varphi)$ at 0.

The next theorem shows that the class of singularities of finite multiplicity coincides with the class of singularities of finite multiplicity on algebraic hypersurfaces.

3.2.A. TOUGERON'S THEOREM (see, for example, [3], §6.3). *If the singularity at 0 of the hypersurface $V_G(\varphi)$ has finite Milnor number μ, then there exist a neighborhood U of 0 in K^n and a diffeomorphism h from this neighborhood onto a neighborhood of 0 in K^n such that $h(V_U(\varphi)) = V_{h(U)}(f_{(\mu+1)})$, where $f_{(\mu+1)}$ is the degree $\mu + 1$ Taylor polynomial of φ.*

The notion of Newton polyhedron carries over in a natural way to power series. The Newton Polyhedron $\Delta(f)$ of the series $f(x) = \sum_{\omega \in \mathbf{Z}_n} a_\omega x^\omega$ (where $x^\omega = x_1^{\omega_1} x_2^{\omega_2} \cdots x_n^{\omega_n}$) is the convex hull of the set $\{\omega \in \mathbf{R}^n | a_\omega \neq 0\}$. (Unlike the case of a polynomial, the Newton polyhedron $\Delta(f)$ of a power series may have infinitely many faces.) But in the theory of singularities the notion of the Newton diagram is of greater importance. The *Newton diagram* $\Gamma(f)$ of a power series f is the union of the compact faces of the Newton polyhedron which face the origin. From the definition of the Milnor number it follows that, if the singularity of $V_G(\varphi)$ at 0 has finite multiplicity, then the Newton diagram of the Taylor series of φ is compact, and its distance from each of the coordinate axes is at most 1. It follows from Tougeron's theorem that in this case adding a monomial of the form $x_i^{m_i}$ to φ with m_i sufficiently large does not change the singularity. Thus, without changing the singularity, one can get the Newton diagram to touch the coordinate axes.

3.3. Dissipating singularities. Now let the function $\varphi: G \to K$ be included as φ_0 in a family of analytic functions $\varphi_t: G \to K$ with $t \in [0, t_0]$, and suppose that this is an analytic family in the sense that the function $G \times [0, t_0] \to K: (x, t) \mapsto \varphi_t(x)$ which it determines is real analytic. If the hypersurface $V_G(\varphi)$ has an isolated singularity at x_0, and if there exists a neighborhood U of x_0 such that the hypersurfaces $V_G(\varphi_t)$ with $t \in [0, t_0]$ do not have singular points in U, then we say that the family of functions φ_t with $t \in [0, t_0]$ *dissipates*(3) the singularity of $V_G(\varphi)$ at x_0.

If the family ϑ_t with $t \in [0, t_0]$ dissipates the singularity of the hypersurface $V_G(\vartheta_0)$ at x_0, then there exists a ball $B \subset K^n$ centered at x_0 such that

(i) for $t \in [0, t_0]$ the sphere ∂B intersects $V_G(\vartheta_t)$ only at nonsingular points of the hypersurface and only transversally;

(ii) for $t \in (0, t_0]$ the ball B contains no singular points of the hypersurface $V_G(\vartheta_t)$;

(iii) the pair $(B, V_B(\varphi_0))$ is homeomorphic to the cone over its boundary $(\partial B, V_{\partial B}(\varphi_0))$.

Then the family of pairs $(B, V_B(\vartheta_t))$ with $t \in [0, t_0]$ is called the *dissipation* of the germ of the hypersurface $V_G(\varphi_0)$ at the point x_0. (Following the accepted terminology in the theory of singularities, we would be more correct in saying not a family of pairs, but rather a family of germs or even germs of a family; however, from a topological point of view, which is more natural in discussing the topology of real algebraic varieties, the distinction between a family of pairs satisfying (i) and (ii) and the corresponding family of germs is of no importance, and so we shall ignore it.)

Conditions (i) and (ii) imply the existence of a smooth isotopy $h_t: B \to B$ with $t \in (0, t_0]$, such that $h_{t_0} = \mathrm{id}$ and $h_t(V_B(\varphi_{t_0})) = V_B(\varphi_t)$, so that the pairs $(B, V_B(\varphi_t))$ with $t \in (0, t_0]$ are diffeomorphic to one another.

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(3)This word (Russian: распускание) has not been used before in the literature. Instead, the expressions "removing singularities" and "perturbing singularities" are used. The first term does not seem to me to be a good choice, since what occurs is not so much an annihilation of the singularity as its replacement by a rather complicated object, and another way of removing a singularity is to resolve it. The second expression is also unfortunate, since the perturbed singularity is no longer a singularity, while in other situations (perturbation of curves, operators, etc.) one does not leave the class of objects under consideration (a perturbed operator is still an operator, for instance). This terminology presumably arose because one has a perturbation of the singular hypersurface. The term "dissipation" is close in meaning to the word "unfolding," which refers to a versal deformation of a singularity. An unfolding is a deformation from which all deformations of the singularity, including the dissipations in our sense, can be obtained. Since the term "unfolding" has already been used, and the word "dissipation" is available and has much the same meaning, it seems to me to be an appropriate term in this context. The word "smoothing" is also less suitable, since it implies the introduction of a differentiable structure.

If we have two germs determining the same singularity, then a dissipation of one of them obviously corresponds to a diffeomorphic dissipation of the other germ. Thus, we may speak not only of dissipations of germs, but also of *dissipations of singularities* of a hypersurface.

The following three topological classification questions arise in connection with dissipations.

3.3.A. *Up to homeomorphism, what manifolds can appear as* $V_B(\varphi_t)$ *in dissipations of a given singularity*?

3.3.B. *Up to homeomorphism, what pairs can appear as* $(B, V_B(\varphi_t))$ *in dissipations of a given singularity*?

Two dissipations $(B, V_B(\varphi_t))$ with $t \in [0, t_0]$ and $(B', V_{B'}(\varphi'_t))$ with $t \in [0, t'_0]$ are said to be *topologically equivalent* if there exists an isotopy $h_t \colon B \to B'$ with $t \in [0, \min(t_0, t'_0)]$, such that h_0 is a diffeomorphism and $V_{B'}(\varphi'_t) = h_t V_B(\varphi_t)$ for $t \in [0, \min(t_0, t'_0)]$.

3.3.C. *Up to topological equivalence, what are the dissipations of a given singularity*?

These questions are analogous to the classification problems 1.1.A and 1.1.B discussed above. Obviously, 3.3.C is a refinement of 3.3.B, which, in turn, is a refinement of 3.3.A (since in 3.3.C we are interested not only in the type of the pair obtained from a dissipation, but also the manner in which the pair is attached to the link of the singularity).

In the case $K = \mathbf{R}$, with which we are especially concerned, these questions have been answered only for a very small number of singularities. In §4 below we shall examine some of these cases. In general, the topology of dissipations of real singularities has a development which runs parallel to the topology of nonsingular real algebraic varieties. In particular, one encounters prohibitions (see [38]) and constructions (see below).

In the case $K = \mathbf{C}$, the dissipation of a given singularity is unique from all three points of view, and there is an extensive literature (see, for example, [15]) devoted to its topology (i.e., questions 3.3.A and B). Incidentally, if we want to obtain questions for $K = \mathbf{C}$ which are truly analogous to questions 3.3.A–C for $K = \mathbf{R}$, then we have to replace dissipations by deformations with singular fibers and one-dimensional complex bases, and the manifolds $V_B(\varphi_t)$ and the pairs $(B, V_B(\varphi_t))$ have to be considered along with monodromy transformations. it is reasonable to suppose that there are interesting connections between questions 3.3.A–C for a real singularity and their analogues for the complexification of the singularity.

3.4. Nondegenerate r**-fold points.** We return to singularities of plane curves. A point (x_0, y_0) of the curve $f(x, y) = 0$ is said to be *nondegenerate* r*-fold point* if it has multiplicity r (i.e., the partial derivatives of f through order $r-1$ inclusive vanish at the point, but not all rth partials vanish) and if the curve

$$f_{x^r}(x_0, y_0)x^r + r f_{x^{r-1}y}(x_0, y_0)x^{r-1}y + \cdots + f_{y^r}(x_0, y_0)y^r = 0$$

is reduced (i.e., the polynomial $\sum_{k=0}^{r} C_r^k f_{x^k y^{r-k}}(x_0, y_0)x^k y^{r-k}$ is not divisible by the square root of any polynomial of positive degree). This notion is clearly a generalization of the notion of a nondegenerate double point.

When $(x_0, y_0) = (0, 0)$, this definition has the following obvious convenient reformulation in terms of the coefficients of f: the point $(0, 0)$ is a nondegenerate r-fold point of the curve $f(x, y) = 0$ if and only if the Newton polygon $\Delta(f)$ is supported by the part of its boundary facing the origin on the segment Γ joining the points $(r, 0)$ and $(0, r)$ (i.e., the Newton diagram $\Gamma(f)$ lies on Γ), and the curve $f^{\Gamma}(x, y) = 0$ consists of distinct lines.

We also give a geometrical reformulation of the definition.

3.4.A. *A point on a curve is a nondegenerate r-fold point if and only if there are exactly r branches of the curve passing through it, these branches are nonsingular, and they have distinct tangents.*

Before proving this, we make a preliminary remark that is of independent interest.

We consider the homothety $\mathbf{C}^2 \to \mathbf{C}^2: (x, y) \mapsto (tx, ty)$. It takes the curve $f(x, y) = 0$ to the curve $f(t^{-1}x, t^{-1}y) = 0$. The monomial $a_{ij}x^i y^j$ in $f(x, y)$ corresponds to the monomial $a_{ij}t^{-i-j}x^i y^j$ in $f(t^{-1}x, t^{-1}y)$, so that the monomials on the line $i + j = n$ are multiplied by t^{-n} in the homothety $(x, y) \mapsto (tx, ty)$. In addition, the equation of the curve can be multiplied through by any number, in particular by t^{ρ}, without changing the curve. Thus, the homothety $(x, y) \mapsto (tx, ty)$ corresponds to the following transformation of the equation of the curve: for some fixed ρ, multiply the monomials on the line $i + j = n$ (i.e., the monomials $a_{ij}x^i y^j$ with $i + j = n$) by $t^{\rho+n}$.

We now prove the above geometrical reformulation 3.4.A of the definition of a nondegenerate r-fold point. It is sufficient to consider the case when the singularity is at the origin. Suppose that the origin is a nondegenerate r-fold point. We apply the homothety $(x, y) \mapsto (tx, ty)$ to the curve, at the same time performing the above transformation on the equation with $\rho = r$. The monomials in Γ remain unchanged, and the other monomials are multiplied by negative powers of t. We let t approach ∞. Then the equation approaches $f^{\Gamma}(x, y) = 0$, i.e, the equation of a union of r distinct lines through $(0, 0)$. This union intersects any sphere in $\mathbf{C}^2$ centered at $(0, 0)$ transversally in a union of r great circles. Under a small perturbation of the equation, the intersection remains transversal, and it consists of r unknotted circles with pairwise linking coefficients equal to 1. Consequently, if the curve is subjected to the homothety for t sufficiently large, it will have r branches through the origin, and they will be nonsingular and transversal to one another. Thus, the same is true of the branches of the original curve.

Conversely, suppose that the curve $f(x, y) = 0$ has r branches at the origin, and they are nonsingular and transversal to one another. Then the origin is an r-fold point, and the Newton polygon $\Delta(f)$ is situated on the segment Γ. Under the homothety $(x, y) \mapsto (tx, ty)$ with $t \to \infty$ the curve $f(x, y) = 0$ approaches the curve $f^{\Gamma}(x, y) = 0$ in a neighborhood of the origin. On the other hand, each of the branches stretches out into a line (the tangent line to the branch). Consequently, $f^{\Gamma}(x, y) = 0$ is a union of distinct lines through $(0, 0)$, i.e., $(0, 0)$ is a nondegenerate r-fold point of the curve $f(x, y) = 0$.

3.5. Dissipation of a nondegenerate r-fold point. Our next goal is to construct perturbations of a curve with nondegenerate r-fold point under which the topology of the curve in a neighborhood of the point changes in a way that can be controlled.

We first consider a special case—when the curve to be perturbed consists of r distinct lines through the origin. The Newton polygon of this curve is a segment of the line Γ joining the points $(r, 0)$ and $(0, r)$. (It clearly either coincides with Γ or is strictly smaller, and the latter can happen when one or both of the extreme monomials $a_{r,0}x^r$, $a_{0,r}y^r$ are missing.)

The argument used above to prove the equivalence of the two definitions of a nondegenerate r-fold point (3.4.A) gives us an indication of how to construct the perturbations. We take an affine curve of degree r which has r asymptotes whose directions coincide with those of our given lines. The Newton polygon of such a curve is contained in the triangle with vertices $(0, 0)$, $(r, 0)$, $(0, r)$, and the defining polynomial can be normalized in such a way that its Γ-truncation coincides with the polynomial defining our original curve. We apply the homothety $(x, y) \mapsto (tx, ty)$ to the affine curve, where, as before, we also transform the equation, again with $\rho = r$. The monomials in Γ remain unchanged, and the other monomials are multiplied by negative powers of t ($a_{ij}x^iy^j$ is multiplied by t^{r-i-j}). We let t approach zero. Then in the limit we obtain the equation of the original curve, i.e., a union of r lines, while the curves of the family are all images of the same affine curve under different homotheties. Thus, an affine curve of degree r with r distinct asymptotes may be regarded as the result of a perturbation of a union of r lines through a point.

In the more general case—when the curve to be perturbed has degree greater than r and, as above, it has the origin as a nondegenerate r-fold point—the monomials of degree $> r$ do not have a noticeable influence near the origin (compare with §3.4). Hence, it is natural to expect that the same adjustment to the equation as above will have a similar effect. But before examining this generalization, we make more precise what we mean by nearby curves.

We say that a smooth submanifold A of a manifold X *approximates* the smooth submanifold B of X in the open set $U \subset X$ if, for some tubular neighborhood T of $B \cap U$ in U, the intersection $A \cap U$ is contained in T and is a section of the tubular fibration $T \to B \cap U$.

It follows from the implicit function theorem (see 1.5.A(3) and §1.7 above) that, if the degree m curve $a(x_0, x_1, x_2) = 0$ has no singular points in the closure of the open set $U \subset \mathbf{R}P^2$, then in the space $\mathbf{R}C_m$ of real curves of degree m it has a neighborhood all curves of which have no singular points in U and approximate one another in U.

Let $a(x_0, x_1, x_2) = 0$ be a real projective curve of degree m which has no singular points except for the point $(1 : 0 : 0)$, and suppose that $(1 : 0 : 0)$ is a nondegenerate r-fold point. Let $g(x, y) = 0$ be a nonsingular real affine curve of degree r, and suppose that $g(x, y)$ and $a(1, x, y)$ have the same Γ-truncation, where Γ is the line segment joining $(r, 0)$ and $(0, r)$. We set

$$f(x, y) = a(1, x, y),$$
$$h_t(x, y) = f(x, y) + t^r g(t^{-1}x, t^{-1}y) - f^\Gamma(x, y),$$
$$c_t(x_0, x_1, x_2) = a(x_0, x_1, x_2) + t^r x_0^m g(x_1, x_0^{-1}t^{-1}, x_2x_0^{-1}t^{-1}) \\ -a^\Gamma(x_0, x_1, x_2).$$

Since clearly $c_t(1, x, y) = h_t(x, y)$ and $c_0(x_0, x_1, x_2) = a(x_0, x_1, x_2)$, it follows that the family of curves $c_t(x_0, x_1, x_2) = 0$ is a perturbation of the curve $a(x_0, x_1, x_2) = 0$.

3.5.A. *There exist circular neighborhoods $U \supset V$ of the point $(1 : 0 : 0)$ in $\mathbf{R}P^2$ such that for $t > 0$ sufficiently small the curve $c_t(x_0, x_1, x_2) = 0$ is* *approximated by the curve* $a(x_0, x_1, x_2) = 0$ *outside* V, *and is approximated by the curve* $x_0^r g(t^{-1}x_1x_0^{-1}, t^{-1}x_2x_0^{-1}) = 0$ *inside* U (*i.e., the latter curve is the image of the curve* $g(x, y) = 0$ *under the composition of the homothety* $(x, y) \mapsto (tx, ty)$ *and the canonical imbedding* $\mathbf{R}^2 \to \mathbf{R}P^2 : (x, y) \mapsto (1 : x : y)$).

PROOF. We include the family of polynomials h_t in a larger family

$$h_{s,t}(x, y) = s^{-r} f(sx, sy) + t^r g(t^{-1}x, t^{-1}y) - f^{\Gamma}(x, y).$$

The homothety $(x, y) \mapsto (ux, uy)$ takes the curve $h_{s,t}(x, y) = 0$ to the curve $u^r h_{s,t}(u^{-1}x, u^{-1}y) = 0$, but we have

$$u^r h_{s,t}(u^{-1}x, u^{-1}y) = s^{-r}u^r f(su^{-1}x, su^{-1}y) + t^r u^r g(t^{-1}u^{-1}x, t^{-1}u^{-1}y) - f^{\Gamma}(x, y) = h_{su^{-1}, tu}(x, y).$$

Thus, the curves $h_{s,t}(x, y) = 0$ corresponding to points (s, t) of the parameter plane which lie on the hyperbolas $st = \text{const}$ can be obtained from one another by means of homotheties.

We set $c_{s,t}(x_0, x_1, x_2) = x_0^m h_{s,t}(x_1x_0^{-1}, x_2x_0^{-1})$. The curve $c_{0,0}(x_0, x_1, x_2) = 0$ is clearly the union of the $(m-r)$-fold line $x_0^{m-r} = 0$ and the r lines through $(1 : 0 : 0)$ defined by the equation $a^{\Gamma}(x_0, x_1, x_2) = 0$. The origin in the parameter plane has a circular neighborhood P such that for $(s, t) \in P$ the curves $h_{s,t}(x, y) = 0$ approximate one another in the annulus $1 \le x^2 + y^2 \le 4$, and, in particular, they approximate the curve $f^{\Gamma}(x, y) = 0$ there.

We take $(s_0, 0) \in P$ with $s_0 > 0$. The corresponding curve $c_{s_0,0}(x_0, x_1, x_2) = 0$ is obtained from the curve $a(x_0, x_1, x_2) = 0$ by means of the dilatation $(x_0 : x_1 : x_2) \mapsto (x_0 : s^{-1}x_1 : s^{-1}x_2)$ (Figure 15). Like the latter curve, it has a singularity only at $(1 : 0 : 0)$. If we go a sufficiently small distance from $(s_0, 0)$ in the region $t > 0$, this singularity is perturbed, while outside some neighborhood of the singularity (say, the disc $x^2 + y^2 < 1$) the curve $c_{s_0,t}(x_0, x_1, x_2) = 0$ is approximated by the curve $c_{s_0,0}(x_0, x_1, x_2) = 0$.

In exactly the same way, the curve $h_{0,t_0}(x, y) = 0$ corresponding to $(0, t_0) \in P$ with $t_0 > 0$ can be obtained from the curve $g(x, y) = 0$ by means of the contraction $(x, y) \mapsto (t_0x, t_0y)$. If we go a sufficiently small distance from $(0, t_0)$ in the region $s > 0$, the curve $h_{s,t}(x, y) = 0$ experiences only a small isotopy in the disc $x^2 + y^2 \le 4$, and is approximated by the curve $h_{0,t_0}(x, y) = 0$, i.e., by the curve $g(t_0^{-1}x, t_0^{-1}y) = 0$.

We choose points (s_0, t_1) and (s_1, t_0) close to $(s_0, 0)$ and $(0, t_0)$ in the above sense, where $s_0t_1 = s_1t_0$, i.e., they lie on the same hyperbola $st = \text{const}$. When we move from (s_1, t_0) to (s_0, t_1) along this hyperbola, the curve $c_{s,t}(x_0, x_1, x_2) = 0$ is subjected to an isotopy made up of homotheties, i.e., contractions toward the point $(1 : 0 : 0)$, and it turns into the curve $c_{s_0,t_1}(x_0, x_1, x_2) = 0$. Since the point (s, t) does not leave P in the course of this isotopy, it follows that the curve does not change in an essential way in the annulus $1 \le x^2 + y^2 \le 4$; it merely slides along the curve $f^{\Gamma}(x, y) = 0$, at all times approximating that curve. Hence, the curve $h_{s_0,t_1}(x, y) = 0$ approximates the curve $h_{0,t_1}(x, y) = 0$ (i.e., the image of the curve $g(x, y) = 0$ under

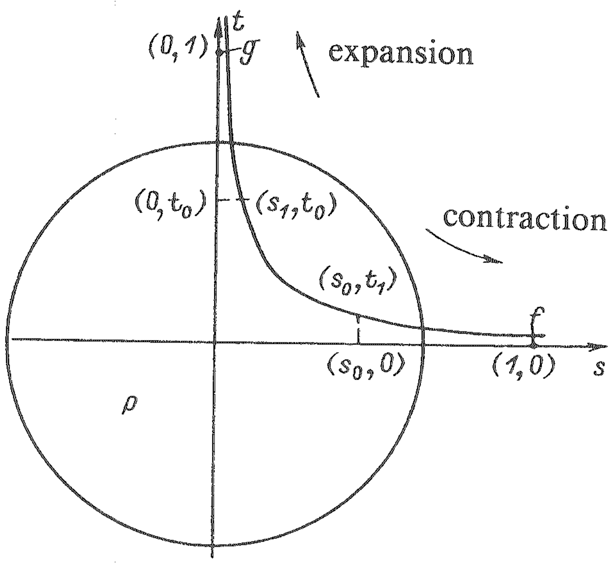


FIGURE 15

the contraction $(x, y) \mapsto (t_1 x, t_1 y))$ not only in the disc $x^2+y^2 \le 4s_1^2 s_0^{-2}$ (i.e., in the image of the disc $x^2+y^2 \le 4$ under the homothety) but even in the disc $x^2+y^2 \le 4$ itself.

We now notice that the curve $c_{1, s_0 t_1}(x_0, x_1, x_2) = 0$ (i.e., the curve $c_{1, s_0 t_1}(x_0, x_1, x_2) = 0$) is the image of the curve $c_{s_0, t_1}(x_0, x_1, x_2) = 0$ under the homothety $(x_0 : x_1 : x_2) \mapsto (x_0 : s_0 x_1 : s_0 x_2)$. Thus, outside the disc $x^2 + y^2 < s_0^2$ the curve $c_{s_0 t_1}(x_0, x_1, x_2) = 0$ is approximated by the curve $c_{1,0}(x_0, x_1, x_2) = 0$, i.e., by the original curve $a(x_0, x_1, x_2) = 0$, and inside the disc $x^2 + y^2 \le 4s_0^2$ it is approximated by the curve $c_{0, s_0 t_1}(x_0, x_1, x_2) = 0$, i.e., by the image of the curve $x_0^r g(x_1 x_0^{-1}, x_2 x_0^{-1})$ under the contraction $(x_0 : x_1 : x_2) \mapsto (x_0 : s_0 t_1 x_1 : s_0 t_1 x_2)$. Hence, if we set $t = s_0 t_1$ and take U to be the disc $x^2 + y^2 \le 4s_0^2$ and V to be the disc $x^2 + y^2 \le s_0^2$, we obtain the objects whose existence is asserted in the theorem. ●

3.6. Quasihomogeneity. The method of perturbing a curve with a nondegenerate r-fold point has an immediate generalization to a much broader class of singularities. Roughly speaking, the generalization comes from replacing the homotheties $(x, y) \mapsto (tx, ty)$ by maps of the form $(x, y) \mapsto (t^u x, t^v y)$ with relatively prime integers u and v—such maps are called *quasihomotheties.* As in the case of homotheties, the quasihomotheties with fixed exponents u and v form a one-parameter group of linear transformations.

Under the action of a quasihomothety $(x, y) \mapsto (t^u x, t^v y)$ the curve given by the equation $f(x, y) = 0$ with $f(x, y) = \sum a_{ij} x^i y^j$, goes to the curve $f(t^{-u}x, t^{-v}y) = 0$, or equivalently, to the curve $t^\rho f(t^{-u}x, t^{-v}y) = 0$. The monomial $a_{ij}x^i y^j$ in $f(x, y)$ correspond to the monomial $a_{ij} t^{\rho - ui - vj} x^i y^j$ in the polynomial $t^\rho f(t^{-u}x, t^{-v}y)$; thus, under the quasihomothety $(x, y) \mapsto (t^u x, t^v y)$ the monomials on the line $ui + vj = n$ are multiplied by $t^{\rho - n}$.

The curves which are defined by *quasihomogeneous* polynomials of weight u, v, i.e., polynomials whose Newton polygon lies on the line $ui + vj = \text{const}$, are invariant relative to all quasihomotheties $(x, y) \mapsto (t^u x, t^v y)$ with fixed u and v. Such a curve is a union of orbits of the action of the group of quasihomotheties with exponents u and v, i.e., a union of curves of the form $\alpha x^v + \beta y^u = 0$. We call the latter curve a *quasiline* of weight u, v.

We now consider the corresponding singularities of plane curves. We shall suppose that the singularity of the curve $f(x, y) = 0$ that is being examined is at the origin. If the Newton polygon $\Delta(f)$ has a side Γ facing the origin such that the Γ-truncation defines a curve with no multiple components (i.e., if $f^\Gamma(x, y)$ is not divisible by the square of any polynomial of nonzero degree), then we say that the curve $f(x, y) = 0$ has a *semi-quasihomogeneous singularity* at the origin. If the segment Γ is on the line $iu + vj = r$ with u and v relatively prime, then we say that the pair u, v is the *weight* of the semi-quasihomogeneous singularity and r is its *degree*.

There is one essential difference between semi-quasihomogeneous singularities and nondegenerate singularities. In the above definition of semi-quasihomogeneity, the choice of coordinate system plays a much more important role than in the definition of a nondegenerate r-fold point. In fact, if a semi-quasihomogeneous singular point is not a nondegenerate singularity, then the coordinate axis corresponding to the smaller weight plays a special role. The singular point will not be semi-quasihomogeneous with respect to an affine coordinate system in which this axis is not a coordinate axis.

Thus, semi-quasihomogeneity of a singularity is closely connected with the coordinate system. When we speak of a semi-quasihomogeneous singularity, we usually mean that it is semi-quasihomogeneous in a suitable coordinate system. If we want to emphasize that the definition of semi-quasihomogeneity is realized with respect to a given affine coordinate system, or with respect to one of the three affine coordinate systems which are canonically associated with a given projective coordinate system, then we say that the singularity is semi-quasihomogeneous *with respect to the given coordinate system.*

Another, perhaps even more fundamental difference between semi-quasihomogeneity and nondegeneracy is that, even when semi-quasihomogeneity is understood in the broader sense, i.e., relative to any affine coordinate system, the property is generally not preserved under local diffeomorphisms. For example, the curve $x^5 - y^2 = 0$ has a semi-quasihomogeneous singularity at the origin; however, its image under the diffeomorphism $(x, y) \mapsto (x, y - x^2)$, i.e., the curve $x^5 - x^4 - 2x^2y - y^2 = 0$, has a singularity at the origin which is not semi-quasihomogeneous relative to any affine coordinate system.

But for our purposes what is important is that many of the features of nondegenerate r-fold singular points are also characteristic of semi-quasihomogeneous singularities. Theorem 3.4.A generalizes to the semi-quasihomogeneous case as follows. In a suitable neighborhood of a semi-quasihomogeneous singular point the curve looks like a union of a number of quasilines. The words "looks like" here mean that there exists a homeomorphism of the neighborhood which takes the curve to a union of quasilines. The union of quasilines is the curve defined by the truncation of the equation of the original curve to the side of the Newton polygon facing the origin. All of this is proved in the same way as Theorem 3.4.A.

3.7. Examples of semi-quasihomogeneous singularities. The simplest singularities are semi-quasihomogeneous (or, more precisely, they become semi-quasihomogeneous after a suitable change of local coordinates). The hierarchy of singularities starts with the simple ones, or equivalently, the zero-modal singularities, i.e., the singularities A_k, D_k, E_6, E_7, E_8, all of which can be taken to semi-quasihomogeneous form by local diffeomorphisms.

A_k *singularities* (with $k \geq 1$). Here one distinguishes between the cases of odd and even k. If k is odd, then there are two nonsingular branches tangent to one another with multiplicity $k-1$ (i.e., with local intersection index equal to k) passing through a point of type A_k. Here either both of the branches are real (with normal form $x^{k+1} - y^2 = 0$), or else they are conjugate imaginary (with normal form $x^{k+1} + y^2 = 0$). If k is even, then there is one branch and it has cusps. If $k = 2$, they are ordinary cusps, but when $k > 2$ they are "sharp" cusps. The normal form is $x^{k+1} - y^2 = 0$.

D_k *singularities* ($k \geq 4$). Topologically, a D_k singularity looks like an A_{k-3} singularity through which one more nonsingular branch of the curve passes, situated in general position with respect to the other branches. In particular, a D_4 singularity is a nondegenerate triple point.

E_6, E_7, E_8 *singularities.* The normal forms are: for E_6, $x^4 - y^3 = 0$; for E_7, $(x^3 - y^2)y = 0$; for E_8, $x^5 - y^3 = 0$.

But we shall need more complicated singularities. The first is the type of singularity which Arnol′d [3] denoted by the symbol J_{10}. In a neighborhood of such a point the curve has three nonsingular branches which have second order tangency to one another at the point. This is a semi-quasihomogeneous singularity of weight $(2, 1)$ and degree 6. We shall only need the real form of the singularity for which all three branches are real. J_{10} singularities are useful in constructing real curves, because curves with a J_{10} singularity can be built up easily using obvious modifications of classical methods of construction, while at the same time they are complicated enough so that interesting new curves appear when one perturbs curves with J_{10} singularities. From this point of view one has good singularities of type N_{16} (nondegenerate 5-fold points), X_{21} (a point where four nonsingular branches have a second order tangency—this is a semi-quasihomogeneous singularity of weight $2, 1$ and degree 8), Z_{15} (a point where three nonsingular branches have second order tangency and a fourth nonsingular branch intersects the other three transversally—this is a semi-quasihomogeneous singularity of weight 1,2 and degree 7). (The symbols N_{16}, X_{21} and Z_{15} are also Arnol′d's notation in [3].)

3.8. Dissipation of semi-quasihomogeneous singularities. Let $a(x_0, x_1, x_2) = 0$ be a real projective curve of degree m with no singular points except for the point $(1:0:0)$, and suppose that this point is a semi-quasihomogeneous singular point of weight u, v and degree r. Further, suppose that the curve is situated (relative to the canonical coordinate system) in such a way that the Newton polygon $\Delta(a)$ has side Γ facing the origin which lies on the line $ui + vj = r$, and the curve $a^\Gamma(1, x, y) = 0$ has no multiple components. Let $g(x, y) = 0$ be a curve having no singularities in $\mathbf{R}^2$. Suppose that $\Delta(g)$ is contained between the origin and the line $ui + vj = r$, and the truncation g^Γ coincides with the Γ-truncation of the polynomial $f(x, y) = a(1, x, y)$. We set

$$h_t(x, y) = f(x, y) + t^r g(t^{-u}x, t^{-v}y) - f^\Gamma(x, y),$$

$$c_t(x_0, x_1, x_2) = a(x_0, x_1, x_2) + t^r x_0^m g(x_1 x_0^{-1} t^{-1}, x_2 x_0^{-1} t^{-1}) - a^\Gamma(x_0, x_1, x_2).$$

It is clear that $c_t(1, x, y) = h_t(x, y)$ and $c_0(x_0, x_1, x_2) = a(x_0, x_1, x_2)$.

3.8.A. *There exist neighborhoods $U \supset V$ of the point $(1:0:0)$ in $\mathbf{R}P^2$ such that for $t > 0$ sufficiently small the curve $c_t(x_0, x_1, x_2) = 0$ is approximated by the curve $a(x_0, x_1, x_2) = 0$ outside V, and it is approximated inside U by the image of the curve $g(x, y) = 0$ under the composition of quasihomothety $(x, y) \mapsto (t^u x, t^v y)$ and the canonical imbedding $\mathbf{R}^2 \to \mathbf{R}P^2: (x, y) \mapsto (1:x:y)$.*

This theorem generalizes Theorem 3.5.A, and its proof, which is a direct generalization of the proof of Theorem 3.5.A, will be left as an exercise for the reader.

The dissipations of a semi-quasihomogeneous singular point which are obtained by means of the construction in this subsection will be called *quasihomogeneous* dissipations.

3.9. Perturbation of curves with several singular points. In Theorems 3.4.A and 3.8.A the curves being perturbed have only one singular point, namely, the singular point which is dissipated and for which the variation of topology in a neighborhood is described in the theorems. If we suppose in Theorem 3.5.A that the curve $a(x_0, x_1, x_2) = 0$ has other singular points as well, then those singularities will generally also dissipate in the family $c_t(x_0, x_1, x_2) = 0$, and some additional information about the polynomial g is needed in order to describe the topology of that dissipation.

However, there is an important special case when, independently of g, the singular points of the curve $a(x_0, x_1, x_2) = 0$ other than $(1:0:0)$ are preserved under the dissipation described above. This is the case when these singular points are $(0:1:0)$ or $(0:0:1)$ or both $(0:1:0)$ and $(0:0:1)$, and they are semi-quasihomogeneous relative to this projective coordinate system.

In fact, the Newton polygons of the polynomials $a = c_0$ and c_t for $t > 0$ on the side of the points $(m, 0)$ and $(0, m)$ coincide, as do the monomials corresponding to points on these parts of the boundary of the Newton polygons.

Thus, *the dissipations described in the previous subsection (i.e., quasihomogeneous ones) can be carried out at two or three semi-quasihomogeneous singular points independently, if the singularities are all semi-quasihomogeneous relative to the same projective coordinate system.*

3.10. Multidimensional generalizations. The definitions in §3.6 of a quasihomothety, a quasihomogeneous polynomial, a quasililne, and a semi-quasihomogeneous singularity generalize in the obvious way to the case of a space of arbitrary dimension. So do the method of dissipating semi-quasihomogeneous singularities in §3.8, Theorem 3.8.A and the remarks in §3.9. The exact statements will be left as an exercise.

§4. Dissipating concrete singularities of curves

This section is devoted to a discussion of dissipations of concrete singularities on plane curves. The topological classification of dissipations has been completed only for certain very simple types of singularities. We begin with simple singularities whose dissipations are completely understood; but such information is of little interest for constructions. We then examine two types of unimodal singularities: J_{10} (three nonsingular branches which are second order tangent to each other at a point) and X_9 (nondegenerate 4-fold singularities). As in the case of simple singularities, dissipations of X_9 give almost nothing of use for constructions of curves. On the other hand, J_{10}—or more precisely, its real form with three real branches—is very useful, and we shall give a detailed discussion of the structure of its dissipations for all possible topological types. We then examine dissipations of nondegenerate 5-fold points and more complicated singularities.

Results on the topology of dissipations of some type of singularity can be divided into three categories. The first consists of prohibitions on the topology of the dissipation. They are similar to the prohibitions on the topology of nonsingular curves, and I shall limit myself to the statement of results for concrete singularities. The second category of results relates to the construction of concrete dissipations. In the case of semi-quasihomogeneous singularities, Theorem 3.8.A reduces the problem of constructing dissipations to the problem of constructing curves. We shall sometimes include proofs of results of this second type; however, as a rule the purpose of the proofs is merely to provide an illustration of new methods and give an idea of how the proofs go. Finally, the third category of results concern how the topology of the dissipations of some family of singularities depends on the parameters which determine a singularity in the family. For example, we consider all nondegenerate r-fold singular points at which all of the branches are real, and we prove that for fixed r the supply of dissipations of a given singularity does not depend on the location of the branches (i.e., the angles between them, their curvature, etc.). In all cases except for the important and first nontrivial case of J_{10}^-, I will limit myself to stating the results.

4.1. Zero-modal singularities. *Singularities of the A_k series with k odd and with two real branches* (A_k^-). Any such singularity can be taken by a local diffeomorphism to the normal form $y^2 - x^{k+1} = 0$. Any dissipation of this singularity is topologically equivalent to one of the dissipations in Figure 16. In this diagram and the ones that follow, the symbol $\langle\alpha\rangle$ replaces a group of α ovals lying outside one another. The dissipations in Figure 16 are constructed as follows: the dissipation on the right is given by the formula $y^2 - x^{k+1} - t = 0$ with $t > 0$; the dissipations shown beneath the original singularity are given by the formulas $y^2 - (x - tx_1)(x - tx_2)\cdots(x - tx_{2a+2})(x^2 + t^2)^{(k-1)/2-a} = 0$, where $x_1, \dots, x_{2a+2}$ are distinct real numbers (and, as usual, t is a parameter which in a given dissipation varies over an interval of the form $[0, t_0]$).

Singularities of the A_k series with k odd and with conjugate imaginary branches (A_k^+). Any such singularity can be taken by a local diffeomorphism to the normal form $y^2 + x^{k+1} = 0$. Any dissipation of this singularity is topologically equivalent to one of the dissipations in Figure 17. These dissipations are given by the formula $y^2 + (x - tx_1)\cdots(x - tx_{2a})(x^2 + t^2)^{(k+1)/2-a} = 0$, where $x_1, \dots, x_{2a}$ are distinct real numbers.

Note that the singularities whose dissipations we have just described include singularities of type A_1, i.e., nondegenerate double points (crossings A_1^- and

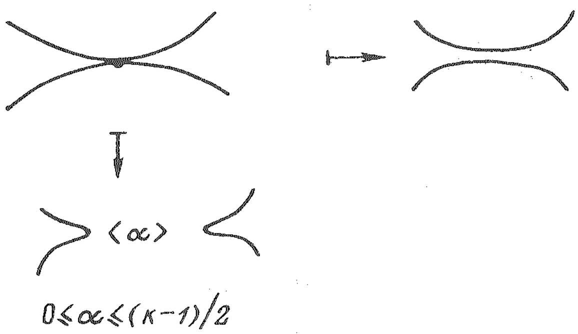


FIGURE 16

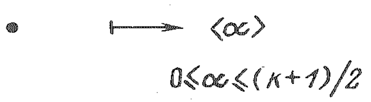


FIGURE 17

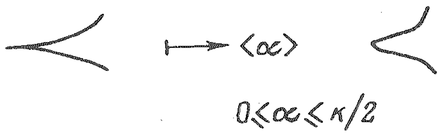


FIGURE 18

isolated double points A_1^+). We considered the removal of such singularities in §1.

Singularities of the A_k *series with* k *even.* Any such singularity can be taken by a local diffeomorphism to the normal form $y^2 - x^{k+1} = 0$. Any dissipation of such a singularity is topologically equivalent to one of the dissipations in Figure 18. They are given by the formula $y^2 - (x - tx_1) \cdots (x - tx_{2a+1})(x^2 + t^2)^{k/2-a} = 0$, where $x_1, \dots, x_{2a+1}$ are distinct real numbers.

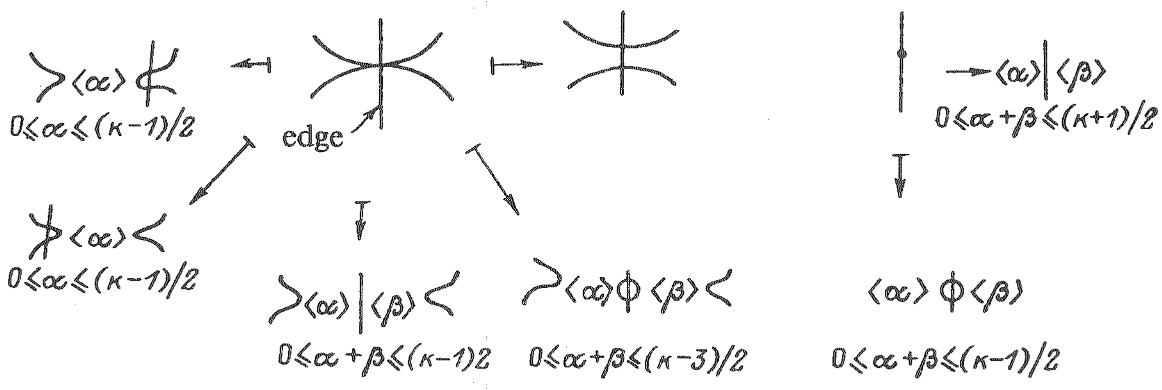
In particular, when $k = 2$ we obtain two types of dissipations of an ordinary cusp.

REMARK. If we make a suitable choice of x_i, we can arrange it so that the curves defined by the polynomials constructed above are situated relative to the y-axis in any of the ways shown in Figure 19. This can be interpreted as constructing all possible (up to topological equivalence) dissipations of the *boundary* singularities of the B_{k+1} series (see §17.4 of [3] concerning boundary singularities). By a dissipation of a boundary singularity we mean a dissipation of the singularity with boundary neglected, in the course of which the hypersurface being perturbed (in our case a curve) is transversal to the boundary (i.e., to a fixed hyperplane, in our case the line $x = 0$).

Singularities of the D_k *series with even* $k \geq 4$ *and with three real branches* (D_k^-). Such a singularity can be taken by a local diffeomorphism to the normal form $xy^2 - x^{k-1} = 0$. Any dissipation is topologically equivalent to one of those in Figure 20.

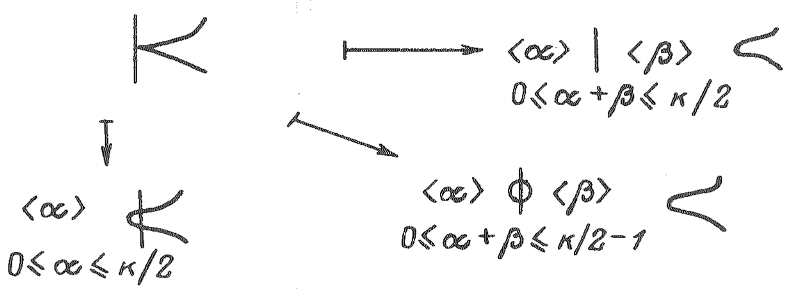
In particular, when $k = 4$ (i.e, when D_4 is a nondegenerate triple point) one has seven dissipations (Figure 21). To construct the dissipations in Figure 20 we note that, since a type D_k^- germ can be obtained from a type A_{k-3}^- germ by adding a line in general position, it follows that a dissipation of a type D_k^- germ can be obtained from a dissipation of the germ of a B_{k-2}^- boundary singularity by adding a boundary line and then making a perturbation. In this way one can obtain all of the dissipations in Figure 20 from the dissipations in the upper left part of Figure 19.

Singularities of the D_k *series with even* $k \geq 4$ *and one real branch* (D_k^+). Such a singularity can be taken by a local diffeomorphism to the normal form



(a) real branches (B^-_{k+1}) (b) imaginary branches (B^+_{k+1})

Dissipation of a B_{k+1} singularity with k odd



Dissipation of a B_{k+1} singularity with k even

FIGURE 19

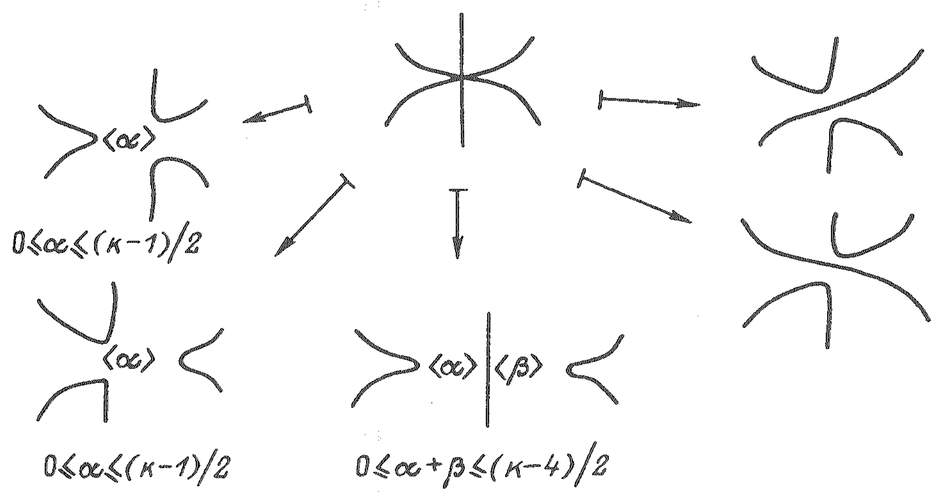


FIGURE 20

$xy^2 + x^{k-1} = 0$. Any dissipation is topologically equivalent to one of those in Figure 22. These dissipations can also be constructed from those in Figure 19 (in the upper right part).

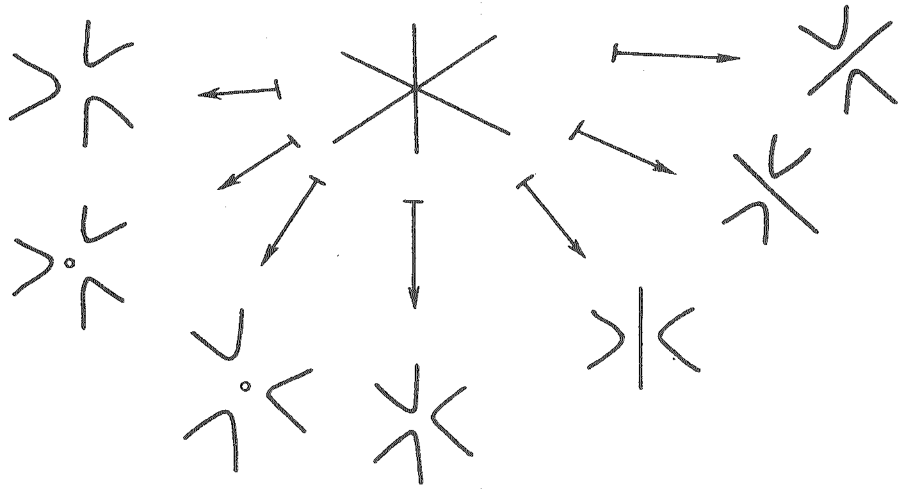


FIGURE 21

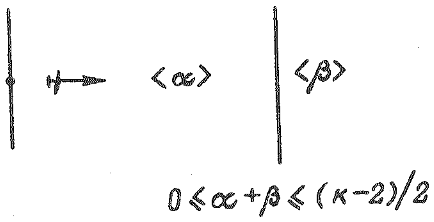


FIGURE 22

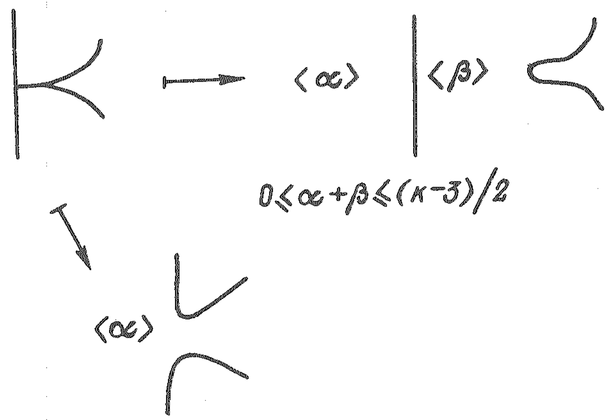


FIGURE 23

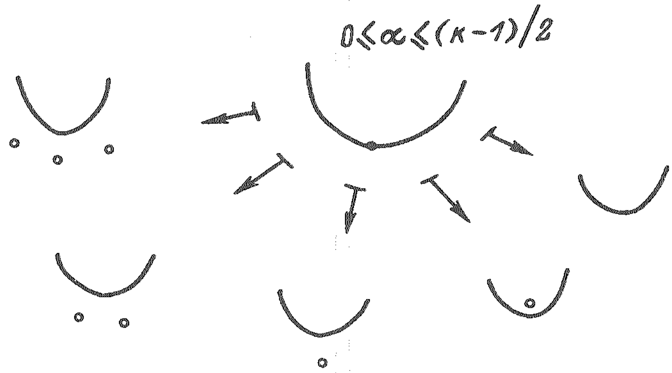


FIGURE 24

Singularities of the D_k series with odd $k \geq 5$. Such a singularity can be taken by a local diffeomorphism to the normal form $xy^2 - x^{k-1} = 0$. Any dissipation is topologically equivalent to one of those in Figure 23. They can be constructed in the same way from the dissipations in the lower part of Figure 19.

E_6 singularities. Such a singularity can be taken by a local diffeomorphism to the normal form $x^4 - y^3 = 0$. We note that all germs of an E_6 singularity are semi-quasihomogeneous, so that by rotating the coordinate axes we can make

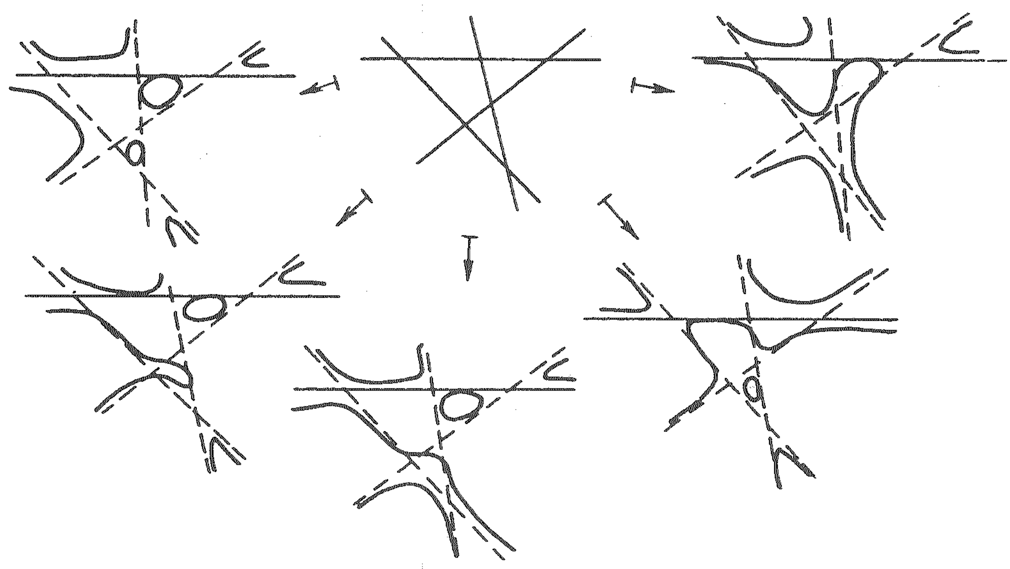


FIGURE 25

the Newton diagram into the segment joining the points (4,0) and (0,3). Any dissipation is topologically equivalent to one of the five dissipations in Figure 24. All of the dissipations in Figure 24 can be obtained as quasihomogeneous dissipations. In this case the curves needed to construct the quasihomogeneous dissipations are nonsingular curves of degree 4 which are tangent to the line at infinity at $(0:0:1)$ with the greatest possible multiplicity (i.e., biquadratic tangency). Such curves can be obtained, for example, by making small perturbations of a curve which splits into four lines. The construction is shown in Figure 25. The perturbation consists each time in adding the product of four linear forms defining lines through $(0:0:1)$ to the equation of the union of four lines, one of which is the line $x_0 = 0$.

E_7 *singularities.* Such a singularity can be taken by a local diffeomorphism to the normal form $y^3 - x^2y = 0$. As in the case of E_6, it is alway semi-quasihomogeneous. Any dissipation is topologically equivalent to one of the ten dissipations in Figure 26. All of the dissipations in Figure 26 can be obtained as quasihomogeneous dissipations. The curves needed for the construction are nonsingular curves of degree 4 which have third order tangency with the line $x_0 = 0$ at the point $(0:0:1)$. For example, as in the E_6 case they can be obtained by small perturbations of a curve which splits into four lines. In Figure 27 we show the construction of one such curve, which gives the dissipations at the top of Figure 26.

E_8 *singularities.* Such a singularity can be taken by a local diffeomorphism to the normal form $x^5 - y^3 = 0$. It is always semi-quasihomogeneous. Any dissipation is topologically equivalent to one of those in Figure 28. All of the dissipations in Figure 28 can be obtained as quasihomogeneous dissipations. The curves needed for the construction are curves of degree 5 with a singular singularity at $(0:0:1)$ which is of type A_4 and is semi-quasihomogeneous relative to the canonical coordinate system. One can obtain such curves, for example, from small perturbations of curves which split into the line $x_0 = 0$ and the degree 4 curves constructed in the dissipations of an E_7-singularity. The perturbation consists in adding to the equation of the curve that splits the product of the equations of five lines distinct from $x_0 = 0$ and passing through $(0:0:1)$.

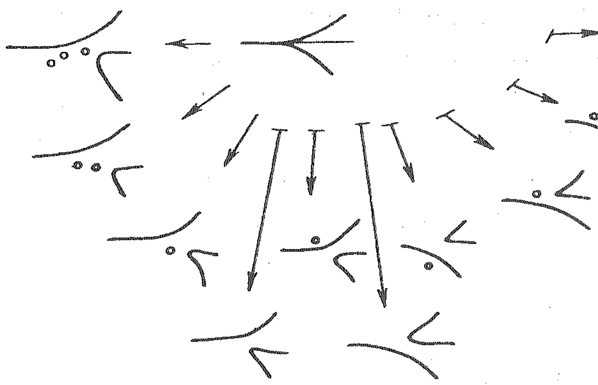


FIGURE 26

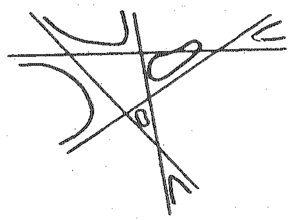


FIGURE 27

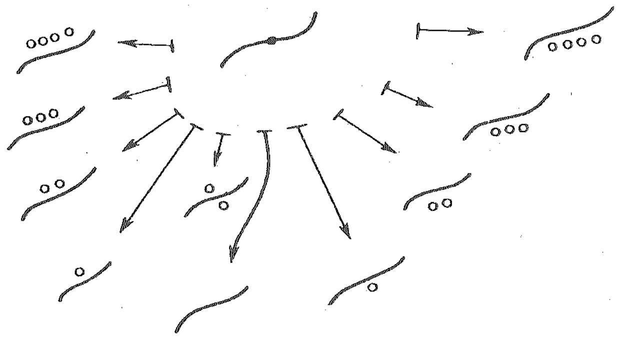
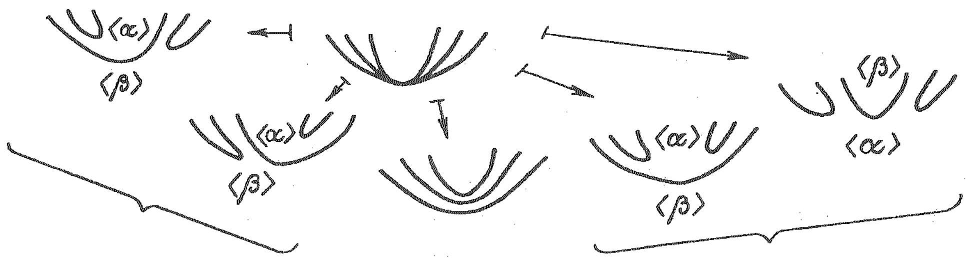


FIGURE 28



α=	4	0	3	0	2	1	0	1	0	0
β=	0	4	0	3	0	1	2	0	1	0

α=	3	2	1	0	0
β=	0	0	0	1	0

FIGURE 29

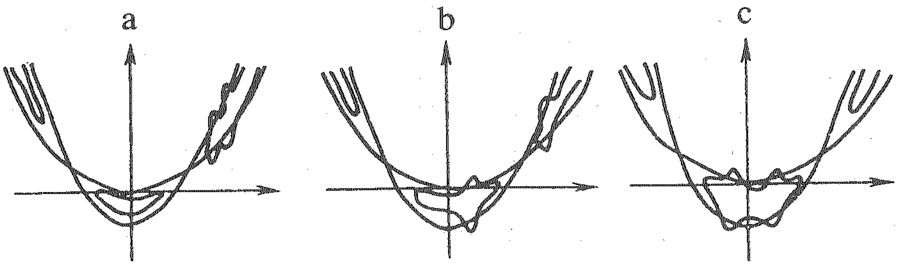


FIGURE 30

4.2. Three branches with second order tangency (J_{10}). The germ of a curve of type J_{10} consists of three nonsingular branches which have second order tangency with one another. Any germ of this type is semi-quasihomogeneous.

Its Newton diagram lies on the segment Γ joining the points $(6, 0)$ and $(0, 3)$ if the x-axis is tangent to all three branches at the origin. From a real viewpoint there are two types of J_{10} singularities: J_{10}^- singularities, where all three branches are real, and J_{10}^+ singularities, where one branch is real and the other two are conjugate imaginary. Let $f(x, y) = 0$ be the equation of a curve with J_{10}^- singularity at the origin, and suppose that the x-axis is tangent at the origin to the branches of the curve $f(x, y) = 0$ which pass through the origin. Then $f^\Gamma(x, y) = \beta(y-\alpha_1 x^2)(y-\alpha_2 x^2)(y-\alpha_3 x^2)$ for some real $\beta \neq 0$, $\alpha_1 > \alpha_2 > \alpha_3$. The curves $y = \alpha_i x^2$ approximate the curve $f(x, y) = 0$ near the origin. The numbers α_i have the following geometric meaning $2\alpha_i$ is the curvature of the ith branch of the curve $f(x, y) = 0$ at $(0, 0)$. The diffeomorphism of the affine plane given by $(x, y) \mapsto (x, ky + lx^2)$ preserves the semi-quasihomogeneity of the germ of the $f(x, y)$ curve relative to the standard coordinate system, but it changes the curvature of the branches, since it takes the curve $y = \alpha_i x^2$ to $y = (k\alpha_i + l)x^2$. Thus, this transformation enables us to make the two curvatures equal to 1 and 2. Moreover, it can be shown that any germ of type J_{10}^- is diffeomorphic to the germ of a curve defined by the equation

$$(y - x^2)(y - 2x^2)(y - \alpha x^2) = 0$$

with $\alpha > 2$. A germ of type J_{10}^+ is diffeomorphic to the germ of a curve defined by the equation

$$(y - x^2)(y^2 + \alpha x^4) = 0$$

with $\alpha > 0$.

The next two theorems give a complete topological classification of dissipations of singularities of type J_{10}^-.

4.2.A. *Any dissipation of a germ of a curve which is of type* J_{10}^- *is topologically equivalent to one of the* 31 *quasihomogeneous dissipations in Figure* 29.

4.2.B. *Any type* J_{10}^- *germ has quasihomogeneous dissipations of all of the* 31 *topological types in Figure* 29.

Theorem 4.2.A is essentially a theorem about prohibitions. We shall not prove it here; however, we shall return to it when we take up the construction of nonsingular curves of degree 6 (see §5.1). At that point we will be able to derive the theorem from the topological prohibitions on the topology of nonsingular curves.

To prove Theorem 4.2.B we must construct curves $g(x, y) = 0$ with Newton polygon contained in the triangle with vertices $(0, 0)$, $(6, 0)$ and $(0, 3)$, such that the truncation $g^\Gamma(x, y)$ is equal to $(y-\alpha_1 x^2)(y-\alpha_2 x^2)(y-\alpha_3 x^2)$, where $\alpha_1 > \alpha_2 > \alpha_3$ are any real numbers prescribed in advance, and such that the set of real points of the curve $g(x, y) = 0$ are situated in $\mathbf{R}P^2$ in the way shown in Figure 29.

We can obtain the curve in the middle of Figure 29 that is beneath the drawing of the singularity to be dissipated, if we take the equation $(y - \alpha_1(x^2 + 1)) \times (y - \alpha_2(x^2 + 1))(y - \alpha_3(x^2 + 1)) = 0$ or a nearby irreducible equation. The other curves are constructed by a method which can be regarded as a version of Hilbert's method in §1.10. We take the union of the parabolas $y = kx^2 - 1$ and $y = lx^2$ with $k > l > 0$, and we perturb it as shown in Figure 30. We then add

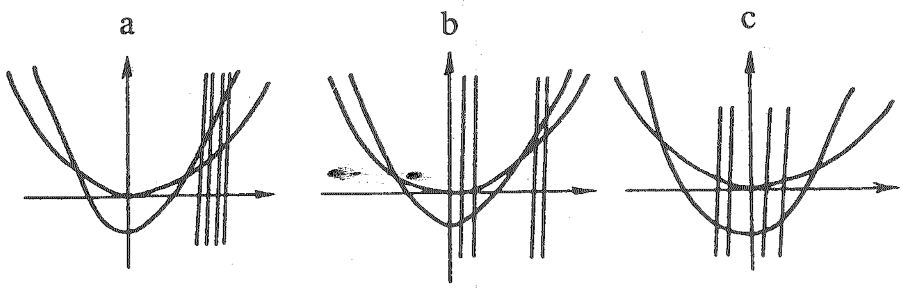


FIGURE 31

one of the original parabolas to the resulting curve and subject the union (which is a curve of degree 6) to a small perturbation. It is easy to see that the other 30 curves in Figure 29 can be obtained using different small perturbations.

It remains to concern ourselves with g^Γ. This requires us practically to go through the above construction once again.

4.2.C. LEMMA. *For any four numbers* $\alpha_0 > \alpha_1 > \alpha_2 > \alpha_3 > 0$ *with* $\alpha_0 + \alpha_3 = \alpha_1 + \alpha_2$ *and for each of the drawings* (a)–(c) *in Figure* 30, *there exists a real polynomial* h *in two variables such that*

(i) *the Newton polygon* $\Delta(h)$ *is the triangle bounded by the coordinate axes and the segment* Γ *joining the points* $(0, 2)$ *and* $(4, 0)$;

(ii) $h^\Gamma(x, y) = (y - \alpha_1 x^2)(y - \alpha_2 x^2)$;

(iii) *the curve* $h(x, y) = 0$ *is nonsingular, and it is situated relative to the parabolas* $y = \alpha_0 x^2 - 1$ *and* $y = \alpha_3 x^2$ *in the manner shown in Figure* 30.

PROOF. We let p_0 and p_3 denote the polynomials $y - \alpha_0 x^2 + 1$ and $y = \alpha_3 x^2$. Clearly the parabolas $p_0(x, y) = 0$ and $p_3^3(x, y) = 0$ intersect at two real points. We set $l_i(x, y) = x - \beta_i$ with $i = 1, \dots, 4$ and $h_t = p_0 p_3 + t l_1 l_2 l_3 l_4$. It is clear that $h_t^\Gamma(x, y) = (y - \alpha_0 x^2)(y - \alpha_3 x^2) + t x^4$. On the other hand, h_t^Γ factors as $h_t^\Gamma(x, y) = (y - \gamma_1 x^2)(y - \gamma_2 x^2)$. Here $\gamma_1 + \gamma_2 = \alpha_0 + \alpha_3$ and $\gamma_1 \gamma_2 = \alpha_0 \alpha_3 + t$. Since $\alpha_0 + \alpha_3 = \alpha_1 + \alpha_2$ and $\alpha_0 > \alpha_1 > \alpha_2 > \alpha_3$, it follows that $\alpha_1 \alpha_2 > \alpha_3 \alpha_0$, and for $t = \alpha_1 \alpha_2 - \alpha_0 \alpha_3 > 0$ the polynomial h_t^Γ is equal to $(y - \alpha_1 x^2)(y - \alpha_2 x^2)$. Thus, $h_{\alpha_1 \alpha_2 - \alpha_0 \alpha_3}$ satisfies conditions (i) and (ii) independently on the choice of $\beta_1, \dots, \beta_4$.

We shall show that the choice of these numbers can be made in such a way that the polynomial also satisfies (iii). If the lines $l_i(x, y) = 0$ are situated relative to the parabolas $p_j(x, y) = 0$ as shown in Figure 31, then there exists $\varepsilon > 0$ such that for $t \in (0, \varepsilon]$ the curve $h_t(x, y) = 0$ consists of three components and is situated relative to the parabolas $p_j(x, y) = 0$ in the way shown in Figure 30. We show that by suitably choosing the lines $l_i(x, y) = 0$ we can arrange it so that the role of ε can be played by any number in the interval $(0, (\alpha_0^2 + \alpha_3^2)/2)$, and in particular by $\alpha_1 \alpha_2 - \alpha_0 \alpha_3$.

Since the Newton polygon $\Delta(h_t)$ has only one interior point, the genus of the curve defined by h_t is at most 1 (see §3.1). Hence, as t increases from zero, the first modification of the curve $h_t(x, y) = 0$ must either decrease the number of components, or else give a curve which decomposes. The latter case cannot occur for $t \in (0, +\infty)$. In fact, by considering the truncation h_t^Γ we see that the curves into which the curve $h_t(x, y) = 0$ can decompose are either two conjugate imaginary curves or else two parabolas. The first is impossible, since for any $t > 0$ a line of the form $x = \gamma$ with $\gamma \in (\beta_1, \beta_2)$ intersects the curve $h_t(x, y) = 0$ at two real points; and the second case is impossible, because the vertical line through a point of intersection of the parabolas $p_0(x, y) = 0$ and $p_3(x, y) = 0$ does not intersect the curve $h_t(x, y) = 0$ for $t > 0$. For $t \in (0, (\alpha_0^2 + \alpha_3^2)/2)$, the branches going out to infinity are preserved. If we place the lines $l_i(x, y) = 0$ near the point of intersection of the parabolas $p_0(x, y) = 0$ and $p_3(x, y) = 0$, we can arrange it so that two branches of the curve $h_t(x, y) = 0$ pass through a prescribed neighborhood of this point for all $t \in (0, (\alpha_0^2 + \alpha_3^2)/2)$, and hence the oval is preserved and no modifications have occurred. ●

END OF THE PROOF OF THEOREM 4.2.B. As we said before, the equation

$$(y - \alpha_1(x^2 + 1))(y - \alpha_2(x^2 + 1))(y - \alpha_3(x^2 + 1)) = 0$$

(and nearby irreducible equations) give the curve that is shown in the middle of Figure 29. The remaining curves in Figure 29 can be realized using polynomials which are obtained from small perturbations of products of the form $p_j h$, where p_j and h are as in 4.2.C. The perturbations involve adding polynomials of the form $\varepsilon \prod_{i=1}^{5}(x - \gamma_i)$. Under such a perturbation there is no change in the terms corresponding to points on the side of the Newton polygon joining $(6, 0)$ and $(0, 3)$.

However, in this way one does not obtain dissipations of all of the type J_{10}^- germs. In the case when the polynomials $p_3 h$ are perturbed, one obtains dissipations of type J_{10}^- germs for which all branches are convex in the same direction and have arbitrary curvature (of the same sign). The point is that the type J_{10}^- germ given by a polynomial with Γ-truncation $(y - a_1x^2) \times (y - a_2x^2)(y - a_3x^2)$, is a union of three branches with curvature $2a_i$. On the other hand, in 4.2.C the numbers $\alpha_1, \alpha_2, \alpha_3$ are subject only to the condition $\alpha_1 > \alpha_2 > \alpha_3 > 0$. In the case when the polynomials $p_0 h$ are perturbed, one obtains dissipations only of type J_{10}^- germs for which all branches are convex in the same direction and, moreover, the curvature satisfies the conditions $\kappa_0 > \kappa_1 > \kappa_2 > 0$ and $\kappa_1 + \kappa_2 - \kappa_0 > 0$, since the numbers $\alpha_0, \alpha_1, \alpha_2$ in 4.2.C must satisfy the inequalities $\alpha_0 > \alpha_1 > \alpha_2 > 0$ and $\alpha_1 + \alpha_2 - \alpha_0 = \alpha_3 > 0$. In the case of type J_{10}^- germ with arbitrary curvature values $\kappa_0 > \kappa_1 > \kappa_2$, we choose δ so that the numbers $k_i = \kappa_i + \delta$ satisfy the inequalities $k_2 > 0$ and (to provide for all cases) $k_1 + k_2 - k_0 > 0$; we then use the above construction to obtain a polynomial which gives the required dissipation of a germ with curvature k_0, k_1, k_2; and, finally, we apply the transformation $(x, y) \mapsto (x, y + \delta/2x^2)$ to this polynomial. It is easy to see that this transformation leaves the Newton polygon inside the triangle with vertices $(0, 0)$, $(6, 0)$ and $(0, 3)$, and it does not affect the topological type of the dissipation. ●

The next two theorems 4.2.D and 4.2.E give a complete topological classification of dissipations of type J_{10}^+ singularities. These theorems are analogous to Theorems 4.2.A and 4.2.B, and, since they will not be needed later, we shall not concern ourselves with the proofs.

4.2.D. *Any dissipation of a type J_{10}^+ germ of a curve is topologically equivalent to one of the ten quasihomogeneous dissipations in Figure* 32.

4.2.E. *Any type J_{10}^+ germ has quasihomogeneous dissipations of all of the ten types in Figure* 32.

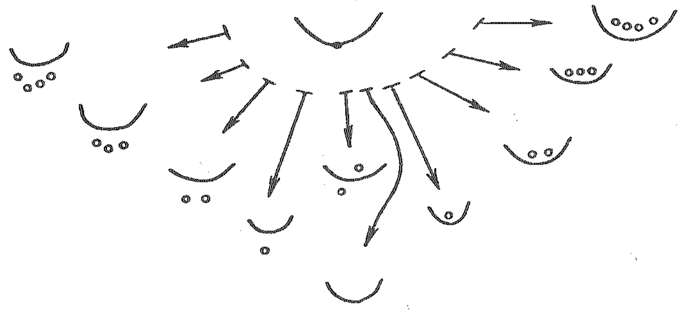


FIGURE 32

4.3. Dissipations of nondegenerate r-fold points. Recall that a nondegenerate r-fold point of a plane curve is a point where the curve has r nonsingular branches which intersect transversally. Any germ of this type is semiquasihomogeneous relative to any coordinate system with origin at the r-fold point. In the cases $r = 2$ and 3, we obtain the singularities of type A_1 and D_4 considered above. Nondegenerate 4-fold singularities are denoted by the symbol X_9, and 5-fold points are denoted N_{16}.

As we showed in §3.5, dissipations of nondegenerate r-fold singularities are closely connected with nonsingular affine real plane algebraic curves of degree r whose projectivization is nonsingular and transverse to the line at infinity. In particular, any such curve gives a quasihomogeneous dissipation of germs of this type. Here the dissipations of a given germ are obtained from affine curves whose asymptotes point in the directions of the tangent lines to the branches of the germ—this is the obvious geometrical meaning of the requirement that the coefficients corresponding to points of the Newton diagram coincide.

There are three types of real nondegenerate 4-fold points: type X_9^2 singularities, where all four branches are real, or where there is a pair of conjugate imaginary branches; and type X_9^0 singularities, where all four branches are imaginary.

The next two theorems give a complete topological classification of dissipations of X_9 singularities.

4.3.A. *Any dissipation of a type X_9 germ of a plane curve is topologically equivalent to one of the quasihomogeneous dissipations in Figure 33.*

4.3.B. *Any type X_9 germ of a plane curve has quasihomogeneous dissipations of all of the topological types in the corresponding part of Figure 33 (with the appropriate number of real branches), and it also has quasihomogeneous dissipations of all of the topological types obtained by rotating the ones in Figure 33 in the plane by multiples of $\pi/4$.*

Theorems 4.3.A and 4.3.B can easily be obtained from the results we have about the topology of curves of degree 4. As in the case of zero-modal singularities, singularities of type X_9 are too simple for their dissipations to be applied directly to give something beyond what the classical methods give in constructing nonsingular projective plane curves. Thus, Theorems 4.3.A and 4.3.B will not be used later, and were only given for the sake of completeness.

But dissipations of nondegenerate 5-fold singularities are of interest from our point of view. The corresponding classification problems for affine real plane curves of degree 5 have been completely solved. Namely, Polotovskii [19], [20] gave a classification up to isotopy for the curves of degree 6 that split into a union of two nonsingular curves of degree 5 and 1 transversal to each other (and

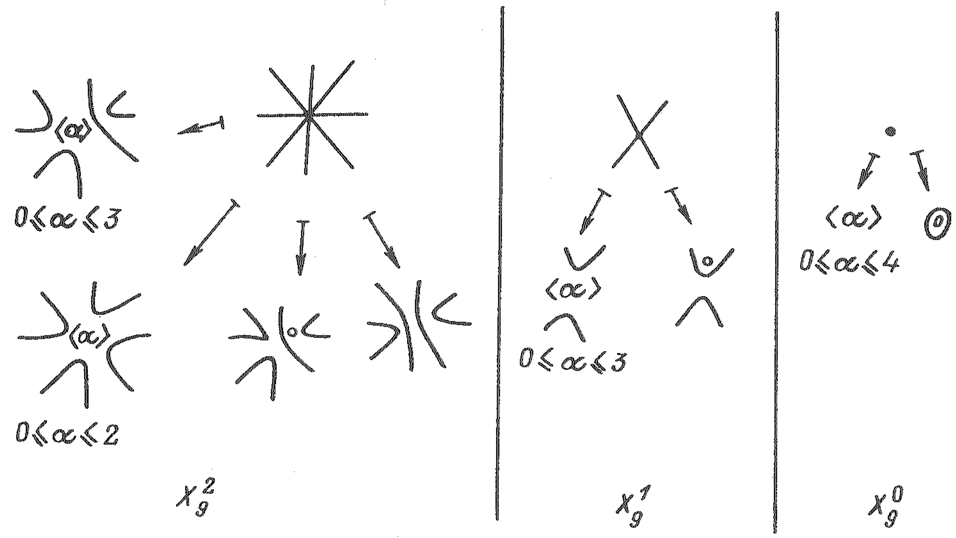


FIGURE 33

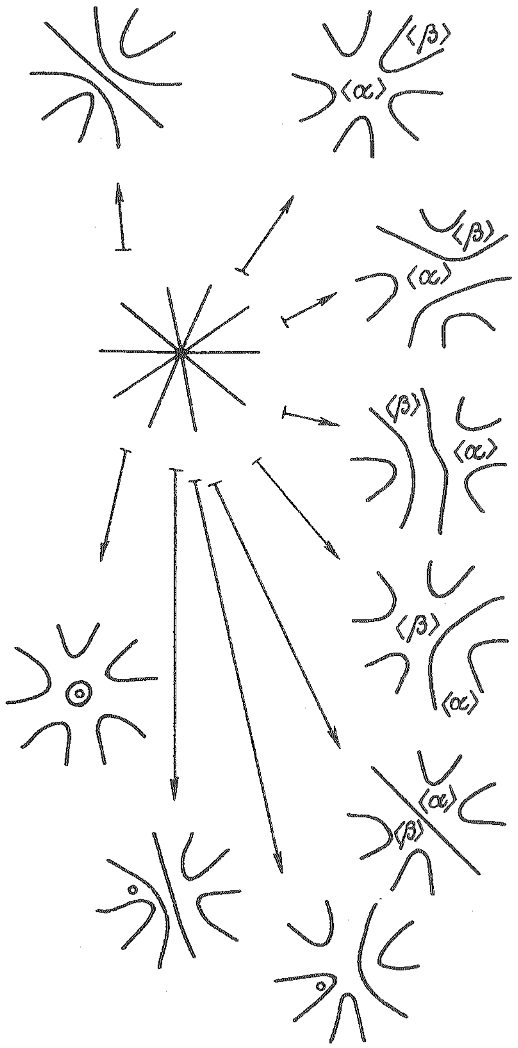
hence for the nonsingular affine curves of degree 5 having 5 (real or imaginary) asymptotes pointing in different directions), and Shustin [31] proved that for any fixed isotopy type of such a split degree 6 curve, all positions of the intersection points on the line are realized (i.e., for a given isotopy type of degree 5 affine curve as above, all sets of directions of the asymptotes are realized). These two results, together with the known prohibitions on nonsingular curves, leads to a complete topological classification of the dissipations of nondegenerate 5-fold points; we shall state the result in Theorems 4.3.C and 4.3.D below.

4.3.C. *Any dissipation of a type* N_{16} *germ of a plane curve is topologically equivalent to one of the quasihomogeneous dissipations in Figure* 34.

4.3.D. *Any type* N_{16} *germ of a plane curve has quasihomogeneous dissipations of all of the topological types in the corresponding part of Figure* 34 (*with the appropriate number of real branches*), *and it also has quasihomogeneous dissipations of all of the topological types which are obtained from these as a result of rotating the plane by multiples of* $2\pi/5$.

A reasonably complete proof of Theorem 4.3.D would take up a lot of space. I shall thus limit myself to a small part: the construction of two affine curves of degree 5 which give two of the four quasihomogeneous dissipations enabling us to construct M-curves. All four of these dissipations are shown in Figure 35. What we construct below are the curves which give the dissipations on the right in Figure 35. I shall give two constructions. One gives a dissipation with $\alpha = 0$, $\beta = 6$ and is carried out by Hilbert's method; the other gives both of the dissipations and is obtained by a new method. The first construction is in some sense contained in the second, and is being considered here mainly for the purpose of illustrating the difference between the methods. It is shown in Figure 36.

For the second construction we take a union of two real conics C_1 and C_2 tangent to one another at two real points and a line L tangent to C_1 and C_2 at one of these two points (Figure 37). We place this union of curves on the plane in such a way that the two common tangent lines are the coordinate axes $x_0 = 0$ and $x_2 = 0$, and the points of intersection of the conics are $(1:0:0)$ and $(0:0:1)$. We then obtain a curve of degree 5 with two singular points of type A_3^- and J_{10}^- which are semi-quasihomogeneous relative to the coordinate sys-



with $0 \le \alpha + \beta \le 6$;
if $\alpha + \beta = 6$, then $\alpha - \beta \equiv 4 \bmod 8$;
if $\alpha + \beta = 5$, then $\alpha - \beta \equiv 4 \pm 1 \bmod 8$.

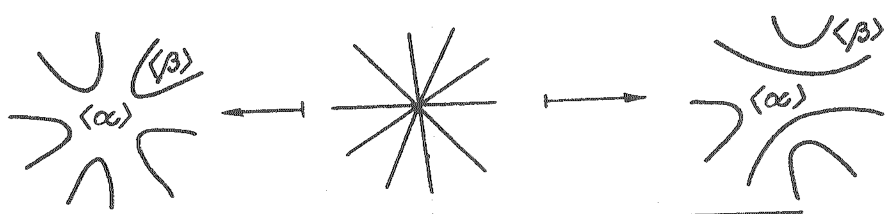
with $0 \le \alpha + \beta \le 6$;
if $\alpha + \beta = 6$, then $\alpha - \beta \equiv 2 \bmod 8$;
if $\alpha + \beta = 5$, then $\alpha - \beta \equiv 2 \pm 1 \bmod 8$.

with $0 \le 2 + \beta \le 5$;
if $\alpha + \beta = 5$, then $\alpha - \beta \equiv 5 \bmod 8$;
if $\alpha + \beta = 4$, then $\alpha - \beta \equiv 5 \pm 1 \bmod 8$.

with $0 \le \alpha + \beta \le 5$

$0 \le \alpha + \beta \le 4$

FIGURE 34



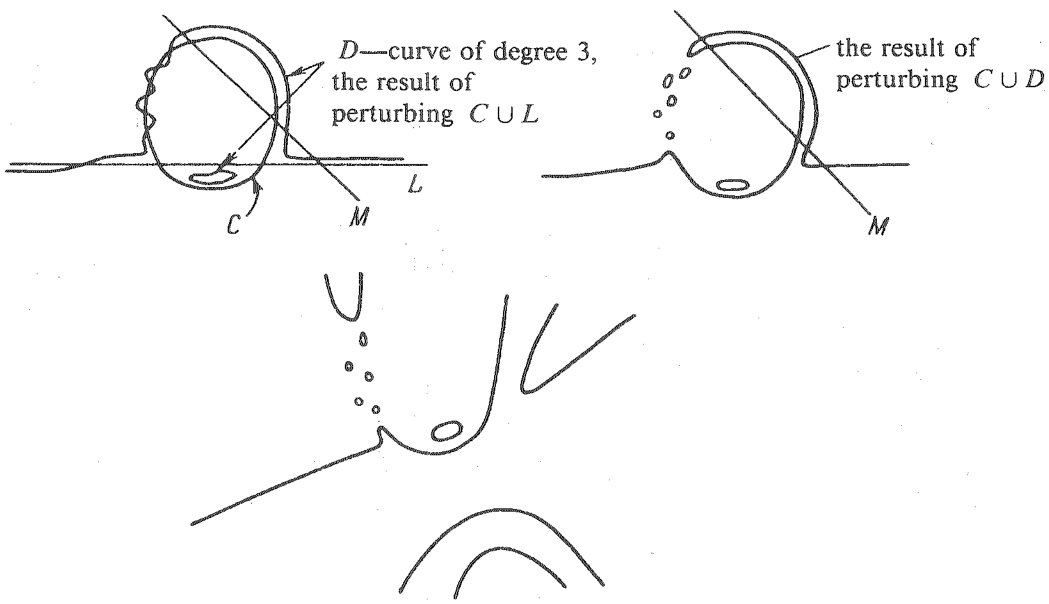
α=	1	5
β=	5	1

α=	0	4
β=	6	2

FIGURE 35

tem. Their quasihomogeneous dissipations give nonsingular projective curves which can be transformed into the required curves by a projective transformation taking the line M to the line at infinity $x_0 = 0$ (Figure 36).

The topological classification problem for dissipations of nondegenerate r-fold singular points on plane curves has not been solved for any $r \ge 6$. Some results for $r = 6$ were obtained by Chislenko [29]. The topological classification problem for dissipations is immense when r is large. However, there are partial results which are within reach and also worthwhile. For example, in §5.4 below we shall need the dissipations given in the following theorem.



after moving the line M to ∞

FIGURE 36

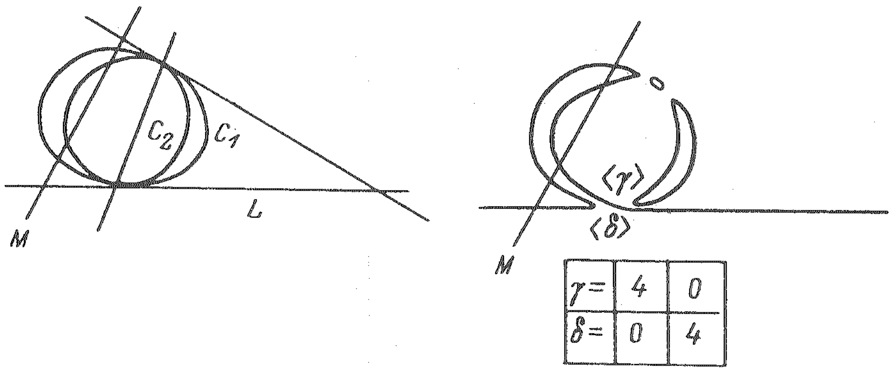


FIGURE 37

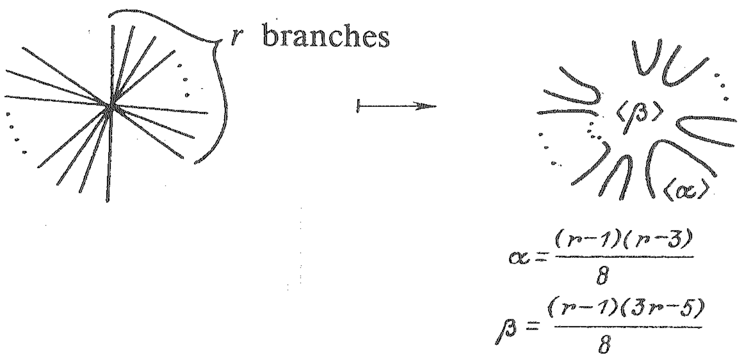


FIGURE 38

4.3.E. *For any odd r, there is a quasihomogeneous dissipation of the form in Figure 38 for any germ of a nondegenerate r-fold singularity on a plane curve at which all branches are real.*

The affine curves of degree r which are needed to prove this theorem can be constructed by Harnack's method (see §1.6). The projective curve of odd degree r with scheme $\langle J \amalg (r-1)(r-2)/2 \rangle$ that can be obtained by Harnack's method is subjected to a projective transformation which takes a generating line to the line at infinity.

4.4. Three crossed-out doubly tangent branches (Z_{15} singularities). In this subsection we examine dissipations of a singular point through which four non-singular branches pass, of which three have a second order tangency at the point, while the fourth intersects the other three transversally. There are two real forms for such singularities: Z_{15}^-, with four real branches, and Z_{15}^+, with two real and two conjugate imaginary branches (clearly, the imaginary branches must be tangent to one another).

A type Z_{15} singularity is semi-quasihomogeneous relative to any coordinate system in which one axis is tangent at the singularity to the branches that are tangent to one another. If this axis is the x-axis and the singularity is of type Z_{15}^-, then the truncation to the line segment from $(7, 0)$ to $(1, 3)$ of the polynomial which in this situation gives the curve has the form $\beta x(y-\alpha_1 x^2) \times (y-\alpha_2 x^2)(y-\alpha_3 x^2)$, where α_1, α_2, α_3 are distinct real numbers, which can be interpreted as half of the curvature of the branches tangent to the x-axis.

Although the complete topological classification of dissipations of points of type Z_{15} is not known, much in this direction has already been done. All of the results I am aware of were obtained by Korchagin [40]. It seems that there is in principle no obstacle to completing the topological classification of dissipations of this type of singularity. Most likely, it remains only to prove a few prohibitions and prove in the Z_{15}^- case that any dissipation is topologically equivalent to a quasihomogeneous dissipation. Here we shall limit ourselves to the statement of a result relating to Z_{15}^-.

4.4.A. *Any germ of type Z_{15}^- has the quasihomogeneous dissipations shown in Figure 39, and also has the quasihomogeneous dissipations which are symmetrical to them relative to the vertical axis.*

A proof of this theorem is contained in Korchagin's article [40], except for one thing: in the case of the dissipation in Figure 40, Korchagin does not prove that it can be applied to a germ with arbitrary curvature of the branches. However, Korchagin's construction enables one to do this without difficulty. In Figure 41 we show a construction of the curves which are needed to obtain some of the dissipations in Figure 39. The construction is carried out by a slight modification of Hilbert's method, followed by dissipation of a type J_{10}^- point.

4.5. Hyperbolism. In the constructions that follow an important role will be played by a certain birational transformation of the plane, the use of which goes back to Huyghens [37] and Newton [16]. Following Newton, we shall call this map a *hyperbolism* and shall denote it by the symbol hy. In homogeneous coordinates it is given by the formula

$$hy(x_0 : x_1 : x_2) = (x_0x_1 : x_1^2 : x_0x_2),$$

and in affine coordinates $x = x_1/x_0$, $y = x_2/x_0$ it is given by the formula

$$hy(x, y) = (x, y/x).$$

The inverse transformation acts according to the formula

$$hy^{-1}(x_0 : x_1 : x_2) = (x_0^2 : x_0x_1 : x_1x_2),$$

or, in affine coordinate,

$$hy^{-1}(x, y) = (x, yx).$$

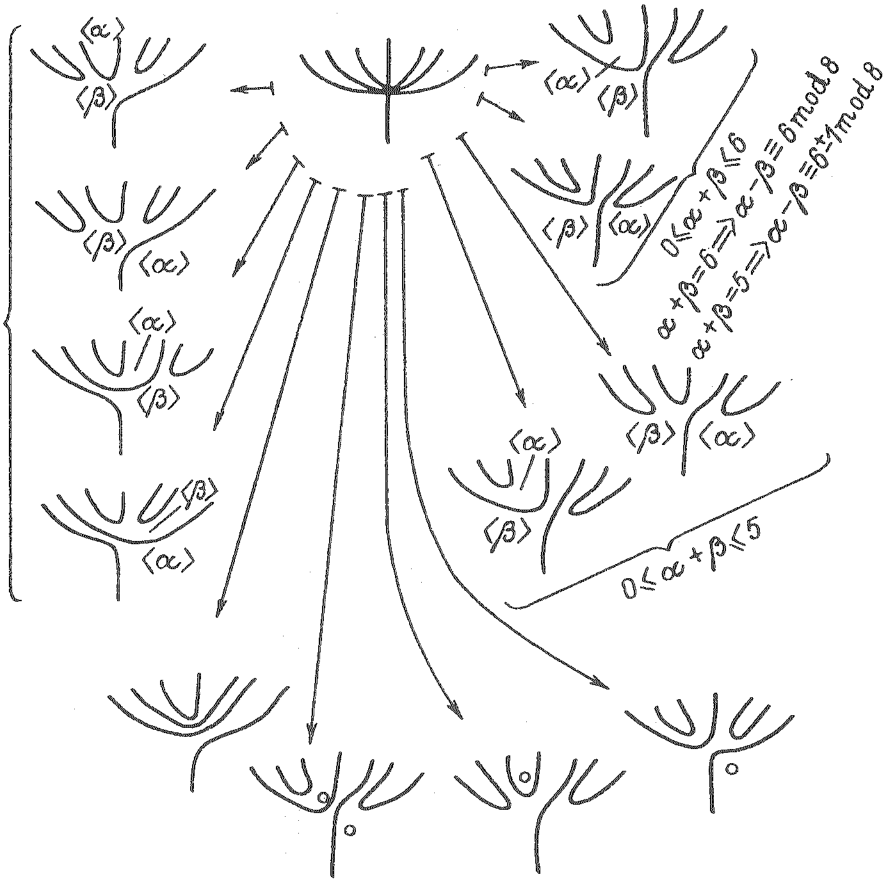


FIGURE 39

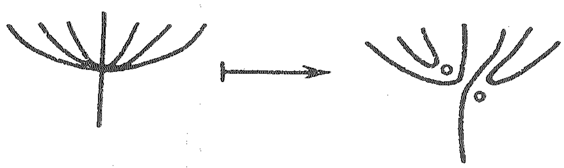


FIGURE 40

hy^{-1} is obtained from hy by conjugation by the projective involution $(x_0 : x_1 : x_2) \mapsto (x_1 : x_0 : x_2)$ (i.e. $(x, y) \mapsto (1/x, y/x)$).

The fundamental points of the hyperbolism (i.e., the points where it is not defined as a map) are $(1 : 0 : 0)$ and $(0 : 0 : 1)$, and the latter point has multiplicity two. The point $(1 : 0 : 0)$ is blown up to the line $x_1 = 0$, and the point $(0 : 0 : 0)$ is blown up to the line $x_0 = 0$. The hyperbolism also contracts two lines: the line $x_1 = 0$ is contracted to the point $(0 : 0 : 1)$, and the line $x_0 = 0$ is contracted to the point $(0 : 1 : 0)$. The set of fixed points of the hyperbolism consists of the line $x_2 = 0$. The first and fourth quadrants are each mapped into themselves, while the second and third quadrants are mapped into each other. The action of the hyperbolism is shown schematically in Figure 42. In Figure 43 we show how the hyperbolism decomposes into a composition of three σ-processes and three inverse σ-processes. In Figure 44 we show what the hyperbolism does to some curves of degree ≤ 2. The name "hyperbolism" comes from the first example in Figure 44: a line being transformed into a hyperbola.

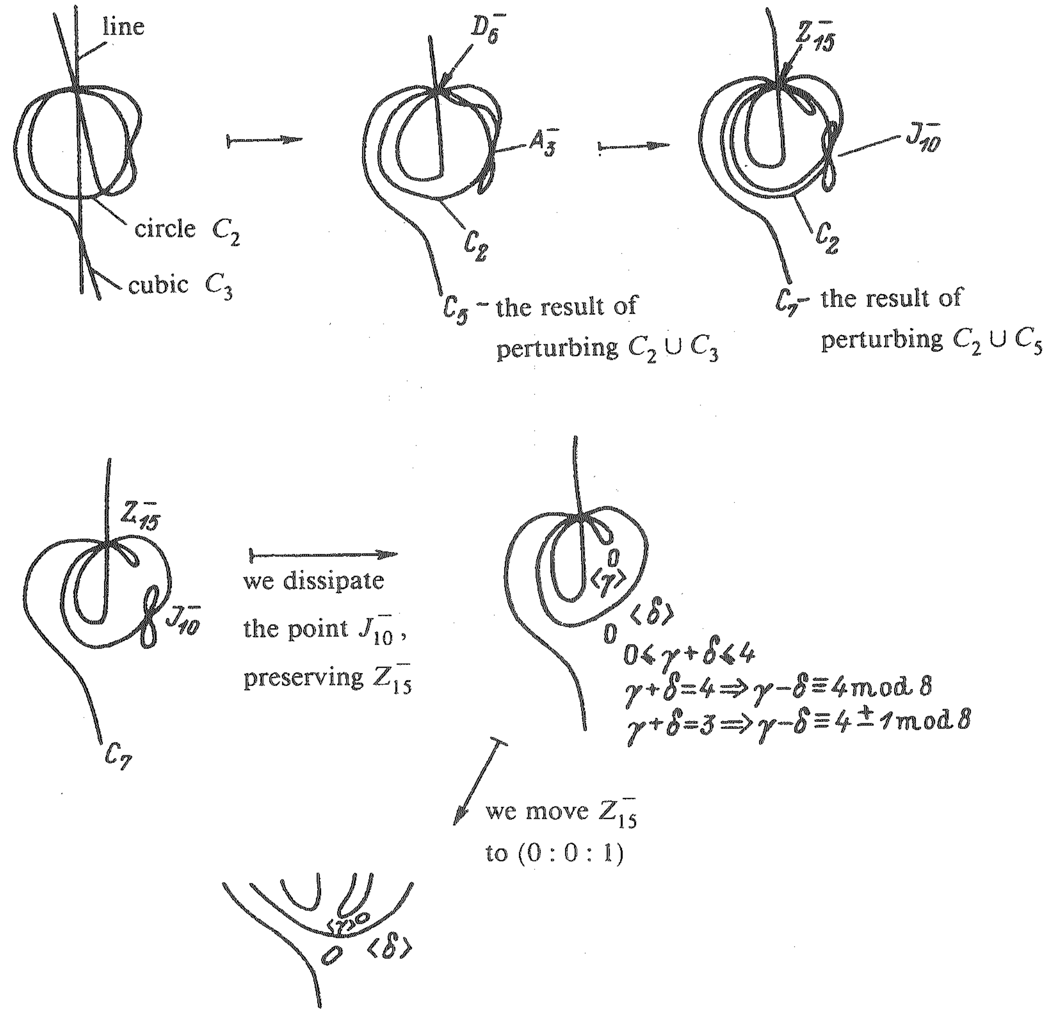


FIGURE 41

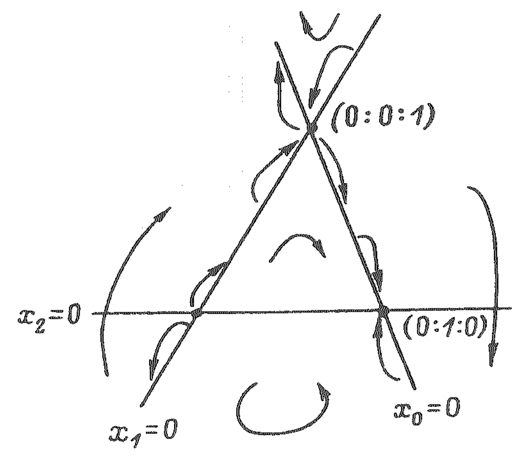


FIGURE 42

The hyperbolism takes the curve $f(x, y) = 0$ to the curve $f(x, yx) = 0$. We note that the Newton polygon of $g(x, y) = f(x, yx)$ is obtained from that of f by means of a shear along the x-axis of the form $(x, y) \mapsto (x, y + x)$.

4.6. Some dissipations of boundary singularities of type $F_{1,0}$ and $F_{2,0}$. In this subsection we construct polynomials which give dissipations of singularities of type D_4 and J_{10}. However, we shall pay special attention to the location of

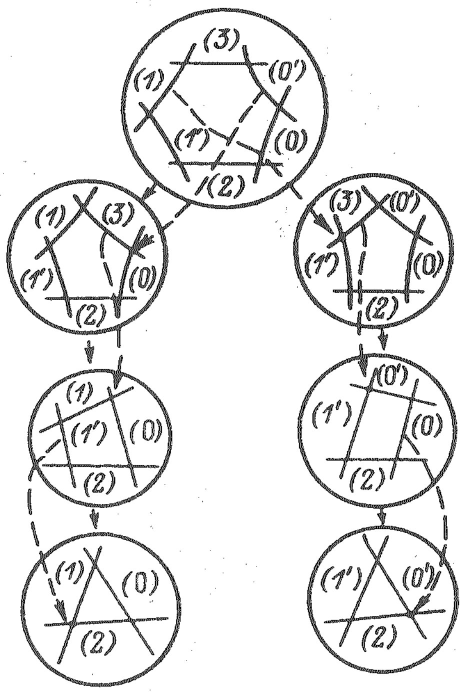


FIGURE 43

the curves relative to a distinguished line through the singular point. Thus, essentially what we are constructing here is dissipations of boundary singularities of type $F_{1,0}$ and $F_{2,0}$ (see [3], §17.4).

4.6.A. *There exists a nonsingular projective cubic curve which is situated relative to the axes as shown in Figure* 45a *and which intersects the axis* $x_1 = 0$ *at any three points given in advance and the axis* $x_0 = 0$ *at any three points given in advance lying in the region* $x_1 x_2 < 0$.

PROOF. We perturb the curve $x_0 x_1 (x_2 - \alpha x_0)$ in the way shown in Figure 45b. The coefficient α and the perturbation can clearly be chosen in such a way that the required curve is obtained. ●

4.6.B. COROLLARY. *Any germ of type* D_4^- *(i.e., a nondegenerate triple point) of a plane curve at the origin whose branches are transversal to the* y*-axis admits a quasihomogeneous dissipation shown in Figure* 46 *such that the perturbed curve intersects the* y*-axis at three points near the origin, with the ratio of distances between neighboring points equal to any preassigned value.* ●

4.6.C. LEMMA. *There exist curves whose Newton polygon is the triangle with vertices* $(0, 0)$, $(6, 0)$ *and* $(0, 3)$ *and which are situated relative to the coordinate axes as shown in Figure* 47.

PROOF. We perturb the union of the circle $(x+5)^2 + (y+5)^2 = 36$ and the lines $y = 0$ and $y = 10$ in the two ways shown in Figure 48. We then add the x-axis to the resulting curves and perturb the union as in Figure 49. We again add the x-axis and make a perturbation; see Figure 50. We now perform the projective transformations $(x_0 : x_1 : x_2) \mapsto (x_2 : -x_1 : x_0)$ for the curve on the left in Figure 50, and $(x_0 : x_1 : x_2) \mapsto (x_2 : x_1 : x_0)$ for the curve on the right. As a result we obtain the curves in Figures 47a–b. We now subject them to the projective transformation $(x_0 : x_1 : x_2) \mapsto (x_0 : x_1 : x_2 - \alpha x_0)$ with α chosen so as to obtain the curves in Figure 51. We now apply the hyperbolism, which

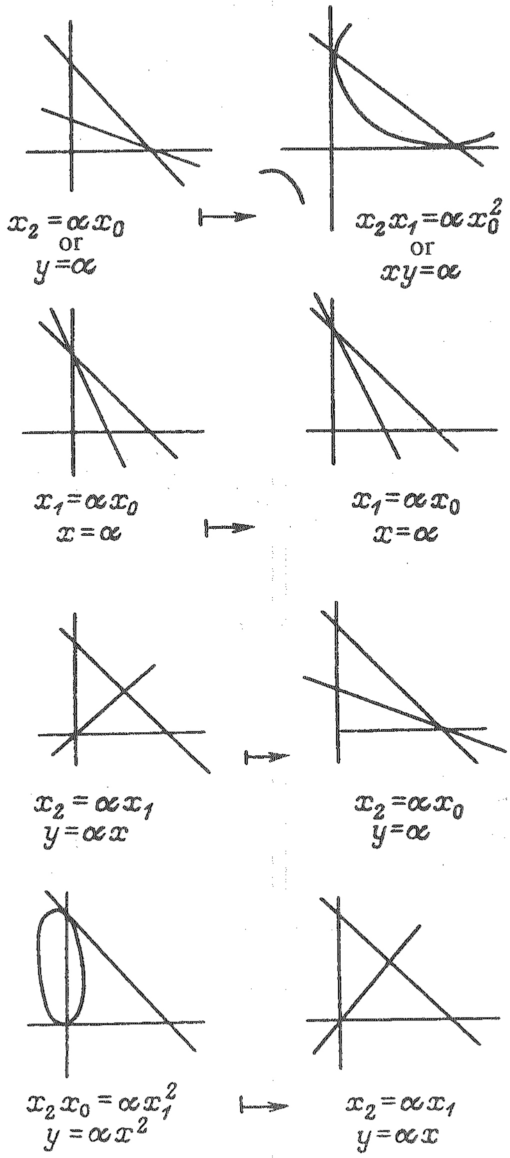


FIGURE 44

gives us the curves in Figure 52. If we apply suitable projective transformations, we obtain the curves in Figures 47c–d. ●

4.6.D. *For any* $\alpha_1 > \alpha_2 > \alpha_3 > 0$ *and* $\beta_1 > \beta_2 > \beta_3$ *there exists a real polynomial* f *in two variables such that*

(i) $\Delta(f)$ *is the triangle with vertices* $(0, 0)$, $(6, 0)$, $(0, 3)$;

(ii) *the curve* $x_0^6 f(x_1/x_0, x_2/x_0) = 0$ *is situated relative to the coordinate axes in the same way as the curve given in advance in Lemma* 4.6.C;

(iii) *if* Γ *is the segment joining the points* $(0, 3)$ *and* $(6, 0)$, *then* $f^\Gamma(x, y) = (y - \alpha_1 x^2)(y - \alpha_2 x^2)(y - \alpha_3 x^2)$;

(iv) *if* Ξ *is the segment joining the points* $(0, 0)$ *and* $(0, 3)$, *then* $f^\Xi(x, y) = (y - \beta_1)(y - \beta_2)(y - \beta_3)$.

PROOF. By Lemma 4.6.C, there exists a polynomial f satisfying (i) and (ii). Using a parallel translation along the x-axis $(x, y) \mapsto (x + c, y)$ (which does not change f^Γ), we can arrange that the curve $f(x, y) = 0$ intersect the y-axis at three points having any preassigned ratio of distances between neighboring points. We choose such a parallel translation so that we obtain the same ratio

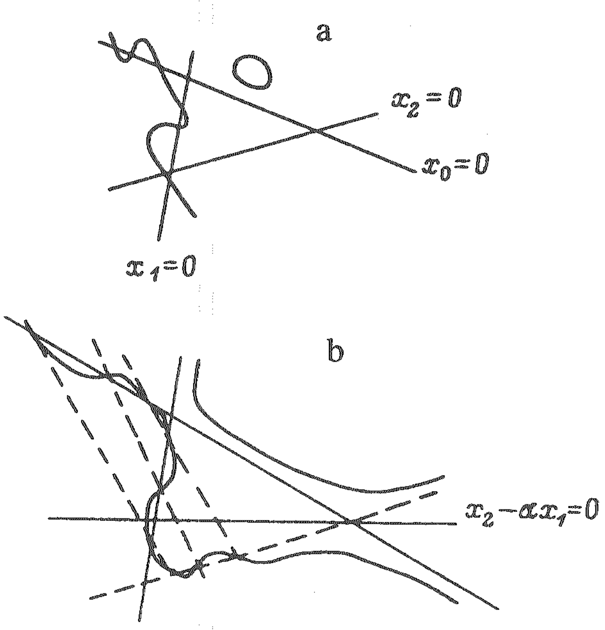


FIGURE 45

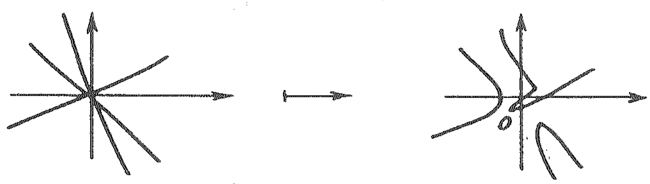


FIGURE 46

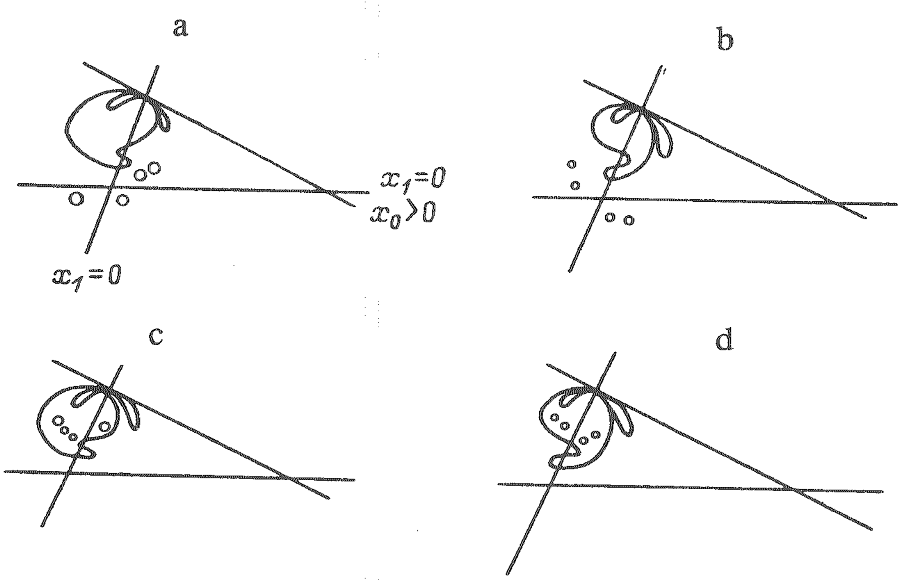


FIGURE 47

as for the points $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}$. Using a parallel translation of the form $(x, y) \mapsto (x, y+c)$ (which again does not affect the polynomial f^{Γ}), we arrange to have $f^{\Xi}(x, y)$ become equal to $\gamma(y-\delta\alpha_1)(y-\delta\alpha_2)(y-\delta\alpha_3)$ for some

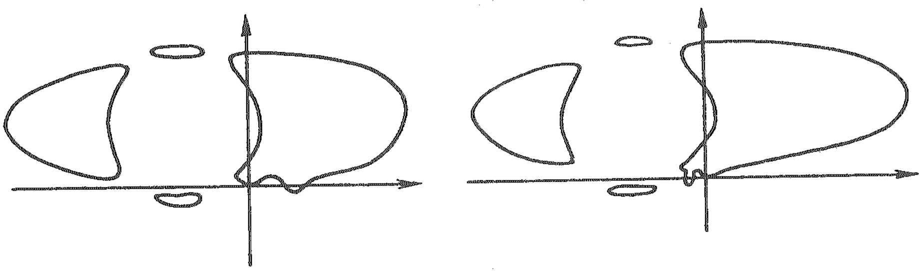


FIGURE 48

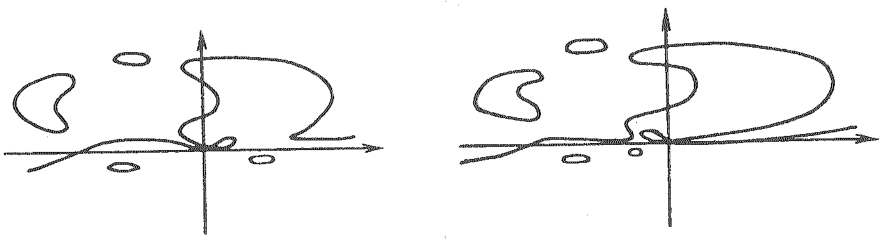


FIGURE 49

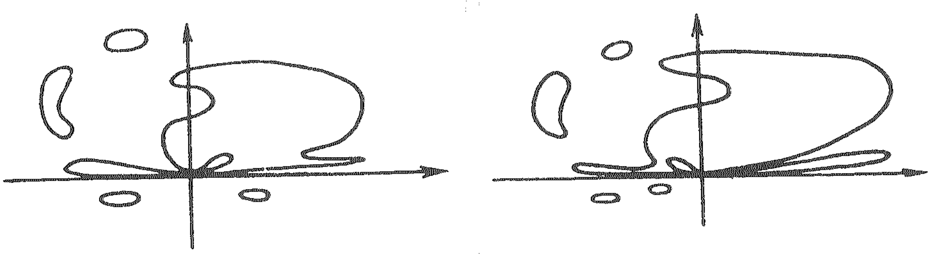


FIGURE 50

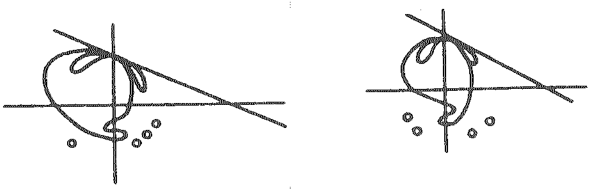


FIGURE 51

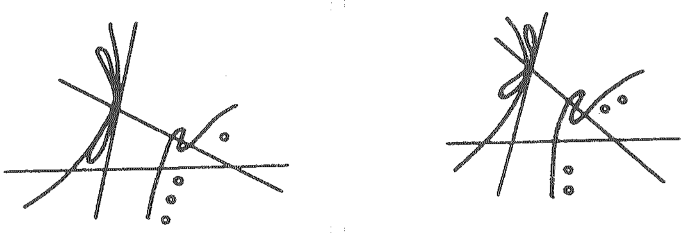


FIGURE 52

$\delta > 0$ and γ. Using the quasihomothety $(x, y) \mapsto (\delta^{-1/2}x, \delta^{-1}y)$ (which causes f^{Γ} to be multiplied by δ^3), we make the polynomial $f^{\Xi}(x, y)$ equal to $\gamma\delta^3(y-\alpha_1)(y-\alpha_2)(y-\alpha_3)$. We divide the resulting polynomial f by $\gamma\delta^3$. We now perform the hyperbolism, followed by the transformation $(x_0 : x_1 : x_2) \mapsto (x_1 : x_0 : x_2)$. The resulting curve turns out to be (topologically) situated as before relative to the coordinate axes—see Figure 53, where this is carried out for the case shown in Figure 47c. The corresponding transformation of the

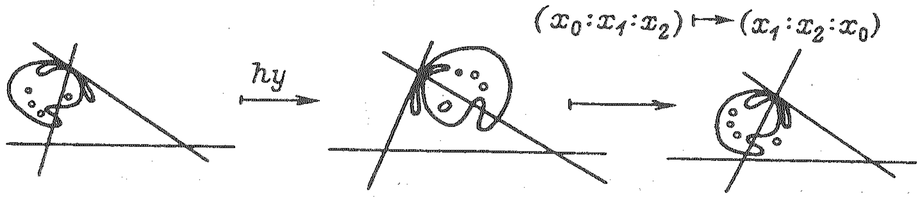


FIGURE 53

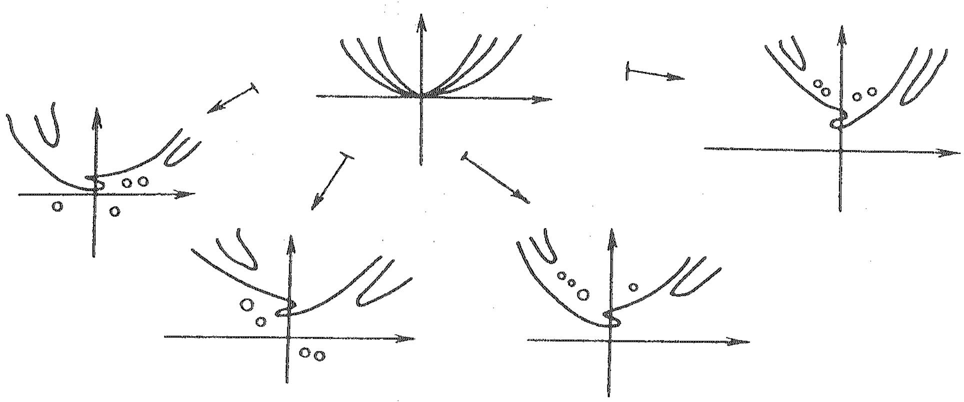


FIGURE 54

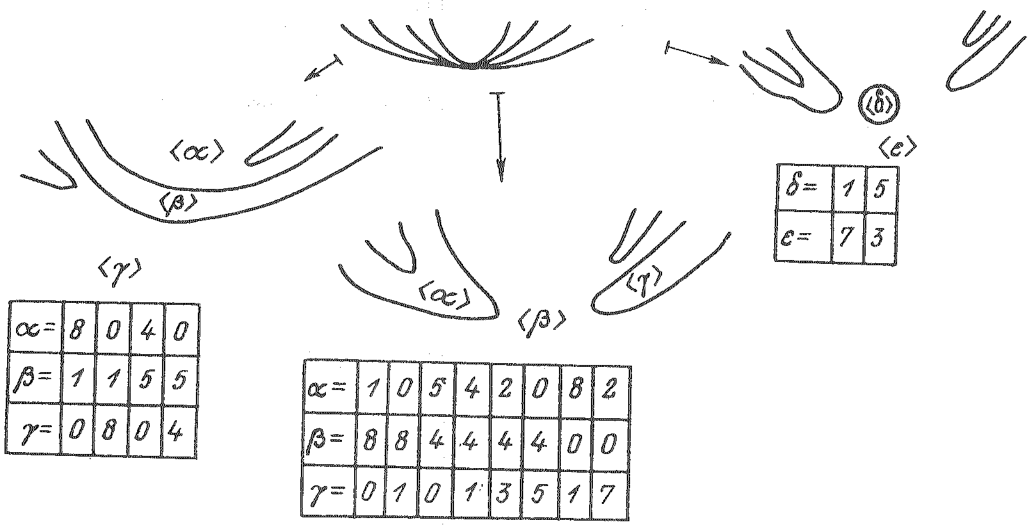


FIGURE 55

Newton polygon and the equation takes Ξ to Γ and f^{Ξ} to f^{Γ}, so that we now have $f^{\Gamma}(x, y) = (y - \alpha_1 x^2)(y - \alpha_2 x^2)(y - \alpha_3 x^2)$. Finally, proceeding as before, we use parallel translations and quasihomotheties which keep the truncation f^{Γ} unchanged and give us a polynomial f satisfying condition (iv) as well. ●

4.6.E. COROLLARY. *Any type J_{10}^{-} germ of a plane curve at the origin whose branches are tangent to the x axis from above*(4) *admits quasihomogeneous dissipations shown in Figure 54 such that the perturbed curve intersects the y-axis at three points near the origin, with ratio of distance between neighboring points equal to any preassigned value.*

4.7. Four branches with second order tangency (X_{21} **singularities).** In this subsection we examine dissipations of singularities of type X_{21}, i.e., singular-

(4)It is not hard to remove the condition that they must be tangent from above.

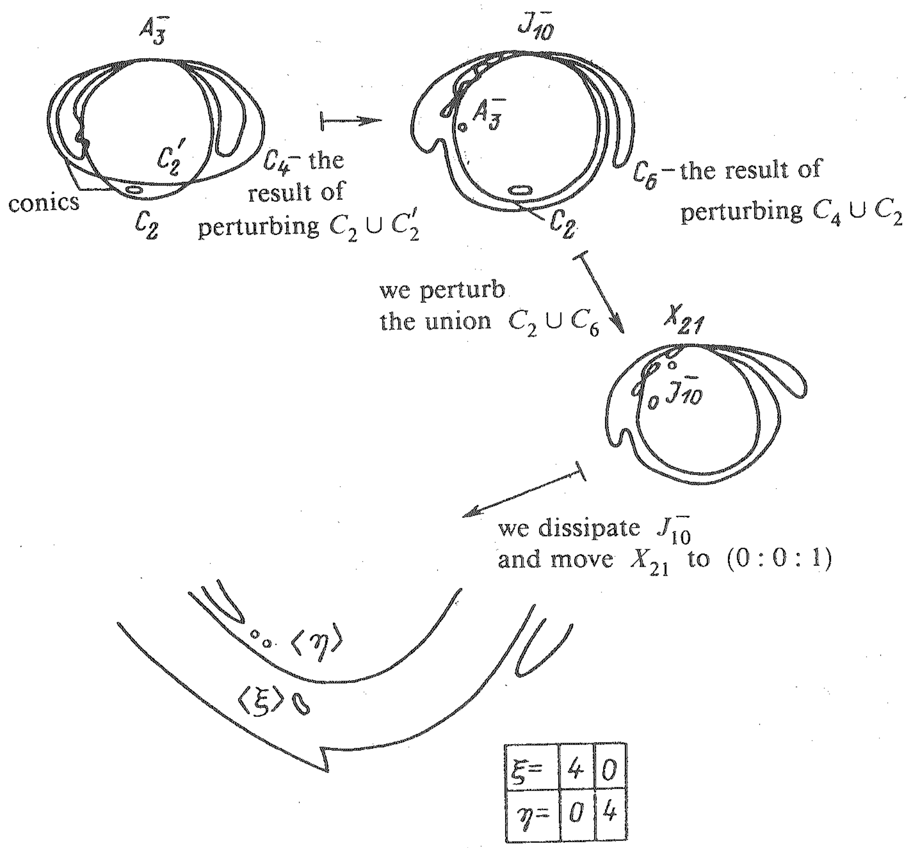


FIGURE 56

ities where four nonsingular branches are tangent to one another with second order tangency. Any germ of this type is semi-quasihomogeneous relative to any coordinate system one of whose axes is tangent to the branches. We shall only be interested in germs for which all of the branches are real. The polynomial defining the curve with type X_{21} point at the origin has truncation to the Newton diagram determined up to a constant of proportionality by the values of the curvature of the branches at the singular point.

We do not yet have a complete topological classification of the dissipations of X_{21} singularities. Shustin [32] proved that all dissipations of type X_{21} singularities with a given number of real branches have the same topological type; however, there is still a big gap between what is given by the constructions and the prohibitions. Curiously, the problem has been completely solved for dissipations that can occur in the construction of nonsingular M-curves. These dissipations are considered in the next theorem. It can be shown that any dissipation of an X_{21} singularity with four real branches in the course of which nine new small ovals appear (this is the maximum possible number) is topologically equivalent to one of the dissipations in Theorem 4.5.A.

4.7.A. *Any type X_{21} germ with four real branches has all of the quasihomogeneous dissipations in Figure 55, and also all of the quasihomogeneous dissipations obtained from them by reflection about the vertical axis.*

I will give a construction of curves realizing these dissipations, where (as above) the curvature of the branches is ignored. In Figure 56 a curve is constructed from the dissipations on the left in Figure 55 with $\alpha = 8$ and 4. If we apply the transformation $(x, y) \mapsto (x, y + ax^2)$ to these curves with a sufficiently large, we obtain curves which give the two remaining dissipations on the left in Figure 55.

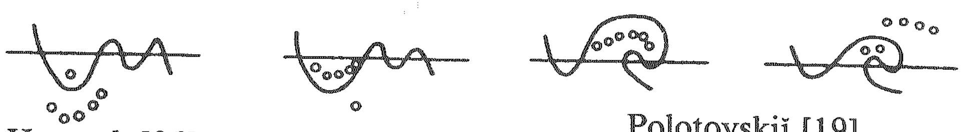


FIGURE 57

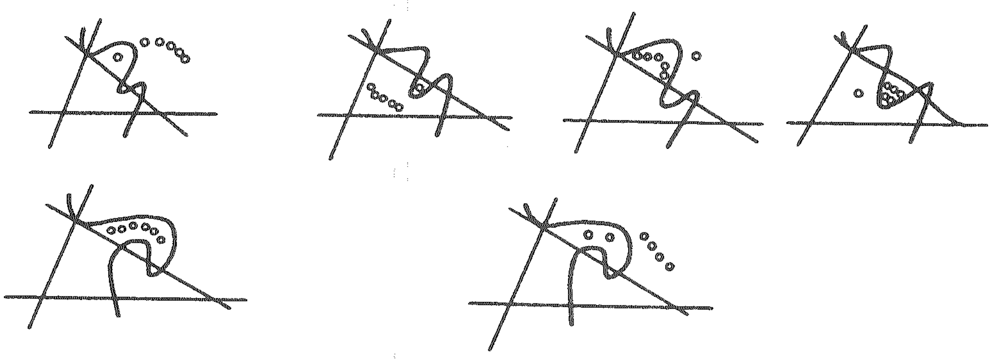


FIGURE 58

Curves giving the rest of the dissipations in Figure 55 can be constructed in a uniform manner. As shown by Polotovskiĭ [19], a nonsingular M-curve of degree 5 whose one-sided component intersects some line in five points can be situated relative to this line in any of four possible ways (see Figure 57, where the papers in which the cases were first realized are indicated). If we rotate the line around one of its points of intersection with the one-sided component and then apply projective transformations which take this line to the line $x_0 = 0$, from the curves in Figure 57 we can obtain curves of degree 5 which are situated relative to the projective coordinate system in the manner shown in Figure 58.

We apply a hyperbolism to all of these curves, obtaining curves of degree 8 with a nondegenerate triple point at $(0:1:0)$ and with a type X_{21} singularity that is solitary (i.e., it lies only on imaginary branches) at the point $(0:0:1)$ (Figure 59).

We next apply the transformation $(x_0 : x_1 : x_2) \mapsto (-x_0 : x_1 : x_2)$ to the first, third, fifth and sixth of these curves; the results are shown in Figure 60.

We now dissipate the singularity at $(0:1:0)$ in the resulting ten curves in Figures 59–60. By 4.6.B, there exist quasihomogeneous dissipations of these triple points such that the resulting curves have the form in Figure 61. It is essential that in place of the singularity one obtains three points of intersection of the perturbed curve with the line $x_0 = 0$ which lie on the same branch. If we now apply the transformation hy^{-1} to these curves, we obtain curves of degree 8 which give the required quasihomogeneous dissipations of a point of type X_{21}. ●

4.8. Dissipation of a point of second order tangency of $2k-1$ nonsingular real branches. Of course, the topological classification problem for such dissipations

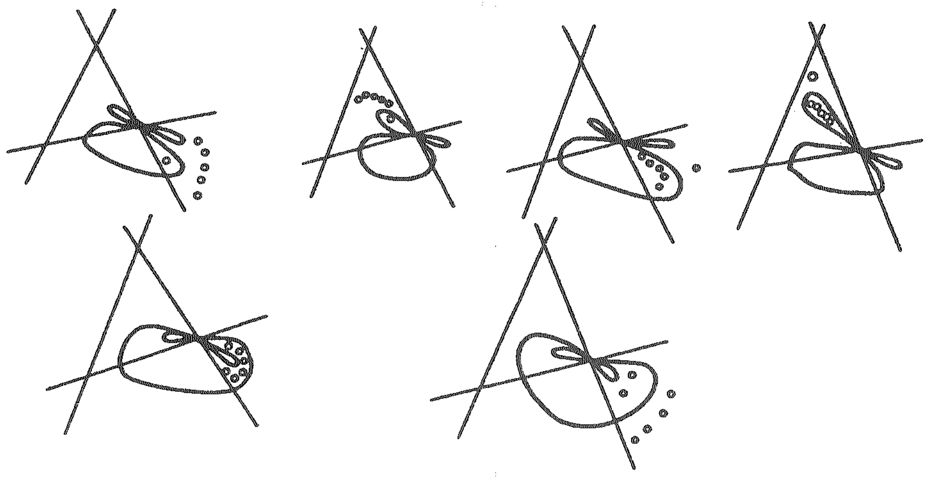


FIGURE 59

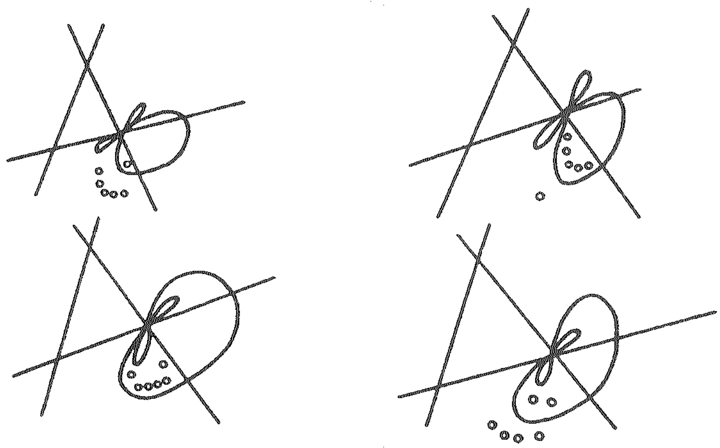


FIGURE 60

very quickly becomes hopeless as k increases. However, Theorem 3.8.A enables us to construct dissipations for some concrete types with k arbitrary.

4.8.A. *Any germ of a curve consisting of* $2k-1$ *nonsingular real branches* ($k>1$) *which have second order tangency to one another and to the* x*-axis at the origin admits a quasihomogeneous dissipation shown in Figure* 62.

PROOF. We begin the construction with the parabola $y=(x-\alpha)(x-\beta)$, where $\alpha>\beta>0$, or, in homogeneous coordinates $x_2x_0-(x_1-\alpha x_0)(x_1-\beta x_0)=0$. Next, we take the union of this curve with the y-axis and perturb by means of the polynomial $(x_2-\gamma_{1,0}x_0)x_0^2$, i.e., we construct the curve

$$(x_2x_0-(x_1-\alpha x_0)(x_1-\beta x_0))x_1+\varepsilon(x_2-\gamma_{1,1}x_0)x_0^2=0.$$

Here $\gamma_{1,0}<0$, and ε is a small positive number. In the projective plane this looks like the drawing in Figure 63. To understand what happens at the point $(0:0:1)$ it suffices to look at the Newton polygon. The degree 3 polynomial we constructed has as its Newton polygon a trapezoid with two sides meeting at $(0,3)$. It is clear that the singularity at $(0:0:1)$ is of type A_1; but one of the branches is tangent to the line $x_0=0$. Now we again add the line $x_1=0$ and perturb by means of the polynomial $(x_2-\delta_{1,1}x_0)(x_2-\delta_{1,2}x_0)x_0^2$ with $\delta_{1,1}>\delta_{1,2}>0$. We now have a type A_3 singularity at $(0:0:1)$. We

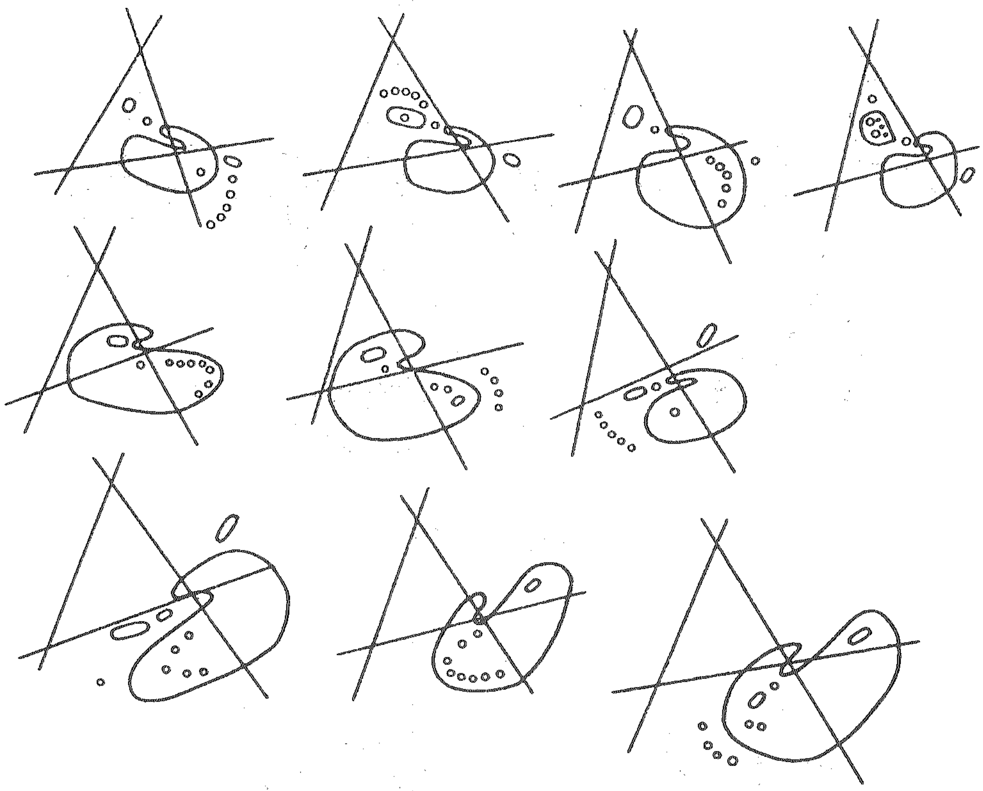


FIGURE 61

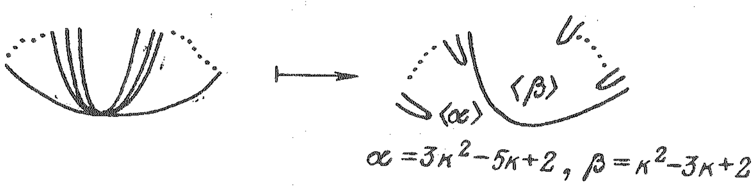


FIGURE 62

continue in the same way. Suppose that we have already constructed a curve of degree $4r - 2$ having $2r - 1$ nonsingular branches at $(0:0:1)$ tangent to the axis $x_0 = 0$ and situated as in Figure 64 on the left. After twice adding a line and perturbing, we obtain an analogous curve of greater by 2 in degree. This curve gives the dissipation whose existence is claimed in the theorem. However, this dissipation is not suitable for any germ, only for a germ with a certain fixed set of values of curvature of the branches, and we have no control over this set of values. Nevertheless, we note that our curve intersects the y-axis at points which depend only on the last step of the construction and can be made to be whatever we want. We apply the hyperbolism to this curve, followed by the symmetry $(x_0 : x_1 : x_2) \mapsto (x_1 : x_0 : x_2)$. A curve of the same form is obtained. But the branches of our original curves which intersect the y-axis become branches tangent to the line $x_0 = 0$; and the y-coordinates of the points of intersection are proportional to the values of the curvature of the branches of the new curve which pass through $(0:0:1)$. Since these values are arbitrary distinct positive numbers, we have obtained the required dissipation. ●

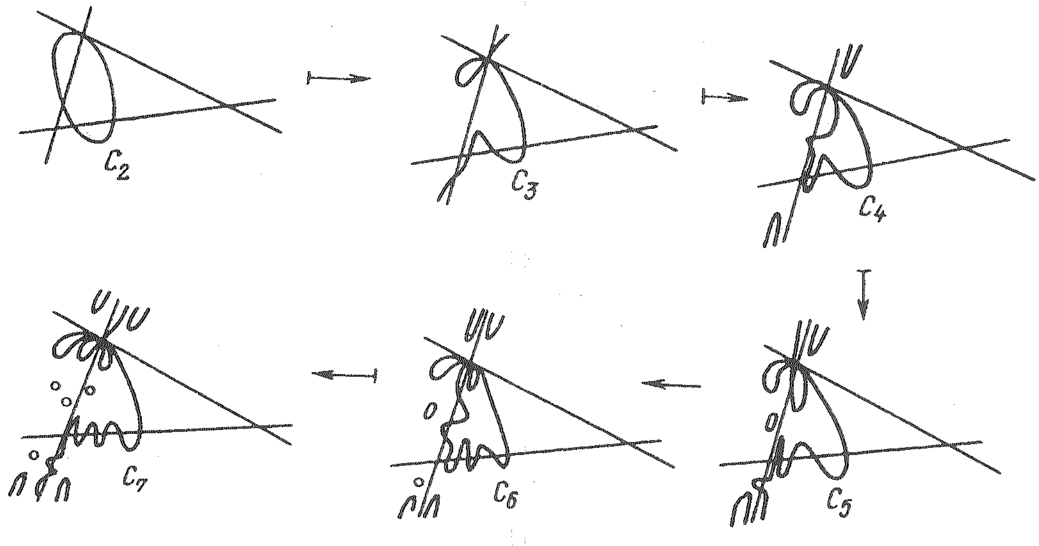


FIGURE 63

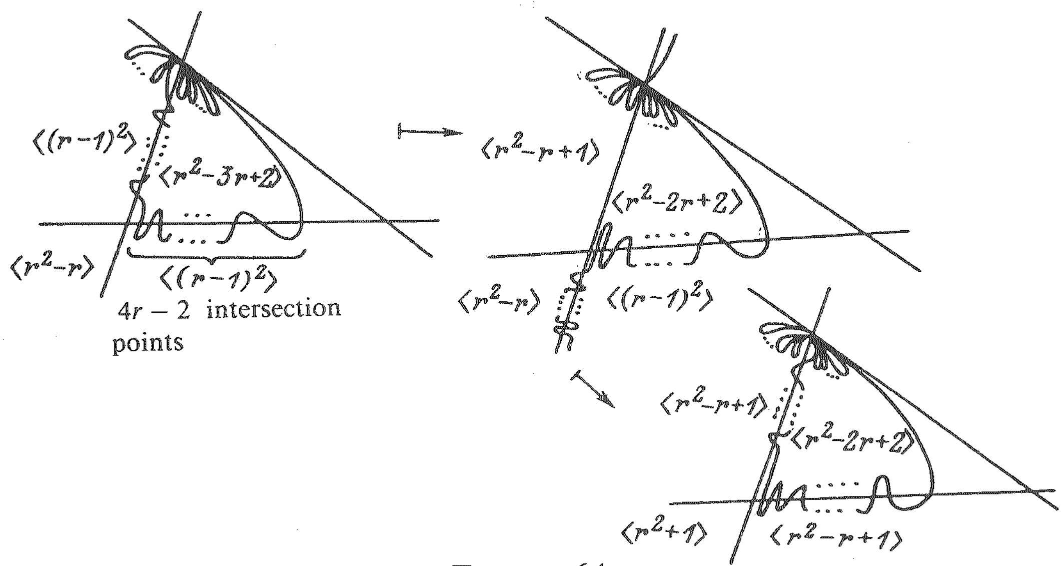


FIGURE 64

§5. Construction of nonsingular curves

5.1. Curves of degree 6. Gudkov's theorem on the isotopy classification of nonsingular projective curves of degree 6 has already been stated in §1.13.

5.1.A. *Any nonsingular curve of degree* 6, *except for an empty curve or a curve with the scheme* $\langle 10 \rangle$ *or* $\langle 1\langle 9 \rangle\rangle$, *is isotopic to the curve resulting from a small perturbation of a union of three ellipses which are tangent to one another at two points, as shown in Figure* 65.

To prove this is sufficient, using Theorem 3.8.A and Remark 3.9, to perturb the union by means of the dissipations in Theorem 4.2.B.

5.1.B. REMARK. Curiously, perturbations of the curve in Figure 66 can also be used to prove prohibitions on the topology of dissipations of a type J_{10}^- singularity. In fact, Theorem 4.2.B along with the prohibitions on the topology

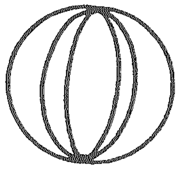


FIGURE 65

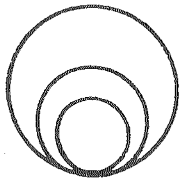


FIGURE 66

of nonsingular projective curves of degree 6 imply the prohibitions in Theorem 4.2.A. The point is that prohibitions on the topology of curves of degree 6 come from topology, and any dissipation of a singularity of a curve obtained from Theorem 4.2.B at least gives a flexible curve of degree 6.

Degree 6 curves with the schemes $\langle 10\rangle$ and $\langle 1\langle 9\rangle\rangle$ can easily be constructed by Hilbert's method (the former can also be constructed by Harnack's method); see §§1.9–1.10.

In my article [46] it is erroneously stated that all nonsingular curves of degree 6 are isotopic to curves resulting from small perturbations of the curve in Figure 65. It can be shown that this is the case for a union of three ellipses which are tangent to one another with multiplicity 3 at a single point and which are situated as in Figure 66. But we shall not prove this, since we have not examined dissipations of a second order point of tangency of three branches (J_{24}^-).

5.2. Curves of degree 7. Since the isotopy classification problem has not yet been examined separately for the case of nonsingular curves of degree 7, I will begin with prohibitions. Unlike the lower degree cases, here the prohibitions coming from topology have so far turned out to be much weaker than the simplest corollaries of Bézout's theorem. Corollary 1.3.C (i.e., the consideration of intersections with auxiliary lines) implies that the real schemes of nonsingular curves of degree 7 have the form $\langle J\amalg\alpha\rangle$, $\langle J\amalg\alpha\amalg 1\langle\beta\rangle\rangle$ or $\langle J\amalg 1\langle 1\langle 1\rangle\rangle\rangle$, where $\alpha > 0$ and $\beta \geq 1$. By Harnack's inequality, the total number of components is at most 16, so that the number of ovals is ≤ 15.

This is almost all. In 1979, when all of the real schemes satisfying these prohibitions except for $\langle J\amalg 1\langle 14\rangle\rangle$ had been realized by nonsingular degree 7 curves, I was able to show that $\langle J\amalg 1\langle 14\rangle\rangle$ is prohibited (thereby completing the isotopy classification of nonsingular curves of degree 7) (see [7]). This was done using auxiliary curves of degree 2 and the theory of complex orientations.

Thus, the solution of the isotopy classification problem for nonsingular real projective algebraic plane curves of degree 7 can be stated as follows.

5.2.A. *There exist nonsingular plane curves of degree* 7 *with the following real schemes*:

(i) $\langle J\amalg\alpha\rangle$ with $0\leq\alpha\leq 15$;

(ii) $\langle J\amalg\alpha\amalg 1\langle\beta\rangle\rangle$ with $\alpha+\beta\leq 14$, $0\leq\alpha\leq 13$, $1\leq\beta\leq 13$;

(iii) $\langle J\amalg 1\langle 1\langle 1\rangle\rangle\rangle$.

Any nonsingular plane curve of degree 7 *has one of these* 121 *real schemes.*

We now consider how to construct the curves needed to prove this theorem. The following real schemes can be realized using Harnack's method: $\langle J\amalg\alpha\rangle$ with $0\leq\alpha\leq 15$; $\langle J\amalg\alpha\amalg 1\langle 1\rangle\rangle$ with $0\leq\alpha\leq 13$ (see Figure 7, which shows a realization of the scheme $\langle J\amalg 13\amalg 1\langle 1\rangle\rangle$) and also the schemes $\langle J\amalg\alpha\amalg 1\langle\beta\rangle\rangle$ with $0\leq\alpha\leq 9$ and $0\leq\beta\leq 4$.

The following real schemes can be obtained by Hilbert's method: $\langle J\amalg\alpha\amalg 1\langle\beta\rangle\rangle$ with $\alpha+\beta\leq 12$; $\langle J\amalg\alpha\amalg 1\langle\beta\rangle\rangle$ with $\alpha+\beta\leq 14$, $\alpha\leq 2$, $\beta\leq 13$ and $\langle J\amalg\alpha\amalg 1\langle\beta\rangle\rangle$ with $\alpha+\beta\leq 14$, $\alpha\leq 12$, $\beta\leq 3$. Of course, the scheme

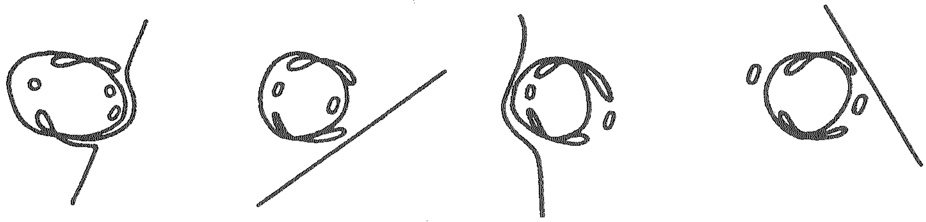


FIGURE 67

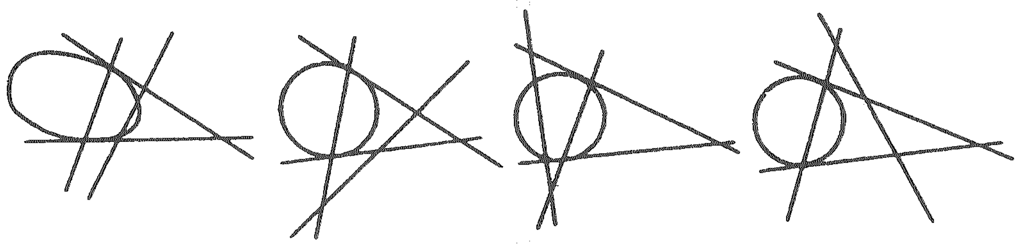


FIGURE 68

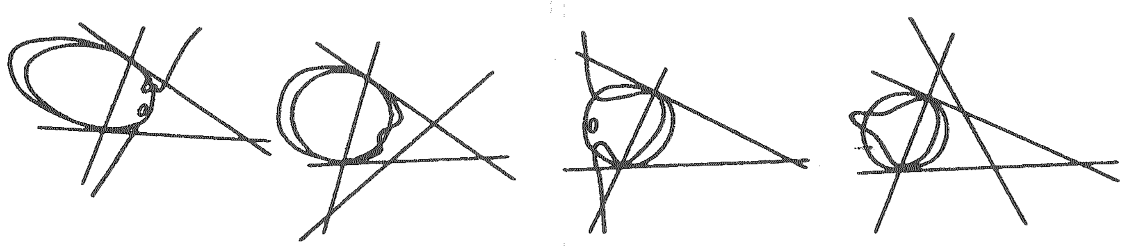


FIGURE 69

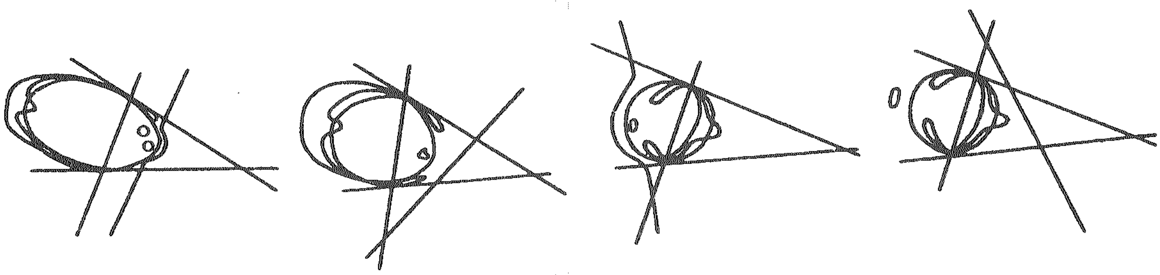


FIGURE 70

$\langle J \amalg 1\langle 1\langle 1\rangle\rangle\rangle$ can be realized as well, but it can also be realized using a split curve.

Gudkov's construction [11], which we alluded to in §1.12, gives the schemes $\langle J \amalg \alpha \amalg 1\langle \beta\rangle\rangle$ with $\alpha \leq 9$, $\beta \leq 5$.

These are all of the schemes which have been realized by means of small perturbations of curves which split into a transverse union of nonsingular curves of lower degree. The theory of complex orientations has made it possible to determine some limitations on this method. In particular, Zvonilov and Fidler showed that the scheme $\langle J \amalg 4 \amalg 1\langle 10\rangle\rangle$ cannot be realized in this way.

The next construction gives all of the missing schemes. The construction described here immediately gives $\langle J \amalg \alpha \amalg 1\langle \beta\rangle\rangle$ with $6 \leq \alpha + \beta \leq 14$, $\alpha \geq 1$, $\beta \geq 2$; small modifications then enable one to obtain many other real schemes (including the ones we have already obtained).

5.2.B. LEMMA. *There exist four curves of degree* 7 *which each have two type* J_{10}^- *singular points and are situated as shown in Figure* 67.

PROOF. We use Hilbert's method, adapted to the construction of singular curves. Using small perturbations of the various unions of the conic $x_2x_0-x_1^2=0$ and a line shown in Figure 68, we construct four nonsingular curves of degree 3 which are situated relative to the coordinate axes and the conic $x_2x_0-x_1^2=0$ as shown in Figure 69. We perturb the unions of these degree 3 curves with the conic $x_2x_0-x_1^2=0$ in such a way as to obtain the curves in Figure 70.

The unions of the resulting degree 5 curves with the conic $x_2x_0-x_1^2=0$ can obviously be perturbed in such a way as to obtain the required curves. ●

We now subject each of the curves in Lemma 5.2.B to perturbations which dissipate the singular points. We do this using the quasihomogeneous dissipations in Theorem 4.2.B. Even just the dissipations on the left in Figure 46 already give all of the real schemes $\langle J \amalg \alpha \amalg 1\langle\beta\rangle\rangle$ with $6 \le \alpha+\beta \le 14$, $\alpha \ge 1$, $\beta \ge 2$. ●

Another realization of almost real schemes of nonsingular curves of degree 7 is given by the following theorem of Korchagin [40].

5.2.C. *Every nonsingular curve of degree* 7 *except for a curve with scheme* $\langle J\amalg 1\langle 13\rangle\rangle$ *is isotopic to a curve resulting from a small perturbation of the union of three ellipses which are tangent to one another at two points and the line through these points, as shown in Figure* 71.

To prove this it suffices, by Theorem 3.8.A and Remark 3.9, to perturb this union by means of the dissipations in Theorem 4.4.A.

5.3. Curves of degree 8. The isotopy classification of nonsingular real projective algebraic plane curves of degree 8 has not yet been completed, although it is reasonable to think that it will be completed within the next few years. In any case, the last ten years have seen much progress, and no diminishing of the intensity of work on the subject.

I will list the prohibitions currently known on the real schemes of nonsingular curves of degree 8.

5.3.A. COROLLARY OF BÉZOUT'S THEOREM. *The real scheme of a nonsingular curve of degree* 8 *has the form* $\langle\alpha\rangle$, *or* $\langle\alpha\amalg 1\langle\beta\rangle\rangle$, *or* $\langle\alpha\amalg 1\langle\beta\rangle\amalg 1\langle\gamma\rangle\rangle$, *or* $\langle\alpha\amalg 1\langle\beta\rangle\amalg 1\langle\gamma\rangle\amalg 1\langle\delta\rangle\rangle$, *or* $\langle 4\langle 1\rangle\rangle$, *or* $\langle\alpha\amalg 1\langle\beta\rangle\amalg 1\langle\gamma\rangle\rangle$, *or* $\langle 1\langle 1\langle 1\langle 1\rangle\rangle\rangle\rangle$.

5.3.B. HARNACK'S INEQUALITY: $p+n \le 22$.

5.3.C. EXTREMAL CONGRUENCES FOR HARNACK'S INEQUALITY. *If* $p+n=22$, *then* $p-n \equiv 0 \mod 8$, *and hence* $p \equiv n \equiv 3 \mod 4$. *If* $p+n=21$, then $p-n \equiv \pm 1 \mod 8$.

5.3.D. COROLLARY OF PETROVSKIĬ'S INEQUALITY. *There is no singular curve of degree* 8 *with real scheme* $\langle 20\rangle$.

5.3.E. (see [7]). *If* $\langle\alpha\amalg 1\langle\beta\rangle\amalg 1\langle\gamma\rangle\amalg 1\langle\delta\rangle\rangle$ *is the real scheme of an* M*-curve of degree* 8 *with* β, γ *and* δ *nonzero, then* β, γ *and* δ *are odd.*

5.3.F. (see [7]). *If* $\langle\alpha\amalg 1\langle\beta\rangle\amalg 1\langle\gamma\rangle\amalg 1\langle\delta\rangle\rangle$ *is the real scheme of an* $(M-2)$*-curve of degree* 8 *with* β, γ *and* δ *nonzero and with* $\beta+\gamma+\delta \equiv 0 \mod 4$, *then two of the number* β, γ, δ *are odd and one is even.*

5.3.G. (see [8], [43]). *If* $\langle 1\langle\alpha\rangle\amalg 1\langle\beta\rangle\amalg 1\langle\gamma\rangle\rangle$ *is the real scheme of an* M*-curve of degree* 8, *then* (α, β, γ) *cannot be any of the following seven triples*: $(1, 3, 15)$, $(1, 5, 11)$, $(1, 9, 9)$, $(3, 3, 13)$, $(3, 5, 11)$, $(3, 7, 9)$, $(5, 5, 9)$. *There is no curve of degree* 8 *with real scheme* $\langle 4\amalg 1\langle 3\rangle\amalg 1\langle 3\rangle\amalg 1\langle 9\rangle\rangle$.

The last three restrictions were proved using Bézout's theorem; thus, they might not come from topology.

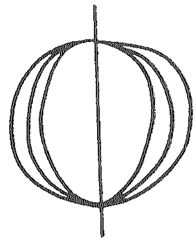


FIGURE 71

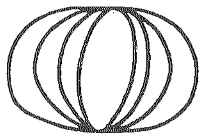


Figure 72

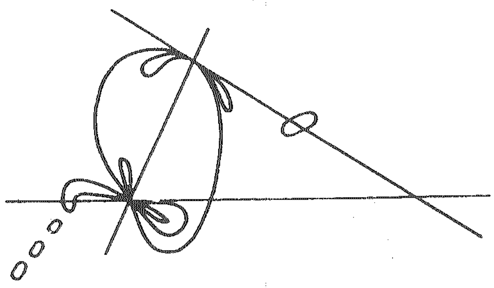


FIGURE 73

Very recently I received a manuscript from Shustin in which he proves two more series of prohibitions of the same sort.

5.3.H. *There are no M-curves of degree* 8 *with scheme* $1\langle\alpha \amalg 1\langle 20-\alpha\rangle\rangle$, *where* $\alpha = 2, 6, 10, 14$ *or* 18.

The second series contains 50 $(M-1)$-curves of degree 8.

The above prohibitions are satisfied by 91 real schemes with 22 ovals and 193 schemes with 21 ovals. Of them 78 schemes with 22 ovals and 171 schemes with 21 ovals have been realized by nonsingular curves of degree 8. I do not have the very latest information on $(M-2)$-schemes. As of a year ago, 337 of the 409 $(M-2)$-schemes that are not prohibited had been realized (largely due to the work of Polotovskiĭ, see his survey [43]), and 332 of the 367 permitted $(M-3)$-schemes had been obtained.

Clearly, it would be unwieldy to examine here all of the schemes realized by nonsingular curves of degree 8. So I shall limit myself to a few examples.

A very large number of schemes are realized by our means of small perturbations of the curve in Figure 72, which is a union of four ellipses having second order tangency at two points. This curve has two type X_{21} singularities. If we dissipate them using all of the known methods (see 4.7.A), we can realize 47 real schemes with 22 ovals, 117 schemes with 21 ovals, 319 schemes with 20 ovals, and 213 schemes with 19 ovals (see Polotovskiĭ [43]). One might think that almost all of the real schemes of degree 8 can be realized by small perturbations of this curve—indeed, in the case of curves of lower degree the analogous curve gives almost all of them (compare with 5.2.C and 5.1.A). However, we shall see that this is far from true.

5.3.I. LEMMA. *There exists a curve of degree* 8 *which has a nondegenerate* 5-*fold singular point at* $(1:0:0)$ *and a type* J_{10}^- *singular point at* $(0:0:1)$ *and which is situated as shown in Figure* 73.

PROOF. We construct a conic C and a line L which are situated relative to one another and relative to the axes in the way shown in Figure 74.

We perturb the union $C \cup L$ in such a way as to obtain a nonsingular curve

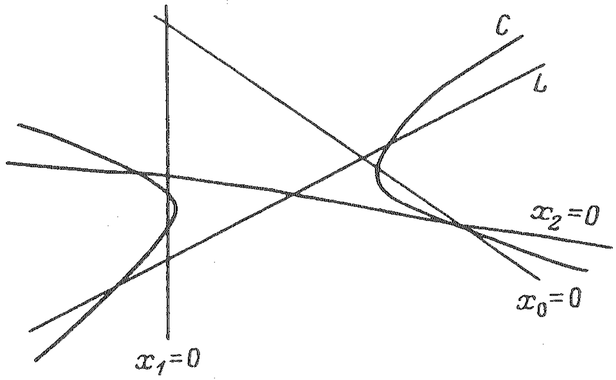


FIGURE 74

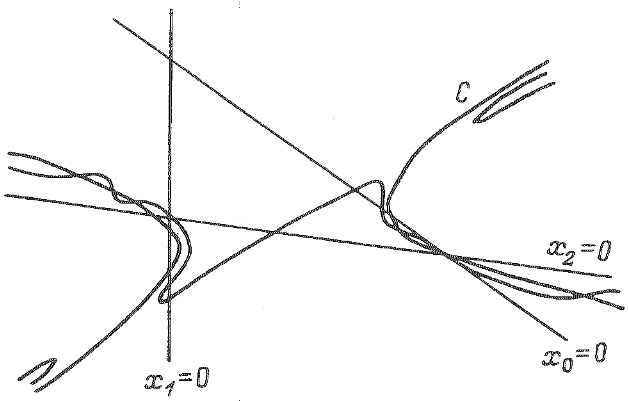


FIGURE 75

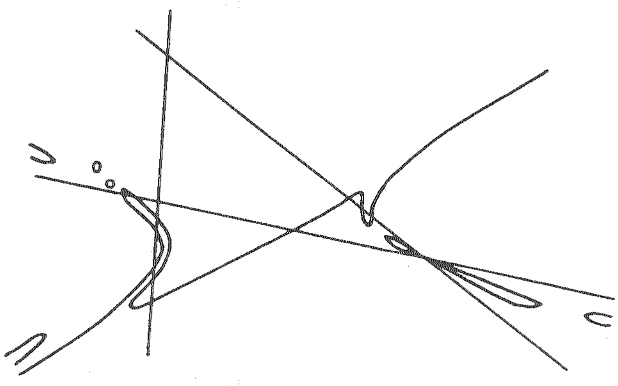


FIGURE 76

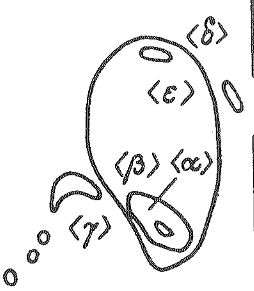
of degree 3 which, like C, passes through the point $(0:1:0)$ and which is situated relative to C and the axes as shown in Figure 75.

We perturb the union of this curve and the conic C in such a way as to obtain a curve of degree 5 which has a nondegenerate double point at $(0:1:0)$ and is situated relative to the coordinate axes as shown in Figure 76.

We apply the transformation hy^{-1} to the resulting curve. Obviously, the result will be the required curve. ●

If we perturb the curve in Lemma 5.3.4 so that dissipations of its germs at the points $(1:0:0)$ and $(0:0:1)$ give six and four new small ovals, respectively (see Theorems 4.3.D and 4.2.B), we obtain M-curves of degree 8 with the schemes in Figure 77.

The real scheme $\langle 1 1 \amalg 1 \langle 2 \amalg 1 \langle 7 \rangle\rangle\rangle$, for example, can be realized in this way



$\delta =$	4	0
$\varepsilon =$	0	4

$\alpha =$	6	0	2	0
$\beta =$	0	0	4	4
$\gamma =$	0	6	0	2

α, β, γ \ δ, ε	4, 0	0, 4
6, 0, 0	$\langle 11 \amalg 1\langle 2 \amalg 1\langle 7\rangle\rangle\rangle$	$\langle 7 \amalg 1\langle 6 \amalg 1\langle 7\rangle\rangle\rangle$
0, 0, 6	$\langle 17 \amalg 1\langle 2 \amalg 1\langle 1\rangle\rangle\rangle$	$\langle 13 \amalg 1\langle 6 \amalg 1\langle 1\rangle\rangle\rangle$
2, 4, 0	$\langle 11 \amalg 1\langle 6 \amalg 1\langle 3\rangle\rangle\rangle$	$\langle 7 \amalg 1\langle 10 \amalg 1\langle 3\rangle\rangle\rangle$
0, 4, 2	$\langle 13 \amalg 1\langle 6 \amalg 1\langle 1\rangle\rangle\rangle$	$\langle 9 \amalg 1\langle 10 \amalg 1\langle 1\rangle\rangle\rangle$

FIGURE 77

and cannot be realized by a small perturbation of the curve in Figure 72.

There is another construction of nonsingular curves of degree 8—involving first constructing a curve of degree 8 with N_{16} and J_{10}^{-} singularities—which leads to interesting curves and can be generalized. It will be described (in a special case) in the next subsection.

5.4. Refinement of Ragsdale's conjecture. Recall (§1.11) that in 1906, based on an analysis of the constructions of Harnack and Hilbert, Ragsdale made a conjecture to the effect that for any nonsingular curve of even degree

$$p \leq (3m^2 - 6m + 8)/8 \quad \text{and} \quad n \leq (3m^2 - 6m)/8.$$

Curiously, in 1938 Petrovskiĭ [42] (independently) proposed the weaker conjecture:

$$p \leq (3m^2 - 6m + 8)/8 \quad \text{and} \quad n \leq (3m^2 - 6m + 8)/8.$$

In the note [6] I announced that there are counterexamples to Ragsdale's second inequality for any $m \geq 8$ with $m \equiv 0 \mod 4$. In this subsection I will describe how these counterexamples are constructed. They are curves with real schemes $\langle (m^2 - 6m)/8 \amalg 1\langle (3m^2 - 6m + 8)/8\rangle\rangle$ of degree $m \geq 8$, $m \equiv 0 \mod 4$. We note that they establish that the strengthened Petrovskiĭ inequality 2.3.I is best possible.

The question of whether the Petrovskiĭ conjecture (or the refined Ragsdale conjecture) is correct, remains open. The question can be generalized as follows (see [6]): Is it true that, if X is the set of fixed points of an antiholomorphic involution of a simply connected nonsingular compact complex surface $\mathscr{X}$, then

$$\dim H_1(X; \mathbf{Z}_2) \leq h^{1,1}(\mathscr{X})?$$

We begin the construction of our counterexamples by constructing some nonsingular curves.

5.4.A. *For every* $k > 1$ *there exists a curve of degree* $4k$ *of the form in Figure 78, where* $\alpha = (k^2 - k - 2)/2$, $\beta = (k^2 - k + 2)/2$, *and* $\gamma = k^2 - 1$, *which has a nondegenerate* $(2k+1)$*-fold singular point at* $(1:0:0)$ *and a point of second order tangency of* $(2k-1)$ *nonsingular branches at* $(0:0:1)$.

PROOF. We construct a nonsingular cubic curve C_3 which is situated relative to the lines L and L_1 as shown in Figure 79. We perturb the union $C \cup L$ in such a way as to obtain a nonsingular curve C_4 of degree 4 which is situated relative to L and L_1 as shown in Figure 80. After that we use a series of Harnack constructions, taking the line L as the generating curve and as bases

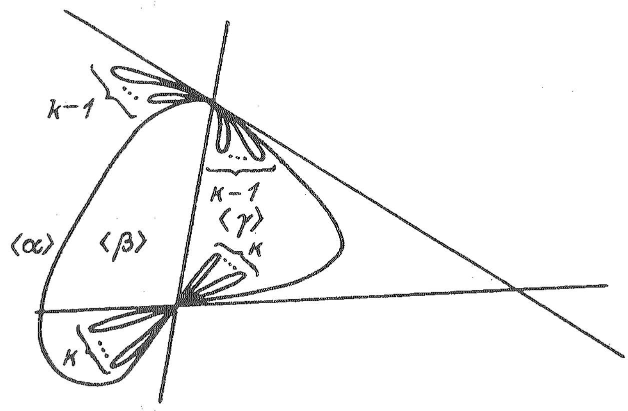


FIGURE 78

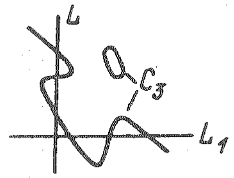


FIGURE 79

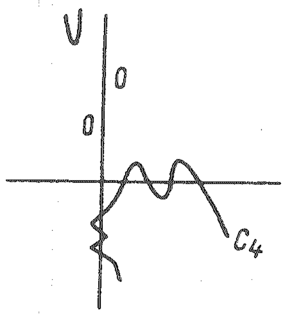


FIGURE 80

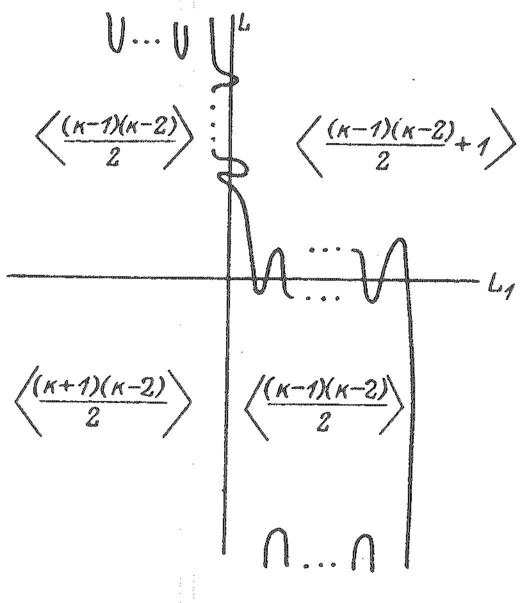


FIGURE 81

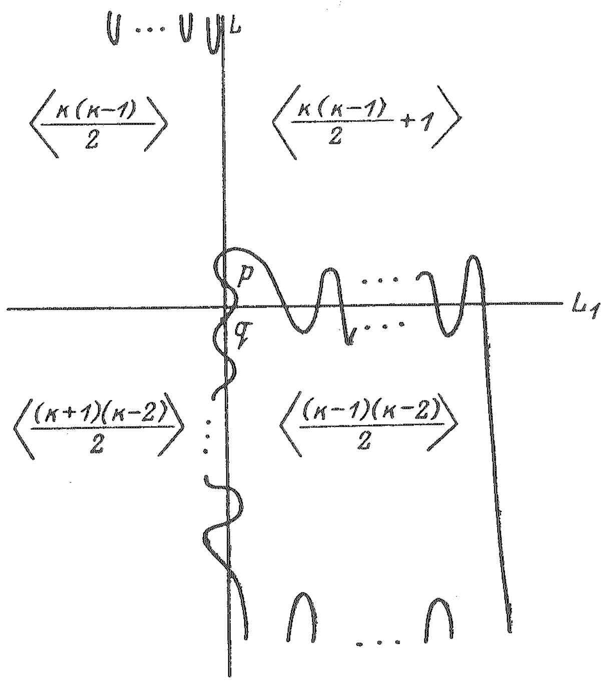


FIGURE 82

taking intervals above and the points of intersection with L_1 below (compare with §1.12). We obtain the curve C_{2k-1} in Figure 81.

We now perturb the union $C_{2k-1} \cup L$ in such a way as to obtain a nonsingular curve C_{2k} of degree $2k$ which is situated relative to L and L_1 as shown in Figure 82. It is important that the nonempty oval of C_{2k} intersect L at $2k$ points, and that the second and third points from the top—we shall denote them by p and q—bound an interval which contains $L \cap L_1$ and is small compared to the other intervals on L cut out by the curve C_{2k}.

We perturb the union $C_{2k} \cup L$, and, by continuing the perturbation, we obtain a contraction of the oval obtained from the lune with vertices p and q to an isolated double point. Because these points are so close to one another, this modification occurs first (before any others that may occur). We have obtained a curve C_{2k+1} of degree $2k+1$ with a single nondegenerate isolated double point, where the curve is situated as shown in Figure 83 relative to the line L and a line L' which is near L_1 and passes through the singular point.

We now perform a projective transformation which takes L to the axis $x_1 = 0$, L' to the axis $x_0 = 0$, and the singular point on C_{2k+1} to the point $(0 : 1 : 0)$. The broken line in Figure 83 shows the preimage of the axis $x_2 = 0$ under this transformation (its location is actually not essential in what follows). Applying the transformation hy^{-1} to the image of our curve, we obtain the required curve. ●

If we perturb the curve in Lemma 5.4.A in such a way that the dissipations of its singularities at $(0 : 0 : 1)$ and $(1 : 0 : 0)$ are as in 4.8.A and 4.3.E, we obtain a nonsingular curve of degree $m = 4k$ with the required real scheme $\langle (m^2 - 6m)/8 \amalg 1\langle (3m^2 - 6m + 1)/8 \rangle \rangle$.

The rest of the survey is expected to be published in the next issue of this journal. The third and final chapter will be devoted to the further development of techniques for constructing real algebraic varieties with controlled topology. The central construction which is studied and applied in Chapter 3 puts to-

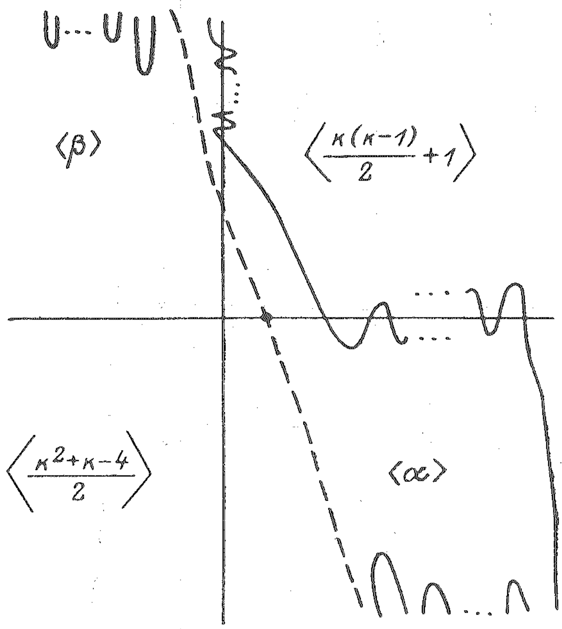


FIGURE 83

gether several algebraic hypersurfaces into a new algebraic hypersurface, which from a topological point of view is obtained by gluing together the original hypersurfaces. This construction arose as a result of analyzing the perturbations of curves with semi-quasihomogeneous singularities which were described and applied in Chapter 2 above. The construction in Chapter 2 of a quasihomogeneous dissipation of singularities is a special case of the gluing construction. The gluing construction can be explained on the same elementary level as the construction of quasihomogeneous dissipations (such an explanation is sketched in §6 of the first section—Chapter 3). However, a much clearer picture emerges if we enlarge our repertory of objects—by including the hypersurfaces of toric manifolds. The basic theory of toric manifolds that is needed will be explained in §7.

Roughly speaking, the transition from quasihomogeneous dissipations to the gluing construction equates the role of the singular curve which is perturbed and that of the curves which determine quasihomogeneous dissipations of its semi-quasihomogeneous singularities. Here a new object appears which is very useful from a technical standpoint—the map of a polynomial. This is a copy of the curve defined by the polynomial, placed in a natural way in a union of four copies of its Newton polygon. In the construction of the map, the singularities governed by the Newton polygon undergo a resolution. From a purely algorithmic point of view, the gluing construction consists in putting together maps of polynomials to form the map of a new polynomial. Of course, the maps that are put together must satisfy some compatibility conditions—analogues of the condition that the tangent directions to the branches of a curve at a nondegenerate r-fold point must coincide with the asymptotic directions of the affine curve which gives a quasihomogeneous dissipation of the point (see §3.5 above). The usefulness of this formulation of constructions of curves can be felt even in the case of quasihomogeneous dissipations. The reader can see this by repeating the above constructions in the language of maps and gluings, as well as by learning the new constructions. In addition, at the end of the article we shall examine constructions of curves with controlled complex scheme, and also some new constructions which do not fit into the framework of the basic methods.

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