Алгебра и анализ
Том 4 (1992), вып. 3

St. Petersburg Math. J.
Vol. 4 (1993), No. 3

AN INEQUALITY FOR THE NUMBER OF NONEMPTY OVALS OF A CURVE OF ODD DEGREE

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ABSTRACT. The authors prove restrictions on the topology of real nonsingular plane projective algebraic curves of odd degree, formulated in [1], inequality (11), and [2], Theorems 3.10 and 3.11. In the proof an essential role is played by results about homology of branched coverings, which are of independent interest: the authors indicate a homological construction that connects the homology class of a cyclic branched covering, constructed from a membrane, which is stretched on the base, and the homology class of the boundary of this membrane in a submanifold of the branching.

INTRODUCTION

Let A be a real nonsingular plane projective algebraic curve of degree m, and let $\mathbb{R}A \subset \mathbb{R}P^2$ be the set of its real points. We recall that its components are homeomorphic to a circle and that if m is odd, then there is exactly one one-sided component. The two-sided components are called *ovals*. The total number of ovals of a curve is denoted by l.

Each oval bounds from the outside a component of the set $\mathbb{R}P^2 \setminus \mathbb{R}A$. Ovals that bound from the outside a component with positive (negative, zero) Euler characteristic are called *elliptic* (*hyperbolic, parabolic*); the number of elliptic ovals is denoted by l^+, the number of hyperbolic ovals by l^-, and the number of parabolic ovals by l^0. Hyperbolic and parabolic ovals envelop domains in $\mathbb{R}P^2$ that contain other ovals, and so they are called *nonempty*.

Theorem 1 (A bound on the number of hyperbolic ovals). *For any odd* m

(1) $$l^- \leq (m-3)^2/4.$$

If the curve has ovals, but not one that envelops all the other ovals, we have an inequality stronger by 1:

(2) $$l^- \leq (m-5)(m-1)/4.$$

Restatement of Theorem 1. *The number of components of the complement of a curve of odd degree* m *that have negative Euler characteristic does not exceed* $(m-3)^2/4$.

Theorem 2 (A bound on the number of nonempty ovals). *For any odd* m

(3) $$l^- + l^0 \leq \frac{(m-3)^2}{4} + \frac{m^2 - h^2}{4h^2},$$

where h *is the highest prime power that divides* m.

1991 *Mathematics Subject Classification*. Primary 14H45, 14N10; Secondary 14H30.
Key words and phrases. Real algebraic curve, cyclic branched covering, real membranes and homology classes of a covering, Smith sequence.

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1061-0022/93 $1.00 + $.25 per page

The inequality (1) is also satisfied for even $m \neq 4$. We also have the stronger inequality

$$l^- + l^0 \leq (m-2)(m-4)/4,$$

which is a consequence of the inequalities of Arnol'd [3]; for the details see [1].

Inequality (3) also admits a strengthening for certain odd m: according to an inequality of Zvonilov (see inequality (9) in [1], and [4]), for any odd m

$$l^- + l^0 \leq (m-1)(m-3)/4. \tag{4}$$

In many cases inequality (4) is weaker than (3), but it may also be stronger. The smallest value of m for which (4) is stronger is 693.

The inequalities (1), (2), and (3) are sharp for $m = 3$ and $m = 5$. For $m \geq 7$ they are far from sharp. Elementary consequences of Bézout's theorem (see [2], inequalities 3.19 and 3.20) and simple constructions show that the maximum value of $l^- + l^0$ for $m = 7$ and $m = 9$ is 2 and 4 respectively, whereas the right-hand side of (3) is equal to 4 and 9 respectively.

Following Klein, we say that a curve A belongs to type I if $\mathbb{R}A$ splits the set $\mathbb{C}A$ of complex points of A, and it belongs to type II if $\mathbb{R}A$ does not split $\mathbb{C}A$. In the first case the natural orientations of the two halves into which $\mathbb{R}A$ splits $\mathbb{C}A$ determine on $\mathbb{R}A$, as on the common boundary of these halves, two opposite orientations. Following Rokhlin [1], we call them *complex.*

We denote the ovals of the curve A by $A_1, \dots, A_l$, its one-sided component by A_0, the components of $\mathbb{R}P^2 \setminus \mathbb{R}A$ bounded from the outside by the ovals $A_1, \dots, A_l$ by $B_1, \dots, B_l$, and the component of $\mathbb{R}P^2 \setminus \mathbb{R}A$ adjoining A_0 by B_0. Let p be the prime number whose power is h. We denote by b_j the class determined in $H_2(\mathbb{R}P^2, \mathbb{R}A; \mathbb{Z}_p)$ by the set B_j endowed with a certain orientation. Clearly the classes b_j form a basis of the space $H_2(\mathbb{R}P^2, \mathbb{R}A; \mathbb{Z}_p)$.

Theorem 3 (Extremal property of inequality (3)). *Suppose equality holds in* (3). *Then the curve A belongs to type* I *and there are numbers $x_0, \dots, x_l \in \mathbb{Z}_p$ such that the image of the class $x = \sum x_j b_j$ under the boundary homomorphism $\partial: H_2(\mathbb{R}P^2, \mathbb{R}A; \mathbb{Z}_p) \to H_1(\mathbb{R}A; \mathbb{Z}_p)$ is the fundamental class $[\mathbb{R}A]$ of the curve $\mathbb{R}A$, endowed with the complex orientation, and $\chi(B_{ij}) = 0$ for all j with $x_j \neq 0$.*

For curves of degree 5 any two of Theorems 1, 2, and 3 together with Harnack's inequality $l \leq 6$ (see [1], for example) form a complete system of restrictions on the topology of the curve. The proof of Theorems 1–3 given below relies only on fundamental topological properties of algebraic curves. It can also be applied to flexible curves (that is, to topological objects that imitate nonsingular plane projective algebraic curves; see [2], §1), and since Harnack's inequality also holds for flexible curves, it follows that the isotopic classification is the same for algebraic and flexible curves of degree 5 (compare [2]).

The general scheme of our proof of Theorem 1 is the same as that of Arnol'd [3] in his proof of the similar inequalities mentioned above for curves of even order. The main difference is that instead of a double covering we work with an m-sheeted or h-sheeted covering of the complex projective plane $\mathbb{C}P^2$, branched over $\mathbb{C}A$. As in Arnol'd's proof, from the components of the set $\mathbb{R}P^2 \setminus \mathbb{R}A$ we construct the homology classes of the branched covering that take up a definite position with respect to the intersection form, and we observe that the structure of the intersection form (which depends only on the degree of the curve) imposes a restriction on the number of such classes.

Theorems 1–3 are proved in §1. The proofs rely essentially on the results of §2, which is devoted to the problem of the linear independence of the homology classes of a cyclic branched covering, which can be constructed from membranes spanning the branching surface at the base. We have discussed these questions in a separate section, having in mind their independent value. In particular, the results of §2 form a basis for a further strengthening of Theorems 1–3, based on various geometrical constructions that give additional membranes. (We observe, however, that constructions of this type known at present rely in one way or another on Bézout's theorem, and therefore cannot be applied to flexible curves.)

§1. Proof of Theorems 1, 2, and 3

For $l = 0$ and $m = 1$ Theorem 1 is obvious, and inequality (3) holds and is not an equality. Therefore in the proof of Theorems 1 and 2 given below, without loss of generality we put $l > 0$ and $m > 1$, and in the proof of Theorem 3 we put $m > 1$.

Let q denote a divisor of m. We use the objects that are defined in §§1.1 and 1.2 and depend on q only when $q = m$ (in §1.4) and $q = h$ (in the rest of the text), so in order not to complicate the notation of these objects we omit the index q in them.

1.1. **Branched coverings of the plane $\mathbb{C}P^2$ with branching over $\mathbb{C}A$.** Suppose that the curve A is given by the equation $f(x_0, x_1, x_2) = 0$. Then the equation $f(x_0, x_1, x_2) = x_3^m$ defines a nonsingular surface Z of degree m in three-dimensional projective space. The formula $(x_0\colon x_1\colon x_2\colon x_3) \mapsto (x_0\colon x_1\colon x_2)$ defines an m-sheeted cyclic covering $\mathbb{C}Z \to \mathbb{C}P^2$, branched over $\mathbb{C}A$. Its automorphism group consists of transformations $(x_0\colon x_1\colon x_2\colon x_3) \mapsto (x_0\colon x_1\colon x_2\colon \exp(2\pi k/m)x_3)$ and contains a subgroup of order m/q consisting of transformations $(x_0\colon x_1\colon x_2\colon x_3) \mapsto (x_0\colon x_1\colon x_2\colon \exp(2\pi kq/m)x_3)$. We denote the space of orbits of this subgroup by Y. The obvious projection $\nu\colon Y \to \mathbb{C}P^2$ is a cyclic q-sheeted covering, branched over $\mathbb{C}A$.

The projection ν determines a diffeomorphism of the set $\nu^{-1}(\mathbb{C}A)$ onto $\mathbb{C}A$, and in what follows we identify $\nu^{-1}(\mathbb{C}A)$ with $\mathbb{C}A$. The automorphism group of the covering $\nu\colon Y \to \mathbb{C}P^2$ (isomorphic to $\mathbb{Z}_q$) acts on the fibers of the normal bundle of the surface $\mathbb{C}A \subset Y$ like the group of rotations of the plane through angles that are multiples of $2\pi/q$. We denote by τ either of the two automorphisms that act on the fibers as a rotation through $2\pi/q$.

In $H_2(Y;\mathbb{C})$ we consider the subspace $M = \ker(\tau_* - \xi^{-1}\,\mathrm{id})$, where id is the identity homomorphism and $\xi = \exp(\pi(q-1)\sqrt{-1}/q)$. From Rokhlin's calculations ([5], §§5.4 and 5.6) it follows that $\dim M = 1 + (m-1)(m-2)$, and the signature $\operatorname{sign} Q$ of the restriction Q to M of the Hermitian intersection form of the manifold Y is equal to $1 - m^2(q^2-1)/2q^2$.

1.2. **The classes $\beta_0, \dots, \beta_l$.** Let conj denote one of the antiholomorphic involutions of the manifold Y that cover the involution of the complex conjugate of $\mathbb{C}P^2$. Let $C_0, \dots, C_l$ be the connected components of the set $\mathrm{fix}(\mathrm{conj}) \setminus \mathbb{R}A$ that are taken by the projection ν into $B_0, \dots, B_l$ respectively. We recall that before stating Theorem 3, having defined the classes b_i, we endowed the surfaces B_i with certain orientations. We endow the surfaces C_i with the induced orientations.

Let γ_{ij} denote the class oriented in $H_2(Y;\mathbb{C})$ by the surface $\mathrm{Cl}(C_i \cup \tau^j C_i)$, oriented in accordance with the orientation of the component C_i. For $i = 0, \dots, l$ we put $\beta_i = \sum_{j=1}^{h-1} \xi^j \gamma_{ij}$. Simple calculation shows that $\beta_i \in M$.

1.3. **The intersection numbers of the classes $\beta_0, \dots, \beta_l$.** Let us first calculate $\gamma_{ij} \circ \gamma_{rs}$. Let u be a tangent vector field on $\mathrm{Cl}\,C_i$ with finitely many zeros, whose restriction to the frontier $\mathrm{Fr}\,C_i$ of the set C_i does not have zeros and is tangent to

$\operatorname{Fr} C_i$. The field $\sqrt{-1}u$ is obviously normal to C_i and tangent to $\mathbb{C}A$ on $\operatorname{Fr} C_i$. Hence the differential of the fransformation τ^j takes the field $\sqrt{-1}u$ into the normal vector field v on $\operatorname{Cl} \tau^j C_i$ which coincides with $\sqrt{-1}u$ on $\operatorname{Fr} C_i$. As a result of the standard calculation of the intersection number by means of a small shift of the set $\operatorname{Cl}(C_i \cup \tau^j C_i)$ along the field $\sqrt{-1}u \cup v$ we deduce that

(i) $\gamma_{ij} \circ \gamma_{rs} = 0$ when $r \neq i$,
(ii) $\gamma_{ij} \circ \gamma_{is} = -\chi(C_i)$ when $j \neq s$,
(iii) $\gamma_{ij}^2 = -2\chi(C_i)$

(compare [3]). We observe that $\chi(C_i)$ in (ii) and (iii) can obviously be replaced by $\chi(B_i)$.

From these equalities it is easy to deduce that $\beta_i \circ \beta_r = 0$ when $i \neq r$, and that

$$\beta_i^2 = -\chi(B_i)q. \tag{5}$$

1.4. **Proof of Theorem 1.** In this subsection we put $q = m$, so the subspace M that contains the classes β_i lies in $H_2(\mathbb{C}Z; \mathbb{C})$. We construct the subset of the collection $\beta_0, \dots, \beta_l$ that generates the space on which the intersection form is nonnegative. Since the classes β_i are pairwise orthogonal, in such a set we can include classes β_i with $\beta_i^2 > 0$, and these classes are linearly independent. By (5), this condition selects l^- classes from the classes $\beta_1, \dots, \beta_l$, and if the curve has ovals, but does not have one that envelopes all the remaining ovals, then it also selects the class β_0. The dimension of the subspace generated by these classes is equal to the number of them. It does not exceed $(\dim M + \operatorname{sgn} Q)/2$, since $\beta_i \in M$. Comparing this with the information about M from §1.1, we obtain Theorem 1.

1.5. **The rank of the system $\beta_0, \dots, \beta_l$.** Let d_i denote the homology class in $H_1(\mathbb{C}A; \mathbb{Z}_p)$ realizable by the boundary of the component B_i. Clearly, d_i is the image of b_i under the composition

$$H_2(\mathbb{R}P^2, \mathbb{R}A; \mathbb{Z}_p) \to H_1(\mathbb{R}A; \mathbb{Z}_p) \to H_1(\mathbb{C}A; \mathbb{Z}_p)$$

of the boundary homomorphism and the inclusion homomorphism.

Let $q = h$ everywhere in what follows.

The next two lemmas are proved in §2.

1.5.A. **Lemma.** *The rank of the system $\beta_0, \dots, \beta_l \in H_2(Y; \mathbb{C})$ is not less than the rank of the system $d_0, \dots, d_l \in H_1(\mathbb{C}A; \mathbb{Z}_p)$.*

Let μ denote the ring homomorphism $\mathbb{Z}[\xi] \to \mathbb{Z}_p$ that takes ξ into 1. (The existence of such a homomorphism follows from the fact that the sum of the coefficients of the minimal integer polynomial of ξ is equal to p. We observe that it is essential here that q is a prime power. In fact, if q is not a prime power, then the sum of the coefficients of the minimal polynomial of any primitive qth root ζ of unity is equal to 1, and so there are no nontrivial ring homomorphisms of $\mathbb{Z}[\zeta]$ into $\mathbb{Z}_p$.)

1.5.B. **Lemma.** *If $\lambda_0\beta_0 + \dots + \lambda_l\beta_l = 0$ is a nontrivial linear relation whose coefficients belong to $\mathbb{Z}[\xi]$ and are coprime in $\mathbb{Z}[\xi]$, then $\mu(\lambda_0)d_0 + \dots + \mu(\lambda_l)d_l = 0$.*

In the relation $\mu(\lambda_0)d_0 + \dots + \mu(\lambda_l)d_l = 0$ of Lemma 1.5.B not all the $\mu(\lambda_i)$ are zero. In fact, it is easy to verify that $\ker \mu$ is the ideal generated by $1 - \xi$. Therefore if all the λ_i were to belong to $\ker \mu$, they would not be coprime.

1.5.C. **Corollary.** *The rank of the system $\beta_0, \dots, \beta_l$ is not less than l, and if it is equal to l (that is, if the classes $\beta_0, \dots, \beta_l$ are linearly independent), then the curve A belongs to type* I.

Proof. Obviously the classes $b_0, \ldots, b_l$ form a basis of $H_2(\mathbb{R}P^2; \mathbb{R}A; \mathbb{Z}_p)$. The boundary homomorphism takes it into a basis of $H_1(\mathbb{R}A; \mathbb{Z}_p)$. As we know, the inclusion homomorphism $H_1(\mathbb{R}A; \mathbb{Z}_p) \to H_1(\mathbb{C}A; \mathbb{Z}_p)$ is injective if the curve belongs to type II, and has a one-dimensional kernel if the curve belongs to type I. Therefore the rank of the system of classes $d_0, \ldots, d_l$ (which are the images of the classes $b_0, \ldots, b_l$ under a composition of these homomorphisms) is equal to l in the case of a curve of type I and to $l+1$ in the case of a curve of type II. Lemmas 1.5.A and 1.5.B enable us to carry over this information to the statement to be proved.

1.6. **Proof of Theorem 2.** Consider a maximal subset of the collection $\beta_0, \ldots, \beta_l$ that generates a space whose intersection form is nonnegative. Since the classes β_i are pairwise orthogonal, it consists of β_i with $\beta_i^2 \geq 0$. By (5), $l^- + l^0$ of the classes $\beta_1, \ldots, \beta_l$ occur in it, and also the class β_0 (the latter is ensured by the assumption that $l > 0$, from which it follows that $\chi(B_0) \leq 0$). Let d denote the dimension of the subspace generated by these classes. It does not exceed $(\dim M + \operatorname{sgn} Q)/2$, since $\beta_i \in M$, as we mentioned in §1.2, and according to [5], §4.2, the form Q is nondegenerate. On the other hand, by 1.5.B the number of classes is not greater than $d+1$. Combining these two inequalities and the information about M from §1.1, we obtain

$$l^0 + l^- + 1 \leq (\dim M + \operatorname{sgn} Q)/2 + 1 = (m-3)^2/4 + (m^2 - h^2)/4h^2 + 1,$$

which is equivalent to the inequality to be proved.

1.7. **Proof of Theorem 3.** Suppose that equality is attained in (3). The arguments of the previous subsection show that in this case $\operatorname{rk}(\beta_0, \ldots, \beta_l) = \operatorname{rk}(d_0, \ldots, d_l) = l$ and the curve A belongs to type I. Let us endow $\mathbb{R}A$ with a complex orientation. Since $H_1(\mathbb{R}P^2; \mathbb{Z}_p) = H_2(\mathbb{R}P^2; \mathbb{Z}_p) = 0$, the boundary homomorphism $H_2(\mathbb{R}P^2, \mathbb{R}A; \mathbb{Z}_p) \to H_1(\mathbb{R}A; \mathbb{Z}_p)$ is an isomorphism. Therefore there is a class $x = \sum x_i b_i \in H_2(\mathbb{R}P^2, \mathbb{R}A; \mathbb{Z}_p)$ with $\partial x = [\mathbb{R}A]$.

Clearly, the classes $\beta_0, \ldots, \beta_l$ belong to the image of the homomorphism λ_*: $H_2(Y; \mathbb{Z}[\xi]) \to H_2(Y; \mathbb{C})$ induced by the inclusion $\lambda: \mathbb{Z}[\xi] \to \mathbb{C}$. Since $H_1(Y) = 0$, by virtue of duality and the universal coefficient formulas we have $H_3(Y) = 0$ and $\operatorname{Tors} H_2(Y) = 0$, and consequently $H_3(Y; \mathbb{C}/\mathbb{Z}[\xi]) = 0$. Therefore λ_* is a monomorphism. Consequently, from the linear dependence of the classes $\beta_0, \ldots, \beta_l$ in $H_2(Y; \mathbb{C})$ it follows that there is a nontrivial linear relation

(6) $$\lambda_0 \beta_0 + \cdots + \lambda_l \beta_l = 0$$

with the λ_i belonging to $\mathbb{Z}[\xi]$ and coprime in $\mathbb{Z}[\xi]$. By 1.5.B, $\mu(\lambda_0) d_0 + \cdots + \mu(\lambda_l) d_l = 0$, and by virtue of the definition of complex orientation $x_0, \ldots, x_l$ are proportional to $\mu(\lambda_0), \ldots, \mu(\lambda_l)$. Therefore if $x_i \neq 0$, then $\lambda_i \neq 0$. Finally, multiplying (6) by β_j and bearing in mind that $\beta_0, \ldots, \beta_l$ are pairwise orthogonal and $\beta_j^2 = -\chi(B_j)h$ (see §1.3), we deduce that $\chi(B_j) = 0$ for all j with $x_j \neq 0$.

§2. Homology classes of a branched covering and homology classes of a branching set

2.1. **Statement of the question.** Let $\nu: Y \to X$ be a cyclic h-sheeted covering of a smooth closed n-dimensional manifold X, branched over a smooth closed $(n-2)$-dimensional subset A of it. Let $\tau: Y \to Y$ be a generator of the automorphism group of ν.

A smooth compact submanifold B of X is called a *membrane on* A if $\partial B = A \cap B$ and along ∂B it does not touch A in the sense that the normal bundle of ∂B in B intersects the tangent bundle of A only at zero vectors. If ν is trivial over

B, we say that B *can be lifted to* Y. In this case $\nu^{-1}(B)$ consists of h copies $C, C_1, \ldots, C_{h-1}$ of B with $\tau^j(C) = C_j$, that are homeomorphically mapped by ν onto B.

Now suppose that B is oriented, can be lifted to Y, and has dimension k. Let γ_j denote the class defined in $H_k(Y; \mathbb{C})$ by the cycle $C \cup C_j$, oriented in accordance with C. Let ζ be an hth root of unity. We put $\beta^\zeta = \sum_{j=1}^{h-1} \zeta^j \gamma_j$. Let M^ζ denote the subspace $\ker(\tau_* - \zeta^{-1}\,\mathrm{id})$ of $H_k(Y; \mathbb{C})$. It is easy to verify that $\beta^\zeta \in M^\zeta$. This is the most natural construction that gives elements of the set M^ζ. We used it in the previous section to construct the classes $\beta_0, \ldots, \beta_l$.

Since the homology of the branching set A is more accessible, as a rule, than the homology of the covering Y, one would like to obtain as much information as possible about the class β^ζ from the homology class realized in A by the manifold ∂B, in particular, information about the linear independence of classes of the form β^ζ; compare §1.

In this section we indicate two ways of attaining this aim. Unfortunately, they can be applied only in the case when h is a prime power. We have been unable to eliminate this condition, which somewhat weakens the results (compare [1]). In both methods it turns out to be essential, though at first glance for different reasons. Possibly the deepest of these reasons is the presence of the homomorphism $\mu: \mathbb{Z}[\zeta] \to \mathbb{Z}_p$ (see §1.5). It enables us to go over to homology with coefficients in $\mathbb{Z}_p$.

2.2. **The homomorphism** v_r. Let us orient the manifold A. For this we observe that the transformation τ^r acts on a fiber of its normal bundle in Y as a rotation other than a rotation through an angle π. We orient this bundle so that τ^r acts as a rotation through an angle in the interval $(0, \pi)$ in the positive direction. We now orient A so that the intersection number for each fiber of its normal bundle is $+1$.

Let ρ denote the inverse Hopf homomorphism (also called the cutting homomorphism) $H_{k+1}(Y; \mathbb{Z}_p) \to H_{k-1}(A; \mathbb{Z}_p)$. We recall that ρ can be defined as a composition of Poincaré duality isomorphisms and a homology inclusion homomorphism:

$$H_{k+1}(Y; \mathbb{Z}_p) \xrightarrow{D^{-1}} H^{n-k-1}(Y; \mathbb{Z}_p) \xrightarrow{\mathrm{in}^*} H^{n-k-1}(A; \mathbb{Z}_p) \xrightarrow{D} H_{k-1}(A; \mathbb{Z}_p),$$

and that it relates the homology class realized by a smooth submanifold transversal to A to the homology class of the intersection of this submanifold with A.

Let H_k^r denote the kernel of the homomorphism $1 - \tau_*^r: H_k(Y; \mathbb{Z}_p) \to H_k(Y; \mathbb{Z}_p)$.

In this subsection, for any natural number r we define the homomorphism

$$v_r: H_k^r \to H_{k-1}(A; \mathbb{Z}_p)/\operatorname{im}\rho.$$

We first describe v_r in geometrical terms (cycles, their transversal intersections with submanifolds, and so on), and then, more formally, in the language of homology and cohomology.

Suppose that $\beta \in H_k^r$, and let x be a cycle representing the class β and in general position relative to A. (We do not go into a discussion of the exact meaning of the words "general position", since this is only an informal description. We only state that in the case when x is a fundamental cycle of a smooth submanifold of Y they signify the transversality of this submanifold to A.) Since $\beta - \tau_*^r\beta = 0$, there is a chain c such that $\partial c = x - \tau^r x$. Suppose that c is also in general position relative to A. Consider the intersection $c \circ [A]$ of the chain c with the fundamental cycle $[A] \in C_{n-2}(A; \mathbb{Z}_p)$ of the manifold A. Since $\partial A = \varnothing$ and $\tau(A) = A$, we have

$$\partial(c \circ [A]) = (\partial c) \circ [A] = (x - \tau^r x) \circ A = x \circ [A] - \tau^r x \circ [A] = x \circ [A] - x \circ \tau^{-r}[A] = 0.$$

Thus $c \circ [A]$ is a cycle. If c_1 is another chain with $\partial c_1 = x - \tau^r x$, then $c \circ [A] - c_1 \circ [A] = (c - c_1) \circ [A]$ is a cycle that represents the image of the class of the cycle $c - c_1$ under the homomorphism ρ. Finally, if the cycle x is homologous to zero (that is, there is a chain z with $\partial z = x$), then for c we can take the chain $z - \tau^r z$, and then obviously $c \circ [A] = z \circ [A] - z \circ \tau^{-r}[A] = 0$. Consequently, the image of the class represented by the cycle $c \circ [A]$ under the natural projection $H_{k-1}(A; \mathbb{Z}_p) \to H_{k-1}(A; \mathbb{Z}_p)/\operatorname{im}\rho$ is uniquely determined by the class β. We put $v_r(\beta)$ equal to this image.

We note that there is an obvious cohomology version of this construction. We need only replace the chains by cochains, the boundary homomorphism ∂ by the coboundary homomorphism δ, and the operation $\circ[A]$ of intersection with $[A]$ by the operation of restriction of a cochain to A. This gives the homomorphism

$$\ker(\tau^{*r} - 1 \colon H^k(Y; \mathbb{Z}_p) \to H^k(Y; \mathbb{Z}_p)) \to H^{k+1}(A; \mathbb{Z}_p)/\operatorname{in}^* H^{k+1}(Y; \mathbb{Z}_p).$$

Without appeal to cycles we can more easily define the class $v_r(\beta)$ by the condition that β belongs to the image of the inclusion homomorphism $i_* \colon H_k(Y \setminus A; \mathbb{Z}_p) \to H_k(Y; \mathbb{Z}_p)$. (In the situation described in §2.1 this condition is satisfied; see the proof of Lemma 2.2.A.) Consider the action of the homomorphism $1 - \tau^r$ in the homology sequence of the pair $(Y, Y \setminus A)$ with coefficients in $\mathbb{Z}_p$ (in the diagram the notation of the group of coefficients is omitted):

$$\begin{array}{ccccccc}
 & & H_{k+1}(Y, Y\setminus A) & \xrightarrow{\partial} & H_k(Y\setminus A) & \xrightarrow{i_*} & H_k(Y) \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 1-\tau^r_*} & & \downarrow{\scriptstyle 1-\tau^r_*} \\
H_{k+1}(Y) & \longrightarrow & H_{k+1}(Y, Y\setminus A) & \xrightarrow{\partial} & H_k(Y\setminus A) & \xrightarrow{i_*} & H_k(Y) \\
 & & \downarrow{\scriptstyle [A]\circ} & & & & \\
 & & H_{k-1}(A) & & & &
\end{array}$$

Since $1 - \tau^r = 0$ in $H_{k+1}(Y, Y \setminus A; \mathbb{Z}_p)$, a homomorphism $(1 - \tau^r_*)i_*^{-1} \colon \operatorname{im} i_* \to H_k(Y \setminus A; \mathbb{Z}_p)$ is defined. The image of $H^r_k \cap \operatorname{im} i_*$ under this homomorphism lies in the image of the boundary homomorphism $\partial \colon H_{k+1}(Y, Y \setminus A; \mathbb{Z}_p) \to H_k(Y \setminus A; \mathbb{Z}_p)$. Clearly, the intersection $[A] \circ \partial^{-1}(1 - \tau^r_*)i_*^{-1}(\beta)$ of the homology classes $[A]$ and $\partial^{-1}(1 - \tau^r_*)i_*^{-1}(\beta)$ (see, for example, [6], Chapter VIII, §13) is defined modulo $\operatorname{im}\rho$. We put

$$v_r(\beta) = [A] \circ \partial^{-1}(1 - \tau^r_*)i_*^{-1}(\beta) \mod \operatorname{im}\rho. \tag{7}$$

Clearly, v_r is a homomorphism.

The domain of definition of this homomorphism extends to the whole of H^r_k as follows. The kernel of the homomorphism $1 - \tau^r \colon C(Y; \mathbb{Z}_p) \to C(Y; \mathbb{Z}_p)$, where $C(Y; \mathbb{Z}_p)$ is the chain complex of Y, obviously contains $C(A; \mathbb{Z}_p)$. Therefore there is a chain homomorphism $z_r \colon C(Y, A; \mathbb{Z}_p) \to C(Y; \mathbb{Z}_p)$ such that $z_r j = 1 - \tau^r$, where j is the homomorphism induced by the inclusion $(Y, \varnothing) \to (Y, A)$. We put

$$v_r(\beta) = [A] \circ \partial^{-1} D z^*_{h-r} D_Y^{-1} \beta \mod \operatorname{im}\rho, \tag{8}$$

where $D \colon H^{n-k}(Y, A; \mathbb{Z}_p) \to H_k(Y \setminus A; \mathbb{Z}_p)$, and $D_Y \colon H^{n-k}(Y; \mathbb{Z}_p) \to H_k(Y; \mathbb{Z}_p)$ is the duality isomorphism. It is easy to verify that when $\beta \in H^r_k \cap \operatorname{im} i_*$ the right-hand sides of (7) and (8) coincide.

2.2.A. **Lemma.** *Let B be an oriented k-dimensional membrane that can be lifted to Y, and let β be the class defined in $H_k(Y; \mathbb{Z}_p)$ by the cycle $\nu^{-1}(B)$, oriented in agreement with B. Then $\beta \in H_k^1$, and the class $v_1\beta$ coincides with the class d realized in A by the manifold ∂B.*

Proof. Clearly, $\tau_*\beta = \beta$, so $\beta \in H_k^1$. Let T be a tubular neighborhood of A in Y, invariant with respect to τ. Consider the components of the set $\mathrm{pr}^{-1}(\partial C)\setminus(C \cup C_1 \cup \cdots \cup C_{h-1})$, where $\mathrm{pr}\colon \partial T \to A$ is a projection. Let T_1 denote that component whose boundary is contained in $C \cup C_1$; clearly, the remaining components are obtained from T_1 by the action of a power of τ. We put $\tilde{\beta} = \sum_{j=1}^{h-1} \tilde{\gamma}_j$, where $\tilde{\gamma}_j$ is the class in $H_k(Y \setminus A; \mathbb{Z}_p)$ determined by the set $(\bigcup_{s=0}^{j-1} \tau^s \operatorname{Cl} T_1) \cup (C \setminus T) \cup (C_j \setminus T)$, oriented according to the orientation of C_j. Clearly, $i_*\tilde{\beta} = \beta$, where i, as in §2.2, is the inclusion $Y \setminus A \to Y$. According to §2.2, $v_1\beta = [A] \circ \partial^{-1}(1-\tau_*)\tilde{\beta}$. It is easy to verify that the class $(1-\tau_*)\tilde{\beta}$ is realized by the manifold $\mathrm{pr}^{-1}(\partial C)$, and therefore $[A] \circ \partial^{-1}(1-\tau_*)\tilde{\beta}$ coincide with d. ●

2.3. **Another approach: application of Smith's theory.** Smith's theory is a small collection of homological facts about spaces with the action of the group $\mathbb{Z}_p$ (see [7], Chapter III, §3). It is easy to verify that this theory can be generalized from the case of spaces with the action of $\mathbb{Z}_p$ to the case of spaces with the action of the group $\mathbb{Z}_h$ (such as Y); the coefficients of all the homology and cohomology groups under consideration are taken from $\mathbb{Z}_p$ as before; this is also assumed in this subsection.

In the notation of §2.1 let B be an oriented k-dimensional membrane that is lifted to Y, let b be the class determined in $H_k(X, A)$ by the membrane B, and β the class determined in $H_k(Y)$ by the cycle $\nu^{-1}(B)$, oriented in accordance with B. Consider the homomorphism α_k from the Smith homology sequence ([7], Chapter III, §§3.3 and 3.4)

$$\cdots \longrightarrow H_{k+1}^{1-\tau}(Y) \xrightarrow{\partial} H_k(X, A) \oplus H_k(A) \xrightarrow{\alpha_k} H_k(Y) \xrightarrow{1-\tau_*} H_k^{1-\tau}(Y) \longrightarrow \cdots. \tag{9}$$

2.3.A. **Lemma.** *The restriction $\tilde{\alpha}_k$ of the homomorphism α_k to $H_k(X, A)$ takes b to β. The homomorphism α_{n-1} is a monomorphism if $H_n^{1-\tau}(Y) = 0$; $\tilde{\alpha}_{n-2}$ is a monomorphism if X is connected and $H_{n-1}(Y) = 0$; and α_k, where $[(n+1)/2] \le k < n-2$, is a monomorphism if X and A are connected and $H_i(Y) = 0$ for $k+1 \le i \le n-1$.*

Proof. The equality $\tilde{\alpha}_k b = \beta$ follows immediately from the definition of the homomorphism α_k. If $H_n^{1-\tau}(Y) = 0$, then the fact that α_{n-1} is monomorphic follows from the fact that (9) is an exact sequence.

When $[(n+1)/2] \le k \le n-2$ we use the commutative diagram of Smith homology sequences that includes the sequence (9) and the short exact sequences of chain complexes obtained from the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^{\sigma}(Y) \oplus C(A) & \longrightarrow & C(Y) & \xrightarrow{1-\tau} & C^{1-\tau}(Y) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \mathrm{in} \oplus 1} & & \big\downarrow{\scriptstyle 1} & & \big\downarrow{\scriptstyle (1-\tau)^{h-2}} & & \\
0 & \longrightarrow & C^{1-\tau}(Y) \oplus C(A) & \longrightarrow & C(Y) & \xrightarrow{\sigma} & C^{\sigma}(Y) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle (1-\tau)^{h-2} \oplus 0} & & \big\downarrow{\scriptstyle (1-\tau)^{h-2}} & & \big\downarrow{\scriptstyle \mathrm{in}} & & \\
0 & \longrightarrow & C^{\sigma}(Y) \oplus C(A) & \longrightarrow & C(Y) & \xrightarrow{1-\tau} & C^{1-\tau}(Y) & \longrightarrow & 0
\end{array}$$

where $\sigma = \sum_{i=0}^{h-1} \tau^i$, $C^\rho(Y) = \operatorname{im}(\rho\colon C(Y) \to C(Y))$, and in is inclusion (see [7], Chapter III, §3.6). We first prove that $\tilde{\alpha}_{n-2}$ is monomorphic. Clearly, this follows from the fact that the component $H^{1-\tau}_{n-1}(Y) \to H_{n-2}(X, A)$ of the homomorphism ∂ of the sequence (9) is monomorphic. By [7], Chapter III, §§3.4, 3.5, and 3.7, this component is the composition

$$H^{1-\tau}_{n-1}(Y) \xrightarrow{(1-\tau)^{h-2}} H^\sigma_{n-1}(Y) \xrightarrow{\cong} H_{n-1}(X, A) \xrightarrow{\partial} H_{n-2}(A),$$

in which ∂ is a monomorphism, since by hypothesis $H_{n-1}(X) = 0$. Let us prove that $(1-\tau)^{h-2}\colon H^{1-\tau}_{n-1}(Y) \to H^\sigma_{n-1}(Y)$ is an isomorphism. Consider the square

$$\begin{array}{ccccc} 0 & \longrightarrow & H^\sigma_n(Y) & \longrightarrow & H_n(Y) \\ & & \big\downarrow \text{in} & & \big\downarrow 1 \\ 0 & \longrightarrow & H^{1-\tau}_n(Y) & \longrightarrow & H_n(Y) \end{array}$$

of the homology diagram. Since $H^\sigma_n(Y) \cong H_n(X, A)$ (see [7], Chapter III, §3.4), $H_n(X, A) \cong H_n(X)$ (since the homology sequence of the pair (X, A) is exact), and $H_n(X) \cong H_n(Y) \cong \mathbb{Z}_p$ (since the manifold X is connected), this square consists of isomorphisms. Therefore in the square

$$\begin{array}{ccc} H^\sigma_n(Y) & \xrightarrow{\partial} & H^{1-\tau}_{n-1}(Y) \\ \big\downarrow \text{in} & & \big\downarrow (1-\tau)^{h-2} \\ H^{1-\tau}_n(Y) & \xrightarrow{\partial} & H^\sigma_{n-1}(Y) \end{array}$$

of the same diagram the homomorphisms ∂ are monomorphisms, and since by hypothesis $H_{n-1}(Y) = 0$, they are also isomorphisms. Consequently, $(1-\tau)^{h-2}$ is an isomorphism.

We now prove that α_k is a monomorphism (when $[(n+1)/2] \le k < n-2$). From the fact that the horizontal segments

$$H_{n-1}(Y) \longrightarrow H^{1-\tau}_{n-1}(Y) \xrightarrow{\partial} H^\sigma_{n-2}(Y) \oplus H_{n-2}(A) \longrightarrow H_{n-2}(Y),$$

$$H_{n-1}(Y) \longrightarrow H^\sigma_{n-1}(Y) \xrightarrow{\partial} H^{1-\tau}_{n-2}(Y) \oplus H_{n-2}(A) \longrightarrow H_{n-2}(Y)$$

of the diagram are exact we deduce that $H^\sigma_{n-2}(Y) = H^{1-\tau}_{n-2}(Y) = 0$, since by hypothesis $H_{n-1}(Y) = H_{n-2}(Y) = 0$, $H_{n-2}(A) \cong \mathbb{Z}_p$ as A is connected, and $H^{1-\tau}_{n-1}(Y) \cong H^\sigma_{n-1}(Y) \cong \mathbb{Z}_p$ by what we said above. Shifting to the right along the horizontals of the same diagram and using the equalities $H_{n-3}(Y) = \cdots = H_{k+1}(Y) = 0$, we obtain successively $H^\sigma_{n-3}(Y) = H^{1-\tau}_{n-3}(Y) = \cdots = H^\sigma_{k+1}(Y) = H^{1-\tau}_{k+1}(Y) = 0$. Therefore α_k is a monomorphism, because the sequence (9) is exact.

2.4. **Proof of Lemmas 1.5.A and 1.5.B.** As we mentioned in §1.6, the classes $\beta_1, \ldots, \beta_l$ lie in the image of the monomorphism λ_*. Let $\beta_i' = \mu_*\lambda_*^{-1}\beta_i$. Since $\operatorname{Tors} H_2(Y) = 0$, the $\mathbb{Z}[\xi]$-module $H_2(Y, \mathbb{Z}[\xi])$ is free. We can therefore choose a basis of the space $H_2(Y, \mathbb{C})$ that goes over under the action of $\mu_*\lambda_*^{-1}$ to a basis of the $\mathbb{Z}_p$-space $H_2(Y; \mathbb{Z}_p)$. Here $\mu_*\lambda_*^{-1}$ takes the coordinates of the vector β_i into the corresponding coordinates of the vector β_i'. Regarding the rank of the system of vectors as the highest order of nonzero minors of the matrix consisting of the coordinates of these vectors, we deduce that $\operatorname{rk}(\beta_0, \ldots, \beta_l) \ge \operatorname{rk}(\beta_0', \ldots, \beta_l')$.

We can now complete the proof by two methods, either by means of Lemma 2.2.A, applying the homomorphism v_1 to the classes β_j', or by means of Lemma 2.3.A, applying the composition $\partial \circ \tilde{\alpha}_2^{-1}$ to these classes, where $\partial: H_2(\mathbb{C}P^2, \mathbb{C}A; \mathbb{Z}_p) \to H_1(\mathbb{C}A; \mathbb{Z}_p)$ is a boundary homomorphism.

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Received 13/Oct/91

Translated by E. J. F. PRIMROSE