COMPACT FOUR-DIMENSIONAL EXOTICA WITH SMALL HOMOLOGY

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Abstract. An infinite family of homeomorphic, but pairwise non-diffeomorphic smooth compact simply-connected four-dimensional manifolds with the second Betti number 2 bounded by the Poincare homology sphere is constructed. Also it is constructed an infinite family of spheres embedded into $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$ such that each of them has only one point where it is not smooth, and they are ambiently homeomorphic (via homeomorphisms smooth on some their neighbourhoods), but are not ambiently diffeomorphic. It is proved that some pairs of topological logarithmic transformations, which chainge smooth type of $\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}$ without changing its topological type, do preserve the smooth type of $S^2 \times S^2$.

1. Exotic objects

In this paper the word "exotic" means "homeomorphic but not diffeomorphic". It has been known (since Moise's work in the early fifties) that such phenomena do not occur in dimensions < 4. Although in higher dimensions the existence of exotic objects had been known also for a long time (since Milnor's exotic 7-spheres, constructed in the fifties), in dimension four it had only been discovered in the eighties. The first examples of four-dimensional exotic manifolds were either non compact or not simply-connected. Their construction and the proof of non-existence of a diffeomorphism rely on these properties. On the other hand the specific effects of dimension 4 seem to appear in the most distilled form for compact simply-connected manifolds without boundary. Examples of this kind of exotica have appeared recently: Donaldson [D1], [D2] proved that many well known simply-connected compact complex algebraic surfaces provide exotica: some of them are homeomorphic, but not diffeomorphic to each other, some are homeomorphic, but not diffeomorphic to corresponding connected sums of copies of $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$, and the K3-surface.

For simply-connected closed 4-manifolds a natural rough measure of their size and complexity is the second Betti number, i.e. the rank of the intersection form (this form is the only homeomorphism invariant for manifolds of this type). Now the lowest known value of the second Betti number for exotic simply-connected

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compact 4-manifolds is 9: Kotschick [K] proved that the Barlow surface (which was constructed by Barlow [B] in 1985 and was known to be homeomorphic to $\mathbb{C}P^2\#8\overline{\mathbb{C}P^2}$) is not diffeomorphic to $\mathbb{C}P^2\#8\overline{\mathbb{C}P^2}$. For the next value 10 there is an infinite family of homeomorphic, but pairwise not diffeomorphic compact simply-connected complex algebraic surfaces. These are the Dolgachev surfaces homeomorphic to $\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}$.

Four-dimensional objects of another kind, which can be exotic, are knottings, i.e. pairs consisting of a smooth 4-manifold and a smooth 2-submanifold. Finashin, Kreck and Viro [FKV] constructed an infinite family of homeomorphic but not diffeomorphic knottings of the connected sum of g copies of $\mathbb{R}P^2$ in the 4-sphere for each $g > 9$.

2. New exotica

In this work there are no new exotic objects of the two types described above (namely closed manifolds and smooth knottings). The exotic objects constructed here are compact simply-connected 4-manifolds with non empty boundary,[1]knottings of surfaces in manifolds with boundary and knottings in closed manifold but with one point of non-smoothness where the surface is locally knotted with a prescribed local knot. However these objects have smaller homology and are closely related to the previous ones. Perhaps they give an opportunity for better understanding of the phenomena of being exotic.

Theorem 1. *There exists an infinite family of smooth simply-connected compact 4-manifolds $V_{p,q}$ where p, q are relatively prime integers such that:*

(1) *the boundary of $V_{p,q}$ is homeomorphic to the Poincare homology sphere,*
(2) *the intersection form of $V_{p,q}$ is isomorphic to $< +1 > + < -1 >$,*
(3) *if $p = r = 2$ and $q \neq s$, then the manifolds $V_{p,q}$ and $V_{r,s}$ are not diffeomorphic,*
(4) *all the $V_{p,q}$ are homeomorphic.*

Theorem 2. *There exists an infinite family of subsets S_p of $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$ with p any integer > 1 such that:*

(1) *S_p is homeomorphic to the 2-sphere,*
(2) *S_p has only one point where it is not a smooth submanifold,*
(3) *at its singular point S_p is locally knotted as a cone over the trefoil knot,*
(4) *S_p has the self-intersection number -1,*
(5) *for any p, q there exists a homeomorphism of $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$ which is smooth on some neighbourhood of S_p and sends S_p onto S_q,*
(6) *for $p \neq q$ there is no diffeomorphism of $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$ mapping S_p onto S_q,*
(7) *the complement of an open regular neighbourhood of S_p in $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$ is diffeomorphic to the manifold $V_{2,2p+1}$ of Theorem 1.*

Theorem 1 will be proved in Section 4 and Theorem 2 in Section 9.

Remark. Removing a small 4-ball centered at the singular point of S_p, one obtains an obvious reformulation of Theorem 2 for smooth knottings, but in the manifold with boundary.

[1] *Added in proof* R.Gompf informed me that he had constructed a family of exotic manifolds containing the manifolds $V_{p,q}$ which are constructed below. S.Akbulut sent me his preprint where he constructed exotic manifolds with boundary having even smaller Betti numbers than $V_{p,q}$.

3. Topological logarithmic transformations

The Dolgachev surfaces mentioned above are obtained from each other (and in particular from $\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}$) by constructions belonging to algebraic geometry in nature. These constructions are called *Kodaira logarithmic transformations.* From the topological point of view (i.e. ignoring complex algebraic structures) this transformation is a regluing of a tubular neighbourhood of a torus smoothly embedded into the manifold which is the object of the transformation.

Consider more carefully this topological version of the Kodaira logarithmic transformation. Let X be a smooth 4-manifold, T a smooth submanifold of X homeomorphic to the 2-torus. Let T have trivial normal bundle. Let U be a tubular neighbourhood of T. Fix some trivialization

$$\theta : S^1 \times S^1 \times D^2 \to U$$

of U. Let j, k, l, m be integers with $jm - kl = 1$. Denote by h the diffeomorphism of the boundary of U defined by

$$h(\theta(x, y, z)) = \theta(x, y^j z^k, y^l z^m)$$

Denote by Y the 4-manifold obtained from $X - Int(U)$ and U by gluing by h. It is called the result of the *topological logarithmic transformation of* X *along* T *with multiplicity* m. To define Y up to diffeomorphism, it is sufficient to fix X, T, θ, m, k and the homology class d of the circle $\theta(1 \times (circle) \times 0)$ in T. The integer k is called the *supplementary multiplicity* and d *the direction* of the logarithmic transformation.

The Dolgachev surfaces are organized into families $D_{p,q}$ with p, q relatively prime numbers (there are also families $D_{p,q}$ with $g.c.d.(p, q) > 1$, but they consist of non simply-connected manifolds, and we will not consider them). The $D_{1,1}$-surfaces are diffeomorphic to $\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}$. The homology class in $\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}$ with coordinates $(3, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ in the natural basis has the zero self-intersection number and is realized naturally by a smoothly embedded torus. The $D_{p,q}$-surfaces are obtained from $\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}$ by pairs of logarithmic transformations with multiplicities p and q respectively along a pair of disjoint tori smoothly isotopic to the torus above. As it was proved by Friedman and Morgan [FM], if $p = r = 2$ and $q \neq s$, then no $D_{p,q}$-surface is diffeomorphic to a $D_{r,s}$-surface.

If one takes a simply-connected smooth 4-manifold X with a smoothly embedded torus T with $T \circ T = 0$ and if the complement of T in X is simply-connected then a pair of topological logarithmic transformations with relatively prime multiplicities p, q along T and some non zero section of its tubular neighbourhood gives a new smooth 4-manifold, which is simply-connected. Varying the supplementary multiplicities and directions of the transformations, the result of the transformations can be made homeomorphic to X.

For X in this construction one can take a 4-manifold with the second Betti number considerably smaller than 10. In fact one can take for X the manifolds $S^2 \times S^2$ and $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ with the second Betti number 2. But it is unknown whether this gives exotica or not. Unfortunately the proofs in the case of the Dolgachev surfaces rely heavily on methods of algebraic geometry and hence on the fact that the manifolds under consideration are complex algebraic surfaces. For X with the second Betti number < 10 the transformations can not be done respecting any complex analytic structure and there is no obvious way for introducing such a structure in the result of the transformations.

The Finashin - Kreck - Viro exotic knottings mentioned above are closely related with the Dolgachev surfaces. Namely they are obtained as a pair (orbit space, fixed point set) for some antiholomorphic involutions acting in Dolgachev surfaces. These involutions appear since the logarithmic transformations can be done equivariantly with respect to the standard complex conjugation involution of $\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}$. The corresponding transformation of the pair (orbit space, fixed point set) is the knotting construction of Finashin - Kreck - Viro [FKV]. The exotic knottings of [FKV] are obtained from each other by such constructions. The constructions can be applied in considerably simpler situations. But as in the case of manifolds there is no proof of non existence of diffeomorphisms. Thus we have constructions, which at least sometimes give exotic objects. This work appeared as a result of attempts to investigate effects of these constructions in some other cases.

4. Manifolds $V_{p,q}$

All the manifolds $V_{p,q}$ are obtained from $V_{1,1}$ by pairs of topological logarithmic transformations. Thus let us begin with constructing $V_{1,1}$. It is a regular neighbourhood of the 2-dimensional CW-complex K which is a wedge sum of a 2-sphere S and a torus R with two disc membranes P, Q, spanning a meridian and a longitude of R. S and R are smoothly embedded into $V_{1,1}$ with the self-intersection numbers $S\circ S=-1$, $R\circ R=0$, they have exactly one common point and intersect each other transversally. The membranes P and Q are also smooth, have indices -1, have only one common point (on their boundaries) and are pairwise transversal. They do not meet S and meet R only along their boundaries. These properties determine $V_{1,1}$ up to diffeomorphism. This manifold was constructed by Matumoto [M] and Guillou and Marin [GM, Appendice D, p.47]. They proved that it admits a smooth embedding into $\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}$ which maps R to one of the usual representatives of the class $(3,1,1,1,1,1,1,1,1,1)$. They also proved that the closure of the complement of the image of this embedding can be obtained by plumbing according to the graph E_8 weighted by -2, and therefore the boundary of $V_{1,1}$ is homeomorphic to the Poincare homology sphere.

To construct $V_{p,q}$ take R and a non-zero section R' of its tubular neighbourhood and make the pair of topological logarithmic transformations of $V_{1,1}$ along R and R' of multiplicities p, q, supplementary multiplicities 1, 1 and with coinciding (in the obvious sense) directions.

The Matumoto - Guillou - Marin embedding of $V_{1,1}$ into $\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}$ gives obviously an embedding of $V_{p,q}$ into one of the simplest $D_{p,q}$-surfaces. The closure of the complement of the image of this embedding does not depend on p, q since it is the same plumbing. Thus the $D_{p,q}$-surface is presented as a result of gluing of $V_{p,q}$ and this constant part independent of p, q. As it was proved by Boileau and Otal [BO], any two homeomorphisms of the Poincare homology sphere are isotopic. Therefore the result of gluing is determined up to diffeomorphism by the parts which are glued. Thus the assertion (3) of Theorem 1 above follows from the known results on diffeomorphism types of the Dolgachev surfaces stated above.

To finish off the proof of Theorem 1, note that assertion (2) follows immediately from the construction of $V_{p,q}$ and (4) follows from (2), (1) and triviality of the fundamental group of $V_{p,q}$ by the homeomorphism classification of compact simply-connected 4-manifolds bounded by homology spheres. This classification theorem is essentially due to Freedman [F], see Vogel [V] and Boyer [Br].

5. Inactive topological logarithmic transformations

The known results on the Dolgachev surfaces and Theorem 1 show that sometimes a pair of topological logarithmic transformations can change the differential type of a 4-manifold without changing its topological type. Here is a result in the opposite direction.

Theorem 3. *Let T be a torus smoothly embedded in $S^2 \times S^2$ and obtained by adding a trivial handle to a fibre $S^2 \times (pt)$, and T' a non-zero section of a tubular neighbourhood of T. For any relatively prime p, q let Y be the result of the pair of topological logarithmic transformations of $S^2 \times S^2$ along T and T' with multiplicities p and q, supplementary multiplicities 1 and 1 and directions being along the handle adjoined to the $S^2 \times (pt)$ to form T. Then Y is diffeomorphic to the $S^2 \times S^2$.*

Note that the same pair of the topological logarithmic transformations changes the differential types of $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$ and $V_{1,1}$ without changing their topological types.

This theorem has an analog (Theorem 4 below) for knottings, since everything in it can be taken to be invariant with respect to the involution which acts in each factor of $S^2 \times S^2$ as the reflection in a plane. That involution can be interpreted as the non-identity covering transformation of the two-fold covering of the 4-sphere branched over the standardly embedded torus. Thus the counterpart of Theorem 3 states that some Finashin - Kreck - Viro knotting construction does not change the smooth type of the torus standardly embedded into the 4-sphere. In fact a special case of Theorem 3 had been obtained in the first version of this paper as a corollary of Theorem 4 below. The proof of Theorem 3 presented below uses a result by Moishezon [Mo] instead (this result had been pointed out to me by Gompf).

6. Knotting along an almost disc annulus membrane

Now I recall the Finashin-Kreck-Viro knotting construction from [FKV]. Let X be a smooth 4-manifold and F a smooth closed 2-submanifold of X. Let M be a smooth annulus membrane in X on F intersecting F only along the boundary and having index zero. In other words, M is a smooth 2-submanifold of X homeomorphic to an annulus and having a regular neighbourhood N in X such that there exists a diffeomorphism $f : N \to S^1 \times D^3$ mapping the intersection of N and F onto $S^1 \times$ (0-tangle), where the 0-tangle is a pair of segments unknotted and unlinked in D^3. For any relatively prime p, q denote by $K(F, M, f, p, q)$ the new submanifold of X obtained from F by replacing the 0-tangle by the sum of $(1/p)$- and $(1/q)$-tangles.

In Theorem 3 each of the tori which are the cores of the topological logarithmic transformations is obtained from an embedded sphere by an embedded surgery of index 1. Consider corresponding knotting constructions. Let F be again a smooth closed 2-submanifolds of a smooth 4-manifold X and let D be a smooth disc membrane in X on F with zero index. Let C be a product $D^2 \times [0,1]$ smoothly embedded into X in such a way that its intersection with D is the image of $D^2 \times 0$ and lies near the center of D and its intersection with F is the image of $D^2 \times 1$.

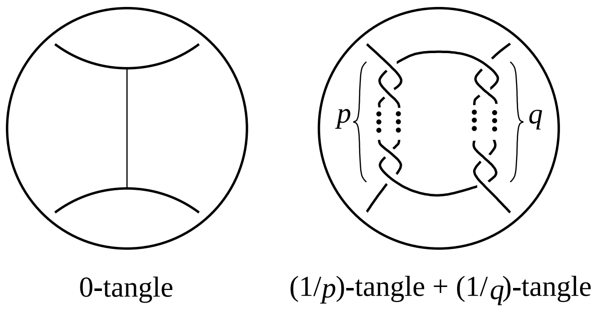

FIGURE 1

Denote by Z the image of $\partial D^2 \times [0,1]$. Denote by M an annulus membrane on F obtained from the union of Z and $D \setminus$ (image of $D^2 \times 0$) by smoothing the corner along the image of $S^1 \times 0$.

In this situation the knotting construction along M can be decomposed (up to diffeomorphism) into some knotting construction along D (or, more precisely, a kind of embedded knotted surgery of index 2 or 0) and an embedded surgery of index 1 along C. The knotting along D is also a composition: first, it creates a new unknotted 2-sphere near the center of D intersected D in a circle and adjoins it to F (i.e. makes an embedded surgery of index 0 of F), and second, it makes the knotting construction along the annulus which is the part of D between the new sphere and the old F. Without this second step, the first 0-surgery would cancel with the subsequent 1-surgery. The isotopy making this cancellation moves the result of the composition of the knotting along D and the 1-surgery into the result of the knotting along M.

7. Knotting the standard torus in the 4-sphere

The torus T standardly embedded into the 4-sphere can be considered as the trace of an unknotted circle lying in a 3-hemisphere under rotation of this 3-hemisphere around its boundary i.e. around the 3-space intersecting the 4-sphere in this boundary (the trace of the whole 3-hemisphere under this rotation is just the 4-sphere). Let D be a disc traced by an unknotted arc connecting a point on the circle and a point P on the boundary of the 3-hemisphere under the same rotation. It is a disc membrane on T with index 0.

The knottings along it of the type described in the preceding section give the Artin spun-knots and links, i.e. 2-knots and 2-links which are traces of sets of arcs and circles under the rotation above. Actually, the 0-surgery can be presented as the result of adding a small unknotted arc with the end-points near P on the boundary of the 3-hemisphere to the rotating circle tracing T. The subsequent knotting construction along an annulus membrane is equivalent to inserting the tangle which defines this knotting into the 3-hemisphere between the circle and the arc to join them, see the picture:

It is clear that if the inserting tangle is a sum of $(1/p)$-tangle and $(1/q)$-tangle with odd p+q then the result is the 2-knot obtained by the Artin construction applied to the torus $(p+q,2)$-knot. In the case of $q = 1-p$, since this torus knot is the unknot, the result of the knotting along D of the standardly embedded torus T is an unknotted 2-sphere.

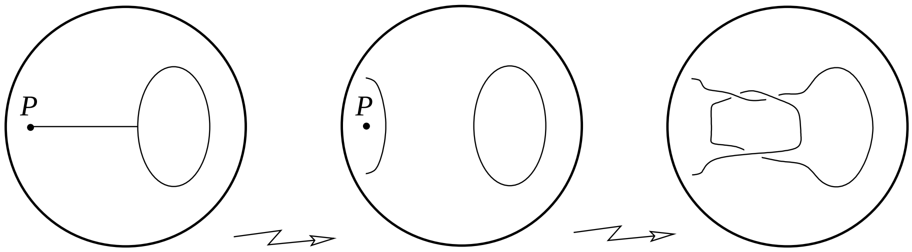

FIGURE 2

Theorem 4. *Let T and D be the simplest torus embedded into the 4-sphere and disc membrane on it as above, let M be any annulus membrane obtained from D in the way described in Section 6 above and f be an appropriate trivialization of its regular neighbourhood. Then $K(T, M, f, p, 1-p)$ is smoothly isotopic to T.*

Proof. As it was shown in Section 6 the knotting along M is the composition of the knotting along D and the subsequent 1-surgery. As it was shown above, the first one gives the 2-unknot. The second is unique up to diffeomorphism since it preserves the orientation and any two arcs which can be cores of such a surgery are isotopic (because their interiors, lying in the complement of the 2-unknot, are homotopic). It is clear that the simplest of such 1-surgeries gives a torus smoothly isotopic to T.

Thus the Finashin - Kreck - Viro knotting construction applied to the torus standardly embedded into the 4-sphere and the simplest annulus membrane with zero index gives a torus of the same smooth isotopy class, if the knotting tangle is the sum of $1/p$-tangle and $(1/1-p)$-tangle. Note that the same knotting tangles in the case of the connected sum of 10 copies of $\mathbb{R}P^2$ give exotic knottings, see [FKV].

8. PROOF OF THEOREM 3

The manifold Y of Theorem 3 is the 2-fold covering space of the 4-sphere branched over the torus which is obtained from the standardly embedded torus by the surgeries described in the preceding section. Since the 2-fold covering space of the 3-sphere branched over the torus $(p+q, 2)$-link is the lens space $L(p+q, 1)$, the 2-fold covering space of the 4-sphere branched over link, which is obtained from the $(p+q, 2)$-torus link by the Artin construction, is diffeomorphic to $L(p+q, 1) \times S^1$ surgered along $pt \times S^1$. This 2-dimensional link is transformed into the torus $K(T, M, f, p, q)$ by a surgery of index 1. The corresponding surgery of index 2 of the 2-fold branched covering gives Y. Thus Y can be obtained from $L(p+q, 1) \times S^1$ by two surgeries of index 2, one of which is along $pt \times S^1$. Since Y is simply-connected and spin (the latter follows from orientability of the branch locus), then it is diffeomorphic to $S^2 \times S^2$, as Moishezon has proved [Mo, Lemma 13, p. 208].

9. PROOF OF THEOREM 2

Lemma 1. *If X is a smooth 4-manifold with boundary obtained by adjoining several 2-handles to the 4-ball, then the double of it (i.e. the result of gluing two copies of X by the identity map of the boundary) is diffeomorphic to a connected sum of several copies of $S^2 \times S^2$ or $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.*

Proof. The natural framed link presentation of the double of X is obtained from a framed link presentation of X by adding new components with zero framing which are meridians of components of the previous part of the link, one meridian for each component. By isotopies and adding of these new components to the old ones it is easy to make the old components to be unknotted and unlinked. After these moves the new components have indices 0 and are meridians of the old ones as before. Now the link becomes a disjoint sum of the Hopf links. It is well known that the Hopf link with framing, which is 0 on one of the components determines $S^2 \times S^2$ or $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ depending on the framing of the other component. □

It is well known that the Poincare homology sphere bounds a compact simply connected 4-manifold with the second Betti number 1. This manifold is obtained for example as a result of adjoining of one 2-handle to the 4-ball along a trefoil knot with framing 1. Denote this manifold by W. It can be presented also as a regular neighbourhood of a smoothly embedded torus J together with two disc membranes spanning meridian and longitude where the torus has self-intersection $+1$ and the membranes have indices -1. Cf. [M] and [GM]. It is clear that $V_{1,1}$ is obtained from W by one blow up and that W contains a deformation retract homeomorphic to the 2-sphere and smoothly embedded everywhere except at a point where it is locally knotted as a cone over trefoil. This deformation retract can easily be seen in the first presentation of W : it is the union of a core of the 2-handle and a cone over the boundary of the core with the center inside the 4-ball. Denote it by S.

Lemma 2. *The 4-manifold obtained by gluing the manifolds $V_{1,1}$ and $-W$ by the identity map of the boundary Poincare homology sphere is diffeomorphic to $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$.*

Proof. Since W is not a Spin-manifold, its double is not Spin too. Therefore by Lemma 1, the double of W is diffeomorphic to $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. Now the result follows from the fact that $V_{1,1}$ can be obtained from W by a blow up (i.e. by adding $\overline{\mathbb{C}P^2}$). □

Lemma 3. *There exists a smooth embedding of the complement of a point in $S^2 \times S^2$ into the result of gluing of $V_{1,1}$ and $-W$ along the boundary such that a torus obtained from a fiber $S^2 \times pt$ by adding a trivial handle is mapped by this embedding onto the torus R in $V_{1,1}$.*

Proof. The complement of a point in $S^2 \times S^2$ is a regular neighbourhood of the bouquet of $S^2 \times pt$ and $pt \times S^2$. A tubular neighbourhood of $S^2 \times pt$ can be obtained from a tubular neighbourhood of the torus T which is the fibre $S^2 \times pt$ with a small trivial handle adjoined by adding a tubular neighbourhoods of the small membranes spanning a meridian and a longitude of the handle. The indices of these membranes are zero. Thus the complement of a point in $S^2 \times S^2$ can be presented as a regular neighbourhood of the two- dimensional CW-complex which is a bouquet sum of a 2-sphere D and a torus C with two disc membranes A and B spanning meridian and longitude of C. Each of A, B, C, D is smoothly embedded, C and D have exactly one common point and are pairwise transversal, A, B have exactly one common point (on their boundaries) and intersect at this point transversally, all the self-intersections and indices are 0. The membranes A, B do not intersect D and intersect C only along their boundaries. These properties of A, B, C, D determine a regular neighbourhood of their union up to diffeomorphism. Thus to define a smooth embedding of the complement of a point in $S^2 \times S^2$ into some 4-manifold (up to smooth isotopy) it is sufficient to find in this manifold a collection of embedded surfaces with these properties.

To construct the desired embedding, one should take the torus R for C. For D let us take a sphere H which is the union of a fibre of the tubular neighbourhood of R in $V_{1,1}$ and the corresponding fibre of the tubular neighbourhood of J in W. (Now we consider the tubular neighbourhoods which participate in the constructing $V_{1,1}$ and W above; their boundaries intersect with the boundaries of $V_{1,1}$ and W.) The membranes P and Q (see the construction of $V_{1,1}$ in Section 4) can not be taken for A and B, since the indices of P, Q are -1. To make an appropriate membranes, let us take two disjoint non-zero section H' and H'' of the tubular neighbourhood of H (remember $H \circ H = 0$), produce two disc membranes from H' and H'' by flattening small discs on H' and H'' to R in such a way that the indices of the new membranes be +1 (there are two ways of the flattening - one gives index +1, the other one -1, see [R]), and then add the new membranes by connected sums along the boundary to P and Q . It is easy to check that the resulting membranes, together with R and H satisfy all the conditions posed above on A, B, C, D. □

Lemma 4. [2]*For any relatively prime p, q the 4-manifold obtained by gluing the manifolds* $V_{p,q}$ *and* $-W$ *by the identity map of the boundary Poincare homology sphere is diffeomorphic to* $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$.

Proof. The result of gluing of $V_{p,q}$ and $-W$ is obtained from the result of gluing of $V_{1,1}$ and $-W$ by a pair of topological logarithmic transformations along two torus closed to R. By Lemma 3 these transformations are produced in an embedded punctured $S^2 \times S^2$ in such a way that by Theorem 3 they do not change the smooth type of it. Thus the results of gluing of $V_{p,q}$ and $-W$ is diffeomorphic to the result of gluing of $V_{1,1}$ and $-W$. The latter is diffeomorphic to $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$ by Lemma 2. □

To finish the proof of Theorem 2, denote by S_p the image of the sphere S (remember that it is a deformation retract of W) under the diffeomorphism of the result of gluing of $V_{p,q}$ and $-W$ onto $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$. The assertions (1)-(4) and (7) are the straightforward consequences of the construction of S_p. The assertions (5) and (6) follow from the assertions (4) and (3) of Theorem 1 respectively.

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References

[B] R.Barlow, *A simply connected surface of general type with* $p_g = 0$, Invent. Math. **79** (1985), 293–301.

[BO] M.Boileau, J.-P.Otal, *Groupe des diffeotopies de certaines varietes de Seifert*, C. R. Acad. Sc. Paris Ser.I **303** (1986), no. 1, 19–22.

[Br] S.Boyer, *Simply-connected 4-manifolds with a given boundary*, Trans. Amer. Math. Soc. **298** (1986), no. 1, 331–357.

[2]*Added in proof* As R.Gompf informs me, this Lemma had been obtained by him in summer 1988 by another method and will be included into his paper on connected sums of elliptic surfaces.

[D1] S.K.Donaldson, *Irrationality and the h-cobordism conjecture*, J.Diff. Geom. **26** (1987), 141168.

[D2] S.K.Donaldson, *Polynomial invariants for smooth 4-manifolds*, J. Diff. Geom. (to appear).

[F] M.H.Freedman, *The topology of four-dimensional manifolds*, J. Diff. Geom. **17** (1982), 357-453.

[FKV] S.M.Finashin, M.Kreck, O.Ya.Viro,, *Non-diffeomorphic but homeomorphic knottings of surfaces in the 4-sphere*, Lect. Notes in Math., vol. 1346, 1988, pp. 157–198.

[FM] R.Friedman, J.W.Morgan, *On the diffeomorphism types of certain algebraic surfaces I*, J. Diff. Geom. **27** (1988), no. 2, 297– 369.

[GM] L.Guillou, A.Marin, *Commentaires sur quatres articles precedent de V.A.Rohlin*, A la Recherche de la Topologie Perdue (Progr.in Math. vol 62), Birkhauser, 1986, pp. 25–95.

[K] D.Kotschick, *On manifolds homeomorphic to* $\mathbb{C}P^2\#8\overline{\mathbb{C}P^2}$, Invent. math. (1989).

[M] T.Matumoto, *Extension problem of diffeomorphisms of a 3-torus over some 4-manifolds*, Hiroshima Math.J. **14** (1984), no. 1, 189–201.

[Mo] B.Moishezon, *Complex surfaces and connected sums of complex projective planes*, Lect. Notes in Math., vol. 603, 1977.

[OV] C.Okonek, A.Van de Ven, *Stable bundles and differentiable structures on certain elliptic surfaces*, Invent. Math. **86** (1986), 357–373.

[R] V.A.Rohlin, *New examples of 4-manifolds*, Doklady AN SSSR **162** (1965), no. 2, 273–276.

[V] P.Vogel, *Simply-connected 4-manifolds*, Sem. Notes Aarhus Univ., vol. 1, 1982, pp. 116–119.

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